Analytic Stacks

People

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1.4 Replete and locally weakly contractible topoi

Throuought this section the word "topos" refers to the category of sheaves on some site

Definition 1.1 (Replete Topos)**.** A topos X is *replete* if epimorphisms are closed under sequential limits, that is for a functor $F : \mathbb{N}^{op} \to X$ with epimorphic transition maps $F_{n+1} \twoheadrightarrow F_n$, then the map $\lim F \to F_n$ is epic for each n

Lemma 1.1.

1. *X is replete then for $x \in X$, X/x is replete*
2. *X is replete if and only if there is a surjection $x \to 1$ and X/x is replete*

Proof. To prove (1) we observe that the canonical functor $X/x \to X$ preserves sequential limits and epimorphisms so in X/x an epi sequential limit corresponds to an epi sequential limit in X and the maps $\lim F \to F_n$ are epi in X and hense epi in X/x

To prove (2) we note that (1) proves the forward direction, then for the other direction the map $g\colon x \to 1$ induced a base change map $g^*\colon X \cong X/1 \to X/x$ defined by $t \mapsto (t \times x \to x)$ which preserves limits and epimorphisms so the same argument at (1) applies $\square$

Proposition 1.2 (Exactness in replete topoi)**.**

1. *For X a replete topos and $H\colon F \to G$ a map in $\mathrm{Fun}(\mathbb{N}^{op}, X)$ where the components $h_i\colon F_i \to G_i$ and the induced maps $(f_{i+1}, h_{i+1})\colon F_{i+1} \to F_i \times_{G_i} G_{i+1}$ are epimorphisms for all i. Then $\lim F \to \lim G$ is an epimorphism*[1]
2. *Countable products are exact in a replete topos*
3. *If X is a replete topos then the functor of N^{op}-indexed limits has cohomological dimension 1*
4. *If X is a replete topos and $F : \mathbb{N}^{op} \to \mathcal{Ab}(X)$ is a diagram with $F_{n+1} \to F_n$ surjective for every n then $\lim F_n \cong R\lim F_n$*

Proof. To prove (1) Take any $A \in X$ and any map $s\colon A \to \lim G$ we want to find an epimorphism $A' \to A$ that lifts s. To do so we construct a tower of epimorphisms

$$\cdots \to A_i \to A_{i-1} \to \cdots \to A_0 \to A$$

Such that there are maps $t_n\colon A_i \to F_i$ that lift the maps $s_n\colon A \to G_i$. We can construct one as follows, due to the fact that the pullback of an epimorphism is an epimorphism in a topos let $X_0 = X \times_{G_0} F_0$ we get an epimorphism $X_0 \to X$

[1] can this be reduced to $F_0 \to G_0$ surjective and the product thing?

that lifts s_0, we can continue inductively with $X_{i+1} = X_i \times_{F_i \times_{G_i} \times G_{i+1}} F_{i+1}$. Diagramatically

$$\begin{tikzcd}
A_{i+1} \arrow[r, "t_{i+1}"] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & F_{i+1} \arrow[d, two heads, "{(f_{i+1}, h_{i+1})}"] & \\
A_i \arrow[r, "{(t_i, s_i \circ \pi_i)}"] \arrow[d, two heads, "\pi_i"] & F_i \times_{G_i} G_{i+1} \arrow[r] \arrow[d, two heads] \arrow[dr, phantom, "\lrcorner", very near start] & F_i \arrow[d, two heads, "h_i"] \\
A \arrow[r, "s_{i+1}"] & G_{i+1} \arrow[r, "g_{i+1}"] & G_i
\end{tikzcd}$$

Since X is replete taking the limit over this tower gives an epimorphism $\lim A_i \to A$ so that s factors through h, that is

$$\begin{tikzcd}
\lim A_i \arrow[r] \arrow[d, two heads] & \lim F \arrow[d] \\
A \arrow[r] & \lim G
\end{tikzcd}$$

Taking then $A = \lim G$ and $s = id$ proves the claim

This allows us to prove (2) quite simply as products are already left exact so we just need to check that for epimorphisms $h_n : F_n \to G_n$ we have $\prod_n F_n \to \prod_n G_n$ is epi. This follows from (1) by taking $\prod_n F_n = \lim \prod_{i<n} F_i$ and noting that finite products preserve epis already and the map

$$\prod_{i<n+1} F_i \to \prod_{i<n} F_i \times_{\prod_{i<n} G_i} \prod_{i<n+1} G_i$$

Can be checked to be epimorphic using the internal language of the topos as the fiber product is just $\{((g_0, ..., g_n), (f_0, ..., f_{n-1})) | g_i = h_i f_i\}$ so surjectivity is clear

To prove (3) we've already shown that the product agrees with the derived product so for any limit for a diagram with $t_n : f_{n+1} \to f_n$ we can write it as the limit of

$$\prod_n F_n \to^{t-id} \prod_n F_n$$

Hense giving an exact triangle

$$R \lim F_n \to \prod_n F_n \to \prod_n F_n \to$$

and so as the products are concentrated in degree 0, $R \lim F_n$ is concentrated in degrees $0, 1$

To prove (4) then we use the same exact triangle from before

$$R \lim F_n \to \prod_n F_n \to \prod_n F_n \to$$

It suffices to show that this $t - id$ is surjective as then $R\lim F_n$ will be concentrated in degree 0. For the sake of making this easier to read we suggestively define $F_n = G_n$ and $t - id$ is assumed to be a map

$$\prod_n F_n \to \prod_n G_n$$

So maps $F \to G$ are defined by this, and maps $F \to F, G \to G$ are understood to be identities/projections .The surjectivity of t means that the map induced by $t - id$

$$\prod_{i \leq n+1} F_i \to \prod_{i \leq n} G_i$$

is surjective as we can show this inductively using the internal language. Additionally as we did before the induced map

$$\prod_{i \leq n+2} F_i \to \prod_{i \leq n+1} G_i \times_{\prod_{i \leq n} G_i} \prod_{i \leq n+1} F_i$$

Is also surjective, this means by (1) the whole map is surjective $\square$

Definition 1.2. An object in a a topos is called

1. Compact if the 'underlying geometric structure' is compact, ie if the geometric morphism $X/a \to \mathrm{Sh}(*) = \mathcal{S}et$ is proper
2. Stable if for all morphisms $Y \to X$ with Y compact, the domain of the kernel pair $R \rightrightarrows Y$ of f is also compact
3. Coherent if it is compact and stable

I dont understand this notion so this section is hard

Definition 1.3 (Locally Weakly Contractible Topos)**.** An object F in a topos X is called weakly contractible if every epimorphism $G \twoheadrightarrow F$ has a section. We say that X is *locally weakly contractible* if each $a \in X$ admits an epimorphism $\bigsqcup Y_i \twoheadrightarrow X$ with Y_i coherent and weakly contractible

Proposition 1.3. *Let X be a locally weakly contractible topos. Then*

1. *X is replete*
2. *The derived category $D(X, \mathbb{Z})$ is compactly generated*
3. *Postnikov towers converge in the associated hypercomplete ∞-topos*

Proposition 1.4. *If X is a replete topos, then $D(X)$ is left-complete*

Proposition 1.5. *Let $f \colon A_\bullet \to A$ by a hyper cover in a replete topos X, then*

1. *The adjunction $id \to f_* f^*$ is an equivalence on $D(A)$*

2. *The adjunction $f_! f^* \to id$ is an eqivalence on $D(X)$*

3. *f^* induces an equivalence $D(X) \cong D_{cart}(X_\bullet) \subset D(X_\bullet)$ where $D(X_\bullet)$ is the derived category of the simplicial topos defined by $X_\bullet$ and $D_{cart}(X_\bullet)$ is the full subcategory spanned by cartesian complexes*

2 Lawvere Theories on Topoi

Definition 2.1. A *Lawvere Theory* is a category C with finite products and a distinguished element x so that each object is isomorphic to a finite power of x, ie $\forall y, y \cong x^{\times n}$

Definition 2.2. For a Laevere theory C, and a category with products S, *a model in S* is a product preserving functor $C \to S$. We define the category of such models as a full subcategory of the category of functors $C \to S$ and denote it

$$\mathrm{Fun}^{\times}(C, S)$$

Example 2.1. 1. On the category of finitely generated free groups we have that $F(n) \sqcup F(m) \cong F(m+n)$ so the opposite category of finitely generated free groups is a Lawvere theory with distinguished object $\mathbb{Z}$. A functor $F : \mathcal{FiniteFreeGroups}^{op} \to \mathcal{Set}$ then picks out a specific set $F(\mathbb{Z})$ along with a multiplication map $F(\mathbb{Z}) \times F(\mathbb{Z}) \to F(\mathbb{Z})$ induced by the map $1 \mapsto ab$ that satisfies the group laws by the pushforward of these laws in $\mathcal{FiniteFreeGroups}^{op}$

2. Similarly a ring is just a model in sets of the Lawvere theory of integer polynomial rings

3. Additionally for a field k a k algebra is a model in sets of the k-polynomial rings

Lemma 2.1. *For C a Lawvere theory. Suppose in addition it's coextensive, ie for any m, n the diagram*

$$\begin{tikzcd} x^0 & x^n \arrow[l] \\ x^m \arrow[u] & x^m \times x^n \arrow[l] \arrow[u] \end{tikzcd}$$

Is a pushout square.[2]
For any topos $X = \mathrm{Sh}(S)$ the following categories are equivalent

[2] I'm fairly sure you can assume this is the case always as x^0 doesn't affect the models for the theory and hense you can formally give it this property without having to worry

1. $\mathrm{Fun}^{\times}(C, X)$

2. $\mathrm{Sh}(S, \mathrm{Fun}^{\times}(C, \mathcal{S}et))$

Proof. First we put a topology on C^{op} so that everything here can be considered as sheaves. Define the topology τ_G on C^{op} the covers of A are finite families of morphisms $\{A_\lambda \leftarrow A\}_{\lambda \in \Lambda}$ so that the induced map

$$A \to \prod_{\Lambda} A_\lambda A_\lambda$$

Is an isomorphism, this is known as the Gaeta toplogy on C^{op}. Sheaves for this topology are thus functors $F\colon C \to \mathcal{S}et$ so that for any finite family $\{A_\lambda\}_{\lambda\in\Lambda}$ we have the equaliser

$$F(A) \to \prod_{\lambda\in\Lambda} F(A_\lambda) \rightrightarrows \prod_{\mu\lambda\in\Lambda} F(A_\mu \sqcup_A A_\lambda)$$

Where $A \cong \prod_{\lambda\in\Lambda} A_\lambda$. If each $F(A_\mu \sqcup_A A_\lambda)$ is terminal then we're ready as this is an equiliser if and only if $F(\prod_{\lambda\in\Lambda} A_\lambda) \cong \prod_{\lambda\in\Lambda} F(A_\lambda)$. Note that x^0 admits the empty cover so must map to a terminal object. So inductively it suffices to show that $x^0 \cong \mathrm{colim}(x^n \leftarrow x^n \times x^m \to x^m)$, this is true by assumptopm[3]

From here the proof is clear as the product of sites is well behaved so

$$\begin{aligned}\mathrm{Fun}^{\times}(C, X) &\cong \mathrm{Sh}(C^{op}, \mathrm{Sh}(S, \mathcal{S}et)) \\ &\cong \mathrm{Sh}(C^{op} \times S, \mathcal{S}et) \\ &\cong \mathrm{Sh}(S, \mathrm{Sh}(C^{op})) & \cong \mathrm{Sh}(S, \mathrm{Fun}^{\times}(C, \mathcal{S}et))\end{aligned}$$

$\square$

This allows us to define algebraic structures internal to a topos

Lemma 2.2. *As written for all the examples*

Proof. Any object F of (4) gives an object of (3) by taking as an object

$$F(\mathrm{Free}(1))$$

and for an n-ary operation $(\cdot, \ldots, \cdot)$, the morphism induced by

$$\mathrm{Free}(1) \to \mathrm{Free}(n)$$

that is in turn induced by by

$$1 \mapsto (1, \ldots, n)$$

[3] I'm not 100% on if this needs to be assumed, i can concieve of a world where x is a cone over the diagram but there is no morphism $x^0 \to x$ but I haven't written it down

The diagrams are satisfied as they hold in the Lawvere Theory. In addition any object of the Lawvere theory embeds in (3) as the image of the hom functor under this association and so (3) embeds in (4) by the restricted internal hom functor giving an equivalence. Note these embeddings are mutual inverses as the double compositions preserve the fundamental object and the maps on this object are precicely the images of the maps on the free objects by pulling back the map

$$\mathrm{hom}(\underline{\mathrm{Free}(n)}, M^k) \to \mathrm{hom}(\underline{\mathrm{Free}(m)}, M^j)$$

to a map

$$\mathrm{hom}(\underline{\mathrm{Free}(nk)}, M) \to \mathrm{hom}(\underline{\mathrm{Free}(mj)}, M)$$

Which by yoneda is just the data of some map in our original category so double application is an isomorphism on (3), and double application on (4) is precicely the yoneda mapping so an isomorphism

(3) is equivalent to (2) by pushing forward and pulling back the structure morphisms through hom.

(4) is equivalent to (1) by lemma $\square$