MA206, Lesson 23 - Properties of Probability

Define **Sample Space** ($\mathcal{S}$).
Sample Space is the list of all possible outcomes of a random process

Define **Event**.
A collection of specific outcomes from the Sample Space

Define **probability**.
The **probability** of an event is the long-run proportion of times the event would occur if the random process were repeated indefinitely (under identical conditions).
Another way is the proportion of the number of outcomes specified in our **Event** divided by the total number of outcomes in our **Sample Space**.
$\boldsymbol{probability} = \frac{count(Event)}{count(\mathcal{S})}$.

Define **complement** and list its notation.
^{c} The complement of an event A is the probability that A does *not* happen.
$A^c =$ "A complement" $\approx$ "Not A"
Of note, $P(A) + P(A^c) = 1$

Define **Union** and list its notation.
$\bigcup$ The Union of two events A and B is the probability that *either* event A *or* event B happened.
$A \cup B =$ "A union B" $\approx$ "A or B"

Define **Intersection** and list its notation.
$\bigcap$ The Intersection of two events A and B is the probability that *both* event A *and* event B happened.
$A \cap B =$ "A intersect B" $\approx$ "A and B"

Define the **Addition Rule** for probability.
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Define **disjoint**.
Disjoint means mutually exclusive. This means it is impossible for both events to occur simultaneously.
$P(A) \cap P(B) = P(\varnothing) = 0$.

1) To answer the following questions, we assume the probability that a family has a boy or a girl is the same (0.50) and that a family will have three children.

a) What is the sample space?

GGG, GGB, GBG, GBB, BBB, BBG, BGB, BGG

b) What is the probability that there are exactly 2 girls?

$\frac{3}{8}$ (GGB, GBG, BGG)

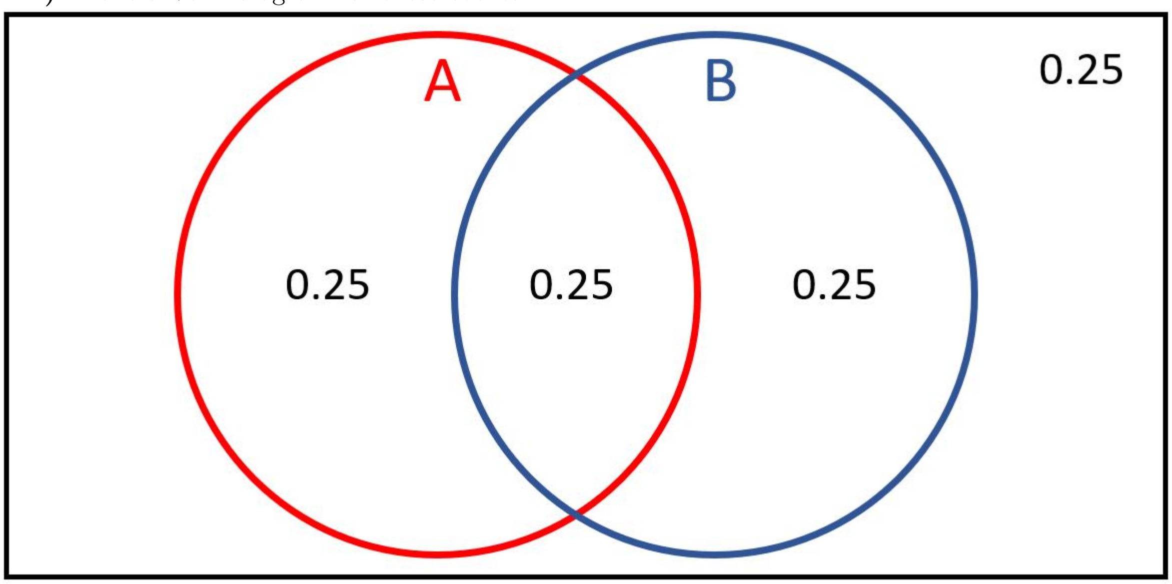
c) What is the probability that there are at least 2 girls?

$\frac{4}{8}$ (GGG, GGB, GBG, BGG)

d) Make a probability table for the events **A** = the first child is a girl, **B** = the second child is a girl.

	$P(A)$	$P(A^c)$	Total
$P(B)$	0.25	0.25	0.5
$P(B^c)$	0.25	0.25	0.5
Total	0.5	0.5	1

e) Make a Venn diagram for these events.



f) Two events are called independent if $P(A \cap B) = P(A) \times P(B)$. Are the two events **A** and **B** independent?

Yes. $P(A) = 0.5$. $P(B) = 0.5$. $P(A \cap B) = 0.25 = P(A) \times P(B)$. Thus, the two events, that the first child is a girl and that the second child is a girl, are independent.

2) Roulette is a game found in most American casinos where players make bets on where a ball will land on a wheel with 38 evenly spaced locations, numbered 1 through 36 with a 0 and 00 space. Each of the spaces 1 through 36 is colored either black or red, while the 0 and 00 spaces are neither red nor black (typically they are green). Eager to get rich at this seemingly simple and easy to win game, let's look at the probability of your bets.

a) What is the sample space for the possible results?

All 38 unique spaces, from 00 to 36, with their matching colors.

b) Betting on a single number has a payout of 35:1. Let event **A** be the event that your single number pick (1 through 36) is the winning number after the roll. After paying in, what is the probability that your number wins, using proper notation? What is the probability the Casino wins? Who wins in the long run, you or the casino?

$P(A) = \frac{1}{38} = 0.02631579.$

$P(A^c) = \frac{37}{38} = 0.9736842.$

In the long run, the Casino wins.

c) Betting on a color, either Red or Black, is seen as a safer bet and as such only pays out 1:1. Let event **B** be that the roulette wheel lands on your chosen color (Red or Black). What is the probability that you win? That the casino wins? In the long run, who wins, you or the Casino?

$P(B) = \frac{18}{38} = 0.4736842.$

$P(B^c) = \frac{20}{38} = 0.5263158.$

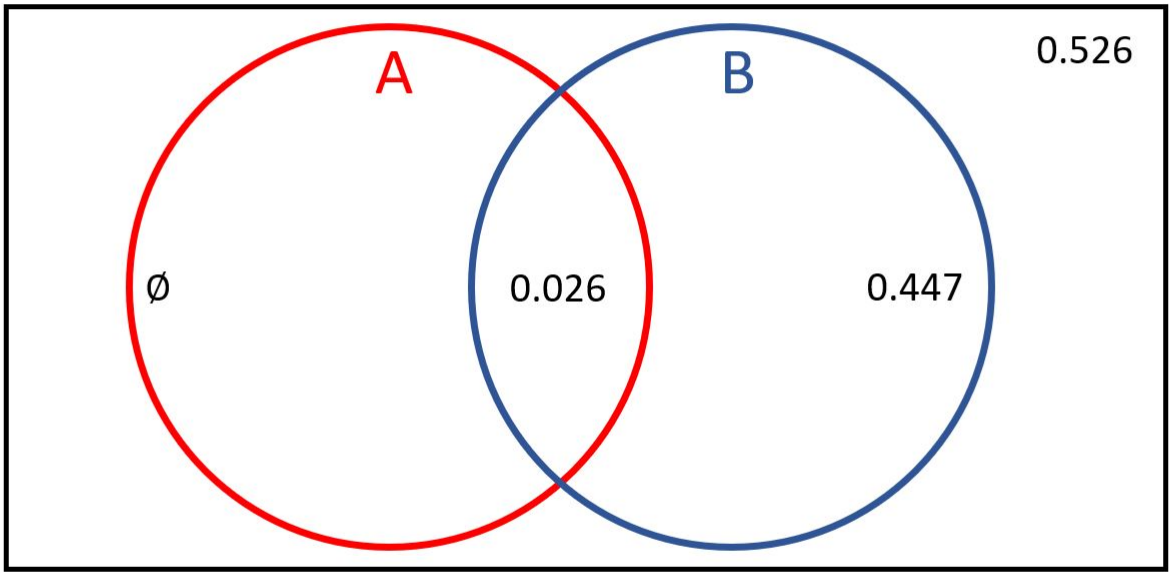
In the long run, the Casino wins.

3) Dissatisfied with your chances, you decide to make both bets, risking twice as much but confident you can win big! You decide to bet on a specific number (**A**) and a color (**B**), choosing your color so that you match your given number (use the image above as reference).

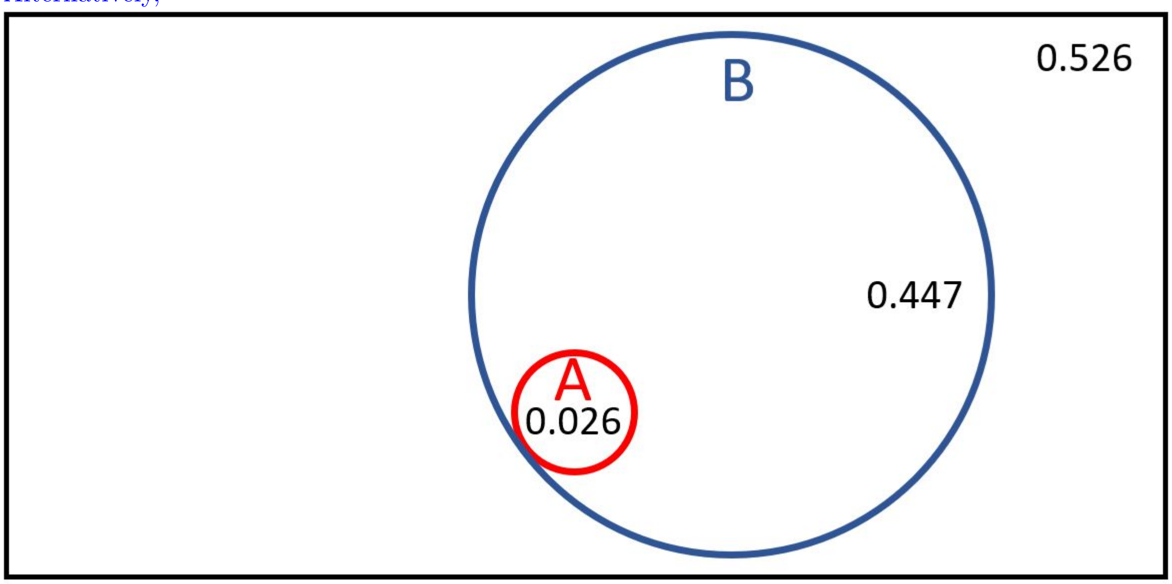
a) Generate a probability table for these events.

	$P(A)$ (My Number)	$P(A^c)$ (Not My Number)	Total
$P(B)$ (My Color)	0.02631579	0.4473684	0.4736842
$P(B^c)$ (Not My Color)	0.00000000	0.5263158	0.5263158
Total	0.02631579	0.9736842	1

b) Draw a Venn Diagram of the above scenario, including probabilities.



Alternatively,



c) What is the probability that you win big? That is, $P(A \cap B)$?

$P(A \cap B) = \frac{1}{38} = 0.02631579$

d) What is the probability that you (and the casino) break even? That is, $P(A^c \cap B)$, you win the color but not the number?

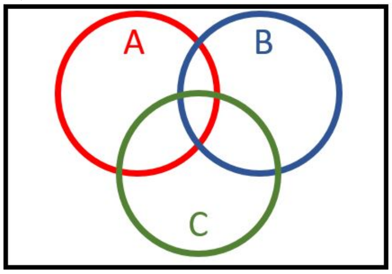
$P(A^c \cap B) = \frac{17}{38} = 0.4473684$

e) What is the probability that you win the number but not the color? That is, $P(A \cap B^c)$?
$P(\varnothing) = 0$. We bet such that these two events are disjoint (mutually exclusive) and cannot both happen.

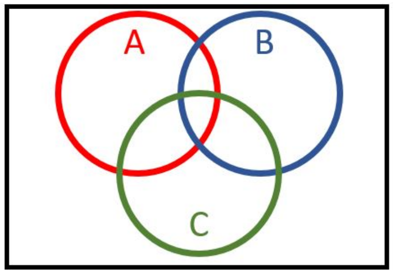
f) What is the probability that you win *anything* and feel good? That is, $P(A \cup B)$?
$P(A \cup B) = \frac{18}{38} = 0.4736842$

g) What is the probability that, despite your gamble, you lose big and the casino pockets your money for both bets? That is, $P(A^c \cap B^c)$?
$P(A^c \cap B^c) = \frac{20}{38} = 0.5263158$.

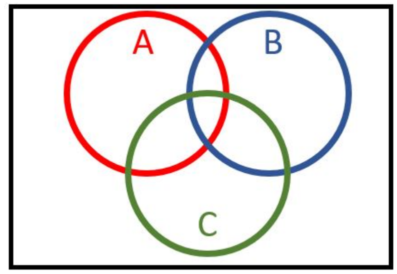
5) Color in the indicated areas of the Venn Diagrams.



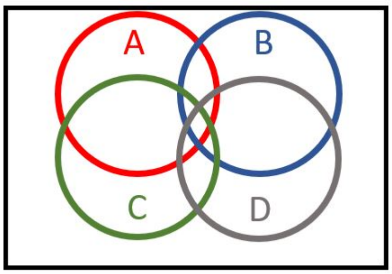
$\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C})$



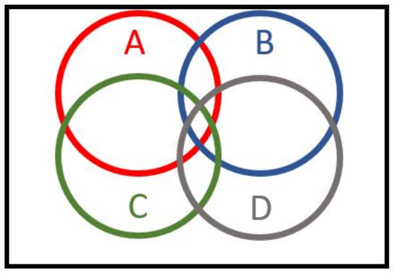
$(\boldsymbol{A} \cup \boldsymbol{B}) \cap \boldsymbol{C}$



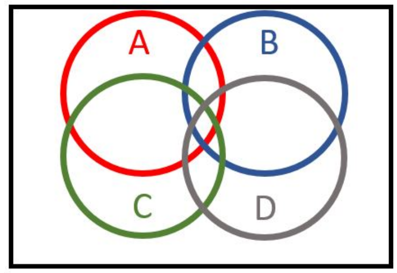
$\boldsymbol{A} \cap (\boldsymbol{B}^{\boldsymbol{c}} \cap \boldsymbol{C}^{\boldsymbol{c}})$



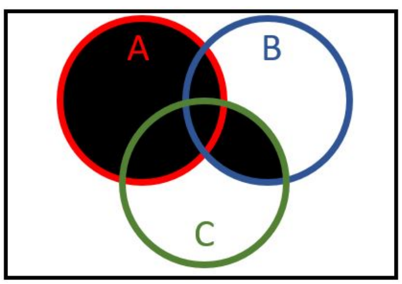
$\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C} \cap \boldsymbol{D}$



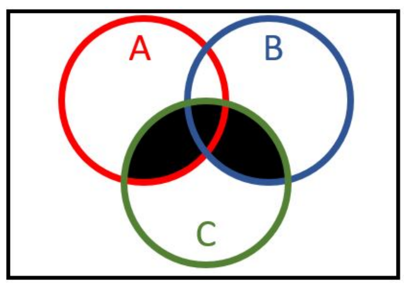
$(\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{C} \cap \boldsymbol{D})$



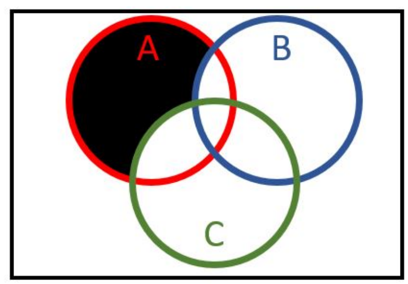
$(\boldsymbol{C} \cap \boldsymbol{B}^{\boldsymbol{c}}) \cap (\boldsymbol{A}^{\boldsymbol{c}} \cup \boldsymbol{D})$



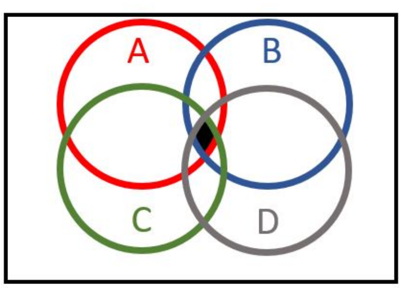
$\boldsymbol{A} \cup (\boldsymbol{B} \cap \boldsymbol{C})$



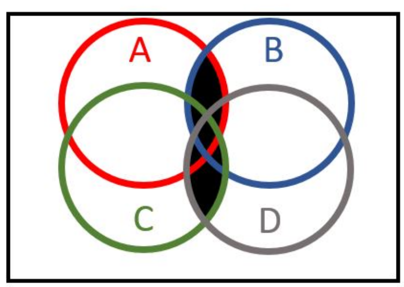
$(\boldsymbol{A} \cup \boldsymbol{B}) \cap \boldsymbol{C}$



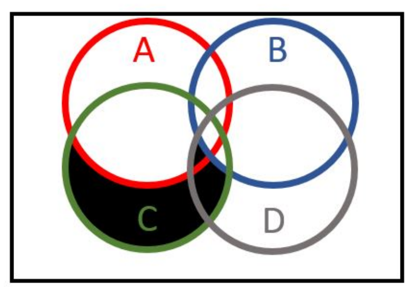
$\boldsymbol{A} \cap (\boldsymbol{B}^{\boldsymbol{c}} \cap \boldsymbol{C}^{\boldsymbol{c}})$



$\boldsymbol{A} \cap \boldsymbol{B} \cap \boldsymbol{C} \cap \boldsymbol{D}$



$(\boldsymbol{A} \cap \boldsymbol{B}) \cup (\boldsymbol{C} \cap \boldsymbol{D})$



$(\boldsymbol{C} \cap \boldsymbol{B}^{\boldsymbol{c}}) \cap (\boldsymbol{A}^{\boldsymbol{c}} \cup \boldsymbol{D})$