Homework Three

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1.5.2

Problem

Find the sup and inf of the following sets. Tell whether each set has a maximum or a minimum.

$(a)(-2, 8]$.

$(b)\left\{\frac{n+2}{n^2+1}\right\}$.

$(c)\left\{n/m\colon n, m \in \mathbb{Z},\ n^2 < 5m^2\right\}$.

Solution

(a) Let A be the set $(-2, 8]$. Then clearly the $\inf A = -2$ and $\sup A = 8$. Since $-2 \notin A$, A has no minimum element. By the same logic, since $8 \in A$, 8 is the maximum element for the set A. $\blacksquare$

(b) Let B be the set $\left\{\frac{n+2}{n^2+1}\right\}$. Because the domain of B is $\mathbb{N}$ and the first element of $\mathbb{N}$ is 1 by Peano's first Axiom,

$$\sup B = \frac{1+2}{1^2+1} = \frac{3}{2} . \frac{3}{2} \in B$$

so $\frac{3}{2}$ is the maximum element for the set B because B grows inversely with n.

We can multiply by the reciprocal of the term with the largest degree in the denominator of B to get

$$B = \left\{\frac{\frac{n}{n^2} + \frac{2}{n^2}}{\frac{n^2}{n^2} + \frac{1}{n^2}}\right\}$$

This simplifies to:

$$\left\{\frac{\frac{1}{n} + \frac{2}{n^2}}{1 + \frac{1}{n^2}}\right\}$$

From Example 1.5.3:

$$\left\{\frac{\frac{1}{n}+\frac{2}{n^2}}{1+\frac{1}{n^2}}\right\} = \left\{\frac{0+0}{1+0}\right\} = 0$$

Thus, $\inf(B) = 0$. The set B has no minimum element because $0 \notin B$.$\blacksquare$

(c) Let C be the set $\{n/m\colon n, m \in \mathbb{Z},\ n^2 < 5m^2\}$. Given some $n, m \in C, m \neq 0$, then we have $\frac{n}{m}$.

Square the fraction to get:

$$(\tfrac{n}{m})^2 = \tfrac{n^2}{m^2}$$

Now make a substitution using the inequality that defines C. That is, substitute $5m^2$ for n to get:

$$\tfrac{n^2}{m^2} < \tfrac{5m^2}{m^2}$$

Simplify to get:

$$\tfrac{5\cancel{m^2}}{\cancel{m^2}} = 5$$

This gives:

$$\tfrac{n^2}{m^2} < 5$$
$$\Rightarrow \tfrac{n}{m} < \sqrt{5}$$

Using a similar argument, we can substitute $\frac{n^2}{5}$ for m to get:

$$\tfrac{n^2}{m^2} > \frac{\frac{n^2}{n^2}}{5} = \tfrac{n^2}{1} * \tfrac{5}{n^2}$$

After cancelling n^2, we have

$$\tfrac{n^2}{m^2} > 5$$
$$\Rightarrow \tfrac{n}{m} > \sqrt{5}$$

Finally, from these two substitutions we have $\sup(C) = \sqrt{5}$ and $\inf(C) = \sqrt{5}$ $\blacksquare$

1.5.3

Problem

Prove that if $\sup A < \infty$, then for each $n \in \mathbb{N}$ there is an element $a_n \in A$ such that $\sup(A) - 1/n < a_n \leq \sup(A)$.

Solution

The statement can be proven by induction. Let P_n be the statement

$$\text{“if } \sup(A) < \infty, \exists\, a_n \in A \text{ such that } \sup(A) - 1/n < a_n \leq \sup(A)\text{”}$$

for some arbitrary $n \in \mathbb{N}$. P_1 is the base case which states that

$$\exists\, a_1 \in A \text{ such that } \sup(A) - 1 < a_1 \leq \sup(A)$$

By definition, $\sup(A) \geq a_1$. Then because $\sup(A) < \infty$, $\sup(A) > \sup(A) - 1$. Thus, $\sup(A) - 1$ must be less than a_1 and P_1 is true.

Assume P_k is true for some $k \in \mathbb{N}$. That is,

$$\exists\, a_k \in A \text{ such that } \sup(A) - 1/k < a_k \leq \sup(A)$$

is true. For the inductive step: P_{k+1} states that

$$\exists\, a_{k+1} \in A \text{ such that } \sup(A) - 1/(k+1) < a_{k+1} \leq \sup(A).$$

By definition, $\sup(A) \geq a_{k+1}$. Because $\sup(A) < \infty$,

$$\sup(A) - 1/(k+1) < \sup(A) - 1/k$$

Since we know that $\sup(A) - 1/k < a_k$, we can conclude that $\sup(A) - 1/(k+1) < a_{k+1}$. Thus, the statement has been proven by induction. $\blacksquare$

1.5.6

Problem

Prove part (d) of Theorem 1.5.7: $\sup(A - B) = \sup(A) - \inf(B)$.

Solution

From Theorem 1.5.7 part (c), which is proven in the textbook, $\sup A + B = \sup A + \sup B$. Then we can define $-B$ to be the set $\{-b \colon b \in B\}$ From Theorem 1.5.7 part (b), which is proven in the textbook, $\sup{(-A)} = -\inf A$

Thus,

$$\sup(A + (-B)) = \sup A + \sup(-B) \quad \text{by part (c)}$$
$$\sup A + \sup(-B) = \sup A + (-\inf B) \quad \text{by part (b)}$$
$$\sup A + \sup(-B) = \sup A - \inf B \; \blacksquare$$

1.5.9

Problem

Find $\sup_I f$ and $\inf_I f$ for the following functions f and sets I. Which of these is actually the maximum of the minimum of the function f on I?

$(a) f(x) = x^2, I = [-1, 1]$.

$(b) f(x) = \frac{x+1}{x-1}, I = (1, 2)$.

$(c) f(x) = 2x - x^2, I = [0, 1)$.

Solution

(a) By definition, we have $\sup_I f = \sup\left\{x^2 \colon x \in I\right\}$. Because $\sup I = 1$, $\sup_I f = f(1) = 1^2 = 1$. This is the maximum of f on I because $1 \in I$.

Similarly, by definition, we have $\inf_I f = \inf\left\{x^2 \colon x \in I\right\}$. Because $\inf I = -1$, $\inf_I f = f(-1) = (-1)^2 = 1$. Although $\inf I = -1$, $\inf_I f$ is not a minimum for f on I.$\blacksquare$

(b) By definition, $\sup_I f = \sup\left\{\frac{x+1}{x-1} \colon x \in I\right\}$ Because $\sup I = 2$, $\sup_I f = f(2) = \frac{3}{1} = 3$. This is not the maximum of f on I because $3 \notin I$.

Similarly, $\inf_I f = \inf\left\{\frac{x+1}{x-1} \colon x \in I\right\}$. Because $\inf I = 1$, $\inf_I f = f(1)$. f is undefined at $x = 1$, so $\inf_I f$ does not exist. $\blacksquare$

(c) By definition, $\sup_I f = \sup\left\{2x - x^2 \colon x \in I\right\}$. Because $\sup I = 1$, $\sup_I f = f(1) = 1$. This is not the maximum of f on I because $1 \notin I$.

By definition, $\inf_I f = \inf\left\{2x - x^2 \colon x \in I\right\}$. Because $\inf I = 0$, $\inf_I f = f(0) = 0$. This is the minimum of f on I because $0 \in I$. $\blacksquare$

2.1.1

Problem

Show that

(a) if $|x-5|<1$, then x is a number greater than 4 and less than 6;

(b) if $|x-3|<1/2$ and $|y-3|<1/2$, then $|x-y|<1$;

(c) if $|x-a|<1/2$ and $|y-b|<1/2$, then $|x+y-(a+b)|<1$.

Solution

(a) From Theorem 2.1.1 which is proven in the textbook, $|x-5|<1$ iff $5-1<x<5+1$. Thus, $|x-5|<1$ iff $4<x<6$ $\blacksquare$

(b) Using the Triangle Inequality, which is proven in the textbook, we can combine the two inequalities to get

$$||x-3|-|y-3|| \leq |(x-3)-(y-3)|$$

Since $(x-3)$ and $(y-3)$ are both real numbers, we can use Theorem 2.1.1 with $\epsilon=1$ to get

$$\begin{gathered} |(x-3)-(y-3)|<1 \quad \text{which gives} \\ y-3-1<x-3<y-3+1 \quad \text{which simplifies to} \\ y-1<x<y+1 \end{gathered}$$

By Theorem 2.1.1, $|x-y|<1$ $\blacksquare$

(c) Using the Triangle Inequality, which is proven in the textbook, we can combine the two inequalities to get

$$||x-a|-|y-b|| \leq |(x-a)-(y-b)|$$

Since $(x-a)$ and $(y-b)$ are both real numbers, we can use Theorem 2.1.1 with $\epsilon=1$ to get

$$\begin{gathered} |(x-a)-(y-b)|<1 \quad \text{which gives} \\ y-b-1<x-a<y-b+1 \quad \text{which simplifies to} \\ (y-b)+a-1<x<(y-b)+a+1 \end{gathered}$$

By Theorem 2.1.1, this gives $|x+y-(a+b)|<1$ $\blacksquare$

2.1.3

Problem

Put each of the following sequences in the form $a_1, a_2, a_3, \ldots, a_n, \ldots$. This requires that you compute the first 3 terms and find an expression for the nth term.

(a) The sequence of positive odd integers.

(b) The sequence defined inductively by $a_1 = 1$ and $a_{n+1} = -\frac{a_n}{2}$.

(c) The sequence defined inductively by $a_1 = 1$ and $a_{n+1} = \frac{a_n}{n+1}$.

Solution

(a) $a_1 = 1, a_2 = 3, a_3 = 5$. $a_n = 2k + 1$ for some $k \in \mathbb{N}$.

This gives the sequence: $1, 3, 5, ..., 2k + 1, ...$ ■

(b) $a_1 = 1, a_2 = -\frac{1}{2}, a_3 = \frac{1}{4}$. $a_n = -\dfrac{a_{n-1}}{2}$.

This gives the sequence: $1, -\frac{1}{2}, \frac{1}{4}, ..., -\dfrac{a_{n-1}}{2}, ...$ ■

(c) $a_1 = 1, a_2 = \frac{1}{2}, a_3 = \frac{1}{8}$. $a_n = \dfrac{a_{n-1}}{n}$.

This gives the sequence: $1, \frac{1}{2}, \frac{1}{8}, ..., \dfrac{a_{n-1}}{n}, ...$ ■

2.1.5

Problem

$\lim \frac{2n-1}{3n+1}$.

Solution

Educated guess: $\lim \frac{2n-1}{3n+1} = \frac{2}{3}$.

Justification: Let $\epsilon > 0$ and let $N = \frac{5}{9\epsilon} - \frac{1}{3}$. If $n > N$, then $n > \frac{5}{9\epsilon} - \frac{1}{3}$ and $\frac{5}{9n+3} < \epsilon$. By definition of convergence, we have $|\frac{2n-1}{3n+1} - \frac{2}{3}| < \epsilon$. Multiplying by a common denominator gives

$$|\frac{6n-3}{9n+3} - \frac{6n+2}{9n+3}| = |\frac{-5}{9n+3}| = \frac{5}{9n+3} < \epsilon$$

By definition, the limit converges to $\frac{2}{3}$ ■

2.2.3

Problem

$\lim \frac{1}{\sqrt{n}}$.

Solution

Educated guess: $\lim \frac{1}{\sqrt{n}} = 0$.

Justification: Let $\epsilon > 0$ and let $N = \frac{1}{\epsilon^2}$ By definition of convergence, we have $|\frac{1}{\sqrt{n}} - 0| < \epsilon$. Plugging in N, we have $|\frac{1}{\epsilon}| < \epsilon$, which gives $\frac{1}{\epsilon} < \epsilon$. This holds for all $n > N$, so the proof is done by the definition of convergence. ■

2.2.5

Problem

$\lim \sqrt{n^2 + n} - n$

Solution

Educated guess: $\lim \sqrt{n^2 + n} - n = \frac{1}{2}$.

Justification: Multiplying by $\dfrac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n}$ gives $\dfrac{n}{\sqrt{n^2+n}+n}$. Let $\epsilon > 0$ and let $N = \dfrac{\epsilon + \sqrt{\epsilon^2 + \epsilon}}{2\epsilon + 2\sqrt{\epsilon^2 + \epsilon}}$. By definition of convergence, we have

$$|\frac{n}{n + \sqrt{n^2 + n}} - \frac{1}{2}| < \epsilon$$

Multiplying by a common denominator gives:

$$|\frac{2n}{2n + 2\sqrt{n^2 + n}} - \frac{n + \sqrt{n^2 + n}}{2n + 2\sqrt{n^2 + n}}| < \epsilon$$

This simplifies to

$$|\frac{n + \sqrt{n^2 + n}}{2n + 2\sqrt{n^2 + n}}| < \epsilon$$

Using the N value stated earlier, this gives $N < \epsilon \; \forall \; n > N$. Thus the proof is done by the definition of convergence. ■

2.2.9

Problem

Does the sequence $\{\cos(n\pi/3)\}$ have a limit? Justify your answer.

Solution

The sequence $\{\cos(n\pi/3)\}$ does not have a limit.

Let A be the sequence $\{\cos(n\pi/3)\}$. Then let $A' = \{\cos(2k\pi)\}$ where k is defined such that $3(2k) = n$ (n is 3 times some even number). Finally, let $A'' = \{\cos(2k+1)\pi\}$ where k is defined such that $3(2k+1) = n$ (n is 3 times some odd number).

For A to converge to some number c, given $\epsilon > 0$, there must be $N \in \mathbb{R}$ such that $|A - c| < \epsilon \; \forall \; n > N$. Using the fact that $A' = 1 \; \forall \; k$ and the fact that $A'' = -1 \; \forall \; (k+1)$, we have a contradiction by Theorem 2.1.6, which is proven in the textbook. A, A', and A'' are equivalent formulations of the same sequence, but they converge to different values. ■

2.2.10

Problem

Give an example of a sequence $\{a_n\}$ which does not converge but for which the sequence $\{|a_n|\}$ does converge.

Solution

Based on the work showed in the solution of 2.2.9 and the fact that the absolute value of the codomain of cos is $[0, 1]$, the sequence in 2.2.9 satisfies this condition. ■