

Analytic Stacks

People

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1.4 Replete and locally weakly contractible topoi

Throughout this section the word "topos" refers to the category of sheaves on some site

Definition 1.1 (Replete Topos). A topos X is *replete* if epimorphisms are closed under sequential limits, that is for a functor $F : \mathbb{N}^{op} \rightarrow X$ with epimorphic transition maps $F_{n+1} \twoheadrightarrow F_n$, then the map $\lim F \rightarrow F_n$ is epic for each n

- Lemma 1.1.**
1. X is replete then for $x \in X$, X/x is replete
 2. X is replete if and only if there is a surjection $x \rightarrow 1$ and X/x is replete

Proof. To prove (1) we observe that the canonical functor $X/x \rightarrow X$ preserves sequential limits and epimorphisms so in X/x an epi sequential limit corresponds to an epi sequential limit in X and the maps $\lim F \rightarrow F_n$ are epi in X and hence epi in X/x

To prove (2) we note that (1) proves the forward direction, then for the other direction the map $g: x \rightarrow 1$ induced a base change map $g^*: X \cong X/1 \rightarrow X/x$ defined by $t \mapsto (t \times x \rightarrow x)$ which preserves limits and epimorphisms so the same argument at (1) applies \square

- Proposition 1.2** (Exactness in replete topoi).
1. For X a replete topos and $H: F \rightarrow G$ a map in $\text{Fun}(\mathbb{N}^{op}, X)$ where the components $h_i: F_i \rightarrow G_i$ and the induced maps $(f_{i+1}, h_{i+1}): F_{i+1} \rightarrow F_i \times_{G_i} G_{i+1}$ are epimorphisms for all i . Then $\lim F \rightarrow \lim G$ is an epimorphism¹
 2. Countable products are exact in a replete topos
 3. If X is a replete topos then the functor of \mathbb{N}^{op} -indexed limits has cohomological dimension 1
 4. If X is a replete topos and $F : \mathbb{N}^{op} \rightarrow \mathcal{A}\mathcal{B}(X)$ is a diagram with $F_{n+1} \rightarrow F_n$ surjective for every n then $\lim F_n \cong R\lim F_n$

Proof. To prove (1) Take any $A \in X$ and any map $s: A \rightarrow \lim G$ we want to find an epimorphism $A' \rightarrow A$ that lifts s . To do so we construct a tower of epimorphisms

$$\cdots \rightarrow A_i \rightarrow A_{i-1} \rightarrow \cdots \rightarrow A_0 \rightarrow A$$

Such that there are maps $t_n: A_i \rightarrow F_i$ that lift the maps $s_n: A \rightarrow G_i$. We can construct one as follows, due to the fact that the pullback of an epimorphism is an epimorphism in a topos let $X_0 = X \times_{G_0} F_0$ we get an epimorphism $X_0 \rightarrow X$

¹can this be reduced to $F_0 \rightarrow G_0$ surjective and the product thing?

that lifts s_0 , we can continue inductively with $X_{i+1} = X_i \times_{F_i \times_{G_i} G_{i+1}} F_{i+1}$. Diagrammatically

$$\begin{array}{ccc}
A_{i+1} & \xrightarrow{t_{i+1}} & F_{i+1} \\
\downarrow & \lrcorner & \downarrow (f_{i+1}, h_{i+1}) \\
A_i & \xrightarrow{(t_i, s_i \circ \pi_i)} & F_i \times_{G_i} G_{i+1} \longrightarrow F_i \\
\downarrow \pi_i & & \downarrow \lrcorner \\
A & \xrightarrow{s_{i+1}} & G_{i+1} \xrightarrow{g_{i+1}} G_i
\end{array}$$

Since X is replete taking the limit over this tower gives an epimorphism $\lim A_i \rightarrow A$ so that s factors through h , that is

$$\begin{array}{ccc}
\lim A_i & \longrightarrow & \lim F \\
\downarrow & & \downarrow \\
A & \longrightarrow & \lim G
\end{array}$$

Taking then $A = \lim G$ and $s = id$ proves the claim

This allows us to prove (2) quite simply as products are already left exact so we just need to check that for epimorphisms $h_n : F_n \rightarrow G_n$ we have $\prod_n F_n \rightarrow \prod_n G_n$ is epi. This follows from (1) by taking $\prod_n F_n = \lim \prod_{i < n} F_i$ and noting that finite products preserve epis already and the map

$$\prod_{i < n+1} F_i \rightarrow \prod_{i < n} F_i \times_{\prod_{i < n} G_i} \prod_{i < n+1} G_i$$

Can be checked to be epimorphic using the internal language of the topos as the fiber product is just $\{((g_0, \dots, g_n), (f_0, \dots, f_{n-1})) | g_i = h_i f_i\}$ so surjectivity is clear

To prove (3) we've already shown that the product agrees with the derived product so for any limit for a diagram with $t_n : f_{n+1} \rightarrow f_n$ we can write it as the limit of

$$\prod_n F_n \xrightarrow{t - id} \prod_n F_n$$

Hence giving an exact triangle

$$R\lim F_n \rightarrow \prod_n F_n \rightarrow \prod_n F_n \rightarrow$$

and so as the products are concentrated in degree 0, $R\lim F_n$ is concentrated in degrees 0, 1

To prove (4) then we use the same exact triangle from before

$$R\lim F_n \rightarrow \prod_n F_n \rightarrow \prod_n F_n \rightarrow$$

It suffices to show that this $t - id$ is surjective as then $R\lim F_n$ will be concentrated in degree 0. For the sake of making this easier to read we suggestively define $F_n = G_n$ and $t - id$ is assumed to be a map

$$\prod_n F_n \rightarrow \prod_n G_n$$

So maps $F \rightarrow G$ are defined by this, and maps $F \rightarrow F, G \rightarrow G$ are understood to be identities/projections. The surjectivity of t means that the map induced by $t - id$

$$\prod_{i \leq n+1} F_i \rightarrow \prod_{i \leq n} G_i$$

is surjective as we can show this inductively using the internal language. Additionally as we did before the induced map

$$\prod_{i \leq n+2} F_i \rightarrow \prod_{i \leq n+1} G_i \times_{\prod_{i \leq n} G_i} \prod_{i \leq n+1} F_i$$

Is also surjective, this means by (1) the whole map is surjective \square

Definition 1.2. An object in a a topos is called

1. Compact if the ‘underlying geometric structure’ is compact, ie if the geometric morphism $X/a \rightarrow \text{Sh}(*) = \mathcal{S}\text{et}$ is proper
2. Stable if for all morphisms $Y \rightarrow X$ with Y compact, the domain of the kernel pair $R \rightrightarrows Y$ of f is also compact
3. Coherent if it is compact and stable

I dont understand this notion so this section is hard

Definition 1.3 (Locally Weakly Contractible Topos). An object F in a topos X is called weakly contractible if every epimorphism $G \twoheadrightarrow F$ has a section. We say that X is *locally weakly contractible* if each $a \in X$ admits an epimorphism $\bigsqcup Y_i \twoheadrightarrow X$ with Y_i coherent and weakly contractible

Proposition 1.3. Let X be a locally weakly contractible topos. Then

1. X is replete
2. The derived category $D(X, \mathbb{Z})$ is compactly generated
3. Postnikov towers converge in the associated hypercomplete ∞ -topos

Proposition 1.4. If X is a replete topos, then $D(X)$ is left-complete

Proposition 1.5. Let $f: A_\bullet \rightarrow A$ by a hyper cover in a replete topos X , then

1. The adjunction $\text{id} \rightarrow f_* f^*$ is an equivalence on $D(A)$
2. The adjunction $f_! f^* \rightarrow \text{id}$ is an equivalence on $D(X)$
3. f^* induces an equivalence $D(X) \cong D_{\text{cart}}(X_\bullet) \subset D(X_\bullet)$ where $D(X_\bullet)$ is the derived category of the simplicial topos defined by X_\bullet and $D_{\text{cart}}(X_\bullet)$ is the full subcategory spanned by cartesian complexes