Homework 6

Kyle Kazemini

October 9, 2020

3.1.4

Problem

Show that the function $f(x) = |x|$ is continuous on all of $\mathbb{R}$.

Solution

From the definition for $|x|$: $f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$

There are 3 cases to prove: $x > 0, x = 0$, and $x < 0$.

For $x > 0$, $f(x) = x$ and $\lim_{x \to a} f(x) = a$ because $\lim_{x \to a} x = a$. Thus, $\lim_{x \to a} f(x) = f(a)$.

Similarly for $x < 0$, $f(x) = -x$ and $\lim_{x \to a} f(x) = -a$ because $\lim_{x \to a} -x = -a$. Thus, $\lim_{x \to a} f(x) = f(a)$.

Finally, for $x > 0$ we have: $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x = 0$
and for $x < 0$ we have: $\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} -x = 0$
Hence, $\lim_{x \to 0} f(x) = f(0) = 0$.

Thus, since $f(x) = |x|$ is continuous for $x > 0, x = 0$, and $x < 0, f(x)$ is continuous on all of $\mathbb{R}$ ■

3.1.6

Problem

Prove (d) of Theorem 3.1.9: f/g is continuous at a, provided $g(a) \neq 0$.

Solution

If f and g are continuous at a and $\{x_n\}$ is a sequence in D which converges to a, then Theorem 3.1.6, which is proven in the textbook, says that $\{f(x_n)\}$ converges to $f(a)$ and $\{g(x_n)\}$ converges to $g(a)$. By the Main Limit Theorem, which is proven in the textbook and exercises, $\{f(x_n)/g(x_n)\}$ converges to $f(a)/g(a)$. (Note: $g(a) \neq 0$ is covered in the Main Limit Theorem). Thus, f/g is continuous at a. ■

3.1.7

Problem

Prove Theorem 3.1.11: With f and g as above, let a be in the domain of $f \circ g$. Then $f \circ g$ is continuous at a if g is continuous at a and f is continuous at $g(a)$.

Solution

Since f is continuous at $g(a)$, for all $\epsilon > 0$, there exists some $\delta > 0$ such that

$$|x - a| < \delta \Rightarrow |f(g(x)) - f(g(a))| < \epsilon$$

Since g is continuous at a, there exists some δ_0 (named differently to show it's distinct from δ) such that

$$|x - a| < \delta_0 \Rightarrow |g(x) - g(a)| < \delta$$

Now simply take $\epsilon = \delta$ to get:

$$|x - a| < \delta_0 \Rightarrow |g(x) - g(a)| < \delta \Rightarrow |f(g(x)) - f(g(a))| < \epsilon$$

Thus, $f(g(x))$ is continuous. ■

3.1.11

Problem

Prove that the function $f(x) = \begin{cases} \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is not continuous at 0.

Solution

We have $f(0) = 0$. It is sufficient to show $\lim_{x \to 0} f(x) \neq 0$. Suppose FSOC that $\lim_{x \to 0} f(x) = 0$. Then by definition, for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - 0| < \delta \Rightarrow |f(x) - 0| < \epsilon$. Take $\epsilon = \frac{1}{100}$. Then for an arbitrary $\delta > 0$, we have $-1 \leq f(\delta) \leq 1$. Thus, $|f(\delta) - 0| \geq \epsilon$, meaning f is discontinuous at $x = 0$. ■

3.2.2

Problem

Prove that if f is a continuous function on a closed bounded interval I and if $f(x)$ is never 0 for $x \in I$, then there is a number $m > 0$ such that $f(x) \geq m$ for all $x \in I$ or $f(x) \leq -m$ for all $x \in I$.

Solution

WLOG, take $I = [a, b] \subset \mathbb{R}$. Then because f is continuous, $f([a, b]) = [m, n]$. Then assume FSOC that $m < 0$. Then by the Intermediate Value Theorem, the value 0 is taken on by f. This contradicts the statement $f(x) \neq 0$ for $x \in I$.

Similarly, assume FSOC that $n < 0$. Then by the Intermediate Value Theorem, the value 0 is taken on by f. This contradicts the statement $f(x) \neq 0$ for $x \in I$. Thus, $m > 0$ satisfies the desired properties. ■

3.2.4

Problem

Find an example of a function which is continuous on a bounded (but not closed) interval I but is not bounded. Then find an example of a function which is continuous and bounded on a bounded interval I but does not have a maximum value.

Solution

Take $I = (0, 1)$ and $f(x) = \frac{1}{x}$ The continuous function f on the interval I is not bounded because it tends towards infinity as $x \to 0$. Thus, f and I satisfy the conditions for the first example.

Take $I = (0, 1)$ and $f(x) = e^x \sin(\frac{1}{x})$. I is a bounded interval and f is a continuous and bounded function on I. Thus, f and I satisfy the conditions for the second example. ■

3.2.8

Problem

Show that if f and g are continuous functions on the interval $[a, b]$ such that $f(a) < g(a)$ and $g(b) < f(b)$, then there is a number $c \in (a, b)$ such that $f(c) = g(c)$.

Solution

By the Intermediate Value Theorem, for some y between $f(a)$ and $f(b)$, there exists c such that $f(c) = y$. Again, by the Intermediate Value Theorem, for some z between $g(a)$ and $g(b)$, there exists d such that $g(d) = z$. Thus since f and g have the same domain, we can take $z = y$ and $c = d$. This gives:

$$f(c) = y = z = g(d)$$

Again, since $c = d, f(c) = g(c)$ ■

3.2.10

Problem

Use the Intermediate Value Theorem to prove that, if n is a natural number, then every positive number a has a positive nth root.

Solution

Let $f(x) = x^n$ and $c \in \mathbb{R}$ such that $c > a$. By the Intermediate Value Theorem, $0 < a < c < f(c)$ and $f(x)$ is continuous. Also by the Intermediate Value Theorem, there exists a point p such that $f(p) = a$, since $f(x)$ is continuous. Thus, by the Intermediate Value Theorem, every $a > 0$ has a positive nth root. ■