DALHOUSIE UNIVERSITY

PHYC 3640 Quantum Physics I: Tutorial 3

Dirac Delta Functions & Matrix Eigenvalues

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Dirac Delta Functions

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Matrix Eigenvalues

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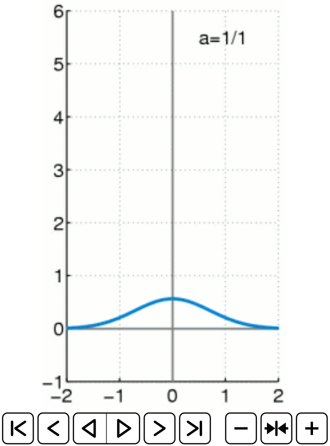
What is the Dirac Delta function?

There are three key definitions normally used when talking about the Delta function.

$$\delta(x) = \begin{cases} \infty & x = 0 \\ 0 & x \neq 0 \end{cases} \qquad (1)$$

$$\int_{-\infty}^{\infty} \delta(x) dx = 1 \qquad (2)$$

$$\delta(x) = \lim_{a \to 0} \frac{1}{|a|\sqrt{\pi}} e^{-(x/a)^2} \qquad (3)$$



https://en.wikipedia.org/wiki/Dirac_delta_function

Some History

The Delta Function was probably first introduced by Fourier, and was then heavily elaborated on by Cauchy at a later date.

Lots of other names contributed over the years, like Poisson, Kirchhoff, Helmholtz, Kelvin, and Heaviside.

Finally, Dirac popularized it by adding it to his textbook, *The Principles of Quantum Mechanics*, as the continuous analogue of the Kronecker delta function. Thus the name "Dirac" Delta function.

https://en.wikipedia.org/wiki/Dirac_delta_function#History

List of Properties

$$\int_{-\infty}^{\infty} f(x)\delta(x-x_0)dx = f(x_0) \tag{4}$$

$$\int_{a}^{b} f(x)\delta(x-x_0)dx = \begin{cases} f(x_0) & \text{if } x_0 \in (a,b) \\ 0 & \text{if } x_0 \in (-\infty,a) \cup (b,\infty) \end{cases} \tag{5}$$

$$\delta(x) = \delta(-x) \tag{6}$$

$$x\delta(x) = 0 \tag{7}$$

$$\delta(ax) = \frac{1}{|a|}\delta(x)\,, \qquad a \neq 0 \tag{8}$$

$$f(x)\delta(x-a) = f(a)\delta(x-a) \tag{9}$$

$$\int \delta(a-x)\delta(x-b)dx = \delta(a-b) \tag{10}$$

Problem Set 1

a) $\displaystyle\int_{-\infty}^{\infty} 1\delta(x)dx$

b) $\displaystyle\int_{-\infty}^{\infty} 2\delta(x)dx$

c) $\displaystyle\int_{-\infty}^{\infty} 2\delta(x-2)dx$

d) $\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{x}\delta(x)dx$

e) $\displaystyle\int_{-\infty}^{\infty} f(x)\delta(x-1)\delta(x+1)dx$

f) $\displaystyle\int_{-5}^{5} 2^{x}\delta(x-3)dx$

g) $\displaystyle\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \mathrm{e}^{-r}r^{2}\sin(\theta)\delta(r-2)\delta(\phi-\pi)drd\theta d\phi$

h) $\displaystyle\int_{-\infty}^{\infty} \delta\big(x^{2}-1\big)dx$

Problem Set 1

a) $\int_{-\infty}^{\infty} 1\delta(x)dx$ — 1

b) $\int_{-\infty}^{\infty} 2\delta(x)dx$ — 2

c) $\int_{-\infty}^{\infty} 2\delta(x-2)dx$ — 2

d) $\int_{-\infty}^{\infty} e^{x}\delta(x)dx$ — 1

e) $\int_{-\infty}^{\infty} f(x)\delta(x-1)\delta(x+1)dx$ — 0

f) $\int_{-5}^{5} 2^{x}\delta(x-3)dx$ — 8

g) $\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} e^{-r}r^{2}\sin(\theta)\delta(r-2)\delta(\phi-\pi)dr\,d\theta\,d\phi$ — $\frac{8}{e^{2}}$

h) $\int_{-\infty}^{\infty} \delta\left(x^{2}-1\right)dx$ — ?

Solution to (g)

g) $\displaystyle\int_{r=0}^{\infty}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi} \mathrm{e}^{-r} r^2 \sin(\theta)\delta(r-2)\delta(\phi-\pi)\,dr\,d\theta\,d\phi$

$$
\begin{aligned}
&= \int_{r=0}^{\infty} \mathrm{e}^{-r} r^2 \delta(r-2)\,dr \times \int_{\theta=0}^{\pi} \sin(\theta)\,d\theta \times \int_{\phi=0}^{2\pi} \delta(\phi-\pi)\,d\phi \\
&= \left[\mathrm{e}^{-2} 2^2\right] \left[-\cos(\theta)\right]_0^{\pi} [1] \\
&= \left[\mathrm{e}^{-2} 2^2\right] [2]\,[1] \\
&= \frac{8}{\mathrm{e}^2}
\end{aligned}
$$

Solution to (h)

$$\int_{-\infty}^{\infty} \delta\big(x^2 - 1\big)dx = ?$$

What do you think it is?

a) -1
b) 0
c) 1
d) 2
e) Undefined

Problem Set 2

$$\delta\big(g(x)\big) = \sum_{x_i} \frac{\delta\big(x - x_i\big)}{|g'(x_i)|}\,,$$

where x_i are the zeroes of $g(x)$

a) $\displaystyle\int_{-\infty}^{\infty} \delta\big(x^2 - 1\big)dx$

b) $\displaystyle\int_{-\infty}^{\infty} \delta\big(x^2 - 2\big)dx$

c) $\displaystyle\int_{-\infty}^{\infty} \delta\big(x^2 + 1\big)dx$

d) $\displaystyle\int_{-\infty}^{\infty} \delta\big(x^2 + 4x + 3\big)dx$

e) $\displaystyle\int_{-4}^{4} \sin(x)\,\delta\big(\cos(x)\big)dx$

f) $\displaystyle\int_{0}^{\infty} \delta(x)dx$

g) $\displaystyle\int_{0}^{\infty} \delta(r)dr$

Problem Set 2

$$\delta\big(g(x)\big) = \sum_{x_i} \frac{\delta\big(x - x_i\big)}{|g'(x_i)|}\,, \qquad \text{where } x_i \text{ are the zeroes of } g(x)$$

a) $\int_{-\infty}^{\infty} \delta\big(x^2 - 1\big)dx$	1
b) $\int_{-\infty}^{\infty} \delta\big(x^2 - 2\big)dx$	$1/\sqrt{2}$
c) $\int_{-\infty}^{\infty} \delta\big(x^2 + 1\big)dx$	0
d) $\int_{-\infty}^{\infty} \delta\big(x^2 + 4x + 3\big)dx$	1
e) $\int_{-4}^{4} \sin(x)\,\delta\big(\cos(x)\big)dx$	0
f) $\int_{0}^{\infty} \delta(x)dx$	$1/2$
g) $\int_{0}^{\infty} \delta(r)dr$	Either $1/2$ or 1

Solution to (b)

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta\big(x^2 - 2\big) dx &= \int_{-\infty}^{\infty} \delta\big((x-\sqrt{2})(x+\sqrt{2})\big) dx \\
&= \int_{-\infty}^{\infty} \left[\sum_{x_i} \frac{\delta\big(x - x_i\big)}{|g'(x_i)|}\right] dx \\
&= \int_{-\infty}^{\infty} \frac{\delta(x-\sqrt{2})}{\big|2\sqrt{2}\big|} + \frac{\delta(x+\sqrt{2})}{\big|-2\sqrt{2}\big|} dx \\
&= \frac{1}{2\sqrt{2}} + \frac{1}{2\sqrt{2}} \\
&= \frac{1}{\sqrt{2}}
\end{aligned}$$

Solution to (d)

$$\begin{aligned}
\int_{-\infty}^{\infty} \delta\big(x^2 + 4x + 3\big)dx &= \int_{-\infty}^{\infty} \delta\big((x+1)(x+3)\big)dx \\
&= \int_{-\infty}^{\infty} \left[\sum_{x_i} \frac{\delta\big(x - x_i\big)}{|g'(x_i)|}\right] dx \\
&= \int_{-\infty}^{\infty} \frac{\delta(x+1)}{|2(-1)+4|} + \frac{\delta(x+3)}{|2(-3)+4|}dx \\
&= \frac{1}{2} + \frac{1}{2} \\
&= 1
\end{aligned}$$

Solution to (e)

$$\begin{aligned}
\int_{-4}^{4} \sin(x)\,\delta\big(\cos(x)\big)\,dx &= \int_{-4}^{4} \left[\sum_{x_i} \frac{\delta\big(x - x_i\big)}{|g'(x_i)|}\right] dx \\
&= \int_{-4}^{4} \sin(x) \left[\frac{\delta\big(x + \pi/2\big)}{|-\sin(-\pi/2)|} + \frac{\delta\big(x - \pi/2\big)}{|-\sin(\pi/2)|}\right] dx \\
&= \frac{\sin(-\pi/2)}{|-(-1)|} + \frac{\sin(\pi/2)}{|-(1)|} \\
&= \frac{-1}{1} + \frac{1}{1} \\
&= 0
\end{aligned}$$

List of Properties

$$\int_a^b f(x)\delta(x-x_0)dx = \begin{cases} f(x_0) & \text{if } x_0 \in (a,b) \\ \frac{1}{2}f(x_0) & \text{if } x_0 \in \{a,b\} \\ 0 & \text{otherwise} \end{cases} \tag{11}$$

However, there is a conflicting definition. The "radial" Dirac Delta Function is defined as

$$\int_0^\infty f(r)\delta(r-r_0)dr = \begin{cases} f(r_0) & \text{if } r_0 \in [0,\infty) \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

If this confuses you, it should. There are "strong" and "weak" definitions of the Delta function, and you need to know which one you're working with. Bottom line: Always know the notation of your field before making any assumptions. Here are some more properties.

$$\delta\big(g(x)\big) = \sum_{x_i} \frac{\delta\big(x-x_i\big)}{|g'(x_i)|}, \qquad \text{for zeroes: } x_i \tag{13}$$

$$\delta\big[(x-a)(x-b)\big] = \frac{1}{|a-b|}\left[\delta(x-a)+\delta(x-b)\right] \tag{14}$$

$$\delta\big[(x^2-a^2)\big] = \frac{1}{2|a|}\left[\delta(x-a)+\delta(x+a)\right] \tag{15}$$

List of Properties

Lastly, some derivative properties

$$\int_a^b f(x)\delta'(x-x_0)dx = \begin{cases} -f'(x_0) & \text{if } x_0 \in (a,b) \\ -\frac{1}{2}f'(x_0) & \text{if } x_0 \in \{a,b\} \\ 0 & \text{otherwise} \end{cases} \tag{16}$$

$$\delta'(x) = -\delta'(-x) \tag{17}$$

$$x\delta'(x) = -\delta(x) \tag{18}$$

$$x^2\delta'(x) = 0 \tag{19}$$

$$\int \delta'(a-x)\delta(x-b)dx = \delta'(a-b) \tag{20}$$

$$\delta'(x) = \frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty} k\mathrm{e}^{\mathrm{i}kx}dk \tag{21}$$

Multidimensional Delta Functions

We also have the Delta function expanded into higher dimensions.
2D:

$$\delta^2(\vec{r} - \vec{r}_0) = \begin{cases} \delta(x - x_0)\delta(y - y_0) & \text{Cartesian} \\ \frac{1}{r}\delta(r - r_0)\delta(\theta - \theta_0) & \text{Polar} \end{cases} \tag{22}$$

3D:

$$\delta^3(\vec{r} - \vec{r}_0) = \begin{cases} \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) & \text{Cartesian} \\ \frac{1}{\rho}\delta(\rho - \rho_0)\delta(\theta - \theta_0)\delta(z - z_0) & \text{Cylindrical} \\ \frac{1}{r^2 \sin(\theta)}\delta(r - r_0)\delta(\theta - \theta_0)\delta(\phi - \phi_0) & \text{Spherical} \end{cases} \tag{23}$$

In general, transforming between coordinates x and ξ is

$$\delta(x - x_0) = \frac{1}{|J|}\delta(\xi - \xi_0) \tag{24}$$

where J is the Jacobian of the transformation.

Strong vs Weak

Table 1: Comparison between the weak and strong definitions of the Dirac delta function.

Quantity	Weak value	Strong value
$\int_{-\infty}^{\infty} \delta(x)dx$	1	1
$\int_{0}^{\infty} \delta(x)dx$	$\frac{1}{2}$	1
$\delta^3(r)$	$\frac{\delta(r)}{2\pi r^2}$	$\frac{\delta(r)}{4\pi r^2}$
$\triangle \frac{1}{r}$	$-4\pi\delta^3(\vec{r})$	$-4\pi\delta^3(\vec{r})$

For more information, check the link below.

https://www.scielo.br/j/rbef/a/xkcBhffNHP3yb6jzHjrcC7h/?format=pdf&lang=en

Problem Set 3

1) What is the Fourir Transform of the Dirac Delta function?

2) Prove $\delta(ax) = \frac{\delta(x)}{|a|}$

3) Prove $x\delta(x) = 0$

4) Prove $x\delta'(x) = -\delta(x)$

5) Prove $x^2\delta''(x) = 2\delta(x)$

6) Prove $\delta(\sin(x)) = \sum_{n=-\infty}^{\infty} \delta(x - n\pi)$

Eigenvalues (Matrix Style)

The “eigen” in “eigenvalues” and “eigenvectors” comes from German, and it means “proper”. These are special numbers and vectors for the system you’re studying that are invariant to linear transformations. Basically, they don’t bend, rotate, grow, shrink, etc...

Here’s a cool Stack-Exchange article on them
https://math.stackexchange.com/questions/300145/what-exactly-are-eigen-things

Quantized systems are sometimes modelled with matrices. A square matrix **A** have eigenvalues λ_i and eigenvectors $\mathbf{x}_i$ that can be found with diagonalization.

$$\mathbf{A}\mathbf{x}_i = \lambda_i \mathbf{x}_i$$

Reminder: What a Determinant Is

$$\det(\mathbf{A}) = |\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

$$\begin{bmatrix}1&-2&3\\2&0&3\\1&5&4\end{bmatrix} = \begin{bmatrix}\boxed{1}&-2&3\\2&0&3\\1&5&4\end{bmatrix} - \begin{bmatrix}1&\boxed{-2}&3\\2&0&3\\1&5&4\end{bmatrix} + \begin{bmatrix}1&-2&\boxed{3}\\2&0&3\\1&5&4\end{bmatrix}$$

$$= \boxed{1} \times \begin{vmatrix}0&3\\5&4\end{vmatrix} - \boxed{-2} \times \begin{vmatrix}2&3\\1&4\end{vmatrix} + \boxed{3} \times \begin{vmatrix}2&0\\1&5\end{vmatrix}$$

$$= 1\times(0-15)+2\times(8-3)+3\times(10-0)$$

$$= 1(-15)+2(5)+3(10)$$

$$= -15+10+30$$

$$= 25$$

Matrix Eigenvalue Example

Let's work through an example. Take

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & -1 \\ 1 & 3 & -2 \end{bmatrix}$$

The eigenvalues can be found by solving for λ in the following equation

$$\det(\mathbf{A} - \lambda \mathbf{I})$$

where $\mathbf{I}$ is the identity matrix

$$\mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Matrix Eigenvalue Example

Thus

$$\begin{vmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & -1 \\ 1 & 3 & -2-\lambda \end{vmatrix} = 0$$

Taking the determinant yields

$$\begin{aligned} (2-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} - 0\begin{vmatrix} \cdots \end{vmatrix} + 0\begin{vmatrix} \cdots \end{vmatrix} &= 0 \\ (2-\lambda)\Big[(2-\lambda)(-2-\lambda) - (-1)(3)\Big] &= 0 \\ \lambda^3 - 2\lambda^2 - \lambda + 2 &= 0 \\ (\lambda-2)(\lambda-1)(\lambda+1) &= 0 \end{aligned}$$

So the eigenvalues are $\{\lambda_1,\, \lambda_2,\, \lambda_3\} = \{2,\, 1,\, -1\}$

Matrix Eigenvalue Example

To find the eigenvectors, we solve the following

$$(\mathbf{A} - \lambda_i \mathbf{I})\, \mathbf{x}_i = 0$$

Consider $\lambda_1 = 2$, we can write the above as an augmented matrix and row reduce it to find the eigenvector

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 3 & -4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The first line indicates $x - z = 0$, and the second indicates $y - z = 0$. z is an independent variable so let's set it to 1 for convenience. Solving the above two equations gives

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Selecting $\lambda_2 = 1$ and $\lambda_3 = -1$ yield eigenvectors of $\mathbf{x}_2 = [0 \;\; 1 \;\; 1]^{\mathrm{T}}$ and $\mathbf{x}_3 = [0 \;\; 1 \;\; 3]^{\mathrm{T}}$ respectively.

Problem 4

1) Find the Eigenvalues and Eigenvectors of $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Problem 4

1) Find the Eigenvalues and Eigenvectors of $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$.

Solution

Questions?

Want My Slides?

https://github.com/KyleBryenton/slides-posters

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