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# Semi-Parametric Time-Series Models: Computation and Simulation


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# Contents

## 1 Abstract

A commonly used class of models in analysing count time series are INGARCH (Integer-valued Generalised Autoregressive Conditional Heteroscedatic) models. Many authors have explored parametric INGARCH models which make assumptions of the conditional response distribution, but model misspecification can create biased inferences in the resultant parametric models. Motivating this, the report proposes a semi-parametric INGARCH model for count time-series data which treats the unknown distribution instead as a infinite-dimensional parameter, which is estimated simultaneously with the model's finite dimensional parameters using a maximum likelihood approach. Numerical simulation results are presented for various distributions and suggests the proposed model has more robust inferences than parametric models in situations of model misspecification, while also performing well in estimation and inference compared to correctly-specified parametric models. The proposed model shows better or on-par model adequacy when applied to a variety of common datasets, compared to parametric models in INGARCH literature.

## 2 Introduction

Count time series are commonly observed in fields such as health, finance and epidemiology, often involving non-Gaussian responses. The analysis of count time-series can be done using a variety of models, usually being classified as one of two types. The first of which transfer an auto-regressive moving average (ARMA) model to count data utilising thinning operators. The second is a class of conditional regression models, of which, *INGARCH (Integer-valued Generalised Autoregressive Conditional Heteroscedatic) Models* are a popular choice that are widely explored in time-series literature. INGARCH models impose constraints on the conditional means to obtain an ARMA like auto-correlation function, with an assumed conditional distribution $F$. Common choices for this distribution include Poisson and Negative Binomial distributions, however, in real applications it is often difficult to correctly specify the underlying conditional distribution generating the data.

In these difficult situations, model misspecification in parametric models leads to biased inferences on model parameters, motivating for a general and flexible modelling framework for various conditional response distributions. Huang and Fung (2015) addressed this for modelling time-series using a generalised linear model by treating the distribution as a infinite dimensional parameter and jointly estimating it with the model parameters. The proposed model is a semi-parametric model and motivated by this, one could find a similar framework useful for modelling count time-series using INGARCH models to address issues mentioned prior. Thus, the project implements a semi-parametric INGARCH model in MATLAB for count time-series data that correctly adapts to a wide range of conditional distributions. Numerical simulations of the proposed model will be presented to explore the performance of estimation and inference compared to correctly specified parametric models, and misspecified parametric models. Finally, applicability and usefulness of the proposed model is explored by fitting it to various datasets used in INGARCH literature for a comparative analysis.

# 3 Statement of Authorship

The workload was divided as follows:

- Alan Huang formulated the project outline, provided the base MATLAB code from past work, guided project direction, assisted with result interpretation and theoretical understanding, and proofread the report.
- Kyle Macaskill developed the model in MATLAB by building on existing MATLAB code, located and studied references, performed various numerical simulations, fitted the model to real-world applications, reported and interpreted results, and wrote the report.

# 4 Semi-Parametric INGARCH Model Formulation

## 4.1 Background

INGARCH models were first introduced by Rydberg and Shephard (2000), Heinen (2003) and Ferland et al. (2006), with an assumed conditional Poisson distribution. Adaptations of these initial Poisson INGARCH models to other conditional response distributions have been explored by other authors, but the structure of the conditional mean has remained the same. See appendix A.1 for a overview of notation choices and terminology.

**Definition 4.1.1** (General INGARCH Model)**.** Suppose $Y_t$ is a count observation at time $t$, conditioned on previous observations be distributed by an underlying conditional distribution $F$.

$$\mu_t = d + \sum_{i=1}^{q} a_i \mu_{t-i} + \sum_{j=1}^{p} b_j Y_{t-i} \qquad t = 2, 3, ..., n \tag{1}$$

Where $d > 0, a_1, \cdots, a_q, b_1, \cdots, b_p \geq 0$. This is known as a INGARCH(p,q) model, which Ferland et al. (2006) showed that if $\sum_{i=1}^{q} a_i + \sum_{j=1}^{p} b_j \leq 1$, it guarantees a strictly stationary solution with finite first and second order moments. The model provides a method of parametrising long term memory as all observations are influenced by past observations in the prior conditional means, with their weight decreasing exponentially as the lag increases.

Semi-parametric time-series model was proposed by Huang and Fung (2015), introducing a flexible and parsimonious approach for modelling based on the exponential tilt representation of the exponential family proposed in Rathouz and Gao (2009) given below.

**Semi-parametric Time Series Model**

Suppose

$$Y_t | F_{t-1} \sim \text{ExpFam}(\mu_t), \qquad t = 1, 2, ... \tag{2}$$

Then, we can re-write this as follows:

$$Y_t | F_{t-1} \sim \exp(b_t + \theta_t y) dF(y), \qquad t = 1, 2, ..., \tag{3}$$

where $F$ is the underlying distribution that generates the exponential family, $\{b_t\}$ are normalisation constants satisfying

$$b_t = -\log \int \exp(\theta_t y) dF(y) \qquad t = 1, 2, ..., \tag{4}$$

and the tilts $\{\theta_t\}$ are given to satisfy

$$0 = \int (y - \mu_t) \exp(\theta_t y) dF(y) \qquad t = 1, 2, ..., \tag{5}$$

We can find thus express the log-likelihood function of the model (3) - (5) as

$$l(\beta, \gamma, F) = \sum_{t=1}^{n} \log (dF(Y_t)) + b_t + \theta_t Y_t \tag{6}$$

### 4.2 Implementation via maximum empirical likelihood

As suggested by Huang and Fung (2015), to create the semi-parametric model, rather than assuming the unknown distribution $F$ prior we treat $F$ as an infinite dimensional parameter within the model (3) - (6). Thus, it is proposed to replace the unknown $dF$ with a probability mass function $\{p_1, ..., p_n\}$ on the observations $Y_1, ..., Y_n$, with $\sum_{j=1}^{n} p_j = 1$.

Extending this framework to handle INGARCH models is possible by changing the definition of the conditional mean to that defined in Definition 4.1.1. The other components, constraints and parameters of the model given can be left unchanged as they are associated with the exponential tilt representation.

The unknown distribution $F$ and model parameters $\beta$ are then jointly estimated. This joint estimation will be performed using a maximum empirical likelihood approach in MATLAB using the fmincon optimisation package, which translates the given problem into the following constrained optimisation problem.

**Constrained Optimisation Problem**

$$\begin{array}{llll}
\text{maximise} & l(\beta, p; b, \theta) = \displaystyle\sum_{t=1}^{n} \{\log (p_t) + b_t + \theta_t Y_t\} & \text{over } \beta, p, b, \theta & \\
\text{subject to} & 0 = \displaystyle\sum_{j=1}^{n} (Y_j - \mu_t) \exp(\theta_t Y_j) p_j & t = 1, ..., n & \text{(mean constraints)} \\
 & b_t = -\log \displaystyle\sum_{j=1}^{n} \exp(\theta_t Y_j) p_j & t = 1, ..., n & \text{(normalisation constraints)} \\
\text{where} & \mu_t = d + \displaystyle\sum_{i=1}^{q} a_i \mu_{t-i} + \sum_{j=1}^{p} b_j Y_{t-i} & t = 2, ..., n & \text{(INGARCH constraints)}
\end{array}$$

Note that Definition 4.1.1 defines the recursive relationship for the conditional mean except for the initial means, eg. $\mu_1$ in an INGARCH(1,1) model. The value of the $\mu_1$ can be treated in different ways, which have an impact on the model's parameter estimates as explored by Weiß (2018). A common choice is to set this to be the sample mean $\bar{Y}$, but instead we follow the suggestion of Ferland et al. (2006) and treat all initial means as a further parameter within our model and estimate it.

Regarding the consistency of the estimators, as stated by Huang and Fung (2015), if the process is stationary and uniformly ergodic then the semi-parametric model is consistent whenever a correctly specified parametric model is. The Semi-Parametric INGARCH process is strictly stationary, but ergodicity is difficult to prove for a general exponential family and is usually proven on a per distribution basis. For example, Fokianos, Rahbek, et al. (2009) utilises a perturbed model to prove ergodicity for the proposed Poisson Autoregression Model which is a Poisson INGARCH(1,1) model. Thus, we can consider the semi-parametric model to be consistent for parametric models with proven ergodicity, but in other cases without this proof, we generally find the semi-parametric model to converge towards the true parameters as $n \to \infty$ as will be explored in Section 5.

### 4.3 Inference Methodology

Parametric INGARCH models provide inference of model parameters $\beta$ directly by either inverting the information matrix, or utilising a sandwich operator involving the conditional information matrix. For the semi-parametric log-likelihood, this is difficult to find analytically and potentially computationally expensive to compute. Instead, we suggest utilising the inference method proposed by Huang and Fung (2015) which uses the likelihood ratio test (LRT).

Supposing that we are interesting in considering testing $H_0 : \beta_k = \beta^*$ where $\beta^*$ is a chosen null value for a component of $\beta$. We have that the LRT is given as follows.

$$LRT = 2\{l(\hat{\beta}, \hat{p}) - sup_{\beta_k=\beta^*} l(\beta, p)\} \tag{7}$$

Thus, we can find the equivalent standard error $se_{eq}$ using the following formula.

$$LRT = \left( \frac{\hat{\beta}_k - \beta^*}{se_{eq}} \right)^2 \tag{8}$$

$$se_{eq} = \sqrt{\left( \frac{(\hat{\beta}_k - \beta^*)^2}{LRT} \right)} \tag{9}$$

Here it is important to note that this equivalent standard error results in the corresponding z-test to have the same significance level compared to a LRT against a $\chi_1$ distribution. In the general case, it is suggested to use a null value of 0 which we will adopt for the mean lag parameters $a_i$ as this null model still has a meaningful interpretation. However, an issue arises if we set the observation lag parameters $b_j = 0$ as in a INGARCH(1,1) model, it becomes deterministic and not dependent on the response variable $Y$. Through simulations, we find that if the null value of $b$ is established far away from the true parameter, the estimated standard errors are not accurate. Thus, we suggest setting the null value $b = 0.5$ as it lies in the middle of the parameter space for $b \in (0, 1]$.

When considering inference, we can also consider the wald-type 95% confidence intervals as these are just as meaningful as the standard error. We can construct these intervals by using the LRT, grid searching various null values around the parameter estimate in either direction until the we reject the null hypothesis against a $\chi_1$ test. For a 95% confidence interval, the upper and lower bounds surrounding the estimated parameter $\hat{\beta}_k$ are the null values $\beta^*$ where $LRT > \chi^2_{1,0.05}$ ($\chi^2_{1,0.05} = 3.841$).

This methods can creates asymmetric confidence intervals and does not require calculating the standard errors, but is computationally expensive for each parameter. In situations where the simulated parameters estimates are normally distributed and symmetric, we can constructed a good approximation using the usual $\hat{\beta}_k \pm z_{\frac{\alpha}{2}} se_{eq}$. As we will provide evidence for in the following section, we recommend using the normal approximation 95% confidence interval for all parameters except the intercept $d$, where instead we recommend constructing the 95% likelihood ratio confidence interval, without finding the standard error. We will explore why by first using the proposed LRT inference method for equivalent standard errors with a null value $d = 0$ in simulations.

# 5 Simulation Study

For the purpose of model testing, we will perform numerical simulations of a semi-parametric INGARCH(1,1) model defined below, as these are most common in literature.

$$\mu_t = d + a\mu_{t-1} + bY_{t-1} \qquad t = 2, 3, ..., n \tag{10}$$

Furthermore, a comparison with a Poisson and Negative Binomial model fitted using the tscount (Liboschik et al. 2017) package in R will be made to consider performance compared to correctly specified parametric models, and misspecified parametric models. The body of the report will consider two numerical simulations but for a broader range of simulations from various common distributions, see appendix A.2-A.5. The terms we are measuring are as follows: Bias = difference between true parameter and mean maximum likelihood estimate, SD = simulation standard deviation, SE = equivalent standard error, Coverage = coverage (%) of the true parameter of symmetric 95% confidence intervals for 500 simulations.

We performed 500 simulations for each sample size as this was found to be large enough to achieve asymptotic behaviour. Note that the analysis of both given simulations will be given afterwards in section 5.3

## 5.1 Negative Binomial Distribution: Overdispersed Case

There exists varying parametisations for an Negative Binomial INGARCH(p,q), but we will simulate from the one proposed by Weiß (2018) which extends on the model by Zhu (2011), to align it with Definition 4.1.1.

$$r\frac{1-p_t}{p_t} = \mu_t = d + \sum_{i=1}^{q} a_i\mu_{t-i} + \sum_{j=1}^{p} b_j Y_{t-i} \tag{11}$$

$$Y_t|F_{t-1} \sim NB\left(r, \frac{r}{r+\mu_t}\right) \tag{12}$$

| Model | n | Estimates | Bias | SD | SE | Coverage | LRT |
|---|---|---|---|---|---|---|---|
| Semi-Parametric | 150 | d | 0.3979 | 0.8556 | 0.5331 | 81.0% | 92.6% |
| | | a | -0.0073 | 0.1050 | 0.1032 | 93.8% | - |
| | | b | -0.0306 | 0.0892 | 0.0852 | 90.8% | - |
| | | $\mu_1$ | 0.0074 | 2.0130 | - | - | - |
| | 500 | d | 0.1502 | 0.4377 | 0.3125 | 84.8% | 93.2% |
| | | a | -0.0034 | 0.0533 | 0.0531 | 95.8% | - |
| | | b | -0.0105 | 0.0468 | 0.0468 | 95.0% | - |
| | | $\mu_1$ | 0.0509 | 1.1526 | - | - | - |
| Poisson | 150 | d | 0.4870 | 1.0647 | 0.4333 | 61.0% | - |
| | | a | -0.0381 | 0.1046 | 0.0463 | 60.0% | - |
| | | b | -0.0205 | 0.0900 | 0.0360 | 58.2% | - |
| | 500 | d | 0.1975 | 0.5398 | 0.2120 | 54.4% | - |
| | | a | -0.0073 | 0.0607 | 0.0227 | 56.6% | - |
| | | b | -0.0151 | 0.0526 | 0.0190 | 51.8% | - |
| Negative Binomial | 150 | d | 0.4870 | 1.0647 | 1.0276 | 95.4% | - |
| | | a | -0.0381 | 0.1046 | 0.1200 | 95.0% | - |
| | | b | -0.0205 | 0.0900 | 0.1051 | 95.4% | - |
| | 500 | d | 0.1975 | 0.5398 | 0.5463 | 95.6% | - |
| | | a | -0.0073 | 0.0607 | 0.0651 | 95.4% | - |
| | | b | -0.0151 | 0.0526 | 0.0606 | 95.0% | - |

Table 1: True model parameter $\beta = (d, a, b, \mu_1, r)^T = (2, 0.3, 0.6, 1, 4)^T$

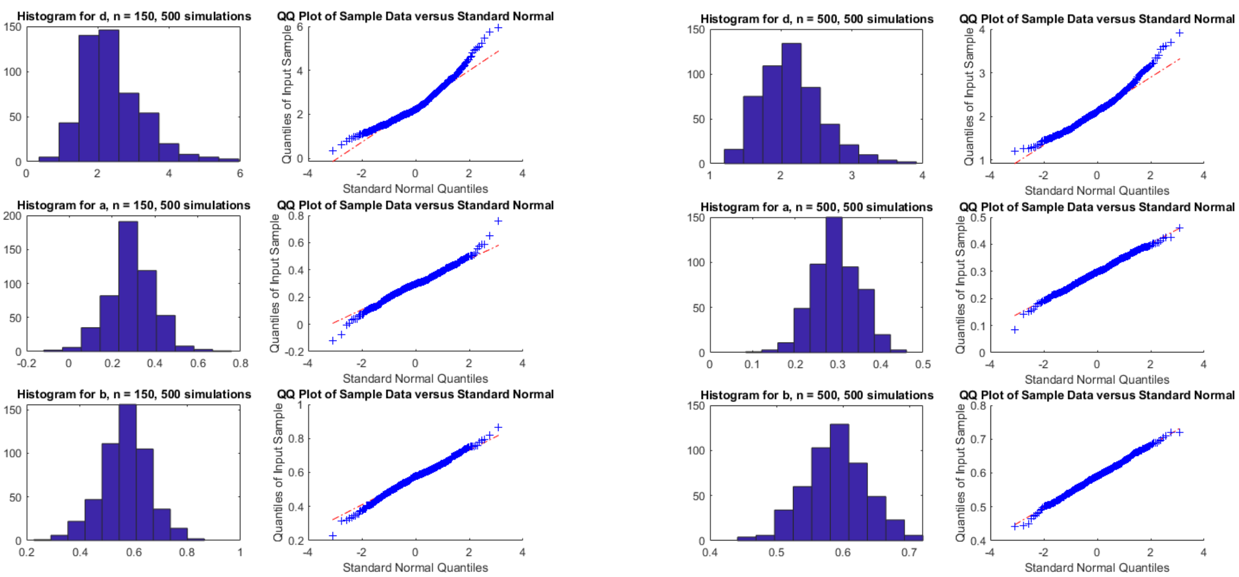



Figure 1: Histogram and QQ-plot of Semi-Parametric Model on NB data. (Left) n = 150, (Right) n = 500

## 5.2 Conway-Maxwell-Poisson Distribution: Underdispsered Case

Underdispersion is also common occurrence in time-series data, which cannot be modelled using using Poisson or Negative Binomial INGARCH models. We will simulate underdispersed data using a mean-parametrised

Conway-Maxwell-Poisson (CMP) distribution (Huang 2017) from the R package mpcmp (Fung et al. 2020).

$$Y_t|F_{t-1} \sim CMP_{\mu}\left(\mu_t, \nu\right) \tag{13}$$

Here, $\nu$ is a diserpsion parameter where $\nu < 1$ implies overdispersion and $\nu > 1$ implies underdispersion.

| Model | n | Estimates | Bias | SD | SE | Coverage | LRT |
|---|---|---|---|---|---|---|---|
| Semi-parametric | 150 | d | 0.1800 | 0.6656 | 0.4389 | 84.8% | 93.6% |
| | | a | -0.0104 | 0.1368 | 0.1389 | 96.4% | - |
| | | b | -0.0174 | 0.0877 | 0.0832 | 92.8% | - |
| | | $\mu_1$ | 0.0031 | 0.8492 | - | - | - |
| | 500 | d | 0.0570 | 0.3642 | 0.2448 | 86.8% | 94.6% |
| | | a | -0.0044 | 0.0752 | 0.0764 | 96.4% | - |
| | | b | -0.0041 | 0.0434 | 0.0444 | 96.0% | - |
| | | $\mu_1$ | -0.0142 | 0.7750 | - | - | - |
| Poisson | 150 | d | 0.4507 | 0.6639 | 0.9945 | 99.2% | - |
| | | a | -0.1015 | 0.1283 | 0.1915 | 99.6% | - |
| | | b | 0.0255 | 0.0880 | 0.1061 | 99.2% | - |
| | 500 | d | 0.0346 | 0.3694 | 0.4980 | 99.4% | - |
| | | a | -0.0070 | 0.0760 | 0.1021 | 99.6% | - |
| | | b | -0.0007 | 0.0435 | 0.0596 | 99.4% | - |
| Negative Binomial | 150 | d | 0.4507 | 0.6639 | 0.9945 | 99.2% | - |
| | | a | -0.1015 | 0.1283 | 0.1915 | 99.6% | - |
| | | b | 0.0255 | 0.0880 | 0.1061 | 99.2% | - |
| | 500 | d | 0.0346 | 0.3694 | 0.4980 | 99.4% | - |
| | | a | -0.0070 | 0.0760 | 0.1021 | 99.6% | - |
| | | b | -0.0007 | 0.0435 | 0.0596 | 99.4% | - |

Table 2: True model parameter $\beta = (d, a, b, \mu_1, \nu)^T = (2, 0.2, 0.5, 1, 1.9)^T$

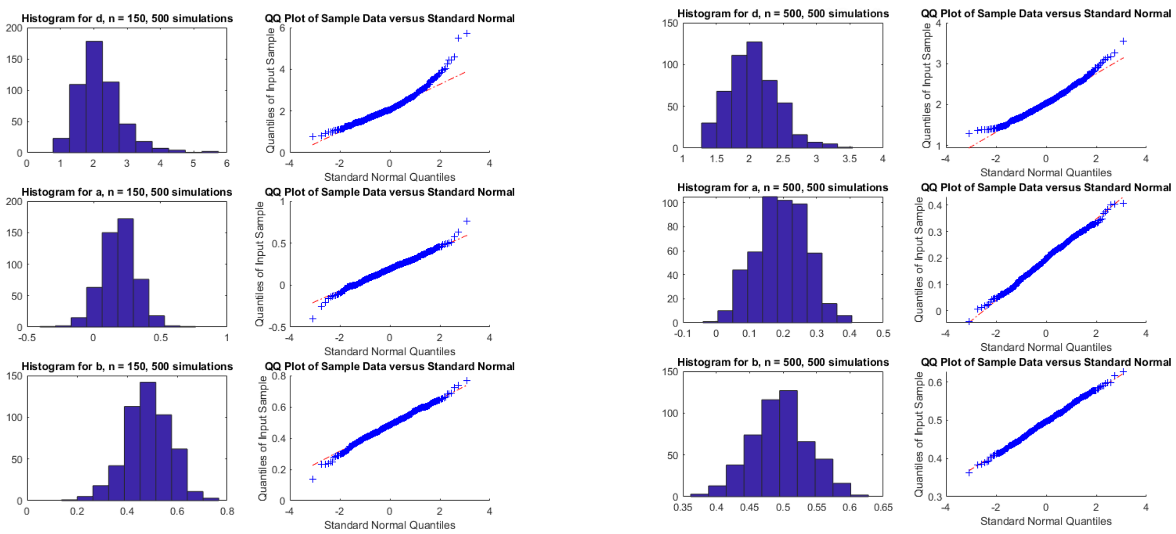



Figure 2: Histogram and QQ-plot of Semi-Parametric Model on CMP data. (Left) n = 150, (Right) n = 500

### 5.3 Summary of simulation results

For the semi-parametric model, we find small biases for the parameters $a$ and $b$ across all sample sizes, similar to the consistent parametric models. Note that the semi-parametric model bias for the parameter 'd' is larger, with more variation in the estimates, but similar biases are also found in the parametric models. As the sample size increases, the semi-parametric model's biases decreases as expected. However, while the semi-parametric model maintains small biases for smaller sample sizes, misspecified models such as in the COM-Poisson case have larger biases for $n = 150$, but similar estimates for $n = 500$ due to model consistency. This suggests that the semi-parametric model estimates adapts well even for smaller sample sizes, resulting in potentially faster convergence to the true parameters compared to misspecified models.

The simulation standard deviation when the number of simulation is large provides a good approximation for the estimates true standard error. We find that for parameters $a$ and $b$ in both examples, the semi-parametric model's standard errors ($se_{eq}$) are close to the standard deviation. Therefore, as the histogram and qq-plots suggest that these parameters estimates are distributed approximately by a normal distribution, the true parameters exhibit close to 95% coverage as $n$ increases for both examples. This performance is similar to the correctly specified Negative Binomial model in 5.1, indicating that the suggested inference method can be used similar to inferences in parametric models.

Contrasting this, misspecified models such as the Poisson model in 5.1 and both parametric models in 5.2 have biased inferences and non-optimal coverage, reinforcing the benefit of the semi-parametric model. The Poisson Model in 5.1 gives low coverage and standard errors indicating that the model is underdispersed relative to the true model. Both parametric models in the underdispersed COM-Poisson example give very high coverage, indicating that they are overdispersed relative to the true model.

Finally, note that the parameter $d$ in the semi-parametric model using the null value $d = 0$ finds non-expected standard errors and coverage under 95%. As the intercept must be positive, using a null value of 0 lies on the boundary of the parameter space which may have implications on the null distribution. Proposing other general null values are difficult is $d$ is unbounded, and using any proposed null values to construct symmetric 95% confidence intervals will not improve coverage as the estimates for $d$ are right-skewed, deviating from normality. Therefore, we can correct this by instead constructing the likelihood ratio confidence intervals as proposed in section 4. Doing so increases the coverage close to roughly 92-94%, indicating much better performance across various distributions, but at the cost of not being able to give an estimated standard error. However, we argue that providing the 95% confidence interval is just as meaningful as providing correct standard errors.

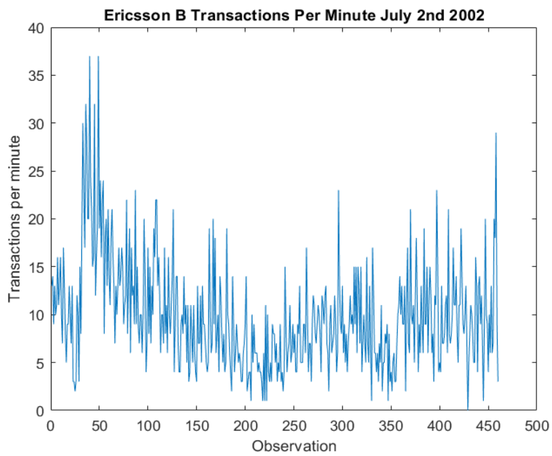
## 6 Real-world Applications

We will now apply the semi-parametric INGARCH model to two commonly analysed datasets in INGARCH literature. Note that a more application studies can be found in the appendix A.6 - A.7 with similar analysis.

To assess model adequacy, we will use a Probability Inverse Transform (PIT) which considers the conditional moments and accesses distribution assumptions of the model. We will consider both a randomised and non-randomised (mean) PIT considered by Cazdo et al. (2009), who suggested these adjustments to handle discrete distributions. The definitions can be found in appendix A.8, with both transforming the fitted model to a uniform distribution if the fitted model is the true model of the data.

## 6.1 Number of Transactions of Ericsson B Stock

Here, we will apply the proposed semiparametric INGARCH model to the number of transactions per minute of the stock Ericsson B on July 2nd 2002 which was initially published by Brännäs and Quoreshi (2010), containing the number of transactions per minute from July 2-22 2002. The dataset is commonly used across time series literature, including Fokianos, Rahbek, et al. (2009), Christou and Fokianos (2014), Davis and Liu (2016), Zhang (2017) among others. The dataset contains 460 observations between 09:35 - 17:14 evident in the figure below, with a sample mean 9.9087, and sample variance 32.8370, indicating that the data is conditionally overdispersed. Below we will fit various INGARCH(1,1) models, noting that the model proposed by Zhu (2011) does not express the conditional mean in the form given in 4.1.1. Thus, we will fit the adjusted model proposed by Weiß (2018) for the comparison below.



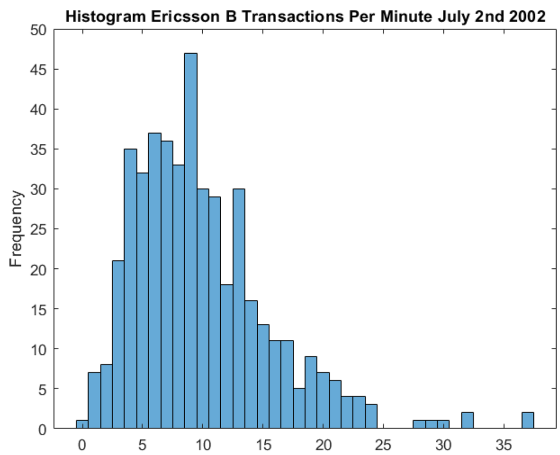




Figure 3: (Left) Data plot, (Right) Histogram

**Model Estimates**

| **Model** | $\hat{d}$ | $\hat{a}$ | $\hat{b}$ | $\hat{\mu}_1$ | $\hat{r}$ |
|---|---|---|---|---|---|
| Our Model SP | 0.273 ( ) | 0.844 (0.105) | 0.129 (0.047) | 12.311 | - |
| Zhang (2017) SP | 0.278 ( ) | 0.839 ( ) | 0.133 ( ) | - | - |
| David and Liu (2016) Pois | 0.291 (0.100) | 0.831 (0.024) | 0.140 (0.019) | - | - |
| David and Liu (2016) NB | 0.268 (0.141) | 0.848 (0.035) | 0.128 (0.027) | - | 8.000 |
| Zhu / Weiß (2018) NB | 0.270 (0.142) | 0.845 (0.034) | 0.127 (0.026) | 12.038 | 7.861 |
| Pois INGARCH(1,1) | 0.292 (0.100) | 0.832 (0.023) | 0.139 (0.018) | 12.058 | - |
| tscount NB (2020) | 0.286 (0.157) | 0.832 (0.039) | 0.140 (0.031) | - | 7.317 |

Table 3: INGARCH(1,1) Model Estimates, Standard Errors in Parenthesis

| Model | $\hat{d}$ | $\hat{a}$ | $\hat{b}$ |
|---|---|---|---|
| Our SP Model | [ 0.0450, 0.6405] | [0.6382, 1.0498] | [0.0369, 0.2211] |
| David and Liu (2016) Pois | [ 0.0950, 0.4870] | [0.7839, 0.8780] | [0.1028, 0.1772] |
| David and Liu (2016) NB | [-0.0084, 0.5444] | [0.7794, 0.9166] | [0.1028, 0.1772] |
| Zhu / Weiß (2018) NB | [-0.0083, 0.5483] | [0.7784, 0.9116] | [0.0761, 0.1780] |
| Pois INGARCH(1,1) | [ 0.0960, 0.4880] | [0.7869, 0.8771] | [0.1037, 0.1743] |
| tscount NB | [-0.0217, 0.5937] | [0.7556, 0.9084] | [0.0792, 0.2008] |

Table 4: INGARCH(1,1) Model's 95% Confidence Interval

**Discussion**

We find that the semi-parametric model gives parameter estimates which are similar to those proposed by the parametric negative binomial models. When comparing inferences, we find that the semi-parametric model has wider confidence intervals and larger standard errors. Therefore, this may indicate that the true underlying distribution is some deviation from the negative binomial distribution, which the semi-parametric model is adapting to. Note that Zhang does propose a semi-parametric INGARCH(1,1) model assuming the unknown distribution log-concave, but no inference methods are given.

**Model Adequacy**

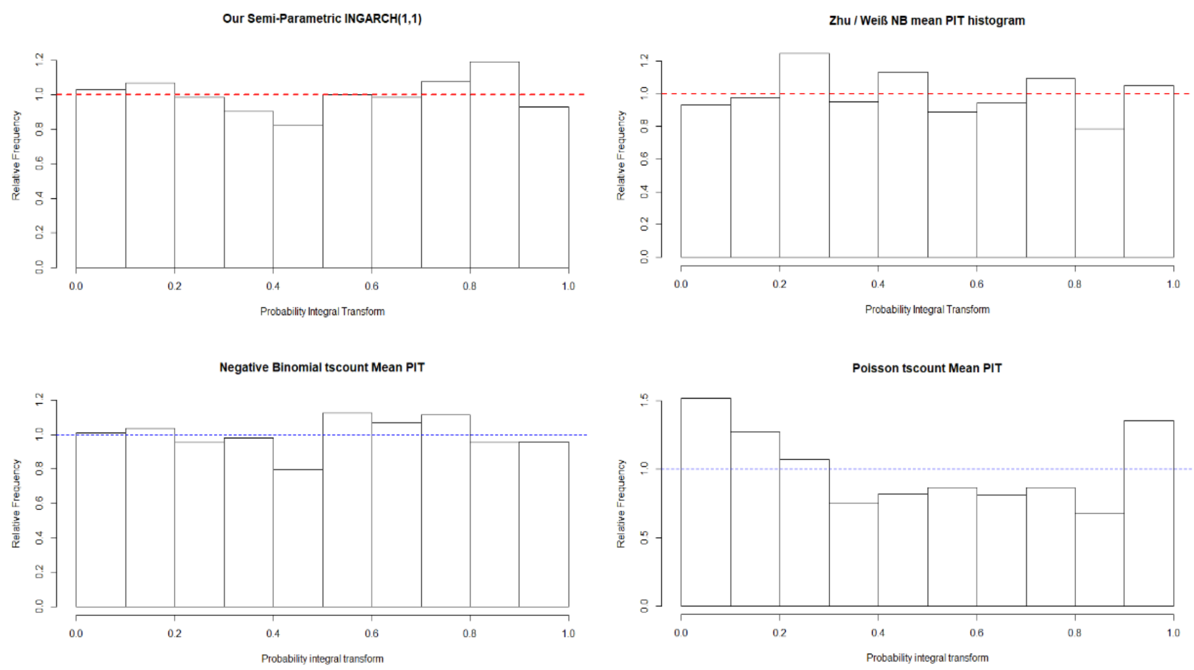



Figure 4: Mean PIT's. (Top Left) Semi Parametric Model, (Top Right) Zhu / Weiß NB Model, (Bottom Left) tscount NB Model, (Bottom Right) tscount Poisson Model

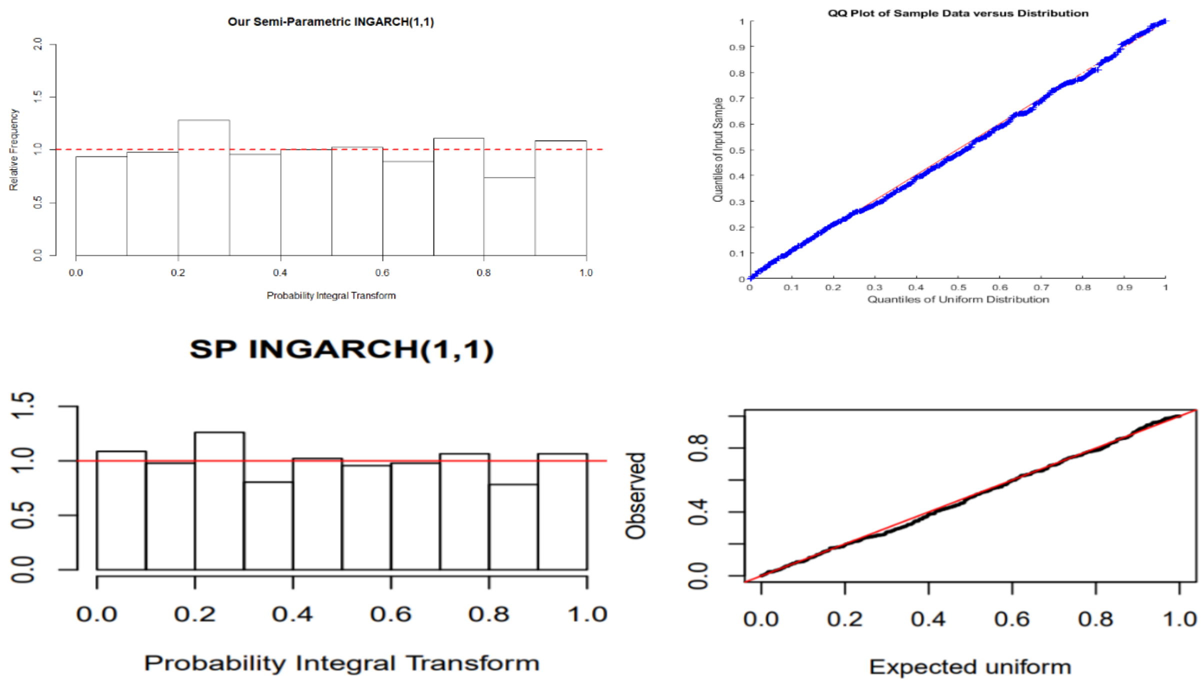



Figure 5: Randomised PIT and Uniform QQ-plots. (Top) Semi Parametric Model, (Bottom) Zhang (2017) Semi-Parametric Model

As evident in Figure 4, we find that the Semi-Parametric Model gives a mean PIT which is similar to the tscount negative binomial model, being close to uniform. The Semi-Parametric model does have slightly smaller deviations, but we can consider it possesses on par model adequacy compared to the parametric model. The Poisson model's U shape indicates underdispersion relative to the true model. When we consider Figure 5, we see that using a randomised PIT we obtain similar model adequacy to the semi-parametric model proposed by Zhang (2017). However, Zhang doesn't consider a mean PIT so a single randomised PIT may not capture it's overall performance like a mean PIT does.

**Final Proposed Model**

$$\begin{aligned} \hat{\mu}_t &= 0.273 + 0.844\hat{\mu}_{t-1} + 0.129Y_{t-1} \quad ,t > 2 \\ \hat{\mu}_1 &= 12.311 \end{aligned} \tag{14}$$

## 6.2 Number of cases of Campylobacterosis Infections

Campylobacterosis a gastrointestinal bacterial infection caused by eating contaminated food, causing cramping, fever and abdominal pain. Ferland et al. (2006) modelled the number of Campylobacterosis infections in the North of Québec in Canada from January 1990 to October 2000 using two different Poisson INGARCH models. The first model had considered a past mean lag of 7 & 13, where the second only considers a past mean lag of 13. Below we will fit the semi-parametric model to the first proposed model alongside the parametric models from tscount, as the analysis of model adequacy is found to be the same for both models. The dataset is available in the tscount R package (Liboschik et al. 2017) , containing 140 observations with one observation every 28 days.

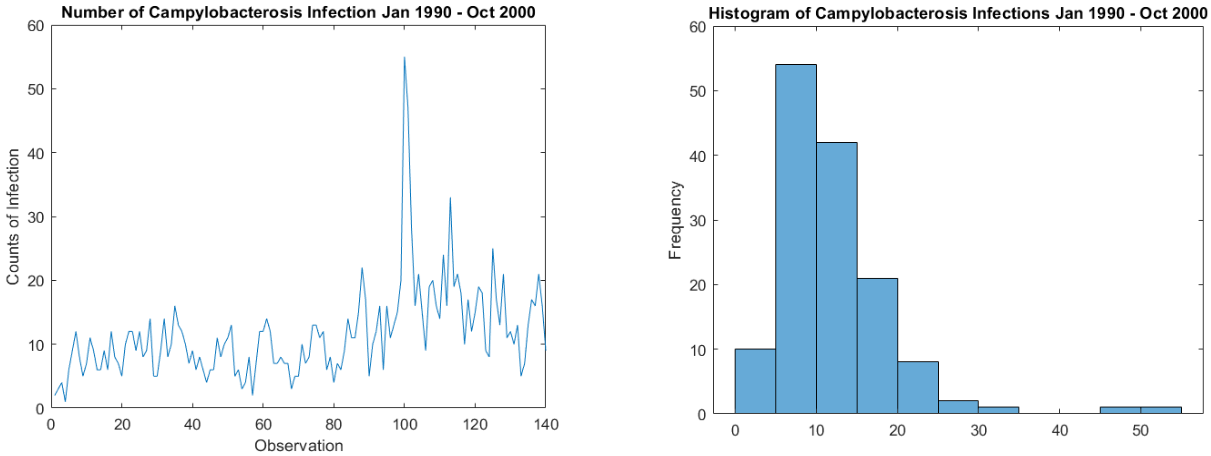



Figure 6: (Left) Data plot, (Right) Histogram

### Proposed INGARCH Model

$$\mu_t = d + a_1\mu_{t-7} + a_2\mu_{t-13} + b_1 Y_{t-1}, \qquad t > 13 \tag{15}$$

### Model Estimates

| Model | $\hat{d}$ | $\hat{a}_1$ | $\hat{a}_2$ | $\hat{b}$ | r |
|---|---|---|---|---|---|
| Our SP Model | 1.8280 ( ) | 0.0880 (0.1065) | 0.2127 (0.0662) | 0.5748 (0.0823) | - |
| Ferland et.al Pois | 1.9280 (0.6764) | 0.0543 (0.0633) | 0.2593 (0.1158) | 0.5437 (0.0828) | - |
| tscount Pois | 1.4965 (0.7100) | 0.0873 (0.0640) | 0.1838 (0.0750) | 0.5856 (0.0529) | - |
| tscount NB | 1.4965 (1.1337) | 0.0873 (0.1042) | 0.1838 (0.1228) | 0.5856 (0.0926) | 8.710 |

Table 5: INGARCH(1,1) Model Estimates, Standard Errors in Parenthesis

| Model | $\hat{d}$ | $\hat{a}_1$ | $\hat{a}_2$ | $\hat{b}_1$ |
|---|---|---|---|---|
| Our SP Model | [-0.4895, 4.2050] | [-0.1207, 0.2967] | [ 0.0829, 0.3425] | [0.4135, 0.7361] |
| Ferland et.al Pois | [ 0.6023, 3.2537] | [-0.0698, 0.1783] | [ 0.0325, 0.4864] | [0.3814, 0.7060] |
| tscount Pois | [ 0.1049, 2.8881] | [-0.0381, 0.2127] | [ 0.0368, 0.3308] | [0.4819, 0.6893] |
| tscount NB | [-0.7256, 3.7185] | [-0.1169, 0.2915] | [-0.0569, 0.4245] | [0.4041, 0.7671] |

Table 6: INGARCH(1,1) Model's 95% Confidence Interval

### Discussion

Above we find that the estimates of the semi-parametric model do show deviations from all other models, with estimates for each parameter lying between the model by Ferland et al. (2006), and the tscount model where estimation is done using the conditional poisson likelihood. The confidence intervals of the semi-parametric model are similar to the Negative Binomial model for parameters $a_1, b_1$, but the semi-parametric model has tighter intervals for the other parameters. Thus, it indicates that the semi-parametric model is adapting to a distribution which differs from the Poisson and Negative Binomial distribution. Note for the semi-parametric model, the initial means are approximately the response at the initial times.

**Model Adequacy**

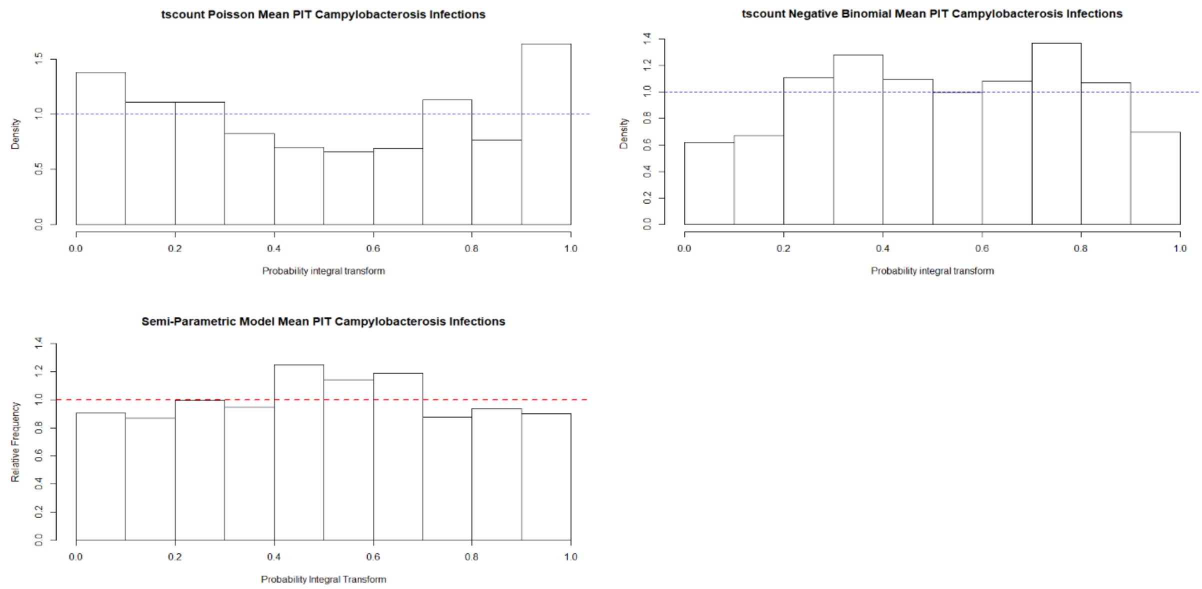



Figure 7: Mean PIT's. (Top Left) tscount Poisson Model, (Top Right) tscount Negative Binomial Model, (Bottom) Semi-Parametric Model

In Figure 7 we find both parametric models are not adequate, with the Poisson model shows signs of relative underdispersion with a U shape, whereas the negative binomial model shows signs of relative overdispersion with an inverse U shape. The Semi-Parametric model's mean PIT possess noticeably less deviations from uniformity, indicating that the model possess good fit to the data compared to the proposed parametric models.

**Final Proposed Model**

$$\begin{aligned} &\hat{\mu}_t = 1.8280 + 0.0880\hat{\mu}_{t-7} + 0.2127\hat{\mu}_{t-13} + 0.5748Y_{t-1} \quad , t > 13 \\ &\hat{\mu}_t \approx Y_t \quad t \leq 13 \end{aligned} \tag{16}$$

## 7 Conclusion

The misspecification of the assumed probability distribution in parametric time series models leads to biased inferences. Thus, we have proposed an extension on the semi-parametric framework proposed by Huang and Fung (2015) to handle INGARCH models, which can correctly adapt to a wide range of distributions with good approximations in estimation and inference. Through a numerical simulation study, we find that the semi-parametric model seem to be adapting well to a wide range of distributions, providing on-par performance with correctly specified parametric models. Furthermore, the semi-parametric model exhibits better performance with unbiased inferences compared to misspecified parametric models. As a result, common applications of

INGARCH models have been analysed where model adequacy of the semi-parametric model was found to be similar or better than other proposed models when considering a Probability Inverse Transform.

Future efforts should be focused on speeding up the semi-parametric model as it's current implementation in MATLAB using fmincon is relatively slow. This can be done by implementing it in R using similar methods to those implemented in the R package gldrm (Wurm and Rathouz 2018) which implements a semi-parametric generalised linear model. It would also be interesting to explore transformations of the INGARCH model such as the log-linear Poisson autoregression model by Fokianos and Tjøstheim (2011) to see how the semi-parametric framework adapts. Finally, developing and implementing the semi-parametric framework to handle multi-variate INGARCH models would be interesting to further generalise the modelling approach.

## 8 Acknowledgements

I would like to give my special thanks to Alan Huang for supervising and guiding me through the research project, for always being available for regular consultations and making it an overall enjoyable experience. I would also like to extend my special thanks to AMSI for the amazing opportunity to be involved in a Vacation Research Scholarship (VRS), and to the University of Queensland for facilitating the completion of the project.

# A Appendix

## A.1 INGARCH Terminology

Note that INGARCH models which provide restrictions on the conditional mean, are an adaptation of GARCH models which instead have restrictions on the conditional variance. The extension from GARCH to INGARCH model seems appropriate for a Poisson INGARCH model, but once other conditional response distributions were considered, literature has split on the terminology. For example, Fokianos, Rahbek, et al. (2009) uses the terminology Poisson autoregresisve models, but we take the suggestion of Weiß (2018) and utilise the INGARCH terminology defined in Ferland et al. (2006).

## A.2 Simulation results for a Poisson Distribution

Here we will consider simulations from an Poisson INGARCH(1,1) model, with the same parameters as those considered by Fokianos, Rahbek, et al. (2009). This is a simple case where we expect good performance from all models.

| Model | n | Estimates | Bias | SD | SE | Coverage | LRT |
|---|---|---|---|---|---|---|---|
| Semi-Parametric Model | 150 | d | 0.0829 | 0.1680 | 0.0895 | 76.8% | 92.6% |
| | | a | -0.0208 | 0.1066 | 0.1197 | 96.4% | - |
| | | b | -0.0154 | 0.0794 | 0.0814 | 94.2% | - |
| | | $\mu_1$ | 0.0168 | 0.5430 | - | - | - |
| | 500 | d | 0.0247 | 0.0743 | 0.0458 | 80.0% | 94.6% |
| | | a | -0.0046 | 0.0561 | 0.0617 | 96.6% | - |
| | | b | -0.0061 | 0.0455 | 0.0440 | 94.2% | - |
| | | $\mu_1$ | 0.0228 | 0.5405 | - | - | - |
| Poisson | 150 | d | 0.0799 | 0.1813 | 0.1526 | 95.0% | - |
| | | a | -0.0479 | 0.1041 | 0.1092 | 93.8% | - |
| | | b | 0.0001 | 0.0800 | 0.0810 | 95.0% | - |
| | 500 | d | 0.0235 | 0.0751 | 0.0737 | 95.4% | - |
| | | a | -0.0119 | 0.0572 | 0.0556 | 95.6% | - |
| | | b | -0.0036 | 0.0460 | 0.0438 | 94.6% | - |
| Negative Binomial | 150 | d | 0.0799 | 0.1813 | 0.1563 | 94.4% | - |
| | | a | -0.0479 | 0.1041 | 0.1128 | 94.4% | - |
| | | b | 0.0001 | 0.0800 | 0.0810 | 95.8% | - |
| | 500 | d | 0.0235 | 0.0751 | 0.0764 | 95.4% | - |
| | | a | -0.0119 | 0.0572 | 0.0566 | 95.8% | - |
| | | b | -0.0036 | 0.0460 | 0.0447 | 94.8% | - |

Table 7: True model parameter $\beta = (d, a, b, \mu_1)^T = (0.3, 0.4, 0.5, 0.3)^T$

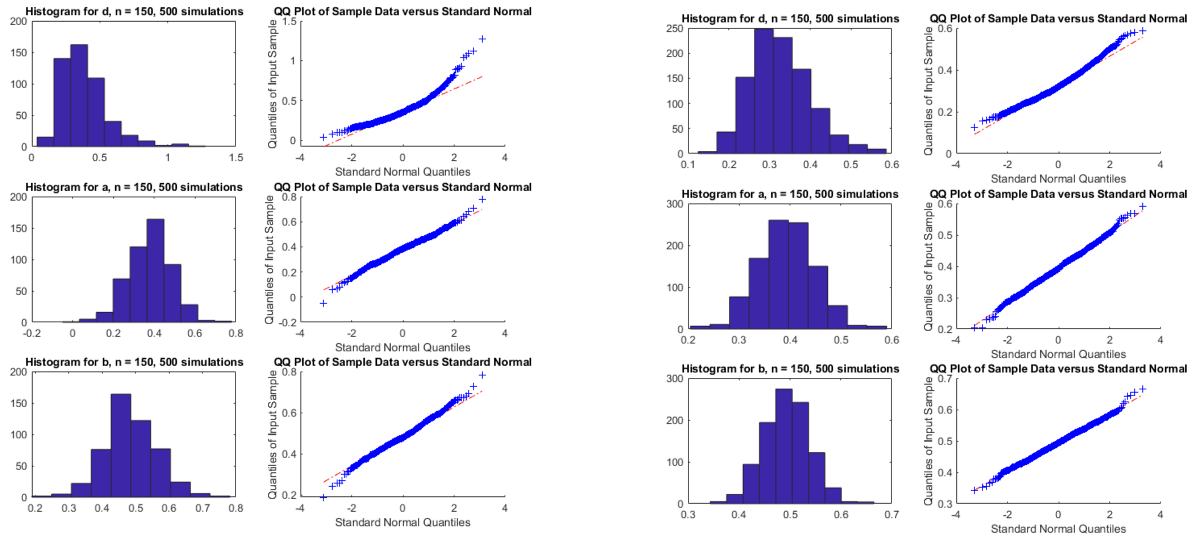



Figure 8: Histogram and QQ-plot of Semi-Parametric Model on Poisson data. (Left) n = 150, (Right) n = 500

### A.3 Simulation results for a Negative Binomial Distribution: Geometric Case

Here we consider the negative binomial distribution with the dispersion parameter $r = 1$, which is a Geometric INGARCH Model. Note that again we use the parametrisation given by Weiß (2018).

| Model | n | Estimates | Bias | SD | SE | Coverage | LRT |
|---|---|---|---|---|---|---|---|
| Semi-Parametric | 150 | d | 0.2530 | 0.8579 | 0.4927 | 76.2% | 91.6% |
| | | a | -0.0181 | 0.1559 | 0.1416 | 91.4% | - |
| | | b | -0.0363 | 0.1158 | 0.1100 | 92.8% | - |
| | | $\mu_1$ | 0.2079 | 2.3405 | - | - | - |
| | 500 | d | 0.1049 | 0.4410 | 0.2677 | 79.4% | 93.8% |
| | | a | -0.0036 | 0.0745 | 0.0704 | 94.2% | - |
| | | b | -0.0193 | 0.0674 | 0.0644 | 92.0% | - |
| | | $\mu_1$ | 0.3729 | 3.0830 | - | - | - |
| Poisson | 150 | d | 0.3484 | 0.9208 | 0.3075 | 50.2% | - |
| | | a | -0.0300 | 0.1589 | 0.0477 | 40.0% | - |
| | | b | -0.0414 | 0.1190 | 0.0311 | 38.0% | - |
| | 500 | d | 0.1931 | 0.5433 | 0.1532 | 38.0% | - |
| | | a | -0.0179 | 0.0933 | 0.0166 | 37.6% | - |
| | | b | -0.0251 | 0.0792 | 0.0232 | 28.8% | - |
| Negative Binomial | 150 | d | 0.3484 | 0.9208 | 1.0188 | 96.6% | - |
| | | a | -0.0300 | 0.1589 | 0.1735 | 93.2% | - |
| | | b | -0.0414 | 0.1190 | 0.1494 | 92.6% | - |
| | 500 | d | 0.1931 | 0.5433 | 0.5689 | 95.8% | - |
| | | a | -0.0179 | 0.0933 | 0.0975 | 93.8% | - |
| | | b | -0.0251 | 0.0792 | 0.0908 | 94.2% | - |

Table 8: True model parameter $\beta = (d, a, b, \mu_1, r)^T = (2, 0.3, 0.5, 1, 1)^T$

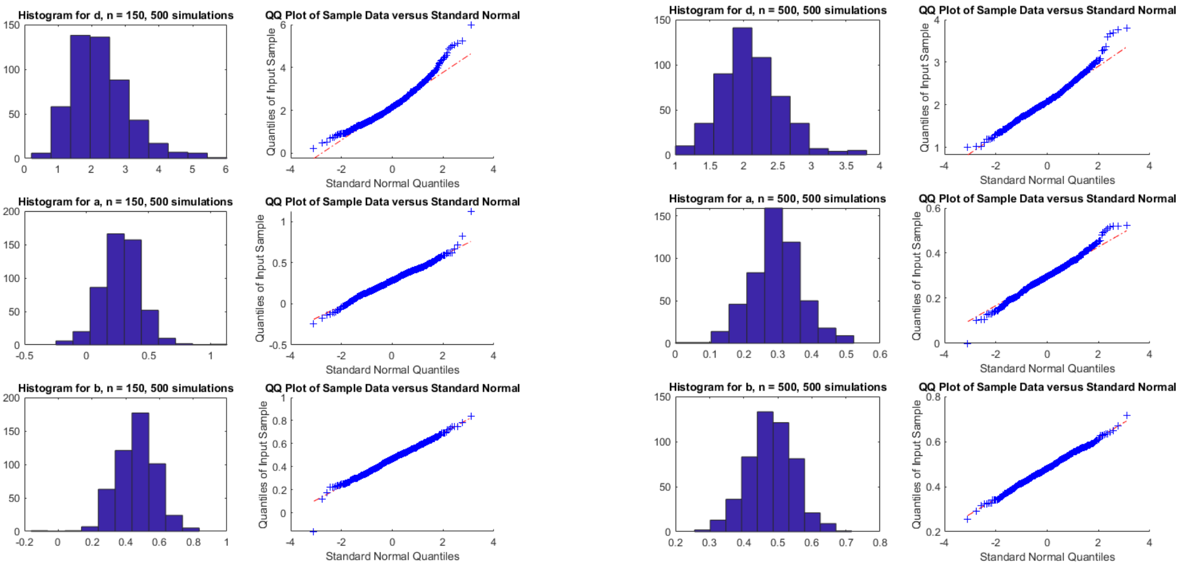



Figure 9: Histogram and QQ-plot of Semi-Parametric Model on Geometric data. (Left) n = 150, (Right) n = 500

## A.4 Simulation results for a Zero-Inflated Poisson (ZIP) Distribution

Finally, we will explore the robustness of the semi-parametric model by considering modelling from a Zero-Inflated Poisson Distribution. A ZIP($\lambda$, $\omega$) has a probability mass function of the form

$$P(X = k) = \omega\delta_{k,0} + (1-\omega)\frac{\lambda^k \exp -\lambda}{k!}$$

Where $0 < \omega < 1$, $\delta_{k,0}$ is the Kronecker delta.

| Model | n | Estimates | Bias | SD | SE | Coverage | LRT |
|---|---|---|---|---|---|---|---|
| Semi-Parametric | 150 | d | 0.0325 | 0.4528 | 0.1927 | 65.6% | 94.0% |
| | | a | -0.0446 | 0.1998 | 0.1996 | 95.0% | - |
| | | b | -0.0483 | 0.0877 | 0.0835 | 91.6% | - |
| | | $\mu_1$ | -0.0241 | 1.0520 | - | - | - |
| | 500 | d | -0.0596 | 0.1996 | 0.1210 | 71.2% | 94.4% |
| | | a | -0.0135 | 0.0938 | 0.1045 | 96.4% | - |
| | | b | -0.0424 | 0.0451 | 0.0458 | 84.6% | - |
| | | $\mu_1$ | -0.1080 | 1.0076 | - | - | - |
| Poisson | 150 | d | 0.0014 | 0.3600 | 0.3595 | 91.8% | - |
| | | a | -0.0404 | 0.1593 | 0.1657 | 93.4% | - |
| | | b | -0.0457 | 0.0836 | 0.0722 | 84.4% | - |
| | 500 | d | -0.0641 | 0.1972 | 0.1807 | 88.4% | - |
| | | a | -0.0143 | 0.0923 | 0.0847 | 91.6% | - |
| | | b | -0.0415 | 0.0446 | 0.0394 | 78.6% | - |
| Negative Binomial | 150 | d | 0.0014 | 0.3600 | 0.4075 | 93.6% | - |
| | | a | -0.0404 | 0.1593 | 0.1886 | 96.4% | - |
| | | b | -0.0457 | 0.0836 | 0.0835 | 91.8% | - |
| | 500 | d | -0.0641 | 0.1972 | 0.2042 | 92.6% | - |
| | | a | -0.0143 | 0.0923 | 0.0961 | 95.2% | - |
| | | b | -0.0415 | 0.0446 | 0.0454 | 83.4% | - |

Table 9: True model parameter $\beta = (d, a, b, \mu_1, \omega)^T = (1, 0.3, 0.4, 1, 0.1)^T$

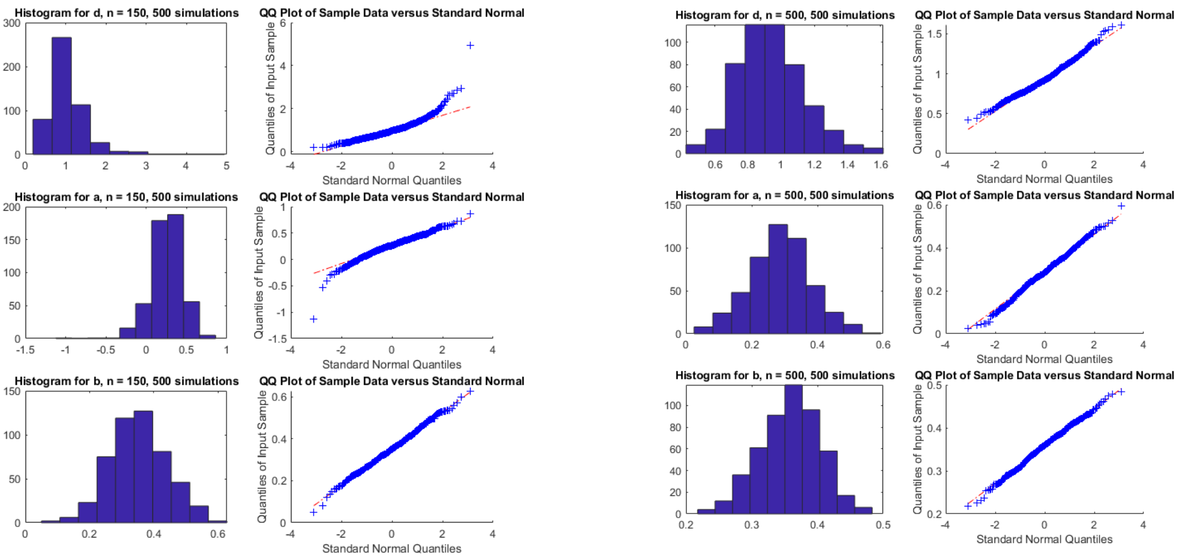



Figure 10: Histogram and QQ-plot of Semi-Parametric Model on ZIP data. (Left) n = 150, (Right) n = 500

### A.5 Appendix Simulation Discussion

Let us discuss the simulation results given in the appendix. Note that the analysis of these simulations finds the same overall results regarding the performance of the semi-parametric model as in the main report body for the Poisson and Geometric Distribution examples.

However, we do find interesting results for the Zero-Inflated Poisson distribution which we explored to access the semi-parametric model's robustness. For smaller sample size we find the bias to be similar to all other cases, with coverage rates close to 95% for all parameters. The standard error estimates and coverage does deviate for the Poisson model, but the negative binomial model provides similar performance to the semi-parametric model. However, interestingly we find for all models that the biases only improve for $a$ as $n$ increases. The semi-parametric model is not giving optimal coverage for $b$ despite the estimates distribution being close to a normal, and mean standard errors being close to the standard deviation. As the standard errors are smaller and the bias is unchanged, we find that the symmetric confidence intervals are tighter, with their center not close to the true parameter 0.4. As a result, all intervals that do not contain the true parameter lie below it, explaining the low coverage. We have similar findings and results for the Negative Binomial model which is interesting considering it is a misspecified model.

### A.6 Application: Polio Dataset

Our next example is the polio dataset discussed in Zeger (1988), which contains n = 168 observations of monthly poliomyelitis cases in the United States between 1970 and 1983 as reported by the Centres for Disease Control. The dataset has a sample mean of 1.3333 and sample variance of 3.5050 indicating conditional overdispersion, with a large proportion of zeros. Authors such as Zhu (2011), Aknouche et al. (2018) and others have all fitted INGARCH(1,1) models to the dataset, some of which we will compare with below.

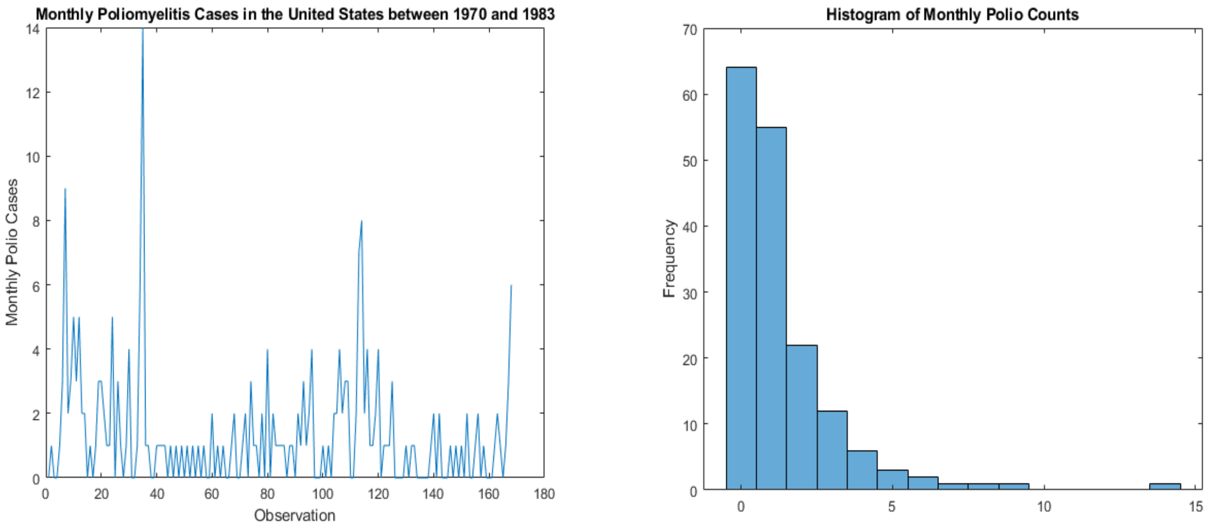



Figure 11: (Left) Data plot, (Right) Histogram

**Model Estimates**

| Model | $\hat{d}$ | $\hat{a}$ | $\hat{b}$ | $\hat{\mu}_1$ | $\hat{r}$ |
|---|---|---|---|---|---|
| Our SP Model | 0.6169 ( ) | 0.2030 (0.1790) | 0.3441 (0.1120) | 3.6e-07 | - |
| Zhu / Weiß NB | 0.6061 (0.2300) | 0.2000 (0.1905) | 0.3637 (0.1039) | 1.0946 | 1.6348 |
| tscount NB | 0.6321 (0.2403) | 0.1840 (0.2019) | 0.3489 (0.1074) | - | 1.8075 |
| tscount Pois | 0.6321 (0.1780) | 0.1840 (0.1460) | 0.3489 (0.0686) | - | - |
| Aknouche et.al NB | 0.6564 (0.2050) | 0.1511 (0.0935) | 0.3743 (0.1580) | 1.3333 | 2.6023 |

Table 10: INGARCH(1,1) Model Estimates, Standard Errors in Parenthesis

| Model | $\hat{d}$ | $\hat{a}$ | $\hat{b}$ |
|---|---|---|---|
| Our SP Model | [0.1469, 1.1469] | [-0.1478, 0.5538] | [0.1246, 0.5636] |
| Zhu / Weiß NB | [0.1553, 1.0569] | [-0.1734, 0.5734] | [0.1601, 0.5673] |
| tscount NB | [0.1611, 1.1030] | [-0.2117, 0.5797] | [0.1384, 0.5594] |
| tscount Pois | [0.2833, 0.9809] | [-0.1021, 0.4702] | [0.2144, 0.4833] |
| Aknouche et.al NB | [0.2546, 1.0582] | [-0.0322, 0.3344] | [0.0646, 0.6840] |

Table 11: INGARCH(1,1) Model's 95% Confidence Interval

**Discussion**

We find that the model estimates for the semi-parametric model closely resemble those proposed by Zhu / Weiß's Negative Binomial model and the tscount Negative Binomial model. The standard errors of the semi-parametric model for $a$ and $b$ are close to the parametric models except for the Poisson model. Thus, when we consider the 95% confidence intervals, we again find that the semi-parametric model is close to the parametric Negative Binomial models which will be reinforced in the next section.

**Model Adequecy**

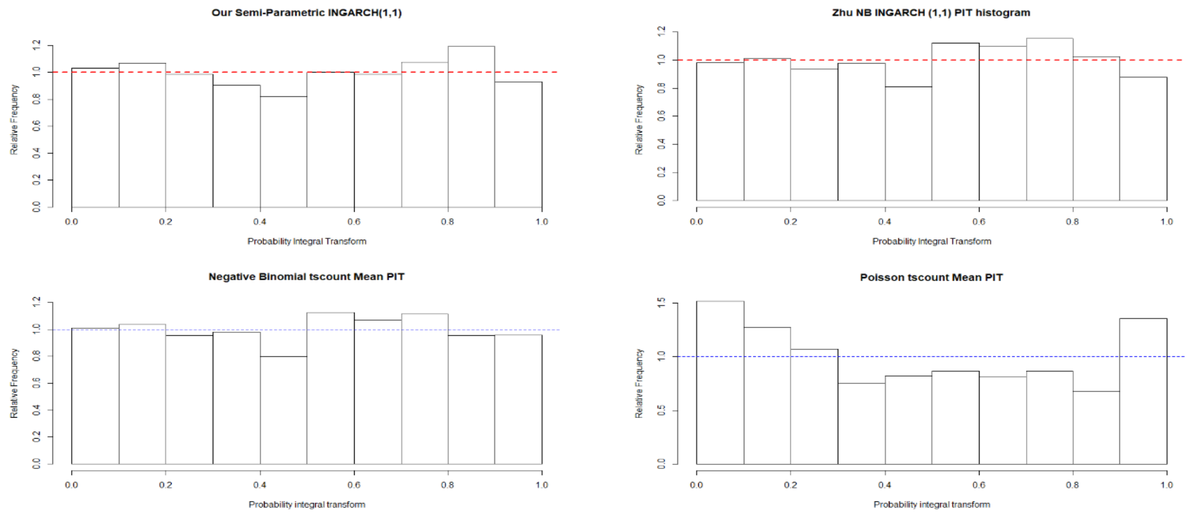



Figure 12: Mean PIT's. (Top Left) Semi Parametric Model, (Top Right) Zhu / Weiß NB Model, (Bottom Left) tscount NB Model, (Bottom Right) tscount Poisson Model

The U shaped mean PIT of the Poisson model indicates a poor fit tot he data as it exhibits conditional underdispersion relative to the true model. The proposed Semi-Parametric model is close to uniform, resembling both Negative Binomial model's mean PIT's. There is slightly less variation in the Semi-Parametric model from a uniform distribution but as this difference is small, we can assume that the Semi-Parametric model gives very similar model adequacy compared to the Negative Binomial models.

**Final Proposed Model**

$$\begin{aligned} \hat{\mu}_t &= 0.6169 + 0.2030\hat{\mu}_{t-1} + 0.3441Y_{t-1}, \quad t > 1 \\ \hat{\mu}_1 &= 3.6e - 07 \end{aligned} \tag{17}$$

## A.7 Application: Arson Counts Dataset

Let us consider a crime dataset analysed by Zhu (2012) of arson counts in a month in the 13th police car beat plus in Pittsburgh, from January 1990 to December 2010. The dataset consists of 144 observations with 42.36% of the series being zero, indicating possible zero inflation.

Zhu fitted various models including a Zero Inflated Poisson (ZIP) and Zero Inflated Negative Binomial (ZINB) INARCH(2) model, finding that the Zero-Inflated Poisson INARCH(2) model was the best fit to the data. Note that INARCH(p) models as a subfamily of a INGARCH(p,q) model, which do not recurse on the prior conditional mean (p = 0). Thus, the proposed semi-parametric model can be adapted to fit a semi-parametric INARCH(2) model for comparison alongside a standard Poisson and Negative Binomial model from tscount

(Liboschik et al. 2017), and a ZIP INARCH(2) model from the R package izpir (Huang and Fung 2021).

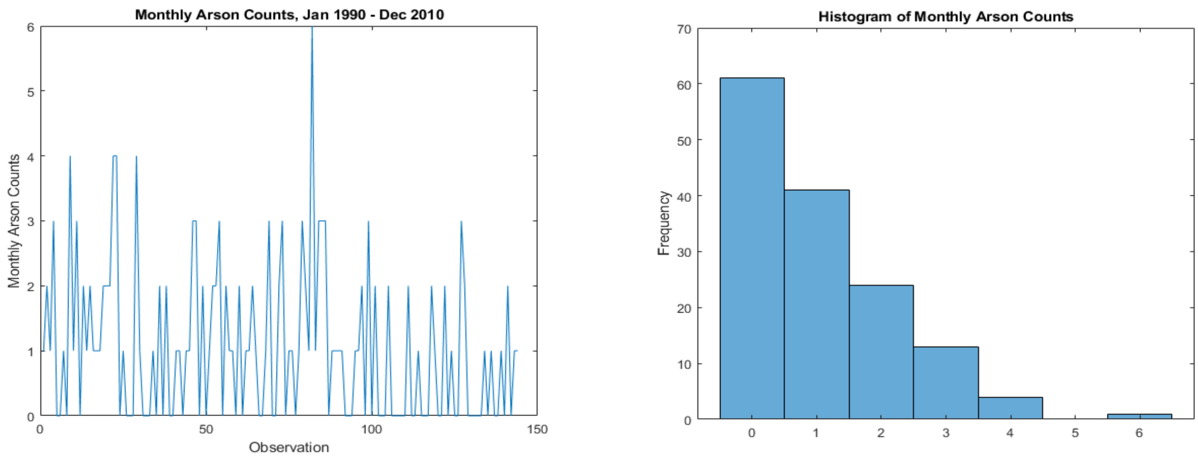



Figure 13: (Left) Data plot, (Right) Histogram

**Model Estimates**

| Model | $\hat{d}$ | $\hat{b}_1$ | $\hat{b}_2$ | $(\hat{\mu}_1, \hat{\mu}_2)$ | $\hat{\omega}$ |
|---|---|---|---|---|---|
| Our SP Model | 0.8256 ( ) | 0.0265 (0.1017) | 0.1746 (0.1016) | (1,2) | - |
| ZIP (Zhu 2012) | 1.0220 ( ) | 0.0560 ( ) | 0.2324 ( ) | - | 0.2149 |
| izipr | 0.8308 ( ) | 0.0277 ( ) | 0.1753 ( ) | - | - |
| tscount Pois | 0.8308 (0.1281) | 0.0277 (0.0737) | 0.1753 (0.0788) | - | - |
| tscount NB | 0.8308 (0.1438) | 0.0277 (0.0839) | 0.1753 (0.0915) | - | - |

Table 12: INARCH(2) Model Estimates, Standard Errors in Parenthesis

| Model | $\hat{d}$ | $\hat{b}_1$ | $\hat{b}_2$ |
|---|---|---|---|
| Our SP Model | [0.5706, 1.1306] | [-0.3243, 0.3773] | [-0.0449, 0.3941] |
| izipr | [0.5516, 1.1099] | [-0.1340, 0.1894] | [ 0.0032, 0.3476] |
| tscount Pois | [0.5797, 1.0818] | [-0.1167, 0.1721] | [ 0.0287, 0.3299] |
| tscount NB | [0.5489, 1.1126] | [-0.1367, 0.1921] | [-0.0040, 0.3547] |

Table 13: INARCH(2) 95% Confidence Intervals

**Discussion**

We find that the model estimates the ZIP proposed by Zhu (2012) are very different to the other models, indicating potentially biased estimates. The confidence intervals are all similar, with the semi-parametric estimates lying comfortably in all constructed intervals. However, it is evident that the semi-parametric model's confidence intervals are wider, indicating potentially an underlying model with more variability than proposed by other models.

**Model Adequacy**

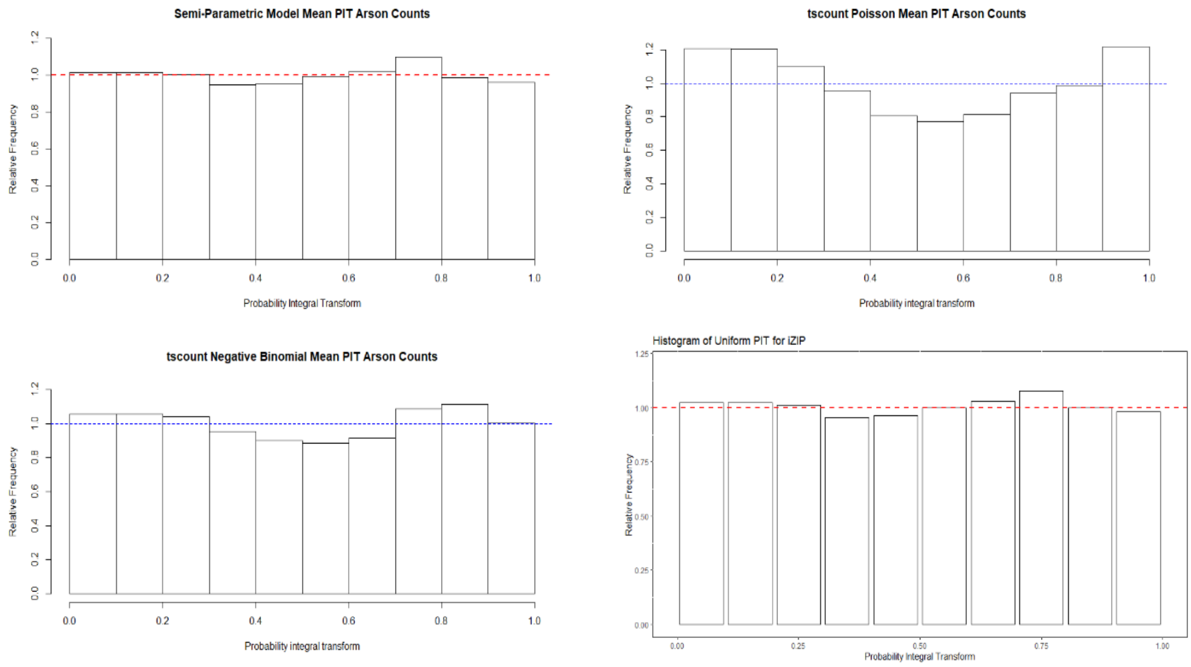



Figure 14: Mean PIT's. (Top Left) Semi Parametric Model, (Top Right) tscount Poisson Model, (Bottom Left) tscount Negative Binomial Model, (Bottom Right) izip Model

From Figure 14 we find that proposed semi-parametric model is very close to uniform, indicating a very good fit to the data, especially compared to the tscount models. We also find similar model adequacy for the izipr model, indicating that either model is a good fit compared to the tscount models. It is interesting that in a zero inflation example that the semi-parametric model gives good model adequacy, considering the bias of $b$ found in the simulation study.

**Final Proposed Model**

$$\begin{aligned} \hat{\mu}_t &= 0.8256 + 0.0265 Y_{t-1} + 0.1746 Y_{t-2}, \quad t > 3 \\ \hat{\mu}_1 &= 1.0000,\ \hat{\mu}_2 = 2.0000 \end{aligned} \tag{18}$$

## A.8 Probability Inverse Transform Definitions

Below we provide the definitions for the randomised and non-randomised PIT used in the report to analyse model adequacy. (Cazdo et al. 2009)

**Definition 8.0.1** (Non Randomised (Mean) PIT)**.** This can be considered as the average PIT if the randomised PIT was carried out a large number of times.

Let $P_t(.)$ is the predictive cumulative distribution of $Y_t$. Now, let

$$F_t(u|Y_t = y) := \begin{cases} 0 & \text{if } u \leq P_t(y-1) \\ \dfrac{u - P_t(y-1)}{P_t(y) - P_t(y-1)} & \text{if } P_t(y-1) \leq u \leq P_t(y) \\ 1 & \text{if } u \geq P_t(y) \end{cases} \tag{19}$$

Then, we obtain the mean PIT by aggregating over the predictions.

$$\bar{F}(u) = \frac{1}{N-1} \sum_{t=2}^{N} F_t(u \mid Y_t) \tag{20}$$

Then, we select $J$ (eg. 10) bins for the histogram and calculate the height of bin $j$, $f_j$ using

$$f_j = \bar{F}(\frac{j}{J}) - \bar{F}(\frac{j-1}{J}) \tag{21}$$

**Definition 8.0.2** (Randomised PIT)**.**

$$U_t := P_t(Y_t - 1) + \nu_t \left(P_t(Y_t) - P_t(Y_t - 1)\right)$$

Where $\nu_t$ is a iid sequence drawn from a standard uniform distribution, and $P_t(.)$ is the MELE for the cdf of a particular response $Y_t$. Then we plot $U_t$ as a histogram with $J$ bins. Note that as we have $t$ variables, it is simple to create to uniform QQ-plot to access uniformity.

## References

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