Homework 8

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4.1.1

Problem

Find the indicated limit and prove that your answer is correct.
$\lim_{x\to 1} \frac{x^2-1}{x-1}$

Solution

$\lim_{x\to 1} \frac{x^2-1}{x-1} = 1$

Proof: From the definition of the limit of a function, $\lim_{x\to a} f(x) = L$ if $\forall\, \epsilon > 0, \exists\, \delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in \mathbb{R} \text{ and } 0 < |x-a| < \delta$$

This gives the following

$$\left|\frac{x^2-1}{x-1} - 1\right| = \left|\frac{(x-1)(x+1)}{x-1} - 1\right| = |x|$$

Thus we can take $x = 1$ to get $|x| < \epsilon$ where $|x-a| < \delta$. This holds for any arbitrary $\epsilon, \delta > 0$. Thus $\lim_{x\to 1} \frac{x^2-1}{x-1} = 1$ ■

4.1.10

Problem

Prove Theorem 4.1.7: Let I be an open interval and let a be a point of I. If f is defined on I except possibly at a, then

$$\lim f(x) = L \text{ if and only if } \lim_{x\to a^+} f(x) = L = \lim_{x\to a^-} f(x)$$

Solution

From the definition for a limit, for $f(x)$ defined on I

$$\lim_{x \to a} f(x) = L$$

if for each $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta$$

Similarly, using the definition for a one-sided limit, we can choose δ_1 such that

$$|f(x) - L| < \epsilon \text{ whenever } a < x < \delta_1$$

Again, using the definition for a one-sided limit, we can choose δ_2 such that

$$|f(x) - L| < \epsilon \text{ whenever } a < x < \delta_2$$

Now we can take $\delta = \min(\delta_1, \delta_2)$. This gives the following

$$|f(x) - L| < \epsilon \text{ whenever } x \in I \text{ and } 0 < |x - a| < \delta$$

This implication holds in both directions, so we have shown that

$$\lim f(x) = L \text{ if and only if } \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x) \blacksquare$$

4.1.11

Problem

Let f be defined on a bounded interval (a, b) and let u be a^+, b^-, or a point of (a, b). Prove that if $\lim_{x \to u} f(x)$ exists and is positive, then there is a $\delta > 0$ such that $f(x) > 0$ whenever $|x - u| < \delta$ and $x \in (a, b)$. Hint: recall the proof of Theorem 2.2.3.

Solution

If $c \in (a, b)$ and $u = b^-$ then $u - c > 0$. Since $\lim_{x \to u} f(x) = L$ for some positive number L,

$$\exists\, \delta > 0 \text{ such that } f(x) > 0 \text{ whenever } |x - u| < \delta$$

Take $\epsilon = u - c$. This implies that

$$c - u + c < f(x) < c + u - c \text{ whenever } |x - u| < \delta \text{ and } x \in (a, b)$$

Thus, $f(x) > 0$ for some arbitrary $\delta > 0$.

Similarly, take $c \in (a, b)$ and $u = a^+$. Then $c - u > 0$. Since $\lim_{x \to u} f(x) = L$ for some positive number L,

$$\exists\, \delta > 0 \text{ such that } f(x) > 0 \text{ whenever } |x - u| < \delta$$

Take $\epsilon = c - u$. This implies that

$$u - c + u < f(x) < u + c - u \text{ whenever } |x - u| < \delta \text{ and } x \in (a, b)$$

Thus, $f(x) > 0$ for some arbitrary $\delta > 0$. We've shown that for any possible $u, \exists\, \delta > 0$ such that $f(x) > 0$ whenever $|x - u| < \delta$ and $x \in (a, b)$. ■

4.1.15

Problem

Prove Theorem 4.1.15: Let f be defined on (a, b) and let $u = a^+$ or b^- or a point in the interval (a, b). If f is positive on (a, b), then

$$\lim_{x \to u} f(x) = \infty \text{ if and only if } \lim \frac{1}{f(x)} = 0$$

Solution

From the definition of the limit of a function, $\lim_{x \to u} f(x) = L$ if $\forall\, \epsilon > 0, \exists\, \delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (a, b) \text{ and } 0 < |x - u| < \delta$$

Assuming that $\lim_{x \to u} f(x) = \infty$, take $N = \dfrac{1}{\epsilon}$. This gives the following

$$|f(x)| > M \text{ whenever } 0 < |x - u| < \delta$$

This implies that

$$\left|\tfrac{1}{f(x)}\right| < \epsilon \text{ whenever } 0 < |x - u| < \delta$$

Thus for some arbitrary δ, we have $\lim_{x \to u} \dfrac{1}{f(x)} = 0$.

Now for the other side of the if and only if statement. From the definition of the limit of a function, $\lim_{x \to u} f(x) = L$ if $\forall\, \epsilon > 0, \exists\, \delta > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in (a, b) \text{ and } 0 < |x - u| < \delta$$

Assuming that $\lim_{x \to u} \dfrac{1}{f(x)} = 0$, take $N = \epsilon$. This gives the following

$$\left| \frac{1}{f(x)} \right| < N \text{ whenever } 0 < |x - u| < \delta$$

This implies that

$$|f(x)| > N \text{ whenever } 0 < |x - u| < \delta$$

Thus, for some arbitrary δ, we have $\lim_{x \to u} f(x) = \infty$. Now that we've proven both directions of the if and only if statement, we have proven that $\lim_{x \to u} f(x) = \infty$ if and only if $\lim \dfrac{1}{f(x)} = 0$. $\blacksquare$

4.2.1

Problem

Using just the definition of the derivative, show that the derivative of $1/x$ is $-1/x^2$

Solution

The definition of the derivative states that for a function f defined on an open interval containing $a \in \mathbb{R}$ if

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. For $f(x) = \frac{1}{x}$ and $a \neq 0$ we have

$$\lim_{x \to a} \frac{\frac{1}{x} - \frac{1}{a}}{x - a} = \lim_{x \to a} \frac{\frac{a - x}{ax}}{x - a} = \lim_{x \to a} \frac{-1}{ax} = \frac{-1}{a^2}$$

Since we chose some arbitrary $a \neq 0$, we've shown the derivative of $f(x) = \dfrac{1}{x}$ is $f'(x) = \dfrac{-1}{x^2}$ using only the definition. $\blacksquare$

4.2.4

Problem

Using theorems from this section, find the derivative of $\tan\left(\dfrac{x}{x^2 + 1}\right)$

Solution

By Theorem 4.2.7, which is proven in the textbook, we have

$$\left(\tan\left(\frac{x}{x^2+1}\right)\right)' = \sec^2\left(\frac{x}{x^2+1}\right) * \left(\frac{x}{x^2+1}\right)'$$

By Theorem 4.2.6 part (d), $\left(\frac{x}{x^2+1}\right)' = \frac{x^2+1-2x^2}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2}$. Now we can put both parts together to get the following

$$\left(\tan\left(\frac{x}{x^2+1}\right)\right)' = \sec^2\left(\frac{x}{x^2+1}\right)\left(\frac{1-x^2}{(x^2+1)^2}\right) \blacksquare$$

4.2.9

Problem

Prove that if f is defined on an open interval I and has a positive derivative at a point $a \in I$, then there is an open interval J, containing a and contained in I, such that $f(x) < f(a) < f(y)$ whenever $x, y \in J$ and $x < a < y$. Hint: see Exercise 4.1.11.

Solution

Let $J \subset I$ be some arbitrary interval with $a, x, y \in J$ and $x < a < y$. From Exercise 4.1.11, we know that $f(x), f(a), f(y) > 0$ because f is defined on J since $J \subset I$. From the definition of the derivative, we know that for the given point $a \in I$

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists and is finite. We can use the fact that $f'(a) > 0$ to say that $f(x) < f(a) < f(y)$ when $x < a < y$. Thus, we have proven that for a function f defined on an open interval I with a positive derivative $f'(a)$ at a point $a \in I$, then there is an open interval $J \subset I$, containing a such that $f(x) < f(a) < f(y)$ whenever $x, y \in J$ and $x < a < y$. $\blacksquare$

4.2.12

Problem

Is the function defined by

$$f(x) = \begin{cases} x^2 & x > 0 \\ 0 & x \leq 0 \end{cases}$$

differentiable at 0?

Solution

From the definition of the derivative, f is said to be differentiable at 0 if

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x} = \lim_{x \to 0} \frac{f(x)}{x}$$

exists and is finite. By Theorem 4.1.7, which is proven in these exercises, it is sufficient to show that $\lim_{x \to 0^-} \frac{f(x)}{x} = \lim_{x \to 0^+} \frac{f(x)}{x} = L$.

It's easy to see that $\lim_{x \to 0^-} \frac{f(x)}{x} = 0$ and $\lim_{x \to 0^+} \frac{f(x)}{x} = 0$. Thus, by Theorem 4.1.7, $\lim_{x \to 0} \frac{f(x)}{x} = 0$. This limit exists and is finite, so f is differentiable at 0. ■