3. On replete topoi

A topos is the category of sheaves on a site, up to equivalence, as in [**SGA72a**]. We will study in §3.1 a general property of topoi that implies good behaviour for the lim and R lim functors, as well as unbounded cohomological descent, as discussed in §3.3. A special subclass of such topoi with even better completeness properties is isolated in §3.2; this class is large enough for all applications later in the paper. In §3.4 and §3.5, with a view towards studying complexes of ℓ-adic sheaves on the pro-étale site, we study derived completions of rings and modules in a replete topos; the repleteness ensures no interference from higher derived limits while performing completions, so the resulting theory is as good as in the punctual case.

3.1. Definition and first consequences. — The key definition is:

Definition 3.1.1. — *A topos $\mathcal{X}$ is* replete *if surjections in $\mathcal{X}$ are closed under sequential limits,* i.e., *if $F : \mathbf{N}^{\mathrm{op}} \to \mathcal{X}$ is a diagram with $F_{n+1} \to F_n$ surjective for all n, then $\lim F \to F_n$ is surjective for each n.*

Before giving examples, we mention two recognition mechanisms for replete topoi:

Lemma 3.1.2. — *If $\mathcal{X}$ is a replete topos and $X \in \mathcal{X}$, then $\mathcal{X}_{/X}$ is replete.*

Proof. — This follows from the fact that the forgetful functor $\mathcal{X}_{/X} \to \mathcal{X}$ commutes with connected limits and preserves surjections. □

Lemma 3.1.3. — *A topos $\mathcal{X}$ is replete if and only if there exists a surjection $X \to 1$ and $\mathcal{X}_{/X}$ is replete.*

Proof. — This follows from two facts: (a) limits commute with limits, and (b) a map $F \to G$ in $\mathcal{X}$ is a surjection if and only if it is so after base changing to X. □

Example 3.1.4. — The topos of sets is replete, and hence so is the topos of presheaves on a small category. As a special case, the classifying topos of a finite group G (which is simply the category of presheaves on $B(G)$) is replete.

Example 3.1.5. — Let k be a field with a fixed separable closure $\overline{k}$. Then $\mathcal{X} = \mathrm{Shv}(\mathrm{Spec}(k)_{\acute{e}t})$ is replete if and only if $\overline{k}$ is a finite extension of k [3]. One direction is clear: if $\overline{k}/k$ is finite, then $\mathrm{Spec}(\overline{k})$ covers the final object of $\mathcal{X}$ and $\mathcal{X}_{/\mathrm{Spec}(\overline{k})} \simeq \mathrm{Set}$, so $\mathcal{X}$ is replete by Lemma 3.1.3. Conversely, assume that $\mathcal{X}$ is replete with $\overline{k}/k$ infinite. Then there is a tower $k = k_0 \hookrightarrow k_1 \hookrightarrow k_2 \hookrightarrow \dots$ of strictly increasing finite separable extensions of k. The associated diagram $\dots \to \mathrm{Spec}(k_2) \to \mathrm{Spec}(k_1) \to \mathrm{Spec}(k_0)$ of surjections has an empty limit in $\mathcal{X}$, contradicting repleteness.

3. Recall that this happens only if k is algebraically closed or real closed; in the latter case, $k(\sqrt{-1})$ is an algebraic closure of k.

Remark 3.1.6. — Replacing $\mathbf{N}^{\mathrm{op}}$ with an arbitrary small cofiltered category in the definition of replete topoi leads to an empty theory: there are cofiltered diagrams of sets with surjective transition maps and empty limits. For example, consider the poset I of finite subsets of an uncountable set T ordered by inclusion, and $F : I^{\mathrm{op}} \to \mathrm{Set}$ defined by

$$F(S) = \{f \in \mathrm{Hom}(S, \mathbf{Z}) \mid f \text{ injective}\}.$$

Then F is a cofiltered diagram of sets with surjective transition maps, and $\lim F = \varnothing$.

Example 3.1.5 shows more generally that the Zariski (or étale, Nisnevich, smooth, fppf) topoi of most schemes fail repleteness due to "finite presentation" constraints. Nevertheless, there is an interesting geometric source of examples:

Example 3.1.7. — The topos $\mathcal{X}$ of fpqc sheaves on the category of schemes[(4)] is replete. Given a diagram $\cdots \to F_{n+1} \to F_n \to \cdots \to F_1 \to F_0$ of fpqc sheaves with $F_n \to F_{n-1}$ surjective, we want $\lim F_n \to F_0$ to be surjective. For any affine $\mathrm{Spec}(A)$ and a section $s_0 \in F_0(\mathrm{Spec}(A))$, there is a faithfully flat map $A \to B_1$ such that s_0 lifts to an $s_1 \in F_1(\mathrm{Spec}(B_1))$. Inductively, for each $n \geqslant 0$, there exist faithfully flat maps $A \to B_n$ compatible in n and sections $s_n \in F_n(\mathrm{Spec}(B_n))$ such that s_n lifts s_{n-1}. Then $B = \mathrm{colim}_n B_n$ is a faithfully flat A-algebra with $s_0 \in F_0(\mathrm{Spec}(A))$ lifting to an $s \in \lim F_n(\mathrm{Spec}(B))$, which proves repleteness as $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ is an fpqc cover.

The next lemma records a closure property enjoyed by surjections in a replete topos.

Lemma 3.1.8. — *Let $\mathcal{X}$ be a replete topos, and let $F \to G$ be a map in $\mathrm{Fun}(\mathbf{N}^{\mathrm{op}}, \mathcal{X})$. Assume that the induced maps $F_i \to G_i$ and $F_{i+1} \to F_i \times_{G_i} G_{i+1}$ are surjective for each i. Then $\lim F \to \lim G$ is surjective.*

Proof. — Fix an $X \in \mathcal{X}$ and a map $s : X \to \lim G$ determined by a compatible sequence $\{s_n : X \to G_n\}$ of maps. By induction, one can show that there exists a tower of surjections $\cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to X$ and maps $t_n : X_n \to F_n$ compatible in n such that t_n lifts s_n. In fact, one may take $X_0 = X \times_{G_0} F_0$, and

$$X_{n+1} = X_n \times_{F_n \times_{G_n} G_{n+1}} F_{n+1}.$$

The map $X' := \lim_i X_i \to X$ is surjective by repleteness of $\mathcal{X}$. Moreover, the compatibility of the t_n's gives a map $t : X' \to \lim F$ lifting s, which proves the claim. $\square$

We now see some of the benefits of working in a replete topos. First, products behave well:

Proposition 3.1.9. — *Countable products are exact in a replete topos.*

4. To avoid set-theoretic problems, one may work with countably generated affine schemes over a fixed affine base scheme.

Proof. — Given surjective maps $f_n : F_n \to G_n$ in $\mathcal{X}$ for each $n \in \mathbf{N}$, we want $f : \prod_n F_n \to \prod_n G_n$ to be surjective. This follows from Lemma 3.1.8 as $f = \lim \prod_{i<n} f_i$; the condition from the lemma is trivial to check in this case. $\square$

In a similar vein, inverse limits behave like in sets:

Proposition 3.1.10. — *If $\mathcal{X}$ is a replete topos and $F : \mathbf{N}^{\mathrm{op}} \to \mathrm{Ab}(\mathcal{X})$ is a diagram with $F_{n+1} \to F_n$ surjective for all n, then $\lim F_n \simeq \mathrm{R}\lim F_n$.*

Proof. — By Proposition 3.1.9, the product $\prod_n F_n \in \mathcal{X}$ computes the derived product in $D(\mathcal{X})$. This gives an exact triangle

$$\mathrm{R}\lim F_n \longrightarrow \prod_n F_n \xrightarrow{t-\mathrm{id}} \prod_n F_n,$$

where $t : F_{n+1} \to F_n$ is the transition map. It thus suffices to show that $s := t - \mathrm{id}$ is surjective. Set $G_n = \prod_{i \leqslant n} F_n$, $H_n = G_{n+1}$, and let $s_n : H_n \to G_n$ be the map induced by $t - \mathrm{id}$. The surjectivity of t shows that s_n is surjective. Moreover, the surjectivity of t also shows that $H_{n+1} \to G_{n+1} \times_{G_n} H_n$ is surjective, where the fibre product is computed using $s_n : H_n \to G_n$ and the projection $G_{n+1} \to G_n$. In fact, the fibre product is $H_n \times F_{n+1}$ and $H_{n+1} \to H_n \times F_{n+1}$ is $(\mathrm{pr}, t - \mathrm{id})$. By Lemma 3.1.8, it follows that $s = \lim s_n$ is also surjective. $\square$

Proposition 3.1.11. — *If $\mathcal{X}$ is a replete topos, then the functor of $\mathbf{N}^{\mathrm{op}}$-indexed limits has cohomological dimension* 1.

Proof. — For a diagram $F : \mathbf{N}^{\mathrm{op}} \to \mathrm{Ab}(\mathcal{X})$, we want $\mathrm{R}\lim F_n \in D^{[0,1]}(\mathcal{X})$. By definition, there is an exact triangle

$$\mathrm{R}\lim F_n \longrightarrow \prod_n F_n \longrightarrow \prod_n F_n$$

with the last map being the difference of the identity and transition maps, and the products being derived. By Proposition 3.1.9, we can work with naive products instead, whence the claim is clear by long exact sequences. $\square$

Question 3.1.12. — Do Postnikov towers converge in the hypercomplete ∞-topos of sheaves of spaces (as in [**Lur09**, §6.5]) on a replete topos?

3.2. Locally weakly contractible topoi. — We briefly study an exceptionally well-behaved subclass of replete topoi:

Definition 3.2.1. — *An object F of a topos $\mathcal{X}$ is called* weakly contractible *if every surjection $G \to F$ has a section. We say that $\mathcal{X}$ is* locally weakly contractible *if it has enough weakly contractible coherent objects,* i.e., *each $X \in \mathcal{X}$ admits a surjection $\cup_i Y_i \to X$ with Y_i a coherent weakly contractible object.*

The pro-étale topology will give rise to such topoi. A more elementary example is:

Example 3.2.2. — The topos $\mathcal{X} = \mathrm{Set}$ is locally weakly contractible: the singleton set S is weakly contractible coherent, and every set is covered by a disjoint union of copies of S.

The main completeness and finiteness properties of such topoi are:

Proposition 3.2.3. — *Let $\mathcal{X}$ be a locally weakly contractible topos. Then*

1. *$\mathcal{X}$ is replete.*
2. *The derived category $D(\mathcal{X}) = D(\mathcal{X}, \mathbf{Z})$ is compactly generated.*
3. *Postnikov towers converge in the associated hypercomplete ∞-topos. (Cf.* [**Lur09**].*)*

Proof. — For (1), note that a map $F \to G$ in $\mathcal{X}$ is surjective if and only if $F(Y) \to G(Y)$ is so for each weakly contractible Y; the repleteness condition is then immediately deduced. For (2), given $j : Y \to 1_{\mathcal{X}}$ in $\mathcal{X}$ with Y weakly contractible coherent, one checks that $\mathrm{Hom}(j_!\mathbf{Z}, -) = H^0(Y, -)$ commutes with arbitrary direct sums in $D(\mathcal{X})$, so $j_!\mathbf{Z}$ is compact; as Y varies, this gives a generating set of $D(\mathcal{X})$ by assumption on $\mathcal{X}$, proving the claim. For (3), first note that the functor $F \mapsto F(Y)$ is exact on sheaves of spaces whenever Y is weakly contractible. Hence, given such an F and point $* \in F(Y)$ with Y weakly contractible, one has $\pi_i(F(Y), *) = \pi_i(F, *)(Y)$. This shows that $F \simeq \lim_n \tau_{\leqslant n} F$ on $\mathcal{X}$, which proves hypercompleteness. (Cf. [**Lur09**, Proposition 7.2.1.10].) $\square$

3.3. Derived categories, Postnikov towers, and cohomological descent

We first recall the following definition:

Definition 3.3.1. — *Given a topos $\mathcal{X}$, we define the* left-completion $\widehat{D}(\mathcal{X})$ *of $D(\mathcal{X})$ as the full subcategory of $D(\mathcal{X}^{\mathbf{N}})$ spanned by projective systems $\{K_n\}$ satisfying:*

1. *$K_n \in D^{\geqslant -n}(\mathcal{X})$.*
2. *The map $\tau^{\geqslant -n} K_{n+1} \to K_n$ induced by the transition map $K_{n+1} \to K_n$ and (1) is an equivalence.*

We say that $D(\mathcal{X})$ is left-complete *if the map $\tau : D(\mathcal{X}) \to \widehat{D}(\mathcal{X})$ defined by $K \mapsto \{\tau^{\geqslant -n} K\}$ is an equivalence.*

Left-completeness is extremely useful in accessing an unbounded derived category as Postnikov towers converge:

Lemma 3.3.2. — *The functor* $\mathrm{R}\lim : \widehat{D}(\mathcal{X}) \hookrightarrow D(\mathcal{X}^{\mathbf{N}}) \to D(\mathcal{X})$ *provides a right adjoint to τ. In particular, if $D(\mathcal{X})$ is left-complete, then $K \simeq \mathrm{R}\lim \tau^{\geqslant -n} K$ for any $K \in D(\mathcal{X})$.*

Proof. — Fix $K \in D(\mathcal{X})$ and $\{L_n\} \in \widehat{D}(\mathcal{X})$. Then we claim that

$$\begin{aligned}\mathrm{RHom}_{D(\mathcal{X})}(K, \mathrm{R}\lim L_n) \simeq \mathrm{R}\lim \mathrm{RHom}_{D(\mathcal{X})}(K, L_n) &\simeq \mathrm{R}\lim \mathrm{RHom}_{D(\mathcal{X})}(\tau^{\geqslant -n} K, L_n) \\ &\simeq \mathrm{RHom}_{\widehat{D}(\mathcal{X})}(\tau(K), \{L_n\}).\end{aligned}$$

This clearly suffices to prove the lemma. Moreover, the first two equalities are formal. For the last one, recall that if $F, G \in \mathrm{Ab}(\mathcal{X}^{\mathbf{N}})$, then there is an exact sequence

$$1 \longrightarrow \mathrm{Hom}(F, G) \longrightarrow \prod_n \mathrm{Hom}(F_n, G_n) \longrightarrow \prod_n \mathrm{Hom}(F_{n+1}, G_n),$$

where the first map is the obvious one, while the second map is the difference of the two maps $F_{n+1} \to F_n \to G_n$ and $F_{n+1} \to G_{n+1} \to G_n$. One can check that if $F, G \in \mathrm{Ch}(\mathcal{X}^{\mathbf{N}})$, and G is chosen to be K-injective, then the above sequence gives an exact triangle

$$\mathrm{RHom}(F, G) \longrightarrow \prod_n \mathrm{RHom}(F_n, G_n) \longrightarrow \prod_n \mathrm{RHom}(F_{n+1}, G_n).$$

In the special case where $F, G \in \widehat{D}(\mathcal{X})$, one has $\mathrm{RHom}(F_{n+1}, G_n) = \mathrm{RHom}(F_n, G_n)$ by adjointness of truncations, which gives the desired equality. $\square$

Classically studied topoi have left-complete derived categories only under (local) finite cohomological dimension constraints; see Proposition 3.3.7 for a criterion, and Example 3.3.5 for a typical example of the failure of left-completeness for the simplest infinite-dimensional objects. The situation for replete topoi is much better:

Proposition 3.3.3. — *If $\mathcal{X}$ is a replete topos, then $D(\mathcal{X})$ is left-complete.*

Proof. — We repeatedly use the following fact: limits and colimits in the abelian category $\mathrm{Ch}(\mathrm{Ab}(\mathcal{X}))$ are computed termwise. First, we show that $\tau : D(\mathcal{X}) \to \widehat{D}(\mathcal{X})$ is fully faithful. By the adjunction from Lemma 3.3.2, it suffices to show that $K \simeq \mathrm{R}\lim \tau^{\geqslant -n} K$ for any $K \in D(\mathcal{X})$. Choose a complex $I \in \mathrm{Ch}(\mathrm{Ab}(\mathcal{X}))$ lifting $K \in D(\mathcal{X})$. Then $\prod_n \tau^{\geqslant -n} I \in \mathrm{Ch}(\mathrm{Ab}(\mathcal{X}))$ lifts the derived product $\prod_n \tau^{\geqslant -n} K \in D(\mathcal{X})$ by Proposition 3.1.9. Since $I \simeq \lim \tau^{\geqslant -n} I \in \mathrm{Ch}(\mathrm{Ab}(\mathcal{X}))$, it suffices as in Proposition 3.1.10 to show that

$$\prod_n \tau^{\geqslant -n} I \xrightarrow{t-\mathrm{id}} \prod_n \tau^{\geqslant -n} I$$

is surjective in $\mathrm{Ch}(\mathrm{Ab}(\mathcal{X}))$, where we write t for the transition maps. Since surjectivity in $\mathrm{Ch}(\mathrm{Ab}(\mathcal{X}))$ can be checked termwise, this follows from the proof of Proposition 3.1.10 as $\tau^{\geqslant -n} I \overset{t-\mathrm{id}}{\to} \tau^{\geqslant -(n-1)} I$ is termwise surjective.

For essential surjectivity of τ, it suffices to show: given $\{K_n\} \in \widehat{D}(\mathcal{X})$, one has $K_n \simeq \tau^{\geqslant -n} \mathrm{R}\lim K_n$. Choose a K-injective complex $\{I_n\} \in \mathrm{Ch}(\mathrm{Ab}(\mathcal{X}^{\mathbf{N}}))$ representing $\{K_n\}$. Then $\prod_n I_n \in \mathrm{Ch}(\mathrm{Ab}(\mathcal{X}))$ lifts $\prod_n K_n$ (the derived product). Moreover, by K-injectivity, the transition maps $I_{n+1} \to I_n$ are (termwise) surjective. Hence, the map

$$\prod_n I_n \xrightarrow{t-\mathrm{id}} \prod_n I_n$$

in $\mathrm{Ch}(\mathrm{Ab}(\mathcal{X}))$ is surjective by the argument in the proof of Proposition 3.1.10, and its kernel complex K computes $\mathrm{R}\lim K_n$. We must show that $H^i(K) \simeq H^i(K_i)$ for each $i \in \mathbf{N}$. Calculating cohomology and using the assumption $\{K_n\} \in \widehat{D}(\mathcal{X}) \subset D(\mathcal{X}^{\mathbf{N}})$ shows that

$$H^i\Big(\prod_n I_n\Big) = \prod_n H^i(I_n) = \prod_{n \geqslant i} H^i(I_n) = \prod_{n \geqslant i} H^i(K_i)$$

for each $i \in \mathbf{N}$; here we crucially use Proposition 3.1.9 to distribute H^i over $\prod$. The map $H^i(t - \mathrm{id})$ is then easily seen to be split surjective with kernel $\lim H^i(K_n) \simeq \lim H^i(K_i) \simeq H^i(K_i)$, which proves the claim. ☐

If repleteness is dropped, it is easy to give examples where $D(\mathcal{X})$ is not left-complete.

Example 3.3.4. — Let $G = \prod_{n \geqslant 1} \mathbf{Z}_p$, and let $\mathcal{X}$ be the topos associated to the category $B(G)$ of finite G-sets (topologized in the usual way). We will show that $D(\mathcal{X})$ is not left-complete. More precisely, we will show that $K \to \widehat{K} := \mathrm{R}\lim \tau^{\geqslant -n} K$ does not have a section for $K = \oplus_{n \geqslant 1} \mathbf{Z}/p^n[n] \in D(\mathcal{X})$; here $\mathbf{Z}/p^n$ is given the trivial G-action.

For each open subgroup $H \subset G$, we write $X_H \in B(G)$ for the G-set G/H given the left G-action, and let $I^{\mathrm{op}} \subset B(G)$ be the (cofiltered) full subcategory spanned by the X_H's. The functor $p^*(\mathcal{F}) = \mathrm{colim}_I\, \mathcal{F}(X_H)$ commutes with finite limits and all small colimits, and hence comes from a point $p : * \to \mathcal{X}$. Deriving gives $p^*L = \mathrm{colim}_I\, \mathrm{R}\Gamma(X_H, L)$ for any $L \in D(\mathcal{X})$, and so $H^0(p^*L) = \mathrm{colim}_I\, H^0(X_H, L)$. In particular, if $L_1 \to L_2$ has a section, so does

$$\operatorname*{colim}_I H^0(X_H, L_1) \longrightarrow \operatorname*{colim}_I H^0(X_H, L_2).$$

If $\pi : \mathcal{X} \to \mathrm{Set}$ denotes the constant map, then $K = \pi^* K'$ where $K' = \oplus_{n \geqslant 1} \mathbf{Z}/p^n[n] \in D(\mathrm{Ab})$, so

$$\operatorname*{colim}_I H^0(X_H, K) = H^0(p^*K) = H^0(p^*\pi^*K') = H^0(K') = 0.$$

Since $\tau^{\geqslant -n} K \simeq \oplus_{i \leqslant n} \mathbf{Z}/p^i[i] \simeq \prod_{i \leqslant n} \mathbf{Z}/p^i[i]$, commuting limits shows that $\widehat{K} \simeq \prod_{n \geqslant 1} \mathbf{Z}/p^n[n]$ (where the product is derived), and so $\mathrm{R}\Gamma(X_H, \widehat{K}) \simeq \prod_{n \geqslant 1} \mathrm{R}\Gamma(X_H, \mathbf{Z}/p^n[n])$. In particular, it suffices to show that

$$H^0(p^*\widehat{K}) = \operatorname*{colim}_I \prod_{n \geqslant 1} H^n(X_H, \mathbf{Z}/p^n)$$

is not 0. Let $\alpha_n \in H^n(X_G, \mathbf{Z}/p^n) = H^n(\mathcal{X}, \mathbf{Z}/p^n)$ be the pullback of a generator of $H^n(B(\prod_{i=1}^n \mathbf{Z}_p), \mathbf{Z}/p^n) \simeq \otimes_{i=1}^n H^1(B(\mathbf{Z}_p), \mathbf{Z}/p^n)$ under the projection $f_n : G \to \prod_{i=1}^n \mathbf{Z}_p$. Then α_n has exact order p^n as f_n has a section, so $\alpha := (\alpha_n) \in \prod_{n \geqslant 1} H^n(\mathcal{X}, \mathbf{Z}/p^n)$ has infinite order. Its image α' in $H^0(p^*\widehat{K})$ is 0 if and only if there exists an open normal subgroup $H \subset G$ such that α restricts to 0 in $\prod_n H^n(X_H, \mathbf{Z}/p^n)$. Since $X_H \to X_G$ is a finite cover of degree $[G : H]$, a transfer argument then implies that α is annihilated by $[G : H]$, which is impossible, whence $\alpha' \neq 0$.

Remark 3.3.5. — The argument of Example 3.3.4 is fairly robust: it also applies to the étale topos of $X = \mathrm{Spec}(k)$ with k a field provided there exist $M_n \in \mathrm{Ab}(X_{\text{ét}})$ for infinitely many $n \geqslant 1$ such that $H^n(\mathcal{X}, M_n)$ admits a class α_n with $\lim \mathrm{ord}(\alpha_n) = \infty$. In particular, this shows that $D(\mathrm{Spec}(k)_{\text{ét}})$ is not left-complete for $k = \mathbf{C}(x_1, x_2, x_3, \dots)$.

Thanks to left-completeness, cohomological descent in a replete topos is particularly straightforward:

Proposition 3.3.6. — *Let $f : X_\bullet \to X$ be a hypercover in a replete topos $\mathcal{X}$. Then*

1. *The adjunction* $\mathrm{id} \to f_* f^*$ *is an equivalence on $D(X)$.*
2. *The adjunction* $f_! f^* \to \mathrm{id}$ *is an equivalence on $D(X)$.*
3. *f^* induces an equivalence $D(X) \simeq D_{\mathrm{cart}}(X_\bullet)$.*

Here we write $D(Y) = D(\mathrm{Ab}(\mathcal{X}_{/Y}))$ for any $Y \in \mathcal{X}$. Then $D(X_\bullet)$ is the derived category of the simplicial topos defined by $X_\bullet$, and $D_{\mathrm{cart}}(X_\bullet)$ is the full subcategory spanned by complexes K which are *Cartesian*, *i.e.*, for any map $s : [n] \to [m]$ in Δ, the transition maps $s^*(K|_{X_n}) \to K|_{X_m}$ are equivalences. The usual pushforward then gives $f_* : D(X_\bullet) \to D(X)$ right adjoint to the pullback $f^* : D(X) \to D(X_\bullet)$ given informally *via* $(f^*K)|_{X_n} = K|_{X_n}$. By the adjoint functor theorem, there is a left adjoint $f_! : D(X_\bullet) \to D(X)$ as well. When restricted to $D_{\mathrm{cart}}(X_\bullet)$, one may describe $f_!$ informally as follows. For each Cartesian K and any map $s : [n] \to [m]$ in Δ, the equivalence $s^*(K|_{X_n}) \simeq K|_{X_m}$ has an adjoint map $K|_{X_m} \to s_!(K|_{X_n})$. Applying !-pushforward along each $X_n \to X$ then defines a simplicial object in $D(X)$ whose homotopy-colimit computes $f_! K$.

Proof. — We freely use that homotopy-limits and homotopy-colimits in $D(X_\bullet)$ are computed "termwise." Moreover, for any map $g : Y \to X$ in $\mathcal{X}$, the pullback g^* is exact and commutes with such limits and colimits (as it has a left adjoint $g_!$ and a right adjoint g_*). Hence $f^* : D(X) \to D(X_\bullet)$ also commutes with such limits and colimits.

1. For any $K \in \mathrm{Ab}(X)$, one has $K \simeq f_* f^* K$ by the hypercover condition. Passing to filtered colimits shows the same for $K \in D^+(X)$. For general $K \in D(X)$, we have $K \simeq \mathrm{R}\lim \tau^{\geqslant -n} K$ by repleteness. By exactness of f^* and repleteness of each X_n, one has $f^* K \simeq \mathrm{R}\lim f^* \tau^{\geqslant -n} K$. Pushing forward then proves the claim.
2. This follows formally from (1) by adjunction.
3. The functor $f^* : D(X) \to D_{\mathrm{cart}}(X_\bullet)$ is fully faithful by (1) and adjunction. Hence, it suffices to show that any $K \in D_{\mathrm{cart}}(X_\bullet)$ comes from $D(X)$. The claim is well-known for $K \in D^+_{\mathrm{cart}}(X_\bullet)$ (without assuming repleteness). For general K, by repleteness, we have $K \simeq \mathrm{R}\lim \tau^{\geqslant -n} K$. Since the condition of being Cartesian on a complex is a condition on cohomology sheaves, the truncations $\tau^{\geqslant -n} K$ are Cartesian, and hence come from $D(X)$. The claim follows as $D(X) \subset D(X_\bullet)$ is closed under homotopy-limits. ∎

We end by recording a finite dimensionality criterion for left-completeness:

Proposition 3.3.7. — *Let $\mathcal{X}$ be a topos, and fix $K \in D(\mathcal{X})$.*

1. *Given $U \in \mathcal{X}$ with $\Gamma(U,-)$ exact, one has $\mathrm{R}\Gamma(U,K) \simeq \mathrm{R}\lim \mathrm{R}\Gamma(U,\tau^{\geqslant -n}K)$.*
2. *If there exists $d \in \mathbf{N}$ such that $\mathcal{H}^i(K)$ has cohomological dimension $\leqslant d$ locally on $\mathcal{X}$ for all i, then $D(\mathcal{X})$ is left-complete.*

Proof. — For (1), by exactness, $\mathrm{R}\Gamma(U,K)$ is computed by $I(U)$ where $I \in \mathrm{Ch}(\mathcal{X})$ is *any* chain complex representing K. Now $D(\mathrm{Ab})$ is left-complete, so $I(U) \simeq \mathrm{R}\lim \tau^{\geqslant -n} I(U)$. As $\Gamma(U,-)$ is exact, it commutes with truncations, so the claim follows. (2) follows from [**Sta**, Tag 0719]. □

3.4. Derived completions of f-adic rings in a replete topos. — In this section, we fix a replete topos $\mathcal{X}$, and a ring $R \in \mathcal{X}$ with an ideal $I \subset R$ that is locally finitely generated, *i.e.*, there exists a cover $\{U_i \to 1_{\mathcal{X}}\}$ such that $I|_{U_i}$ is generated by finitely many sections of $I(U_i)$. Given $U \in \mathcal{X}$, $x \in R(U)$ and $K \in D(\mathcal{X}_{/U}, R)$, we write $T(K,x) := \mathrm{R}\lim(\dots \xrightarrow{x} K \xrightarrow{x} K \xrightarrow{x} K) \in D(\mathcal{X}_{/U}, R)$.

Definition 3.4.1. — *We say that $M \in \mathrm{Mod}_R$ is* classically I-complete *if $M \simeq \lim M/I^n M$; write $\mathrm{Mod}_{R,\mathrm{comp}} \subset \mathrm{Mod}_R$ for the full subcategory of such M. We say that $K \in D(\mathcal{X}, R)$ is* derived I-complete *if for each $U \in \mathcal{X}$ and $x \in I(U)$, we have $T(K|_U, x) = 0$; write $D_{\mathrm{comp}}(\mathcal{X}, R) \subset D(\mathcal{X}, R)$ for the full subcategory of such K.*

It is easy to see that $D_{\mathrm{comp}}(\mathcal{X}, R)$ is a triangulated subcategory of $D(\mathcal{X}, R)$. Moreover, for any $U \in \mathcal{X}$, the restriction $D(\mathcal{X}, R) \to D(\mathcal{X}_{/U}, R)$ commutes with homotopy-limits, and likewise for R-modules. Hence, both the above notions of completeness localise on $\mathcal{X}$. Our goal is to compare these completeness conditions for modules, and relate completeness of a complex to that of its cohomology groups. The main result for modules is:

Proposition 3.4.2. — *An R-module $M \in \mathrm{Mod}_R$ is classically I-complete if and only if it is I-adically separated and derived I-complete.*

Remark 3.4.3. — The conditions of Proposition 3.4.2 are not redundant: there exist derived I-complete R-modules M which are not I-adically separated, and hence not classically complete. In fact, there exists a ring R with principal ideals I and J such that R is classically I-complete while the quotient R/J is not I-adically separated; note that $R/J = \mathrm{cok}(R \to R)$ is derived I-complete by Lemma 3.4.14.

The result for complexes is:

Proposition 3.4.4. — *An R-complex $K \in D(\mathcal{X}, R)$ is derived I-complete if and only if each $H^i(K)$ is so.*

Remark 3.4.5. — For $\mathcal{X} = \mathrm{Set}$, one can find Proposition 3.4.4 in [**Lur11**].