Supplementary Material

September 14, 2018

Contents

1 One-Dimensional Condensates with Contact Interactions

The action functional for the condensate wavefunction $\psi(\boldsymbol{x},t)$ in the absence of quantum fluctuations, and for a non-dipolar condensate is given by

$$\mathcal{S} = \int \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,\psi^*(\boldsymbol{x},t)\left\{i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\nabla^2 - \mathcal{V}_{\text{trap}}(\boldsymbol{x}) - \frac{1}{2}gN|\psi(\boldsymbol{x},t)|^2\right\}\psi(\boldsymbol{x},t). \tag{1}$$

where $\mathcal{V}_{\text{trap}}(\boldsymbol{x})$ is the external trapping potential for the condensate, and g is the s-wave scattering coupling constant, where $g = 4\pi\hbar^2 a_s/m$. To physically create a quasi-one-dimensional condensate, an external trapping potential of the following form must be applied,

$$\mathcal{V}_{\text{trap}}(\boldsymbol{x}) = \frac{1}{2}m\omega_\perp^2(x^2+y^2) + \frac{1}{2}m\omega_z^2 z^2, \tag{2}$$

where $\omega_\perp$ is the frequency of the harmonic trapping in the transverse plane, and ω_z is the frequency of the harmonic trapping in the axial direction. With this potential we write our condensate wavefunction as

$$\psi(\boldsymbol{x},t) = f(z,t)\varphi(x,y,\sigma(z,t)), \tag{3}$$

where $f(z,t)$ is the condensate wavefunction in the axial direction, and $\varphi(x,y,\sigma(z,t))$ is the wavefunction in the transverse direction, and $\sigma(z,t)$ describes the width of the condensate in the transverse direction, i.e., the width of the 'cigar,' if you will. The wavefunction in the transverse direction takes the usual Gaussian form, leaving our condensate wavefunction as

$$\psi(\boldsymbol{x},t) = \frac{1}{\sqrt{\pi}\sigma(z,t)}\exp\left\{-\frac{(x^2+y^2)}{2\sigma(z,t)^2}\right\}f(z,t). \tag{4}$$

We can then substitute our wavefunction in Eq. 4 into the action functional in Eq. 1, and then integrate over x and y, leaving a time-integral over a

Lagrangian density in z. We can do this term by term for simplicity. Firstly, the time-derivative of the wavefunction is given by

$$\begin{aligned} i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t} &= i\hbar\frac{\partial}{\partial t}\left\{\frac{f(z,t)}{\sqrt{\pi}\sigma(z,t)}\exp\left\{-\frac{(x^2+y^2)}{2\sigma(z,t)^2}\right\}\right\} \\ &= i\hbar\exp\left\{-\frac{(x^2+y^2)}{2\sigma(z,t)^2}\right\}\left[\frac{\dot{f}(z,t)\sqrt{\pi}\sigma(z,t)-f(z,t)\sqrt{\pi}\dot{\sigma}(z,t)}{\pi\sigma(z,t)^2}\right] \\ &\quad + i\hbar\frac{f(z,t)\dot{\sigma}(z,t)(x^2+y^2)}{\sqrt{\pi}\sigma(z,t)^4}\exp\left\{-\frac{(x^2+y^2)}{2\sigma(z,t)^2}\right\} \end{aligned} \tag{5}$$

For applying the Laplacian to our wavefunction, we first notice that we assume that the transverse wavefunction, $\phi(x,y,\sigma(z,t))$, is slowly varying along the axial, or z, direction, and therefore we make the approximation

$$\nabla^2\phi \approx \nabla_\perp^2\phi, \tag{6}$$

Now, we can compute the integral term-by-term. Firstly, we have

$$\int \mathrm{d}x\,\mathrm{d}y\,\psi^*(\boldsymbol{x},t)\left\{i\hbar\frac{\partial\psi(\boldsymbol{x},t)}{\partial t}\right\} = i\hbar f^*(z,t)\frac{\partial f(z,t)}{\partial t},$$

$$\int \mathrm{d}x\,\mathrm{d}y\,\psi^*(\boldsymbol{x},t)\left[\frac{\hbar^2}{2m}\nabla^2\psi(\boldsymbol{x},t)\right] = \frac{\hbar^2}{2m}\left[f(z,t)^*\frac{\partial^2 f(z,t)}{\partial z^2} - \frac{|f(z,t)|^2}{\sigma(z,t)^2}\right],$$

$$\int \mathrm{d}x\,\mathrm{d}y\,\psi^*(\boldsymbol{x},t)\mathcal{V}_{\mathrm{trap}}\psi(\boldsymbol{x},t) = \frac{1}{2}m\omega_\perp^2|f(z,t)|^2\sigma(z,t)^2 + \frac{1}{2}m\omega_z^2 z^2|f(z,t)|^2,$$

$$\int \mathrm{d}x\,\mathrm{d}y\,\psi^*(\boldsymbol{x},t)\left[-\frac{1}{2}gN|\psi(\boldsymbol{x},t)|^2\right]\psi(\boldsymbol{x},t) = -\frac{gN}{4\pi}\frac{|f(z,t)|^4}{\sigma(z,t)^2}.$$

Then, the action functional, now being an integral over t and z, can be expressed as

$$\begin{aligned} \mathcal{S} = \int \mathrm{d}t \int \mathrm{d}z\, f^*(z,t)\Bigg\{ & i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} - \frac{1}{2}m\omega_z^2 z^2 - \frac{gN}{4\pi}|f(z,t)|^2\sigma(z,t)^{-2} \\ & -\frac{\hbar^2}{2m}\sigma(z,t)^{-2} - \frac{m\omega_\perp^2}{2}\sigma(z,t)^2\Bigg\} f(z,t). \end{aligned} \tag{7}$$

This can be written as $\mathcal{S} = \int \mathrm{d}t\,\mathcal{L}[f,f^*,\sigma]$ where $\mathcal{L}$ is the Lagrangian of our system. Then, we can apply the Euler-Lagrange equations with respect to $f^*(z,t)$ and $\sigma(z,t)$. We have,

$$\begin{aligned} \frac{\partial\mathcal{L}[f,f^*,\sigma]}{\partial f^*} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}[f,f^*,\sigma]}{\partial \dot{f}^*} &= \Bigg\{i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} - V(z) - \frac{gN}{2\pi}|f(z,t)|^2\sigma(z,t)^{-2} \\ &\qquad -\frac{\hbar^2}{2m}\sigma(z,t)^{-2} - \frac{m\omega_\perp^2}{2}\sigma(z,t)^2\Bigg\} f(z,t) \\ &= 0. \end{aligned} \tag{8}$$

Also,

$$\frac{\partial \mathcal{L}[f,f^*,\sigma]}{\partial \sigma} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}[f,f^*,\sigma]}{\partial \sigma} = \frac{g}{2\pi}|f(z,t)|^2\sigma(z,t)^{-3} + \frac{\hbar^2}{m}\sigma(z,t)^{-3} - m\omega_\perp^2\sigma(z,t)$$
$$= 0. \tag{9}$$

Thus, the final equations governing the axial component of the condensate wavefunction, $f(z,t)$, and the width in the transverse direction, $\sigma(z,t)$, are given by

$$i\hbar\frac{\partial f(z,t)}{\partial t} = \left\{-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} + V(z) + \frac{gN}{2\pi}\frac{|f(z,t)|^2}{\sigma(z,t)^2} + \frac{\hbar^2}{2m}\frac{1}{\sigma(z,t)^2} + \frac{m\omega_\perp^2}{2}\sigma(z,t)^2\right\} f(z,t), \tag{10}$$

$$0 = \frac{\hbar^2}{2m}\frac{1}{\sigma(z,t)^3} - \frac{1}{2}m\omega_\perp^2\sigma(z,t) + \frac{gN}{4\pi}\frac{|f(z,t)|^2}{\sigma(z,t)^3}. \tag{11}$$

We can then de-dimensionalise with respect to the units dictated by the trapping potential in the least-confined direction, i.e., the z-direction. We scale according to,

$$\tilde{t} = t\omega_z,\ \tilde{z} = z\ell_z^{-1},\ \tilde{a_s} = a_s\ell_z^{-1},\ \tilde{\sigma} = \sigma\ell_z^{-1},\ |\tilde{f}|^2 = |f|^2\ell_z, \tag{12}$$

where $\ell_z = \sqrt{\hbar/(m\omega_z)}$. Then, de-dimensionalising term-by-term, we have

$$i\hbar\frac{\partial f}{\partial t} \to i\hbar\frac{\partial f}{\partial(\tilde{t}/\omega_z)}$$
$$= i\hbar\omega_z\frac{\partial f}{\partial \tilde{t}}, \tag{13}$$

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial z^2} \to -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial(\tilde{z}^2\ell_z^2)}$$
$$= -\frac{\hbar^2}{2m}\frac{m\omega_z}{\hbar}\frac{\partial^2}{\partial \tilde{z}^2}$$
$$= \frac{1}{2}\hbar\omega_z\frac{\partial^2}{\partial \tilde{z}^2}, \tag{14}$$

$$\frac{1}{2}m\omega_z^2 z^2 \to \frac{1}{2}m\omega_z^2(\tilde{z}\ell_z)^2$$
$$= \frac{1}{2}m\omega_z^2\frac{\hbar}{m\omega_z}\tilde{z}^2,$$
$$= \frac{1}{2}\hbar\omega_z\tilde{z}^2, \tag{15}$$

$$\frac{gN}{2\pi}\frac{|f(z,t)|^2}{\sigma(z,t)^2} = \frac{4\pi\hbar^2 a_s N}{2\pi m}\frac{|f(z,t)|^2}{\sigma(z,t)^2}$$
$$= \frac{2\hbar^2 a_s N}{m}\frac{|f(z,t)|^2}{\sigma(z,t)^2}$$

$$\begin{aligned}
&\to \frac{2\hbar^2\tilde{a}_s N\ell_z}{m}\frac{|\tilde{f}|^2\ell_z^{-1}}{\tilde{\sigma}^2\ell_z^2} \\
&= \frac{2\hbar^2\tilde{a}_s N}{m}\frac{|\tilde{f}|^2}{\tilde{\sigma}^2}\frac{m\omega_z}{\hbar} \\
&= 2\hbar\omega_z\tilde{a}_s N\frac{|\tilde{f}|^2}{\tilde{\sigma}^2},
\end{aligned} \tag{16}$$

$$\begin{aligned}
\frac{\hbar^2}{2m}\frac{1}{\sigma(z,t)^2} &\to \frac{\hbar^2}{2m}\frac{1}{\tilde{\sigma}^2}\frac{m\omega_z}{\hbar} \\
&= \frac{1}{2}\hbar\omega_z\frac{1}{\tilde{\sigma}^2},
\end{aligned} \tag{17}$$

$$\begin{aligned}
\frac{m\omega_\perp^2}{2}\sigma(z,t)^2 &\to \frac{m\omega_\perp^2}{2}\tilde{\sigma}(z,t)^2\frac{\hbar}{m\omega_z} \\
&= \frac{1}{2}\frac{\hbar\omega_\perp^2}{\omega_z}\tilde{\sigma}^2.
\end{aligned} \tag{18}$$

Now, for the algebraic equation, to make matters simpler, we first multiply through by $\sigma(z,t)$, giving

$$0 = \frac{\hbar^2}{2m}\frac{1}{\sigma(z,t)^2} - \frac{1}{2}m\omega_\perp^2\sigma(z,t)^2 + \frac{gN}{4\pi}\frac{|f(z,t)|^2}{\sigma(z,t)^2}. \tag{19}$$

Then, de-dimensionalising in the same manner,

$$\begin{aligned}
\frac{\hbar^2}{2m}\frac{1}{\sigma(z,t)^2} &\to \frac{\hbar^2}{2m}\frac{m\omega_z}{\hbar}\frac{1}{\tilde{\sigma}^2} \\
&= \frac{1}{2}\hbar\omega\frac{1}{\tilde{\sigma}^2},
\end{aligned} \tag{20}$$

$$\begin{aligned}
-\frac{1}{2}m\omega_\perp^2\sigma(z,t)^2 &\to -\frac{1}{2}m\omega_\perp^2\frac{\hbar}{m\omega_z}\tilde{\sigma}^2 \\
&= -\frac{1}{2}\frac{\hbar\omega_\perp^2}{\omega_z}\tilde{\sigma}^2,
\end{aligned} \tag{21}$$

$$\begin{aligned}
\frac{gN}{4\pi}\frac{|f(z,t)|^2}{\sigma(z,t)^2} &= \frac{4\pi\hbar^2 a_s N}{4\pi m}\frac{|f(z,t)|^2}{\sigma(z,t)^2} \\
&= \frac{\hbar^2 a_s N}{m}\frac{|f(z,t)|^2}{\sigma(z,t)^2} \\
&\to \frac{\hbar^2\tilde{a}_s N\ell_z}{m}\frac{|\tilde{f}|^2\ell_z^{-1}}{\tilde{\sigma}^2\ell_z^2} \\
&= \frac{\hbar^2\tilde{a}_s N}{m}\frac{|\tilde{f}|^2}{\tilde{\sigma}^2}\frac{m\omega_z}{\hbar} \\
&= \hbar\omega_z\tilde{a}_s N\frac{|\tilde{f}|^2}{\tilde{\sigma}^2}.
\end{aligned} \tag{22}$$

Then, putting everything together, we have the equations,

$$i\frac{\partial \tilde{f}}{\partial \tilde{t}} = \left\{ -\frac{1}{2}\frac{\partial^2}{\partial \tilde{z}^2} + \frac{1}{2}\tilde{z}^2 + 2\tilde{a}_s N \frac{|\tilde{f}|^2}{\sigma^2} + \frac{1}{2\tilde{\sigma}^2} + \frac{1}{2}\frac{\omega_\perp^2}{\omega_z^2}\tilde{\sigma}^2 \right\} \tilde{f}, \tag{23}$$

$$0 = \frac{1}{2\tilde{\sigma}^2} - \frac{1}{2}\frac{\omega_\perp^2}{\omega_z^2}\tilde{\sigma}^2 + \tilde{a}_s N \frac{|\tilde{f}|^2}{\tilde{\sigma}^2}, \tag{24}$$

We now set $\gamma = \omega_\perp/\omega_z$, and $\tilde{g}_{1\mathrm{D}} = 2\tilde{a}_s N$, and then multiply the algebraic equation through by $2\tilde{\sigma}^2$, we can write the equations in the simple form,

$$i\partial_{\tilde{t}}\tilde{f} = \left\{ -\frac{1}{2}\partial_{\tilde{z}}^2 + \frac{1}{2}\tilde{z}^2 + \frac{1}{2\tilde{\sigma}^2} + \frac{\gamma^2}{2}\tilde{\sigma}^2 + \tilde{g}_{1\mathrm{D}}\frac{|\tilde{f}|^2}{\tilde{\sigma}^2} \right\} \tilde{f}, \tag{25}$$

$$0 = 1 + \tilde{g}_{1\mathrm{D}}|\tilde{f}|^2 - \gamma^2\tilde{\sigma}^4. \tag{26}$$

2 The Fourier Transform of the Dipole-Dipole Interaction

The Gross-Pitaevskii equation for a dipolar condensate in the presence of contact interactions describes the time-evolution of the condensate wavefunction, $\psi(\boldsymbol{x}, t)$, and has the form

$$i\hbar\frac{\partial \psi(\boldsymbol{x}, t)}{\partial t} = \left\{ -\frac{\hbar^2}{2m}\nabla^2 + \mathcal{V}_{\mathrm{trap}}(\boldsymbol{x}) + gN|\psi(\boldsymbol{x}, t)|^2 + \Phi_{\mathrm{dd}} \right\} \psi(\boldsymbol{x}, t). \tag{27}$$

Here, $\Phi_{\mathrm{dd}} = \int \mathrm{d}\boldsymbol{x}'\, \mathcal{V}_{\mathrm{dd}}(\boldsymbol{x} - \boldsymbol{x}')|\psi(\boldsymbol{x}', t)|^2 N$, where

$$\mathcal{V}_{\mathrm{dd}}(\boldsymbol{x}) = \frac{\mu_0\mu^2}{4\pi}\frac{(1 - 3\cos^2\beta)}{|\boldsymbol{x} - \boldsymbol{x}'|^3}. \tag{28}$$

It is interesting to see where this particular dipole-dipole interaction potential comes from. Consider the multipole expansion of a current loop [1]

$$\begin{aligned}
\boldsymbol{A}(\boldsymbol{x}) &= \frac{\mu_0 I}{4\pi}\oint \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}\,\mathrm{d}\boldsymbol{\ell}' \\
&= \frac{\mu_0 I}{4\pi}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\oint (r')^n P_n(\cos\alpha) \\
&= \frac{\mu_0 I}{4\pi}\left\{ \frac{1}{r}\oint \mathrm{d}\boldsymbol{\ell}' + \frac{1}{r^2}\oint r'\cos\alpha\,\mathrm{d}\boldsymbol{\ell}' + \cdots \right\},
\end{aligned} \tag{29}$$

where the point we are evaluating the vector potential is $\boldsymbol{x}$, $\mathrm{d}\boldsymbol{\ell}'$ is the line element of the loop, $\boldsymbol{x}'$ is the vector directed from the origin to the line element $\mathrm{d}\boldsymbol{\ell}'$, I is the current in the loop, μ_0 is the magnetic permeability of free space, and α is the angle between $\boldsymbol{x}$ and $\boldsymbol{x}'$.

Now, the first term in this series expansion is the monopole term, which is zero, since $\nabla \cdot \boldsymbol{B} = 0$. The second term is the one we are interested in, i.e., the dipole term, which is the dominant term in the series expansion. Hence, we can write the magnetic vector potential of a magnetic dipole as

$$\boldsymbol{A}_{\mathrm{dip}}(\boldsymbol{x}) = \frac{\mu_0 I}{4\pi r^2}\oint r'\cos\alpha\,\mathrm{d}\boldsymbol{\ell}' = \frac{\mu_0 I}{4\pi r^2}\oint (\hat{\boldsymbol{x}} \cdot \boldsymbol{x}')\,\mathrm{d}\boldsymbol{\ell}'. \tag{30}$$

We then borrow a from Griffiths (2013) [1] and express the result as

$$\boldsymbol{A}_{\text{dip}}(\boldsymbol{x}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\hat{\boldsymbol{x}}}{r^2}, \tag{31}$$

where $\boldsymbol{m}$ is the *magnetic dipole moment*, which has the form

$$\boldsymbol{m} = I\int \mathrm{d}\boldsymbol{a}. \tag{32}$$

Now, to find the magnetic field from this, we recall that the magnetic field is the curl of the magnetic vector potential. It can be shown (this is actually a in Griffiths) that we can write

$$\boldsymbol{B}_{\text{dip}}(\boldsymbol{x}) = \frac{\mu_0}{4\pi r^3}\Big[3(\boldsymbol{m}\cdot\hat{\boldsymbol{x}})\hat{\boldsymbol{x}} - \boldsymbol{m}\Big]. \tag{33}$$

Now, to find out the interaction energy between two dipoles, we need to find the magnetic field on one dipole due to the other, and then simply calculate $U = -\boldsymbol{m}\cdot\boldsymbol{B}$, when $\boldsymbol{m}$ is the dipole moment of the first, and $\boldsymbol{B}$ is the field caused by the latter. Let us have a dipole at position $\boldsymbol{x}$, with a field as above in Eq. 33, then the field at $\boldsymbol{x}-\boldsymbol{x}'$ is given by

$$\boldsymbol{B}_{\text{dip}}(\boldsymbol{x}-\boldsymbol{x}') = \frac{\mu_0}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|^3}\left[3\left(\frac{\boldsymbol{m}\cdot(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\frac{\boldsymbol{x}-\boldsymbol{x}'}{|\boldsymbol{x}-\boldsymbol{x}'|} - \boldsymbol{m}\right] \tag{34}$$

Then, if we have another dipole at $\boldsymbol{x}-\boldsymbol{x}'$, then interaction energy is just $U = -\boldsymbol{m}\cdot\boldsymbol{B}(\boldsymbol{x}-\boldsymbol{x}')$. This then gives us

$$\begin{aligned} U &= -\boldsymbol{m}\cdot\boldsymbol{B}(\boldsymbol{x}-\boldsymbol{x}') \\ &= -\frac{\mu_0}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|^3}\left[3\left(\frac{\boldsymbol{m}\cdot(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\frac{\boldsymbol{m}\cdot(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} - \boldsymbol{m}\cdot\boldsymbol{m}\right]. \end{aligned} \tag{35}$$

Now, let $|\boldsymbol{m}| = \mu$ be the magnitude of the magnetic dipole moment. Also, consider Fig. 2.1, we can see that the angle between $\boldsymbol{m}$ and $\boldsymbol{x}-\boldsymbol{x}'$ is β, and hence $m\cdot(\boldsymbol{x}-\boldsymbol{x}') = |\boldsymbol{x}-\boldsymbol{x}'|\mu\cos\beta$. This gives us

$$\begin{aligned} U &= -\frac{\mu_0}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|^3}\left[3\left(\frac{\boldsymbol{m}\cdot(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\frac{\boldsymbol{m}\cdot(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|} - \boldsymbol{m}\cdot\boldsymbol{m}\right] \\ &= -\frac{\mu_0}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|^3}\left[3\left(\frac{|\boldsymbol{x}-\boldsymbol{x}'|\mu\cos\beta}{|\boldsymbol{x}-\boldsymbol{x}'|}\right)\frac{|\boldsymbol{x}-\boldsymbol{x}'|\mu\cos\beta}{|\boldsymbol{x}-\boldsymbol{x}'|} - \mu^2\right] \\ &= -\frac{\mu_0}{4\pi|\boldsymbol{x}-\boldsymbol{x}'|^3}\left[3\mu^2\cos^2\beta - \mu^2\right] \\ &= \frac{\mu_0\mu^2}{4\pi}\left[\frac{1-3\cos^2\beta}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\right]. \end{aligned} \tag{36}$$

This is how the formula most often appears in the literature regarding dipolar condensates. For example see B. C. Mulkerin, *et. al.* (2013) [2].

Now, when evaluating the energy functional for the condensed system, we come across the integral

$$\Phi_{\text{dd}} = \int \mathrm{d}\boldsymbol{x}'\left\{\frac{\mu_0\mu^2}{4\pi}\frac{(1-3\cos^2\beta)}{|\boldsymbol{x}-\boldsymbol{x}'|^3}\right\}N|\psi(\boldsymbol{x}',r)|^2, \tag{37}$$

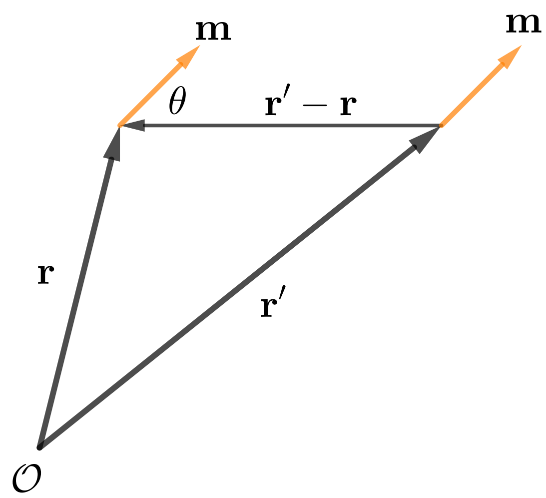

Figure 2.1: Two dipoles with the same dipole moment, and the definition of θ and $\boldsymbol{x}' - \boldsymbol{x}'$.

which we recognise as the convolution between two functions, allowing us to use Fourier analysis and write $\Phi_{\mathrm{dd}} = \mathcal{F}^{-1}\{\tilde{U}_{\mathrm{dd}}|\psi(\tilde{\boldsymbol{x}}, t)|^2$ where $\mathcal{F}^{-1}$ is the inverse Fourier transform, and˜represents the Fourier transform. Thus, it is convenient to know the Fourier transform of the dipole-dipole interaction term.

Consider Fig. 2.2. We have a dipole at the origin, and another at $\boldsymbol{x}$, and hence the inter-dipole position vector is given by $\boldsymbol{x}$. We align the dipole moment $\boldsymbol{m}$ in the xz-plane, at an angle α to the positive z axis. Let β be the angle between $\boldsymbol{m}$ and $\boldsymbol{x}$, as before, and thus we can see that

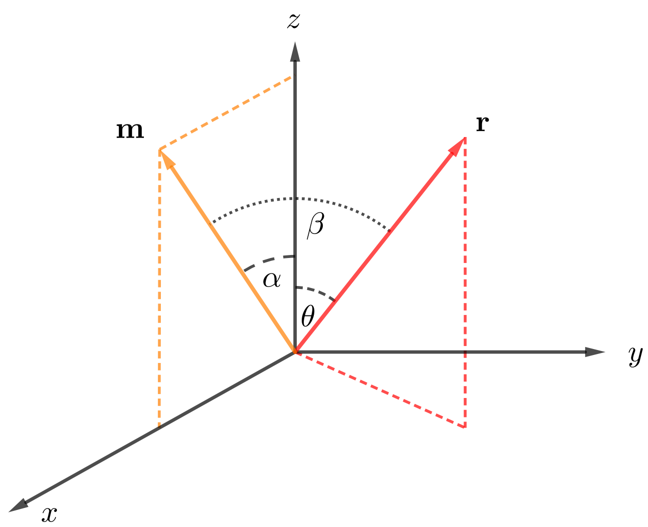

Figure 2.2: The dipole moment $\boldsymbol{m}$ and a given position vector $\boldsymbol{x}$ are separated by an angle β. Furthermore, the dipole moment lies in the xz-plane at an angle α to the positive z-axis.

$$\begin{aligned}\cos\beta &= \frac{\boldsymbol{m}\cdot\boldsymbol{x}}{|\boldsymbol{m}||\boldsymbol{x}|} \\ &= \frac{(m\sin\alpha, 0, m\cos\alpha)\cdot(r\sin\theta\cos\varphi, r\sin\theta\sin\varphi, r\cos\theta)}{rm}\end{aligned}$$

$$= \sin\alpha \sin\theta \cos\varphi + \cos\alpha\cos\theta. \tag{38}$$

Here, we used the fact that $\boldsymbol{x}$ is an arbitrary position vector, and $\boldsymbol{m}$ is represented in spherical coordinates. Then, we take the Fourier transform,

$$\mathcal{F}\left\{\mathcal{V}_{\mathrm{dd}}(\boldsymbol{x})\right\} = \int \mathrm{d}\boldsymbol{x}\, e^{i\boldsymbol{x}\cdot\boldsymbol{k}} \left\{ \frac{\mu_0\mu^2}{4\pi} \frac{(1-3\cos^2\beta)}{|\boldsymbol{x}-\boldsymbol{x}'|^3} \right\} \tag{39}$$

Conveniently, the system is invariant under rotation, and hence we can choose $\boldsymbol{k}\cdot r = kr\cos\theta$, where $k = |\boldsymbol{k}|$ and $r = |\boldsymbol{x}|$. Then, our integral becomes

$$\begin{aligned}
\tilde{\mathcal{V}}_{\mathrm{dd}}(\boldsymbol{k}) &= \frac{\mu_0\mu^2}{4\pi}\int e^{-ikr\cos\theta}\left\{\frac{1-3(\sin\alpha\sin\theta\cos\varphi+\cos\alpha\cos\theta)^2}{r}\right\}\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi,\\
&= \frac{\mu_0\mu^2}{4\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^{\infty} e^{-ikr\cos\theta}\left\{\frac{1-3(\sin\alpha\sin\theta\cos\varphi+\cos\alpha\cos\theta)^2}{r}\right\}\\
&\qquad\qquad\times\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}\varphi\\
&= -\frac{\mu_0\mu^2}{4}\int_0^{\pi}\int_0^{\infty}\frac{e^{-ikr\cos\theta}}{r}\left\{(3\cos^2-1)(3\cos^2\alpha-1)\right\}\sin\theta\,\mathrm{d}r\,\mathrm{d}\theta.
\end{aligned} \tag{40}$$

The integrals are exceedingly difficult to do analytically, and therefore I used *Mathematica*, giving

$$\tilde{\mathcal{V}}_{\mathrm{dd}}(\boldsymbol{k}) = \mu_0\mu^2(1-3\cos^3\alpha)\int_{kb}^{\infty}\left(\frac{\sin u}{u^2}+\frac{3\cos u}{u^3}-\frac{4\sin u}{u^4}\right)\mathrm{d}u, \tag{41}$$

where $u = kr$ and $r = b$ is a small cutoff to stop the divergence at $1/r$ when $r = 0$. This integral can be done, although I used *Mathematica* again, giving

$$\tilde{\mathcal{V}}_{\mathrm{dd}}(\boldsymbol{k}) = \mu_0\mu^2(1-3\cos^2\alpha)\left\{\frac{kb\cos(kb)-\sin(kb)}{(kb)^3}\right\}. \tag{42}$$

We can now remove the artificial cutoff parameter and take the limit as b tends to zero, giving the nice analytic form

$$\tilde{\mathcal{V}}_{\mathrm{dd}}(\boldsymbol{k}) = \mu_0\mu^2(\cos^2\alpha - 1/3). \tag{43}$$

Now, referring back to Fig. 2.2 we can see that the angle α is between $\boldsymbol{m} = \mu\hat{\boldsymbol{m}}$, and the z-axis, and hence it is also the angle between $\boldsymbol{m}$ and $\boldsymbol{k}$, since we assumed that $\boldsymbol{k} = k\hat{\boldsymbol{k}}_z$. We can then rotate out arbitrarily again to get $\boldsymbol{k} = (k_x, k_y, k_z)$ where k_i are not all zero, in which case we garner

$$\begin{aligned}
\cos\alpha &= \frac{\boldsymbol{m}\cdot\boldsymbol{k}}{|\boldsymbol{m}|\cdot|\boldsymbol{k}|}\\
&= \frac{\mu(\sin\alpha\hat{\boldsymbol{k}}_x+\cos\alpha\hat{\boldsymbol{k}}_z)\cdot(k_x\hat{\boldsymbol{k}}_x+k_y\hat{\boldsymbol{k}}_y+k_z\hat{\boldsymbol{k}}_z)}{\mu\sqrt{k_x^2+k_y^2+k_z^2}}\\
&= \frac{k_x\sin\alpha+k_z\cos\alpha}{\sqrt{k_x^2+k_y^2+k_z^2}}.
\end{aligned} \tag{44}$$

Hence,

$$\cos^2\alpha = \frac{k_x^2\sin^2\alpha + k_z^2\cos^2\alpha + 2k_xk_z\cos\alpha\sin\alpha}{k_x^2+k_y^2+k_z^2}$$
$$= \frac{k_x^2\sin^2\alpha + k_xk_z\sin(2\alpha) + k_z^2\cos^2\alpha}{k_x^2+k_y^2+k_z^2}. \quad (45)$$

Thus, the final, general, result of the Fourier transform is given by

$$\tilde{\mathcal{V}}_{\mathrm{dd}}(\boldsymbol{k}) = \frac{\mu_0\mu^2}{3}\left\{\frac{3(k_x^2\sin^2\alpha + k_xk_z\sin(2\alpha) + k_z^2\cos^2\alpha)}{k_x^2+k_y^2+k_z^2} - 1\right\}. \quad (46)$$

3 One-Dimensional Condensates with Contact and Dipole-Dipole Interactions

The relevant component of the action function that pertains to the dipole-dipole interactions is given by

$$\mathcal{S}_{\mathrm{dd}} = -\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,\psi^*(\boldsymbol{x},t)\,\{\Phi_{\mathrm{dd}}\}\,\psi(\boldsymbol{x},t)$$
$$= -\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,\psi^*(\boldsymbol{x},t)\left\{\int \mathrm{d}\boldsymbol{x}'\mathcal{V}_{\mathrm{dd}}(\boldsymbol{x}-\boldsymbol{x}')N|\psi(\boldsymbol{x}',t)|^2\right\}\psi(\boldsymbol{x},t), \quad (47)$$
$$= -\int \mathrm{d}\boldsymbol{x}\,\mathrm{d}t\,\psi^*(\boldsymbol{x},t)\mathcal{F}^{-1}\left\{\tilde{\mathcal{V}}_{\mathrm{dd}}(\boldsymbol{k})N|\mathcal{F}\{|\psi(\boldsymbol{x}',t)|^2\}\right\}\psi(\boldsymbol{x},t). \quad (48)$$

To begin with, let us take the Fourier transform of the modulus of our wave-function squared. This gives us,

$$\mathcal{F}\{|\psi(\boldsymbol{x}',t)|^2\} = \int \mathrm{d}x'\,\mathrm{d}y'\,\mathrm{d}z'\,e^{-2\pi i(k_xx'+k_yy'+k_zz')}\frac{|f(z',t)|^2}{\sigma(z',t)^2\pi}\exp\left\{-\frac{x'^2+y'^2}{\sigma(z',t)^2}\right\}$$
$$= \int \mathrm{d}z'\,|f(z',t)|^2e^{-2\pi ik_zz'}\int \mathrm{d}x'\mathrm{d}y'e^{-2\pi i(k_xx'+k_yy')}\frac{e^{-(x^2+y^2)/(\sigma^2)}}{\sigma^2\pi}$$
$$= e^{-(k_x^2+k_y^2)\pi^2\sigma^2}\int \mathrm{d}z'e^{-2\pi ik_zz'}|f(z',t)|^2$$
$$= e^{-(k_x^2+k_y^2)\pi^2\sigma^2}\mathcal{F}_{z'}\{|f(z',t)|^2\}. \quad (49)$$

Then, substituting this into the relevant component of the action functional,

$$\mathcal{S}_{\mathrm{dd}} = -\int \mathrm{d}x\,\mathrm{d}y\,\frac{f^*(z,t)}{\pi^{1/2}\sigma(z,t)}\exp\left\{-\frac{(x^2+y^2)}{2\sigma(z,t)^2}\right\}\int \mathrm{d}k_x\,\mathrm{d}k_y\,\mathrm{d}k_ze^{2\pi i(k_xx+k_yy+k_zz)}$$
$$\times\left\{C_{\mathrm{dd}}N\left(\frac{k_z^2}{k_x^2+k_y^2+k_z^2}-\frac{1}{3}\right)e^{-(k_x^2+k_y^2)\pi^2\sigma^2}\mathcal{F}_{z'}\{|f(z',t)|^2\}\right\}$$
$$\times\frac{f(z,t)}{\pi^{1/2}\sigma(z,t)}\exp\left\{-\frac{(x^2+y^2)}{2\sigma(z,t)^2}\right\}$$
$$= -\frac{|f(z,t)|^2}{\pi\sigma(z,t)^2}\int \mathrm{d}x\,\mathrm{d}y\,\exp\left\{-\frac{(x^2+y^2)}{\sigma(z,t)^2}\right\} \quad (50)$$

References

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