Session 9

FACTORIAL DESIGN

DEFINITIONS AND PRINCIPLES

- The experiment involves two or more factors

  Consider two factors A and B

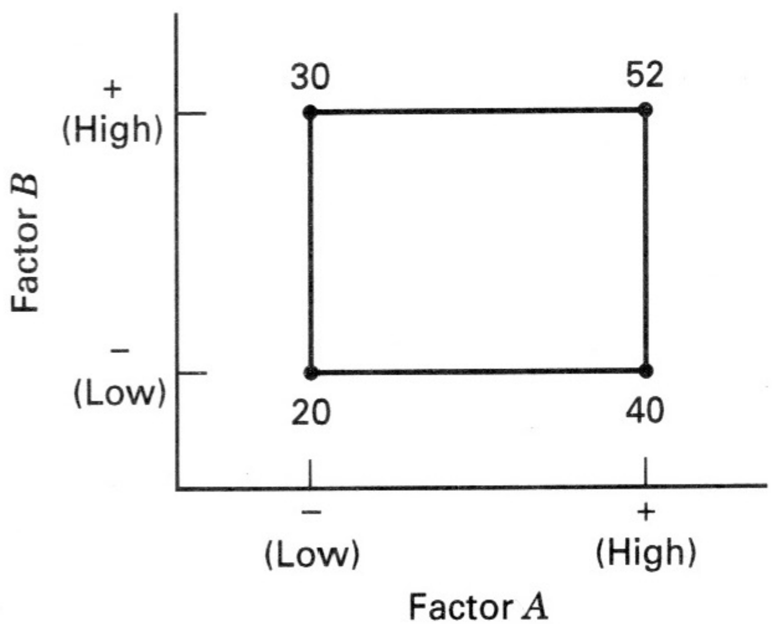
  - Factor A has a levels
  - Factor B has b levels

  $\Rightarrow ab$ treatment combinations in one replication of the experiment

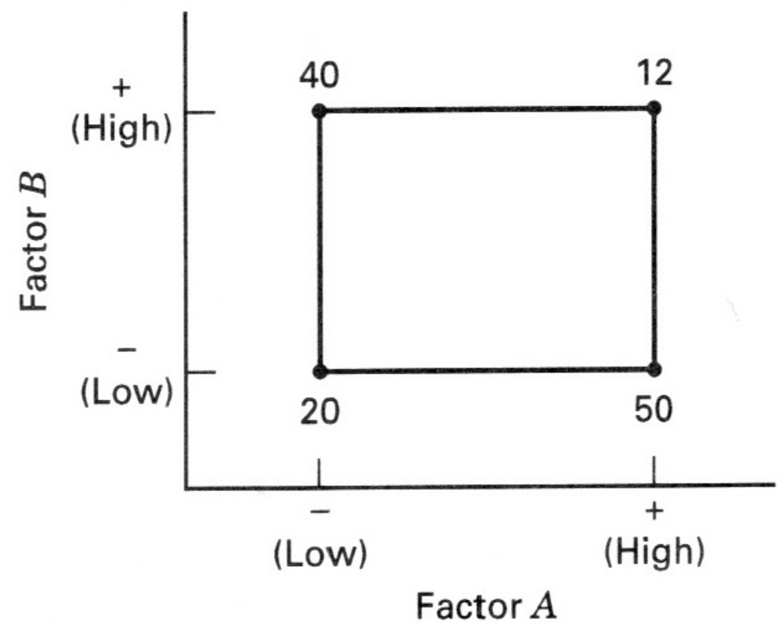
DEFINITIONS AND PRINCIPLES

Main effect:

The change in response due to change in the level of the factor



NO INTERACTION



INTERACTION

DEFINITIONS AND PRINCIPLES

Example: Main effect of factors A and B (in case of no interaction)

$$A = \frac{40+52}{2} - \frac{20+30}{2} = 21; B = \frac{30+52}{2} - \frac{20+40}{2} = 11$$

Interaction:

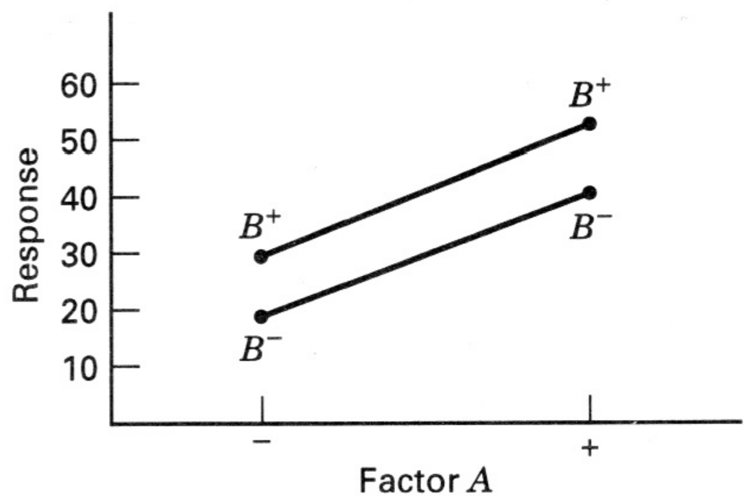
The difference in response between the levels of one factor is not the same at all levels of other factors

DEFINITIONS AND PRINCIPLES

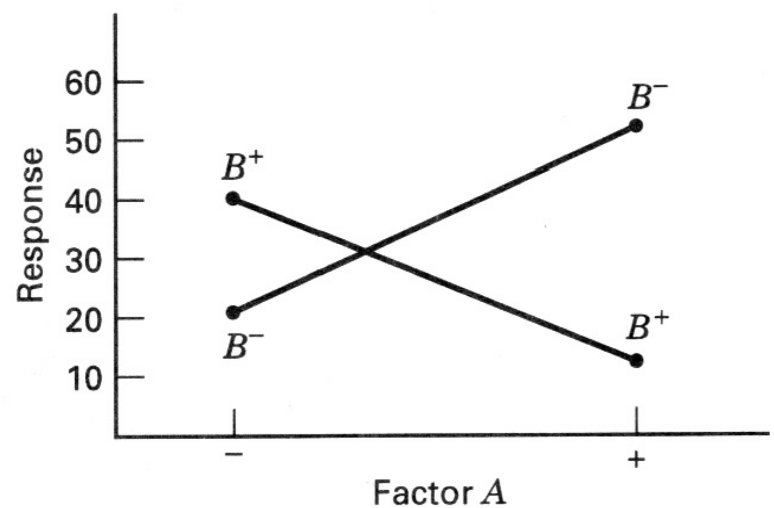
<u>*Example*</u>:

The main effect of factor A at low level of B: $A = 50 - 20 = 30$
The main effect of factor A at high level of B: $A = 12 - 40 = -28$

$$\Rightarrow \text{Interaction effect: } AB = \frac{(-28)-(30)}{2} = -29$$



NO INTERACTION



INTERACTION

DEFINITIONS AND PRINCIPLES

Advantage of factorial designs

1. Reduce the required number of observations needed

Consider two factors with two levels: $\left(A^{+}, A^{-}\right);\left(B^{+}, B^{-}\right)$

Factorial Experiment is not used:

- Number of combinations needed to detect effects of A and B: 3

$$A^{-}B^{+}, A^{-}B^{-}, A^{+}B^{-}$$

- Number of observations in each combination: *at least* 2 (due to error)

$\Rightarrow$ At least 6 observations are needed.

DEFINITIONS AND PRINCIPLES

With Factorial Experiment:

- Number of combinations needed to detect effects of A and B: 4

$$A^-B^+, A^-B^-, A^+B^-, A^+B^+$$

- Number of observations in each combination: *at least* 1 because two estimates of A (and B) have been made:

$$A^+B^- - A^-B^-, A^+B^+ - A^-B^+$$

$\Rightarrow$ Only 4 observations are needed.

2. Detect the interaction effect correctly

$$\boxed{A^-B^+, A^+B^- \gg A^-B^- \Rightarrow A^+B^+ : \text{the best!}}$$

This conclusion may be wrong!

TWO-FACTOR FACTORIAL DESIGN

Example: Lifetime of battery

Factors: *Plate Material* and *Environmental Temperature*

Material Type	Temperature (^{0}F)					
	15		70		125	
1	130	155	34	40	20	70
	74	180	80	75	82	58
2	150	188	136	122	25	70
	159	126	106	115	58	45
3	138	110	174	120	96	104
	168	160	150	139	82	60

Problems:

1. Effect of material type and temperature on lifetime
2. Choice of material such that the effect of temperature can be ignored $\Rightarrow$ *Robust Design*!

TWO-FACTOR FACTORIAL DESIGN

Structure of Completely Randomized Design

General Arrangement for a Two-Factor Factorial Design

		Factor B 1	2	...	b
Factor A	1	$y_{111}, y_{112}, \ldots, y_{11n}$	$y_{121}, y_{122}, \ldots, y_{12n}$		$y_{1b1}, y_{1b2}, \ldots, y_{1bn}$
	2	$y_{211}, y_{212}, \ldots, y_{21n}$	$y_{221}, y_{222}, \ldots, y_{22n}$		$y_{2b1}, y_{2b2}, \ldots, y_{2bn}$
	⋮				
	a	$y_{a11}, y_{a12}, \ldots, y_{a1n}$	$y_{a21}, y_{a22}, \ldots, y_{a2n}$		$y_{ab1}, y_{ab2}, \ldots, y_{abn}$

y_{ijk} : observed response for the k^{th} replication $(k=1,2,\ldots,n)$ when

$$\begin{cases} A: & \text{at the } i^{th} \text{ level } (i=1,2,\ldots,a) \\ B: & \text{at the } j^{th} \text{ level } (j=1,2,\ldots,b) \end{cases}$$

TWO-FACTOR FACTORIAL DESIGN

- The order of the abn observations should be *random*.

Effect Model:

$$\boxed{y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1, 2\ldots., a \\ j = 1, 2, \ldots, b \\ k = 1, 2, \ldots, n \end{cases}}$$

μ: overall mean

τ_i: effect of the ith level of row factor A

β_j: effect of the jth level of column factor B

$(\tau\beta)_{ij}$: interaction effect between τ_i and β_j

ε_{ijk}: NID$(0, \sigma^2)$ random error term

TWO-FACTOR FACTORIAL DESIGN

Note:

$$\sum_{i=1}^{a} \tau_i = 0 \qquad \sum_{j=1}^{b} \beta_j = 0$$

$$\sum_{i=1}^{a} (\tau\beta)_{ij} = \sum_{j=1}^{b} (\tau\beta)_{ij} = 0$$

The Mean Model:

$$\boxed{y_{ijk} = \mu_{ij} + \varepsilon_{ijk} \quad \begin{cases} i = 1, 2 \dots, a \\ j = 1, 2, \dots, b \\ k = 1, 2, \dots, n \end{cases}}$$

in which $\mu_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij}$: Mean of the ijth cell.

TWO-FACTOR FACTORIAL DESIGN

Hypotheses of Interest:

- Testing the equality of row treatment effects:

$$\boxed{\begin{array}{l} H_0: \tau_1 = \tau_2 = \ldots = \tau_a = 0 \\ H_1: \tau_i \neq 0 \text{ for at least one } i \end{array}}$$

- Testing the equality of column treatment effects:

$$\boxed{\begin{array}{l} H_0: \beta_1 = \beta_2 = \ldots = \beta_b = 0 \\ H_1: \beta_j \neq 0 \text{ for at least one } j \end{array}}$$

- Testing the interaction between row and column treatments:

$$\boxed{\begin{array}{l} H_0: (\tau\beta)_{ij} = 0 \ \forall i, j \\ H_1: \text{at least one } (\tau\beta)_{ij} \neq 0 \end{array}}$$

TWO-FACTOR FACTORIAL DESIGN

Notation:

$$y_{i..} = \sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{i..} = \frac{y_{i..}}{bn}$$

$$y_{.j.} = \sum_{i=1}^{a}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{.j.} = \frac{y_{.j.}}{an}$$

$$y_{ij.} = \sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{ij.} = \frac{y_{ij.}}{n}$$

$$y_{...} = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk} \qquad \bar{y}_{...} = \frac{y_{...}}{abn}$$

TWO-FACTOR FACTORIAL DESIGN

<u>Sums of Squares Definitions</u>

- Total corrected sum of squares = Sum of squares due to row A + Sum of squares due to column factor B + Sum of square to interaction + Error sum of square

$$\boxed{SS_T = SS_A + SS_B + SS_{AB} + SS_E}$$

in which:

$$\boxed{SS_A = bn\sum_{i=1}^{a}\left(\bar{y}_{i..} - \bar{y}_{...}\right)^2}$$

$$\boxed{SS_B = an\sum_{j=1}^{b}\left(\bar{y}_{.j.} - \bar{y}_{...}\right)^2}$$

$$\boxed{SS_{AB} = n\sum_{i=1}^{a}\sum_{j=1}^{b}\left(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}\right)^2}$$

$$\boxed{SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk} - \bar{y}_{...}\right)^2}$$

$$\boxed{SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n}\left(y_{ijk} - \bar{y}_{ij.}\right)^2}$$

TWO-FACTOR FACTORIAL DESIGN

- Computing Formulas

$$\boxed{SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{\dots}^2}{abn}}$$

$$\boxed{SS_A = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{\dots}^2}{abn}}$$

$$\boxed{SS_B = \frac{1}{an}\sum_{j=1}^{b} y_{.j.}^2 - \frac{y_{\dots}^2}{abn}}$$

$$\boxed{SS_{AB} = SS_{Subtotals} - SS_A - SS_B} \quad \text{with} \quad \boxed{SS_{Subtotals} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{y_{\dots}^2}{abn}}$$

$$\boxed{SS_E = SS_T - \left(SS_A + SS_B + SS_{AB}\right) = SS_T - SS_{Subtotals}}$$

TWO-FACTOR FACTORIAL DESIGN

Effect	Degrees of Freedom
A	$(a-1)$
B	$(b-1)$
AB	$(a-1)(b-1)$
Error	$ab(n-1)$
Total	$abn-1$

TWO-FACTOR FACTORIAL DESIGN

- If two factors are fixed then:

$$E[MS_A] = E\left[\frac{SS_A}{a-1}\right] = \sigma^2 + \frac{bn\sum_{i=1}^{a}\tau_i^2}{a-1}$$

$$E[MS_B] = E\left[\frac{SS_B}{b-1}\right] = \sigma^2 + \frac{an\sum_{j=1}^{b}\beta_j^2}{b-1}$$

$$E[MS_{AB}] = E\left[\frac{SS_{AB}}{(a-1)(b-1)}\right] = \sigma^2 + \frac{n\sum_{i=1}^{a}\sum_{j=1}^{b}(\tau\beta)_{ij}^2}{(a-1)(b-1)}$$

$$E[MS_E] = E\left[\frac{SS_E}{ab(n-1)}\right] = \sigma^2$$

TWO-FACTOR FACTORIAL DESIGN

Statistical Analysis

If the null hypotheses of no difference in levels of factor A, B and of no interaction effect are true, the ratios (the test statistics):

$$F_0 = \frac{MS_A}{MS_E} \qquad F_0 = \frac{MS_B}{MS_E} \qquad F_0 = \frac{MS_{AB}}{MS_E}$$

are distributed as $F_{(a-1),ab(n-1)}$, $F_{(b-1),ab(n-1)}$ & $F_{(a-1)(b-1),ab(n-1)}$, respectively.

Rejection Criteria:

$$\boxed{F_0 = \frac{MS_A}{MS_E} > F_{\alpha,(a-1),ab(n-1)}} \qquad \boxed{F_0 = \frac{MS_B}{MS_E} > F_{\alpha,(b-1),ab(n-1)}}$$

$$\boxed{F_0 = \frac{MS_{AB}}{MS_E} > F_{\alpha,(a-1)(b-1),ab(n-1)}}$$

TWO-FACTOR FACTORIAL DESIGN

ANOVA Table – Two-Factors Factorial Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
A Treatment	SS_A	$a-1$	MS_A	$F_0 = \frac{MS_A}{MS_E}$
B Treatment	SS_B	$b-1$	MS_B	$F_0 = \frac{MS_B}{MS_E}$
Interaction	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$F_0 = \frac{MS_{AB}}{MS_E}$
Error	SS_E	$ab(n-1)$	MS_E	
Total	SS_T	$abn-1$		

TWO-FACTOR FACTORIAL DESIGN

Example: The Battery Design Experiment

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{n} y_{ijk}^2 - \frac{y_{...}^2}{abn} = 77646.97$$

$$SS_{Material} = \frac{1}{bn}\sum_{i=1}^{a} y_{i..}^2 - \frac{y_{...}^2}{abn} = 10683.72$$

$$SS_{Temperature} = \frac{1}{an}\sum_{j=1}^{b} y_{.j.}^2 - \frac{y_{...}^2}{abn} = 39118.72$$

$$SS_{Interaction} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij.}^2 - \frac{y_{...}^2}{abn} - SS_{Material} - SS_{Temperature} = 9613.78$$

$$SS_E = SS_T - \left(SS_{Material} + SS_{Temperature} + SS_{Interaction}\right) = 18230.75$$

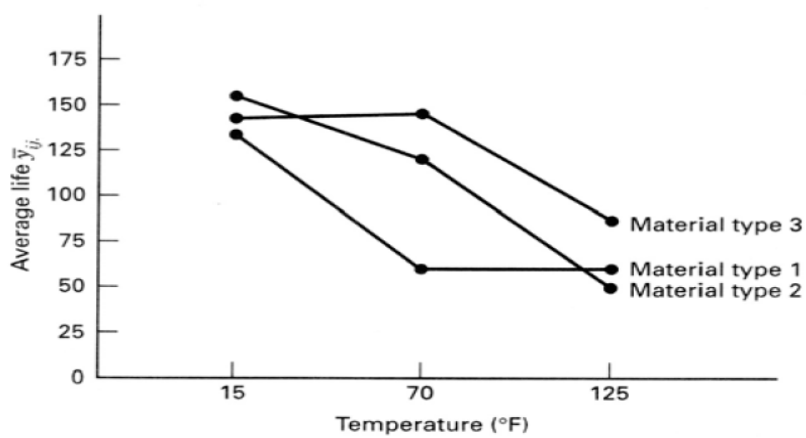
TWO-FACTOR FACTORIAL DESIGN

Source of Variation	Sum of Squares	DF	Mean Square	F_0	p-value
Material types	10683.72	2	5341.86	7.91	0.0020
Temperature	39118.72	2	19559.36	28.97	0.0001
Interaction	9613.78	4	2403.44	3.56	0.0186
Error	18230.75	27	675.21		
Total	77646.97	35			

Conclusions:

$F_{0.05,4,27} = 2.73 \Rightarrow$ There exists a significant interaction

$F_{0.05,2,27} = 3.35 \Rightarrow$ Effects of material & temperature are significant



Material type 3 seems to give the *best result*.

TWO-FACTOR FACTORIAL DESIGN

Multiple Comparisons

- The multiple comparison methods discussed before can be used to discover differences between row or column means when ANOVA indicates that these means are differed.

Tukey's Test

If interaction effect is significant, one factor should be *fixed* to investigate the difference in means of the other factor

Example: The Battery Design Experiment

Fix temperature at the second level (70^0F), then

$$\bar{y}_{12.} = 57.25 \qquad \text{(material type 1)}$$
$$\bar{y}_{22.} = 119.75 \qquad \text{(material type 2)}$$
$$\bar{y}_{32.} = 145.75 \qquad \text{(material type 3)}$$

TWO-FACTOR FACTORIAL DESIGN

$$T_{0.05} = q_{0.05}(3,27)\sqrt{\frac{MS_E}{n}} = 3.50\sqrt{\frac{675.21}{4}} = 45.47$$

Comparison results:

$$\begin{aligned}
&\text{3 vs 1:} \quad && 145.75 - 57.25 = 88.50 > T_{0.05} \\
&\text{3 vs 2:} \quad && 145.75 - 119.75 = 26.00 < T_{0.05} \\
&\text{2 vs 1:} \quad && 119.75 - 57.25 = 62.50 > T_{0.05}
\end{aligned}$$

$\Rightarrow$ At 70^0F, the mean battery life is the same for material types 2&3 and the mean battery life for material type 1 is lower.

TWO-FACTOR FACTORIAL DESIGN

Model Adequacy Checking

- The adequacy of the model can be checked by use of *Residual Analysis* as before.

Estimating the Model Parameters

$$\boxed{\begin{aligned}
&\hat{\mu} = \bar{y}_{...} \\
&\hat{\tau}_i = \bar{y}_{i..} - \bar{y}_{...} \qquad i = 1,2,\ldots,a \\
&\hat{\beta}_j = \bar{y}_{.j.} - \bar{y}_{...} \qquad j = 1,2,\ldots,b \\
&\widehat{(\tau\beta)}_{ij} = \bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...} \quad \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \end{cases}
\end{aligned}}$$

TWO-FACTOR FACTORIAL DESIGN

The fitted value of y_{ijk} :

$$\boxed{\begin{aligned}\hat{y}_{ijk} &= \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j + \widehat{(\tau\beta)}_{ij} \\ &= \bar{y}_{...} + \left(\bar{y}_{i..} - \bar{y}_{...}\right) + \left(\bar{y}_{.j.} - \bar{y}_{...}\right) \\ &\quad + \left(\bar{y}_{ij.} - \bar{y}_{i..} - \bar{y}_{.j.} + \bar{y}_{...}\right) \\ &= \bar{y}_{ij}\end{aligned}}$$

TWO-FACTOR FACTORIAL DESIGN ONE OBSERVATION PER CELL

- The effect model for one observation per cell:

$$\boxed{y_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ij} \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \end{cases}}$$

Analysis of Variance for a Two-Factor Model, One Observation per Cell

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square
Rows (A)	$\sum_{i=1}^{a} \frac{y_{i.}^2}{b} - \frac{y_{..}^2}{ab}$	$a - 1$	MS_A	$\sigma^2 + \frac{b \sum \tau_i^2}{a-1}$
Columns (B)	$\sum_{j=1}^{b} \frac{y_{.j}^2}{a} - \frac{y_{..}^2}{ab}$	$b - 1$	MS_B	$\sigma^2 + \frac{a \sum \beta_j^2}{b-1}$
Residual or AB	Subtraction	$(a-1)(b-1)$	MS_{Residual}	$\sigma^2 + \frac{\sum\sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$
Total	$\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{ab}$	$ab - 1$		

TWO-FACTOR FACTORIAL DESIGN ONE OBSERVATION PER CELL

- In this situation, the interaction effect $(\tau\beta)_{ij}$ and the error term cannot be separated $\Rightarrow$ cannot estimate the error variance σ^2.
- Tests on main effects cannot be conducted unless *the interaction effect is zero*.

<u>Tukey's Test</u>: in determining the existence of interaction effect

The test is conducted based on the assumption that:

$$\boxed{(\tau\beta)_{ij} = \gamma\tau_i\beta_j}$$

TWO-FACTOR FACTORIAL DESIGN ONE OBSERVATION PER CELL

Procedure:

Divide the residual sum of squares into two components:

1. A *single-degree-of-freedom* component due to nonadditive (interaction)

$$SS_N = \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}y_{i.}y_{.j} - y_{..}\left(SS_A + SS_B + \frac{y_{..}^2}{ab}\right)\right]^2}{abSS_ASS_B}$$

2. Error component with $(a-1)(b-1)-1$ degrees of freedom

$$SS_{Error} = SS_{\text{Residual}} - SS_N$$

3. Compute the test statistic:

$$F_0 = \frac{SS_N}{SS_{Error}/\left[(a-1)(b-1)-1\right]}$$

4. If $F_0 > F_{\alpha,1,(a-1)(b-1)-1}$: Reject the null hypothesis of *no interaction*.

TWO-FACTOR FACTORIAL DESIGN ONE OBSERVATION PER CELL

Example: The impurity present in a chemical product is affected by pressure and temperature. The data from a single replicate of a factorial design are as follows:

Temperature (^{0}F)	Pressure 25	30	35	40	45	$y_{i.}$
100	5	4	6	3	5	23
125	3	1	4	2	3	13
150	1	1	3	1	2	8
$y_{.j}$	9	6	13	6	10	$y_{..} = 44$

The sums of squares:

$$SS_A = \frac{1}{b}\sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{ab} = 23.33 \qquad SS_B = \frac{1}{a}\sum_{i=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{ab} = 11.60$$

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{ab} = 36.93 \qquad SS_{\text{Residual}} = SS_T - SS_A - SS_B = 2.00$$

TWO-FACTOR FACTORIAL DESIGN ONE OBSERVATION PER CELL

❖ Test for the presence of interaction effect:

$$SS_N = \frac{\left[\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij} y_{i.} y_{.j} - y_{..}\left(SS_A + SS_B + \frac{y_{..}^2}{ab}\right)\right]^2}{abSS_A SS_B} = 0.0985$$

$$SS_{Error} = SS_{\text{Residual}} - SS_N = 1.9015$$

$$F_0 = \frac{SS_N}{SS_{Error} \big/ \left[(a-1)(b-1)-1\right]} = 0.36$$

$\Rightarrow$ No evidence of interaction effect

TWO-FACTOR FACTORIAL DESIGN ONE OBSERVATION PER CELL

Results:

Source of Variation	Sum of Squares	DF	Mean Square	F_0	p-value
Temperature	23.33	2	11.67	42.97	0.0001
Pressure	11.60	4	2.90	10.68	0.0042
Nonadditive	0.0985	1	0.0985	0.36	0.5674
Error	1.9015	7	0.2716		
Total	36.93	14			

The main effects of temperature and pressure are significant.

GENERAL FACTORIAL DESIGN

GENERAL FACTORIAL DESIGN

Structure:

- a levels of factor A
- b levels of factor B
- c levels of factor C $\Rightarrow$ $abc...n$ observations
- ...
- n replications ($n \geq 2$)

The effect model for Three-Factor Analysis of Variance:

$$\boxed{y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijkl}}$$

$$i = 1,2,...,a \qquad j = 1,2,...,b$$

$$k = 1,2,...,c \qquad l = 1,2,...,n$$

GENERAL FACTORIAL DESIGN

The Analysis of Variance Table for the Three-Factor Fixed Effects Model

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	Expected Mean Square	F_0
A	SS_A	$a-1$	MS_A	$\sigma + \frac{bcn \sum \tau_i^2}{a-1}$	$F_0 = \frac{MS_A}{MS_E}$
B	SS_B	$b-1$	MS_B	$\sigma^2 + \frac{acn \sum \beta_j^2}{b-1}$	$F_0 = \frac{MS_B}{MS_E}$
C	SS_C	$c-1$	MS_C	$\sigma^2 + \frac{abn \sum \gamma_k^2}{c-1}$	$F_0 = \frac{MS_C}{MS_E}$
AB	SS_{AB}	$(a-1)(b-1)$	MS_{AB}	$\sigma^2 + \frac{cn \sum \sum (\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$F_0 = \frac{MS_{AB}}{MS_E}$
AC	SS_{AC}	$(a-1)(c-1)$	MS_{AC}	$\sigma^2 + \frac{bn \sum \sum (\tau\gamma)_{ik}^2}{(a-1)(c-1)}$	$F_0 = \frac{MS_{AC}}{MS_E}$
BC	SS_{BC}	$(b-1)(c-1)$	MS_{BC}	$\sigma^2 + \frac{an \sum \sum (\beta\gamma)_{jk}^2}{(b-1)(c-1)}$	$F_0 = \frac{MS_{BC}}{MS_E}$
ABC	SS_{ABC}	$(a-1)(b-1)(c-1)$	MS_{ABC}	$\sigma^2 + \frac{n \sum \sum \sum (\tau\beta\gamma)_{ijk}^2}{(a-1)(b-1)(c-1)}$	$F_0 = \frac{MS_{ABC}}{MS_E}$
Error	SS_E	$abc(n-1)$	MS_E	σ^2	
Total	SS_T	$abcn-1$			

GENERAL FACTORIAL DESIGN

<u>Computing Formulas</u>

Total sum of squares:

$$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c}\sum_{l=1}^{n} y_{ijkl}^2 - \frac{y_{....}^2}{abcn}$$

Sums of squares of the main effects:

$$SS_A = \frac{1}{bcn}\sum_{i=1}^{a} y_{i...}^2 - \frac{y_{....}^2}{abcn}$$

$$SS_B = \frac{1}{acn}\sum_{j=1}^{b} y_{.j..}^2 - \frac{y_{....}^2}{abcn}$$

$$SS_C = \frac{1}{abn}\sum_{k=1}^{c} y_{..k.}^2 - \frac{y_{....}^2}{abcn}$$

GENERAL FACTORIAL DESIGN

The two-factor sums of squares:

$$\boxed{SS_{AB} = SS_{Subtotals(AB)} - SS_A - SS_B}$$

$$\boxed{SS_{Subtotals(AB)} = \frac{1}{cn}\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij..}^2 - \frac{y_{....}^2}{abcn}}$$

$$\boxed{SS_{AC} = SS_{Subtotals(AC)} - SS_A - SS_C}$$

$$\boxed{SS_{Subtotals(AC)} = \frac{1}{bn}\sum_{i=1}^{a}\sum_{k=1}^{c} y_{i.k.}^2 - \frac{y_{....}^2}{abcn}}$$

$$\boxed{SS_{BC} = SS_{Subtotals(BC)} - SS_B - SS_C}$$

$$\boxed{SS_{Subtotals(BC)} = \frac{1}{an}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{.jk.}^2 - \frac{y_{....}^2}{abcn}}$$

GENERAL FACTORIAL DESIGN

The three-factor sum of squares:

$$\boxed{SS_{ABC} = SS_{Subtotals(ABC)} - SS_A - SS_B - SS_C - SS_{AB} - SS_{AC} - SS_{BC}}$$

with $$\boxed{SS_{Subtotals(ABC)} = \frac{1}{n}\sum_{i=1}^{a}\sum_{j=1}^{b}\sum_{k=1}^{c} y_{ijk.}^2 - \frac{y_{....}^2}{abcn}}$$

The error sum of square:

$$\boxed{SS_E = SS_T - SS_{Subtotals(ABC)}}$$

GENERAL FACTORIAL DESIGN

Example: Soft Drink Bottling System

Target: Uniform fill height in the bottles produced

Controllable variables:

- Percent carbonate (A)
- Operating pressure in the filler (B)
- Line speed – bottles produced per minute (C)

GENERAL FACTORIAL DESIGN

Fill Height Deviation Data

Percent Carbonation (A)	Operating Pressure (B) 25 psi, Line Speed (C) 200	25 psi, 250	30 psi, 200	30 psi, 250	$y_{i...}$
10	−3, −1 (−4)	−1, 0 (−1)	−1, 0 (−1)	1, 1 (2)	−4
12	0, 1 (1)	2, 1 (3)	2, 3 (5)	6, 5 (11)	20
14	5, 4 (9)	7, 6 (13)	7, 9 (16)	10, 11 (21)	59
$B \times C$ Totals $y_{.jk.}$	6	15	20	34	$75 = y_{....}$
$y_{.j..}$	21		54		

$A \times B$ Totals $y_{ij..}$

A	B: 25	30
10	−5	1
12	4	16
14	22	37

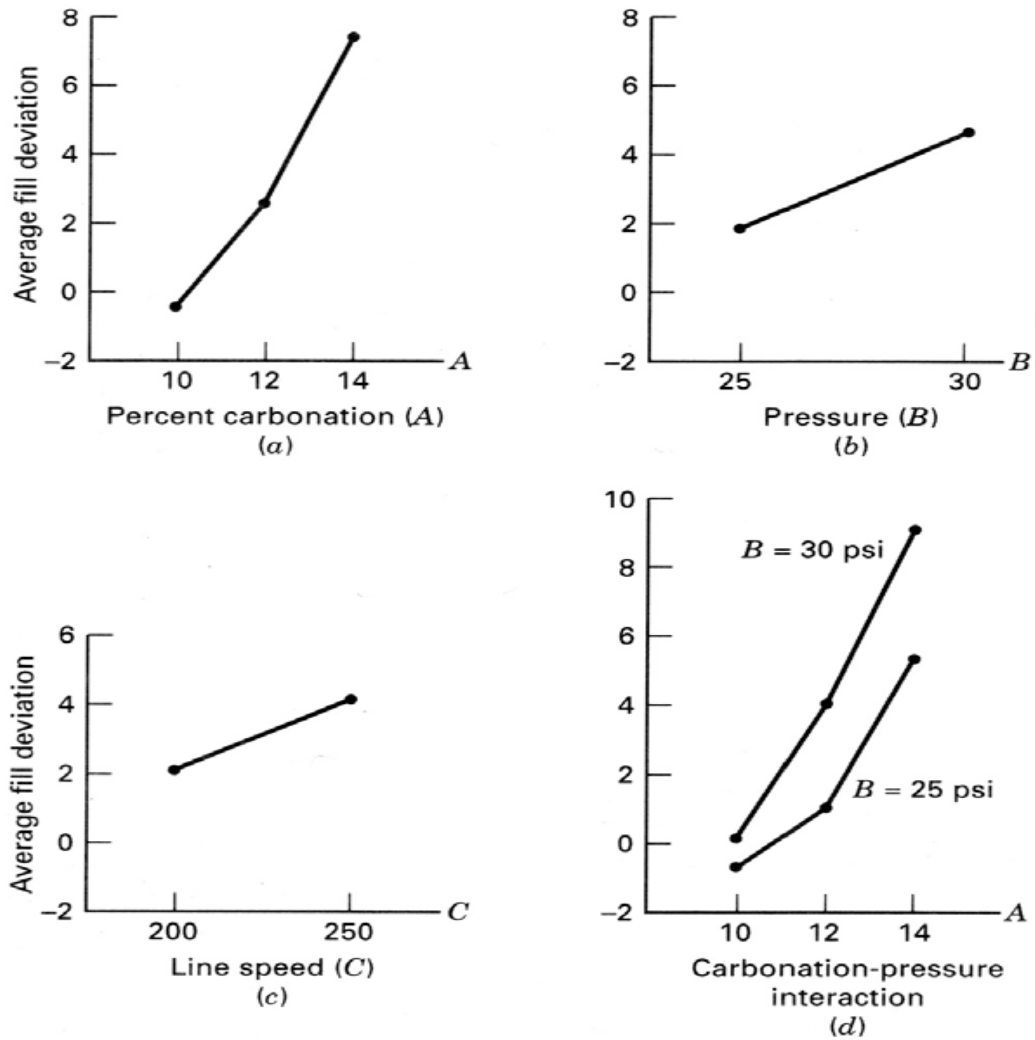
$A \times C$ Totals $y_{i.k.}$

A	C: 200	250
10	−5	1
12	6	14
14	25	34

GENERAL FACTORIAL DESIGN

Source of Variation	Sum of Squares	DF	Mean Square	F_0	p
Perc. of carbonation (A)	252.750	2	126.375	178.412	<0.0001
Operating pressure (B)	45.375	1	45.375	64.059	<0.0001
Line speed (C)	22.042	1	22.042	31.118	0.0001
AB	5.250	2	2.625	3.706	0.0558
AC	0.583	2	0.292	0.412	0.6713
BC	1.042	1	1.042	1.471	0.2485
ABC	1.083	2	0.542	0.765	0.4867
Error	8.500	12	0.708		
Total	336.625	23			

GENERAL FACTORIAL DESIGN



GENERAL FACTORIAL DESIGN

Recommendations:

- Low pressure & High line speed (max. production rate)
- Control temperature to reduce fluctuation in carbonation level and keep it at the low level.

BLOCKING IN FACTORIAL DESIGN

- The presence of nuisance factor may require *blocking* technique.

Consider a factorial design with two factors A, B:

$$y_{ij} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk} \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \\ k = 1,2,\ldots,n \end{cases}$$

Suppose that

- ❖ Running this experiment requires a particular material
- ❖ The material is available in batches that are not large enough for all *abn* treatment combinations

$\Rightarrow$ Run each of the *n* replicates using one batch.

$\Rightarrow$ **BLOCKING!**

BLOCKING IN FACTORIAL DESIGN

The effect model :

$$\boxed{y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \delta_k + \varepsilon_{ijk} \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,b \\ k = 1,2,\ldots,n \end{cases}}$$

δ_k : effect of the kth block.

Assumption: *No interaction between blocks and treatments.*

BLOCKING IN FACTORIAL DESIGN

ANOVA Table:

Analysis of Variance for a Two-Factor Factorial in a Randomized Complete Block

Source of Variation	Sum of Squares	Degrees of Freedom	Expected Mean Square	F_0
Blocks	$\frac{1}{ab}\sum_k y_{..k}^2 - \frac{y_{...}^2}{abn}$	$n-1$	$\sigma^2 + ab\sigma_\delta^2$	
A	$\frac{1}{bn}\sum_i y_{i..}^2 - \frac{y_{...}^2}{abn}$	$a-1$	$\sigma^2 + \frac{bn\sum \tau_i^2}{a-1}$	$\frac{MS_A}{MS_E}$
B	$\frac{1}{an}\sum_j y_{.j.}^2 - \frac{y_{...}^2}{abn}$	$b-1$	$\sigma^2 + \frac{an\sum \beta_j^2}{b-1}$	$\frac{MS_B}{MS_E}$
AB	$\frac{1}{n}\sum_i\sum_j y_{ij.}^2 - \frac{y_{...}^2}{abn} - SS_A - SS_B$	$(a-1)(b-1)$	$\sigma^2 + \frac{n\sum\sum(\tau\beta)_{ij}^2}{(a-1)(b-1)}$	$\frac{MS_{AB}}{MS_E}$
Error	Subtraction	$(ab-1)(n-1)$	σ^2	
Total	$\sum_i\sum_j\sum_k y_{ijk}^2 - \frac{y_{...}^2}{abn}$	$abn-1$		

BLOCKING IN FACTORIAL DESIGN

Example:

An engineer is studying method for improving the ability to detect target on a radar scope.

Two important factors:

Background noise on the scope:	Three levels
Type of Filter placed over the screen:	Two types

Procedure:

Increase the intensity of a signal (target) until it is detected.

Blocking variable:

Operator - different in skill and ability to use the scope

BLOCKING IN FACTORIAL DESIGN

Test results:

Operator (blocks)	1		2		3		4	
Filter Type	1	2	1	2	1	2	1	2
Ground clutter								
Low	90	86	96	84	100	92	92	81
Medium	102	87	106	90	105	97	96	80
High	114	93	112	91	108	95	98	83

ANOVA Table:

Source of Variation	Sum of Squares	DF	Mean Square	F_0	*p*-value
Ground Clutter	335.58	2	167.79	15.13	0.0003
Filter Type	1066.67	1	1066.67	96.19	<0.0001
Interaction	77.08	2	38.54	3.48	0.0573
Blocks (Operators)	402.17	3	134.06		
Error	166.33	15	11.09		
Total	2047.83	23			

BLOCKING IN FACTORIAL DESIGN

Conclusions:

1. Ground clutter & Filter type are significant at $\alpha = 0.01$

$\Rightarrow$ They affect the operator's ability to detect the target.

2. Interaction between ground clutter & filter type is significant only at $\alpha = 0.1$

BLOCKING IN FACTORIAL DESIGN

Special Case

In dealing with two randomization restrictions, each with p levels:

If *the number of treatment combinations in a k-factor factorial design equals the number of restriction level*:

$$p = ab...m$$

then *the factorial design may be run in a* $p \times p$ *Latin square*.

BLOCKING IN FACTORIAL DESIGN

Example: The Radar Target Detection Experiment

Suppose that 6 workers are available and only six runs can be made per day.

Blocking variables: Operator – Day

$\Rightarrow 6 \times 6$ Latin square design:

- Each operator would be used only one on each day
- A = $f_1 g_1$, B = $f_1 g_2$, C = $f_1 g_3$, D = $f_2 g_1$, E = $f_2 g_2$, F = $f_2 g_3$
  (f_i, g_j: *i*th level of filter type, *j*th level of ground clutter)

BLOCKING IN FACTORIAL DESIGN

Radar Detection Experiment Run in a 6 × 6 Latin Square

	Operator					
Day	1	2	3	4	5	6
1	$A(f_1g_1 = 90)$	$B(f_1g_2 = 106)$	$C(f_1g_3 = 108)$	$D(f_2g_1 = 81)$	$F(f_2g_3 = 90)$	$E(f_2g_2 = 88)$
2	$C(f_1g_3 = 114)$	$A(f_1g_1 = 96)$	$B(f_1g_2 = 105)$	$F(f_2g_3 = 83)$	$E(f_2g_2 = 86)$	$D(f_2g_1 = 84)$
3	$B(f_1g_2 = 102)$	$E(f_2g_2 = 90)$	$F(f_2g_3 = 95)$	$A(f_1g_1 = 92)$	$D(f_2g_1 = 85)$	$C(f_1g_3 = 104)$
4	$E(f_2g_2 = 87)$	$D(f_2g_1 = 84)$	$A(f_1g_1 = 100)$	$B(f_1g_2 = 96)$	$C(f_1g_3 = 110)$	$F(f_2g_3 = 91)$
5	$F(f_2g_3 = 93)$	$C(f_1g_3 = 112)$	$D(f_2g_1 = 92)$	$E(f_2g_2 = 80)$	$A(f_1g_1 = 90)$	$B(f_1g_2 = 98)$
6	$D(f_2g_1 = 86)$	$F(f_2g_3 = 91)$	$E(f_2g_2 = 97)$	$C(f_1g_3 = 98)$	$B(f_1g_2 = 100)$	$A(f_1g_1 = 92)$

Effect Model:

$$\boxed{y_{ijkl} = \mu + \alpha_i + \tau_j + \beta_k + (\tau\beta)_{jk} + \theta_l + \varepsilon_{ijkl} \begin{cases} i,l = 1,2,\ldots,6 \\ j = 1,2,3 \\ k = 1,2 \end{cases}}$$

τ_j, β_k: effects of ground clutter and filter type

α_i, θ_l: represent the randomization restrictions of days & operators

BLOCKING IN FACTORIAL DESIGN

<u>Calculation:</u>

Ground Clutter	Filter type 1	Filter type 2	$y_{.j..}$
Low	560	512	1072
Medium	607	528	1135
High	646	543	1189
$y_{..k.}$	1813	1583	$y_{....} = 3396$

Row Totals	$y_{i...}$:	563	568	568	568	565	564
Column Totals	$y_{...l}$:	572	579	597	530	561	557

BLOCKING IN FACTORIAL DESIGN

ANOVA Table:

Analysis of Variance for the Radar Detection Experiment Run as a 3×2 Factorial in a Latin Square

Source of Variation	Sum of Squares	Degrees of Freedom	General Formula for Degrees of Freedom	Mean Square	F_0	P-Value
Ground clutter, G	571.50	2	$a - 1$	285.75	28.86	<0.0001
Filter type, F	1469.44	1	$b - 1$	1469.44	148.43	<0.0001
GF	126.73	2	$(a - 1)(b - 1)$	63.37	6.40	0.0071
Days (rows)	4.33	5	$ab - 1$	0.87		
Operators (columns)	428.00	5	$ab - 1$	85.60		
Error	198.00	20	$(ab - 1)(ab - 2)$	9.90		
Total	2798.00	35	$(ab)^2 - 1$			