CL Tutorial 4

Q1) Derive $\neg(a \wedge b) \equiv \neg a \vee \neg b$

LHS

$$\dfrac{\dfrac{\neg a \models c}{\neg c \models a}\,\text{contraposition} \qquad \dfrac{\neg b \models c}{\neg c \models b}\,\text{contra position}}{\dfrac{\neg c \models a \wedge b}{\neg(a \wedge b) \models c}\,\text{contraposition}}\,\wedge R$$

RHS

$$\dfrac{\neg a \models c \qquad \neg b \models c}{\neg a \vee \neg b \models c}\,\vee L$$

Thus $\dfrac{\neg(a \wedge b) \models c}{\neg a \vee \neg b \models c}$

$$\dfrac{\dfrac{}{\neg a \vee \neg b \models \neg a \vee \neg b}\,\text{immediate}}{\neg a \vee \neg b \models \neg(a \wedge b)} \qquad \dfrac{\dfrac{}{\neg(a \wedge b) \models \neg(a \wedge b)}\,\text{immediate}}{\neg(a \wedge b) \models \neg a \vee \neg b}$$

$\therefore\ \neg a \vee \neg b \subseteq \neg(a \wedge b)$ and $\neg(a \wedge b) \subseteq \neg a \vee \neg b$

↑ is the subset of

$\therefore\ \cancel{\neg(a \wedge b)} \models$

$\therefore\ \neg(a \wedge b) \equiv \neg a \vee \neg b$

$\therefore\ \cancel{[\neg a \wedge b)}$

Ex 2

As every branch derives into immediate rules, we see it is universally valid.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$x, x \models x, y$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$x, z \models x, y$}
\RightLabel{$\vee L$}
\BinaryInfC{$x, (x \vee z) \models x, y$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$y, x \models x, y$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$y, z \models x, y$}
\RightLabel{$\vee L$}
\BinaryInfC{$y, (x \vee z) \models x, y$}
\RightLabel{$\vee L$, $\wedge L$}
\BinaryInfC{$(x \vee y) \wedge (x \vee z) \models x, y$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$x, x \models x, z$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$x, y \models x, z$}
\RightLabel{$\vee L$}
\BinaryInfC{$x, (x \vee y) \models x, z$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$z, x \models x, z$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$z, y \models x, z$}
\RightLabel{$\vee L$}
\BinaryInfC{$z, (x \vee y) \models x, z$}
\RightLabel{$\vee L$}
\BinaryInfC{$(x \vee y), (x \vee z) \models x, z$}
\RightLabel{$\wedge L$}
\UnaryInfC{$(x \vee y) \wedge (x \vee z) \models x, z$}
\RightLabel{$\wedge R$}
\BinaryInfC{$(x \vee y) \wedge (x \vee z) \models x, (y \wedge z)$}
\RightLabel{$\vee R$}
\UnaryInfC{$(x \vee y) \wedge (x \vee z) \models x \vee (y \wedge z)$}
\end{prooftree}
$$

Ex3

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$x, y \models x, z$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$x, y \models y, z$}
\RightLabel{$\wedge R$}
\BinaryInfC{$x, y \models (x \wedge y), z$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$z, y \models x, z$}
\AxiomC{}
\RightLabel{I}
\UnaryInfC{$z, y \models y, z$}
\RightLabel{$\wedge R$}
\BinaryInfC{$z, y \models (x \wedge y), z$}
\RightLabel{$\vee L$}
\BinaryInfC{$(x \vee z), y \models (x \wedge y), z$}
\RightLabel{$\neg R$}
\UnaryInfC{$x \vee z \models x \wedge y, \neg y, z$}
\RightLabel{$\vee R$}
\UnaryInfC{$x \vee z \models x \wedge y, \neg y \vee z$}
\RightLabel{$\neg R$}
\UnaryInfC{$\models (x \wedge y), \neg(x \vee z), \neg y \vee z$}
\RightLabel{$\vee R$}
\UnaryInfC{$\models (x \wedge y), (\neg(x \vee z) \vee (\neg y \vee z))$}
\RightLabel{$\vee R$}
\UnaryInfC{$\models (x \wedge y) \vee (\neg(x \vee z) \vee (\neg y \vee z))$}
\end{prooftree}
$$

As every leaf is an I-rule we can see that it is Universally valid, and if a statement has the left-hand side empty, and is also valid, then it is a tautology. Thus, the statement is a tautology.