Homework One

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August 29, 2020

1.1 Question 1

Problem

If $a, b \in \mathbb{R}$ and $a < b$, give a description in set theory notation for each of the intervals $(a, b), [a, b], [a, b)$, and $(a, b]$ (see Example 1.1.1).

Solution

$(a, b) = \{x \in \mathbb{R} \colon a < x < b\}$
$[a, b] = \{x \in \mathbb{R} \colon a \leq x \leq b\}$
$[a, b) = \{x \in \mathbb{R} \colon a \leq x < b\}$
$(a, b] = \{x \in \mathbb{R} \colon a < x \leq b\}$

1.1 Question 4

Problem

What is the intersection of all the open intervals containing the closed interval $[0, 1]$? Justify your answer.

Solution

The intersection of all the open intervals containing the closed interval $[0, 1]$ is $[0, 1]$.

Let $\epsilon > 1$ be an element of an open interval containing $[0, 1]$. Then by the completeness of $\mathbb{R}$, there exists a midpoint between ϵ and 1 of the form $\frac{\epsilon - 1}{2}$. Therefore, the intersection of all open intervals containing the closed interval $[0, 1]$ cannot contain an element > 1.

Let $\delta < 0$ be an element of an open interval containing $[0, 1]$. Then by the completeness of $\mathbb{R}$, there exists a midpoint between δ and 0 of the form $\frac{\delta - 1}{2}$. Therefore, the intersection of all open intervals containing the closed interval $[0, 1]$ cannot contain an element < 0.

Since there can be no element < 0 or > 1, the intersection must be $[0, 1]$.

1.1 Question 6

Problem

What is the union of all of the closed intervals contained in the open interval $(0, 1)$? Justify your answer.

Solution

Let $A = \{\cup[a, b]\colon a, b \in \mathbb{R},\ 0 < a < b < 1\}$
Let $\epsilon > 0, \delta < 1$.

By the completeness of $\mathbb{R}$, there exists a midpoint between ϵ and 0 of the form $\frac{\epsilon+1}{2}$.

By the same logic, there exists a midpoint between δ and 1 of the form $\frac{\delta+1}{2}$.

Therefore, the union of all closed intervals contained in the open interval $(0, 1)$ must be $[0 + \epsilon, 1 - \delta]$.

1.1 Question 8

Problem

Which of the following functions $f\colon \mathbb{R} \to \mathbb{R}$ are one-to-one and which ones are onto? Justify your answer.
$(a) f(x) = x^2$;
$(b) f(x) = x^3$;
$(c) f(x) = e^x$.

Solution

Definition: A function $f\colon A \to B$ is **one-to-one** (injective) if $\forall\, x, y \in A, f(x) = f(y)$ then $x = y$.
Definition: A function $f\colon A \to B$ is **onto** (surjective) if $\forall\, y \in B, \exists\, x \in A$ such that $f(x) = y$.

$(a) f(x) = x^2$ is not a one-to-one function because $f(-2) = 4$ and $f(2) = 4$ i.e. $f(x) = f(y)$ but $x \neq y$.
Let $y = f(x) = x^2$. Then $x = \sqrt{y}$.
$\Rightarrow$ Since the function $x = \sqrt{y}$ is valid for any pair $(x, y) \in \mathbb{R}$, $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}$ such that $f(x) = y$. By definition, the function is onto.

(b) Let $x, y \in \mathbb{R}$ such that $f(x) = f(y)$. Then $x^3 = y^3$. Because we can take the cube root of both sides, $x = y$. By definition $f(x) = x^3$ is one-to-one.
Let $y = f(x) = x^3$, then $x = \sqrt[3]{y}$.
$\Rightarrow$ Since the function $x = \sqrt[3]{y}$ is valid for any pair $(x, y) \in \mathbb{R}$, $\forall y \in \mathbb{R}\ \exists x \in \mathbb{R}$ such that $f(x) = y$. By definition, the function is onto.

(c) Let $x, y \in \mathbb{R}$ such that $f(x) = f(y)$. Then $e^x = e^y$. We can take the natural logarithm of both sides to get $y = x$. So by definition, e^x is one-to-one.
Let $y = f(x) = e^x$, then $x = ln(y)$.
$\Rightarrow$ Since the function $x = ln(y)$ is valid for any pair $(x, y) \in \mathbb{R}, \forall y \in \mathbb{R}\ \exists x \in \mathbb{R}$ such that $f(x) = y$. By definition, the function is onto.

1.2 Question 2

Problem

Prove that if $n, m \in \mathbb{N}$, then $m + n \neq n$. Hint: Use induction on n.

Solution

Let P_n be the statement "$m + n \neq n$" for some n.
Let $s(n)$ be the successor function, i.e. $s(n) = n + 1$.

Base case: $P_1 =$ "$m+1 \neq 1$" is true by the Peano's third Axiom. (1 is not the successor of any element in $\mathbb{N}$).

Inductive step: Assume P_k is true. That is, $m + k \neq k$ for some $k \in \mathbb{N}$. Then $P_{k+1} = m + s(k) \neq s(k)$. By the definition of addition on $\mathbb{N}, m + s(k) = s(m + k)$. Then $s(m + k) \neq s(k)$.

By Peano's fourth Axiom, if any two elements of $\mathbb{N}$ have the same successor, they are equal. Since $m + k$ and k don't have the same successor, they're not equal. Hence $m + n \neq n$ for $n, m \in \mathbb{N}$.

1.2 Question 3

Problem

Use the preceding exercise to prove that if $n, m \in \mathbb{N}, n \leq m$, and $m \leq n$, then $n = m$. This is the *reflexive* property of an order relation.

Solution

Case 1: $n = m, m < n$
By Axiom four, if two elements have the same successor, they're equal.

$$n = m$$
$$\Rightarrow s(n) = s(m)$$
$$\therefore m \text{ cannot be less than } n.$$

Case 2: $n = m, n < m$
By Axiom four, if two elements have the same successor, they're equal.

$$n = m$$
$$\Rightarrow s(n) = s(m)$$
$$\therefore n \text{ cannot be less than } m.$$

Case 3: $n < m, m < n$

$$\text{By defintion, } s(n) \leq m \text{ and } s(m) \leq n.$$
$$\Rightarrow n < s(n) \leq m < s(n) < m.$$

This is a contradiction.
Case 4: $n = m, m = n$

By Axiom four, if two elements have the same successor, they're equal.

$$n = m \text{ and } m = n$$
$$\Rightarrow s(n) = s(m) \text{ and } s(m) = s(n)$$

$\therefore$ if $n, m \in \mathbb{N}, n \leq m$, and $m \leq n$, then $n = m$.

1.2 Question 10

Problem

Using induction, prove that $\sum_{k=1}^{n} (2k - 1) = n^2 \; \forall n \in \mathbb{N}$.

Solution

Let P_n be the statement $\sum\limits_{k=1}^{n}(2k-1) = n^2$ for $n, k \in \mathbb{N}$.

Base case: $P_1 = 2(1) - 1 = 1^2$ is true.

Inductive step: Assume P_n is true. That is, $\sum\limits_{k=1}^{n}(2k-1) = n^2$ for some $n \in \mathbb{N}$. Then we must show that P_{n+1} is true to complete the inductive step of the proof.

$$\begin{gathered}
P_{n+1} : \sum_{k=1}^{n+1}(2k-1) = (n+1)^2 \\
\sum_{k=1}^{n}(2k-1) + 2(n+1) - 1 \\
= 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 \\
= 2\tfrac{n(n+1)}{2} - n + 2n + 1 \\
= n(n+1) + n + 1 \\
= n^2 + 2n + 1 = (n+1)^2
\end{gathered}$$

$\therefore P_{n+1}$ is true and the statement has been proven by induction.

1.2 Question 13

Problem

Let a sequence $\{x_n\}$ of all numbers be defined recursively by

$$x_1 = 0 \quad \text{and} \quad x_{n+1} = \tfrac{x_n+1}{2}.$$

Prove by induction that $x_n \leq x_{n+1}\ \forall n \in \mathbb{N}$. Would this conclusion change if we set $x_1 = 2$?

Solution

Let P_n be the statement $x_n \leq x_{n+1}$.

Base case P_1: $x_2 = \frac{0+1}{2} > 0$ is true.

Inductive step: Assume P_n is true for some $n \in \mathbb{N}$. That is, $x_k \leq x_{k+1}$. Then P_{n+1} is the statement

$$x_{n+2} \geq x_{n+1}.$$

Add 1 to both sides and divide both sides by 2.

$$\tfrac{x_{n+2}+1}{2} \geq \tfrac{x_{n+1}+1}{2}$$

This shows that P_{n+1} is true and the statement has been proven by induction.