CHAPTER - 1 RELATIONS AND FUNCTIONS

- Empty relation is the relation R and X given by R = $\emptyset \subset X \times X$.
- Universal relation is the relation R and X given by R = X$\times X$.
- Reflexive relation R in X is a relation with (a , a) $\in R\ \forall\ a \epsilon X$.
- Transitive relation R in X is a relation satisfying (a , b) $\epsilon\ R\ implies\ (b, a)\ \epsilon\ R$.
- Symmetric relation R in X is a relation satisfying (a , b) $\epsilon\ R\ implies\ (b, c)\ \epsilon\ R$ implies that $(a, c)\ \epsilon\ R$
- Equivalence relation R in X is a relation which is reflexive, symmetric and transitive.
- Equivalence class [a] containing $a \in X$ for an equivalence relation R in X is the subset of X containing all elements b related to a.
- A function $f: X \to Y$ is one – one (or injective) if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2\ \forall\ x_1, x_2 \in X$.
- A function $f: X \to Y is\ onto$ (or surjective) if given any $y \in Y, \exists\ x\ \in X\ such\ that\ f(x) = y$.
- A function $f: X \to Y\ is\ one - one\ and\ onto\ (or\ bijective), if\ f\ is\ both\ one - one\ and\ onto.$
- The composition of functions $f: A \to B\ and\ g: B \to C$ is the function $gof: A \to C\ given\ by\ gof\ (x) = g\big(f(x)\big)\ \forall\ x \in A$.
- A function $f: X \to Y\ is\ invertible\ if\ \exists g: Y \to X\ such\ that\ gof = I_x$ and $fog = I_y$ is called inverse of f and denoted by f^{-1}
- A function $f: X \to Y\ is\ invertible\ if$ and only if f is one- one and onto
- Given a finite set X, a function $f: X \to X$ is one – one if and only if f is onto. This is the characteristic property of a finite set. This is not true for infinite set

- A binary operation* on a set A is a function * from $A \times A$ *to* A.
- An element $e \in X$ is the identify element for binary operation* $X \times X\ t \rightarrow\ X$

  If a*e = a = e*a $\forall\ a\ \in X$.

- An element $a \in X$ is invertible for binary operation*: $X \times X \rightarrow X$, if there exists $b \in X$ *such that* $a * b = e = b * a$ where, e is the identify for the binary operation*. The element b is called inverse of *and is denoted by* a^{-1}.
- An operation* on X is commutative if a*b = b * a $\forall\ a, b$ *in* X.
- An operation* on X is associative if $(a * b) * c = a * (b * c) \forall\ a, b, c$ *in* X.

CHAPTER - 2 INVERSE TRIGONOMETRIC FUNCTIONS

- The domains and range of inverse trigonometric functions are given in the following table:

Functions	Domain	Range(Principle Value Branches)
$y = sin^{-1}x$	[-1,1]	$[\frac{-\pi}{2}, \frac{\pi}{2}]$
$y = cos^{-1}x$	[-1,1]	$[0,\pi]$
$y = cosec^{-1}x$	R – (-1,1)	$[\frac{-\pi}{2}, \frac{\pi}{2}]$-{0}
$y = sec^{-1}x$	R – (-1,1)	$[0,\pi]$- $\{\frac{\pi}{2}\}$
$y = tan^{-1}x$	R	$(\frac{-\pi}{2}, \frac{\pi}{2})$
$y = tan^{-1}x$	R	$(0,\pi)$

- $sin^{-1}x$ should not be confused with $(sinx)^{-1}$. In fact $sinx)^{-1} = \frac{1}{\sin x}$ *and* similarly for other trigonometric functions.
- The value of an inverse trigonometric functions which lies in its principal value branch is called the Principle value of that inverse trigonometric functions.

For suitable values of domain, we have

- $y = sin^{-1}x \Rightarrow x = \sin y$
- $\sin(sin^{-1}x) = x$
- $sin^{-1}\frac{1}{x} = cosec^{-1}x$

- $cos^{-1}\frac{1}{x} = sec^{-1}x$
- $tan^{-1}\frac{1}{x} = cot^{-1}x$
- $sin^{-1}(-x) = -sin^{-1}x$
- $tan^{-1}x + cot^{-1}x = \frac{\pi}{2}$
- $tan^{-1}x + tan^{-1}y = tan^{-1}\frac{x+y}{1-xy}$
- $2tan^{-1}x = sin^{-1}\frac{2x}{1-x^2} = cos^{-1}\frac{1-x^2}{1+x^2}$
- $tan^{-1}x + tan^{-1}y = \pi + tan^{-1}\left(\frac{x+y}{1-xy}\right), xy > 1; x, y > 0$
- $x = \sin y \Rightarrow y = sin^{-1}x$
- $sin^{-1}(\sin x) = x$
- $cos^{-1}(-x) = \pi - cos^{-1}x$
- $cot^{-1}(-x) = \pi - cot^{-1}x$
- $sec^{-1}(-x) = \pi - sec^{-1}x$
- $tan^{-1}(-x) = tan^{-1}x$
- $cosec^{-1}(-x) = -cosec^{-1}x$
- $sin^{-1}x + cos^{-1}x = \frac{\pi}{2}$
- $cosec^{-1}x + sec^{-1}x = \frac{\pi}{2}$
- $2tan^{-1}x = tan^{-1}\frac{2x}{1-x^2}$

CHAPTER - 3 MATRICES

- A matrix is an ordered rectangular array of numbers or functions.
- A matrix having m rows and n columns is called a matrix of order $m \times n$
- $[a_{ij}]_{m\times 1}$ *is* a column matrix.
- $[a_{ij}]_{1\times n}$ *is* a row matrix.
- An $m \times n$ matrix is a square matrix if m = n.
- A = $[a_{ij}]_{m\times m}$ is a diagonal matrix if $a_{ij} = 0, when\ i \neq j$.
- A = $[a_{ij}]_{n\times n}$ is a scalar matrix if $a_{ij} = 0, when\ i \neq j, a_{ij} = k,$ (k is some constant), when $i = j$.
- A = $[a_{ij}]_{n\times n}$ is a identify matrix if $a_{ij} = 1, when\ i = j, a_{ij} = 0$, when $i \neq j$.

- A zero matrix has all its elements as zero.
- A = $[a_{ij}] = [b_{ij}]$ = B if (i) A and B are of some order, (ii) $a_{ij} = b_{ij}$ for all possible values of i and j.
- $kA = k[a_{ij}]_{m\times n} \quad = [\mathrm{k}(a_{ij})]_{m\times n}$
- $-A = (-1)A$
- $A - B = A + (-1)B$
- A+B = B+A
- (A + B) + C = A + (B + C), where A, B and C are of same order.
- $k(A + B) = kA + kB$, where A and B are of same order,k is constant.
- $(k + l)A = kA + lA, where\ k\ and\ l\ are\ constant.$
- $If\ A = \ [a_{ij}]_{m\times n}$ and B $= [b_{jk}]_{n\times p}$, then AB $=$ C $= [c_{ik}]_{m\times p}$, where $c_{ik}\sum_{j=1}^{n} a_{ij}b_{jk}$
- (i) A(BC) = (AB)C, (ii) A(B+C) = AB+AC, (iii) (A+B)C = AC+BC
- If $A = \ [a_{ij}]_{m\times n}\ then\ A'or\ A^T = \ [a_{ji}]_{n\times m}$
- (i) $(A')' = A$, (ii) $(kA)' = kA'$, (iii) $(A + B)' = A' + B'$, (iv) $(AB)' = B'A'$
- A is a symmetric matrix if $A' = A$
- A is a skew symmetric matrix if $A' = -A$
- Any square matrix can be represented as the sum of a symmetric and a skew symmetric matrix.
- Elementary operations of a matrix are as follow:
  $(\boldsymbol{i})R_i \leftrightarrow R_j or\ C_i \leftrightarrow C_j$
  $(\boldsymbol{ii})R_i \rightarrow kR_i or\ C_i \rightarrow kC_i$
  $(\boldsymbol{iii})R_i \rightarrow R_i + kR_j or\ C_i \rightarrow C_i + kC_j$
- If A and B are two square matrices such that AB = BA = I, then B is the inverse matrix of A and is denoted by A^{-1} and A is the inverse of B.
- Inverse of square matrix, if it exists, is unique.

CHAPTER - 4 DETERMINANTS

- Determinant of a matrix A = $[a_{11}]_{1\times 1}$ is given by $|a_{11}| = a_{11}$
- Determinant of a matrix A = $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ is given by
  $$|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$
- Determinant of a matrix A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is given by (expanding along R_1)

$$|A| = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

For any square matrix A, the $|A|$ satisfy following properties.

- $|A'| = |A|, where\ A' = transpose\ of\ A.$
- If we interchange ant two rows (or columns), then sign of determinant changes.
- If any two rows or any two columns are identical or proportional, then value of determinant is zero.
- If we multiply each element of a row or a column of a determinant by constant k, then value of determinant is multiplied by k.
- Multiplying a determinant by k means multiply elements of only one row by k.
- If A = $[a_{ij}]_{3\times3}, then\ |k.A| = |k^3||A|$
- If elements of a row or a column in a determinant can be expressed as sum of two or more elements, then the given determinant can be expressed as sum of two or more determinants.
- If to each element of a row or a column of a determinant the equimultiples of corresponding elements of other rows or columns are added, then value of determinant remains same.
- Area of a triangle with vertices (x_1, y_1),(x_2, y_2), $and (x_3, y_3)$ is given by

$$\Delta = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_2 & 1 \end{vmatrix}$$

- Cofactor of a_{ij} of given by $A_{ij} = (-1)^{i+j}M_{ij}$
- Value of determinant of a matrix A is obtained by sum of product of element of row (or a columns) with corresponding cofactors. For examples,

$$|A| = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}.$$

- If elements of one row(or column) are multiplied with cofactor of elements of any other row (or column), then their sum is zero. For example, $a_{11}A_{21} + a_{12}A_{22} + a_{13}A_{23} = 0$
- If A $= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ $, then\ adj$ A $= \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$ $, where\ A_{ij}$ is cofactor of a_{ij}.

- A $(adj\ A)= (adj\ A)A = |A|I, where\ A\ is\ square\ matrix\ of\ order\ n.$
- A square matrix A is said to be singular or non-singular according as $|A| = 0$ or $|A| \neq 0$.
- If AB = BA = I, where B is square matrix, then B is called inverse of A. Also $A^{-1} = B\ or\ B^{-1} = A\ and\ hence\ (A^{-1})^{-1} = A.$
- A square matrix A has inverse if and only if A is non-singular.
- $A^{-1} = \frac{1}{|A|}(adjA)$
- If $\begin{matrix} a_1x & b_1y & c_1z = d_1 \\ a_2x & b_2y & c_2z = d_2 \\ a_3x & b_3y & c_3z = d_3 \end{matrix}$

  Then these equations can be written as A X = B, where

  A = $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} and\ B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$
- Unique solution of equation AX = B is given by X = $A^{-1}B, where\ |A| \neq 0$

CHAPTER - 5 CONTINUITY AND DIFFERENTIABILITY

- A real valued function is **continuous** at a point in its domain if the limit of the function at that point equals the value of the function at that point. A function is continuous if it is continuous on the whole of its domain.
- Sum, difference, product and quotient of continuous functions are continuous. i.e., if $f\ and\ g$ are continuous functions, then

  $(f \pm g)(x)$ = f(x) $\pm\ g(x)\ is\ continuous.$

  $(f\ .\ g)(x) = f(x).g\ (x) is\ continuous.$
- Every differentiable function is continuous, but the converse is not true.
- Chain rule is rule to differentiate composites of functions. If $f = v\ o\ u, t = u(x)\ and\ if\ both\ \frac{dt}{dx}\ and\ \frac{dv}{dt}$ exist then,

$$\frac{df}{dx} = \frac{dv}{dt} \cdot \frac{dt}{dx}$$

- Following are some of the standard derivatives (in appropriate domains):

$$\frac{d}{dx}(sin^{-1}x) = \frac{1}{\sqrt{1-x^2}} \qquad \frac{d}{dx}(cos^{-1}\ x) = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(tan^{-1}x) = \frac{1}{1-x^2} \qquad \frac{d}{dx}(cot^{-1}\ x) = -\frac{1}{1-x^2}$$

$$\frac{d}{dx}(sec^{-1}x) = \frac{1}{x\sqrt{x^2-1}} \qquad \frac{d}{dx}(cosec^{-1}\ x) = -\frac{1}{x\sqrt{x^2-1}}$$

- Logarithmic differentiation is a powerful technique to differentiate functions of the form $f(x) = [u(x)]^{v\ (x)}$. Here both $f(x)$ and $u(x)$ need to be positive for this technique to make sense.
- **Rolle's Theorem:** If $f: [a, b] \rightarrow \boldsymbol{R}$ is continuous on [a, b] and differentiable on (a, b) such that $f(a) = f(b), then$ there exists some c in (a, b) such that $f'(c)$ = 0.
- **Mean Value Theorem:** If $f: [a, b] \rightarrow \boldsymbol{R}$ is continuous on [a, b] and differentiable on (a, b).$Then$ there exists some c in (a, b) such that $f'(c)\ \frac{f(b)-f(a)}{b-a}$

CHAPTER - 6 APPLICATION OF DERIVATIVES

- If quantity y varies with another quantity x, satisfying some rule $y = f(x)$, then $\frac{dy}{dx}$ $(or\ f'(x))$ represent the rate of change of y with respect to x and $\frac{dy}{dx}\ _{x=x_0}$ $(or\ f'(x_0))$ represent the rate of change of y with respect to x at $x = x_0$.
- If two variables x and y are varying with respect to another variable t, i.e., if $x = f(t)$ and $y = g(t)$, then by chain Rule $\frac{dy}{dx} = \frac{dy}{dt}/\frac{dx}{dt}, if\ \frac{dy}{dx} \neq 0$.
- A function f is said to be
  (a) increasing on an interval (a, b) if $x_1 < x_2\ in\ (a,\ b) \Rightarrow f(x_1) < f(x_2) fro\ all\ x_1, x_2\ \in (a, b)$.
  Alternatively, if $f'(x) \geq 0$ for each x in (a, b).
  (b) decreasing on (a, b) is $x_1 < x_2\ in\ (a,\ b) \Rightarrow f(x_1) > f(x_2) fro\ all\ x_1, x_2\ \in (a, b)$.
  (c) constant in $(a, b), if\ f(x) = c\ for\ all\ x\ \in (a,\ b)$ where c is constant.
- Let $y = f(x), \Delta x$ be a small increment in $x\ and\ \Delta y$ be the increment in y corresponding to the increment in $x, i.e. \Delta y = f(x + \Delta x) - f(x)$. Then dy given by
  $dy = f'(x)dx or\ dy = \left(\frac{dy}{dx}\right)\Delta x$. is a good approximation of Δy when $dx = \Delta x$ is relatively small and we denote it by $dy = \Delta y$.

CHAPTER - 7 INTEGRALS

- **Some Properties of indefinite integrals are as follows:-**
  1. $\int [f(x) + g(x)]dx = \int f(x)dx + \int g(x)dx$
  2. For any real number k, $\int k\ f(x)dx = k \int f(x)dx$
     More generally, if $f_1, f_2, f_3 \ldots., f_n$ are functions and $k_1, k_2, k_3 \ldots\ldots k_n$ are real number. Then $\int [k_1 f_1(x) + k_2 f_2(x) + .. + k_n f_n(x)]\ dx = k_1 \int f_1(x)\ dx + k_2 \int f_2(x)\ dx + \cdots + k_n \int f_n(x)\ dx$

- **Some standard integrals:**
  1. $\int x^n\ dx = \frac{x^{n+1}}{n+1} + C\ , n \neq -1. Particularly, \int dx = x + C$
  2. $\int \cos x\ \ dx = \ \sin x + C$
  3. $\int \sin x\ \ dx = \ -\cos x + C$
  4. $\int sec^2 x\ dx = \ \tan x + C$
  5. $\int cosec^2 x\ dx = \ -cotx\ \ +$C
  6. $\int secx\ tanx\ dx = secx + C$
  7. $\int cosecx. cotx\ dx = \ -cosecx + C$
  8. $\int \frac{dx}{\sqrt{1-x^2}} = \ \sin^{-1} x + C$
  9. $\int \frac{dx}{\sqrt{1-x^2}} = \cos^{-1} x + C$
  10. $\int \frac{dx}{1+x^2}\ = \ \tan^{-1} x + C$
  11. $\int \frac{dx}{1+x^2}\ = \ \cot^{-1} x + C$
  12. $\int e^x\ dx = \ e^x + C$
  13. $\int a^x\ dx = \ \frac{a^x}{\log a} + C$
  14. $\int \frac{dx}{x\sqrt{x^2-1}} = \ \sec^{-1} x + C$
  15. $\int \frac{dx}{x\sqrt{x^2-1}} = \ \csc^{-1} x + C$
  16. $\int \frac{1}{x}$dx $= \log x + C$

- **Integration by partial fractions:**

1. $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}, a \neq b$
2. $\frac{px+q}{(x-a)^2} = \frac{A}{(x-a)} + \frac{B}{(x-a)^2}$
3. $\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$
4. $\frac{px^2+qx+r}{(x-a0)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$
5. $\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$

Where $x^2 + bx + c$ can not be factorized further.

- **Integration by substitution:**

1. $\int \tan x\, dx = \log|secx| + C$
2. $\int cotx\, dx = \log|sinx| + C$
3. $\int sex\, dx = \log|secx + tanx| + C$
4. $\int cosecx\, dx = \log|cosecx - cotx| + C$

- **Integrals some special functions:**

1. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left|\frac{x-a}{x+a}\right| + C$
2. $\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\log\left|\frac{a+x}{a-x}\right| + C$
3. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
4. $\int \frac{dx}{\sqrt{x^2-a^2}} = \log\left|x + \sqrt{x^2-a^2}\right| + C$
5. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$
6. $\int \frac{dx}{\sqrt{x^2+a^2}} = \log\left|x + \sqrt{x^2+a^2}\right| + C$

- **Integration by Parts:**

1. For given functions f_1 and f_2 we have
2. $\int f_1(x).f_2(x)\, dx = f_1(x)\int f_2(x)\, dx - \int\left[\frac{d}{dx}f_1(x).\ \int f_2(x)dx\right]dx$. i. e. the integral of the product of two functions = first function × integral of the second function – integral of {differential coefficient of the first function × integral of the second function}. Care must be taken in choosing the first function and the second function. Obviously, we must take that function as the second function whose integral is well known to us.

$$\int e^x[f(x) + f'(x)]dx = \int e^x\, f(x)\ dx + C$$

- **Some special types of integrals:**

1. $\int\sqrt{x^2-a^2}\, dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\log\left|x + \sqrt{x^2-a^2}\right| + C$
2. $\int\sqrt{x^2+a^2}\, dx = \frac{x}{2}\sqrt{x^2+a^2} + \frac{a^2}{2}\log\left|x + \sqrt{x^2+a^2}\right| + C$
3. $\int\sqrt{a^2-x^2}\, dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{b} + C$
4. Integrals of the types $\int\frac{dx}{x^2+bx+c}$ *or* $\int\frac{dx}{\sqrt{x^2+bx+c}}$ can be transformed into standard from by expressing

$$ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x + \frac{c}{a}\right] = a[\left(x + \frac{b}{2a}\right)^2 + (\frac{c}{a} - \frac{b^2}{4a^2})]$$

5. Integrals of the types $\int\frac{px+q\, dx}{ax^2+bx+c}$ *or* $\int\frac{px+q\, dx}{\sqrt{ax^2+bx+c}}$ can be transformed into standard form by expressing $px + q = A\frac{d}{dx}(ax^2 + bx + c) + B = A(2ax + b) + B$, where A and B are determined by comparing coefficients on both sides.

- We have define $\int_a^b f(x)\ dx$ as the area of the region bounded by the curve $y = f(x), a \leq x \leq b$, the $x - axis$ and the ordinates $x = a$ and $x = b$. Let x be a given point in [a, b]. Then $\int_a^b f(x)\ dx$ represent the **Area function** A(x).

- **First fundamental theorem of integral calculus:**

  Let the area function be defined by A(x) = $\int_a^x f(x)\ dx$ for all x$\geq a$, where the f is assumed to be continuous on [a, b]. Then A'(x) = $f(x)$ for all x ∈ [a, b].

- **Second fundamental theorem of integral calculus:**

  Let f be a continuous function of x defined on the closed interval [a, b] and let F be another function such that $\frac{d}{dx}F(x) = f(x)$ for all x in the domain of f, then

  $\int_a^b f(x)dx = [F(x) + C]_a^b = F(b) - F(a)$.

  This is called the definite integral of f over the range [a, b], where a and b are called the limits of integration, a being the lower limit and b the upper limit.

CHAPTER - 8 APPLICATION OF INTEGRALS

- The area of the region bounded by the curve $y = f(x), x - axis$ and the lines $x = a$ and $x = b$ (b > a) is given by the formula: $Area = \int_a^b y\ dx = \int_a^b f(x)dx$.
- The area of the region bounded by the curve $x = \emptyset\ (y), y - axis$ and the lines $y = c$, $y = d$ is given by the formula: $Area = \int_c^d x\ dy = \int_c^d \emptyset\ (y)dy$.
- The area of the region enclosed between two curves $y = f(x), y = g(x)\ and$ the lines $x = a, x = b$ is given by the formula ,

  $Area = \int_b^a [f(x) - g(x)]dx$, where $f(x) \geq g(x) in\ [a, b]$
- If $f(x) \geq g(x) in\ [a, c]$ and $f(x) \leq g(x) in\ [c, b], a < c < b$, then

  $$Area = \int_a^c [f(x) - g(x)]dx + \int_c^b [g(x) - f(x)]dx.$$

CHAPTER - 9 DIFFRENTIAL EQUATIONS

- An equation involving derivatives of the dependent variable with respect to independent variable (variables) is known as a differential equation.
- Order of a differential equation is the order of he highest order derivatives occurring in the differential equations.
- Degree of a differential is defined if it is a polynomial equation in its derivatives.
- Degree (when defined) of a differential equation is the highest power (positive integer only) of the highest order derivative in it.
- A function which satisfies the given differential equation is called its solution. The solution which contains as many arbitrary constants as the order of the differential equation is called a general solution and the solution free from arbitrary constants is called particular solution.
- A differential equation which can be expressed in the form $\frac{dy}{dx} = f(x, y) or\ \frac{dy}{dx} = g(x, y)$ where, ○ $f(x, y) and\ g(x, y)$ are homogenous functions of degree zero is called a homogenous differential equation.
- A differential equation of the form $\frac{dy}{dx} + Py = Q$, where P and Q are constants or functions of x only is called a first order linear differential equation.

CHAPTER - 10 VECTOR ALGEBRA

- Position vector of a point P(x, y, z) is given as $\overrightarrow{OP}(=\vec{r})= x\hat{\imath}+y\hat{\jmath}+z\hat{k}$, and its magnitude by $\sqrt{x^2+y^2+z^2}$.
- The scalar components of a vector are its direction ratios, and represent its projections along the respective axes.
- The magnitude (r), direction ratios (a, b, c) and direction cosines (l, m, n) of any vector are related as: $l=\frac{a}{r},\ m=\frac{b}{r},\ n=\frac{c}{r}$
- The vector sum if the three sides of a triangle taken in order is $\vec{0}$.
- The vector sum of two coinitial vectors is given by the diagonal of the parallelogram whose adjacent sides are the given vectors.
- The multiplication of a given vector by a scalar λ, changes the magnitude of the vector by the multiple $|\lambda|$, and keeps the direction same according as the value of λ is positive.
- For a given vector $\vec{a}$, the vector $\hat{a}=\frac{\vec{a}}{|\vec{a}|}$ gives the unit vector in the direction of $\vec{a}$.
- The position vector of a point R dividing a line segment joining the points P and Q whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, in the ratio m:n

  (i) internally, is given by $\frac{n\vec{a}+m\vec{b}}{m+n}$ (ii) externally, is given by $\frac{m\vec{b}-n\vec{a}}{m-n}$
- The scalar product of two given vectors $\vec{a}$ and $\vec{b}$ having angle θ between them is defined as $\vec{a}\,.\,\vec{b}=|\vec{a}||\vec{b}|\cos\theta$

  Also, when $\vec{a}\,.\,\vec{b}$ is given, the angle $'\theta'$ between the vectors $\vec{a}$ and $\vec{b}$ may be determined by $\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$
- If θ is the angle between two vectors $\vec{a}$ and $\vec{b}$, then their cross product is given as

  $$\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n},$$

  when $\hat{n}$ is a unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$, such that $\vec{a},\vec{b},\hat{n}$ from right handed system of coordinates axes.
- If we have two vectors $\vec{a}$ and $\vec{b}$, given in component from as $\vec{a}=a_1\hat{\imath}+a_2\hat{\jmath}+a_3\hat{k}$ and $\vec{b}=b_1\hat{\imath}+b_2\hat{\jmath}+b_3\hat{k}$ and λ any scalar then $\vec{a}+\vec{b}=(a_1+b_1)\hat{\imath}+(a_2+b_2)\hat{\jmath}+(a_3+b_3)\hat{k}$

  $$\lambda\vec{a}=(\lambda a_1)\hat{\imath}+(\lambda a_2)\hat{\jmath}+(\lambda a_3)\hat{k}$$

  $$\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$$

  and $\vec{a}\times\vec{b}=\begin{vmatrix}\hat{\imath} & \hat{\jmath} & \hat{k}\\ a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\end{vmatrix}$

CHAPTER - 11 THREE DIMENSIONAL GEOMETRY

- Direction cosines of a line are the cosines of the angels made by the line with the positive directions of the coordinates axes.
- If l, m, n are the direction cosines of a line, then $l^2+m^2+n^2=1$.

- Direction cosines of a line joining two points P (x_1, y_1, z_1) and Q (x_2, y_2, z_2) are $\frac{x_2-x_1}{PQ}, \frac{y_2-y_1}{PQ}, \frac{z_2-z_1}{PQ}$
  where $PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$
- Direction ratios of a line are the numbers which are proportional to the direction cosines of a line.
- If l, m, n are the direction cosines and a, b, c are the direction ratios of a line then

$$l = \frac{a}{\sqrt{a^2 + b^2 + c^2}}\ ; m = \frac{b}{\sqrt{a^2 + b^2 + c^2}}\ ; \ n = \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

- Skew lines are lines in space which are neither parallel nor intersecting. They lie in different planes.
- Angle between skew lines is the angle between two intersecting lines drawn from any point (preferable through the origin) parallel to each of the skew lines.
- If l_1, m_1, n_1 and $l_2, m_2\ n_2$ are the direction cosines of two lines; and θ is the acute angle between the two lines; then $\cos\theta = |l_1 l_2 + m_1 m_2 + n_1 n_2|$
- If a_1, b_1, c_1 and a_2, b_2, c_2 are the direction ratios of two lines and θ is the acute angle between the two line then $\cos\theta = \left|\frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1{}^2 + b_1{}^2 + c_1{}^2}\ \sqrt{a_2{}^2 + b_2{}^2 + c_2{}^2}}\right|$
- Vector equation of a line that passes through the given point whose position vector is $\vec{a}$ and parallel to given vector $\vec{b}$ is $\vec{r} = \vec{a} + \lambda\vec{b}$
- Equation of a line through a point (x_1, y_1, z_1) and having direction cosines l, m, n is $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$
- If θ is a acute angle between $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \lambda\overrightarrow{b_2}$, then $\cos\theta = \left|\frac{\overrightarrow{b_1}.\overrightarrow{b_2}}{|\overrightarrow{b_1}|\ |\overrightarrow{b_2}|}\right|$
- If $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_1}{l_2} = \frac{y-y_1}{m_2} = \frac{z-z_1}{n_2}$
  are the equations of two lines, then the acute angle between the two lines is given by $\cos\theta = ||l_1 l_2 + m_1 m_2 + n_1 n_2||$.
- Shortest distance between two skew lines is the line segment perpendicular to both the lines.
- Shortest distance between $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\overrightarrow{b_2}$ is $\left|\frac{(\overrightarrow{b_1}\times\overrightarrow{b_2})(\overrightarrow{a_2}-\overrightarrow{a_1})}{|\overrightarrow{b_1}\times\overrightarrow{b_2}|}\right|$
- Shortest distance between the lines : $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_1}{a_2} = \frac{y-y_1}{b_2} = \frac{z-z_1}{c_2}$ is

$$\frac{\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\sqrt{(b_1 c_2 - b_2 c_1)^2} + \sqrt{(c_1 a_2 - c_2 a_1)^2} + \sqrt{(a_1 b_2 - a_2 b_1)^2}}$$

- Distance between parallel lines $\vec{r} = \overrightarrow{a_1} + \lambda\vec{b}$ and $\vec{r} = \overrightarrow{a_2} + \mu\vec{b}$ is $\left|\frac{\vec{b}\times(\overrightarrow{a_2}-\overrightarrow{a_1})}{|\vec{b}|}\right|$
- In the vector form, equation of a plane which is at a distance *d* from the origin, and $\hat{n}$ is
- Cartesian equation of a line that passes through two points (x_1, y_1, z_1) and (x_2, y_2, z_2) is $\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y-y_1} = \frac{z-z_1}{z_2-z_1}$.

the unit vector normal to the plane through the origin is $\hat{r}.\hat{n} = d$.

- Equation of a plane which is at a distance of d from the origin and the direction cosines of the normal to the plane as l, m, n is $lx + my + nz = d$.
- The equation of a plane through a point whose position vestor is $\vec{a}$ and perpendicular to the vector $\vec{N}$ is $(\vec{r} - \vec{a}).\vec{N} = 0$.
- Equation of a plane perpendicular to a given line with direction ratios A, B, C and passing through a given point (x_1, y_1, z_1) is $A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$
- Equation of a plane passing through three non – collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) is $\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$.
- Vector equation of a plane that contains three non collinear points having position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is $(\vec{r} - \vec{a}).[(\vec{b} - \vec{a}) \times \overrightarrow{(c - \vec{a})}] = 0$
- Equation of a plane that cuts the coordinates axes at $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ is
$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$
- Vector equation of a plane that passes through the intersection of planes $\vec{r}.\overrightarrow{n_1} = d_1$ and $\vec{r}.\overrightarrow{n_2} = d_2$ is $\vec{r}.(\overrightarrow{n_1} + \lambda\overrightarrow{n_2}) = d_1 + \lambda d_2$ where λ is any nonzero constant.
- Cartesian equation of a plane that passes through the intersection of two given planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is
$$(A_1x + B_1y + C_1z + D_1) + \lambda(A_2x + B_2y + C_2z + D_2) = 0$$
- Two lines $\vec{r} = \overrightarrow{a_1} + \lambda\overrightarrow{b_1}$ and $\vec{r} = \overrightarrow{a_2} + \mu\overrightarrow{b_2}$ are coplanar if $(\overrightarrow{a_2} - \overrightarrow{a_1}).(\overrightarrow{b_1} \times \overrightarrow{b_2}) = 0$
- In the Cartesian form two lines= $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are coplanar if $\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$
- In the vector form, if θ is the angle between the two planes $\vec{r}.\overrightarrow{n_1} = d_1$ and $\vec{r}.\overrightarrow{n_2} = d_2$, then $\theta = \cos^{-1}\frac{|\overrightarrow{n_1}.\overrightarrow{n_2}.|}{|\overrightarrow{n_1}|.|\overrightarrow{n_2}|}$
- The angle ϕ between the lines $\vec{r} = \vec{a} + \lambda\vec{b}$ and the plane $\vec{r}\,\hat{n} = d$ is $\sin\phi = \left|\frac{\vec{b}\,\hat{n}}{|\vec{b}|\,|\hat{n}|}\right|$
- The angle θ between the planes $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ is given by $\cos\theta = \left|\frac{A_1A_2 + B_1B_2 + C_1C_2}{\sqrt{A_1{}^2 + B_1{}^2 + C_1{}^2}\,\sqrt{A_2{}^2 + B_2{}^2 + C_2{}^2}}\right|$
- The distance of a point whose position vectors is a $\vec{a}$ from the plane $\vec{r}\,\hat{n} = d$ is $|d - \vec{a}\,\hat{n}|$
- The distance from a point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is $\left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$

- **Linear Programming Problems –** Problems which concern with finding the minimum or maximum value of a linear function Z (called objective function) of several variables (say x and y), subject to certain conditions that the variables are non-negative and satisfy a set of linear inequalities (called linear constraints) are known as linear programming problems.
- **Feasible Region –** The common region determined by all the constraints including non–negative constraints $x, y \geq 0$ of linear programming problem is known as feasible region (or solution region) If we shade the region according to the given constraints, then the shaded areas is the feasible region which is the common area of the regions drawn under the given constraints.
- Points within and on the boundary of the feasible region represent feasible solutions of the constraints.
  Any point outside the feasible region is an **infeasible solution.**
- Any point in the feasible region that gives the optimal value (maximum or minimum) of the objective function is called an **optimal solution.**
- **Theorem 1** – Let R be the feasible region (convex polygon) for a linear programming problem and let $Z = ax + by$ be the objective function. When Z has an optimal value (maximum or minimum), where the variables x and y are subject to constraints described by linear inequalities, the optimal value must occur at a corner point of the feasible region.
- **Theorem 2** – Let R be the feasible region for a linear programming problem, and let Z = ax + by be the objective function. If R is bounded them the objective function Z has both maximum and minimum value on R and each of these occurs at a corner point of R.
- **Corner point method** for solving a linear programming problem. The method comprises of the following steps:-
  (i) Find the feasible region of the linear programming problem and determine its corner points (vertices).
  (ii) Evaluate the objective function $Z = ax + by$ at each corner point. Let M and m respectively be the largest and smallest values at these points.
  (iii) If the feasible region is bounded, M and m respectively are the maximum and minimum values of the objective function.
  If the feasible region is unbounded, then
  (i) M is the maximum value of the objective function, if the open half plane determined by $ax + by > M$ has no point in common with the feasible region. Otherwise, the objective function has no maximum value.
  (ii) m is the maximum value of the objective function, if the open half plane determined by $ax + by < m$ has no point in common with the feasible region. Otherwise, the objective function has no minimum value.

- The conditional probability of an event E, given the occurrence of the event F is given by $P(E|F) = \frac{P(E \cap F)}{P(F)}, P(F) \neq 0$
- $0 \leq P(E|F) \leq 1, P(E'|F) = 1 - P(E|F)$
- $P\big((E \cup F)\big|G\big) = P(E|G) + (F|G) - P\big((E \cap F)\big|G\big)$
- $P(E \cap F) = P(E)P(F|E), P(E) \neq 0;\ P(E \cap F) = P(F)P(E|F), P(F) \neq 0$
- If E and F are independent ,then $P(E \cap F) = P(E)\,P(F)$
- $P(E|F) = P(E), P(F) \neq 0;\ P(F|E) = P(F), P(E) \neq 0$
- **Theorem of total probability**
- Let $(E_1, E_2 \ldots, E_n)$ be a partition of a sample space and suppose that each of $E_1, E_2 \ldots, E_n$ has non zero probability. Let A be any event associated with S, then P(A) = P(E_1) $P(A|E_1)$ + P(E_2) $P(A|E_2)$ + ………… + P(E_n) $P(A|E_n)$
- **Bayes' theorem** If $E_1, E_2, \ldots\ldots E_3$ are events which constitute a partition of sample space S, i.e. $E_1, E_2, \ldots\ldots E_n$ are pairwise disjoint and $E_1 \cup E_2 \cup \ldots \ldots \cup E_n = S$ and A be any event with nonzero probability, then $P(E_i|A) = \frac{P(E_i)P(A|E_i)}{\sum_{j=1}^{n} P(E_j)\, P(A|E_j)}$
- A random variable is a real valued function whose domain is the sample space of a random experiment.
- The probability distribution of a random variable X is the system of numbers

X	: x_1	x_2	…….	x_n
P(X)	: p_1	p_2	…….	p_n

  Where, $p_i > 0, \sum_{i=1}^{n} p_i = 1\ \ i = 1, 2, \ldots., n$
- Let X be a random variable whose possible values $x_1, x_2, x_3, \ldots, x_n$ occur with probabilities $p_1, p_2, p_3, \ldots, p_n$respectively. The mean of X, denoted by μ, is the number $\sum_{i=1}^{n} x_i\, p_i$

  The mean of random variable X is also called the expectation of X, denoted by E(X).
- Let X be a random variable whose possible values $x_1, x_2, \ldots., x_n$ occur with probabilities $p(x_1), p(x_2) \ldots.. p(x_n)$

  Let $\mu = E(X)$ be the mean of X. The variance of X, denoted by Var (X) or ${\sigma_x}^2$, is defined as ${\sigma_x}^2 = Var\ (X) = \sum_{i=1}^{n}(x_i - \mu)^2\, p(x_i)$ or equivalently number ${\sigma_x}^2 = E\ (X - \mu)^2$

  The non-negative number $\sigma_x = \sqrt{Var\ (X)} = \sqrt{\sum_{i=1}^{n}(x_i - \mu)^2\, p(x_i)}$ is called the standard deviation of the random variable X.
- Var (X) = E(X^2) $- [E(X)]^2$
- For binomial distribution B(n, p), P(X = x) = ${}^{n}C_r\, q^{n-x} p^x, x = 0, 1, \ldots\ldots, n$