Homework Four

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2.3.3

Problem

Use the Main Limit Theorem to find $\lim \dfrac{2^n}{2^n+1}$.

Solution

Begin by manipulating the sequence algebraically. After simplifying, we can use the Main Limit Theorem to find our solution.

Multiply the sequence by $\frac{\frac{1}{2^n}}{\frac{1}{2^n}}$ to eliminate the dominating terms. This gives

$$\lim \frac{\frac{1}{2^n}}{\frac{1}{2^n}} * \frac{2^n}{2^n+1} = \lim \frac{\frac{2^n}{2^n}}{\frac{2^n}{2^n} + \frac{1}{2^n}} = \lim \frac{1}{1+\frac{1}{2^n}}$$

Using part (b) of the main Limit Theorem, which is proven in the textbook, we can split up the denominator to get:

$$\lim(1+\frac{1}{2^n}) = \lim(1) + \lim \frac{1}{2^n}$$

$\lim(1) = 1$. $\lim \frac{1}{2^n} = 0$ these statements are both proven as Corollaries below.

Now we have $\lim \dfrac{1}{1} = \lim(1)$. Again, by the proof in Corollary 1, $\lim(1) = 1$.

Thus, $\lim \frac{2^n}{2^n+1} = 1$ ■

Corollary 1: $\lim(1) = 1$ Using the definition of convergence, given $\epsilon > 0$, there exists a real number N such that $|1-1| < \epsilon$. This inequality obviously holds for any value of $n > N$, so by the definition of convergence,

$\lim(1) = 1$.

Corollary 2: $\lim \frac{1}{2^n} = 0$ Using the definition of convergence, given $\epsilon > 0$, there exists a real number N such that $|\frac{1}{2^n}| < \epsilon$. Since $\epsilon, n > 0$, Take $N = \log_2(\frac{1}{\epsilon})$. This gives:

$$|\log_2(\frac{1}{\epsilon})| < \epsilon$$

This inequality holds for any $n > N$. Thus, by the definition of convergence, $\lim \frac{1}{2^n} = 0$

2.3.5

Problem

Prove Theorem 2.3.2: Let $\{a_n\}$ be a sequence of real numbers such that $\lim a_n = 0$, and let $\{b_n\}$ be a bounded sequence. Then $\lim a_n b_n = 0$.

Solution

Due to the fact that b_n is bounded, $\exists\ k > 0$ such that $|b_n| \leq k\ \forall\ n$. This gives:

$$0 \leq |a_n \cdot b_n| \leq k|a_n|$$

By the definition of $a_n \to 0$, given $\epsilon > 0$, $\exists\ N$ such that $|a_n| < \epsilon\ \forall\ n > N$. Using the facts that $N > 0$ and for any ϵ, $|a_n| < \frac{\epsilon}{k}$ given $n > N$ for some N, we have:

$$0 \leq |a_n \cdot b_n| \leq k|a_n| \leq k \cdot \frac{\epsilon}{k} = \epsilon$$

Thus, $\lim a_n b_n \to 0$ ■

2.3.8

Problem

Prove that if $\{b_n\}$ is a sequence of positive terms and $b_n \to b > 0$, then there is a number $m > 0$ such that $b_n \geq m$ for all n.

Solution

By Corollary 2.2.4, if a sequence converges, then it is bounded. b_n converges so it's bounded. It must also be bounded below by definition. Thus, there is a real number m such that $m \leq b_n$ for all n.

Now we need to show that $m > 0$. Since we know that $b_n \to b$, we can use the definition of convergence with $\epsilon = m$. That is, since $b_n \to b$, for every $m > 0$, there is a real number N such that $|b_n - b| < m$ whenever $n > N$. Thus, because of the fact that $b_n \to b > 0$, the definition holds for $m > 0$.

This shows that there is a number $m > 0$ such that $b_n \geq m$ for all n. ■

2.3.9

Problem

Prove part (d) of Theorem 2.3.6: $a_n/b_n \to a/b \ if \ b \neq 0 \ and \ b_n \neq 0 \ for \ all \ n.$ Hint: Use the previous exercise.

Solution

Let $c_n = \dfrac{a_n}{b_n}$ and let $c = \dfrac{a}{b}$. If $b \neq 0$ and $b_n \neq 0$, then by Exercise 2.3.8, there is a number $m > 0$ such that $c_n \geq m$ for all n. Thus, we can use this m to show that $c_n \to c$. Given a real number N, $|c_n - c| < m$ whenever $n > N$.

$\therefore\ c_n \to c$ if $b \neq 0$ and $b_n \neq 0$. This is equivalent to $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$ ■

2.4.1

Problem

Tell which of these sequences are non-increasing, non-decreasing, bounded? Justify your answers.

$(a) \left\{n^2\right\}$

$(b) \left\{\frac{1}{\sqrt{n}}\right\}$

$(c) \left\{\frac{(-1)^n}{n}\right\}$

(*d*) $\left\{\frac{n}{2^n}\right\}$

(*e*) $\left\{\frac{n}{n+1}\right\}$

Solution

Note: a sequence is said to be non-decreasing if $n_1 < n_2 \Rightarrow a_{n_1} < a_{n_2}$ for all terms in the sequence. A sequence is said to be non-increasing if $n_1 > n_2 \Rightarrow a_{n_1} > a_{n_2}$ for all terms in the sequence.

(a) Let $a_n = \{n^2\}$. The sequence is non-decreasing. This can be proven using simple induction.

Base case - $a_1 : 1 < 2 \Rightarrow 1^2 < 2^2$ is true.

Inductive step: assuming a_k is true, a_{k+1} is true because $n < m \Rightarrow n^2 < m^2$ for all $n, m \in \mathbb{N}$.

The sequence is unbounded because for any number k, there is a number greater than k, (call it m) by the Archimedean Property. Because a_n is non-decreasing, a_m is also greater than k. $\blacksquare$

(b) Let $b_n = \left\{\frac{1}{\sqrt{n}}\right\}$. The sequence is non-decreasing. This can be proven using simple induction.

Base case - $b_1 : \frac{1}{\sqrt{1}} < \frac{1}{\sqrt{2}}$ is true.

Inductive step: assuming b_k is true, b_{k+1} is also true because $n < m \Rightarrow \sqrt{n} < \sqrt{m}$ for all $n, m \in \mathbb{N}$.

The sequence is unbounded because for any number k, there is a number greater than k (call it m) by the Archimedean Property. Because b_n is non-decreasing, b_m is also greater than k. $\blacksquare$

(c) Let $c_n = \left\{\frac{(-1)^n}{n}\right\}$. The sequence c_n is neither non-increasing or non-decreasing because the numerator of the sequence causes oscillations between negative and positive terms.

Similarly, the sequence c_n is unbounded because of the oscillations between negative and positive terms caused by $(-1)^n$. $\blacksquare$

(d) Let $d_n = \left\{\frac{n}{2^n}\right\}$. The sequence is non-increasing because for all $n, m \in \mathbb{N}$, the following holds:

$$n < m \Rightarrow \frac{n}{2^n} < \frac{m}{2^m}$$

The sequence is bounded above by its first term d_1 and bounded below by the number it converges to: 0. Thus, d_n is a bounded sequence. $\blacksquare$

(e) Let $e_n = \left\{\frac{n}{n+1}\right\}$. The sequence e_n is non-decreasing because of the fact that $n+1 > n$ for all n. Thus the denominator of the sequence is bigger than the numerator for all n.

The sequence is bounded below by its first term e_1 and bounded above by the number it converges to: 1. So e_n is bounded. ■

2.4.3

Problem

If $a_1 = 1$ and $a_{n+1} = (1 - 2^{-n})a_n$, prove that $\{a_n\}$ converges.

Solution

To prove this sequence converges to 0, we must show that for some $\epsilon > 0$, there exists a real number N such that $|a_n - a| < \epsilon$ for all $n > N$. Take $N = \log_2(\frac{1}{\frac{2\epsilon}{a_{n-1}} - 2})$. This gives:

$$|\log_2(\frac{1}{\frac{2\epsilon}{a_{n-1}} - 2})| < \epsilon$$

Because this inequality holds for all $n > N$, the sequence converges by definition. ■

2.4.4

Problem

Let $\{d_n\}$ be a sequence of 0's and 1's and define a sequence of numbers $\{a_n\}$ by

$$a_n = d_1 2^{-1} + d_2 2^{-2} + ... + d_n 2^{-n}$$

Prove that this sequence converges to a number between 0 and 1.

Solution

To prove this sequence converges to a number between 0 and 1, we must show that for some $\epsilon > 0$, there exists a real number N such that $|\frac{d_n}{2^n} - a| < \epsilon$ for all $n > N$ (given that $a \in [0,1]$). Take $N = \log_2(\frac{d_n}{\epsilon - a})$. This gives:

$$|\log_2(\frac{d_n}{\epsilon - a})| < \epsilon$$

This inequality is true for all $n > N$, so the sequence converges to a by definition. Since $a \in [0, 1]$, the proof is complete. $\blacksquare$

2.4.9

Problem

Prove that $\lim \dfrac{2^n}{n} = \infty$

Solution

By Definition 2.4.4, $\lim a_n = \infty$ if for every real number M, there is a number N such that $a_n > M$ whenever $n > N$.

Take $N = \dfrac{2^M}{M}$. Then for $n > \dfrac{2^M}{M}$, we have $\dfrac{2^n}{n} > M$. This inequality holds for any real number M.

Thus, $\lim \dfrac{2^n}{n} = \infty$ by Definition 2.4.4 $\blacksquare$

2.4.10

Problem

Prove Theorem 2.4.6: Every monotone sequence has a limit.

Solution

By the Monotone Convergence Theorem, which is proven in the textbook, each bounded monotone sequence converges. Thus, we need to show that any unbounded monotone sequence has a limit, whether finite or infinite.

First we'll show that all unbounded, non-decreasing sequences have a limit. Let $\{a_n\}$ be an unbounded, non-decreasing sequence. By the definition of non-decreasing, a_n must be unbounded above because it's bounded below by its first term a_1. Given this fact, for any real number M, there exists N such that $a_n > M$ for all $n > N$. Take $M = N$, then due to monotonicity, $a_n \geq a_N > M$ for all $n > N$. Thus by definition, $\lim a_n = \infty$.

Now we'll show that all unbounded, non-increasing sequences have a limit. Let $\{b_n\}$ be an unbounded, non-increasing sequence. By the definition of non-increasing, b_n must be unbounded below because it's bounded above by its first element b_1. Given this fact, for any real number K, there exists N_K such that $b_n < K$ for all $n > N_K$. Take $K = N_K$, then due to monotonicity, $b_n \leq a_{N_K} < K$ for all $n > N_K$. Thus by definition, $\lim b_n = -\infty$.

Thus, we've shown that every monotone sequence has a (not necessarily finite) limit. ■