Homework 5

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2.5.1

Problem

Give an example of a nested sequence of bounded open intervals that does not have a point in its intersection.

Solution

Let $I_n = (0, \frac{1}{n})$ be a nested sequence of open bounded intervals. $I_{n+1} \subset I_n\ \forall\ n$. Any element of the intersection $\cap_n I_n$ must be strictly greater than 0 and strictly less than $\frac{1}{n}$ for all n. By the Archimedean Property, no such element can exist. Thus, this is an example of a nested sequence of bounded open intervals such that $\cap_n I_n = \varnothing$ $\blacksquare$

2.5.6

Problem

For each of the following sequences $\{a_n\}$, find a subsequence which converges.

(a) $a_n = (-1)^n$.

(b) $a_n = \sin n\pi/4$.

(c) $a_n = \frac{n}{2^{k_n}} - 1$ with k_n the largest integer k so that $2^k \leq n$.

Solution

(a) $\{1, 1, 1, ...\}$ converges, but it is not a subsequence of a_n because by Definition 2.5.3, the sequence of natural numbers $1, 1, 1, ...$ is not strictly increasing. Similarly, $\{-1, -1, -1, ...\}$ converges, but it is not a subsequence of a_n. Thus, there is no convergent subsequence for $a_n = (-1)^n$. ■

(b) Let $b_k = sin(n\pi/8)$. b_k is a subsequence of a_n because for all $k = 2n$, $b_k = a_n$. However b_k does not converge. There is no subsequence of a_n that converges because for any $x \in \mathbb{Q}, \sin x\pi$ does not converge. ■

(c) Let $b_n = \frac{n}{2^n} - 1$. b_n is a subsequence of a_n because b_n doesn't have the same restriction for k_n to be the largest integer such that $2^n \leq n$. So we need to show that b_n converges to -1. That is, for any $\epsilon > 0$ there exists a real number N such that $|b_n| < \epsilon$ for any $n > N$. Take $N = \frac{\epsilon}{2}$. This gives:

$$|\frac{2^\epsilon}{\epsilon}| < \frac{\epsilon}{2}$$

This clearly holds for all $n > N$, so b_n converges to -1. Thus, it is a convergent subsequence of a_n. ■

2.5.9

Problem

Prove that a sequence which satisfies $|a_{n+1} - a_n| < 2^{-n}$ for all n is a Cauchy sequence.

Solution

By the definition of a Cauchy sequence, we need to show that for every $\epsilon > 0$, there is some real number N such that for all $n > N$

$$|a_{n+1} - a_n| < 2^{-n}$$

Take $N = -\log \epsilon$. This gives:

$$|a_{n+1} - a_n| < \epsilon$$

Since N is a real number and not a natural number, the definition of a Cauchy sequence holds for all $n \in \mathbb{N}$. ■

2.5.11

Problem

Let $s_n = \sum_{k=1}^{n} \frac{1}{k2^k}$ be the sequence of partial sums of the series $\sum_{k=1}^{\infty} \frac{1}{k2^k}$. Prove that $\{s_n\}$ converges. Hint: Show that it is a Cauchy sequence.

Solution

By Theorem 2.5.8, to show that s_n converges, it is sufficient to show that it is a Cauchy sequence. That is, if for every $\epsilon > 0$, there exists a real number N such that

$$|s_m - s_n| < \epsilon \text{ whenever } n, m > N$$

For $m > n$, we have:

$$|s_m - s_n| = \left| \sum_{k=n+1}^{m} \frac{1}{k2^k} \right| \leq \frac{1}{2^{n+1}(n+1)} \sum_{k=0}^{\infty} \frac{1}{2^k} = \frac{1}{2^n}$$

Since $\lim \frac{1}{2^n} = 0$, given $\epsilon > 0$, there is an N such that $\frac{1}{2^n} < \epsilon$ for all $n > N$. Hence $|s_m - s_n| < \epsilon$ for all n, m with $m > n > N$. Thus s_n is Cauchy and, hence, converges. ■

2.5.12

Problem

Given a series $\sum_{k=1}^{\infty} a_k$, set $s_n = \sum_{k=1}^{n} a_k$ and $t_n = \sum_{k=1}^{n} |a_k|$. Prove that $\{s_n\}$ converges if $\{t_n\}$ is bounded.

Solution

$m \leq t_n \leq p$ for some m, p because t_n is bounded. Then for every $\epsilon > 0$ there exists N such that for $n > N, |s_n - a| < \epsilon$. That is:

$$|\sum_{k=1}^{n} a_k - a| < \epsilon$$

Take $\epsilon = p - m \geq 0$. Then it's easy to see that $|\sum_{k=1}^{n} a_k - a| < \epsilon$ for all $n > N$ because ϵ is strictly greater than 0 and $p - m \geq 0$. Thus if t_n is bounded , s_n converges to some value a. ■

2.6.1

Problem

Find $\limsup a_n$ and $\liminf a_n$ for the following sequences:

(a) $a_n = (-1)^n$;

(b) $a_n = (-1/n)^n$;

(c) $a_n = \sin n\pi/3$.

Solution

(a) $\liminf a_n = \lim i_n$ where $i_n = \inf \{a_k : k \geq n\}$. In this case, $i_n = -1$ so $\lim i_n = -1$. Thus $\liminf a_n = -1$.

$\limsup a_n = \lim s_n$ where $s_n = \sup \{a_k : k \geq n\}$. In this case, $s_n = 1$ so $\lim s_n = 1$ also. Thus $\limsup a_n = 1$. ■

(b) $\liminf a_n = \lim i_n$ where $i_n = \inf \{a_k : k \geq n\}$. In this case, $i_n = -1$ so $\lim i_n = -1$. Thus $\liminf a_n = -1$.

$\limsup a_n = \lim s_n$ where $s_n = \sup \{a_k : k \geq n\}$. In this case, $s_n = \infty$ so $\lim s_n = \infty$ as well. Thus $\limsup a_n = \infty$. ■

(c) $\liminf a_n = \lim i_n$ where $i_n = \inf \{a_k : k \geq n\}$. In this case, $i_n = \sin \pi/3 = 0$ so $\lim i_n = 0$. Thus $\liminf a_n = 0$.

$\limsup a_n = \lim s_n$ where $s_n = \sup \{a_k : k \geq n\}$. In this case, $s_n = \infty$ so $\lim s_n = \infty$ as well. Thus $\limsup a_n = \infty$. ■

2.6.2

Problem

Find $\liminf$ and $\limsup$ for the sequence of Exercise 2.5.6(c).

Solution

$\liminf a_n = \lim i_n$ where $i_n = \inf \{a_k : k \geq n\}$. a_n is bounded below by 0, thus $i_n = 0$ and $\liminf a_n = 0$ (this is equivalent to the justification given in Exercise 2.6.1)

$\limsup a_n = \lim s_n$ where $s_n = \sup \{a_k : k \geq n\}$. In this case, $s_n = 0$ so $\lim s_n = 0$ as well. Thus, $\limsup a_n = 0$. ■

Note: by Theorem 2.6.6, this is equivalent to $\lim a_n = 0$.

2.6.4

Problem

If $\limsup a_n$ and $\limsup b_n$ are finite, prove that

$$\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n.$$

Solution

Let $C = \limsup(a_n + b_n),\ A = \limsup a_n$, and $B = \limsup b_n$. Suppose FSOC that $C > A + B$ and suppose that $\exists\ \epsilon > 0$ such that $C - \epsilon > A + B + \epsilon$. Because A and B are finite, by the definition of A and B, for some real number N, $\exists\ n > N$ such that

$$a_n + b_n > C - \epsilon$$

By that same logic, for some real number N_1, $\exists\ n > N_1$ such that

$$a_n < A + \tfrac{\epsilon}{2}$$

Similarly, for some real number N_2, $\exists\ n > N_2$ such that

$$b_n < B + \tfrac{\epsilon}{2}$$

Now if we take $N = max(N_1, N_2)$, then this gives:

$$a_n + b_n < A + B + \tfrac{\epsilon}{2} + \tfrac{\epsilon}{2} < C - \epsilon$$

Thus, contradicting the earlier statement:

$$a_n + b_n > C - \epsilon$$

So by contradiction, we've shown that $\limsup(a_n + b_n) \leq \limsup a_n + \limsup b_n$ ■