

Analytic Stacks

People

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1.4 Replete and locally weakly contractible topoi

Throuought this section the word "topos" refers to the category of sheaves on some site

Definition 1.1 (Replete Topos). A topos X is *replete* if epimorphisms are closed under sequential limits, that is for a functor $F : \mathbb{N}^{op} \rightarrow X$ with epimorphic transition maps $F_{n+1} \twoheadrightarrow F_n$, then the map $\lim F \rightarrow F_n$ is epic for each n

Lemma 1.1. 1. X is replete then for $x \in X$, X/x is replete

2. X is replete if and only if there is a surjection $x \rightarrow 1$ and X/x is replete

Lemma 1.2. Let X be a replete topos, and let $F \rightarrow G$ be a map in $\text{Fun}(\mathbb{N}^{op}, X)$, assume that the induced map $F_i \rightarrow G_i$ and $F_{i+1} \rightarrow F_i \times_{G_i} G_{i+1}$ are surjective for each i . Then $\lim F \rightarrow \lim G$ is surjective

Proposition 1.3. 1. Countable products are exact in a replete topos

2. If X is a replete topos and $F : \mathbb{N}^{op} \rightarrow \mathcal{A}\mathcal{B}(X)$ is a diagram with $F_{n+1} \rightarrow F_n$ surjective for every n then $\lim F_n \cong R\lim F_n$

3. If X is a replete topos then the functor of \mathbb{N}^{op} -indexed limits has cohomological dimension 1

Definition 1.2. An object in a topos is called

1. Compact if the "underlying geometric structure" is compact, ie if the geometric morphism $X/a \rightarrow \text{Sh}(\ast) = \text{Set}$ is proper
2. Stable if for all morphisms $Y \rightarrow X$ with Y compact, the domain of the kernel pair $R \rightrightarrows Y$ of f is also compact
3. Coherent if it is compact and stable

Definition 1.3 (Locally Weakly Contractible Topos). An object F in a topos X is called weakly contractible if every epimorphism $G \twoheadrightarrow F$ has a section. We say that X is *locally weakly contractible* if each $a \in X$ admits an epimorphism $\bigsqcup Y_i \twoheadrightarrow X$ with Y_i coherent and weakly contractible

Proposition 1.4. Let X be a locally weakly contractible topos. Then

1. X is replete
2. The derived category $D(X, \mathbb{Z})$ is compactly generated
3. Postnikov towers converge in the associated hypercomplete ∞ -topos

Proposition 1.5. *If X is a replete topos, then $D(X)$ is left-complete*

Proposition 1.6. *Let $f: A_\bullet \rightarrow A$ by a hyper cover in a replete topos X , then*

1. *The adjunction $id \rightarrow f_* f^*$ is an equivalence on $D(A)$*
2. *The adjunction $f_! f^* \rightarrow id$ is an equivalence on $D(X)$*
3. *f^* induces an equivalence $D(X) \cong D_{\text{cart}}(X_\bullet) \subset D(X_\bullet)$ where $D(X_\bullet)$ is the derived category of the simplicial topos defined by X_\bullet and $D_{\text{cart}}(X_\bullet)$ is the full subcategory spanned by cartesian complexes*