PHYS 420: Computational Physics

Homework 3

Due Tuesday, Oct 3, 2017

Here are some comments on coding and presenting your work.

PLEASE READ THEM CAREFULLY.

1. The answers to the questions may include the computer code and output, in addition to any writing that might be needed. If you are not sure what is required please ask me, preferably before the last evening before the homework is due.

2. Please assemble all parts of a question together. I don't want to have the code for all the questions together, followed by all the output, followed by all the explanations. Rather, I require all parts of question 1, followed by all parts of question 2, etc.

3. In some problems, there will be many lines of output only a few of which are relevant. To save paper just print the relevant lines. You could indicate the missing lines by a line with something like ……..

4. In problems where you have to show that the error goes as a certain power of a small parameter h, choose values for h which decrease in a geometric (rather than arithmetic) sequence (i.e. divide by 2 or 10, whichever is most appropriate, each time).

5. Keep things simple.

6. Make sure that constants are defined with the desired precision.

Now we start the questions.

1. Root Finding

   a. Plot the function: $x^3 - 5x + 3 = 0$, such that all roots (y=0) are visible. Please add gridlines onto the plot and label axes appropriately (even if it is just x or y). Make a title that shows the function being plotted.

   b. Identify 2 x-values that bracket (fall on either side) each of the 3 roots to this equation.

   c. Using the 2 bracketing x-values (a, b), find the corresponding root using the **bisection method**:

      i. Calculate the midpoint, $c = \frac{a+b}{2}$
      ii. Calculate the function value at the midpoint, f(c).
      iii. If f(c) has the same sign as f(a), then $a_{new} = c, b_{new} = b$
      iv. If f(c) has the same sign as f(b), then $a_{new} = a, b_{new} = c$
      v. Repeat until you converge to predefined precision.

Using the 2 bracketing x-values (a, b) as your 2 previous steps, find the corresponding root using the **secant method**:

i. $x_0 = a, \quad x_1 = b$

ii. $x_n = \dfrac{x_{n-2}f(x_{n-1}) - x_{n-1}f(x_{n-2})}{f(x_{n-1}) - f(x_{n-2})}$

iii. Repeat until you converge to predefined precision. Be careful, you will eventually reach values that agree within numerical precision, at which time, you'll be dividing by zero. Stop before you hit numerical precision.

2. Transcendental Equations

Oftentimes, in physics, we left with an equation that cannot be simplied.

$$\sqrt{x+2} = \tan(x)$$

If we define a function, $f(x) = \sqrt{x+2} - \tan(x)$, we can find solutions to the above equation, by simply finding the zeroes to *f(x)*. Please use the **Newton-Raphson method** to find the first 10 solutions to this equation.

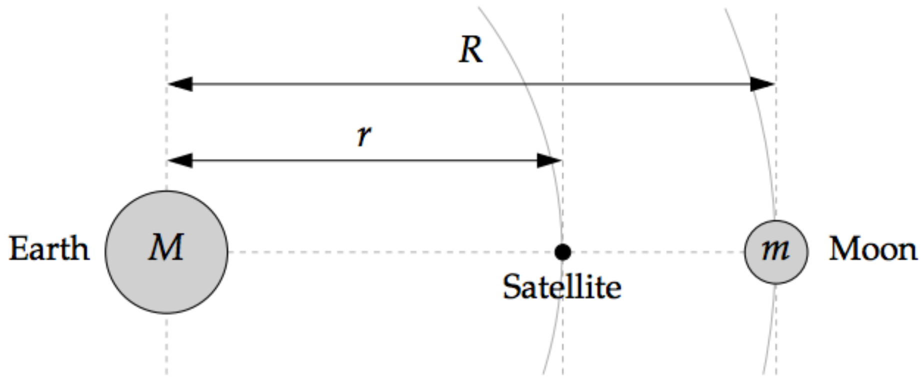
i. Plot the two sides of the equation and see where they overlap the first 10 times ($LHS = \sqrt{x+2}, \quad RHS = \tan(x)$).

ii. Guess an initial value close to the zero you are interested in (x_0).

iii. Calculate a better guess with: $x_n = x_{n-1} - \dfrac{f(x_{n-1})}{f'(x_{n-1})}$

iv. Iterate until you reach the desired precision.

3. The Lagrange point: There is a magical point between the Earth and the Moon, called the L_1 Lagrange point, at which a satellite will orbit the Earth in perfect synchrony with the Moon, staying always in between the two. This works because the inward pull of the Earth and the outward pull of the Moon combine to create exactly the needed centripetal force that keeps the satellite in its orbit. Here's the setup:



a. Assuming circular orbits, and assuming that the Earth is much more massive than either the Moon or the satellite, show that the distance r from the center of the Earth to the L1 point satisfies:

$$\frac{GM}{r^2} - \frac{Gm}{(R-r)^2} = \omega^2 r,$$

where M and m are the Earth and Moon masses, G is Newton's gravitational constant, and ω is the angular velocity of both the Moon and the satellite.

b. The equation above is a fifth-order polynomial in r (also called a quantic equation). Such equations cannot be solved exactly in closed form, but it's straight-forward to solve them numerically. Write a program to solve for the distance r from the Earth to the L_1 point. You can use any of the methods we have studied, bisection, secant and Newton-Raphson. Compute a solution accurate to at least four significant figures.

The values of the various parameters are:

$$G = 6.674x10^{-11}\ m^3\ kg^{-1}\ s^{-2},$$
$$M = 5.974x10^{24}\ kg,$$
$$m = 7.348x10^{22}\ kg,$$
$$R = 3.844x10^{8}\ m,$$
$$\omega = 2.662x10^{-6}\ s^{-1}.$$

4. In many situations, we may be trying to find an appropriate function that has certain properties that make it well suited for problem solving. In some realm of physics, it is useful to define a kernel function that represents a nugget or element that is not a fundamental chunk of matter, but rather just a representative chunk. For instance, a mass element in a hydrodynamic problem. One can say that a fluid element of mass M, has some density profile given by this kernel function and that other physical properties stem from this kernel. Let's try to develop our own kernel.

$$f(x) = 1 + ax^2 + bx^4 + cx^6.$$

Here, we have demanded certain smoothness criteria at the origin (even-powers of x). Let us now worry about smoothness criteria at the edge of the element (at x=1). This is a finite element and kernels and their derivatives should vanish at the boundary. It is up to us to decide up to what order (or derivative) should be identically zero. Let us consider the case such that up to second derivatives are zero.

$$f'(x) = 2ax + 4bx^3 + 6cx^5$$
$$f''(x) = 2a + 12bx^2 + 30cx^4$$

Evaluating our kernel at the boundary we find:

$$1 + a + b + c = 0$$
$$0 + 2a + 4b + 6c = 0$$
$$0 + 2a + 12b + 30c = 0$$

$$a + b + c = -1$$
$$2a + 4b + 6c = 0$$
$$2a + 12b + 30c = 0$$

Which can be written in Matrix form: $\overleftrightarrow{A} \cdot \vec{x} = \vec{b}$

Where: $\overleftrightarrow{A} = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 4 & 6 \\ 2 & 12 & 30 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$.

Review the page (http://www.bogotobogo.com/python/python_numpy_matrix_tutorial.php) and write a python script that can solve for $\vec{x}$. With these coefficient values, is there a simpler form for the kernel than the expanded polynomial listed above?