Homework Two

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1.3.3

Problem

Prove that if $\mathbb{Z}$ satisfies the axioms for a commutative ring, then $\mathbb{Q}$ satisfies **A1** and **M1**.

Solution

A1: $x + y = y + x \; \forall x, y \in \mathbb{R}$.

Assuming $\mathbb{Z}$ satisfies **A1**, then let $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}, b, d \neq 0$.

Addition on $\mathbb{Q}$ is defined in the familiar way: $\frac{n}{m} + \frac{p}{q} = \frac{nq+mp}{mq}$ (from the textbook).

Then

$$x + y = \frac{a}{b} + \frac{c}{d}$$

By definition,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

Since ad, bc, and $bd \in \mathbb{Z}$, then by the Commutative Law of Multiplication and the Commutative Law of Addition for integers (**M1** and **A1**):

$$\frac{ad+bc}{bd} = \frac{cb+da}{db}$$

By definition,

$$\frac{cb+da}{db} = \frac{c}{d} + \frac{a}{b}$$

$\therefore \mathbb{Q}$ satisfies **A1**.

M1: $xy = yx \; \forall x, y \in \mathbb{R}$.

Assuming $\mathbb{Z}$ satisfies **M1**, then let $x = \frac{a}{b}$ and $y = \frac{c}{d}$ for some $a, b, c, d \in \mathbb{Z}, b, d \neq 0$.

Multiplication is defined in the familiar way: $\frac{n}{m} * \frac{p}{q} = \frac{np}{mq}$ (from the textbook).

Then

$$x * y = \frac{a}{b} * \frac{c}{d}$$

And by definition,

$$\frac{a}{b} * \frac{c}{d} = \frac{ac}{bd}$$

Since $ac, bd \in \mathbb{Z}$, then by the Commutative Law of Multiplication for integers (**M1**):

$$\frac{ac}{bd} = \frac{ca}{db}$$

And by definition,

$$\frac{ca}{db} = \frac{c}{d} * \frac{a}{b}$$

$\therefore \mathbb{Q}$ satisfies **M1**. $\square$

1.3.5

Problem

In the next three exercises you are to prove the given statement assuming x, y, z are elements of a field. You may use the results of examples and theorems from this section.

5. $(-x)(-y) = xy$.

Solution

By axiom **A4**, each non-zero element of a field has an additive inverse. I.e. $x + (-x) = 0$. This holds because a field satisfies all conditions for commutative rings.

Then

$$\begin{aligned}
(-x)(-y) &= (-x)(-y) + \underbrace{(-x+x)y}_{=0} = \quad \text{(By A4)} \\
&= (-x)(-y) + (xy + (-x)y) \quad \text{(By D)} \\
&= ((-x)(-y) + (-x)y) + xy \\
&= (-x)\underbrace{(-y+y)}_{=0} + xy \\
&= (-x)0 + xy \\
&= xy
\end{aligned}$$

$\therefore (-x)(-y) = xy$. $\square$

1.3.7

Problem

7. $xy = 0$ implies $x = 0$ or $y = 0$.

Solution

Assume that $xy = 0$

Then we have two cases to show that this implies $x = 0$ or $y = 0$.

First: we have $x = 0$. From Example 1.3.2 in the book, for any element a of a commutative ring F, the following holds: $a * 0 = 0$. Therefore, $x = 0$ implies that $xy = 0$ because x, y are elements of some field and a field satisfies all axioms for commutative rings.

Next: we have $x \neq 0$. Then we are allowed to divide both sides of the equation by x. This gives:

$$y = \frac{0}{x}$$

By definition of $0 \in F$, and because x, y are elements of a field, we have $y = 0$.

Therefore, $xy = 0$ implies $x = 0$ or $y = 0$. $\square$

1.3.10

Problem

In the next three exercises you are to prove the given statement assuming x, y, z are elements of an ordered field. Again, you may use the results of examples and theorems from this section.

10. $0 < x < y$ implies $y^{-1} < x^{-1}$.

Solution

Take

$$x < y$$

Divide both sides by y since $y > 0$.

$$\frac{x}{y} < 1$$

Now divide both sides by x since $x > 0$. Then

$$\frac{1}{y} < \frac{1}{x}$$

It's easy to see that

$$\frac{1}{y} * y = 1 \text{ and } \frac{1}{x} * x = 1$$

By definition, $\frac{1}{y} = y^{-1}$ and $\frac{1}{x} = x^{-1}$.

Therefore, $0 < x < y$ implies $y^{-1} < x^{-1}$ $\square$

1.4.1

Problem

For each of the following sets, describe the set of all upper bounds for the set:

(a) of odd integers;
(b) $\{1 - 1/n \colon n \in \mathbb{N}\}$;
(c) $\left\{r \in \mathbb{Q} \colon r^3 < 8\right\}$;
(d) $\{\sin x \colon x \in \mathbb{R}\}$.

Solution

(a) Let A be the set of odd integers. There are no upper bounds for the set A.

Suppose, FSOC, A has an upper bound $m \in \mathbb{Z}$. By definition,

$$m > x \; \forall x \in A$$

This means that $m \notin A$. Since $m \in \mathbb{Z}, m$ must be even. Since the integers have cardinality $\aleph$, $m + 1$ is also an integer and it must be odd. Because $m + 1 > m, m$ cannot be an upper bound for the set A.

(b) Let $A = \{1 - 1/n \colon n \in \mathbb{N}\}$. The set of all upper bounds for A is $[1, \infty)$.

If $m < 1$ then $\exists x \in A$ such that $x \geq m$. That is, $\exists x \in A$ such that $1 - 1/n \geq m$. Then $1/n \leq 1 - m$. By the Archimedean Property, the set of all upper bounds for A is $[1, \infty)$.

(c) Let $A = \left\{r \in \mathbb{Q} \colon r^3 < 8\right\}$. The set of all upper bounds for A is $[2, \infty)$.

Let $m < 2$ be an element of A. By theorem, $\exists x \in A$ such that $m < x < 2$. Therefore, $x \in A$ and $x > m$, so m is not an upper bound for A. This means that the set of all upper bounds includes 2 and any number greater than 2. That is: $\sup(A) = 2$.

(d) Let $A = \{\sin x \colon x \in \mathbb{R}\}$. The set of all upper bounds for A is $[1, \infty)$.

Let $m \in A$. It's easy to see that if $m \leq -1, m$ is not an upper bound for A because m is a lower bound for A and it can't be both. For $-1 < m$, by theorem, $\exists x \in A$ such that $m < x < 1$. Therefore, $x \in A$ and $x > m$, so m is not an upper bound for A. This means that the set of all upper bounds includes 1 and any number greater than 1. That is: $\sup(\sin x) = 1$.

1.4.2

Problem

For each of the sets $(a), (b), (c)$ of the preceding exercise, find the least upper bound of the set, if it exists.

Solution

(a) From Exercise 1.4.1 part (a), there is no least upper bound for the set of odd integers because there are no upper bounds for the set.

(b) From Exercise 1.4.1 part (b), the set of all upper bounds for the set $A = \{1 - 1/n \colon n \in \mathbb{N}\}$ is $[1, \infty)$. Then it's easy to see that $\sup(A) = 1$.

(c) From Exercise 1.4.1 part (c), the set of all upper bounds for the set $A = \left\{r \in \mathbb{Q} \colon r^3 < 8\right\}$ is $[2, \infty)$. Then it's easy to see that $\sup(A) = 2$.

1.4.7

Problem

Prove that if $x < y$ are two real numbers, then there is a rational number r with $x < r < y$. Hint: Use the result of Example 1.4.9.

Solution

Let $\delta = y - x > 0$. By the Archimedean Property, $\exists n \in \mathbb{N}$ such that $0 < \frac{1}{n} < \delta$. This gives

$$\begin{gathered} \tfrac{1}{n} < (y - x) \\ ny - nx > 1 \end{gathered}$$

Then by the well-ordering property, $\exists r \in \mathbb{Q}$ such that $nx < r < ny$. This shows that $x < r < y$ where $x, y \in \mathbb{R}$ and $r \in \mathbb{Q}, r = \frac{m}{n}$ for some $m, n \in \mathbb{Z}$. $\square$

1.4.10

Problem

Show that if L_x and L_y are Dedekind cuts defining the real numbers x and y, then

$$L_x + L_y = \{r + s \colon r \in L_x \textit{ and } s \in \ L_y\}$$

is also a Dedekind cut (this is the Dedekind cut determining the sum $x + y$).

Solution

Definitions:
$L_x = \{r \in \mathbb{Q} \colon r < x\}$
$L_y = \{s \in \mathbb{Q} \colon s < y\}$

To show that $L_x + L_y$ is a Dedekind cut, we must show that it satisfies the following properties:

(a)$L \neq \varnothing$ and $L \neq \mathbb{Q}$.
(b)L has no largest element.
(c) if $x \in L$, then so is every $y \in \mathbb{Q}$ with $y < x$.

$L_x + L_y \neq \varnothing$ because L_x and L_y are valid Dedekind cuts, so their sum cannot be empty.
$L_x + L_y \neq \mathbb{Q}$ because by the definitions of L_x and L_y, there are elements of $\mathbb{Q}$ that are greater than x and y respectively.

$L_x + L_y$ has no largest element because if $\frac{n}{m}$ is any positive element of L, then we can always choose a larger rational number of the form $\frac{kn+1}{km} > \frac{n}{m}$ $\forall k \in \mathbb{N}$ (from Example 1.4.2).

Lastly, if $x \in L_x + L_y$, then by the respective definitions of L_x and L_y, any $z \in \mathbb{Q}$ will be in $L_x + L_y$ because $z \in L_x$ means that $x \in \mathbb{Q}$ such that $z < x$.
Similarly, $z \in L_y$ means that $y \in \mathbb{Q}$ such that $z < y$.

This shows that $L_x + L_y$ is a Dedekind cut which determines the sum $x + y$. $\square$

1.4.12

Problem

If L is the Dedekind cut of Example 1.4.2 and L determines the real number x (so that $L = L_x$), prove that $L_{x^2} = L_2$. Thus, the real number corresponding to L has square 2.

Solution

From Example 1.4.2, $L = \left\{r \in \mathbb{Q} \colon r \geq 0 \textit{ and } r^2 < 2\right\} \cup \{r \in \mathbb{Q} \colon r < 0\}$

Similarly to the argument given in Example 1.4.2, let r be a real number such that $L = L_r$. r is a positive rational number not in L so $r^2 \geq 2$. Because there are numbers in L arbitrarily close to r and each of them has a square less than two, $r^2 \leq 2$. By Axiom **O2** for ordered fields, $r = 2$.

This shows that the real number corresponding to L has square 2. $\square$