HW1 Report

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1 Introduction

1.1 The problem to solve

In this report, we are going to use ODE to model the growth of the population of a yeast culture. We will use four different growth rates to fit the data and compare the performance of them.

1.2 Brief summary of my results

I use four different growth rates to fit the data. The best one is $k(P_t) = r(M - P_t)$, which all most catch the precise trend of the growth. Besides, I also find that with the population increasing, a huge population may resist the growth of itself. We need to model the resistance effect in our ODEs.

2 Data

The data we use in this report with title of *Growth of a Yeast Culture Versus Time in Hours*, from R. Pearl, "The Growth of Population", Quart. Rev. Biol. 2(1927): 532-548. The following is the plot of this data: From the plot we can see the curve just like the shape of sigmoid function. It is convex in the first stage and then becomes concave in the next stage. It also has a upper bound and increases all the time.

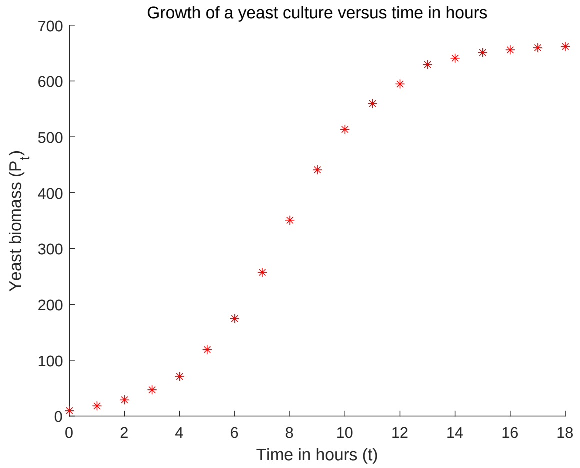

Figure 1: Plot of data

3 Model

3.1 Build the ODEs

I choose four different growth rates to fit the data and them are shown as follows:

1. $k(P_t) = \eta$
2. $k(P_t) = r(M - P_t)$
3. $k(P_t) = \kappa ln(M/P_t)$
4. $k(P_t) = \theta(M^2 - P_t^2)$

where $\eta, r, M, \kappa, \theta$ are parameters to be estimated.

Then I build four ODEs to describe the growth of the population, respectively:

1. $P_t' = \eta P_t$
2. $P_t' = r(M - P_t)P_t$
3. $P_t' = \kappa ln(M/P_t)P_t$
4. $P_t' = \theta(M^2 - P_t^2)P_t$

For the η in the first ODE, it is the differece between the birth rate and the death rate without capacity limit. From the data, we assume that $\eta > 0$ in this report.

For the M in the last three ODEs, it is the capacity limit of the population. In this report, we take M as 665.

$r(M - P_t)$ is the growth rate with capacity limit in the second ODE. So, r is growth rate divide the residual capacity of the population. We can take $M - P_t$ (the residual capacity of the population) as a resistance to the growth of population. From the data, we assume that $r > 0$ in this report.

$\kappa ln(M/P_t)$ is the growth rate with capacity limit in the third ODE. So, κ is the growth rate divide the napierian logarithm of the capacity ratio. We can take $ln(M/P_t)$ (the napierian logarithm of the capacity ratio) as a resistance to the growth of the population. From the data, we assume that $\kappa > 0$ in this report.

$\theta(M^2 - P_t^2)$ is the growth rate with capacity limit in the fourth ODE. So, θ is the growth rate divide the differece of square limit capacity and square P_t of the population. We can take $M^2 - P_t^2$ (the differece of square limit capacity and square P_t of the population) as a resistance to the growth of the population. From the data, we assume that $\theta > 0$ in this report.

3.2 Analyzes the ODEs

3.2.1 The First ODE

Let $P_t' = \eta P_t = 0$ to find the equilibrium point. Then with the assumption $\eta > 0$, we can get $P_t = 0$. With $P_t' = \eta P_t > 0$ and $P_t'' = \eta P_t' > 0$, we can get that $P_t = 0$ is an unstable equilibrium point and the convexity of this ODE is convex for P_t from 0 to ∞.
The plot of an example of the first ODE is shown as follow:

3.2.2 The Second ODE

Let $P_t' = r(M - P_t)P_t = 0$ to find the equilibrium point. Then with the assumption $r > 0$, we can get $P_t = 0$ and $P_t = M$. When $0 < P_t < M/2$, we have $P_t' = r(M - P_t)P_t > 0$ and

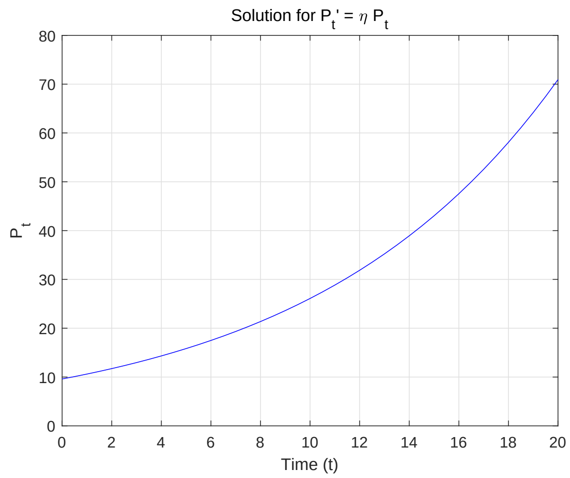

Figure 2: An example of the first ODE

$P_t'' = r(M - 2P_t)P_t' > 0$. When $M/2 < P_t < M$, we have $P_t' = r(M - P_t)P_t > 0$ and $P_t'' = r(M - 2P_t)P_t' < 0$. When $P_t > M$, we have $P_t' = r(M - P_t)P_t < 0$ and $P_t'' = r(M - 2P_t)P_t' > 0$. So, $P_t = 0$ is an unstable equilibrium point and $P_t = M$ is a stable equilibrium point. The convexity of this ODE is convex for P_t from 0 to $M/2$ and M to $infty$, and concave for P_t from $M/2$ to M.

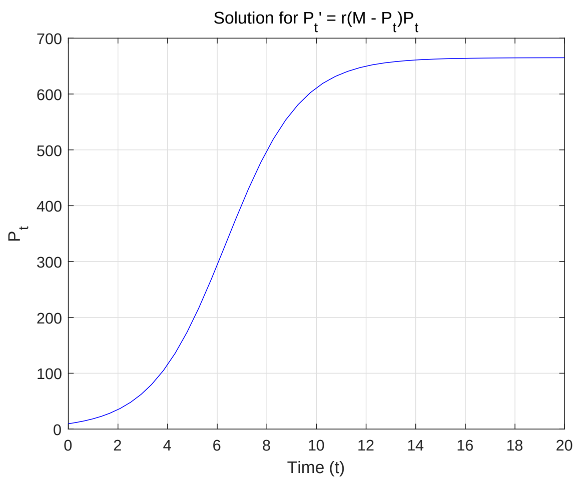

Figure 3: An example of the second ODE

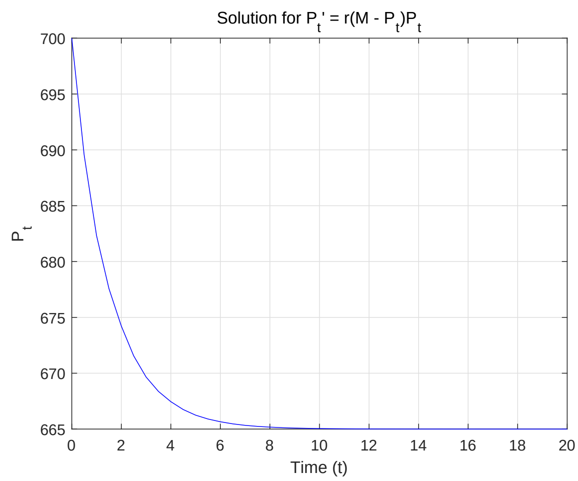

Figure 4: Another example of the second ODE

3.2.3 The Third ODE

Let $P_t' = \kappa ln(M/P_t)P_t = 0$ to find the equilibrium point. Then with the assumption $\kappa > 0$, we can get $P_t = M$. When $0 < P_t < M/e$, we have $P_t' = \kappa ln(M/P_t)P_t > 0$ and $P_t'' =$

$\kappa P_t'(P_t^2/M + P_t' ln(M/P_t)) > 0$. When $M/e < P_t < M$, we have $P_t' = \kappa ln(M/P_t)P_t > 0$ and $P_t'' = \kappa(ln(M/P_t) - 1)P_t' < 0$. When $P_t > M$, we have $P_t' = \kappa ln(M/P_t)P_t < 0$ and $P_t'' = \kappa(ln(M/P_t) - 1)P_t' > 0$. So, $P_t = M$ is a stable equilibrium point. The convexity of this ODE is convex for P_t from 0 to M/e and from M to ∞, and concave for P_t from M/e to M.

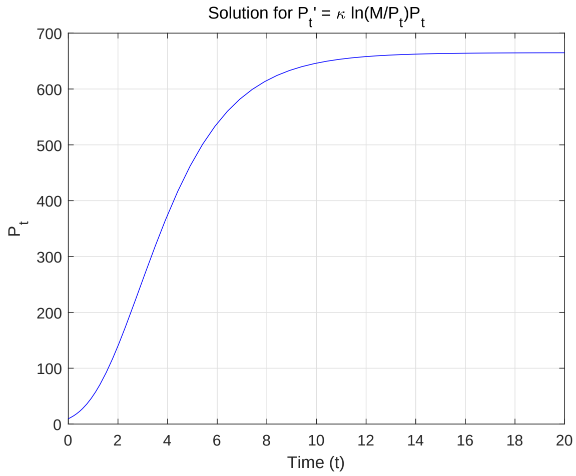

Figure 5: An example of the third ODE

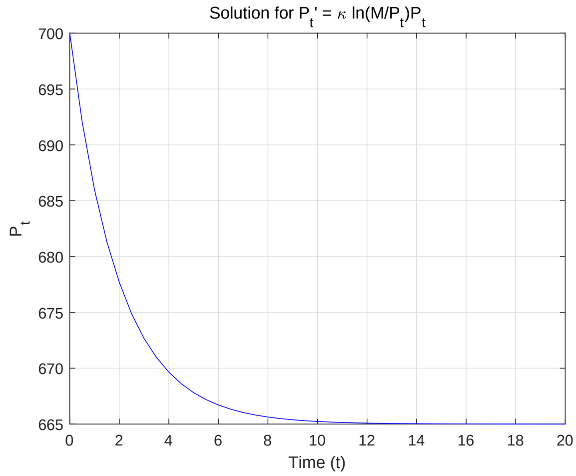

Figure 6: Another example of the third ODE

3.2.4 The Fourth ODE

Let $P_t' = \theta(M^2 - P_t^2)P_t = 0$ to find the equilibrium point. Then with the assumption $\theta > 0$, we can get $P_t = 0$ and $P_t = M$. When $0 < P_t < M/\sqrt{3}$, we have $P_t' = \theta(M^2 - P_t^2)P_t > 0$ and $P_t'' = \theta(M^2 - 3P_t^2)P_t' > 0$. When $M/\sqrt{3} < P_t < M$, we have $P_t' = \theta(M^2 - P_t^2)P_t > 0$ and $P_t'' = \theta(M^2 - 3P_t^2)P_t' < 0$. When $P_t > M$, we have $P_t' = \theta(M^2 - P_t^2)P_t < 0$ and $P_t'' = \theta(M^2 - 3P_t^2)P_t' > 0$. So, $P_t = M$ is a stable equilibrium point. The convexity of this ODE is convex for P_t from 0 to $M/\sqrt{3}$ and from M to ∞, and concave for P_t from $M/\sqrt{3}$ to M.

3.3 Analytical Solutions

3.3.1 The First ODE

As $\frac{d(ln(P_t))}{dt} = \frac{P_t'}{P_t}$, $\frac{d(ln(P_t))}{dt} = \eta$. We can get $ln(P_t) - ln(P_0) = \eta t$, which means $P_t = P_0 e^{\eta t}$.

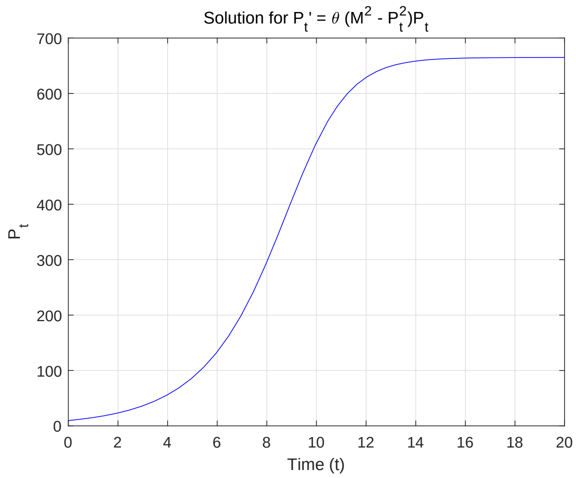

Figure 7: An example of the fourth ODE

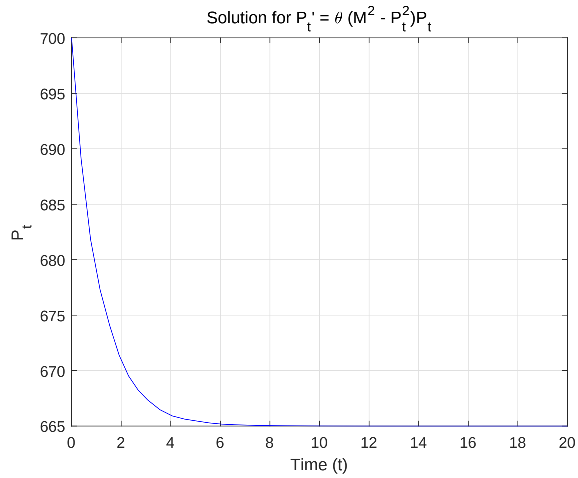

Figure 8: Another example of the fourth ODE

3.3.2 The Second ODE

From the second ODE we can get $\frac{dP_t}{(M-P_t)P_t} = rdt$. So that, $dP_t(\frac{1}{M-P_t} + \frac{1}{P_t}) = Mrdt$ As $\frac{d(ln(P_t))}{dt} = \frac{P_t'}{P_t}$, we integrate from 0 to t to the both sides to get $ln\frac{P(t)}{P0} - ln\frac{M-P_t}{M-P_0} = Mrt$. The solution is $P_t = \frac{MP_0}{P_0+(M-P_0)e^{-Mrt}}$

3.3.3 The Third ODE

From the third ODE we can get $\frac{1}{ln(M/P_t)P_t}dP_t = \kappa dt$. We can integrate from 0 to t to the both sides to get $\int_{P_0}^{P_t} \frac{1}{ln(M/\tau)\tau}d\tau = \kappa t$. Here, let us simpfy this integral:

$$\begin{aligned}
&\int_{P_0}^{P_t} \frac{1}{ln(M/\tau)\tau}d\tau \\
&= \int_{1/P_0}^{1/P_t} \left(-\frac{1}{uln(Mu)}\right) du \quad (u = 1/\tau) \\
&= [-ln(ln(Mu))]_{1/P_0}^{1/P_t} \\
&= ln(ln(M/P_0)) - ln(ln(M/P_t))
\end{aligned}$$

So, we can get $ln(ln(M/P_0)) - ln(ln(M/P_t)) = \kappa t$. The solution is $P_t = Me^{-e^{-\kappa t + ln(ln(M/P_0))}}$

3.4 The fourth ODE

From the fourth ODE we can get $\frac{1}{(M^2-P_t^2)P_t}dP_t = \theta dt$. We can integrate from 0 to t to the both sides to get $\int_{P_0}^{P_t} \frac{1}{(M^2-\tau^2)\tau}d\tau = \theta t$. Here, let us simpfy this integral:

$$\begin{aligned}
&\int_{P_0}^{P_t} \frac{1}{(M^2-\tau^2)\tau}d\tau \\
&= \int_{P_0^2}^{P_t^2} \frac{1}{2(M^2-u)u}du \quad (u=\tau^2) \\
&= \frac{1}{2M^2}\int_{P_0^2}^{P_t^2} \left(\frac{1}{M^2-u}+\frac{1}{u}\right)du \\
&= \frac{1}{2M^2}\left[-ln(M^2-u)+ln(u)\right]_{P_0^2}^{P_t^2} \\
&= \frac{1}{2M^2}\left[ln\frac{P_t^2}{M^2-P_t^2} - ln\frac{P_0^2}{M^2-P_0^2}\right]
\end{aligned}$$

So, we can get $ln\frac{P_t^2}{M^2-P_t^2} - ln\frac{P_0^2}{M^2-P_0^2} = 2M^2\theta t$. The solution is $P_t = \sqrt{\frac{M^2P_0^2e^{2M^2\theta t}}{P_0^2e^{2M^2\theta t}+M^2-P_0^2}}$

4 Empirical Results

For the method to estimate the parameters, I use Curve Fitting Toolbox of MATLAB.

4.1 All data results

4.1.1 First ODE

The estimated η is 0.2578 and the plot of the first ODE is shown as follow: Here are some performance metrics of the first ODE:

- RMSE (Root Mean Squared Error): 238.9313
- MeAE (Mean Absolute Error): 190.7347
- MaAE (Maximum Absolute Error): 396.0639

4.1.2 Second ODE

The estimated r is 0.0008 and the plot of the second ODE is shown as follow:

- RMSE (Root Mean Squared Error): 3.3794

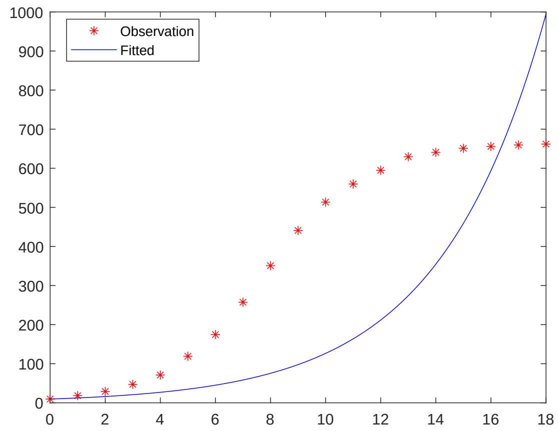

Figure 9: Fitted result of the first ODE

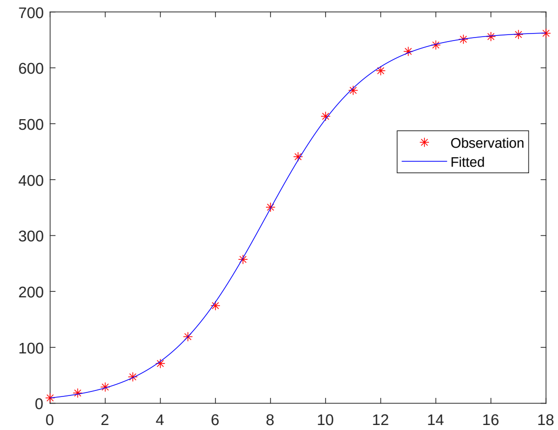

Figure 10: Fitted result of the second ODE

- MeAE (Mean Absolute Error): 2.5871

- MaAE (Maximum Absolute Error): 7.1878

4.1.3 Third ODE

The estimated κ is 0.2519 and the plot of the third ODE is shown as follow:

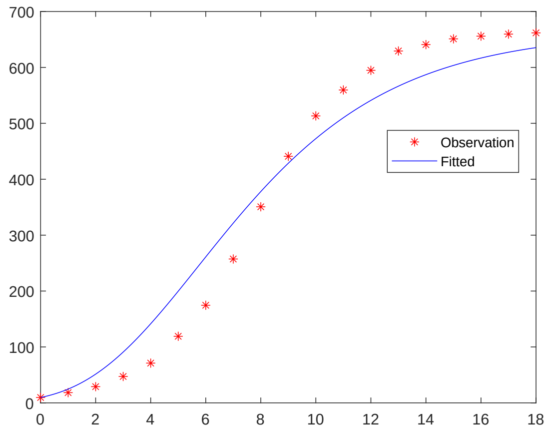

Figure 11: Fitted result of the third ODE

- RMSE (Root Mean Squared Error): 49.0928

- MeAE (Mean Absolute Error): 43.1541

- MaAE (Maximum Absolute Error): 86.4596

4.1.4 Fourth ODE

The estimated θ is 0.0000010 and the plot of the fourth ODE is shown as follow:

- RMSE (Root Mean Squared Error): 22.6785

- MeAE (Mean Absolute Error): 17.6920

- MaAE (Maximum Absolute Error): 46.4288

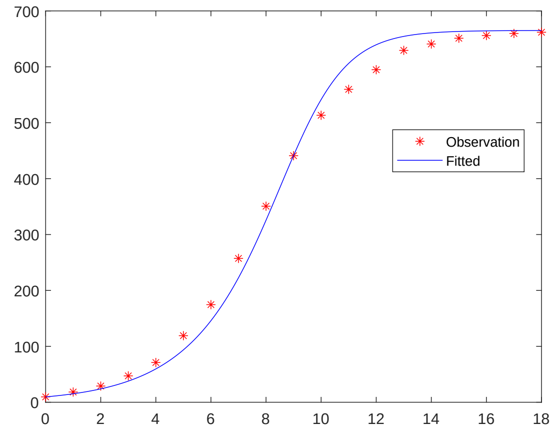

Figure 12: Fitted result of the fourth ODE

4.2 Separated data results

4.2.1 First ODE

The estimated η is 0.2527 and the plot of the first ODE is shown as follow: Here are some perfor-

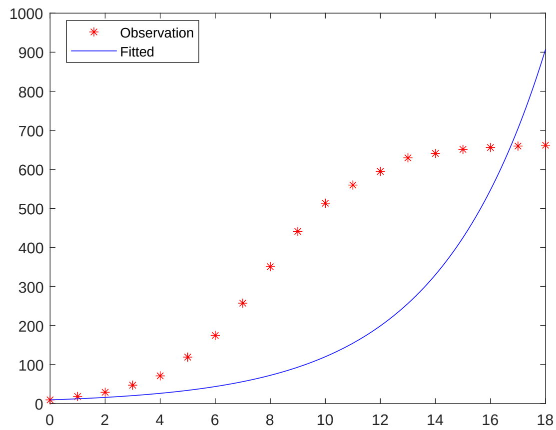

Figure 13: Fitted result of the first ODE with separated data

mance metrics of the first ODE:

- RMSE (Root Mean Squared Error) of training data: 240.0645
- MeAE (Mean Absolute Error) of training data: 192.0341
- MaAE (Maximum Absolute Error) of training data: 395.6352

- RMSE (Root Mean Squared Error) of test data: 241.6478
- MeAE (Mean Absolute Error) of test data: 190.6102
- MaAE (Maximum Absolute Error) of test data: 405.0082

4.2.2 Second ODE

The estimated r is 0.0008 and the plot of the second ODE is shown as follow:

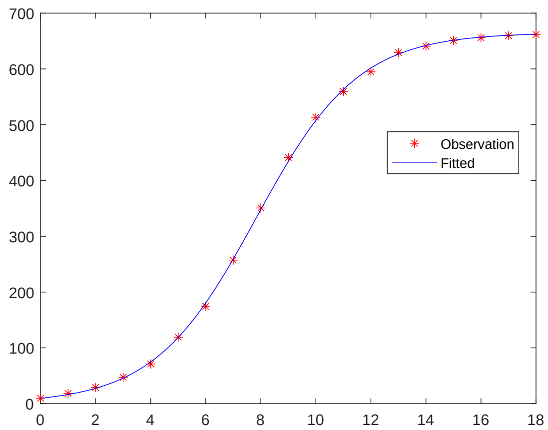

Figure 14: Fitted result of the second ODE with separated data

- RMSE (Root Mean Squared Error) of training data: 3.7603
- MeAE (Mean Absolute Error) of training data: 2.9338
- MaAE (Maximum Absolute Error) of training data: 6.7647
- RMSE (Root Mean Squared Error) of test data: 2.9627
- MeAE (Mean Absolute Error) of test data: 2.2347
- MaAE (Maximum Absolute Error) of test data: 6.9038

4.2.3 Third ODE

The estimated κ is 0.2515 and the plot of the third ODE is shown as follow:

- RMSE (Root Mean Squared Error) of training data: 48.3042

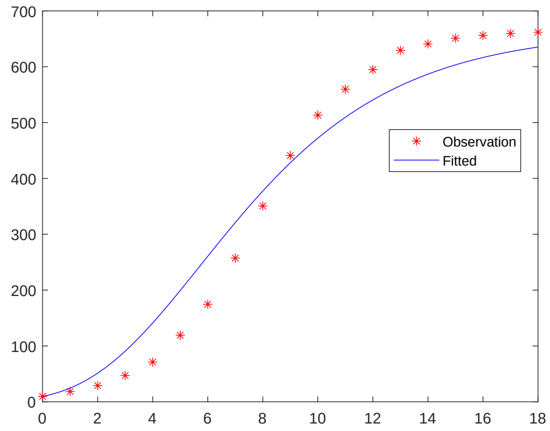

Figure 15: Fitted result of the third ODE with separated data

- MeAE (Mean Absolute Error) of training data: 42.0506
- MaAE (Maximum Absolute Error) of training data: 85.9372
- RMSE (Root Mean Squared Error) of test data: 49.9586
- MeAE (Mean Absolute Error) of test data: 44.5132
- MaAE (Maximum Absolute Error) of test data: 80.1920

4.2.4 Fourth ODE

The estimated θ is 0.0000010 and the plot of the fourth ODE is shown as follow:

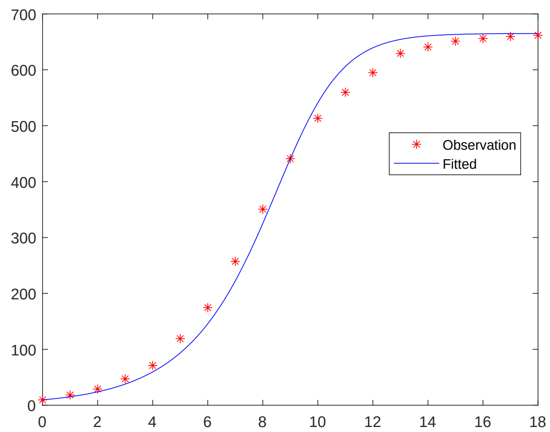

Figure 16: Fitted result of the fourth ODE with separated data

- RMSE (Root Mean Squared Error) of training data: 22.0695
- MeAE (Mean Absolute Error) of training data: 17.4612
- MaAE (Maximum Absolute Error) of training data: 44.5020
- RMSE (Root Mean Squared Error) of test data: 23.3352
- MeAE (Mean Absolute Error) of test data: 17.9521
- MaAE (Maximum Absolute Error) of test data: 46.1727

5 Discussions

5.1 Compare the performance

From the above plot and performance metrics, we can see the second ODE best present the growth of the yeast population no matter the data is separated or not. The first ODE is the worst one to present the growth of the yeast population. The third and fourth ODEs are better than the first ODE but worse than the second ODE.

5.2 Possible reasons

For why the first ODE is the worst one, I think it is because the first ODE does not consider the capacity limit of the population. For why the second one is the best one, I think it is because the the $M - P_t$ is better to present the resistance in the growth of the population than other two resistances.