Homework 11

Kyle Kazemini

November 24, 2020

6.1.3

Problem

In each of the following six exercises, determine whether the indicated series converges. Justify your answer.

$\sum_{k=0}^{\infty} \frac{2^{k+1}}{3^k}$

Solution

By the properties of series and exponents we have the following.

$$\sum_{k=0}^{\infty} \frac{2^{k+1}}{3^k} = \sum_{k=0}^{\infty} 2\frac{2^k}{3^k} = \sum_{k=0}^{\infty} 2\left(\frac{2}{3}\right)^k$$

Since $2, \frac{2}{3} \in \mathbb{R}$ and $2 \neq 0$ and $|\frac{2}{3}| < 1$, we can use Theorem 6.1.6 for Geometric Series, which is proven in the textbook, to get the desired result.

$$\sum_{k=0}^{\infty} 2\left(\frac{2}{3}\right)^k = \frac{2}{1-\frac{2}{3}} = \frac{2}{\frac{1}{3}} = 6$$

Hence, we've shown that the series $\sum_{k=0}^{\infty} \frac{2^{k+1}}{3^k}$ converges to 6 by Theorem 6.1.6. ■

6.1.5

Problem

$\sum_{k=1}^{\infty} \frac{k^2}{4^k}$

Solution

Since $\lim_{k\to\infty} \frac{k^2}{4^{k/2}} = 0$ by L'Hopital's Rule, there exists some N such that

$$\frac{k^2}{4^{k/2}} < 1 \ \ \forall k > N$$

Thus

$$\frac{k^2}{4^{k/2}} < \frac{1}{4^{k/2}} = \frac{1}{2^k} \ \ \forall k > N$$

Thus we have the series $\sum_{k=0}^{\infty} \frac{1}{2^k}$. This is a geometric series. By Theorem 6.1.6, it converges to $\frac{1}{1-1/2} = 2$. Since $\frac{k^2}{4^k} < \frac{k^2}{4^{k/2}}$, by Theorem 6.1.9 (comparison test), the series $\sum_{k=0}^{\infty} \frac{k^2}{4^k}$ converges. ■

6.1.6

Problem

$$\sum_{k=1}^{\infty} \frac{k}{k^2 - k + 2}$$

Solution

Since $k^2 - k + 2 \leq 2k^2 \ \ \forall k \in \mathbb{N}$, we have the following.

$$\frac{1}{k} \leq \frac{2k}{k^2 - k + 2} \ \ \forall k \in \mathbb{N}$$

We proved in class that the harmonic series $\sum_{k=0}^{\infty} \frac{1}{k}$ diverges. Thus, $\sum_{k=1}^{\infty} \frac{k}{k^2 - k + 2}$ also diverges by the comparison test. ■

6.1.8

Problem

Determine whether the indicated series converges absolutely. Justify your answer.

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$$

Solution

From the definition, a series $\sum_{k=1}^{\infty} a_k$ converges absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges. So we have the following.

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k+1}}{\sqrt{k}} \right| = \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$$

We know that $\frac{1}{k} \leq \frac{1}{\sqrt{k}} \ \ \forall k \in \mathbb{N}$. Hence, $\sum_{k=1}^{\infty} \frac{1}{k} \leq \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$. By the comparison test, $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges. Thus, the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{\sqrt{k}}$ does not converge absolutely. ■

6.2.1

Problem

In each of the following eight exercises, determine whether the indicated series converges. Justify your answer by indicating what test to use and then carrying out the details of the application of that test.

$\sum_{k=2}^{\infty} \frac{1}{k \ln k}$

Solution

Since $a_k = f(k) \ \forall k \in \mathbb{N}$ and since f is a positive, non-increasing function on $[2, \infty)$, the series $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ converges iff $\int_2^{\infty} \frac{1}{x \ln x} \, dx$ converges. So we have the following.

$$\int_2^{\infty} \frac{1}{x \ln x} \, dx$$

A simple u-substitution where $u = \ln x$ and $du = \frac{1}{x} dx$ gives the following.

$$\int_{\ln 2}^{\infty} \frac{1}{u} \, du = \ln \infty - \ln \sqrt{2} = \infty$$

Note: the absolute value inside the natural log function can be omitted. Since this integral diverges, the series $\sum_{k=2}^{\infty} \frac{1}{k \ln k}$ also diverges. ■

6.2.3

Problem

$\sum_{k=1}^{\infty} \frac{k2^k}{3^k}$

Solution

The ratio test says that
$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} > 1 \Rightarrow$ divergence and
$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} < 1 \Rightarrow$ absolute convergence. Thus, we have the following.

$$\lim_{k\to\infty} \frac{|(k+1)2^{k+1}|}{|3^{k+1}|} \frac{|3^k|}{|k2^k|} = \lim_{k\to\infty} \frac{2k+2}{3k} = \frac{2}{3}$$

The absolute value signs can be taken out because all of the terms are strictly positive. So we have $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = \frac{2}{3} < 1$ which means that the series $\sum_{k=1}^{\infty} \frac{k2^k}{3^k}$ converges absolutely. ■

6.2.4

Problem

$\sum_{k=0}^{\infty} \frac{5^k}{k!}$

Solution

The ratio test says that
$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} > 1 \Rightarrow$ divergence and
$\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} < 1 \Rightarrow$ absolute convergence. Thus, we have the following.

$$\lim_{k\to\infty} \frac{|5^{k+1}|}{|(k+1)!|} \frac{|k!|}{|5^k|} = \lim_{k\to\infty} \frac{5}{k+1} = 0$$

The absolute value signs can be taken out because all of the terms are strictly positive. So we have $\lim_{k\to\infty} \frac{|a_{k+1}|}{|a_k|} = 0 < 1$ which means that the series $\sum_{k=0}^{\infty} \frac{5^k}{k!}$ converges absolutely. ■

6.2.11

Problem

Prove that if $\sum_{k=1}^{\infty} a_k$ converges absolutely and $\{b_k\}$ is a bounded sequence, then $\sum_{k=1}^{\infty} a_k b_k$ also converges absolutely.

Solution

If $\sum_{k=1}^{\infty} a_k$ converges absolutely, we know that $\sum_{k=1}^{\infty} |a_k|$ converges from the definition. Thus, we have the following series.

$$\sum_{k=1}^{\infty} |a_k| b_k$$

Since $\{b_k\}$ is a bounded sequence, we know that there exists some $N \in \mathbb{R}$ such that $|b_k| \leq N\ \forall k \in \mathbb{N}$. So we can say the following.

$$\sum_{k=1}^{\infty} |a_k| b_k \leq N \sum_{k=1}^{\infty} |a_k|$$

We know that $\sum_{k=1}^{\infty} |a_k|$ converges by the definition of absolute convergence, so $N \sum_{k=1}^{\infty} |a_k|$ also converges. Since the series $\sum_{k=1}^{\infty} |a_k| b_k$ is less than or equal to a convergent series, it must also converge. Thus, by definition of absolute convergence, $\sum_{k=1}^{\infty} a_k b_k$ converges. $\blacksquare$

6.3.1

Problem

In each of the next five exercises, determine whether the given series converges absolutely, converges conditionally, or diverges. Justify your answer.

$$\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/3}}$$

Solution

From Theorem 6.3.2, if $\{a_k\}$ is a non-increasing sequence of non-negative numbers which converges to 0, then the series

$$\sum_{k=1}^{\infty} (-1)^k a_k$$

converges. In this case, the sequence $a_k = \frac{1}{k^{1/3}}$ is indeed non-negative because $k^{1/3} > 0\ \forall k \in \mathbb{N}$. Also, a_k is non-increasing because $a_k > a_{k+1}\ \forall k \in \mathbb{N}$.
Finally, a_k converges to 0 because $\lim_{k\to\infty} \frac{1}{k^{1/3}} = 0$. Hence, we can apply Theorem 6.3.2 to say that $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/3}}$ converges.

By definition, a series $\sum_{k=1}^{\infty} a_k$ converges absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges. This gives the following.

$$\sum_{k=1}^{\infty} \frac{|(-1)^k|}{k^{1/3}} = \sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$$

The series $\sum_{k=1}^{\infty} \frac{1}{k^{1/3}}$ is a p-series with $p = 1/3 < 1$ so it diverges by Example 6.2.2. Thus, $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^{1/3}}$ converges but it does not converge absolutely. ■

6.3.2

Problem

$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$

Solution

From Theorem 6.3.2, if $\{a_k\}$ is a non-increasing sequence of non-negative numbers which converges to 0, then the series

$$\sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

converges. In this case, the sequence $a_k = \frac{1}{k^2}$ is indeed non-negative because $k^2 > 0\ \forall k \in \mathbb{N}$. Also, a_k is non-increasing because $a_k > a_{k+1}\ \forall k \in \mathbb{N}$.
Finally, a_k converges to 0 because $\lim_{k\to\infty} \frac{1}{k^2} = 0$. Hence, we can apply Theorem 6.3.2 to say that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ converges.

By definition, a series $\sum_{k=1}^{\infty} a_k$ converges absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges. This gives the following.

$$\sum_{k=1}^{\infty} \frac{|(-1)^{k+1}|}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

The series $\sum_{k=1}^{\infty} \frac{1}{k^2}$ is a p-series with $p = 2 > 1$ so it converges by Example 6.2.2. Thus, $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^2}$ converges absolutely by the definition. ■

Note: The first part of this proof could have been omitted because absolute convergence implies convergence but I left it in for the sake of completeness.