Homework 7

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3.3.1

Problem

Is the function $f(x) = x^2$ uniformly continuous on (0, 1)? Justify your answer.

Solution

By definition, a function is uniformly continuous if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } x, a \in (0, 1) \text{ and } |x - a| < \delta$$

This gives the following

$$|x^2 - a^2| = |x - a||x + a|$$

Since $x, a \in (0, 1)$, $|x - a| < 2$ Thus, take $\delta = \dfrac{\epsilon}{2}$. Then $\forall\, x, a \in (0, 1)$ with $|x - a| < \delta$ we have

$$|x^2 - a^2| < |x - a||x + a| < 2|x - a| < 2\delta = 2\tfrac{\epsilon}{2} = \epsilon$$

Thus, $f(x) = x^2$ is uniformly continuous on (0, 1) by the definition. ■

3.3.2

Problem

Is the function $f(x) = \dfrac{1}{x^2}$ uniformly continuous on $(0, 1)$? Justify your answer.

Solution

By definition, a function is uniformly continuous if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } x, a \in (0,1) \text{ and } |x - a| < \delta$$

This gives the following

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \frac{|x^2 - a^2|}{a^2x^2} = \frac{|x-a||x+a|}{a^2x^2}$$

By the same logic from the previous exercise, take $\epsilon = \dfrac{2\delta}{a^2x^2}$. This gives

$$\frac{|x-a||x+a|}{a^2x^2} \leq \frac{2|x-a|}{a^2x^2} < \frac{2\delta}{a^2x^2} = \epsilon$$

Thus $f(x) = \frac{1}{x^2}$ is uniformly continuous on (0, 1) by the definition. ■

3.3.4

Problem

Using only the $\epsilon - \delta$ definition of uniform continuity, prove that the function $f(x) = \frac{x}{x+1}$ is uniformly continuous on $[0, \infty)$.

Solution

By definition, a function is uniformly continuous if for each $\epsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(a)| < \epsilon \text{ whenever } x, a \in (0,1) \text{ and } |x - a| < \delta$$

This gives the following

$$\left|\frac{x}{x+1} - \frac{a}{a+1}\right| = \left|\frac{ax+x}{(x+1)(a+1)} - \frac{ax+a}{(x+1)(a+1)}\right| = \frac{|x-a|}{(x+1)(a+1)}$$

Take $\epsilon = \dfrac{\delta}{(x+1)(a+1)}$ to get

$$\frac{|x-a|}{(x+1)(a+1)} < \frac{\delta}{(x+1)(a+1)} = \epsilon$$

Thus, by definition, $f(x) = \dfrac{x}{x+1}$ is uniformly continuous on $[0, \infty)$ ■

3.3.7

Problem

Prove that if I is a bounded interval and f is an unbounded function defined on I, then f cannot be uniformly continuous.

Solution

WLOG take I to be (a, b). Suppose FSOC that f is uniformly continuous. Then take $\epsilon = 1$, there must be a $\delta > 0$ such that

$$|f(x) - f(y)| < \epsilon \text{ whenever } x, y \in I \text{ and } |x - y| < \delta$$

Take n to be some number such that $n\delta > |b - a|$. Then $\forall\ x, y \in I, |f(x) - f(y)| \leq n$ by the Triangle Inequality. Since this difference is finite, the function f is bounded on I, which contradicts our initial assumption. Thus, if f is an unbounded function defined on a bounded interval I, f cannot be uniformly continuous. $\blacksquare$

3.3.9

Problem

Is the function $f(x) = \sin(1/x)$ continuous on $(0, 1)$? Is it uniformly continuous on $(0, 1)$? Justify your answers.

Solution

From the textbook, we know that $\sin(x)$ is continuous on $\mathbb{R}$ and we know that $\frac{1}{x}$ is continuous on $\mathbb{R}\setminus\{0\}$. By Theorem 3.1.11, since both of these functions are continuous, the composition $f(x) = \sin(1/x)$ is also continuous on (0, 1).

Now we must show that $f(x) = \sin(1/x)$ is not uniformly continuous on (0, 1). That is, we must show that $\exists\ \epsilon > 0$ such that $\forall\ \delta > 0$ with $x, y \in (0, 1)$ such that $|x - a| < \delta$ implies $|f(x) - f(y)| \geq \epsilon$. Take $\epsilon = 2$, $x = \frac{2}{4n\pi + \pi}$, and $y = \frac{2}{4n\pi - \pi}$. Then for some large enough n, we have $0 < x, y < \delta \Rightarrow |x - y| < \delta$. This gives the following

$$|f(x) - f(y)| = |1 - (-1)| \geq \epsilon = 2$$

Thus, we have shown that $f(x) = \sin(1/x)$ is continuous on (0, 1) but not uniformly continuous on the same interval. ■

3.4.1

Problem

Prove that the sequence $\{x/n\}$ converges uniformly to 0 on each bounded interval but does not converge uniformly on $\mathbb{R}$.

Solution

Let I be some arbitrary bounded interval such that $I \subset \mathbb{R}$. Then by definition, $\{x/n\}$ converges uniformly to 0 on I if for each $\epsilon > 0, \exists\, N$ such that

$$|f(x) - 0| < \epsilon \text{ whenever } x \in I \text{ and } n > N$$

This gives $|x/n|$. Now choose N such that $x/n < \epsilon$. This is possible because $n \neq 0$. Thus, $\{x/n\}$ converges to 0 on each bounded interval. However, it cannot converge on $\mathbb{R}$ because for an unbounded interval, say $(0, \infty)$, because as $x \to \infty$ and $n \to \infty, \{x/n\}$ becomes an indeterminate form $\frac{\infty}{\infty}$. Thus, $\{x/n\}$ converges uniformly to 0 on a bounded interval $I \subset \mathbb{R}$, but not on $\mathbb{R}$. ■

3.4.3

Problem

Prove that the sequence $\{\sin(x/n)\}$ converges to 0 pointwise on $\mathbb{R}$ but it does not converge uniformly on $\mathbb{R}$.

Solution

By definition, a sequence of functions converges pointwise on I if for each $\epsilon > 0, \exists\, N$ such that

$$|f(x) - f_n(x)| < \epsilon \text{ whenever } n > N$$

Since $n \neq 0$, we have $\lim \sin(x/n) = 0$. Thus, for N sufficiently large, $|f(x) - f_n(x)| < \epsilon$ is satisfies for any $\epsilon > 0$.

Now, by definition, $\{\sin(x/n)\}$ converges uniformly to $\sin x$ on $\mathbb{R}$ if $\forall\, \epsilon > 0, \exists\, N$ such that

$$|\sin x - \sin(x/n)| < \epsilon \text{ whenever } x \in \mathbb{R} \text{ and } n > N$$

Take $\epsilon = 2$, $x = \frac{2}{4n\pi+\pi}$, and $y = \frac{2}{4n\pi-\pi}$. Then for some large enough n, we have $0 < x, y < \delta \Rightarrow |x-y| < \delta$. This gives the following

$$|f(x) - f(y)| = |1 - (-1)| \geq \epsilon = 2$$

Thus, by definition, we have shown that $\{\sin(x/n)\}$ converges to 0 pointwise on $\mathbb{R}$ but it does not converge uniformly on $\mathbb{R}$. ■

3.4.9

Problem

For $x \in (-1,1)$ set $s_n(x) = \sum_{k=0}^{n} x^k$. This is the nth partial sum of a geometric series. Prove that $s_n(x) = \dfrac{1-x^{n+1}}{1-x}$.

Solution

This statement can be proven by induction. Let P_n be the statement given.
$P_1 : \displaystyle\sum_{k=0}^{1} x^k = \frac{1-x^2}{1-x} = \frac{(1+x)(1-x)}{1-x} = 1 + x. P_1$ is true.

Assume P_i is true. That is, for some $i \in \mathbb{N}$, we have $\displaystyle\sum_{k=0}^{i} x^k = \frac{1-x^{i+1}}{1-x}$.

Now P_{i+1} is the statement $\displaystyle\sum_{k=0}^{i+1} x^k = \frac{1-x^{i+2}}{1-x} = \frac{1-x^i x^2}{1-x}$.

P_{i+1} is true by P_1 and P_i. Thus, we have proven $s_n(x) = \dfrac{1-x^{n+1}}{1-x}$ by induction. ■

3.4.10

Problem

Prove that the sequence $\{s_n\}$ of the previous exercise converges uniformly to $\dfrac{1}{1-x}$ on each interval of the form $[-r, r]$ with $r < 1$ but it does not converge uniformly on $(-1, 1)$.

Solution

By definition, a sequence of functions converges uniformly on I if for each $\epsilon > 0, \exists\, N$ such that

$$|f(x) - f_n(x)| < \epsilon \text{ whenever } x \in I \text{ and } n > N$$

This gives the following

$$\left|\frac{1}{1-x} - \frac{1-x^{n+1}}{1-x}\right|$$

For N sufficiently large we have

$$\left|\frac{1-x^{n+1}}{1-x}\right|$$

Since $x < 1$, the x^{n+1} term in the numerator goes to 0. Therefore we have

$$\left|\frac{1-x}{1-x}\right| < \epsilon$$

for any $\epsilon > 0$. Thus, by definition, $\{s_n\}$ converges uniformly on $[-r, r]$ with $r < 1$.

FSOC, let $r = 1$. This gives $\lim s_n = \dfrac{1-x^{\infty}}{1-x}$. This is an indeterminate form. The same thing is true for $-r = 1$. Thus, $\{s_n\}$ converges uniformly on $[-r, r]$ with $r < 1$ but it does not converge uniformly on (-1, 1).

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