IL NUOVO CIMENTO VOL. XX, N. 3 1° Maggio 1961

Structure of a Quantized Vortex in Boson Systems (*).

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(ricevuto il 9 Gennaio 1961)

Summary. — For a system of weakly repelling bosons, a theory of the elementary line vortex excitations is developed. The vortex state is characterised by the presence of a finite fraction of the particles in a single particle state of integer angular momentum. The radial dependence of the highly occupied state follows from a self-consistent field equation. The radial function and the associated particle density are essentially constant everywhere except inside a core, where they drop to zero. The core size is the de Broglie wavelength associated with the mean interaction energy per particle. The expectation value of the velocity has the radial dependence of a classical vortex. In this Hartree approximation the vorticity is zero everywhere except on the vortex line. When the description of the state is refined to include the zero point oscillations of the phonon field, the vorticity is spread out over the core. These results confirm in all essentials the intuitive arguments of ONSAGER and FEYNMAN. The phonons moving perpendicular to the vortex line are coherent excitations of equal and opposite angular momentum relative to the substratum of moving particles that constitute the vortex. The vortex motion resolves the degeneracy of the Bogoljubov phonons with respect to the azimuthal quantum number.

1. – Introduction.

The idea that liquid helium permits macroscopic vortex type motions, as does an ordinary liquid, has played a key role in suggesting and interpreting a large number of recent experiments. The experiments of VINEN (1) provide

(*) Work supported by the Office of Scientific Research, U.S. Air Force and by the National Science Fundation.

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(1) H. E. HALL: *Adv. in Phys.*, **9**, 89 (1960); K. R. ATKINS: *Liquid Helium* (Cambridge, 1959); W. F. VINEN: *Physica Suppl.*, **24**, 13 (1958).

convincing evidence for the existence of free vortex lines with a circulation quantized in units of h/m. Superposed on a field of vortex motions is the general phonon field. The idea of ONSAGER [2] and FEYNMAN [3], that circulation is quantized, meets the objection that if vortex excitations of arbitrary circulation and energy were permitted, there would be no superfluidity. On the other hand if there were only phonon-type excitations, it follows from Landau's well known argument based on Galilean invariance, that the critical flow velocity would be much higher than is observed experimentally. With the assumption of quantized vortex lines and rings a qualitative explanation of the low critical velocity becomes possible. The same is true of the behaviour of rotating helium and of many other phenomena.

The description of a vortex quantum mechanically has a number of puzzling aspects. Arguments, for and against the existence of vorticity have been made. None have been precise enough to be universally convincing. Against the existence of vorticity one can argue as follows. In the wave function $\Psi(x_1 \dots x) = R \exp[i(S/\hbar)]$, S plays the role of a velocity potential when the Schrödinger equation is written in hydrodynamic form. The existence of a potential seems to imply that there can be no vorticity. This has been one objection to attempts to relate macroscopic continuum quantum hydrodynamics with vortex motions to the properties of liquid helium. But it has not been shown, with reasonable definitions of the velocity and vorticity, that the vorticity is everywhere zero. For example, if we define the density as

$$n(\boldsymbol{x}) = \int \Psi^* \Psi \sum_{i=1}^{N} \delta(\boldsymbol{x} - \boldsymbol{x}_i)\, \mathrm{d}\tau ,$$

the velocity,

$$\boldsymbol{v}(\boldsymbol{x}) = \frac{1}{n(\boldsymbol{x})} \frac{1}{M} \frac{i\hbar}{2} \sum_{i=1}^{N} \int \Psi^* \left\{ \delta(\boldsymbol{x} - \boldsymbol{x}_i) \frac{\partial}{\partial x_i} + \frac{\partial}{\partial x_i} \delta(\boldsymbol{x} - \boldsymbol{x}_i) \right\} \Psi \, \mathrm{d}\tau ,$$

one would have to show that the vorticity, curl $\boldsymbol{v}(\boldsymbol{x})$, is zero. At best one can make it plausible that if S is slowly varying, and R not very different from the ground state in certain spatial regions, the vorticity is zero. There is an essential distinction between the existence of a potential function $S(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)$ in $3N$ dimensional configuration space and potential flow of average values of operators in ordinary space.

On the other hand, there is the more convincing reasoning of ONSAGER and FEYNMAN, which indicates that there may be vorticity in concentrated regions. We will try to paraphrase their argument. One may start by con-

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(2) L. ONSAGER: *Suppl. Nuovo Cimento*, **6**, 2, 249 (1949).

(3) R. P. FEYNMAN: *Prog. Low Temp. Phys.*, Vol. I (Amsterdam, 1957), ch. 2; *Physica Suppl.*, **24**, 18 (1958).

sidering the flow in a multiply connected region, for example, between concentric cylinders. Corresponding to a given state, say the ground state, there should be other states in which S differs by $\mu \sum_{j=1}^{N} \hbar\vartheta_j$, where μ is an integer. The new motions are those in which each particle has μ units of angular momentum, *i.e.* the system has a total angular momentum $N\hbar\mu$. Without vorticity anywhere, there is nevertheless a circulation. When the inner radius, ϱ, of the region is made small, if it is assumed that the real part of the wave function is still essentially the ground state function, the expectation value of the azimuthal velocity should be proportional to $1/\varrho$, and the vorticity still essentially zero. But the situation becomes unclear as ϱ shrinks to atomic dimensions. An ideal classical line vortex has a vorticity zero everywhere, except on the singular vortex line; and has a characteristic $1/\varrho$ value of the velocity. Quantum mechanically, the $1/\varrho$ behaviour of the velocity can persist as $\varrho \to 0$ only if the real part of the wave function drops essentially to zero on the vortex line. Otherwise the kinetic energy would become infinite. But there must be limitations on the definition of the position of the singular line implied by the uncertainty principle. So, one expects a core in which the density will become small but not necessarily zero, and in which the velocity will be finite. The vorticity should be spread out over the core and should drop continuously and rapidly to zero, as one moves out. That these things happen has not been shown in a detailed theoretical treatment.

Starting from the assumption that there exist such macroscopic excitations with a core of atomic dimensions and with a quantized circulation, but otherwise behaving like a classical vortex, FEYNMAN (3) interpreted a large number of phenomena occurring in helium. This point of view was extended and applied with great success by HALL and VINEN (1) and others. In spite of the fundamental importance of these physical ideas, both practically and conceptually, very little work has been done to relate the ideas to basic quantum mechanics. Instead attention has been concentrated recently on how the phonon-type excitations, which are relatively well understood as a result of the work of LANDAU, BOGOLJUBOV and FEYNMAN, follow from the many-body Hamiltonian. The purpose of the present paper is to construct a theory of the structure of the simple line vortex, and of the superposed phonon-type excitations when a vortex is present. We shall do this for the case of weakly repelling bosons (or for a dilute gas of hard spheres) in the quantized field description of the many-body problem. In this limit a systematic theory is constructed which follows closely, and is essentially an application of the work of references (4a) and (4b). It is close in spirit to Bogoljubov's fundamental

(4) E. P. GROSS: *Ann. Phys.*, *a*) **4**, 57 (1958); *b*) **9**, 292 (1960).

paper ([5]). The results agree in all essentials with the ideas of FEYNMAN and ONSAGER.

The main features are already apparent in the Hartree approximation. That is to say, the possibility of describing a vortex motion in a Bose fluid is contained in this approximation. Each particle is in the same single particle state with angular momentum $\hbar\mu$ about the z-axis. The single particle state is a solution $f^{\mu}(\varrho)\exp[i\mu\vartheta]$ of a self-consistent field (or semiclassical self-interacting field; ref. ([4a])) equation. It has the remarkable property that the density $|f^{\mu}(\varrho)|^2$ is substantially constant at distances greater than a length a. Near the vortex core the density tends to zero like $(\varrho/a)^{2\mu}$. In the semiclassical approximation the expectation value of the azimuthal velocity is strictly proportional to $1/\varrho$, just as for a classical elementary vortex. The core energy per unit length of vortex line is finite. The radius of the core is given very simply as the de Broglie wavelength $a=\hbar/\sqrt{2ME}$, where E is the mean interaction energy per particle $n\int V(x)\,\mathrm{d}^3x$. For a gas of dilute hard spheres, described by a pseudopotential, this is $.15\sqrt{\pi}(\alpha/\sqrt{n\alpha^3})$, where α is the sphere radius and n is the density. These results are obtained in Section **2** from an exact solution of the semiclassical field equation. The connection of such a solution with a Hartree wave function is discussed in ref. ([4a]).

In Section **3** we consider the small oscillations of the semiclassical field about the exact solution, *i.e.* the phonon spectrum in the presence of a vortex. The phonons are described as coherent excitations of pairs of particles of equal and opposite angular momentum relative to the Hartree state of angular momentum $N\hbar\mu$. Far from the cortex core, where the fluid is hardly turning at all, the phonons go over to the usual excitations of Bogoljubov ([5]). The vortex motion removes the degeneracy of the Bogoljubov spectrum with respect to the azimuthal quantum number.

In Section **4** the classical considerations are put in a fully quantum mechanical form. One result is the lowering of the self consistent field estimate of the energy of a vortex line because of the shift in zero point energy of the phonons provided by the normal mode analysis. In addition, consideration of the form of the wave function of a line vortex shows that the zero point motions of the oscillations smear out the density pattern of the Hartree field, yielding a small finite value at the vortex line. At the same time the expectation value of the velocity drops to a finite value (in fact to zero in our approximation), at the vortex line. There is now a definite finite vorticity differing from zero mainly in the core.

The systematic quantum theory has been obtained by first studying in some detail the associated semi classical field theory. Notably, we use special solutions of the semi classical theory to suggest an appropriate single particle

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([5]) N. N. BOGOLJUBOV: *Journ. of Phys. USSR*, **9**, 292 (1960).

basis, in terms of which the quantized field is expanded. We use the small oscillation analysis of the classical field to suggest a suitable quasi-particle transformation. This pattern of analysis can be extended to study more general types of vortex excitations. In order to obtain insight into the reason for the success of the hydrodynamic arguments of Feynman, we transcribe the semi-classical theory into hydrodynamic form in Section **5**. The main differences from the usual equations of compressible flow are the non-local pressure-density functional relation, and a quantum mechanical pressure term. The latter is responsible for the vortex core structure. Provided the cores are well separated, complicated solutions of the equations can be obtained in the same way as in usual hydrodynamic theory. Each pattern of flow represents a wave packet of macroscopic duration. This opens the way to a consideration of general hydrodynamic flows of a system of bosons, on a basis which has a clear and definite quantum counterpart.

2. – Semi-classical theory [4a].

Consider the boson fluid governed by the Hamiltonian

$$(2.1)\qquad H=\frac{\hbar^2}{2M}\int\nabla\psi^+\nabla\psi a^3x+\frac{1}{2}\iint\psi^+(\boldsymbol{x})\,\psi^+(\boldsymbol{x}')\,V(|\boldsymbol{x}-\boldsymbol{x}'|)\,\psi(\boldsymbol{x})\,\psi(\boldsymbol{x}')\,\mathrm{d}^3x\,\mathrm{d}^3x'\,.$$

The equation of motion of the field ψ is

$$(2.2)\qquad i\hbar\dot{\psi}=-\frac{\hbar^2}{2M}\nabla^2\psi+\psi(\boldsymbol{x})\int V(|\boldsymbol{x}-\boldsymbol{x}'|)/|\psi(\boldsymbol{x}')|^2\,\mathrm{d}^3x'\,,$$

and will be studied as a classical field equation, of the self consistent field type in this section.

Let us look for special exact solutions of the equations of motion possessing cylindrical symmetry. We put

$$(2.3)\qquad \psi(\boldsymbol{x})=f(\varrho)\exp[i\mu\vartheta]\exp\left[-i\,\frac{Et}{\hbar}\right],$$

where ϱ, ϑ, z are the cylindrical co-ordinates. The z component of the field angular momentum associated with such a solution is

$$S_z=\frac{\hbar}{2i}\int\left(\psi^+\frac{\partial\psi}{\partial\vartheta}-\frac{\partial\psi^+}{\partial\vartheta}\,\psi\right)\mathrm{d}^3x=\hbar\mu\int|f|^2(\varrho)\,\mathrm{d}^3x\,.$$

But the total number of particles is $N=\int\psi^+\psi\,\mathrm{d}^3x=\int|f|^2\,\mathrm{d}^3x$. Thus the total angular momentum is $N\hbar\mu$, *i.e.* $\hbar\mu$ per particle. In addition the local angular momentum density is constant in time and equal to $|f|^2(\varrho)\hbar\mu$. The component of the velocity of the field at a given space point may be defined as

$$v_\vartheta = \frac{1}{\psi^+\psi}\frac{1}{2M}\frac{\hbar}{i}\left\{\psi^+\frac{1}{\varrho}\frac{\partial\psi}{\partial\vartheta} - \frac{1}{\varrho}\frac{\partial\psi^+}{\partial\vartheta}\psi\right\},$$

and is equal to $\hbar\mu/M\varrho$. It exhibits the characteristic radial dependence of the flow pattern of a classical line vortex. We shall see that in contrast to the usual incompressible (or even compressible) fluid the density $\psi^+\psi = |f(\varrho)|^2$ tends to zero as $\varrho \to 0$ (at the vortex line). However, as $\varrho \to \infty$ the density tends to a constant as is the case for the usual vortex. The vorticity $w_z = (\operatorname{curl} \boldsymbol{v})_z = \frac{1}{2}(1/\varrho)(\partial/\partial\varrho)(\varrho V_\vartheta) = 0$ everywhere except at the singular line. But the circulation is $\Gamma = \int v_\vartheta(\mathrm{d}\vartheta\cdot\varrho) = \mu h/M \neq 0$, so that we may write $v_\vartheta = \Gamma/2\pi\varrho$. The radial function $f(\varrho)$ must satisfy the equation

$$(2.4)\qquad Ef = -\frac{\hbar^2}{2M}\left(\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho}\right) - \frac{\mu^2}{\varrho^2}\right)f(\varrho) + f(\varrho)\int V(|\boldsymbol{x}-\boldsymbol{x}'|)^2|f(\varrho')|^2\,\mathrm{d}^3x'.$$

Since

$$V(|\boldsymbol{x}-\boldsymbol{x}'|) = V\left(\sqrt{(z-z')^2+\varrho^2+\varrho'^2-2\varrho\varrho'\cos(\vartheta-\vartheta')}\right),$$

is invariant to the rotation $\vartheta\to\vartheta+\alpha$, $\vartheta'\to\vartheta'+\alpha$, the assumed separation of variables is indeed consistent.

To examine the behaviour of the function $f(\varrho)$, we consider first for simplicity the case of a short range potential, in fact $V(|\boldsymbol{x}-\boldsymbol{x}'|) = V\,\delta(\boldsymbol{x}-\boldsymbol{x}')$. We have in mind, more precisely, a pseudopotential [6]. At this stage of the theory it can be treated as a δ-function. Then

$$(2.5)\qquad Ef = -\frac{\hbar^2}{2M}\left(\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho}\right) - \frac{\mu^2}{\varrho^2}\right)f + Vf|f(\varrho)|^2,$$

with

$$\int|f(\varrho)|^2\,\mathrm{d}^3x = N.$$

(For the s-wave pseudopotential $V = 8\pi\alpha(\hbar^2/M)$, where α is the radius of the hard spheres.)

$f(\varrho)$ may be taken as real. For small ϱ, the centrifugal force term dominates and

$$(2.6)\qquad f(\varrho)\to j_\mu\left(\sqrt{\frac{2ME}{\hbar^2}}\,\varrho\right)\qquad\text{as}\qquad \varrho\to 0.$$

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[6] K. HUANG and C. N. YANG: *Phys. Rev.*, **105**, 767 (1957); K. HUANG, T. D. LEE and C. N. YANG: *Phys. Rev.*, **106**, 1136 (1975).

As $\varrho \to \infty$

$$f(\varrho) \to f_0 = \text{constant} . \tag{2.7}$$

We must have $E = V f_0^2$. The last remaining constant, f_0^2, is fixed by the normalization condition $\int |f^2| \, d^3x = N$. This has a small finite contribution from the core of the vortex as $N \to \infty$, $\Omega \to \infty$. In this limit

$$f_0^2 \to \frac{N}{\Omega} = \frac{N}{L\pi R^2} ,$$

where L is the extent of the system in the z direction and R is the radius of the cylinder of « quantization ». Thus f_0^2 is the mean number density. The behaviour of f as $\varrho \to \infty$ is quite remarkable. It is brought about by the non linear term $Vf|f|^2$, *i.e.* the self-consistent field. It is physically clear that the repulsive forces should force the system to uniform density almost everywhere.

The energy of the fluid is obtained by substituting the solution $f^\mu(\varrho)$ in H

$$H = -\frac{\hbar^2}{2M} \int f^\mu(\varrho) \left\{ \frac{1}{\varrho} \frac{d}{d\varrho} \left(\varrho \frac{d}{d\varrho} \right) - \frac{\mu^2}{\varrho^2} \right\} f^\mu(\varrho) \, d^3x + \\ + \frac{1}{2} \iint (f^\mu(\varrho))^2 \, V(\boldsymbol{x} - \boldsymbol{x}') \, (f^\mu(\varrho'))^2 \, d^3x \, d^3x' . \tag{2.8}$$

For the vortex-free case ($\mu = 0$) the lowest state is $f^0 = f_0 = \text{const.}$, with an energy $H = \frac{1}{2} f_0^4 \int V(s) \, d^3s$. When a vortex is present ($\mu^0 \neq 0$), $f^\mu(\varrho) \neq f_0$ everywhere and there is a finite correction to the potential energy per unit length of the vortex, arising from the core. The behaviour of the kinetic energy is, however, more important. If the kinetic energy is evaluated with $f^\mu(\varrho) = f_0$ everywhere, the result is divergent, as $\varrho \to 0$. However, the actual $f^\mu(\varrho) \to 0$, as $\varrho \to 0$, so that the density of fluid tends to zero as the velocity tends to ∞, in such a way that the kinetic energy per unit length of the vortex core is finite. The region outside the core ($\varrho > a$) contributes a kinetic energy

$$\frac{\hbar^2}{2M} \mu^2 \int \frac{f_0^2}{\varrho^2} \, \varrho \, d\varrho \cdot 2\pi L = \frac{\hbar^2 \mu^2}{2M} f_0^2 \cdot 2\pi L \ln \frac{R}{a} = \frac{M\Gamma^2}{8\pi^2} f_0^2 \ln \frac{R}{a} (2\pi L) ,$$

i.e., the characteristic logarithmic dependence on the outer radius of the vortex.

For a more general, but still essentially short-range potential, $f(\varrho)$ still tends to a constant f_0 with

$$E = f_0^2 \int V(S) \, d^3S . \tag{2.9}$$

The behaviour for $\varrho \to 0$ is again the same as for a δ-function potential, and is determined entirely by the centrifugal potential.

It is of course not easy to find exact solutions of the self consistent field equation for $f^\mu(\varrho)$. However, it is easy to find an accurate estimate of the size of the vortex core. We use our knowledge of the exact behaviour at small and large distances and assume that

$$
\begin{aligned}
f^\mu(\varrho) &= A j_\mu\left(\sqrt{\frac{2ME}{\hbar^2}}\, a\right), \qquad & \varrho < a\,, \\
&= f_0\,, & \varrho > a\,.
\end{aligned}
\tag{2.10}
$$

A is fixed by the continuity requirement at $\varrho = a$,

$$
A = f_0 / j_\mu\left(\sqrt{\frac{2ME}{\hbar^2}}\, a\right).
\tag{2.11}
$$

The continuity of the radial derivatives of f requires

$$
\frac{\mathrm{d}j_\mu}{\mathrm{d}\varrho}\left(\sqrt{\frac{2ME}{\hbar^2}}\,\varrho\right)\Big/\varrho = a = 0\,.
\tag{2.12}
$$

This fixes the core size a_μ in terms of the position of the first zero of $\mathrm{d}j^\mu/\mathrm{d}\varrho$. We have

$$
\begin{array}{lll}
\mu = 1 & a_1 = \dfrac{\hbar}{\sqrt{2ME}} & \pi(.59)\,. \\
\mu = 2 & a_2 = & \pi(.97)\,, \\
\mu \text{ large} & a_\mu = & (\mu - .26\pi\mu^{\frac{1}{3}})\,.
\end{array}
$$

For $\mu = 1$ the core size is of the order of the de Broglie wavelength associated with the mean energy of interaction per particle $E = Vf_0^2$. For a dilute gas of hard spheres of radius α, mass M, $V = 8\pi(\alpha\hbar^2/M)$ and we have

$$
a' = \left(\frac{.59}{4}\sqrt{\pi}\right)\frac{\alpha}{\sqrt{\alpha^3 f_0^2}}\,,
$$

i.e.: the vortex radius is larger then the hard sphere radius by the factor

$$
\frac{1}{4}\left(\frac{\text{particle separation}}{\text{sphere radius}}\right)^{\frac{3}{2}}.
$$

This indicates that in a more dense gas, the vortex would have a size about equal to the interparticle spacing. But we cannot be sure for the zero point motions of the phonons are as important as the self-consistent field.

The energy associated with this approximate solution may be obtained readily. We define the pure numbers

$$(2.13)\qquad \begin{cases} C_\mu = \dfrac{\int_0^{S_\mu} j_\mu^2 S\,\mathrm{d}S}{j_\mu^2(S_\mu)}\,, \qquad D_\mu = \dfrac{\int_0^{S_\mu} j_\mu^4 S\,\mathrm{d}S}{j_\mu^4(S_\mu)}\,, \\ C_\mu = \dfrac{S_\mu^2}{2}\left[1 - \dfrac{j_0(S_\mu)\,j_1(S_\mu)}{j_1^2(S_\mu)}\right], \end{cases}$$

where S^μ are the roots $S_1 = (.59)$, $S_2 = (.97)$, ... Then the normalization of $\int f^2\,\mathrm{d}^3x = N$ yields

$$(2.14)\qquad \frac{N}{2\pi L} = f_0^2\left\{\frac{a^2}{S_\mu^2}C_\mu + \frac{R^2 - a^2}{2}\right\}.$$

The energy is

$$(2.15)\qquad \frac{H}{2\pi L} = f_0^2\,\frac{\hbar^2}{2M}\left(C_\mu + \frac{D_\mu}{2}\right) + \frac{Vf_0^4}{2}(R^2 - a^2) + \frac{\hbar^2}{2M}\ln\frac{R}{a}.$$

In addition to the potential energy of the quiescent fluid and the logarithmic vortex energy there is the finite correction (as $R \to \infty$) per unit length of the vortex core. If we write $\ln(R/a) = \ln(R/b) - \ln(a/b)$, we can make the corrections vanish by taking

$$C_\mu + \frac{D_\mu}{2} - \frac{1}{4} = \ln\frac{a}{b}\,; \qquad b_1 \sim \frac{a_1}{5}.$$

Then b is a «pseudo» radius of the vortex core.

We might improve the above estimates by taking a more refined trial function $f^\mu(\varrho)$ and using the variation principle for the energy with f subject to $\int f^2\,\mathrm{d}^3x = N$. But the results differ from the above in only unimportant ways. It is more important to consider the possibility, that for a given μ, there exist other solutions, with nodes, of the self-consistent field equations. This requires more detailed study. If they exist, they would have to be stable to be directly relevant.

We note in conclusion that the results of this section may be obtained directly in configuration space. The Hartree approximation is $\Psi^\mu(\boldsymbol{x}, \ldots, \boldsymbol{x}_N) = (1/\sqrt{N})\prod_{j=1}^{N} f^\mu(\varrho_j)\exp[i\mu\vartheta_j]$, and represents a fixed number of particles. The function $f^\mu(\varrho_j)$ is the same one that is used in the field picture. The advantage of the field point of view appears in the introduction of quasi-particles, as discussed in the next section. This is done at the expense of working with an indefinite number of particles, and an indefinite angular momentum.

3. – Small oscillations of the fluid about a vortex motion.

We have shown that there is an exact solution of the classical equations of motion involving circulation. It differs from a hydrodynamic vortex in that the structure of the core is fully determined, and the density goes to zero at the singular vortex line.

In the classical theory of the wave field, the next step is to examine the small oscillations of the system about the exact solution. We put

$$\psi = \exp\left[-i\,\frac{Et}{\hbar}\right]\{f^{\mu}(\varrho)\exp[i\mu\vartheta] + \varphi(\boldsymbol{x}, t)\}\,, \tag{3.1}$$

and linearize the equations of motion. One finds for a δ-function potential

$$i\hbar\dot{\varphi} = -\frac{\hbar^2}{2M}\nabla^2\varphi + (V|f(\varrho)|^2 - E)\varphi + Vf^2(\varrho)\exp[i\mu\vartheta]\cdot \\ \cdot\{\exp[-i\mu\vartheta]\varphi + \exp[i\mu\vartheta]\varphi^+\}\,, \tag{3.2}$$

together with the complex conjugate equation.

It corresponds to the Hamiltonian

$$H_2 = -\frac{\hbar^2}{2M}\int\varphi^+\nabla^2\varphi\,\mathrm{d}^3x + V\int f^2(\varrho)\,\{\varphi^+\exp[i\mu\vartheta] + \varphi\exp[-i\mu\vartheta]\}^2\,\mathrm{d}^3x\,, \tag{3.3}$$

which is positive definite. The same result holds for more general positive interactions $V(|\boldsymbol{x}-\boldsymbol{y}|)$.

To analyse the normal mode spectrum, we define a complete set of orthonormal functions as the eigenfunctions of the linear operator

$$L^{\mu} = -\frac{\hbar^2}{2M}\nabla^2 + V|f^{\mu}(\varrho)|^2 - E\,. \tag{3.4}$$

We shall continue to work with the δ-function, in spite of the fact that one must use a pseudopotential to avoid a divergent total zero-point energy from the shifted normal modes. The spectrum itself is quite definite and finite.

We write

$$L^{\mu}\varphi^{\mu}_{\varkappa,\sigma,m} = \left(E^{\mu}_{\sigma,m} - E + \frac{\hbar^2\varkappa^2}{2M}\right)\varphi^{\mu}_{\varkappa,\sigma,m}\,, \tag{3.5}$$

where

$$(3.6)\qquad \begin{cases} \varphi_{\varkappa,\sigma,m} = \dfrac{1}{\sqrt{2\pi L}} \exp[i\varkappa z] \exp[im\vartheta] g^{\mu}_{\sigma,m}(\varrho) , \\ \displaystyle\int g^{*\mu}_{\sigma,m}(\varrho) g^{\mu}_{\sigma',m}(\varrho) \varrho \, d\varrho = \delta_{\sigma,\sigma'} \end{cases}$$

The superscript μ is to emphasize that the set of functions depends on the state of vortex excitation. The eigenvalues E^{μ}_{σ} depend on $|m|$ because of the cylindrical symmetry of the potential $V|f^{\mu}(\varrho)|^2$. We will work with standing waves. The functions $g^{\mu}_{\sigma,m}(\varrho)$ can be taken to be real and defined in a large cylindrical region, with the boundary conditions

$$(3.7)\qquad g^{\mu}_{\sigma,m}(\varrho = R) = 0 .$$

We note that

$$g^{\mu}_{0,\mu} = \frac{1}{\sqrt{N}} f^{\mu}(\varrho) , \qquad E^{\mu}_{0,\mu} = E ,$$

where we write $\sigma = 0$ for the lowest possible value of σ. We also have

$$g^{\mu}_{\sigma,m} = g^{\mu}_{\sigma,-m} ; \qquad \varphi^{*\mu}_{\varkappa,\sigma,m} = \varphi^{\mu}_{-\varkappa,\sigma,-m} .$$

Let us now expand

$$(3.8)\qquad \begin{cases} \varphi \;\; = \sum a_{\varkappa,\sigma,m} \varphi_{\varkappa,\sigma,m} , \\ \varphi^{+} = \sum a^{+}_{\varkappa,\sigma,m} \varphi^{*}_{\varkappa,\sigma,m} = \sum a^{+}_{-\varkappa,\sigma-m} \varphi_{\varkappa,\sigma,m} . \end{cases}$$

Then eq. (3.2) is equivalent to

$$(3.9)\qquad \left\{ i\hbar \frac{\partial}{\partial t} - \left(E^{\mu}_{\sigma,m} - E + \frac{\hbar^2 \varkappa^2}{2M} \right) \right\} a_{\varkappa,\sigma,m} = \\ = \sum_{\sigma'} \{ F^{\mu}(\sigma m | \sigma' m) a_{\varkappa,\sigma',m} + F \;(\sigma m | \sigma', 2\mu - m) a^{+}_{-\varkappa,\sigma',2\mu-m} \} ,$$

where

$$F^{\mu}(\sigma m | \sigma' m') = V \int g^{\mu}_{\sigma,m} g^{\mu}_{\sigma',m'} (f^{\mu})^2 \varrho \, d\varrho , \qquad F^{*} = F .$$

It follows that

$$(3.10)\qquad \left\{ - i\hbar \frac{\partial}{\partial t} - \left(E^{\mu}_{\sigma,2\mu-m} - E + \frac{\hbar^2 \varkappa^2}{2M} \right) \right\} a^{+}_{-\varkappa,\sigma,2\mu-m} = \\ = \sum_{\sigma'} \{ F^{\mu}(\sigma, 2\mu - m | \sigma', 2\mu - m) a^{+}_{-\varkappa,\sigma',2\mu-m} + F^{\mu}(\sigma, 2\mu - m | \sigma', m) a_{\varkappa,\sigma',m} \} .$$

The dominant feature of these equations is the strong coupling of $a_{\varkappa,\sigma,m}$ and $a^+_{-\varkappa,\sigma,2\mu-m}$. There is a less important coupling of the radial modes. In the absence of vortex motion ($\mu=0$), the coupling is that of equal and opposite linear momenta, *i.e.* Bogoljubov's theory, here desciibed in cylindrical co-ordinates. When there is a vortex, the annihilation operator of angular momentum $m-\mu$ relative to the vortex angular momentum is strongly coupled to the creation operator of angular momentum $-(m-\mu)$ relative to the vortex. This actually involves a net loss of angular momentum in the small amplitude approximation. In the quantum version the state vector for the vortex contains pairs of particles of equal and opposite angular momentum excited out of the vortex « reservoir ». There is a loss of 2μ units of angular momentum for each pair, since it comes from the moving substratum which is treated as fixed. The state is one of indefinite angular momentum and is analogous to, but in addition to, the description in terms of an indefinite number of particles in the usual boson theory.

Let us consider only the terms in which $\sigma'=\sigma$, thus neglecting the radial mode coupling. We look for solutions where a and a^+ have the time behaviour $\exp[-i\varepsilon t/\hbar]$. (The fact that this violates the requirement that φ^+ be the hermitian conjugate of φ is taken care of by the fact that there is a corresponding solution $\exp[+i\varepsilon t/\hbar]$.)

$$a_{\varkappa,\sigma,m}=A_{\varkappa,\sigma,m}\exp[-i\varepsilon t/\hbar]\,,\qquad a^+_{\varkappa,\sigma,m}=\exp[-i\varepsilon t/\hbar]A^+_{\varkappa,\sigma,m}\,. \tag{3.11}$$

The energy ε is given as

$$\begin{aligned}\varepsilon^2+\varepsilon\{F(\sigma,\,2\mu-m)-F(\sigma m\,|\,\sigma m)\}&+\Delta_{\sigma,\varkappa,2\mu-m}-\Delta_{\sigma,\varkappa,m}\,, =\\ &=\{\Delta_{\sigma,\varkappa,2\mu-m}+F(\sigma,\,2\mu-m\,|\,\sigma,\,2\mu-m)\}\{\Delta_{\sigma,\varkappa,m}+F(\sigma m\,|\,\sigma m)\}-\\ &-F(\sigma m\,|\,\sigma,\,2\mu-m)\,F(\sigma,\,2\mu-m\,|\,\sigma,\,m)\,,\end{aligned} \tag{3.12}$$

$$\Delta_{\sigma,\varkappa,m}=E^{\mu}_{\sigma,m}-E+\frac{\hbar^2\varkappa^2}{2M}\,.$$

The eigenvector is

$$\frac{A_{\varkappa,\sigma,m}}{A^+_{-\varkappa,\sigma,2\mu-m}}=\frac{F(\sigma m\,|\,\sigma,\,2\mu-m)}{\varepsilon-\Delta_{\sigma,\varkappa,m}-F(\sigma m\,|\,\sigma m)}\,. \tag{3.13}$$

In complete analogy with the usual boson theory, the variables

$$\left\{\begin{aligned}&b_{\varkappa,\sigma,m}=\cosh\gamma a_{\varkappa,\sigma,m}+\sinh\gamma a^+_{-\varkappa,\sigma,2\mu-m}\,,\\ &b^+_{-\varkappa,\sigma,2\mu-m}=\sinh\gamma a_{\varkappa,\sigma,m}+\cosh\gamma a^+_{-\varkappa,\sigma,2\mu-m}\,,\\ &\gamma=\gamma_{\varkappa,\sigma,m}\,,\end{aligned}\right. \tag{3.14}$$

are connected by a canonical transform to the a, a^+. They oscillate harmonically as

$$i\hbar\, \dot{b}_{\varkappa,\sigma,m} = \varepsilon_{\varkappa\sigma m}\, b_{\varkappa,\sigma,m}\,.$$

(Compare eq. (4.12) to (4.16) for the values of $\gamma_{\varkappa,\sigma,m}$.)

For the case that there is no vortex, we should recover the results of BOGOLJUBOV. Let us review how this comes about. Our treatment is unfamiliar, first because we are using cylindrical co-ordinates, and second because we have expanded in standing waves. The first step has been the determination of the self-consistent field $f^\mu(\varrho)$. The function $f^0(\varrho)$ obeys equation (2.5) with $\mu = 0$. It is independent of ϱ as $\varrho \to 0$. We have also $E = V(f^0)^2$. But with the boundary condition (3.7), as $\varrho \to R$, f^0 must adjust to our demand $f^0(\varrho = R) = 0$. The function drops to zero within a distance $\hbar/\sqrt{2ME}$ of the cylinder of quantization. This is the same characteristic length that occurs in the core size argument. Thus the ground state contains a finite fraction of the particles in a single-particle state which is essentially a constant except near the boundary. This is substantially the starting point of the usual theory as developed with periodic boundary conditions. The details of $f^0(\varrho)$ near the boundary complicate the analysis, but are not essential for many purposes. It is clear that the semiclassical starting point offers a natural way of treating general boundary conditions. The highly occupied single-particle state, as determined by a self-consistent field equation, tends to uniform density except near the boundary. One can impose the condition that the wave function vanish, just as one can impose periodic conditions, without creating artificial difficulties which occur in other methods. These difficulties arise because the standing-wave solutions of the Schrödinger equation for non-interacting particles have large density variations throughout the box.

When there is no vortex, the functions $g^0_{\sigma,m}(\varrho)$ are ordinary Bessel functions $j_m(\sigma\varrho)$, with the quasi-continuous values of σ determined by $j_m(\sigma R) = 0$ and with $E_\sigma - E = \hbar^2\sigma^2/2M$. The energy normal-mode-frequencies are

$$\varepsilon^2 = \frac{\hbar^2}{2M}(\sigma^2 + \varkappa^2)\left\{\frac{\hbar^2}{2M}(\sigma^2 + \varkappa^2) + 2F^0(\sigma m\,|\,\sigma m)\right\}. \tag{3.15}$$

Now the matrix element

$$F^0(\sigma m\,|\,\sigma m) = \int g^0_{\sigma m}(\varrho)\, g^0_{\sigma' m}(f^0)^2 \varrho\, \mathrm{d}\varrho\,,$$

is just $\delta_{\sigma,\sigma'}(f^0)^2$ and is independent of m. Thus the frequency spectrum of the excitations is degenerate with respect to the azimuthal quantum number. Eq. (3.15) is of course Bogoljubov's spectrum, as written in cylindrical coordinates. In the special case of no vortex, the entire Hamiltonian H_2 is made diagonal by the simple normal mode transformation, *i.e.* there is no radial mode coupling. Thus we see how the usual theory is contained in the present description.

When there is a quantized vortex, there are modifications of the normal mode frequencies and eigenvectors. The most striking feature of eq. (3.12) is the removal of the m degeneracy. The only remaining degeneracy is that eq. (3.12) is invariant to the substitution $\varepsilon \to -\varepsilon$, $m \rightleftarrows 2\mu - m$. This reflects the possibility of forming standing waves relative to the vortex. The modes with different m values represent different motions relative to the vortex and are therefore perturbed differently.

A more detailed study of the normal mode spectrum may be made if one takes a crude approximation to the « self-consistent » potential

$$U(\boldsymbol{x}) = \int V(|x - x'|)\,|f^{\mu}(\varrho')|^2\,\mathrm{d}^3x' ,$$

$$\text{(3.16)} \qquad \begin{cases} U(\boldsymbol{x}) \sim 0 & \text{for } \varrho < a , \\ \qquad E & \text{for } \varrho > a , \end{cases}$$

where a depends on μ.

Thus a cylindrical well will be used to generate an approximation to the orthonormal basis $\varphi_{\varkappa,\sigma,m}$. This leads to

$$\text{(3.17)} \qquad \begin{cases} g^{\mu}_{\sigma,m} = \alpha\, j_m(\sigma' a) , & \varrho < a , \\ \qquad = \beta\, j_m(\sigma\varrho) + \gamma\, n_m(\sigma\varrho) , & \varrho > a , \end{cases}$$

where

$$\sigma' = \left(\frac{2ME_\sigma}{\hbar^2}\right)^{\frac{1}{2}} , \qquad \sigma = \left(\frac{2M}{\hbar^2}\,(E_\sigma - E)\right)^{\frac{1}{2}} ,$$

and α, β, γ depend on σ and m. The conditions that $g_{\sigma,m}$ and its radial derivative be continuous at $\varrho = a$ fix β and γ in terms of α, *i.e.*

$$\text{(3.18)} \qquad \begin{cases} \alpha j_m(\sigma' a) = \beta j_m(\sigma a) + \gamma n_m(\sigma a) , \\ \alpha\sigma' \dfrac{\mathrm{d}j_m}{\mathrm{d}(\sigma' a)} = \beta\sigma \dfrac{\mathrm{d}j_m(\sigma a)}{\mathrm{d}(\sigma a)} + \gamma\sigma \dfrac{\mathrm{d}n_m(\sigma a)}{\mathrm{d}(\sigma a)} . \end{cases}$$

In turn, α is fixed by the requirement that $g_{\sigma,m}$ be normalized. The possible values of σ are determined by

$$\text{(3.19)} \qquad g_{\sigma,m}(R) = \beta\, j_m(\sigma R) + \gamma\, n_m(\eta R) = 0 ,$$

There are of course shifts from the vortex free values in the permitted values of σ of order $1/R$, and associated shifts in $E_\sigma = E + (\hbar^2\sigma^2/2M)$. There will be additional shifts in the normal mode frequencies $\varepsilon_{\varkappa,\sigma,m}$. This gives rise to a change in zero point energy. We can compute the difference in zero-point energies with and without a vortex as

$$\tag{3.20} \tfrac{1}{2}\sum_{\varkappa\sigma m}(\varepsilon^{\mu}_{\varkappa,\sigma,m} - \varepsilon^{0}_{\varkappa,\sigma}),$$

using eq. (3.12).

The examination of the phase-shifts is closely related to the theory of the scattering of a phonon from a vortex. It is sufficiently interesting to warrant separate treatment, which will be carried out in a separate paper.

Let us now sketch the modifications to be expected for a more general potential. The operator L^μ becomes

$$\tag{3.21} L^\mu = -\frac{\hbar^2}{2M}\nabla^2 + \int V(|\boldsymbol{x}-\boldsymbol{x}'|)\,|f^\mu(\varrho')|^2\,\mathrm{d}^3x' - E.$$

The « potential » is again cylindrically symmetrical. We expand

$$\tag{3.22} V(|\boldsymbol{x}-\boldsymbol{x}'|) = \frac{1}{2\pi L}\sum_{\varkappa',m'} W_{\varkappa',m'}(\varrho,\varrho')\exp[i\varkappa(z-z')]\exp[im'(\vartheta-\vartheta')].$$

The « potential » is

$$\int W_{0,0}(\varrho,\varrho')\,|f^\mu(\varrho')|^2\varrho'\,\mathrm{d}\varrho'.$$

The equation for the small oscillations is

$$\tag{3.23} \left(i\hbar\frac{\partial}{\partial t} + \frac{\hbar^2}{2M}\nabla^2\right)\varphi = \left\{\int V(\boldsymbol{x}-\boldsymbol{x}')\,|f^\mu(\varrho')|^2\,\mathrm{d}^3x' - E\right\}\varphi + \\ + f^\mu(\varrho)\exp[i\mu\vartheta]\int V(x-x')f^\mu(\varrho')\left\{\exp[-i\mu\vartheta']\varphi(x') + \exp[i\mu\vartheta']\varphi^+(x')\right\}\mathrm{d}^3x'.$$

Expanding in terms of the basis $\varphi_{\varkappa,\sigma,m}$, we find eq. (3.9) and (3.10) with the replacements

$$F^\mu(\sigma m|\sigma' m') \to \mathscr{F}^\mu_\varkappa(\sigma m|\sigma' m'),$$

$$\tag{3.24} \mathscr{F}^\mu_\varkappa = \iint \varrho\,\mathrm{d}\varrho\,\varrho'\,\mathrm{d}\varrho' f^\mu(\varrho)\,g^\mu_{\sigma,m}(\varrho)\,W_{\varkappa,m-\mu}(\varrho,\varrho')f^\mu(\varrho')\,g_{\sigma',m'}(\varrho').$$

The rest of the theory is then developed as before.

4. – Quantum theory.

There are many, essentially related ways, of developing the quantum theory of weakly interacting bosons which yield identical results in the lowest approximations. We adopt the general approach of Section **4** of reference ([1b]), which is in particularly close correspondence to the semiclassical theory. In this approach the main new point in the quantum theory is that the approximate eigenstates no longer correspond to a sharp value for certain constants of the motion. Thus the classical solution $\exp[-iEt/\hbar]f^{\mu}(\varrho)\exp[i\mu\vartheta]$ corresponds to fixed, time-independent values of the number functional $N_{op}=\int\psi^+\psi\,\mathrm{d}^3x$ and the angular momentum functional $J_z=(\hbar/2i)\int(\psi^+(\partial\psi/\partial\vartheta)-(\partial\psi^+/\partial\vartheta)\psi)\,\mathrm{d}^3x$. In the quantum theory, the approximate eigenstates of H have given expectation values for the operators N_{op} and J_z, but are not exact eigenstates. This is particularly apparent in the case of the number operators. The phase factor $\exp[-iEt/\hbar]$ may be removed from the field operators by the time-dependent canonical transformation $U=\exp[i(Et/\hbar)N_{op}]$ which changes the Hamiltonian to $H-EN_{op}$. This is equivalent, in the quantum theory to introducing the Lagrange multiplier E and determining it by the condition that the expectation value of N_{op} is N. To find approximate eigenstates of the Hamiltonian which are also exact eigenstates of N and J_z, one has to apply projection operators in the present formalism. For the vortex state, they are of the order of N particles in a single-particle state of angular momentum $\hbar\mu$ with a vortex-type radial dependence. This amounts to a total angular momentum of $N\hbar\mu$. In the quantum theory we ought to introduce another Lagrange multiplier ω, to be determined by the requirement that the expectation value of J_z is $N\hbar\mu$

We shall therefore study the effective Hamiltonian $\mathscr{H}=H-EN_{op}-\omega(J_z)_{op}$. The quantized field is expanded in a complete basis similar to that of Section **3**. The Hamiltonian is

$$\begin{aligned}(4.1)\qquad \mathscr{H}=\sum\{(K\sigma m|T|K\sigma' m)-(E+\hbar\omega m)\delta_{\sigma,\sigma'}\}\,a^+_{K\sigma m}a_{K\sigma' m}+\\ +\sum(K\sigma m;\,K'\sigma' m'|G|K''\sigma'' m'';\,K'''\sigma''' m''')\,a^+_{K\sigma m}a^+_{K'\sigma' m'}a_{K''\sigma'' m''}a_{K'''\sigma''' m'''}\,.\end{aligned}$$

Here

$$\begin{aligned}(4.2)\qquad (K\sigma m|T|K'\sigma' m')&=-\frac{\hbar^2}{2M}\int\varphi^*_{K\sigma m}\nabla^2\varphi_{K'\sigma' m'}\,\mathrm{d}^3x=\\ &=\delta_{m,m'}\delta_{K,K'}\left(-\frac{\hbar^2}{2M}\right)\int g^*_{\sigma,m}(\varrho)\left\{D_\varrho-\frac{m^2}{\varrho^2}-K^2\right\}g_{\sigma' m}\,\varrho\,\mathrm{d}\varrho\,,\\ D_\varrho&=\frac{1}{\varrho}\frac{\mathrm{d}}{\mathrm{d}\varrho}\left(\varrho\frac{\mathrm{d}}{\mathrm{d}\varrho}\right),\end{aligned}$$

and

$$\langle K\sigma m;\, K'\sigma' m' \,|G|\, K''\sigma'' m'';\, K'''\sigma''' m''' \rangle = \tag{4.3}$$
$$= \frac{1}{2}\iint V(x-y)\varphi^*_{K\sigma m}(x)\,\varphi^*_{K'\sigma' m'}(y)\,\varphi_{K''\sigma'' m''}(x)\,\varphi_{K'''\sigma''' m'''}(y)\,\mathrm{d}^3x\,\mathrm{d}^3y =$$
$$= \frac{1}{2}\frac{1}{2\pi L}\sum\iint \varrho\,\mathrm{d}\varrho\,\varrho'\,\mathrm{d}\varrho'\,W_{L,n}(\varrho,\varrho')g_{\sigma m}(\varrho)g_{\sigma'' m''}(\varrho)g_{\sigma' m'}(\varrho')g_{\sigma''' m'''}(\varrho')\cdot$$
$$\cdot\delta_{L,K-K''}\delta_{L,K'''-K'}\delta_{n,m-m''}\delta_{n,m'''-m'}\,,$$

G contains factors $\delta_{m+m',m''+m'''}\delta_{K+K',K''+K'''}$.

We now introduce the unitary transformation

$$W_1 = \exp\left[c_{00\mu}a^+_{00\mu} - c^*_{00\mu}a_{00\mu}\right] \tag{4.4}$$

expressing the privileged role of the single particle state. Then

$$W_1 a_{00\mu} W_1^{-1} = c_{00\mu} + a_{00\mu}\,. \tag{4.5}$$

For simplicity we write $c_{00\mu} = c$ and $a_{00\mu} = a_\sigma$.

The transformed Hamiltonian is expressed in the form

$$W_1\mathscr{H}W_1^{-1} = \sum_{i=0}^{4}\mathscr{H}_i\,, \tag{4.6}$$

with

$$\mathscr{H}_0 = \big((00\mu|T|00\mu) - E - \hbar\omega\mu\big)|c|^2 + \big(00\mu;\,00\mu|G|00\mu;\,00\mu\big)|c|^2|c|^2\,,$$
$$\mathscr{H}_1 = \sum{}'\{(0\sigma\mu|T|00\mu) - (E+\hbar\omega\mu)\}a^+_\sigma + \text{h. c.} +$$
$$+\sum\{(0\sigma\mu;\,00\mu|G|00\mu;\,00\mu) + (00\mu;\,0\sigma\mu|G|00\mu;\,00\mu)\}a^+_\sigma c^* c^2 + \text{h. c.}\,,$$
$$\mathscr{H}_2 = \sum\{(K\sigma m|T|K'\sigma' m') - (E+\hbar\omega m)\delta_{\sigma\sigma'}\}a^+_{K\sigma m}a_{K\sigma' m} +$$
$$+2|c|^2\sum\{(K\sigma m;\,00\mu|G|K\sigma' m;\,00\mu) + (K\sigma m;\,00\mu|G|00\mu;\,K'\sigma' m')\}a^+_{K\sigma m}a_{K\sigma' m} +$$
$$+c^2\sum(K\sigma m;\,-K\sigma',\,2\mu-m|G|00\mu;\,00\mu)a^+_{K\sigma m}a_{-K,\sigma',\,2\mu-m} + \text{h. c.}\,,$$
$$\mathscr{H}_3 = 2c\sum(K\sigma m;\,K'\sigma' m'|G|K+K',\,\sigma'',\,m+m+m'-\mu;\,00\mu)a^+_{K\sigma m}a_{K'\sigma' m'}\cdot$$
$$\cdot a_{K+K',\sigma'',m+m'-\mu} + \text{h. c.}\,,$$
$$\mathscr{H}_4 = H_4\,.$$

In the limit of weakly interacting bosons $|c_{00\mu}|^2$ is N, but for finite interactions there is a depletion effect which reduces the value to a finite fraction of N. In the limit of weak interactions the function is determined by the condition that $\mathscr{H}_1$ vanishes. There will then be no constant term in the equation of motion of the $a_{K\sigma m}$. This leads to the requirement

$$(4.7)\qquad -\frac{\hbar^2}{2M}\nabla^2\varphi_{00\mu}+|c|^2\varphi_{00\mu}(\boldsymbol{x})\int V(\boldsymbol{x}-\boldsymbol{y})|\varphi_{00\mu}(\boldsymbol{y})|^2\,\mathrm{d}^3y=(E+\hbar\omega\mu)\varphi_{00\mu}(\boldsymbol{x})\,.$$

This is substantially the same as eq. (2.4) of the semiclassical theory. It is the same condition as $\delta\mathscr{H}_0/\delta\varphi^*_{00\mu}=0$ for fixed $|c|^2$, subject to the multiplier conditions

$$\int|\varphi_{00\mu}|^2\,\mathrm{d}^3x=1\,,\qquad i\hbar\int\varphi^*_{00\mu}\,\frac{\partial\varphi_{00\mu}}{\partial\vartheta}\,\mathrm{d}^3x=\hbar\mu\,.$$

The operator which takes the place of the L^μ of eq. (3.21) is

$$(4.8)\qquad L^\mu=-\frac{\hbar^2}{2M}\nabla^2+|c|^2\int V(|\boldsymbol{x}-\boldsymbol{y}|)|\varphi_{00\mu}(y)|^2\,\mathrm{d}^3y-(E+\hbar\omega\mu)\,.$$

If we use the eigenfunctions and eigenvalues $\varphi_{K\sigma m}$, $E_{K,\sigma}$ of L^μ, and multiply

$$(4.9)\qquad L^\mu\varphi_{K\sigma m}=E_{K\sigma}\varphi_{K\sigma m}=\left(E^\mu_\sigma-E+\hbar\omega\mu+\frac{\hbar^2K^2}{2M}\right)\varphi_{K\sigma m}\,,$$

by $\varphi^*_{K'\sigma'm'}$ and integrate over space, we find

$$(4.10)\qquad \langle K'\sigma'm'|T|K\sigma m\rangle-(E+\hbar\omega m)\,\delta_{K,K'}\,\delta_{\sigma,\sigma'}\,\delta_{m,m'}+ \\ +2|c|^2(K'\sigma'm';\,00\mu|G|K\sigma m;\,00\mu)=E_{K\sigma}\,\delta_{K,K'}\,\delta_{\sigma,\sigma'}\,\delta_{m,m'}\,.$$

$\mathscr{H}_2$ then takes the simple form

$$(4.11)\qquad \mathscr{H}_2=\sum E_{K\sigma}a^+_{K\sigma m}a_{K\sigma m}+2\sum|c|^2(K\sigma m;\,00\mu|G|00\mu;\,K\sigma'm)a^+_{K\sigma m}a_{K\sigma'm}+ \\ +\sum(K\sigma m;\,-K,\,\sigma',\,2\mu-m|G|00\mu;\,00\mu)c^2a^+_{K\sigma m}a_{-K,\sigma',2\mu-m}+\text{h. c.}$$

The main point is the strong coupling of $a_{K,\sigma m}$ to $a_{-K,\sigma,a\mu-m}$. We may again partially diagonalize $\mathscr{H}_2$ (neglecting the coupling of the radial modes) by introducing the normal mode transformation W_2.

$$(4.12)\qquad \begin{cases} W_2a_{K,\sigma,m}W_2^{-1}\equiv a'_{K,\sigma,m}=\cosh\gamma\,a_{K\varrho m}+\sinh\gamma\,a^+_{-K,\sigma,2\mu-m}\,,\\ W_2a^+_{-K,\sigma,2\mu-m}W_2^{-1}\equiv a'^+_{-K,\sigma,2\mu-m}=\sinh\gamma\,a_{K,\sigma,m}+\cosh\gamma\,a^+_{-K,\sigma,2\mu-m}\,,\\ \gamma=\gamma_{K,\sigma,m}=\gamma_{-K,\varrho,2\mu-m}\,.\end{cases}$$

With

$$(4.13)\qquad \begin{cases} \mathscr{E}_{K\sigma m} = E_{K\sigma m} - 2|c|^2(K\sigma m;\, 00\mu\,|G|\,00\mu;\, K\sigma m)\,, \\ \lambda_{K\sigma m} = \varepsilon_{K\sigma m}/c^2(K\sigma m;\, -K\sigma,\, 2\mu - m\,|G|\,00\mu;\, 00\mu)\,, \end{cases}$$

we find for $\gamma_{K\sigma m}$

$$(4.14)\qquad (1 + \operatorname{tgh}^2\gamma_{K\sigma m}) + \lambda_{K\sigma m}\operatorname{tgh}\gamma_{K\sigma m} = 0\,.$$

The zero point energy is

$$(4.15)\qquad \varepsilon_0 = \sum_{K\sigma m}\mathscr{E}_{K\sigma m}\left\{\sinh^2\gamma_{K\sigma m} - \frac{1}{\lambda_{K\sigma m}}\sinh 2\gamma_{K\sigma m}\right\},$$

while the spectrum of excitations is

$$(4.16)\qquad \sum_{K\sigma m} a^+_{K\sigma m}a_{K\sigma m}\varepsilon_{K\sigma m}\,,$$

where

$$\varepsilon_{K\sigma m} = 2\mathscr{E}_{K\sigma m}\left(\cosh^2\gamma_{K\sigma m} - \frac{\sinh^2 2\gamma_{K\sigma m}}{\lambda_{K\sigma m}}\right).$$

Thus, we have

$$(4.17)\qquad \begin{cases} W_2\mathscr{H}_2W_2^{-1} = \varepsilon_0 + \sum a^+_{K\sigma m}a_{K\sigma m}\varepsilon_{K\sigma m} + \mathscr{H}_r\,, \\ \mathscr{H}_r = 2\sum\limits_{\sigma\neq\sigma'}|c|^2(K\sigma m;\, 00\mu\,|G|\,00\mu;\, K\sigma' m)(a^+_{K\sigma m}\,a_{K\sigma' m})' + \\ \qquad + \sum\limits_{\sigma\neq\sigma'} c^2(K\sigma m;\, K\sigma',\, 2\mu - m\,|G|\,00\mu;\, 00\mu)(a^+_{K\sigma m}a^+_{K\sigma',2\mu-m})' + \text{h. c.} \end{cases}$$

In the lowest approximation, neglecting depletion effects and the modifications of the self consistent potential arising from close collisions, $|c_{00\mu}|^2 = N$ and the Lagrange multiplier ω is zero. The results are then exactly the same as for the semi-classical theory, and mainly provide a formal justification for adding the shift zero-point energy of the oscillations to the self-consistent field energy of the vortex state. The state vectors in this approximation are $\Psi_{\text{new}} = W_1\, W_2\Phi_{\text{old}}$, where Φ_{old} runs through a set of states consisting of a vacuum state Φ_0 such that $a_{K\sigma m}\Phi_0 = 0$, and a set of states obtained by operating separately on Φ_0 with the creation operators

$$a^+_{K\sigma m} = \int\psi^+(x)\varphi_{K\sigma m}(x)\,\mathrm{d}^3x\,,$$

$$\Psi_{\text{new}} = \exp\left[\sum_\sigma (ca^+_\sigma - c^*a_\sigma)\right]\cdot\exp\left[\sum\gamma_{K\sigma m}(a^+_{K\sigma m}a^+_{K,\sigma,2\mu-m} - \text{h. c.})\right]\cdot\Phi_{\text{old}}\,.$$

It is of some interest to examine the expectation values of physical quantities to see the modifications of the self consistent field approximation induced by the zero-point motions of the field. The expectation value of the density is

$$(4.18)\qquad \begin{cases} n(\boldsymbol{x}) = \langle \Psi_0,\, \psi^+(x)\,\psi(\boldsymbol{x})\Psi_0\rangle = \langle \Phi_0,\, (W_1W_2)^{-1}\psi^+\psi(W_1W_2)\Phi_0\rangle \\ \qquad = |\varphi_{00\mu}(\boldsymbol{x})|^2\,|c|^2 + \sum \sinh^2\gamma_{K\sigma m}\,|g''_{\sigma m}(\varrho)|^2\,. \end{cases}$$

The main interest is when $\varrho \to 0$. For $m \neq 0$ all terms tend to zero. The only contribution is from $m = 0$ and is $\sum_{K\sigma} \sinh^2\gamma_{K\sigma 0}\,|g^{\mu}_{\sigma 0}(\varrho)|^2 \neq 0$. The zero-point fluctuations of the s-wave phonons give a non-zero density at the vortex line. We may define the expectation value of the velocity as

$$(4.19)\qquad \overline{v_\vartheta} = \frac{1}{n(x)}\,\frac{i\hbar}{2}\left\langle \Psi_0,\, \left(\psi^+\,\frac{1}{\varrho}\,\frac{\partial\psi}{\partial\vartheta} - \frac{1}{\varrho}\,\frac{\partial\psi^+}{\partial\vartheta}\,\psi\right)\Psi_0\right\rangle .$$

The density is now non-zero at $\varrho = 0$. But the numerator is

$$(4.20)\qquad |c_{00\mu}|^2\,\frac{\hbar\mu}{\varrho}\,|g''_{0\mu}(\varrho)|^2 + \sum \sinh^2\gamma_{K\sigma m}\,\frac{\hbar m}{\varrho}\,|g''_{\sigma m}(\varrho)|^2\,.$$

It tends to zero as $\varrho \to 0$, since the $m = 0$ term contains a factor $\hbar m$, and is thus absent from the sum. Hence, the expectation value of the velocity is finite, and is in fact zero at $\varrho = 0$. For $\mu = 1$

$$(4.21)\qquad \overline{v_\vartheta} \xrightarrow[\varrho\to 0]{} \frac{|c_{001}|^2}{n(\varrho = 0)}\,\frac{|g'_{01}(\varrho)|^2}{\varrho}\,,$$

since $g'_{01}(\varrho)$ goes as $\alpha j_1(\sqrt{(2ME/\hbar^2)}\varrho)$, where α is a normalization factor, we have

$$(4.22)\qquad \overline{v_\vartheta} \xrightarrow[\varrho\to 0]{} \frac{|c_{001}|^2}{n(\varrho = 0)}\,\hbar\varrho\left(\frac{2ME}{\hbar^2}\right)\alpha^2\,.$$

Thus the vorticity $\overline{w} = \frac{1}{2}(1/\varrho)(\partial/\partial\varrho)(\varrho\overline{v_\vartheta})$ has the limiting value

$$(4.23)\qquad \overline{w} \xrightarrow[\varrho\to 0]{} 2\,\frac{\overline{v_\vartheta}}{\varrho} \to \frac{2\,|c_{001}|^2}{n(\varrho = 0)}\,\hbar\alpha^2\left(\frac{2ME}{\hbar^2}\right) = \text{constant} \neq 0\,.$$

The vorticity has a finite value in the core.

It is clear that these results depend on the general form of the vortex wave-function $W_1W_2\Phi_0$, and are independent of the precise values of $|c_{00\mu}|^2$ and $\gamma_{K\sigma m}$. As described in ref. ([1b]), in a higher approximation these parameters are shifted from the values given by the theory that considers only $\mathscr{H}_0+\mathscr{H}_1+\mathscr{H}_2$. One must consider the modifications induced when the creation and annihilation operators are reordered after the normal mode transformation. This includes a diagonal contribution to the ground state energy from $W_2\mathscr{H}_4W_2^{-1}$. The best ground state for the chosen type of approximate state vector is obtained variationally. The parameters $c_{00\mu}$, ω, and E, are determined by minimizing the ground state energy expectation value with respect to $c^*_{00\mu}$, and imposing the two Lagrange multiplier conditions. Since we are not working with a plane wave basis, the reordering of $W_2\mathscr{H}_3W_2^{-1}$ leads to a modified self-consistent potential because terms linear in the creation and annihilation operators appear. $W_2\mathscr{H}_4W_2^{-1}$ also contains terms that lead to a modified normal mode problem, *i.e.* a redetermination of the $\gamma_{K\sigma m}$. Following these lines we may obtain an improved value for the energy of a vortex state without phonons. One main effect is the replacement of the total density N/Ω by the superfluid density $|c_{00\mu}|^2/\Omega$ in the estimates. Further improvements must consider a more general normal mode transformation to take into account the radial mode coupling, and also off diagonal contributions of $W_2(\mathscr{H}_3+\mathscr{H}_4)W_2^{-1}$. It would be worthwhile to undertake a systematic calculation to give a clear development of all quantities in the strength of the interparticle potential. A trustworthy calculation for the actual case of liquid helium would be difficult. It is likely that the zero-point oscillation-spread of the core density is comparable to the self-consistent field contribution. It is well known that the Bogoljubov normal mode transformation is too restricted for studying the phonon spectrum in higher approximations. The off diagonal elements of $W_2(\mathscr{H}_3+\mathscr{H}_4)W_2^{-1}$ must be considered to avoid spurious energy gaps.

It is, however, not our concern here to study these refinements. Further development might be preferably undertaken by other methods [7-10]. The main point, however, already appears in the simpler treatment presented here. It is the specification of a special, highly occupied, « vortex » single-particle state $\varphi_{00\mu}(\boldsymbol{x})$. For weakly repelling bosons, where the self-consistent field contribution to the structure of the vortex dominates quantum fluctuation effects, the picture is particularly clear.

The wave function representing an elementary vortex state has an expectation value $N\hbar\mu$ for the angular momentum. The projected state with the exact $N\hbar\mu$ is of course orthogonal to the ground state without a vortex. In our approximation the orthogonality occurs as $N\to\infty$, essentially because the inner product

$$\left\langle \exp\left[f_0\int\psi^+(x)\,\mathrm{d}^0x-\text{c.c.}\right]\Phi_0\Big/\exp\left[\int f^\mu(\boldsymbol{x})\psi^+(\boldsymbol{x})\,\mathrm{d}^3x-\text{c.c.}\right]\Phi_0\right\rangle\to\exp\left[-N\beta\right].$$

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(7) K. A. Brueckner and K. Sawada: *Phys. Rev.*, **106**, 1117 (1957).
(8) S. T. Belyaev: *Soviet Physics JETP*, **7**, 289 (1958).
(9) R. Abe: *Prog. Theor. Phys.*, **19**, 1, 57, 407, 699, 713 (1958).
(10) N. M. Hugenholtz and D. Pines: *Phys. Rev.*, **116**, 489 (1959).

Cf. the similar discussion of the periodic ground states in ref. ([4b]). This argument holds when we consider corresponding states with a small number of excitations. But it breaks down when there are of the order of N excitations, for example at a finite temperature near the λ-point. Furthermore, the terms $W_2\mathscr{H}_3W_2^{-1}$ and $W_2\mathscr{H}_4W_2^{-1}$ create and annihilate only a small number of particles in a single act. Thus an elementary vortex can only decay by a very high order process in perturbation theory, *i.e.* it has an essentially macroscopic lifetime. We do not try to estimate this lifetime here. With a large number of excitations the vortex state becomes seriously depleted and the core grows in size and fluctuates. There are then rapid transitions to a circulation-free state with production of phonons. The problem appears difficult to discuss precisely from the present point of view. We have an overcompleteness of states characteristic of self-consistent field theories. Each $f^\mu(\boldsymbol{x})$ and the associated excitation spectrum presumably spans the same space of state vectors. The approximate wave functions can then be accurate only when are there is a small number of excitations.

5. – Further discussion of the semiclassical theory.

The equation of motion for the classical field $\psi(\boldsymbol{x}, t)$ may be rewritten in hydrodynamic form. Putting $\psi = E\exp[iS/\hbar]$, where R and S are real, one finds

$$\frac{\partial}{\partial t}(R^2) = -\operatorname{div}\left(R^2\frac{\nabla S}{M}\right), \tag{5.1}$$

$$-\frac{\partial S}{\partial t} = \frac{(\nabla S)^2}{2M} - \frac{\hbar^2}{2M}\frac{\nabla^2 R}{R} + \int V(\boldsymbol{x}-\boldsymbol{y})R^2(\boldsymbol{y})\,\mathrm{d}^3y\,. \tag{5.2}$$

We introduce the velocity field $\boldsymbol{v} = \boldsymbol{\nabla}S/M$ and take the gradient of the second equation. Then

$$M\left(\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right) + \boldsymbol{\nabla}\Pi = 0\,, \tag{5.3}$$

Π is given by the functional

$$\Pi = -\frac{\hbar^2}{2M}\frac{\nabla^2 R}{R} + \int V(\boldsymbol{x}-\boldsymbol{y})R^2(\boldsymbol{y})\,\mathrm{d}^3y\,. \tag{5.4}$$

consisting of a « quantum » contribution and a contribution arising from particle interactions. It has the dimensions of energy. In the usual hydrodynamics one assumes the pressure p is a function of the density and $\Pi = \int \mathrm{d}p/n$. But the present hydrodynamics is quite different. It is useful to introduce the density $n = R^2$ and to write (summation convention)

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x_k}(n v_k) = 0\,, \tag{5.6}$$

$$M\frac{\partial}{\partial t}(n v_i) + M\frac{\partial}{\partial x_k}(n v_i v_k) = n\frac{\partial \Pi}{\partial x_i}\,. \tag{5.7}$$

$n(\partial \Pi/\partial x_k)$ can be expressed as [11]

$$n\frac{\partial \Pi}{\partial x_k} = -n\frac{\partial}{\partial x_k}\int V(x-y)n(y)\,\mathrm{d}^3y + \frac{\partial \sigma_{ik}}{\partial x_k}\,, \tag{5.8}$$

with a stress tensor

$$\sigma_{ik} = \frac{\hbar^2}{2M}\,n\,\frac{\partial^2(\log n)}{\partial x_i\,\partial x_k}\,; \qquad \sum\frac{\partial \sigma_{ik}}{\partial x_k} = n\frac{\partial}{\partial x_i}\left(\frac{\hbar^2}{2M}\,\frac{\nabla^2 R}{R}\right). \tag{5.9}$$

This differs from ordinary hydrodynamics in that the stress depends on derivatives of the density, rather than velocity.

The elementary line vortex has the property that $\nabla^2 S = 0$ and $\nabla S \cdot \nabla R = 0$, and that $R = 0$ on the vortex line. The quantum pressure $-(\hbar^2/2M)(\nabla^2 R/R) \to -\infty$ as $\varrho \to 0$ and cancels $(\nabla S)^2/2M$ which $\to +\infty$, leaving a finite quantity. It is clear since the vortex core is small in extent, that we can generalize the argument, and find approximate solutions representing steady patterns of vortices separated by distances greater than the core diameter. One way is to look first for solutions of $\nabla^2 S = 0$, *e.g.* ring vortices, sets of line vortices, etc. Then one finds the appropriate behaviour of R near the lines of singularity. This brings into play the type of classical hydrodynamic argument already used in interpreting experimental properties of superfluid helium. The main role of the quantum mechanics, at least for weakly interacting bosons, is to provide a foundation for this procedure, and to ensure that there is a definite theory of the structure of the vortex core. This type of consideration is foreign to classical (even compressible) hydrodynamics. The structure depends on the quantum pressure term, as is evident from the fact that the characteristic size depends on a De Broglie wavelength. In addition, the study of the quantum state vectors associated with a flow pattern should show that decay is possible only in a macroscopic time.

Formally, each solution of the hydrodynamic equations defines a basis function which is occupied by a finite fraction of the particles. Such a state is not as symmetrical as the elementary cylindrical vortex and would probably not be considered from the point of view of ordinary quantum mechanics. The complete set of functions orthogonal to the basis function permits an

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[11] T. TAKABAYASI: *Prog. Theor. Phys.*, **8**, 143 (1952); **9**, 187 (1953).

analysis of the small oscillation spectrum comparable to that of Sections **3** and **4**.

Phenomenological single fluid hydrodynamic equations have been used to study processes such as scattering of phonons and protons from vortices ([12]). These theories assume a pressure density relation such that $(\mathrm{d}p/\mathrm{d}\varrho)^{\frac{1}{2}}$ agrees with the observed first sound velocity. In contrast the present theory contains the quantum pressure term and is directly related to basic quantum mechanics. However, the defect of our procedure is that only weakly interacting particles can be treated in a clear manner. It should be possible, in the spirit of a pseudopotential method, of Brueckner's ideas, or of Landau's theory of a Fermi liquid to justify replacing the interaction potential *ab initio* to treat strong interactions. Then the semiclassical theory itself would become useful for quantitative applications to superfluid helium.

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The author is indebted to Professor DAVID BOHM for many valuable discussions. He also wishes to express appreciation for the hospitality of the University of Bristol and CERN, and for financial support from the National Science Foundation, and the office of Scientific Research, U.S. Air Force.

([12]) L. P. PITAEVSKI: *Soviet Physics JETP*, **8**, 888 (1958).

RIASSUNTO (*)

Sviluppiamo una teoria delle eccitazioni dei vortici lineari elementari, per un sistema di bosoni debolmente repulsivi. Lo stato di vortice è caratterizzato dalla presenza di una frazione finita di particelle nello stato di particella singola con momento angolare intero. La dipendenza radiale dello stato densamente occupato segue da una equazione di campo autocongruente. La funzione radiale e la densità di particelle associata sono essenzialmente costanti dovunque tranne che nell'interno di un nocciolo, dove si annullano. La dimensione del nocciolo è uguale alla lunghezza d'onda di de Broglie associata con l'energia media di interazione per particella. Il valore previsto per la velocità ha la dipendenza radiale del vortice classico. In questa approssimazione di Hartree la vorticità è nulla dovunque tranne che sulla linea di vortice. Quando si raffina la descrizione dello stato sino ad includere le oscillazioni nel punto zero del campo fononico, la vorticità si estende a tutto nocciolo. Questi risultati confermano nei punti essenziali gli argomenti intuitivi di Onsager e Feynman. I fononi che si muovono normalmente alla linea di vortice hanno eccitazioni congruenti con quantità di moto uguale e contraria rispetto al substrato di particelle in movimento che costituisce il vortice. Il movimento del vortice risolve la degenerazione dei fononi di Bogoljubov rispetto al numero quantico azimutale.

(*) *Traduzione a cura della Redazione.*