Assignment

The mathematical constant π is the ratio of circuference of a circle to its diameter.

`https://en.wikipedia.org/wiki/Pi`

This has been known to be a constant for a long time. We know today that π is transendental (not a solution of any algebraic equation) with no periods for its digits. The first 50 digits after the decimal point are:

$$\pi = 3.\underbrace{1415926535}_{1-10}\underbrace{8979323846}_{11-20}\underbrace{2643383279}_{21-30}\underbrace{5028841971}_{31-40}\underbrace{6939937510}_{41-50}$$

In this assignment, you will investigate the frequency distribution of digits in the fractional part of π. Let $\pi(n)$ denote π with n digits after the decimal point. For example, $\pi(2) = 3.14$, $\pi(3) = 3.142$, $\pi(4) = 3.1416$ and so on. For each representation of $\pi(n)$, we associate a 10-element vector $v(n) = (v_0, v_1, \ldots, v_9)$ where v_i denote the percentage of digits $0, 1, \ldots, 9$ in $\pi(n)$. These percentages should be rounded to a nearest integer.

Example 1: Consider $\pi(2) = 3.14$ and examine $n = 2$ digits after the decimal point. In this case, 50% of all digits is 1 and

50% of all digits is 4. Therefore,

$$v(2) = (\underbrace{0}_{\%0}, \underbrace{50}_{\%1}, \underbrace{0}_{\%2}, \underbrace{0}_{\%3}, \underbrace{50}_{\%4}, \underbrace{0}_{\%5}, \underbrace{0}_{\%6}, \underbrace{0}_{\%7}, \underbrace{0}_{\%8}, \underbrace{0}_{\%9})$$

Example 2: Consdier $\pi(4) = 3.1416$ and examine $n = 4$ digits after the decimal point. 50% of these is 1, and 25% is digits 4 and 6. Therefore,

$$v(4) = (\underbrace{0}_{\%0}, \underbrace{50}_{\%1}, \underbrace{0}_{\%2}, \underbrace{0}_{\%3}, \underbrace{25}_{\%4}, \underbrace{0}_{\%5}, \underbrace{25}_{\%6}, \underbrace{0}_{\%7}, \underbrace{0}_{\%8}, \underbrace{0}_{\%9})$$

We will use $\pi(50)$ as exact value π^* and we will use the notation $v^* = v(50)$. We will consider a number of methods of computing π. All such results should be computed to 50 digits after decimal point. Standard Python computes up tp 15 digits. For example, the *math* module has a stored value for π as math.pi = 3.141592653589793. To have higher precision, we will use *mpmath* Python module to get arbitrary precision in floating numbers. We will limit our precision to 50 digits.

```
import mpmath
mpmath.mp.dps = 52
print("pi with 52 digits:\n", mpmath.pi)
print("pi(Egypt)=22/7:\n",  mpmath.mpf(22/7))
```

This will produce the following output

```
pi with 52 digits:
 3.141592653589793238462643383279502884197169399375106
pi(Egypt)=22/7:
 3.142857142857142793701541449991054832935333251953125
```

We will consider the following "ancient" methods of computing π (take the first 50 digits after the decimal, do not round!).

1. "exact"

$$\pi^* = 3.\underbrace{1415926535}_{1-10}\underbrace{8979323846}_{11-20}\underbrace{2643383279}_{21-30}\underbrace{5028841971}_{31-40}\underbrace{6939937510}_{41-50}$$

2. ancient Egypt (2500 BCE)

$$\pi_{Egypt} = \frac{22}{7}$$

3. Zu Chongzhi (China, 480 AD)

$$\pi_{China} = \frac{355}{113}$$

4. Shatapatha Brahmana (India, 4-th century BCE)

$$\pi_{india} = \frac{339}{108}$$

5. Archimedes (ancient Greece 250 BC) using polygon approximation $223/71 \leq \pi \leq 22/7$. We will use the midpoint of these bounds:

$$\pi_{Greece} = \frac{1}{2}\left(\frac{223}{71} + \frac{22}{7}\right)$$

We will be investigating frequency of digits in the fractional part of π. We are not investigating what is the best approximation for π!

Question 1: For each of the 5 models above

1. compute n - how many first decimal digits are correct when compared with π^*?
2. which method gave you the highest "precision"?
3. compute the vectors v of frequencies
4. what are min and max values of these frequencies?
5. summarize your findings in a table below:

Method	n	$v = (v_0, v_1, \ldots, v_9)$	$\min(v_i)$	$\max(v_i)$
Exact	....	......	...	...
Egypt	....	......	...	...
China	....	......	...	...
India	....	......	...	...
Greece	....	......	...	....

6. examine your table and discuss your findings

For the next question, you will be asked to compare errors in frequency distribution of digits in v vs. v^* for different methods of computing π. Recall that there are a number of ways to measure errors. Let $A = (a_1, \ldots, a_n)$ be the vector of n actual values and let $P = (p_1, \ldots, p_n)$ be the vector of n predicted values. Consider the following ways to measure errors:

(a) max absolute error

$$\max(|a_1 - p_1|, \ldots, |a_n - p_n|)$$

(b) median absolute error

$$\text{median}(|a_1 - p_1|, \ldots, |a_n - p_n|)$$

(c) mean absolute error(MAE)

$$\frac{1}{n}\sum_{i=1}^{n}|a_i - p_i|$$

(d) root mean squared error (RMSE)

$$\sqrt{\frac{1}{n}\sum_{i=1}^{n}(a_i - p_i)^2}$$

Question 2:

1. for each of the above expressions for error, compute the error of your v with v^* for each of the metrics and summarize them in the table (round results to 1 decimal point)

Method	Max Absolute	Median Absolute	Mean Absolute	RMSE
Egypt	......	......	......	......
China	......	......	......	......
India	......	......	......	......
Greece	......	......	......	......

2. discuss your findings

Today, these are still many open and difficult math problems related to π. The following is taken from
`https://mathshistory.st-andrews.ac.uk/HistTopics/Pi_through_the_ages/`

Open questions about the number π:

1. Does each of the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 each occur infinitely often in π?

2. Brouwer's question: In the decimal expansion of π, is there a place where a thousand consecutive digits are all zero?

3. Is π simply normal to base 10? That is, does every digit appear equally often in its decimal expansion in an asymptotic sense?

4. Is π *normal* in any base? That is does every block of digits of a given length appear equally often in the expansion in every base in an asymptotic sense? The concept was introduced by Borel in 1909.

5. Another normal question! We know that π is not rational so there is no point from which the digits will repeat. However, if π is normal then the first million digits 314159265358979... will occur from some point. Even if π is not normal this might hold! Does it? If so, from what point? Note: Up to 200 million the longest to appear is 31415926 and this appears twice.

As a postscript, here is a mnemonic for the decimal expansion of π. Each successive digit is the number of letters in the corresponding word.

How I want a drink, alcoholic of course, after the heavy lectures involving quantum mechanics. All of thy geometry, Herr Planck, is fairly hard...:

3.14159265358979323846264 . . .