Homework 10

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5.2.13

Problem

Let $\{f_n\}$ be a sequence of integrable functions defined on a closed bounded interval $[a, b]$. If $\{f_n\}$ converges uniformly on $[a, b]$ to a function f, prove that f is integrable and

$$\int_a^b f(x)\,dx = \lim \int_a^b f_n(x)\,dx.$$

Solution

Theorem 3.4.4 states that if the sequence of functions $\{f_n\}$ converges uniformly to a function f on an interval $[a, b]$, then f is continuous on $[a, b]$. Theorem 5.2.2 states that if f is a continuous function on a closed, bounded interval $[a, b]$, then f is integrable on $[a, b]$. Thus, we can conclude that f is integrable on $[a, b]$.

From the definition, a sequence of functions $\{f_n\}$ is said to converge uniformly on $[a, b]$ to f if for each $\epsilon > 0$, there is an N such that

$$|f(x) - f_n(x)| < \epsilon \text{ whenever } x \in [a, b] \text{ and } n > N$$

Thus, we can evaluate the limit to get the following.

$$\lim \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$$

Hence, we've shown that if $\{f_n\}$ converges uniformly to f, f is integrable and $\int_a^b f(x)\,dx = \lim \int_a^b f_n(x)\,dx$. ■

5.2.14

Problem

Is the function which is $\sin 1/x$ for $x \neq 0$ and 0 for $x = 0$ integrable on $[0, 1]$? Justify your answer.

Solution

Let $f(x)$ be the function which is $\sin 1/x$ for $x \neq 0$ and 0 for $x = 0$. Let P be a partition of the interval $[0, 1]$. f is said to be integrable on $[0, 1]$ if

$$\underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx$$

Which, from the definitions of lower and upper integrals, is to say that

$$\inf\{U(f, P)\} = \sup\{L(f, P)\}$$

For some arbitrary partition P we have the following.

$$\inf\{U(f, P)\} = 0 \text{ and } \sup\{L(f, P)\} = 0.$$

This is from the definition of supremum and infimum for functions. Thus, the function f is integrable on $[0, 1]$. ■

5.3.1

Problem

Find $\displaystyle\int_{4/\pi}^{2/\pi} (2x \sin 1/x - \cos 1/x)\,dx$. Hint: see example 5.3.2.

Solution

Example 5.3.2 tells us that the function $f'(x) = 2x \sin 1/x - \cos 1/x$ has the anti-derivative $f(x) = x^2 \sin 1/x$. We can use this fact and the first part of the Fundamental Theorem of Calculus to evaluate the integral.

$$\int_{4/\pi}^{2/\pi} (2x \sin 1/x - \cos 1/x)\,dx = x^2 \sin 1/x \,\Big|_{4/\pi}^{2/\pi} = \left(\frac{2}{\pi}\right)^2 \sin\frac{\pi}{2} - \left(\frac{4}{\pi}\right)^2 \sin\frac{\pi}{4} = \frac{4}{\pi^2} - \frac{16}{\pi^2}\frac{\sqrt{2}}{2} = \frac{4 - 8\sqrt{2}}{\pi^2}$$

■

5.3.3

Problem

Find $\frac{d}{dx}\int_0^{2x} \sin t^2 \, dt$.

Solution

We can use the second part of the Fundamental Theorem of Calculus, which says the following.

$$F(x) = \int_a^x f(t)\, dt \text{ where } F'(x) = f(x)$$

We have a composition of functions, so we are finding the derivative of $F(2x)$. Using the chain rule, the derivative of this composition is $2F'(2x)$. The second part of the Fundamental Theorem then tells us that

$$\frac{d}{dx}\int_0^{2x} \sin t^2 \, dt = 2F'(2x) = 2\sin(2x)^2 = 2\sin(4x^2)$$

■

5.3.5

Problem

If $f(x) = -1/x$, then $f'(x) = 1/x^2$. Thus, Theorem 5.3.1 seems to imply that

$$\int_{-1}^{1} 1/x^2 \, dx = f(1) - f(-1) = -1 - 1 = -2.$$

However, $1/x^2$ is a positive function, and so its integral over [-1, 1] should be positive. What is wrong?

Solution

In order for Theorem 5.3.1 to hold in this case, the function f must be continuous on [-1, 1] and differentiable on (-1, 1) with f' integrable on [-1, 1]. There is a discontinuity at $x = 0$ for both f and f'. Thus, f is neither continuous or differentiable at $x = 0$, and f' is not integrable at $x = 0$. This means that f is not differentiable and f' is not integrable at every point in the interval [-1, 1]. Thus, Theorem 5.3.1 can't be used here. ■

5.3.9

Problem

Prove that if f is integrable on $[a, b]$ and $c \in [a, b]$, then changing the value of f at c does not change the fact that f is integrable or the value of its integral on $[a, b]$.

Solution

Define g to be a function such that $f(x) = g(x)\ \forall x \in [a, b]$ except for at the point c. That is, $f(c) \neq g(c)$. Then let $N = |g(c) - f(c)| > 0$. Let $L(f, P)$ and $U(f, P)$ denote the lower and upper sums respectively for some partition P of the interval $[a, b]$. Let $\epsilon > 0$. Since f is integrable, we know that there exists a partition P such that

$$U(f, P) - L(f, P) < \epsilon$$

Let Q be some refinement of this partition P such that $Q = P \cap \{x - \frac{\epsilon}{N}, x + \frac{\epsilon}{N}\}$. We know from the construction of g that $|L(g, Q) - L(f, Q)| < \frac{\epsilon}{N} * N = \epsilon$. Similarly, $|U(g, Q) - U(f, Q)| < \frac{\epsilon}{N} * N = \epsilon$. We can now conclude the following about g based on these statements.

$$|U(g, Q) - L(g, Q)| \leq |U(g, Q) - U(f, Q)| + |L(g, Q) - L(f, Q)| < \epsilon$$

Since g is defined to be the function f except for at the point $c \in [a, b]$, because

$$|U(g, Q) - L(g, Q)| < \epsilon$$

Hence, we have shown that changing the value of f at c does not change the integrabilty of f or the value of the integral on $[a, b]$. ■

5.4.9

Problem

For which values of $p > 0$ does the improper integral $\displaystyle\int_1^\infty \frac{1}{x^p}\,dx$ converge? Justify your answer.

Solution

For $p = 1$ we have the following.

$$\lim_{a\to\infty} \int_1^a \frac{1}{x}\,dx$$

Evaluate the integral to get the following

$$\lim_{a\to\infty}\int_1^a \frac{1}{x}\,dx = \lim_{a\to\infty} \ln|x|\Big|_1^a = \infty$$

Thus, the integral diverges for $p = 1$. Since we know that for any value $0 < p < 1$, that $\frac{1}{x} < \frac{1}{x^p}$, we can conclude that $\int_1^\infty \frac{1}{x^p}\,dx$ diverges.

For $p > 1$, we have the following.

$$\lim_{a\to\infty}\int_1^a \frac{1}{x^p}\,dx$$

Evaluate the integral to get the following

$$\lim_{a\to\infty}\int_1^a \frac{1}{x^p}\,dx = \lim_{a\to\infty} \frac{x^{1-p}}{1-p}\Big|_1^a = \frac{1}{1-p}$$

This is a well defined, finite value since $p > 1$. Thus, the improper integral $\int_1^\infty \frac{1}{x^p}\,dx$ converges for values of $p > 1$.

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