Session 7

ANALYSIS OF VARIANCE

ONE-FACTOR ANALYSIS OF VARIANCE

ANALYSIS OF VARIANCE (ANOVA)

A group of techniques for statistical analyzing the data to determine whether there is a significant difference in two or more levels of a treatment variable.

Targets:

- Why the variation exists?
- Find possible reasons $\Rightarrow$ break down the total variance into possible causes.

ONE-FACTOR ANALYSIS OF VARIANCE

- Types of Experimental Designs

  Completely Randomized Design
  Randomized Block Design (Two-way ANOVA)
  Factorial Design
  …

- Structure of Single-Factor ANOVA

  - a treatments/levels of a single factor
  - y_{ij} : observation taken under factor level or treatment i

ONE-FACTOR ANALYSIS OF VARIANCE

Typical Data for a Single-Factor Experiment

Treatment (level)	Observations				Totals	Averages
1	y_{11}	y_{12}	$\cdots$	y_{1n}	$y_{1.}$	$\bar{y}_{1.}$
2	y_{21}	y_{22}	$\cdots$	y_{2n}	$y_{2.}$	$\bar{y}_{2.}$
$\vdots$	$\vdots$	$\vdots$	$\cdots$ $\cdots$	$\vdots$	$\vdots$	$\vdots$
a	y_{a1}	y_{a2}	$\cdots$	y_{an}	$\underline{y_{a.}}$	$\underline{\bar{y}_{a.}}$
					$y_{..}$	$\bar{y}_{..}$

ONE-FACTOR ANALYSIS OF VARIANCE

The Mean Model: $$\boxed{y_{ij} = \mu_i + \varepsilon_i \quad \begin{cases} i = 1, 2...., a \\ j = 1, 2, ..., n \end{cases}}$$

y_{ij} : the ij^{th} observation

μ_i : mean of the *i*th factor level or treatment

ε_{ij} : random error due to other sources of variability

- *measurement*
- *uncontrolled factors*
- *experimental units*
- *background noise* (variability over time, effects of environmental variables,…)

ONE-FACTOR ANALYSIS OF VARIANCE

The Effect Model:

$$\boxed{y_{ij} = \mu_i + \varepsilon_{ij} = \mu + \tau_i + \varepsilon_{ij} \quad \begin{cases} i = 1, 2...., a \\ j = 1, 2, ..., n \end{cases}}$$

μ: overall mean

τ_i: ith treatment effect

ε_{ij}: random error due to other sources of variability

τ_i represents deviation from μ when treatment i is applied.

ONE-FACTOR ANALYSIS OF VARIANCE

Completely Randomized Design:

- The experiment is performed in random order so that effect of environmental units is as *uniform* as possible.

- Target: test hypotheses about the treatment means

- Assumptions:

  1. Model errors are *normally* and *independently* distributed random variables with mean 0 and variance σ^2.
  2. σ^2 is constant for all levels of the factor

$$\Rightarrow \quad y_{ij} \sim N\left(\mu + \tau_i, \sigma^2\right)$$

ONE-FACTOR ANALYSIS OF VARIANCE

Fixed Effects Model vs. Random Effects Model

- Fixed effects model: The a treatments are prespesified and conclusions are applied only to the factor levels considered. The τ_i is a parameter to be estimated.
- Random effect model: The a treatments are *random sample* from a population of treatments and hence τ_i is a random variable

ONE-FACTOR ANALYSIS OF VARIANCE

ANALYSIS OF THE FIXED EFFECTS MODEL

Hypotheses:

$$\boxed{\begin{aligned} &H_0 : \mu_1 = \mu_2 = \ldots = \mu_a \\ &H_1 : \mu_i \neq \mu_j \text{ for at least one pair } (i, j) \end{aligned}}$$

Notation:

$$y_{i.} = \sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{i.} = \frac{y_{i.}}{n}$$

$$y_{..} = \sum_{i=1}^{a} \sum_{j=1}^{n} y_{ij} \qquad \bar{y}_{..} = \frac{y_{..}}{na} = \frac{y_{..}}{N}$$

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Note that in the effects model:

1. $\mu_i = \mu + \tau_i$ and

2. We think that: $$\mu = \frac{\sum_{i=1}^{a} \mu_i}{a}$$

$\Rightarrow \quad \sum_{i=1}^{a} \tau_i = 0 \quad \Rightarrow$ Hypotheses:
$$H_0 : \tau_1 = \tau_2 = \ldots = \tau_a = 0$$
$$H_1 : \tau_i \neq 0 \text{ for at least one } i$$

ONE-FACTOR ANALYSIS OF VARIANCE

Sums of Squares Definitions

- Total sum of squares = Treatment sum of squares + Error sum of square

$$\boxed{SS_T = SS_{Treatments} + SS_E}$$

$$\boxed{SS_{Treatments} = n\sum_{i=1}^{a}\left(\bar{y}_{i.} - \bar{y}_{..}\right)^2}$$

$$\boxed{SS_E = \sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{i.}\right)^2}$$

$$\boxed{SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{..}\right)^2}$$

ONE-FACTOR ANALYSIS OF VARIANCE

- Computing Formulas

$$\boxed{SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - \frac{y_{..}^2}{N}} \quad \boxed{SS_{Treatments} = \frac{1}{n}\sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N}}$$

ONE-FACTOR ANALYSIS OF VARIANCE

- Mean Square of Treatment

$$\boxed{MS_{Treatments} = \frac{SS_{Treatments}}{df_{Treatemnts}}}$$

$$(df_{Tretments} = a - 1 : \text{treatment degrees of freedom})$$

- Mean Square of Error

$$\boxed{MS_E = \frac{SS_E}{df_E}}$$

$$(df_E = N - a : \text{error degrees of freedom})$$

ONE-FACTOR ANALYSIS OF VARIANCE

<u>Notes:</u>

1. $\dfrac{\sum_{i=1}^{a}\left(\bar{y}_{i.}-\bar{y}_{..}\right)^2}{a-1}$ is an estimate of σ^2/n - The variance of the treatment average.

Hence, $MS_{Treatments}=\dfrac{SS_{Treatments}}{a-1}=\dfrac{n\sum_{i=1}^{a}\left(\bar{y}_{i.}-\bar{y}_{..}\right)^2}{a-1}$ is an estimate of σ^2 if there are no differences in treatment means.

ONE-FACTOR ANALYSIS OF VARIANCE

2. The sample variance of the *i*th treatment:

$$S_i^2 = \frac{\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{i.}\right)^2}{n-1}$$

The pooled estimate of the common variance within each of the a treatments is:

$$\frac{(n-1)S_1^2 + (n-1)S_2^2 + \ldots + (n-1)S_a^2}{(n-1)+(n-1)+\ldots+(n-1)} = \frac{\sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{i.}\right)^2}{N-a} = \frac{SS_E}{N-a} = MS_E$$

ONE-FACTOR ANALYSIS OF VARIANCE

3. It can be shown that:

$$E\left[MS_E\right] = \sigma^2 \quad \text{and} \quad E\left[MS_{Treatments}\right] = \sigma^2 + \frac{n\sum_{i=1}^{a}\tau_i^2}{a-1}$$

Hence, if *treatment means do differ,* $E\left[MS_{Treatments}\right] > \sigma^2$

ONE-FACTOR ANALYSIS OF VARIANCE

- By the assumption that $\varepsilon_{ij} \sim \mathbf{NID}\left(0,\sigma^2\right)$, y_{ij} are normally and independently distributed with mean $\mu+\tau_i$ and variance σ^2.
- SS_T is the sum of squares in normally distributed random variables $\Rightarrow SS_T/\sigma^2$ is chi-square distributed with $N-1$ degrees of freedom.
- SS_E/σ^2 is chi-square with $N-a$ degrees of freedom
- $SS_{Treatments}/\sigma^2$ is chi-square with $a-1$ degrees of freedom if the null hypothesis $H_o : \tau_i = 0$ is true.

ONE-FACTOR ANALYSIS OF VARIANCE

Statistical Analysis

If the null hypothesis of no difference in treatment means is true, the ratio (the test statistic):

$$F_0 = \frac{SS_{Treatments}/(a-1)}{SS_E/(N-a)} = \frac{MS_{Treatments}}{MS_E}$$

Follows Fisher distribution with $a-1$ and $N-a$ degrees of freedom.

Rejection Criterion

$$\boxed{F_0 > F_{\alpha,a-1,N-a}}$$

ONE-FACTOR ANALYSIS OF VARIANCE

ANOVA Table

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Between treatment	$SS_{Treatments} = n\sum_{i=1}^{a}\left(\bar{y}_{i.} - \bar{y}_{..}\right)^2$	$a-1$	$MS_{Treatments}$	$F_0 = \dfrac{MS_{Treatments}}{MS_E}$
Error (within treatment)	$SS_E = SS_T - SS_{Treatments}$	$N-a$	MS_E	
Total	$SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij} - \bar{y}_{..}\right)^2$	$N-1$		

ONE-FACTOR ANALYSIS OF VARIANCE

Example: The Tensile Strength Experiment

Cotton Weight Percentage	Observations 1	2	3	4	5	Total $y_{i.}$	Average $\bar{y}_{i.}$
15	7	7	15	11	9	49	9.8
20	12	17	12	18	18	77	15.4
25	14	18	18	19	19	88	17.6
30	19	25	22	19	23	108	21.6
35	7	10	11	15	11	54	10.8
						$y_{..} = \mathbf{376}$	$\bar{y}_{..} = \mathbf{15.04}$

Hypotheses:

$$H_0: \mu_1 = \mu_2 = \mu_3 = \mu_4 = \mu_5$$

$$H_1: \textit{some means are different}$$

ONE-FACTOR ANALYSIS OF VARIANCE

$$SS_T = \sum_{i=1}^{5}\sum_{j=1}^{5} y_{ij}^2 - \frac{y_{..}^2}{N} = 636.96; \qquad SS_{Treatments} = \frac{1}{5}\sum_{i=1}^{5} y_{i.}^2 - \frac{y_{..}^2}{5*5} = 475.76$$

$$\Rightarrow SS_E = SS_T - SS_{Treatments} = 161.20$$

Test statistic: $F_0 = 14.76 > F_{0.05,4,20} = 2.87$

$\Rightarrow$ Reject H_0: Treatment means differ.

<u>Summary</u>:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	p-value
Cotton weight perc.	475.76	4	118.94	$F_0 = 14.76$	<0.01
Error	161.20	20	8.06		
Total	639.96	24			

ONE-FACTOR ANALYSIS OF VARIANCE

Example: Coding the Observations -The Tensile Strength Experiment

Cotton Weight Percentage	**Observations 1**	**2**	**3**	**4**	**5**	**Total** $y_{i.}$
15	-8	-8	0	-4	-6	-26
20	-3	2	-3	3	3	2
25	-1	3	3	4	4	13
30	4	10	7	4	8	33
35	-8	-5	-4	0	-4	-21
						$y_{..} = \mathbf{1}$

$$SS_T = \sum_{i=1}^{5}\sum_{j=1}^{5} y_{ij}^2 - \frac{y_{..}^2}{N} = 636.96; \qquad SS_{Treatments} = \frac{1}{5}\sum_{i=1}^{5} y_{i.}^2 - \frac{y_{..}^2}{5*5} = 475.76$$

$$\Rightarrow SS_E = SS_T - SS_{Treatments} = 161.20$$

Test statistic: $F_0 = 14.76 > F_{0.05,4,20} = 2.87$

Substracting a constant from the original data does not change the sum of square

ONE-FACTOR ANALYSIS OF VARIANCE

Estimation of the Model Parameters

$$\boxed{y_{ij} = \mu + \tau_i + \varepsilon_{ij}}$$

The estimates of *overall mean* & *treatment effects* are:

$$\hat{\mu} = \bar{y}_{..}$$
$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} \qquad i = 1, 2, \ldots, a$$

A point estimator of the *i*th treatment mean $\mu_i = \mu + \tau_i$ is:

$$\hat{\mu}_i = \hat{\mu} + \hat{\tau}_i = \bar{y}_{i.}$$

ONE-FACTOR ANALYSIS OF VARIANCE

If the errors are normally distributed, each $\bar{y}_{i.}$ is $\mathbf{NID}\left(\mu_i, \sigma^2/n\right)$.

Therefore,

1. If σ is known: confidence interval of μ_i can be defined from normal distribution.
2. If σ is unknown: use MS_E as an estimator of σ^2

$$\boxed{\bar{y}_{i.} - t_{\alpha/2, N-a}\sqrt{\frac{MS_E}{n}} \le \mu_i \le \bar{y}_{i.} + t_{\alpha/2, N-a}\sqrt{\frac{MS_E}{n}}}$$

$$\boxed{\bar{y}_{i.} - \bar{y}_{j.} - t_{\alpha/2, N-a}\sqrt{\frac{2MS_E}{n}} \le \mu_i - \mu_j \le \bar{y}_{i.} - \bar{y}_{j.} + t_{\alpha/2, N-a}\sqrt{\frac{2MS_E}{n}}}$$

ONE-FACTOR ANALYSIS OF VARIANCE

<u>*Example*</u>: The Tensile Strength Data

$$\hat{\mu} = \bar{y}_{..} = 15.04$$
$$\hat{\tau}_1 = \bar{y}_{1.} - \bar{y}_{..} = -5.24$$
$$\hat{\tau}_2 = \bar{y}_{2.} - \bar{y}_{..} = +0.36$$
$$\hat{\tau}_3 = \bar{y}_{3.} - \bar{y}_{..} = -2.56$$
$$\hat{\tau}_4 = \bar{y}_{4.} - \bar{y}_{..} = +6.56$$
$$\hat{\tau}_5 = \bar{y}_{5.} - \bar{y}_{..} = -4.24$$

A 95% percent confidence interval on μ_4:

$$18.95 = 21.60 - 2.086\sqrt{\frac{8.06}{5}} \le \mu_4 \le 21.60 + 2.086\sqrt{\frac{8.06}{5}} = 24.45$$

ONE-FACTOR ANALYSIS OF VARIANCE

Simutanous Confidence Intervals:

- The confidence level $(1-\alpha)$ is applied only to one estimate.
- If there are r estimates of interest at the same time, the probability that *the r intervals will simutaneously correct is* $(1-r\alpha)$ or the *overall confidence coefficient is* $r\alpha$.
- The **Bonferroni method**: when r is not too large, a set of r confidence intervals for which the overall confidence level is at least $(1-\alpha)$ can be contructed by:

$$\alpha/2 \Rightarrow \alpha/2r$$

ONE-FACTOR ANALYSIS OF VARIANCE

Unbalanced Data:

- The number of observation taken under treatment i is n_i.
- Modifications:

$$\boxed{SS_T = \sum_{i=1}^{a}\sum_{j=1}^{n_i} y_{ij}^2 - \frac{y_{..}^2}{N}} \quad \text{and} \quad \boxed{SS_{Treatments} = \sum_{i=1}^{a} \frac{y_{i.}^2}{n_i} - \frac{y_{..}^2}{N}}$$

in which: $N = \sum_{i=1}^{a} n_i$.

ONE-FACTOR ANALYSIS OF VARIANCE

MODEL ADEQUACY CHECKING

Assumptions in ANOVA procedure:

1. The observations are adequately described by: $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$.
2. The errors are *normally* and *independently* distributed with mean 0 and constant variance σ^2 (unknown).

Violations of these basic assumptions and model adequacy can be investigated by the examination of **residuals**

- The residual for observation j in treatment i is: $e_{ij} = y_{ij} - \hat{y}_{ij}$
- The estimate $\hat{y}_{ij}$ of y_{ij}: $\hat{y}_{ij} = \hat{\mu} + \hat{\tau}_i = \bar{y}_{..} + \left(\bar{y}_{i.} - \bar{y}_{..}\right) = \bar{y}_{i.}$

Model Adequacy $\Leftrightarrow$ The residuals should be *structureless*

ONE-FACTOR ANALYSIS OF VARIANCE

The Normality Assumption:

- Check of normality assumption could be made by plotting a *histogram* of the residuals:

  The plot should look like a sample from a normal distribution centered at zero

- Another useful procedure is to construct a *normal probability plot* of the residuals.

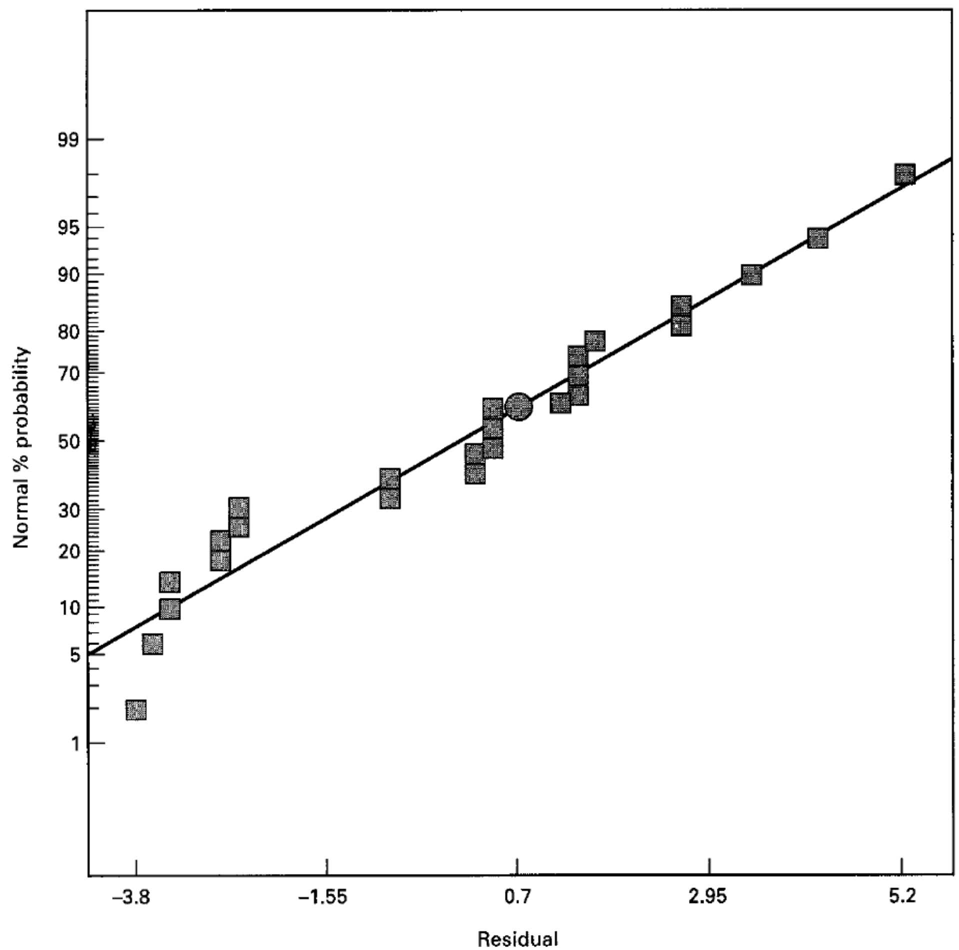
ONE-FACTOR ANALYSIS OF VARIANCE

Example: Tensile strength data

Weight Percentage of Cotton	Observations (j) 1	2	3	4	5	$\hat{y}_{ij} = \bar{y}_{i.}$
15	$\boxed{-2.8}$ 7 (15)	$\boxed{-2.8}$ 7 (19)	$\boxed{5.2}$ 15 (25)	$\boxed{1.2}$ 11 (12)	$\boxed{-0.8}$ 9 (6)	9.8
20	$\boxed{-3.4}$ 12 (8)	$\boxed{1.6}$ 17 (14)	$\boxed{-3.4}$ 12 (1)	$\boxed{2.6}$ 18 (11)	$\boxed{2.6}$ 19 (3)	15.4
25	$\boxed{-3.6}$ 14 (18)	$\boxed{0.4}$ 18 (13)	$\boxed{0.4}$ 18 (20)	$\boxed{1.4}$ 19 (7)	$\boxed{1.4}$ 19 (9)	17.6
30	$\boxed{-2.6}$ 19 (22)	$\boxed{3.4}$ 25 (5)	$\boxed{0.4}$ 22 (2)	$\boxed{-2.6}$ 19 (24)	$\boxed{1.4}$ 23 (10)	21.6
35	$\boxed{-3.8}$ 7 (17)	$\boxed{-0.8}$ 10 (21)	$\boxed{0.2}$ 11 (4)	$\boxed{4.2}$ 15 (16)	$\boxed{0.2}$ 11 (23)	10.8

[a] The residuals are shown in the box in each cell. The numbers in parentheses indicate the order of data collection.

ONE-FACTOR ANALYSIS OF VARIANCE

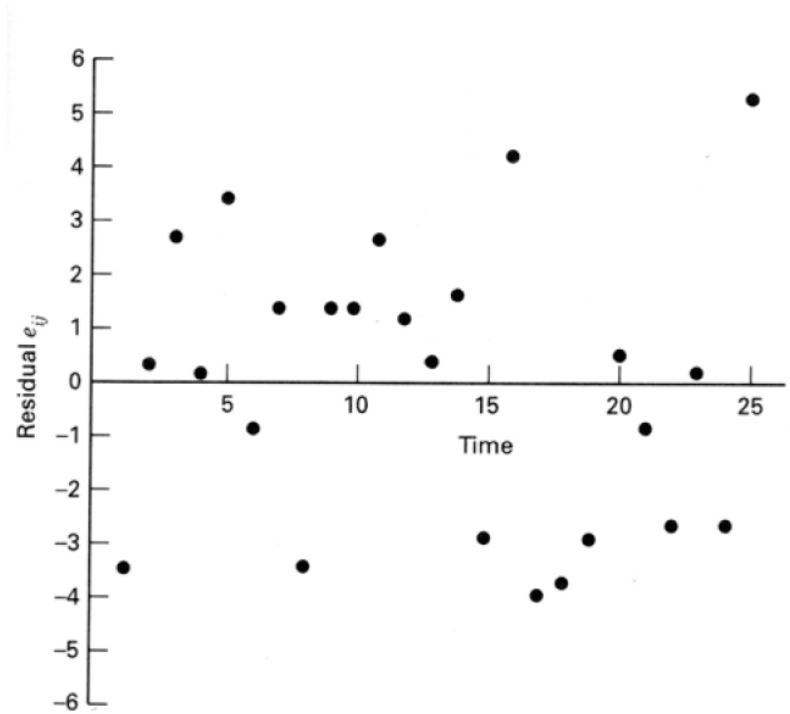


Normal Probability Plot

ONE-FACTOR ANALYSIS OF VARIANCE

<u>Plot of Residuals in Time Sequence</u>

Detect *correlation* between the residuals to check *independence assumption* on the errors



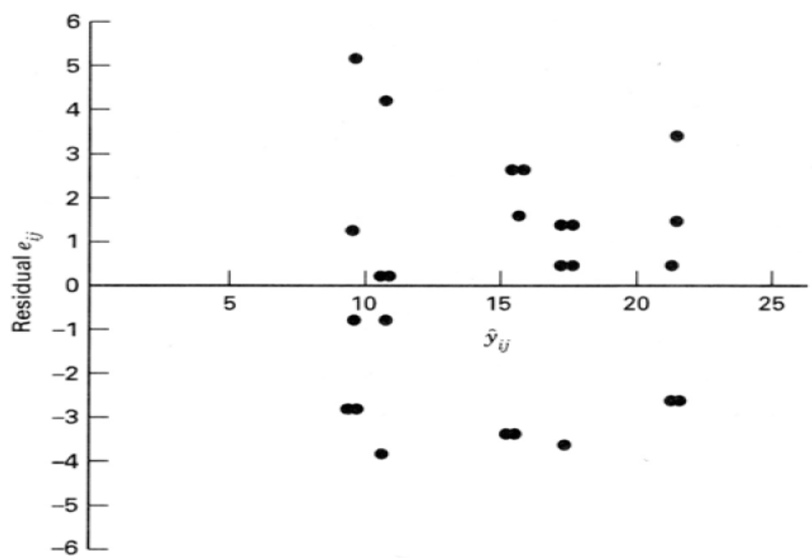
Plot of residuals versus time.

ONE-FACTOR ANALYSIS OF VARIANCE

<u>Plot of Residuals vs. Fitted Value</u>

Check the assumption on *constant variance*.

- Error or background noise may be a constant percentage of the size of the observation (e.g., measuring equipment) $\Rightarrow$ the variance of the observations increases as the magnitude of the observation increases



Plot of residuals versus fitted values.

ONE-FACTOR ANALYSIS OF VARIANCE

- Dealing with nonconstant variance: Apply *Variance-stabilizing transformation* and run ANOVA on transformed data.

For example:

- Observation ~ Poisson Distribution:

  Square root transformation: $y_{ij}^{*} = \sqrt{y_{ij}}$
- Observation ~ Lognormal Distribution:

  Logarithmic transformation: $y_{ij}^{*} = \log y_{ij}$
- Observation ~ Binomial Distribution:

  Arcsin transformation: $y_{ij}^{*} = \arcsin \sqrt{y_{ij}}$

ONE-FACTOR ANALYSIS OF VARIANCE

Statistical Tests for Equality of Variance

Bartlett's Test:

$$H_0: \quad \sigma_1^2 = \sigma_2^2 = \ldots = \sigma_a^2$$
$$H_1: \quad Above\,not\,true$$

The test statistic:

$$\boxed{\chi_0^2 = 2.3026\frac{q}{c}}$$

where:

$$q = (N-a)\log S_p^2 - \sum_{i=1}^{a}(n_i-1)\log S_i^2$$

$$c = 1 + \frac{1}{3(a-1)}\left(\sum_{i=1}^{a}\frac{1}{n_i-1} - \frac{1}{N-a}\right)$$

$$S_p^2 = \frac{\sum_{i=1}^{a}(n_i-1)S_i^2}{N-a} \qquad (pooled\ variance)$$

ONE-FACTOR ANALYSIS OF VARIANCE

Rejection criteria: $\boxed{\chi_0^2 > \chi_{\alpha,a-1}^2}$

$$\left(\chi_{\alpha,a-1}^2 \text{:} \textit{upper } \alpha \textit{ perc. point of the } \chi^2 \textit{ dist. with } (a-1) \textit{ d.f.} \right)$$

*Note that **Bartlett's test** is very sensitive to normality assumption*

<u>*Example*</u>: The Tensile Strength Data

$$S_1^2 = 11.2 \quad S_2^2 = 9.8 \quad S_3^2 = 4.3 \quad S_4^2 = 6.8 \quad S_5^2 = 8.2$$

$\Rightarrow \quad S_p^2 = 8.06$

$\Rightarrow \quad q = 0.45, \quad c = 1.10$

The test statistic: $\chi_0^2 = 2.3026\dfrac{q}{c} = 0.93 < \chi_{0.05,4}^2 = 9.49$

$\Rightarrow$ The null hypothesis cannot be rejected.

ONE-FACTOR ANALYSIS OF VARIANCE

Modified Levene Test

- Not so sensitive to the normality assumption
- To test the hypothesis of equal variances, use the *absolute deviation* of the observations y_{ij} from the treatment median $\tilde{y}_i$:

$$\boxed{d_{ij} = \left| y_{ij} - \tilde{y}_i \right| \qquad \begin{cases} i = 1,2,\ldots,a \\ j = 1,2,\ldots,n_i \end{cases}}$$

ANOVA is then used to evaluate if the *mean of these deviations* are equal for all treatments.

ONE-FACTOR ANALYSIS OF VARIANCE

<u>Plot of Residuals vs. Other Variables</u>

- Residual plot should be constructed on *any other variables* that might affect the response

For example, in the tensile strength experiment, thickness of fiber may be a variable of interest.

MULTIPLE COMPARISON METHODS

- In ANOVA, when the null hypothesis is rejected, we might want to know which means differ from others.
- Several methods will be discussed here.

Contrast

- Many multiple comparison methods use the idea of contrast

Example: Consider the tensile strength experiment,

1. We might suspect that levels 4 and 5 (30% and 35%) of the factor produce the same tensile strength:

$$\Rightarrow \quad H_0 : \mu_4 - \mu_5 = 0$$
$$H_1 : \mu_4 - \mu_5 \neq 0$$

MULTIPLE COMPARISON METHODS

2. If we suspect that the average of the lowest levels (1 and 2) did not differ from the average of the highest levels (3 and 4) of cotton weight percentage:

$$\Rightarrow \quad H_0: \mu_1 + \mu_2 - \mu_4 - \mu_5 = 0$$
$$H_1: \mu_1 + \mu_2 - \mu_4 - \mu_5 \neq 0$$

- Definition of a contrast:

$$\boxed{\Gamma = \sum_{i=1}^{a} c_i \mu_i \ in\, which \sum_{i=1}^{a} c_i = 0}$$

$$\Rightarrow \quad H_0: \sum_{i=1}^{a} c_i \mu_i = 0, \ H_1: \sum_{i=1}^{a} c_i \mu_i \neq 0$$

MULTIPLE COMPARISON METHODS

Testing hypotheses involving contrast can be done in two basic ways:

1. ***t*- Test:**

❖ Write the contrast in term of treatment totals

$$C = \sum_{i=1}^{a} c_i y_{i.}$$

$\Rightarrow$ The variance of C: $\quad V(C) = n\sigma^2 \sum_{i=1}^{a} c_i^2$ (for equal sample sizes)

Note that if H_0 is true then $\quad \dfrac{\sum_{i=1}^{a} c_i y_{i.}}{\sqrt{n\sigma^2 \sum_{i=1}^{a} c_i^2}} \sim N(0,1)$

❖ Test statistic: $\quad t_0 = \dfrac{\sum_{i=1}^{a} c_i y_{i.}}{\sqrt{nMS_E \sum_{i=1}^{a} c_i^2}}$

❖ Rejection criteria: $\quad |t_0| > t_{\alpha/2, N-a}$

MULTIPLE COMPARISON METHODS

2. ***F*- Test:**

❖ Test statistic: $$F_0 = t_0^2 = \frac{\left(\sum_{i=1}^{a} c_i y_{i.}\right)^2}{nMS_E \sum_{i=1}^{a} c_i^2}$$

❖ Rejection criteria: $F_0 > F_{\alpha,1,N-a}$

Note that the contrast sum of squares is: $$SS_C = \frac{\left(\sum_{i=1}^{a} c_i y_{i.}\right)^2}{n \sum_{i=1}^{a} c_i^2}$$

Hence, $F_0 = \dfrac{MS_C}{MS_E} = \dfrac{SS_C/1}{MS_E}$

MULTIPLE COMPARISON METHODS

- Confidence Interval for a Contrast

Write the contrast in term of treatment averages:

$$C = \sum_{i=1}^{a} c_i \bar{y}_{i.} \qquad \Rightarrow \qquad V(C) = \frac{\sigma^2}{n} \sum_{i=1}^{a} c_i^2$$

$$\boxed{\sum_{i=1}^{a} c_i \bar{y}_{i.} - t_{\alpha/2, N-a} \sqrt{\frac{MS_E}{n} \sum_{i=1}^{a} c_i^2} \le \sum_{i=1}^{a} c_i \mu_i \le \sum_{i=1}^{a} c_i \bar{y}_{i.} + t_{\alpha/2, N-a} \sqrt{\frac{MS_E}{n} \sum_{i=1}^{a} c_i^2}}$$

MULTIPLE COMPARISON METHODS

- Standardized Contrast

If the contrast is written in terms of treatment total $C = \sum_{i=1}^{a} c_i y_{i.}$,

dividing by $\sqrt{n\sum_{i=1}^{a} c_i^2}$ will produce a contrast with variance σ^2

$$C = \sum_{i=1}^{a} c_i^* \mu_i \qquad \text{where} \qquad c_i^* = \frac{c_i}{\sqrt{n\sum_{i=1}^{a} c_i^2}}$$

MULTIPLE COMPARISON METHODS

- The case of unequal sizes:

Requirement for contrast: $\sum_{i=1}^{a} n_i c_i = 0$

Other modifications: $t_0 = \frac{\sum_{i=1}^{a} c_i y_{i.}}{\sqrt{MS_E \sum_{i=1}^{a} n_i c_i^2}}$, $SS_C = \frac{\left(\sum_{i=1}^{a} c_i y_{i.} \right)^2}{\sum_{i=1}^{a} n_i c_i^2}$

MULTIPLE COMPARISON METHODS

Orthogonal Contrast

- Two contrasts with coefficients $\{c_i\}$ and $\{d_i\}$ are orthogonal if:

$$\sum_{i=1}^{a} c_i d_i = 0 \qquad \text{(balanced design)}$$

$$\sum_{i=1}^{a} n_i c_i d_i = 0 \qquad \text{(unbalanced design)}$$

- For a treatments, the set of $a-1$ orthogonal contrasts partition the sum of squares due to treatments into $a-1$ *independent single-degree-of-freedom* components.

- The method of contrasts (orthogonal contrasts) is used for *preplanned comparisons* – the contrasts are specified prior to running the experiment.

MULTIPLE COMPARISON METHODS

Example: Tensile Strength Experiment

Number of treatments: 5
Degrees of freedom: 4

Consider the following set of comparisons:

Hypothesis	(Orthogonal) Contrast
$H_0 : \mu_4 = \mu_5$	$C_1 = \quad -y_{4.} + y_{5.}$
$H_0 : \mu_1 + \mu_3 = \mu_4 + \mu_5$	$C_2 = y_{1.} + y_{3.} - y_{4.} - y_{5.}$
$H_0 : \mu_1 = \mu_3$	$C_3 = y_{1.} - y_{3.}$
$H_0 : 4\mu_2 = \mu_1 + \mu_3 + \mu_4 + \mu_5$	$C_4 = -y_{1.} + 4y_{2.} - y_{3.} - y_{4.} - y_{5.}$

$$C_1 = -108 + 54 = -54$$
$$C_2 = 49 + 88 - 108 - 54 = -25$$
$$C_3 = 49 - 88 = -39$$
$$C_4 = -49 + 4 * 77 - 88 - 108 - 54 = 9$$

MULTIPLE COMPARISON METHODS

$$\Rightarrow \qquad SS_{C_1} = \frac{(-54)^2}{5(2)} = 291.60$$

$$SS_{C_2} = \frac{(-25)^2}{5(4)} = 31.25$$

$$SS_{C_3} = \frac{(-39)^2}{5(2)} = 152.10$$

$$SS_{C_4} = \frac{(9)^2}{5(20)} = 0.81$$

MULTIPLE COMPARISON METHODS

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	p-value
Cotton Weight Perc.	475.76	4	118.94	14.76	<0.001
orthogonal contrast					
$C_1 : \mu_4 = \mu_5$	291.60	1	291.60	36.18	<0.001
$C_2 : \mu_1 + \mu_3 = \mu_4 + \mu_5$	31.25	1	31.25	3.88	0.06
$C_3 : \mu_1 = \mu_3$	152.10	1	152.10	18.87	<0.001
$C_4 : 4\mu_2 = \mu_1 + \mu_3 + \mu_4 + \mu_5$	0.81	1	0.81	0.10	0.76
Error	161.20	20	8.06		
Total	636.96	24			

$\Rightarrow$ There are significant differences between levels 4 and 5; 1 and 3 but the avarage of 1&3 does not differ from the average of 4&5. Level 2 does not differ from the average of the other four levels.

MULTIPLE COMPARISON METHODS

❖ **Scheffe's Method for Comparing All Contrasts**

Consider a set of m contrasts in the treament means:

$$\Gamma_u = c_{1u}\mu_1 + c_{2u}\mu_2 + \ldots + c_{au}\mu_a \qquad (u = 1, 2, \ldots, m)$$

Null Hypothesis: $H_0 : \Gamma_u = 0$

In terms of treatment averages:

$$C_u = c_{1u}\bar{y}_{1.} + c_{2u}\bar{y}_{2.} + \ldots + c_{au}\bar{y}_{a.} \qquad (u = 1, 2, \ldots, m)$$

The standard error: $S_{C_u} = \sqrt{MS_E \sum_{i=1}^{a} \left(c_{iu}^2 / n_i \right)}$

Critical value: $S_{\alpha,u} = S_{C_u} \sqrt{(a-1) F_{\alpha,a-1,N-a}}$

Rejection Criteria:

$$|C_u| > S_{\alpha,u}$$

MULTIPLE COMPARISON METHODS

❖ **Scheffe's Method for Comparing All Contrasts**

Example: The Temsile Strength Experiment

Consider the contrasts: $\Gamma_1 = \mu_1 + \mu_3 - \mu_4 - \mu_5; \Gamma_2 = \mu_1 - \mu_4$

The numerical values of these contrasts:

$$C_1 = \bar{y}_{1.} + \bar{y}_{3.} - \bar{y}_{4.} - \bar{y}_{5.} = 5.00$$

$$C_1 = \bar{y}_{1.} - \bar{y}_{4.} = -11.80$$

The standard errors:

$$S_{C_1} = \sqrt{MS_E \sum_{i=1}^{5} \left(c_{iu}^2 / n_i \right)} = \sqrt{8.06(1+1+1+1)/5} = 2.54$$

$$S_{C_2} = \sqrt{MS_E \sum_{i=1}^{5} \left(c_{iu}^2 / n_i \right)} = \sqrt{8.06(1+1)/5} = 1.80$$

MULTIPLE COMPARISON METHODS

❖ **Scheffe's Method for Comparing All Contrasts**

The 1% critical values:

$$S_{0.01,1} = S_{C_1}\sqrt{(a-1)F_{0.01,a-1,N-a}} = 2.54\sqrt{4(4.43)} = 10.69$$

$$S_{0.01,2} = S_{C_2}\sqrt{(a-1)F_{0.01,a-1,N-a}} = 1.80\sqrt{4(4.43)} = 7.58$$

$$\Rightarrow \quad \Gamma_1 = 0; \qquad \Gamma_2 \neq 0$$

- Note that Scheffe method is not the most *sensitive* procedure for comparing pairs of treatment means.

PAIRWISE COMPARISON

Tukey's Test

Hypotheses: $$\begin{array}{l} H_0 : \mu_i = \mu_j \\ H_1 : \mu_i \neq \mu_j \end{array} \quad \text{for all } i \neq j$$

- The overall significant level is *exactly* α with equal sample sizes and *at most* α with unequal sample sizes.
- The procedure is based on the distribution of **Studentized range statistic**

$$\boxed{q = \frac{\bar{y}_{\max} - \bar{y}_{\min}}{\sqrt{MS_E / n}}}$$

$\bar{y}_{\max}, \bar{y}_{\min}$: largest, smallest sample means in group of p sample means.

The upper α percentage point of q with f degrees of freedom, $q_\alpha(p, f)$, is presented in Appendix Table VIII.

PAIRWISE COMPARISON

Tukey's Test

- The two means are different if the absolute value of their sample differences exceeds:

$$\boxed{T_\alpha = q_\alpha(a,f)\sqrt{\frac{MS_E}{n}}} \quad \text{(equal sample sizes)}$$

$$\boxed{T_\alpha = \frac{q_\alpha(a,f)}{\sqrt{2}}\sqrt{MS_E\left(\frac{1}{n_i}+\frac{1}{n_j}\right)}} \quad \text{(unequal sample sizes)}$$

f: the degrees of freedom associated with the error MS_E

- The set of $100(1-\alpha)\%$ confidence intervals:

$$\boxed{\bar{y}_{i.} - \bar{y}_{j.} - T_\alpha \le \mu_i - \mu_j \le \bar{y}_{i.} - \bar{y}_{j.} + T_\alpha}$$

PAIRWISE COMPARISON

Tukey's Test

Example: Tensile Strength Experiment

$$\alpha = 0.05, f = 20 \Rightarrow q_{0.05}(5,20) = 4.23$$

$$T_{0.05} = q_{0.05}(5,20)\sqrt{\frac{MS_E}{n}} = 4.23\sqrt{\frac{8.06}{5}} = 5.37$$

$$\bar{y}_{1.} = 9.8 \quad \bar{y}_{2.} = 15.4 \quad \bar{y}_{3.} = 17.6 \quad \bar{y}_{4.} = 21.6 \quad \bar{y}_{5.} = 10.8$$

PAIRWISE COMPARISON

Tukey's Test

$$\bar{y}_{1.} - \bar{y}_{2.} = -5.6^{*}$$
$$\bar{y}_{1.} - \bar{y}_{3.} = -7.8^{*}$$
$$\bar{y}_{1.} - \bar{y}_{4.} = -11.8^{*}$$
$$\bar{y}_{1.} - \bar{y}_{5.} = -1.0$$
$$\bar{y}_{2.} - \bar{y}_{3.} = -2.2$$
$$\bar{y}_{2.} - \bar{y}_{4.} = -6.2^{*}$$
$$\bar{y}_{2.} - \bar{y}_{5.} = 4.6$$
$$\bar{y}_{3.} - \bar{y}_{4.} = -4.0$$
$$\bar{y}_{3.} - \bar{y}_{5.} = 6.8^{*}$$
$$\bar{y}_{4.} - \bar{y}_{5.} = 10.8^{*}$$

$\bar{y}_{1.}$	$\bar{y}_{5.}$	$\bar{y}_{2.}$	$\bar{y}_{3.}$	$\bar{y}_{4.}$
9.8	10.8	15.4	17.6	21.6

Results of Tukey's test.

PAIRWISE COMPARISON

The Fisher Least Significant Difference (LSD) Method

Hypotheses: $H_0: \mu_i = \mu_j$; $H_1: \mu_i \neq \mu_j$ for all $i \neq j$

- The procedure is based on the F statistic

$$\boxed{t_0 = \frac{\bar{y}_{i.} - \bar{y}_{j.}}{\sqrt{MS_E\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}}}$$

- The two means are different if the absolute value of their sample differences exceeds:

$$\boxed{LSD = t_{\alpha/2, N-a}\sqrt{\frac{2MS_E}{n}}} \quad \text{or} \quad \boxed{LSD = t_{\alpha/2, N-a}\sqrt{MS_E\left(\frac{1}{n_i} + \frac{1}{n_j}\right)}}$$

PAIRWISE COMPARISON

Example: The Tensile Strength Experiment

$$\alpha = 0.05 \Rightarrow LSD = t_{.025,20}\sqrt{\frac{2MS_E}{n}} = 2.086\sqrt{\frac{2(8.06)}{5}} = 3.75$$

$$\bar{y}_{1.} - \bar{y}_{2.} = -5.6^*$$
$$\bar{y}_{1.} - \bar{y}_{3.} = -7.8^*$$
$$\bar{y}_{1.} - \bar{y}_{4.} = -11.8^*$$
$$\bar{y}_{1.} - \bar{y}_{5.} = -1.0$$
$$\bar{y}_{2.} - \bar{y}_{3.} = -2.2$$
$$\bar{y}_{2.} - \bar{y}_{4.} = -6.2^*$$
$$\bar{y}_{2.} - \bar{y}_{5.} = 4.6^*$$
$$\bar{y}_{3.} - \bar{y}_{4.} = -4.0^*$$
$$\bar{y}_{3.} - \bar{y}_{5.} = 6.8^*$$
$$\bar{y}_{4.} - \bar{y}_{5.} = 10.8^*$$

$\bar{y}_{1.}$	$\bar{y}_{5.}$	$\bar{y}_{2.}$	$\bar{y}_{3.}$	$\bar{y}_{4.}$
9.8	10.8	15.4	17.6	21.6

Results of the LSD procedure.

Note that the *overall* α *risk* (Type I error) may be considerably inflated using this method.

PAIRWISE COMPARISON

❖ **Comparing Treatment Means with a Control**

Hypotheses: $H_0: \mu_i = \mu_a$, $H_1: \mu_i \neq \mu_a$ for $i = 1, 2, \ldots, a-1$

Dunnett's Procedure

- For each hypothesis, compute the observed differences in the sample means:

$$\left|\bar{y}_{i.} - \bar{y}_{a.}\right| \quad \text{for } i = 1, 2, \ldots, a-1$$

- The null hypothesis is rejected at Type I error α if

$$\boxed{\left|\bar{y}_{i.} - \bar{y}_{a.}\right| > d_\alpha(a-1, f)\sqrt{MS_E\left(\frac{1}{n_i} + \frac{1}{n_a}\right)}}$$

$d_\alpha(a-1, f)$: Appendix IX; α: *Joint significant level*

PAIRWISE COMPARISON

Example: The Tensile Strength Experiment

$$d_{0.05}(4,20)\sqrt{\frac{2MS_E}{n}} = 2.65\sqrt{\frac{2(8.06)}{5}} = 4.76$$

1 vs. 5: 9.8 – 10.8 = -1.0
2 vs. 5: 15.4 – 10.8 = 4.6
3 vs. 5: 17.6 – 10.8 = 6.8
4 vs. 5: 21.6 – 10.8 = 10.8

$\Rightarrow \quad \mu_3 \neq \mu_5$ and $\mu_4 \neq \mu_5$

REGRESSION APPROACH TO ANOVA

Least Squares Estimation of the Model Parameters

$$y_{ij} = \mu + \tau_i + \varepsilon_{ij}$$

The sum of squares of the errors:

$$L = \sum_{i=1}^{a}\sum_{j=1}^{n}\varepsilon_{ij}^2 = \sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij} - \mu - \tau_i\right)^2$$

The values of parameters that minimizes L are the solution of:

$$\frac{\partial L}{\partial \mu}\bigg|_{\hat{\mu},\hat{\tau}_i} = -2\sum_{i=1}^{a}\sum_{j=1}^{n}\left(y_{ij} - \hat{\mu} - \hat{\tau}_i\right) = 0$$

$$\frac{\partial L}{\partial \tau_i}\bigg|_{\hat{\mu},\hat{\tau}_i} = -2\sum_{j=1}^{n}\left(y_{ij} - \hat{\mu} - \hat{\tau}_i\right) = 0 \qquad i = 1, 2, \ldots, a$$

REGRESSION APPROACH TO ANOVA

The *least squares normal equations*

$$
\begin{aligned}
N\hat{\mu} + n\hat{\tau}_1 + n\hat{\tau}_2 + \ldots + n\hat{\tau}_a &= y_{..} \\
n\hat{\mu} + n\hat{\tau}_1 &= y_{1.} \\
n\hat{\mu} + n\hat{\tau}_2 &= y_{2.} \\
&\vdots \\
n\hat{\mu} + n\hat{\tau}_a &= y_{a.}
\end{aligned}
$$

The first equation is redundant $\Rightarrow$ we need another constraint:

$$\sum_{i=1}^{a} \hat{\tau}_i = 0$$

REGRESSION APPROACH TO ANOVA

Solution:
$$\hat{\mu} = \bar{y}_{..}$$
$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} \qquad i = 1, 2, \ldots, a$$

Depend on the additional constraint, different solutions of $\hat{\tau}_i$ may be obtained. However, some estimators are not changed:

$$\hat{\tau}_i - \hat{\tau}_j = \bar{y}_{i.} - \bar{y}_{j.}$$
$$\hat{\mu}_i = \mu + \tau_i = \bar{y}_{i.}$$

REGRESSION APPROACH TO ANOVA

General Regression Significance Test

Writing the Normal Equations

Rule 1: One normal equation for each parameter in the model

Rule 2: RHS is the sum of all observations that contain the parameter

Rule 3: LHS is the sum of all model parameters multiplied by the number of times it appears in the total on the RHS

For example:

$$\begin{aligned} \mu: &\quad N\hat{\mu} + n\hat{\tau}_1 + n\hat{\tau}_2 + \ldots + n\hat{\tau}_a = y_{..} \\ \tau_1: &\quad n\hat{\mu} + n\hat{\tau}_1 \quad\quad\quad\quad\quad\quad\quad = y_{1.} \end{aligned}$$

REGRESSION APPROACH TO ANOVA

Reduction in the Sum of Squares by Fitting a Model

- The reduction in the unexplained variability is the sum of the parameter estimates, each multiplied by the RHS of the corresponding normal equation

$$\boxed{\begin{aligned} R(\mu,\tau) &= \hat{\mu}y_{..} + \hat{\tau}_1 y_{1.} + \hat{\tau}_2 y_{2.} + \ldots + \hat{\tau}_a y_{a.} \\ &= \hat{\mu}y_{..} + \sum_{i=1}^{a} \hat{\tau}_i y_{i.} \end{aligned}}$$

$R(\mu,\tau)$ is also called *the regression sum of squares* from fitting the model $y_{ij} = \mu + \tau_i + \varepsilon_{ij}$, with d.f. = number of linearly independent normal equations (a).

REGRESSION APPROACH TO ANOVA

- The remaining variability unaccounted for by the model:

$$\boxed{SS_E = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - R(\mu,\tau)}$$

- Expressions (with the constraint: $\sum_{i=1}^{a}\hat{\tau}_i = 0$):

$$\begin{aligned} R(\mu,\tau) &= \hat{\mu} y_{..} + \sum_{i=1}^{a}\hat{\tau}_i y_{i.} = (\bar{y}_{..}) y_{..} + \sum_{i=1}^{a}(\bar{y}_{i.} - \bar{y}_{..}) y_{i.} \\ &= \frac{y_{..}^2}{N} + \sum_{i=1}^{a}\bar{y}_{i.} y_{i.} - \bar{y}_{..}\sum_{i=1}^{a} y_{i.} \\ &= \sum_{i=1}^{a}\frac{y_{i.}^2}{n} \end{aligned} \qquad (\text{d.f.} = a)$$

$$\boxed{SS_E = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - R(\mu,\tau) = \sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - \sum_{i=1}^{a}\frac{y_{i.}^2}{n}} \qquad (\text{d.f.} = N - a)$$

REGRESSION APPROACH TO ANOVA

- To find the sum of squares resulting from the treatment effects τ_i's, restrict the model to the null hypothesis $\tau_i = 0, \forall i$: $y_{ij} = \mu + \varepsilon_{ij}$ (the reduced model)

The only normal equation: $N\hat{\mu} = y_{..}$

Hence: $$\boxed{R(\mu) = \hat{\mu} y_{..} = (\bar{y}_{..}) y_{..} = \frac{y_{..}^2}{N}} \quad \text{(d.f. = 1)}$$

The sum of squares due to τ_i:

$$\boxed{\begin{aligned} R(\tau|\mu) &= R(\text{Full Model}) - R(\text{Reduced Model}) \\ &= R(\mu,\tau) - R(\mu) \\ &= \frac{1}{n}\sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N} \end{aligned}} \quad \text{(d.f.} = a-1\text{)}$$

(This is $SS_{Treatments}$)

REGRESSION APPROACH TO ANOVA

The appropriate statistic for testing $H_0 : \tau_1 = \tau_2 = ...\tau_a$ is:

$$\boxed{F_0 = \frac{R(\tau|\mu)/(a-1)}{\left[\sum_{i=1}^{a}\sum_{j=1}^{n} y_{ij}^2 - R(\mu,\tau)\right]\Big/(N-a)}}$$

which is distributed as $F_{a-1,N-a}$ under the null hypothesis (refer to single-factor ANOVA)

NONPARAMETRIC METHODS IN ANOVA

<u>The Kruskal-Wallis Test</u>

- Used when the normality assumption is unjustified
- A nonparametric alternative to the usual ANOVA
- Procedure:

1. Rank the observations y_{ij} in ascending order and replace y_{ij} by its rank R_{ij}.
2. In the case of ties, use the average rank to each of the tied observations
3. Calculate the test statistic:

$$\boxed{H = \frac{1}{S^2}\left[\sum_{i=1}^{a}\frac{R_{i.}^2}{n_i} - \frac{N(N+1)^2}{4}\right]}$$

NONPARAMETRIC METHODS IN ANOVA

where, the variance of the rank is

$$\boxed{S^2 = \frac{1}{N-1}\left[\sum_{i=1}^{a}\sum_{j=1}^{n_i} R_{ij}^2 - \frac{N(N+1)^2}{4}\right]}$$

Note that if there are no ties:

$$\boxed{S^2 = \frac{N(N+1)}{12}} \qquad \boxed{H = \frac{12}{N(N+1)}\sum_{i=1}^{a}\frac{R_{i.}^2}{n_i} - 3(N+1)}$$

4. If n_i's are reasonable large ($n_i \geq 5$), H is distributed as χ^2_{a-1} under the null hypothesis. Hence, the rejection region is defined as:

$$\boxed{H > \chi^2_{\alpha,a-1}}$$

ONE-FACTOR ANALYSIS OF VARIANCE

Example: The Tensile Strength Experiment

Weight Percentage of Cotton									
15		20		25		30		35	
y_{1j}	R_{1j}	y_{2j}	R_{2j}	y_{3j}	R_{3j}	y_{4j}	R_{4j}	y_{5j}	R_{5j}
7	2.0	12	9.5	14	11.0	19	20.5	7	2.0
7	2.0	17	14.0	18	16.5	25	25.0	10	5.0
15	12.5	12	9.5	18	16.5	22	23.0	11	7.0
11	7.0	18	16.5	19	20.5	19	10.5	15	12.5
9	4.0	18	16.5	19	20.5	23	24.0	11	7.0
$R_{i.}$	27.5		66.0		85.0		113.0		33.5

Value of the test statistic:

$$H = \frac{1}{S^2}\left[\sum_{i=1}^{a}\frac{R_{i.}^2}{n_i} - \frac{N(N+1)^2}{4}\right] = 19.25$$

Note that $H > \chi^2_{0.01,4} = 13.28 \quad \Rightarrow \quad$ Reject the null hypothesis

The treatments differ!

NONPARAMETRIC METHODS IN ANOVA

Rank Transformation

- Rank transformation: replacing the observations by its rank
- We can apply the ordinary F test to the rank other than to the original data

$$F_0 = \frac{H/(a-1)}{(N-1-H)/(N-a)}$$

- Rank transformation is widely used in experimental design problems when no *nonparametric alternative* exists.
- Rank transformation is an approximate procedure with good result and it is less likely to be distorted by *nonnormality* and *unusual observations*.
- If the tests conducted on original data and its rank gives different conclusions, the rank transformation should be preferred.

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

Nuisance factors:

- The designs factor that might have an effect on the response but
- We are not interested in that effect

Design technique related to the nature of nuisance factor:

- *Unknown and uncontrollable*: Randomization
- *Known but uncontrollable*: Analysis of Covariance
- *Known and Controllable*: Blocking

Why blocking? Consider the following experiment:

Measuring hardness of metal: pressing the tip into a metal test coupon and record the depth of resulting depression

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

Determine whether different tips produce different readings on the hardness testing machine

Completely randomized design:

4 tip types * 4 observations for each tip = 16 runs (metal coupons)

Problem: Metal coupons might differ slightly in hardness

$\Rightarrow$ Affect the variability observed in the data

$\Rightarrow$ Experimental error = random error + variability

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

Randomized complete block design (RCBD):

- Remove variability between experimental units from the experimental error by

Test each tip once on each of the four coupons

Type of Tip	Test Coupon			
	1	2	3	4
1	9.3	9.4	9.6	10.0
2	9.4	9.3	9.8	9.9
3	9.2	9.4	9.5	9.7
4	9.7	9.6	10.0	10.2

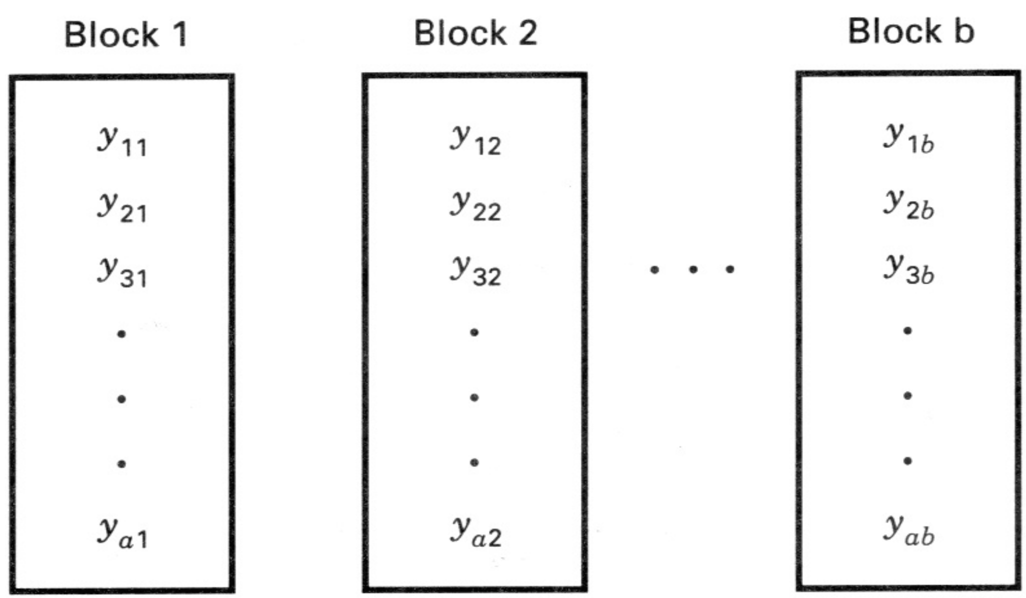
(Measurement unit: *Rockwell C scale hardness* – 40)

Types of tip: Treatments Test coupons: Blocks

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

- Within a block the order in which the four tips are tested is randomly determined ⇒ The blocks represent a ***restriction on randomization***

Analysis of Randomized Complete Block Design



The randomized complete block design.

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

Effect Model:

$$\boxed{y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \quad \begin{cases} i = 1, 2, \ldots, a \\ j = 1, 2, \ldots, b \end{cases}}$$

μ: overall mean

τ_i: ith treatment effect

β_j: jth block effect

ε_{ij}: $\mathrm{NID}(0, \sigma^2)$ random error term

Note: treatment and block effects are usually considered as deviation from the overall mean

$$\Rightarrow \qquad \sum_{i=1}^{a} \tau_i = 0 \quad \text{and} \quad \sum_{j=1}^{b} \beta_j = 0$$

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

The Mean Model:

$$\boxed{y_{ij} = \mu_{ij} + \varepsilon_{ij} \quad \begin{cases} i = 1, 2 \ldots, a \\ j = 1, 2, \ldots, b \end{cases}}$$

in which $\mu_{ij} = \mu + \tau_i + \beta_j = \mu_i + \beta_j$

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

Hypotheses of Interest:

Testing the equality of treatment means:

$$\boxed{\begin{array}{l} H_0 : \mu_1 = \mu_2 = \ldots = \mu_a \\ H_1 : \mu_i \neq \mu_j \text{ for at least one pair } (i, j) \end{array}}$$

or

$$\boxed{\begin{array}{l} H_0 : \tau_1 = \tau_2 = \ldots = \tau_a = 0 \\ H_1 : \tau_i \neq 0 \text{ for at least one } i \end{array}}$$

Notation:

$$y_{i.} = \sum_{j=1}^{b} y_{ij} \qquad \bar{y}_{i.} = \frac{y_{i.}}{b}$$

$$y_{.j} = \sum_{i=1}^{a} y_{ij} \qquad \bar{y}_{.j} = \frac{y_{.j}}{a}$$

$$y_{..} = \sum_{i=1}^{a} \sum_{j=1}^{b} y_{ij} \qquad \bar{y}_{..} = \frac{y_{..}}{ab} = \frac{y_{..}}{N}$$

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

Sums of Squares Definitions

- Total sum of squares = Treatment sum of squares + Block sum of squares + Error sum of square

$$\boxed{SS_T = SS_{Treatments} + SS_{Blocks} + SS_E}$$

in which:

$$\boxed{SS_{Treatments} = b\sum_{i=1}^{a}\left(\bar{y}_{i.} - \bar{y}_{..}\right)^2}$$

$$\boxed{SS_{Blocks} = a\sum_{j=1}^{b}\left(\bar{y}_{.j} - \bar{y}_{..}\right)^2}$$

$$\boxed{SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(y_{ij} - \bar{y}_{..}\right)^2}$$

$$\boxed{SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(y_{ij} - \bar{y}_{.j} - \bar{y}_{i.} + \bar{y}_{..}\right)^2}$$

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

- Computing Formulas

$$\boxed{SS_T = \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{y_{..}^2}{N}}$$

$$\boxed{SS_{Treatments} = \frac{1}{b}\sum_{i=1}^{a} y_{i.}^2 - \frac{y_{..}^2}{N}}$$

$$\boxed{SS_{Blocks} = \frac{1}{a}\sum_{j=1}^{b} y_{.j}^2 - \frac{y_{..}^2}{N}}$$

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

- Mean Square of Treatment

$$\boxed{MS_{Treatments} = \frac{SS_{Treatments}}{df_{Treatments}}}$$

($df_{Treatments} = a - 1$: treatment degrees of freedom)

- Mean Square of Block

$$\boxed{MS_{Blocks} = \frac{SS_{Blocks}}{df_{Blocks}}}$$

($df_{Blocks} = b - 1$: block degrees of freedom)

- Mean Square of Error

$$\boxed{MS_E = \frac{SS_E}{df_E}}$$

($df_E = (a-1)(b-1)$: error degrees of freedom)

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

If treatments and blocks are fixed then:

$$\boxed{E\left[MS_{Treatments}\right]=\sigma^2+\frac{b\sum_{i=1}^{a}\tau_i^2}{a-1}}$$

$$\boxed{E\left[MS_{Blocks}\right]=\sigma^2+\frac{a\sum_{j=1}^{b}\beta_j^2}{b-1}}$$

$$\boxed{E\left[MS_E\right]=\sigma^2}$$

Hence, if *treatment means do differ,* $E\left[MS_{Treatments}\right]>\sigma^2$

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

<u>Statistical Analysis</u>

If the null hypothesis of no difference in treatment means is true, the ratio (the test statistic):

$$F_0 = \frac{MS_{Treatments}}{MS_E}$$

is distributed as $F_{(a-1),(a-1)(b-1)}$.

Rejection Criteria: $\boxed{F_0 > F_{\alpha,(a-1),(a-1)(b-1)}}$

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

ANOVA Table – Randomized Complete Block Design

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0
Between Treatments	$SS_{Treatments}$	$a-1$	$MS_{Treatments}$	$F_0 = \dfrac{MS_{Treatments}}{MS_E}$
Between Blocks	SS_{Blocks}	$b-1$	MS_{Blocks}	$F_0^* = \dfrac{MS_{Blocks}}{MS_E}$
Error	SS_E	$(a-1)(b-1)$	MS_E	
Total	SS_T	$N-1$		

<u>Note</u>: F_0^* can be used to test the equality of block means (*approximate procedure* – due to *randomization restriction*!).

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

Example: Hardness Testing Experiment

Code the data: subtracting 9.5 and multiplying by 10

Type of Tip	Test Coupon 1	2	3	4	$y_{i.}$
1	-2	-1	1	5	3
2	-1	-2	3	4	4
3	-3	-1	0	2	-2
4	2	1	5	7	15
$y_{.j}$	-4	-3	9	18	$y_{..} = 20$

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

Result:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	p-value
Treatments	38.50	3	12.83	14.44	0.0009
Blocks	82.50	3	27.50		
Error	8.00	9	0.89		
Total	129.00	15			

With $\alpha = 0.05$: $F_{0.05,3,9} = 3.86 < F_0$

$\Rightarrow$ The type of tip affects the mean hardness reading

Note that the coupons seem to differ significantly also!

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

Model Adequacy Checking

What to check?

1. Normality assumption
2. Unequal error variance by treatment or block
3. Block – Treatment interaction

By what means? Normal Probability Plot and Residual Analysis

TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN

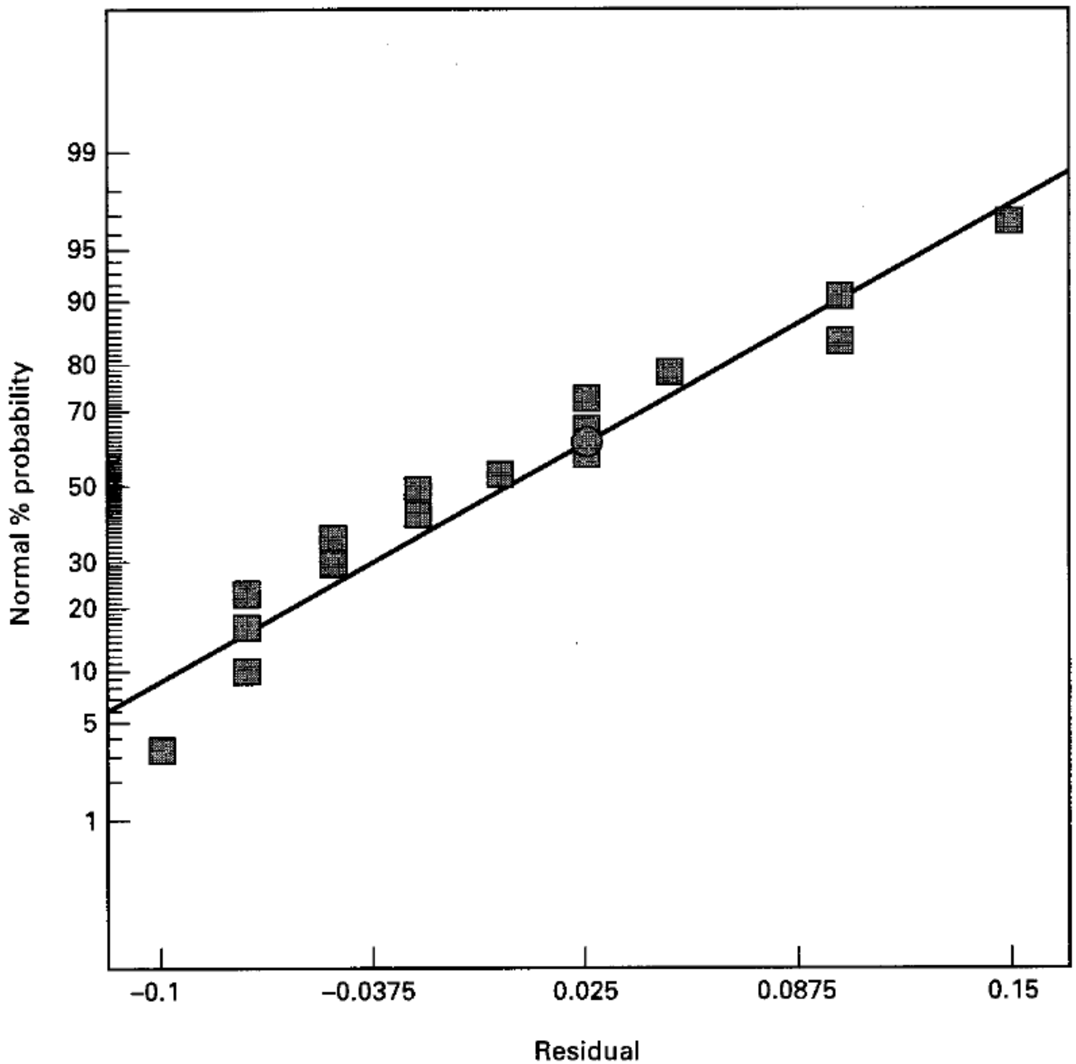
Residuals for the coded hardness testing data:

$$\boxed{e_{ij} = y_{ij} - \hat{y}_{ij}}$$

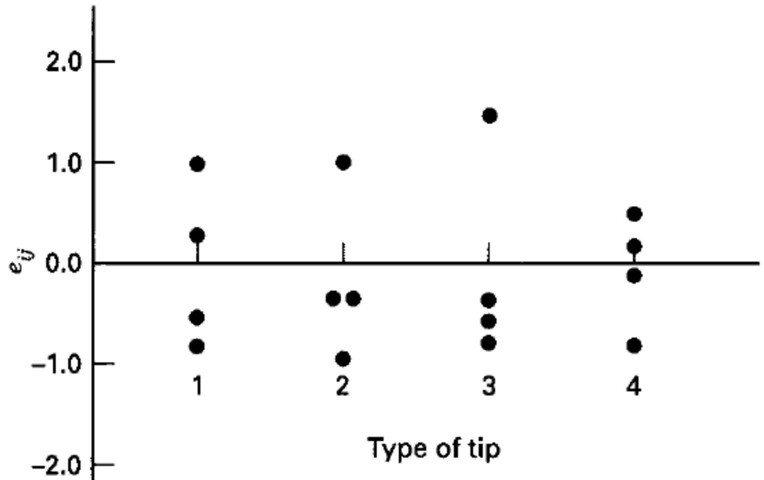
The fitted value: $\boxed{\hat{y}_{ij} = \bar{y}_{i.} + \bar{y}_{.j} - \bar{y}_{..}}$

y_{ij}	-2.00	-1.00	1.00	5.00	-1.00	-2.00	3.00	4.00
$\hat{y}_{ij}$	-1.50	-1.25	1.75	4.00	-1.25	-1.00	2.00	4.25
e_{ij}	-0.50	0.25	-0.75	1.00	0.25	-1.00	1.00	-0.25
y_{ij}	-3.00	-1.00	0.00	2.00	2.00	1.00	5.00	7.00
$\hat{y}_{ij}$	-2.75	-2.50	0.50	2.75	1.50	1.75	4.75	7.00
e_{ij}	-0.25	1.50	-0.50	-0.75	0.50	-0.75	0.25	0.00

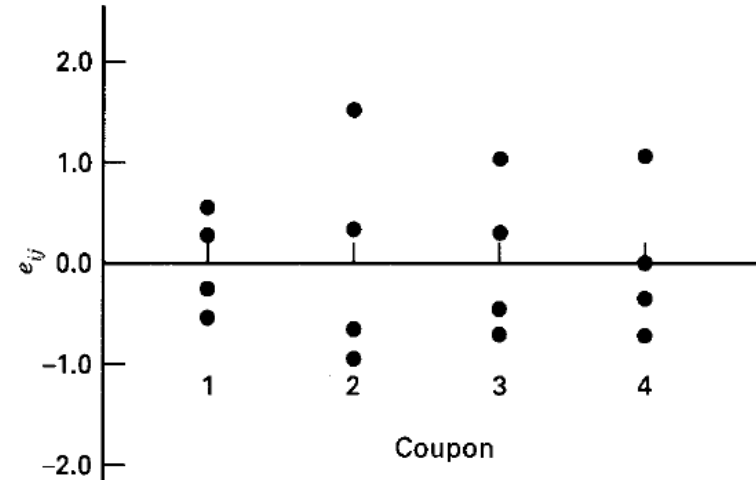
TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN



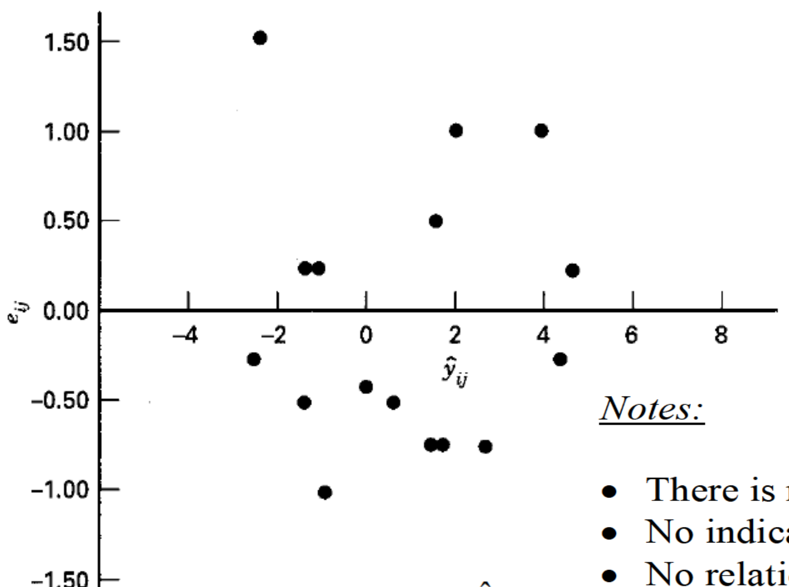
TWO-WAY ANOVA - THE RANDOMIZED COMPLETE BLOCK DESIGN



(a) e_{ij} versus treatment



(b) e_{ij} versus block



e_{ij} versus $\hat{y}_{ij}$

<u>*Notes:*</u>

- There is no severe indication of nonnormality
- No indication of inequality of variance of treatment and block
- No relationship between the residuals and fitted values: *no interaction* between treatments and blocks

ESTIMATE MISSING VALUE

- A missing value causes the treatments no longer orthogonal to blocks

- Two general approaches to deal with missing value:

  1. *Approximate Analysis*: estimate the missing value and use this estimated value in usual ANOVA.

  2. *Exact Analysis*: use General Regression Significant Test

ESTIMATE MISSING VALUE

Approximate Analysis for One Missing Value:

The missing value should be selected so that it has a minimum contribution to the error sum of square

Notation:

x : The missing value

$y'_{..}$: Grand total with one missing observation

$y'_{i.}$: Total for the treatment with one missing observation

$y'_{.j}$: Total for the block with one missing observation

ESTIMATE MISSING VALUE

Approximate Analysis for One Missing Value:

We have

$$SS_E = \sum_{i=1}^{a}\sum_{j=1}^{b}\left(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}\right)$$

$$= \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \frac{1}{b}\sum_{i=1}^{a}\left(\sum_{j=1}^{b} y_{ij}\right)^2 - \frac{1}{a}\sum_{j=1}^{b}\left(\sum_{i=1}^{a} y_{ij}\right)^2 + \frac{1}{ab}\left(\sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}\right)^2$$

$$\Rightarrow \quad SS_E = x^2 - \frac{1}{b}\left(y'_{i.} + x\right)^2 - \frac{1}{a}\left(y'_{.j} + x\right)^2 + \frac{1}{ab}\left(y'_{..} + x\right)^2 + R$$

R: All terms not involving x

From $\dfrac{dSS_E}{dx} = 0 \;\Rightarrow\; \boxed{x = \dfrac{ay'_{i.} + by'_{.j} - y'_{..}}{(a-1)(b-1)}}$

ESTIMATE MISSING VALUE

Approximate Analysis for One Missing Value:

Example: Hardness Testing Experiment

Type of Tip	Test Coupon 1	2	3	4	$y_{i.}$
1	-2	-1	1	5	3
2	-1	-2	$\boxed{x}$	4	1
3	-3	-1	0	2	-2
4	2	1	5	7	15
$y_{.j}$	-4	-3	6	18	$y_{..} = 17$

From the data: $y'_{2.} = 1;\ y'_{.3} = 6;\ y'_{..} = 17 \Rightarrow \quad x = y_{23} = 1.22$

ESTIMATE MISSING VALUE

<u>Approximate Analysis for One Missing Value</u>:

The ANOVA table:

Source of Variation	Sum of Squares	Degrees of Freedom	Mean Square	F_0	p-value
Treatments	39.98	3	13.33	17.12	0.0008
Blocks	79.53	3	26.51		
Error	6.22	**8**	0.78		
Total	125.73	14			

<u>Note</u>:

The error degrees of freedom are reduced by 1.

ESTIMATE MISSING VALUE

More than one missing value:

Approach 1: General

Differentiating the error sum of squares to each missing value.

Approach 2: The case of two missing values

Apply the procedure for the case of one missing value *iteratively*.

GENERAL REGRESSION SIGNIFICANT TEST

The effect model:

$$\boxed{y_{ij} = \mu + \tau_i + \beta_j + \varepsilon_{ij} \quad \begin{cases} i = 1, 2 \ldots, a \\ j = 1, 2, \ldots, b \end{cases}}$$

The set of normal equations:

$$\begin{array}{ll}
\mu: & ab\hat{\mu} + b\hat{\tau}_1 + b\hat{\tau}_2 + \ldots + b\hat{\tau}_a + a\hat{\beta}_1 + a\hat{\beta}_2 + \ldots + a\hat{\beta}_b = y_{..} \\
\tau_1: & b\hat{\mu} + b\hat{\tau}_1 + \hat{\beta}_1 + \hat{\beta}_2 + \ldots + \hat{\beta}_b = y_{1.} \\
\tau_2: & b\hat{\mu} + b\hat{\tau}_2 + \hat{\beta}_1 + \hat{\beta}_2 + \ldots + \hat{\beta}_b = y_{..} \\
\vdots & \qquad \vdots \qquad \vdots \\
\tau_a: & b\hat{\mu} + b\hat{\tau}_a + \hat{\beta}_1 + \hat{\beta}_2 + \ldots + \hat{\beta}_b = y_{a.} \\
\beta_1: & a\hat{\mu} + \hat{\tau}_1 + \hat{\tau}_2 + \ldots + \hat{\tau}_a + a\hat{\beta}_1 = y_{.1} \\
\beta_2: & a\hat{\mu} + \hat{\tau}_1 + \hat{\tau}_2 + \ldots + \hat{\tau}_a + a\hat{\beta}_2 = y_{.2} \\
\vdots & \qquad \vdots \qquad \vdots \\
\beta_b: & a\hat{\mu} + \hat{\tau}_1 + \hat{\tau}_2 + \ldots + \hat{\tau}_a + a\hat{\beta}_b = y_{.b}
\end{array}$$

GENERAL REGRESSION SIGNIFICANT TEST

There are two linear dependencies in the system $\Rightarrow$ we need two more constraints:

$$\sum_{i=1}^{a} \hat{\tau}_i = 0 \quad \text{and} \quad \sum_{j=1}^{b} \hat{\beta}_j = 0$$

The set of normal equation reduced to:

$$\begin{aligned} ab\hat{\mu} &= y_{..} \\ b\hat{\mu} + b\hat{\tau}_i &= y_{i.} \qquad i = 1,2,\ldots,a \\ a\hat{\mu} + a\hat{\beta}_j &= y_{.j} \qquad j = 1,2,\ldots,b \end{aligned}$$

GENERAL REGRESSION SIGNIFICANT TEST

Solution:

$$\hat{\mu} = \bar{y}_{..}$$
$$\hat{\tau}_i = \bar{y}_{i.} - \bar{y}_{..} \qquad i = 1, 2, ..., a$$
$$\hat{\beta}_j = \bar{y}_{.j} - \bar{y}_{..} \qquad j = 1, 2, ..., b$$

The estimated fitted value of y_{ij}:

$$\begin{aligned}\hat{y}_{ij} &= \hat{\mu} + \hat{\tau}_i + \hat{\beta}_j \\ &= \bar{y}_{..} + (\bar{y}_{i.} - \bar{y}_{..}) + (\bar{y}_{.j} - \bar{y}_{..}) \\ &= \bar{y}_{i.} + \bar{y}_{.j} - \bar{y}_{..}\end{aligned}$$

GENERAL REGRESSION SIGNIFICANT TEST

- The regression sum of squares from fitting the full model:

$$\boxed{\begin{aligned} R(\mu,\tau,\beta) &= \hat{\mu}y_{..} + \sum_{i=1}^{a}\hat{\tau}_i y_{i.} + \sum_{j=1}^{b}\hat{\beta}_j y_{.j} \\ &= \sum_{i=1}^{a}\frac{y_{i.}^2}{b} + \sum_{j=1}^{b}\frac{y_{.j}^2}{a} - \frac{y_{..}^2}{ab} \end{aligned}}$$

with d.f. = $(a+b-1)$.

GENERAL REGRESSION SIGNIFICANT TEST

- The error sum of squares:

$$\boxed{\begin{aligned} SS_E &= \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - R(\mu,\tau,\beta) \\ &= \sum_{i=1}^{a}\sum_{j=1}^{b} y_{ij}^2 - \sum_{i=1}^{a}\frac{y_{i.}^2}{b} - \sum_{j=1}^{b}\frac{y_{.j}^2}{a} + -\frac{y_{..}^2}{ab} \\ &= \sum_{i=1}^{a}\sum_{j=1}^{b}\left(y_{ij} - \bar{y}_{i.} - \bar{y}_{.j} + \bar{y}_{..}\right)^2 \end{aligned}}$$

with d.f. $= (a-1)(b-1)$.

GENERAL REGRESSION SIGNIFICANT TEST

- The reduced model to test $H_0 : \tau_i = 0$

$$\boxed{y_{ij} = \mu + \beta_j + \varepsilon_{ij}}$$

(A single-factor ANOVA)

$$\Rightarrow \quad \boxed{R(\mu, \beta) = \sum_{j=1}^{b} \frac{y_{.j}^2}{a}} \qquad (\text{d.f.} = b)$$

The sum of squares due to $\{\tau_i\}$ after fitting μ and $\{\beta_j\}$:

$$\boxed{\begin{aligned} R(\tau | \mu, \beta) &= R(\text{Full Model}) - R(\text{Reduced Model}) \\ &= R(\mu, \tau, \beta) - R(\mu, \beta) \\ &= \sum_{i=1}^{a} \frac{y_{i.}^2}{b} - \frac{y_{..}^2}{ab} \end{aligned}} \qquad (\text{d.f.} = a - 1)$$

(This is $SS_{Treatments}$)

GENERAL REGRESSION SIGNIFICANT TEST

- The block sum of squares is obtained by fitting the reduced model:

$$\boxed{y_{ij} = \mu + \tau_i + \varepsilon_{ij}}$$

(A single-factor ANOVA)

$$\Rightarrow \qquad \boxed{R(\mu,\tau) = \sum_{i=1}^{a} \frac{y_{i.}^2}{b}} \text{(d.f.} = a\text{)}$$

The sum of squares due to $\{\beta_j\}$ after fitting μ and $\{\tau_i\}$:

$$\boxed{\begin{aligned} R(\beta|\mu,\tau) &= R(\text{Full Model}) - R(\text{Reduced Model}) \\ &= R(\mu,\tau,\beta) - R(\mu,\tau) \\ &= \sum_{j=1}^{b} \frac{y_{.j}^2}{a} - \frac{y_{..}^2}{ab} \end{aligned}} \quad \text{(d.f.} = b-1\text{)}$$

(This is SS_{Blocks})

GENERAL REGRESSION SIGNIFICANT TEST

Exact Analysis of the Missing Value Problem

- The missing value causes the design to be *unbalanced* and *not orthogonal*.
- The General Regression Significance Test can be used to exactly analyze the problem (**How?** Let consider the hardness testing experiment!).