Using A Finite Element Method to Solve a Poisson Problem

Kyle Mylonakis

October 26, 2016

1 Introduction

1.1 Electrostatics

In electrostatics, one of the fundamental observed laws of nature is Gauss' law, which states that the divergence of the electric field is proportional to the electric charge density. Let $\mathbf{E} : \mathbb{R}^3 \to \mathbb{R}^3$ denote the time independent electric field, and let $\rho : \mathbb{R}^3 \to \mathbb{R}$ denote the electric charge density. Then Gauss' Law states:

$$\nabla \cdot \mathbf{E} = 4\pi\rho. \tag{1}$$

It turns out that in electrostatics it is often convenient to introduce a potential for the electric field, known as the electric, or scalar, potential. That is there exists a function $\phi : \mathbb{R}^3 \to \mathbb{R}$ such that $\nabla\phi = \mathbf{E}$. One then may substitute this into equation (1) to arrive at

$$\Delta\phi = 4\pi\rho. \tag{2}$$

Part of the phenomenology of metal in electrostatics is that when charge in introduced into a system, the charge carries within the bulk of the metal distributes themselves more to the surface so that the electric field inside a metal is zero. In this case we see that it is always possible to choose a choice of gauge for the scalar potential so that the scalar potential is zero in the metal.

For the purposes of this paper, we will restrict ourselves to two dimensional physical systems inside a thin grounded metal square annulus. Our goal will be to solve for the scalar potential everywhere given some known charge density. See figure one.

From elementary electrostatics we know that the grounding on the outside of the thin metal annulus causes electromagnetic field everywhere outside the the inner boundary of the square annulus to be zero (this is just a routine application of Gauss' law), hence we know that the electric potential everywhere outside the inner boundary of the square annulus may be chosen to be zero.

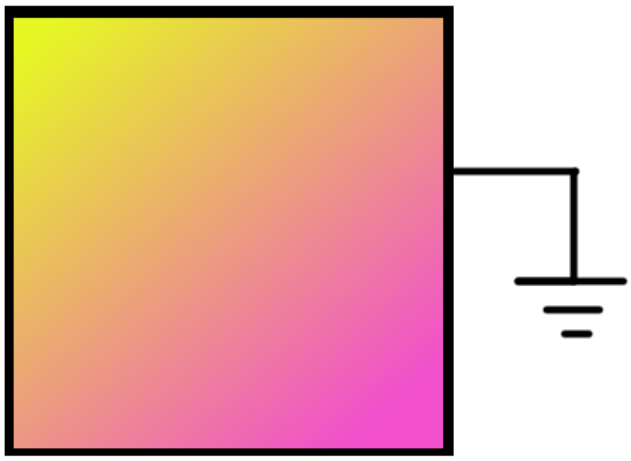

Figure 1: The black line represents the metal square annulus, while the colored interior represents the electric charge on the inside of the metal square annulus

1.2 The Variational Problem

As a result we need only need to solve for the electric potential on the inside of the thin metal annulus, but now subject to the boundary condition that the scalar potential is zero on the inner boundary. Without loss of generality we may assume that the inside of the square annulus is given by the set

$$R = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, 0 \leq y, ||(x, y)||_\infty \leq 1\}.$$

Thus out goal becomes to solve

$$\begin{cases} \Delta\phi = 4\pi\rho \\ \phi|_{\partial R} = 0, \end{cases} \tag{3}$$

for some given charge density $\rho \in L^2(R)$, $\rho|_{\partial R} = 0$, where $\phi \in C^2(R)$. Now let $\psi \in C_0^\infty(R)$. Integrating we find that,

$$\int_R \psi\Delta\phi = \int_R 4\pi\rho.$$

Now using Green's identity we find

$$\int_R \psi\Delta\phi = -\int_R \nabla\phi \cdot \nabla\psi.$$

Thus any solution of (3) must also satisfy

$$\begin{cases} \int\limits_R \nabla\phi \cdot \nabla\psi = -4\pi \int\limits_R \rho\psi \\ \phi|_{\partial R} = 0, \end{cases} \tag{4}$$

for all $\psi \in C_0^\infty(R)$. We may further relax the domains of the problem: in (4) our integrals only appear under integrals, which tells us that the appropriate domain of (4) is $H^1(R)$. For convenience let $a : H^1(R) \times H^1(R) \to \mathbb{R}$ be given by

$$a(u, v) = \int_R \nabla u \cdot \nabla v$$

and let $\langle , \rangle$ be the standard $L^2(R)$ inner product. Then the system (4) may be written as

$$\begin{cases} a(u, v) = -4\pi\langle \rho, v\rangle \\ u|_{\partial R} = 0. \end{cases} \tag{5}$$

for some $\rho \in L^2(R)$, $\rho|_{\partial R} = 0$, and for all $v \in H^1(R)$. Recall that the Dirichlet boundary condition is well posed since our functions are in $H^1(R)$. However, we showed in class that one may wrap the Dirichlet boundary condition into the variational problem itself by restricting the choices of v to the set $H_0^1(R) := \{v \in H^1(R) : v|_{\partial R} = 0\}$. Thus we can finally recast (3) in the variational form as finding the $u \in H^1(R)$ such that

$$a(u, v) = -4\pi\langle \rho, v\rangle \tag{6}$$

for all $v \in H_0^1(R)$, for some fixed $\rho \in L^2(R)$, $\rho|_{\partial R} = 0$. Now as seen in class a is a bounded bi-linear function which is coercive, and $\langle -, \rho\rangle$ is a bounded linear functional. Thus the Lax-Milgram theorem guarantees that there exists a unique u which solves (6).

For the purposes of this paper we will consider $\rho = \dfrac{200(x^2 - x + y^2 - y))}{4\pi}$ which has as an analytical solution to (2), and thus (6), which is $\phi = 100(x^2 - x)(y^2 - y)$.

2 Numerical Methods

2.1 Method and Error Estimates

We will attempt to find a solution to (6) by using quadratic Lagrange elements over a triangulation of R which is a uniform mesh. Let T be the triangulation of R and let $d = \max\{diam(k) : k \in T\}$. See figure two.

Now let ϕ be the solution to (6). Let $P_2(R)$ be the space of piecewise quadratic polynomials defined on R, which are piecewise defined on the elements of the triangulation. Note that $P_2(R)$ is a finite dimensional real vector space. Hence one may form a finite dimensional analog of (6). That is find the $\phi_h \in P_2$ such that for all $v \in P_2(R)$ with $v|_{\partial R} = 0$

$$a(\phi_h, v) = \langle v, -200(x^2 - x + y^2 - y)\rangle. \tag{7}$$

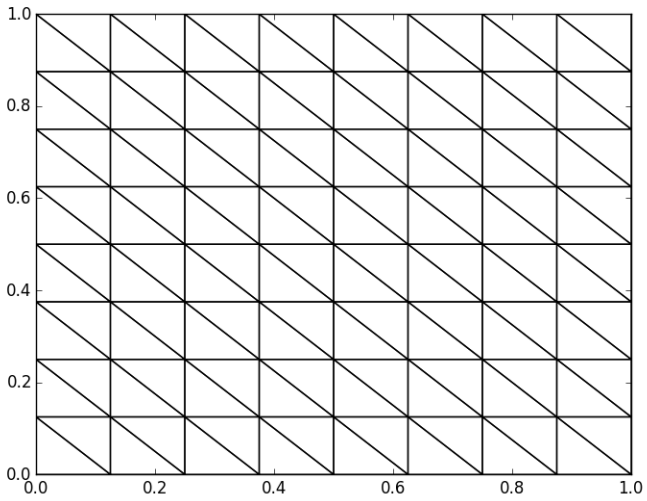

Figure 2: An example of the uniform triangulation used. Here we have triangles of side length 1/8, giving a total of 128 total triangles.

Let ϕ be the actual solution to (6), and let ϕ_h be the solution to (7). Further let $I : H^1(R) \to P_2(R)$ be the operator which interpolates an element of H^1 to piecewise quadratic polynomial on R. Then by Cea's lemma and Remark 4.4.27 of Brenner and Scott,

$$||\phi - \phi_h||_{L^2(R)} \leq ||\phi - I(\phi_h)||_{L^2(R)} \leq \tilde{C} d |\phi|_{H^1(R)},$$

where C is a positive constant dependent only on the continuity and coercivity of a, and where $|\phi|_{H^1(R)}$ denotes the first order semi-norm in $H^1(R)$. Unfortunately, these theorems cannot be used to gain convergence in H^1 or $L^\infty(R)$ as we refine the mesh since Cea's lemma and remark 4.4.27 only gives

$$||\phi - \phi_h||_{H^1(R)} \leq ||\phi - I(\phi_h)||_{H^1(R)} \leq \tilde{C}' |\phi|_{H^1(R)},$$

$$||\phi - \phi_h||_{L^\infty(R)} \leq ||\phi - I(\phi_h)||_{L^\infty(R)} \leq \tilde{C} |\phi|_{L^\infty(R)}$$

for some new positive constants $\tilde{C}$ and $\tilde{C}'$, where $|\phi|_{L^\infty(R)}$ denotes the first order semi-norm in $L^\infty(R)$.

2.2 Implementation and Results

Using a desktop computer with an i7-3770 @ 4.41 Ghz processor we solved (2) via the usual Ritz-Galerkin methods for uniform triangulations with $2n^2$ elements where n is an integer between 2 and 20. We obtain 2.2825 order convergence in the sup norm and 2.7500 order convergence in the L^2 norm, as indicated by the slopes of the lines in the following log-log plot. Thus our error convergence is within the theoretical bounds outlined in the previous section.

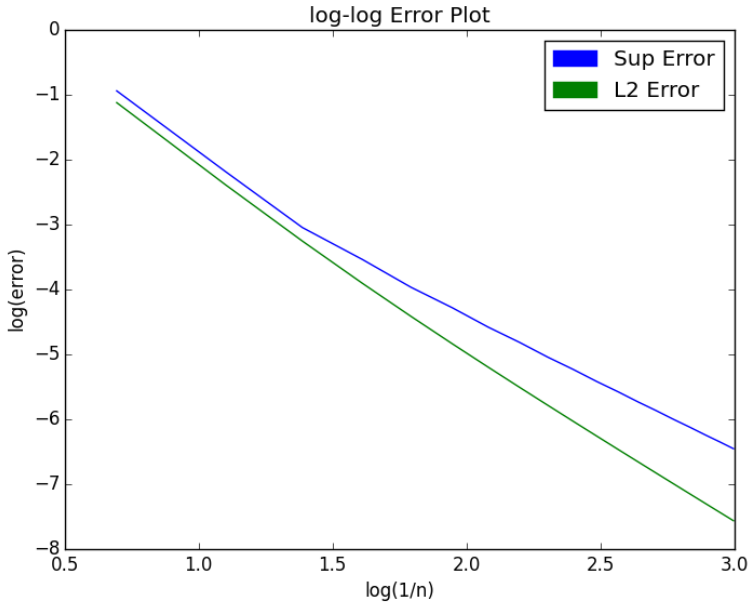

Figure 3: The log-log plot of Error versus the inverse of the lattice constant

Moreover, below is a scatter plot of the approximate solution to (2) with $n = 26$. In this case we find that the L^2 error is bounded above by 2.68×10^{-4} and the sup norm error is bounded above by 9.29×10^{-3}. Thus the approximate solution in the figure is representative of the analytical solution to (2).

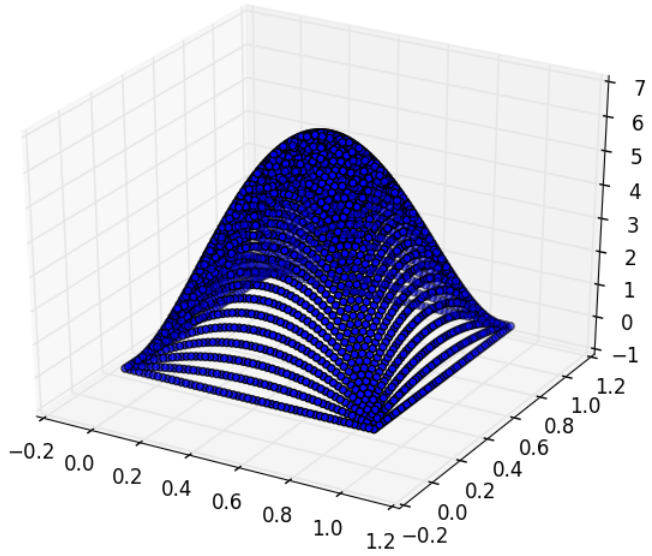

Figure 4: A scatter plot of the approximate solution with $n = 26$. This reveals the overall shape of the solution

3 Future Improvements

While the assembly matrix algorithm is fast, the actual integrals for computation of the mass and stiffness submatrices themselves were computed with SciPy's double integrator function. This definitely bogged down the runtime of the simulation, and would be improved by performing the integrals by hand and then simply evaluating them.

The algorithm used to generate the mesh was not recursive, but simply required one to specify an inverse of the desired lattice constant as an integer. A recursive mesh algorithm would be superior since it would support non-uniform meshes.

The assembly algorithm may easily be parallelized, which would greatly speed up the algorithm.