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Math 3210-001
9/18/20

Exam 1

All solutions will be my own and I will not consult outside resources. I understand that doing otherwise would be unfair to my classmates and a violation of the U's honor code. –

1) Base case $n=1$: $(-1)^1 \cdot 1^2 = \frac{(-1)^1(2)}{2}$ is True ✓

Inductive step: Assume P_J is true for some $J \in \mathbb{N}$.
That is, $\sum_{k=1}^{J} (-1)^k k^2 = \frac{(-1)^J J(J+1)}{2}$ is true.

Then P_{J+1} is the statement $\sum_{k=1}^{J+1} (-1)^k k^2 = \frac{(-1)^{J+1}(J+1)(J+2)}{2}$

P_{J+1} : $\sum_{k=1}^{J+1} (-1)^k k^2 = \frac{(-1)^{J+1}(J^2+3J+2)}{2}$

Since P_J : $\sum_{k=1}^{J} (-1)^k k^2 = \frac{(-1)^J (J^2+J)}{2}$,

P_{J+1} holds because $(J+1)^2 + J + 1 = J^2 + 3J + 2$

So by induction, $\sum_{k=1}^{n} (-1)^k k^2 = \frac{(-1)^n n(n+1)}{2}$ □

2a) Using Axiom M1: $(-1)x = x(-1)$

By axiom A4, (-1) is the additive inverse of 1. ⇒ by Axiom M3,

$(-1)x$ must equal $-x$ because 1 is the multiplicative identity for a field & (-1) is its additive inverse

□

2b) Using the fact that $x \cdot 0 = 0$, this means that if $y=0$, $x>0$, then $xy=0$

The same holds for x:
if $x=0$, $y>0$, then $xy=0$.

$\Rightarrow$ if both $x, y > 0$ then $xy \neq 0$

xy must also be greater than 0 by the order property of the field F.

$\therefore$ if $x, y \in F$, $x>0$ & $y>0 \Rightarrow xy>0$ $\square$

3a) Let $x, y \in \mathbb{Q}$, then $x = \frac{a}{b}$, $y = \frac{c}{d}$, for some $a, b, c, d \in \mathbb{Z}$, $b, d \neq 0$.

If $y < x$ then $\frac{c}{d} < \frac{a}{b}$ & $bc < ad$ since $b, d \neq 0$

So $y^3 < x^3$ means that $\left(\frac{c}{d}\right)^3 < \left(\frac{a}{b}\right)^3$

$\Rightarrow \frac{c^3}{d^3} < \frac{a^3}{b^3}$ multiply by b^3 & d^3 since they're nonzero.

$b^3c^3 < a^3d^3$. $\Rightarrow (bc)^3 < (ad)^3$

which holds because by assumption, $bc < ad$ $\square$

3b) (1) $L \neq \emptyset$ & $L \neq \mathbb{Q}$.
$L = \{x \in \mathbb{Q} : x^3 < 5\}$. $L \neq \emptyset$ because it contains $\frac{1}{2}$. $\frac{1}{2} \in \mathbb{Q}$ & $\left(\frac{1}{2}\right)^3 < 5$.

$L \neq \mathbb{Q}$ because $L \notin \{x \in \mathbb{Q} = x^3 \geq 5\}$ for example: $\frac{40}{3} \in \mathbb{Q}$ & $\left(\frac{40}{3}\right)^3 \geq 5$. So $\frac{40}{3} \notin L$ but $\frac{40}{3} \in \mathbb{Q}$

(3) If $x \in L$, then $\forall y \in \mathbb{Q}$, $y < x$, $y \in L$

$y < x$ implies $y^3 < x^3$ by part (a)

Since $x \in L$, $x^3 < 5$. Thus $y^3 < x^3 < 5$
By the definition of L, $y \in L$ because $y \in \mathbb{Q}$ & $y^3 < 5$

$\square$