Exactly Solvable Anharmonic Potentials with Variable Bumps and Depths

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Come see my talk: Saturday, December 7th in the Turner Room of the Chelsea Hotel, 16:00-16:30

Abstract

A new approach based on Darboux Transformations was introduced to generate new classes of solvable anharmonic potentials with a variable number of bumps and depths. By introducing the concept of a transformation key, a method of controlling the number of bumps and their depths in these potentials is presented. Although this method was applied to a one-dimensional generalized harmonic oscillator potential, it could be easily adapted to generate new exactly-solvable potentials from other known quantum systems.

Generalized Harmonic Oscillator

The 1-D Schrödinger equation is an eigenvalue problem satisfying $\hat{H}\psi_n(x) = E_n\psi_n(x)$ for a wavefunction $\psi_n(x)$ and energy eigenvalues E_n. The Hamiltonian is given by $\hat{H} = \frac{-\hbar^2}{2m}D_x^2 + V(x)$ where m is the mass of the particle, and $V(x)$ is the position-dependent potential that is applied to it. The presented technique transforms a solved Schrödinger equation into a new one with accompanying exact solutions that are isospectral to the original. As a demonstration of this technique, a generalized harmonic oscillator potential was derived and solved to act as a starting point. The potential and wavefunctions are given by

$$V_0(x) = (a\,x+b)^2\,, \quad \psi_{0;n}(x) = \left(\sqrt{\pi}\,a^{-\frac{1}{2}+n}\,2^n\,n!\right)^{-\frac{1}{2}} \exp\left[\frac{(a\,x+b)^2}{2a}\right] D_x^n \exp\left[-\frac{(a\,x+b)^2}{a}\right]\,, \tag{1}$$

where $E_n = a(2n+1)$ and $n \in \mathbb{N}$. The subscript 0 on the potential and waverfunctions indicate they have not yet been transformed. These wavefunctions exhibit orthonormality such that $\langle \psi_{0;m}(x) | \psi_{0;n}(x) \rangle_{\mathscr{H}} = \delta_{mn}$.

Algorithm

The transformation key is defined as a list of non-negative integers $\mathcal{T} \equiv \{\mathcal{T}_1;\, \mathcal{T}_2,\, \mathcal{T}_3,\, \cdots\}$, where $\mathcal{T}_m \in \mathbb{N}$ and $m \in \mathbb{Z}^+$. The seedfunction for the first transformation is obtained by inserting a complex argument into (1): $\psi_{0;n}([a\,x+b]) \to \phi_{0;\mathcal{T}_1}(i[a\,x+b])$. For the general m^{th} transformation, the seedfunction is taken from the previous transformation as given by $\mathcal{T}_m$. For all $m > 1$, no value of $\mathcal{T}_m \geq 0$ may be repeated. It is useful to define an operator $\hat{\mathcal{L}}_m$ as

$$\hat{\mathcal{L}}_m \psi_{m-1;n}(x) = \begin{cases} \left[D_x - \dfrac{D_x[\phi_{0;\mathcal{T}_1}(x)]}{\phi_{0;\mathcal{T}_1}(x)}\right]\psi_{0;n}(x)\,, & m=1\,, \\[2ex] \left[D_x - \dfrac{D_x[\psi_{m-1;\mathcal{T}_m}(x)]}{\psi_{m-1;\mathcal{T}_m}(x)}\right]\psi_{m-1;n}(x)\,, & m>1\,. \end{cases} \tag{2}$$

Using this operator, the m^{th} wavefunction may be recursively defined for all m as

$$\psi_{m;n}(x) = \hat{\mathcal{L}}_m\psi_{m-1;n}(x) = \hat{\mathcal{L}}_m\,\hat{\mathcal{L}}_{m-1}\,\hat{\mathcal{L}}_{m-2}\cdots\hat{\mathcal{L}}_2\,\hat{\mathcal{L}}_1\,\psi_{0;n}(x)\,, \tag{3}$$

where $\psi_{0;n}(x)$ is given in (1). Consequently, the m^{th} potential may be expressed as

$$V_m = \begin{cases} V_0 - 2\,D_x^2 \log\phi_{0;\mathcal{T}_1}(x)\,, & m=1\,, \\ V_1 - 2\sum\limits_{i=1}^{m-1} D_x^2 \log\left[\hat{\mathcal{L}}_i\,\hat{\mathcal{L}}_{i-1}\,\hat{\mathcal{L}}_{i-2}\cdots\hat{\mathcal{L}}_2\,\hat{\mathcal{L}}_1\,\psi_{0;\mathcal{T}_{i+1}}(x)\right]\,, & m>1\,. \end{cases} \tag{4}$$

This formalism permits the expression of a series of transformations using only a transformation key. In the *Results* section, four interesting families of potentials that have been generated using this technique are discussed.

Results

This new technique has revealed four interesting families of solved potentials. The first was the n-bump anharmonic family, defined on the entire real line. The second was the singular n-bump anharmonic family, defined on a half-line. The third was a broadened version of the singular n-bump anharmonic family, while the fourth was the n-bump anharmonic family with a variable-depth central potential well at the origin. These families have been classified as follows:

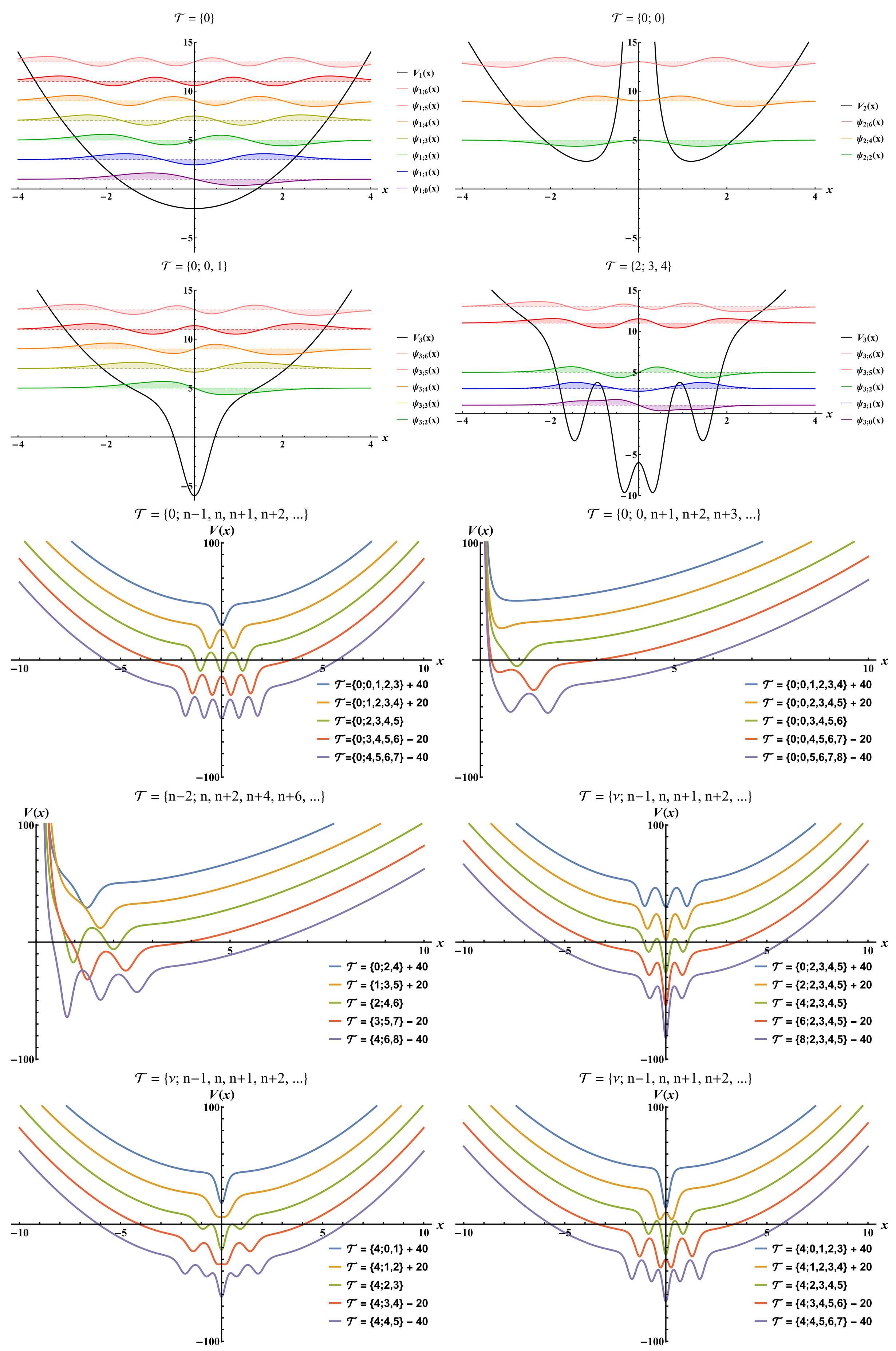
n-Bump Anharmonic Potential Family
$\mathcal{T} \equiv \{0;\, n-1,\, n,\, n+1,\, n+2,\, \ldots\}\,, \qquad n = 1,\, 2,\, 3,\, \ldots$

Singular n-Bump Anharmonic Potential Family
$\mathcal{T} \equiv \{0;\, 0,\, n+1,\, n+2,\, n+3,\, \ldots\}\,, \qquad n = 0,\, 1,\, 2,\, \ldots$

Broadened Singular n-Bump Anharmonic Potential Family
$\mathcal{T} \equiv \{n-2;\, n,\, n+2,\, n+4,\, \ldots\}\,, \qquad n = 2,\, 3,\, 4,\, \ldots$

Central Well n-Bump Anharmonic Potential Family ($\nu = 0,\, 2,\, 4\ldots$)
$\mathcal{T} \equiv \{\nu;\, n-1,\, n,\, n+1,\, n+2,\, \ldots\}\,, \qquad n = 1,\, 2,\, 3,\, \ldots$

The top four images on the right demonstrate interesting wavefunction behaviour, such as domain restriction and removed states. States are removed from the spectrum based on the presence of singularities, and the choice of $\mathcal{T}_m$ for $m > 1$. The bottom six images are examples of the families discussed above.



Conclusion

A method based on Darboux Transformations, explained via the notion of a transformation key, was used to generate new classes of exactly-solvable potentials. Making use of the known solutions of the generalized harmonic oscillator potential, new families of isospectral potentials were constructed. For each potential that was generated, corresponding normalized wavefunctions were as well. The presented technique establishes a method of generating new anharmonic potentials and provides a means of controlling the number of bumps and their depths. This approach can be easily applied to most known solved potentials with discrete spectra, e.g. the Coulomb potential, the Morse potential, and the general cosecant squared potential. This work also gives rise to new orthogonal potentials that require further investigation of their mathematical properties, as well as their connection with the exceptional orthogonal polynomials.

Further Reading

Darboux transformations: G. Darboux, *Lecons sur la theorie generale des surfaces et les application geometriques du calcul infinitesimal*, Deuziem pattie, G.V. et fils, Paris (1889).

Supersymmetry and the generalized harmonic oscillator: K. R. Bryenton *Darboux-Crum Transformations, Supersymmetric Quantum Mechanics, and the Eigenvalue Problem*, (B.Sc Hons) University of Prince Edward Island, Charlottetown, Canada, (2016). DOI: 10.13140/RG.2.2.23129.98408

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