Homework Two

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(1)

Problem

Consider a random variable X with expected value $E[X] = 100$ and variance $Var[X] = 144$. We would like to upper bound the probability $P[X < 75]$.
(a) Which bound can and cannot be used with what we know about X (Markov, Chebyshev, or Chernoff-Hoeffding), and why?
(b) Using that bound, calculate an upper bound for $P[X < 75]$.
(c) Describe a probability distribution for X where the other two bounds are definitely not applicable.

Solution

(a) The Markov Inequality cannot be used because the condition $X \geq 0$ is not satisfied in this problem for the random variable X. The Chernoff-Hoeffding Inequality cannot be used because each X_i must lie in a bounded domain. Thus, the only bound we can use is the Chebyshev Inequality because in this problem, $\epsilon = 75 > 0$ and we know $E[X]$ and $Var[X]$.

(b) The Chebyshev Inequality gives:

$$P[X - 100 < 75] \leq \frac{144}{75^2}$$
$$P[X < 175] \leq \frac{144}{5625}$$
$$P[X < 175] \leq \boxed{0.0256}$$

(c) An example distribution where the the Markov Inequality does not hold is a distribution where the random variable X can be less than 0. An example distribution where the Chernoff-Hoeffding Inequality does not hold is an unbounded distribution. Therefore, a Normal distribution with mean μ and standard deviation σ is an example of a distribution where both the Markov and Chernoff-Hoeffding Inequalities don't hold because the Normal distribution is unbounded and it allows for random variables $X < 0$.

(2)

Problem

Consider n iid random variables $X_1, X_2, ..., X_n$ with expected value $E[X_i] = 7$ and variance $Var[X_i] = 2$. Assume we also know that each X_i must satisfy $1 \leq X_i \leq 13$. We now want to analyze the random variable of their average $\bar{X} = \frac{1}{n} \sum\limits_{i=1}^{n} X_i$.

Assume first that $n = 2$ (the number of random variables).
(a) Use the Chebyshev Inequality to upper bound $P[X > 12]$.
(b) Use the Chernoff-Hoeffding Inequality to upper bound $P[X > 12]$.

Now assume first that $n = 20$ (the number of random variables).
(c) Use the Chebyshev Inequality to upper bound $P[X > 12]$.
(d) Use the Chernoff-Hoeffding Inequality to upper bound $P[X > 12]$.

Solution

(a) The Chebyshev Inequality with $P[X > 12]$ and $n = 2$ gives:

$$P[X - 7 > 12] \leq \frac{2}{2 * 12^2}$$
$$P[X > 19] \leq \frac{1}{144}$$
$$P[X > 19] \leq \boxed{0.0069}$$

This value is the upper bound for $P[X > 12]$ and $n = 2$ random variables.

(b) The Chernoff-Hoeffding Inequality with $P[X > 12]$ and $n = 2$ gives:

$$P[|X - 7| > 12] \leq 2\exp(\frac{-2 * 12^2 * 2}{(13-1)^2})$$
$$P[|X - 7| > 12] \leq 2\exp(-4)$$
$$P[|X - 7| > 12] \leq \boxed{0.36631}$$

This value is the upper bound for $P[X > 12]$ and $n = 2$ random variables.

(c) The Chebyshev Inequality with $P[X > 12]$ and $n = 20$ gives:

$$P[X - 7 > 12] \leq \frac{2}{20 * 12^2}$$
$$P[X > 19] \leq \frac{1}{1440}$$
$$P[X > 19] \leq \boxed{0.0007}$$

This value is the upper bound for $P[X > 12]$ and $n = 20$ random variables.

(d) The Chernoff-Hoeffding Inequality with $P[X > 12]$ and $n = 20$ gives:

$$P[|X - 7| > 12] \leq 2\exp(\frac{-2 * 12^2 * 20}{(13-1)^2})$$
$$P[|X - 7| > 12] \leq 2\exp(-40)$$
$$P[|X - 7| > 12] \leq \boxed{8.49671 * 10^{-18}}$$

This value is the upper bound for $P[X > 12]$ and $n = 20$ random variables.

(3)

Problem

Consider the following two vectors in $\mathbb{R}^4$:

$$p = (1, -2, 4, x)$$
$$q = (2, -4, 8, -2)$$

Report the following:
(a) Choose the value x so that p and q are linearly dependent.
(b) Choose the value x so that p and q are orthogonal.
(c) Calculate $\|q\|_1$
(d) Calculate $\|q\|_2^2$

Solution

(a) Take $\boxed{x = -1}$. Then it's easy to see that $2p = q$. Thus, p and q are linearly dependent because q can be expressed as some multiple of p.

(b) p and q are orthogonal if $\langle p, q \rangle = 0$. So we can calculate $\langle p, q \rangle$, set it equal to 0, and solve for x.

$$\langle p, q \rangle = 1 * 2 + (-2)(-4) + 4 * 8 - 2x = 0$$
$$2 + 8 + 32 - 2x = 0$$
$$-2x = -42$$
$$\boxed{x = 21}$$

(c) $\|q\|_1 = \sum_{i=1}^{4} |q_i| = 2 + 4 + 8 + 2 = \boxed{16}$

(d) $\|q\|_2^2 = (\sqrt{\langle q, q \rangle})^2 = \langle q, q \rangle = 2^2 + (-4)^2 + 8^2 + (-2)^2 = \boxed{88}$

(4)

Problem

Consider the following 2 matrices:

$$A = \begin{bmatrix} 2 & -1 & 4 \\ 0 & -1 & 0 \\ 3 & -2 & 6 \end{bmatrix} \quad B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}$$

Report the following:

(a) $A^T B$
(b) AB
(c) BA
(d) $B + A$
(e) B^T
(f) Which matrices are invertible? For any that are invertible, report the result.

Solution

```
import numpy as np
from numpy import linalg as LA

# Create matrices A and B

A = np.matrix([[2, -1, 4],
             [0, -1, 0],
             [3, -2, 6]])
print(A)

[[ 2 -1  4]
 [ 0 -1  0]
 [ 3 -2  6]]

B = np.matrix([[0, 0, 1],
              [1, 0, 0],
              [0, 2, 0]])
print(B)

[[0 0 1]
 [1 0 0]
 [0 2 0]]


# Transpose the two matrices

AT = A.transpose()
BT = B.transpose()


################# Perform and print the calculations #################

print(AT @ B)

[[ 0  6  2]
 [-1 -4 -1]
 [ 0 12  4]]

print(A @ B)

[[-1  8  2]
 [-1  0  0]
 [-2 12  3]]

print(B @ A)

[[ 3 -2  6]
 [ 2 -1  4]
 [ 0 -2  0]]

print(B + A)

[[ 2 -1  5]
 [ 1 -1  0]
 [ 3  0  6]]
```

```python
print(BT)
```

```
[[0 1 0]
 [0 0 2]
 [1 0 0]]
```

```python
# LA.inv(A) - Singular value error, so A is not invertible.

print(LA.inv(B))
```

```
[[0.  1.  0. ]
 [0.  0.  0.5]
 [1.  0.  0. ]]
```

```python
# LA.inv(AT) - Singular value error, so A transpose is not invertible.

print(LA.inv(BT))
```

```
[[0.  0.  1. ]
 [1.  0.  0. ]
 [0.  0.5 0. ]]
```

```python
# LA.inv(AT @ B) - Singular value error, A transpose * B is not invertible.

# LA.inv(A @ B) - Singular value error, A*B is not invertible.

# LA.inv(B @ A) - Singular value error, B*A is not invertible.

print(LA.inv(B + A))
```

```
[[-0.66666667 0.66666667 0.55555556]
 [-0.66666667 -0.33333333 0.55555556]
 [ 0.33333333 -0.33333333 -0.11111111]]
```