1 $Y_t = \mu + \rho Y_{t-1} + \varepsilon_t$

1a $|\rho|<1$ $E[Y_t] = E[\mu + \rho Y_{t-1} + \varepsilon_t]$

$= E(\mu) + E(\rho Y_{t-1}) + E(\varepsilon_t)$

$= \mu + \rho E(Y_{t-1})$

since $|\rho|<1$ $E(Y_t) = \mu + \rho E(Y_t)$

$\mu = E(Y_t) - \rho E(Y_t)$

$\mu = E(Y_t)(1-\rho)$

$E(Y_t) = \frac{\mu}{(1-\rho)}$ = Unconditional mean

1b Show $E[Y_{t+h}|Y_t] = \rho^h Y_t + \mu \sum_{l=0}^{h-1} \rho^l$

assume $h=1$ $E[Y_{t+1}|Y_t] = \rho Y_t + \mu + \underbrace{E(\varepsilon_{t+1}|Y_t)}_{0} = \rho Y_t + \mu$

assuming the formula works for $h=k$ we will show it works for $h=k+1$

$E[Y_{t+k+1}|Y_t] = \rho E[Y_{t+k}|Y_t] + \mu + 0$

$= \rho(\rho^k Y_t + \mu \sum_{l=0}^{k-1} \rho^l) + \mu$

$= \rho^{k+1} Y_t + \mu \sum_{l=0}^{k} \rho^l$

thus $E[Y_{t+h}|Y_t] = \rho^h Y_t + \mu \sum_{l=0}^{h-1} \rho^l = \rho^h Y_t + \mu/(1-\rho)$

1c given $|\rho|<1$

$\lim_{h\to\infty} E[Y_{t+h}|Y_t] = \lim_{h\to\infty} \rho^h Y_t + \lim_{h\to\infty} \mu/(1-\rho) = \mu/(1-\rho)$

$\lim_{h\to\infty} E[Y_{t+h}|Y_t]$ does not depend on Y_t

1d $E[Y_{t+h}|Y_t] = \rho^h Y_t + \mu \sum_{l=0}^{h-1} \rho^l$ given $|\rho|=1$ $E[Y_{t+h}|Y_t] = Y_t + h\mu$

$$\lim_{h\to\infty} E[Y_{t+h}|Y_t] = \lim_{h\to\infty} Y_t + \lim_{h\to\infty} h\mu = Y_t + \mu \lim_{h\to\infty} h \begin{cases} \text{if } \mu < 0 \to = -\infty \\ \text{if } \mu = 0 \to = 0 \\ \text{if } \mu > 0 \to = \infty \end{cases}$$

No it does not depend on Y_t it depends on μ.

3 $Y_t = \beta_1 + \delta t + \varepsilon_t, t = 1, \dots, T$

Based on the FOC's of the OLS

$$\hat{\delta} = \frac{\sum_{t=1}^{T}(t-\bar{t})(Y_t - \hat{Y}_t)}{\sum_{t=1}^{T}(t-\bar{t})}$$

$$= \frac{\sum_{t=1}^{T}(t-\bar{t})(\beta + \delta t + \varepsilon_t - \hat{Y}_t)}{\sum_{t=1}^{T}(t-\bar{t})}$$

$$= \frac{\sum_{t=1}^{T}[(t-\bar{t})\delta t + \beta_1(t-\bar{t}) + (t-\bar{t})\varepsilon_t - (t-\bar{t})\hat{Y}_t]}{\sum_{t=1}^{T}(t-\bar{t})^2}$$

$$= \frac{\sum_{t=1}^{T}(t-\bar{t})\delta t}{\sum_{t=1}^{T}(t-\bar{t})^2} + \frac{\sum_{t=1}^{T}(t-\bar{t})(\beta_1 - \hat{Y}_t)}{\sum_{t=1}^{T}(t-\bar{t})^2} + \frac{\sum_{t=1}^{T}(t-\bar{t})\varepsilon_t}{\sum_{t=1}^{T}(t-\bar{t})^2}$$

$$= \delta + \frac{\sum_{t=1}^{T}(\varepsilon_t - \bar{\varepsilon}_t)(t - \frac{T+1}{2})}{\sum_{t=1}^{T}(t-\frac{T+1}{2})^2}$$

$$Var(\hat{\delta} - \delta)\sqrt{T} = var\left[\sqrt{T}\frac{\sum_{t=1}^{T}\varepsilon_t(t - \frac{T+1}{2})}{\sum_{t=1}^{T}(t-\frac{T+1}{2})^2} \quad \varepsilon_t \quad \frac{\sqrt{T}\sum_{t=1}^{T}(t - \frac{T+1}{2})}{\sum_{t=1}^{T}(t-\frac{T+1}{2})^2}\right]$$

$$= var\left[\varepsilon_t \quad \frac{T\sum_{t=1}^{T}(t-\frac{T+1}{2})^2}{(\sum_{t=1}^{T}(t-\frac{T+1}{2})^2)^2}\right]$$

$$= var\left[\varepsilon_t \cdot \frac{T}{\sum_{t=1}^{T}(t-\frac{T+1}{2})^2}\right]$$

given $\sum_{t=1}^{T} t^2 = \frac{T(2T+1)(T+1)}{6}$

$$\sum_{t=1}^{T} t^2 - 2\sum_{t=1}^{T}(t \cdot \tfrac{T+1}{2}) + T(\tfrac{T+1}{2})^2 = \frac{T(2T+1)(T+1)}{6} - \frac{T(T+1)^2}{2} + \frac{T(T+1)^2}{4}$$

$$= T\left[\frac{(T+1)(2T+1)}{6} - \frac{(T+1)^2}{2} + \frac{(T+1)^2}{4}\right]$$

$$= \frac{T(T^2-1)}{12}$$

$$Var(\hat{\delta} - \delta) = \frac{\varepsilon_T \, T \, 12}{(T+1)(T-1)T}$$

$$= Var(\varepsilon_T) \cdot \frac{12}{(T+1)(T-1)}$$

$$= \sigma^2 \cdot 0 \text{ when } T \to \infty$$

$$= 0$$

δ is superconsistent if the following model holds

$$E(\hat{\delta}) = E(\delta) + E\left[\sum_{t=1}^{T} \frac{t - \bar{t}}{\sum_{t=1}^{T}(t - \frac{T+1}{2})^2}(\varepsilon_t - \bar{\varepsilon}_T)\right]$$

$$= \delta + \left(\sum_{t=1}^{T} \frac{t - \bar{t}}{\sum_{t=1}^{T}(t - \frac{T+1}{2})^2}\right) E(\varepsilon_t - \bar{\varepsilon}_T)$$

$$= \delta + 0$$

$$= \delta$$

In a finite sample it is unbiased