Team 553
Problem A

Deflection of Asteroid's Trajectory using a Kinetic Impactor Spacecraft by Orbital Maneuver

Abstract

There are around 18000 asteroids hitting Earth's atmosphere every year and fortunately none are able to penetrate through the atmosphere. In the case that there is an asteroid 100 meters in diameter and is traveling with a speed of $25\,\mathrm{km\,s^{-1}}$ with a density of $1.78\,\mathrm{g\,cm^{-3}}$ and 31.01% porosity, Earth's atmosphere alone wouldn't be enough to burn it. We propose a strategy to change a spacecraft's orbital eccentricity in the Low Earth Orbit to hit the asteroid at the spacecraft's apogee perpendicular to the asteroid's velocity. We estimated that with a spacecraft $20\,000\,\mathrm{kg}$ in mass (including the spacecraft's empty weight and fuel) and in orbit $2\times10^{6}\,\mathrm{m}$ above Earth's surface, the minimum time it would need to hit the asteroid would be at least 4.7 years before the asteroid reaches Earth.

1 Introduction

NASA monitors asteroids and other celestial bodies each year and according to data, the number of asteroids hitting Earth's atmosphere is somewhere around 18000 a year [1]. Although these asteroids are small enough to burn up in the atmosphere before it hits earth's surface, there are larger asteroids that have a possibility of hitting our wonderful blue marble. Thankfully, the asteroid belt between Mars and Jupiter and other celestial bodies such as our moon blocks a considerable number of them.

However, the problem arrives when a big enough asteroid where it does not burn up in Earth's atmosphere is heading towards Earth. Say the asteroid has a diameter of 100 m and speed of $25\,\mathrm{km\,s^{-1}}$ coming straight to Earth. If we were to send a spacecraft to deflect the trajectory of the asteroid so that it's just enough to miss the earth then there are several steps needed in order to achieve that goal. This report attempts to determine the amount of time needed before the asteroid reaches Earth.

1.1 Assumptions

To simplify our problem, let us state several assumptions.

1.1.1 Spacecraft

The model used for the spacecraft is taken from the Atlas III-B launch vehicle. Let us assume that the fuel used in the spacecraft is a mix of Liquid Hydrogen and Liquid Oxygen (LOX/LH_2). The thrust of a LOX/LH_2 powered engine is around 99.10 kN with a Specific Impulse (I_{SP}) of 451 s [2]. We also assume that the spacecraft's shape is perfectly spherical to match with reference data from [3] and that the spacecraft is moving smoothly along low earth orbit's trajectory with no interruptions or interactions with debris in lower earth orbit.

In order to reach the needed speed and in turn the needed momentum to alter the trajectory of the asteroid without any interference from other forces, such as the gravity of nearby planets in the solar system (Mars being the closest at an average distance of 78.34 million km from earth [4]), we assume that there are no influence from other forces.

1.1.2 Asteroid

Asteroids varies a lot in its shape, size, and surface [5]. It generally does not have smooth surfaces but for this model we assume it is perfectly spherical. This is done to simplify the model and to help create the final result. If we do not assume the asteroid's shape is perfectly spherical then the model becomes complex with almost negligible differences in the results.

Asteroids are usually comprised of different elements, porosity, and density distributions [6]. The asteroid is assumed to have a uniform density distribution. By doing this we can also safely assume that the center of mass is at the center of the asteroid. We assume the asteroid is made out of Savonniére Limestone which has a density of around $1.78 g/cm^3$ and a porosity of 31.01% [3], this assumption is based off of an average asteroid around this size.

The asteroid's moving at $25\,\mathrm{km\,s^{-1}}$ towards earth. From here we can assume that the position of the asteroid is aligned and moving together with earth in the x axis, so there are no asteroid velocity in the y axis.

The last assumption is to neglect all kinds of interference other than the forces inside and between the spacecraft, earth, and the asteroid.

2 Notations Used

In the following table we list some symbols which will be used in our model. More symbols are to be defined in the subsequent sections.

Symbols	Meaning	Numeric Value
G	Gravitational constant	$6.7 \times 10^{-11}\,\mathrm{m\,kg^{-3}\,s^{-2}}$ [7]
M_E	Earth's mass	$5.9722 \times 10^{24}\,\mathrm{kg}$ [8]
M_S	Spacecraft's mass	$20 \times 10^{3}\,\mathrm{kg}$
r_{LEO}	Low Earth Orbit's height from Earth's surface	$2 \times 10^{6}\,\mathrm{m}$ [9]
I_{sp}	Specific Impulse of Spacecraft (Atlas III-B -Centaur rocket at the Second Stage)	$451.0\,\mathrm{s}$
g_0	Earth's surface gravity	$9.8\,\mathrm{m\,s^{-2}}$ [8]
β	The asteroid's momentum multiplication	1.64 [3]

3 Model

3.1 Journey to the Asteroid

We propose a strategy to approach this problem by changing the spacecraft's orbital eccentricity to meet at the changed spacecraft's orbit's apogee. Change in the spacecraft's eccentricity, assuming the orbit at LEO is circular, can be calculated using the equation [10]

$$\epsilon_n = \left(\frac{v_n}{v_{\mathrm{LEO}}}\right)^2 - 1 \tag{1}$$

where v_n is the spacecraft's speed in the new elliptical orbit at the perigee and v_{LEO} is the spacecraft's speed in the circular LEO that can be derived by equating the gravitational pull between Earth and the spacecraft and its centripetal force

$$v_{\mathrm{LEO}} = \sqrt{\frac{GM_E}{r}} \tag{2}$$

where r is the orbit radius. Since the eccentricity limit for an elliptical orbit is $0 < \epsilon < 1$, this limits our new speed to $v_{\mathrm{LEO}} < v_n < \sqrt{2} \cdot v_{\mathrm{LEO}}$.

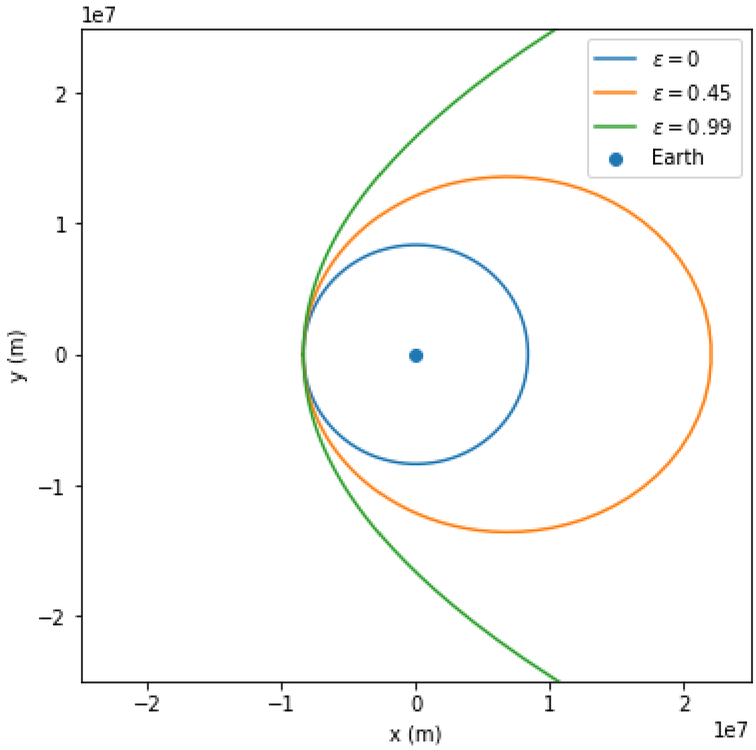

Figure 1: Change of eccentricity from circular orbit to elliptical by changing the orbital speed to $\sqrt{1.45} \cdot v_{\text{LEO}}$(orange) and $\sqrt{1.99} \cdot v_{\text{LEO}}$.

Fuel must be burned for the change of speed to occur. This can be calculated by solving the differential equation of the rocket thrust equation [11]

$$F = I_{\text{sp}} \cdot g_0 \cdot \dot{m} \tag{3}$$

$$m\frac{dv}{dt} = I_{\text{sp}} \cdot g_0 \cdot \frac{dm}{dt}$$

Then, by integrating the speed from the initial (v_i) to final speed (v_f) and integrating the mass from its final (m_f) to initial mass (m_i) because it is decreasing, we get the change in speed

$$v_f - v_i = \Delta v = I_{\text{sp}} \cdot g_0 \cdot \ln\frac{m_i}{m_f} \tag{4}$$

Eq. (4) can be rearranged to find m_f after a boost

$$m_f = \exp\left(-\frac{\Delta v}{I_{\text{sp}} \cdot g_0}\right) m_i \tag{5}$$

The apogee of the new orbit after changing its eccentricity is

$$r_{\text{apo}} = \left[\frac{1+\epsilon_n}{1-\epsilon_n}\right] r_{\text{LEO}} \tag{6}$$

and this shall be the impact point of our spaceship. The corresponding speed of the spaceship when it is at the apogee can be derived from the conservation of angular momentum

$$v_{\text{apo}} = \left[\frac{1-\epsilon_n}{1+\epsilon_n}\right] v_n \tag{7}$$

Since we assumed that the asteroid came from infinity heading straight to Earth with a speed of $25\,\mathrm{km\,s^{-1}}$, its speed would change due to Earth's pull. The change in speed can be derived from the conservation of energy

$$v_A = \sqrt{\frac{2GM_B}{r_{\text{apo}}} + 25000^2} \tag{8}$$

3.2 Momentum transfer

The spacecraft's momentum would be transferred to the asteroid during impact. Other than the spacecraft's momentum, there are debris that are ejected from the impact that contribute to the asteroid's final momentum. This added momentum is contained within the momentum multiplication [12]

$$\beta = 1 + \frac{p_e}{p_S} \tag{9}$$

where p_e is the ejecta momentum. Using the conservation of momentum, assuming that the ejecta momentum direction is in the other direction of the spaceship's impact, we get

$$\vec{p}_S + \vec{p}_A = \vec{p}'_A + \vec{p}_e \tag{10}$$

If we use Earth as our frame of reference, in the case that the asteroid is traveling from the positive x-axis and the spacecraft hit the asteroid from positive y-axis, we would get

$$-p_A\hat{i} + (-p_e - p_S)\hat{j} = \vec{p}_{A'}$$

$$-p_A\hat{i} - \beta p_S\hat{j} = \vec{p}_{A'}$$

From that we get the angle of the asteroid's trajectory between x-axis and $\vec{p}'_A$ is

$$\theta = \arctan\frac{|\beta p_S|}{|p_A|} \tag{11}$$

which is the deflection angle after the impact. It wont matter whether the spacecraft impacted from the above or below or the asteroid came from the negative or positive x-axis, hence the absolute value.

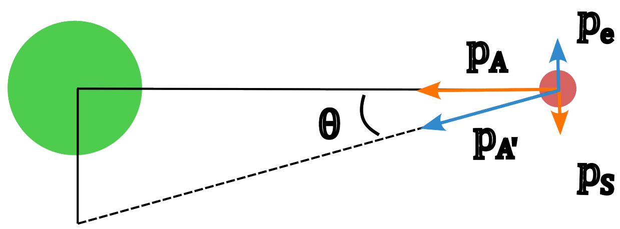

Figure 2: Illustration of the momentum transfer after spacecraft impact. Orange arrows indicates the momentum before impact, blue arrows indicates the momentum after impact, green circle is the earth, red circle is the asteroid.

Additionally, we can use the remaining fuel to increase the speed thereby increasing the momentum. Using Eq. 4 to find momentum as a function of fuel used to boost the spaceship, we get this graph

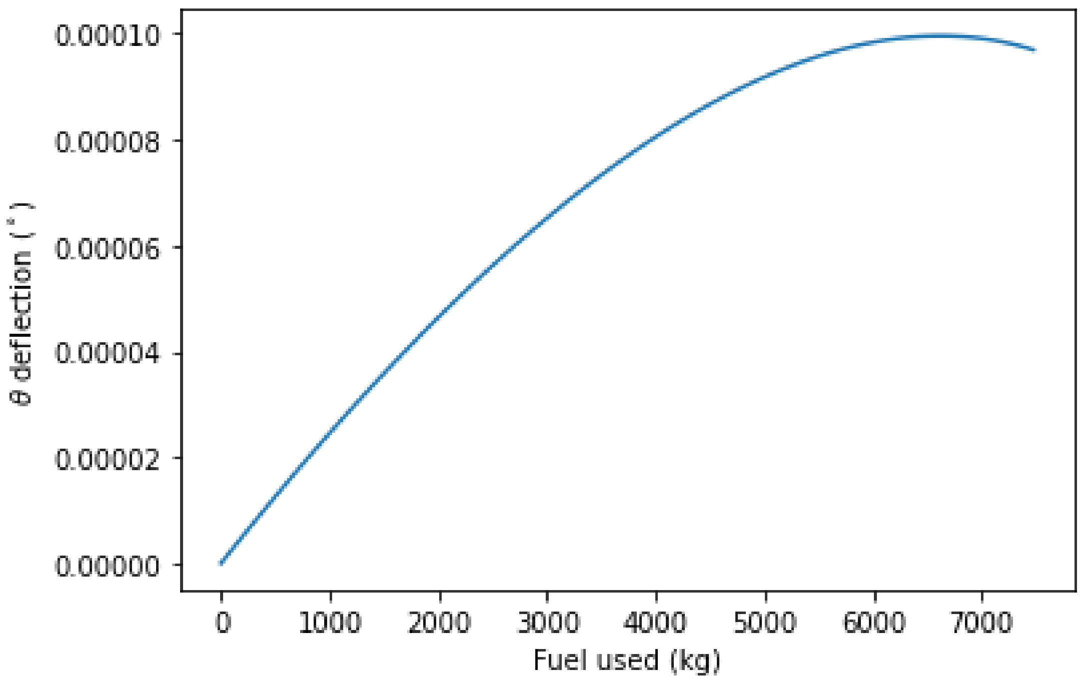

Figure 3: Deflection angle as a function of fuel used to increase the spacecraft's momentum. In this particular example, $\epsilon = 0.99995$.

3.3 Calculating the Deflection Angle

The theoretical minimum angle, not including the effects of gravity, to deflect the asteroid for Earth's safety is given by

$$\theta_{\text{min}} = \arctan\left(\frac{r_E + r_A}{r_{\text{apo}}}\right) \tag{12}$$

and by using the deflection angle from the spacecraft's impact we can get a root finding problem to find the minimum eccentricity.

$$\arctan\left(\frac{r_E + r_A}{r_{\text{apo}}(\epsilon)}\right) - \arctan\left(\frac{\beta|p_{S_{\text{max}}}(\epsilon)|}{|p_A(\epsilon)|}\right) = 0 \tag{13}$$

where $p_{S_{\text{max}}}$ is the maximum momentum that can be delivered by the spacecraft after using fuel near the asteroid to boost itself. Solving this using SciPy's root finding function in Python, we get the minimum eccentricity to be around 0.9999954 with a deflection angle of $0.000\,095\,8°$, apogee distance of 3.7×10^{12} m, and time before the asteroid impacts Earth is around 4.7 years. Since this is an approximation without taking into account the Earth's gravitational pull, the deflection angle should be bigger.

4 Conclusions and Possible Improvements

Using the method of changing the spacecraft's velocity to change its orbital eccentricity and meeting the asteroid at its apogee and assuming that the asteroid is made of mostly Savonniére Limestone, we would need to deflect it by $0.000\,095\,8°$ in 4.7 years' time before impact.

Since we are using a LOX/LH_2 engine, this method could be improved by using an ion propulsion engine that can produce more thrust with less mass though with the sacrifice of time. Using the same specification of a spacecraft with 20 000 kg of mass, it could bring more momentum with its extra residual mass when compared to an LOX/LH_2 engine.

This method can be easily executed in theory due to its low maintenance, only requiring one boost at the beginning and at the end of flight. Although, the speed required to change into a very specific eccentricity leaves no room for error, since a small difference in speed could result in kilometers of error. Using the aforementioned ion propulsion engine with high monitoring to adjust its trajectory incrementally could increase the chance of successfully deflecting the asteroid.

5 References

[1] Elise Kjørstad. How often do asteroids and comets hit the earth? `https://sciencenorway.no/asteroid-astronomy-the-universe/how-often-do-asteroids-and-comets-hit-the-earth/1986794`, 2022. Last accessed 7 November 2022.

[2] Mark Wade. Rl-10a-4-2. `http://www.astronautix.com/r/rl-10a-4-2.html`, 2019. Last accessed 6 November 2022.

[3] Tobias Hoerth, Frank Schäfer, Jan Hupfer, Oliver Millon, and Matthias Wickert. Momentum transfer in hypervelocity impact experiments on rock targets. *Procedia Engineering*, 103:197–204, 2015.

[4] Dave Williams. Mars fact sheet. `https://nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html`, 2021. Last accesed 7 November 2022.

[5] Phil Davis. Asteroids. `https://solarsystem.nasa.gov/asteroids-comets-and-meteors/asteroids/overview/`, 2021. Last accesed 7 November 2022.

[6] Dan T Britt, Don Yeomans, Kevin Housen, and G Consolmagno. Asteroid density, porosity, and structure. 2003.

[7] NIST. Codata value: Newtonian constant of gravitation. `https://physics.nist.gov/cgi-bin/cuu/Value?bg`. Last accesed 7 November 2022.

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[11] Nancy Hall. Specific impulse. `https://www.grc.nasa.gov/WWW/k-12/airplane/specimp.html`, 2021. Last accessed 6 November 2022.

[12] Tobias Hoerth, Frank Schäfer, Jan Hupfer, Oliver Millon, and Matthias Wickert. Momentum transfer in hypervelocity impact experiments on rock targets. *Procedia Engineering*, 103:197–204, 2015. Proceedings of the 2015 Hypervelocity Impact Symposium (HVIS 2015).

6 Appendix

Here we append the Python code used to produce Figure 1.

```
import numpy as np
import matplotlib.pyplot as plt

e_earth = 0
r0_orbit = -6371e3 - 2e6
e_new = 1.45-1
e_new2 = 1.99-1

def r(tetha,e,r0):
  alpha = (1 - e**2)*r0/(1-e)
  return alpha/(1+e*np.cos(tetha))

angles = np.linspace(0,2*np.pi, 100)
r_earth = r(angles, e_earth , r0_orbit)
r_new = r(angles, e_new, r0_orbit)
r_new2 = r(angles, e_new2, r0_orbit)
plt.figure(figsize=(6,6))
plt.plot(r_earth*np.cos(angles),r_earth*np.sin(angles), label="$\epsilon = 0$")
plt.plot(r_new*np.cos(angles),r_new*np.sin(angles), label=f"$\epsilon = {e_new:.2}$")
plt.plot(r_new2*np.cos(angles),r_new2*np.sin(angles), label=f"$\epsilon = {e_new2:.2}$")
plt.scatter([0], [0], label="Earth")
plt.xlim(-2.5e7, 2.5e7)
plt.ylim(-2.5e7, 2.5e7)
plt.xlabel("x (m)")
plt.ylabel("y (m)")
plt.legend()
plt.show()
```

and also for Figure 3.

```
Isp =  451 # s

e_orbit = 0.99995

scaling = (e_orbit+1)**(1/2)
print(scaling)

v_orbit_scaled = scaling*v_orbit

M_S_apo = M_S_i * np.exp(-(v_orbit_scaled-v_orbit)/Isp/g0)

r_apo = (1 + e_orbit)/(1-e_orbit) * r_orbit

v_apo = v_orbit_scaled / (1 + e_orbit) * (1-e_orbit)

fuel_used = np.linspace(0, M_S_apo - 3000, 100)
M_S_boosted = M_S_apo - fuel_used
```

```
dv = Isp * g0 * np.log(M_S_apo/(M_S_boosted))

v_apo_boosted = dv + v_apo
p_apo = v_apo_boosted * (M_S_boosted)

v_A = np.sqrt(2*G*M_E/r_apo + (25000)**2)

thetas = (np.rad2deg(np.arctan(np.abs(1.64*p_apo)/(v_A * M_A))))
print(np.rad2deg(np.arctan((r_E +r_A)/r_apo)))

print(f"{r_apo:.2}")
plt.plot(fuel_used, thetas)
plt.xlabel('Fuel used (kg)' )
plt.ylabel(r'$\theta$ deflection $(^\circ)$')
plt.show()
```