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Exam 3

All solutions will be my own and I will not consult outside resources. I understand that doing otherwise would be unfair to my classmates and a violation of the U's honor code. - [signature]

1.) By definition, $f(x)$ is continuous at $a>0, a \in (0,\infty)$ if $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t.

$$|f(x)-f(a)| < \varepsilon \quad \text{whenever } x \in (0,\infty) \text{ \& } |x-a| < \delta.$$

Equivalently, $f(x)$ is cts at $a>0$ if $f(x_n) \to a$ for any $x_n \to a$. Thus, take $x_n = \frac{a^2+n}{n}$ which $\to a$.

$$f(x_n) = \sqrt{\tfrac{a^2+n}{n}} \cos\left(\frac{1+\frac{a^2+n}{n}}{\frac{a^2+n}{n}}\right)$$ take the limit to get

$$\lim_{n\to\infty} \sqrt{\tfrac{a^2+n}{n}} \cos\left(\frac{1+\frac{a^2+n}{n}}{\frac{a^2+n}{n}}\right)$$ which evaluates to (by the MLT)

$a\cos\left(\frac{1+a^2}{a^2}\right) = a$. Thus, using the formulation described above, $f(x)$ is cts at $a>0$.

Now for $a=0$, we have

$$|f(x)-0| < \varepsilon \quad \text{whenever } x \in [0,\infty) \text{ \& } |x-a| < \delta.$$

In this case, $f(x)=0$ for $x=0$ so we have

$$|0-0| = 0 < \varepsilon$$

which certainly holds for any x, δ.

Thus, we've shown that $f(x)$ is cts on $[0,\infty)$ ∎

2) a) Using a Thm proven in the textbook, it is sufficient to show that the continuous function $f(x) = x^2$ is continuous on the extension $\bar{I} = [0,1]$ to show uniform continuity.

For $x = 0$, from the defn of continuity, we have
$$|x^2 - 0| < \varepsilon \text{ when } |x - 0| < \delta.$$
For x close to 0, $x > x^2$, so $|x| < \delta \Rightarrow |x^2| < \varepsilon$.

For $x = 1$, using the defn, we have
$$|x^2 - 1| < \varepsilon \text{ when } |x - 1| < \delta.$$
This gives
$|x-1||x+1| \leq |x-1|^2$. Take $\delta = \sqrt{\varepsilon}$ to get $|x-1|^2 < \delta^2 = \varepsilon$, which is equivalent to $|x^2 - 1| < \varepsilon$ ∎

b) Using part (a), we only need to show unif cnty for $a > 1$. Using the defn we have
$$|x^2 - a| < \varepsilon \text{ when } a \in (1, \infty) \text{ \& } |x - a| < \delta.$$

$|x^2 - a| = |x - \sqrt{a}||x + \sqrt{a}|$ since a is nonnegative.
This gives
$|x-a||x+a| \leq |x-a|^2$. Take $\delta = \sqrt{\varepsilon}$ to get $|x-a|^2 < \delta^2 = \varepsilon$. Thus, $f(x) = x^2$ is uniformly continuous on $(0,1)$ (part a) & on $[1, \infty)$ ∎

3) a) for $x = 0$, we have

$$\lim_{x \to 0} \frac{nx}{1+nx} = \frac{0}{1+0} = 0 \checkmark$$

For $x > 0$, we have $\lim_{n \to \infty} \frac{nx}{1+nx} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n \to \infty} \frac{x}{\frac{1}{n}+x}$

by the MLT we have $\lim_{n \to \infty} \frac{x}{x} = 1.$

Thus, we've shown that $\{f_n\}$ converges pointwise to the function $f(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \end{cases}$ ■

b) Suppose FSOC that $\exists N$ s.t.

$$|f(x) - f_n(x)| < \varepsilon \quad \text{where } n > N \ \& \ \varepsilon > 0.$$

This would mean that

$$\left| 1 - \frac{nx}{1+nx} \right| < \varepsilon \quad \text{for } n > N.$$

In order for this inequality to hold, N must depend on x. This is where the contradiction occurs. Uniform convergence requires a N which does not depend on x. This is impossible because for N not dependent on x, we have

$$\left| 1 - \frac{Nx}{1+Nx} \right|, \text{ which is not less than } \varepsilon.$$

■

4) From the definition, we have

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.$$

Thus we have

$$f'(0) = \lim_{x \to 0} \frac{\frac{x}{1+x} - 0}{x - 0} = \lim_{x \to 0} \frac{\frac{x}{1+x}}{x} = \lim_{x \to 0} \frac{x}{x + x^2} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

by the MLT, which gives $\lim_{x \to 0} \frac{1}{1+x} = \frac{1}{1+0} = 1$

Thus we've shown that $f'(0) = 1$ using the MLT & the definition. ∎