THE SCHWARZ LEMMA: AN ODYSSEY

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1. The birth of an empire

The Schwarz lemma is one of the pioneering results in the function theory of one complex variable. It states that a holomorphic map between concentric disks, which fixes their center, is distance-decreasing:

1.1. **Theorem.** A holomorphic map $f : \mathbb{D}(R_1) \to \mathbb{D}(R_2)$ fixing the origin, satisfies, for all $z \in \mathbb{D}(R_1)$,

$$|f(z)| \leq \frac{R_2}{R_1}|z|. \tag{1.1}$$

The standard proof that we teach is the following: If $f(0) = 0$, the function $g(z) := f(z)/z$ admits a holomorphic extension to all of $\mathbb{D}(R_1)$. Applying the maximum principle to $g(z)$ on each disk $|z| \leq R_1 - \varepsilon$, and letting $\varepsilon \to 0$ proves the statement.
Let us remark that this was not the proof originally given by Schwarz in [16] (who proved the Schwarz lemma for one-to-one holomorphic maps). The proof here was first presented by Caratheodory [7, p. 114, Note 13], where it is attributed to Erhard Schmidt.
In (1.1), if we keep R_2 fixed, and let R_1 get arbitrarily large, we recover the following well-known corollary:

1.2. **Corollary.** (Liouville's theorem). A bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ assumes at most one value.
The Schwarz lemma is a local, finite statement about holomorphic maps. The Liouville theorem, in contrast, is a global statement obtained from letting $R_1 \to \infty$. This is the prototypical example of the so-called *Bloch principle*; namely, the principle that any global statement concerning holomorphic maps arises from a stronger, finite version:

Nihil est in infinito quod non prius fuerit in finito.[1]

To further illustrate Bloch's principle, let us recall the following generalization of the Liouville theorem: The *Picard theorem* states

1.3. **Corollary.** A non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ assumes all but possibly one value.

The corresponding finite version prophesized by the Bloch principle is the *Schottky theorem*:

1.4. **Theorem.** Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map which omits the values 0 and 1. Then $|f(z)|$ affords a bound in terms of $|f(0)|$ and $|z|$.[2]

The first instance of the Bloch principle was the following *Valiron theorem*:

1.5. **Corollary.** A non-constant entire function has holomorphic branches of the inverse in arbitrarily large Euclidean disks.

In [4], Bloch improved Valiron's arguments and proved the underlying finite agent, which we now call the *Bloch theorem*:

1.6. **Theorem.** Every holomorphic function $f : \mathbb{D} \to \mathbb{C}$ has an inverse branch in some Euclidean disk of radius $\mathcal{B}|f'(0)|$, where $\mathcal{B} > 0$ is an absolute constant.

The constant $\mathcal{B}$ is now called *Bloch's constant*, and its precise value remains unknown.

2010 *Mathematics Subject Classification.* 32Q25, 32Q20, 32Q05.

[1] There is nothing in the infinite that has not previously been in the finite. See [5, p. 84]

[2] The original proof [15] gave no explicit bound for $|f(z)|$.

2. The first divide: The Schwarz Pick Lemma

As it stands, the Schwarz lemma controls the distortion of holomorphic maps between disks. It was observed by Pick [12] that the Schwarz lemma could be given a radically different interpretation.

Pick's first observation was that we do not require $f(0) = 0$. Indeed, suppose $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic self-map of the unit disk. For $\alpha \in \mathbb{D}$, the Möbius transformation

$$\varphi_\alpha : \mathbb{D} \to \mathbb{D}, \qquad \varphi_\alpha(z) := \frac{z - \alpha}{1 - \overline{\alpha} z}$$

defines an automorphism of $\mathbb{D}$ which sends α the origin. The inverse of φ_α is, moreover, $\varphi_\alpha^{-1} = \varphi_{-\alpha}$. If $f(0) \neq 0$, we can produce a holomorphic self-map of $\mathbb{D}$ which fixes the origin by considering the composite map

$$\varphi_{f(z)} \circ f \circ \varphi_{-z}.$$

Setting $w = \varphi_{-z}(\zeta)$, the familar Schwarz lemma gives

$$|\varphi_{f(z)} \circ f(w)| \quad \leq \quad |\varphi_z(w)|.$$

Explicitly, this reads

$$\left| \frac{f(w) - f(z)}{1 - \overline{f(z)} f(w)} \right| \quad \leq \quad \left| \frac{w - z}{1 - \overline{z} w} \right|.$$

The function

$$d_{\mathrm{H}} : \mathbb{D} \times \mathbb{D} \to \mathbb{R}, \qquad d_{\mathrm{H}}(z, w) := \left| \frac{z - w}{1 - \overline{w} z} \right|$$

defines a distance function on $\mathbb{D}$, the *pseudo-hyperbolic distance*. The function d_{H} defines an honest distance function[3] It does not, however, come from integrating a Riemannian metric.

2.1. Reminder: Distance functions from Riemannian metrics.

A Riemannian metric g on a smooth manifold M is a smoothly varying family of positive-definite quadratic forms g_p on each tangent space T_pM. The metric permits us to calculate the length of smooth curves $\gamma : [0, 1] \to M$ in M by integrating their tangent vectors:

$$\text{length}_g(\gamma) := \int_0^1 |\gamma'(t)|_{g, \gamma(t)} dt. \tag{2.1}$$

We can subsequently construct a distance function $\text{dist}_g : M \times M \to \mathbb{R}$ from the formula

$$\text{dist}_g(p, q) \ := \ \inf_\gamma \int_0^1 |\gamma'(t)|_{g, \gamma(t)} dt.$$

Here, the infimum is taken over all smooth curves $\gamma : [0, 1] \to M$ such that $\gamma(0) = p$ and $\gamma(1) = q$.

2.2. The Poincaré metric.

On the unit disk $\mathbb{D} = \{|z| < 1\}$, we have the *Poincaré metric*:

$$\rho \ := \ \frac{|dz|}{(1 - |z|^2)^2}.$$

Let us compute the length of a line segment ℓ connecting 0 with $w \in (0, 1)$, with respect to the Poincaré metric. Indeed, parametrizing ℓ by $\gamma(t) = tw$, we see that

$$\begin{aligned} \text{length}_\rho(\ell) \ &= \ \int_\ell d\rho \ = \ \int_0^1 \frac{w}{1 - w^2 t^2} dt \\ &= \ \frac{1}{2} \log \frac{1 + w}{1 - w}. \end{aligned}$$

It is an elementary exercise to show that the associated *Poincaré distance function* is given by

$$\text{dist}_\rho(z, w) \ = \ \frac{1}{2} \log \left| \frac{z - w}{1 - z \overline{w}} \right|.$$

Summarizing this discussion, we have recovered the theorem of Pick [12]:

2.3. Theorem. (Schwarz–Pick lemma). Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of the unit disk. Then for all $z, w \in \mathbb{D}$,

$$\text{dist}_\rho(f(z), f(w)) \quad \leq \quad \text{dist}_\rho(z, w).$$

That is, with respect to the Poincaré distance, all holomorphic maps are distance-decreasing.

The Schwarz–Pick lemma holds equally well with the Poincaré distance function replaced by the pseudo-hyperbolic distance d_{H}. It will be clear in subsequent

[3] i.e., a symmetric non-degenerate function satisfying the triangle inequality.

[4] Independent of this, however, the statement given by Pick in [12] was in terms of the Poincaré metric, not the pseudo-hyperbolic distance.

developments why we have chosen to state the Schwarz–Pick lemma in terms of the Poincaré distance. [4]

3. Aside: Intrinsic metrics

The Schwarz–Pick lemma cast the focus of the Schwarz lemma away from holomorphic maps. Instead, the Poincaré distance is the object of focus. There are many examples of such (pseudo-)distance functions, generalizing the Poincaré metric:

The *Carathéodory metric* d_Ω is the pseudo-distance function defined on a domain $\Omega \subseteq \mathbb{C}$ by

$$d_\Omega(z,w) := \sup_f \mathrm{dist}_\rho(f(z), f(w)),$$

where, the supremum is taken over all holomorphic maps $f : \Omega \to \mathbb{D}$. Clearly, all holomorphic maps are distance-decreasing wth respect d_Ω. The Carathéodory pseudo-distance does not define a distance function, in general, since it may be degenerate (e.g., when $\Omega = \mathbb{C}$).

The most notable example of such a pseudo-distance (which can be defined on any complex manifold M) is the *Kobayashi pseudo-distance*:

$$d_K(p,q) \quad := \quad \inf \sum_{i=1}^m \mathrm{dist}_\rho(s_j, t_j),$$

where the infimum is taken over all $m \in \mathbb{N}$, all pairs of points $\{s_j, t_j\}$ in $\mathbb{D}$, and all collections of holomorphic maps $f_j : \mathbb{D} \to M$ such that $f_1(s_1) = p$, $f_m(t_m) = q$, and $f_j(t_j) = f_{j+1}(s_{j+1})$.

A complex manifold is said to be *Kobayashi hyperbolic* if the Kobayashi pseudo-distance is an honest distance function, i.e., non-degenerate.

A compact Riemann surface Σ_g of genus $g \geq 2$ is Kobayashi hyperbolic. Note that in this case, the Carathéodory pseudo-distance vanishes identically!

4. The second rift: The Ahlfors Schwarz lemma

In contrast with the pseudo-hyperbolic distance, the Poincaré distance comes from a Riemannian metric. In particular, one can use the machinery of differential geometry to study holomorphic maps. From this viewpoint, the Poincaré metric has some very special properties.

4.1. **Reminder: Gauss Curvature.** Let $g = \lambda |dz|$ be a Hermitian metric on a domain $\Omega \subseteq \mathbb{C}$, or more generally, a Riemann surface. The *Gauss curvature* of g is the function

$$K_g \;=\; -\frac{1}{\lambda^2}\Delta \log \lambda \;=\; -\frac{1}{\lambda^2}\frac{\partial^2}{\partial z \partial \bar{z}} \log \lambda.$$

In particular, if $K_g \leq -C$, then, setting $u = \log(\lambda)$, we have $\Delta u \geq Ce^{2u}$.

Applying the above formula, the curvature of Poincaré metric

$$\rho = \frac{|dz|}{(1-|z|^2)^2}$$

is seen to be $K_\rho \equiv -1$. That is, the Poincaré metric has constant negative Gauss curvature.

If we suppose that $g = e^u |dz|$ is some metric of negative curvature, in the sense that $K_g \leq -1$, it is natural to ask how g compares with the Poincaré metric. If we let $R = 1 - \varepsilon$ for some $\varepsilon \in (0,1)$, the Poincaré metric on $\mathbb{D}(R)$ is given by $\rho_R = e^v |dz| := \frac{R}{R^2 - |z|^2}|dz|$. Since $K_{\rho_R} \leq -1$, we have $\Delta v \geq e^{2v}$. Hence, on the open set $\Omega := \{z \in \mathbb{D}(R) : u(z) > v(z)\}$, the function $u - v$ is subharmonic: $\Delta(u-v) \geq e^{2u} - e^{2v}$. By the maximum principle, $u - v$ cannot achieve an interior maximum, and hence, the supremum must be approached on the boundary. But Ω cannot have boundary points on $|z| = R$, since $v \to \infty$ as $|z| \to R$. By continuity, at an interior boundary point, $u - v = 0$, yielding a contradiction if Ω is non-empty. We therefore deduce that $u(z) \leq v(z)$ for all $|z| < R$. Letting $\varepsilon \to 0$ recovers the following theorem of Ahlfors:

4.2. **Theorem.** (Ahlfors–Schwarz lemma). Let (Σ, g) be a Riemann surface with Gauss curvature $K_g \leq -1$. Then for all holomorphic maps $f : \mathbb{D} \to \Sigma$,

$$\mathrm{dist}_g(f(z), f(w)) \;\leq\; \mathrm{dist}_\rho(z,w),$$

for all $z, w \in \mathbb{D}$.

In the collected works of Ahlfors [2, p. 341], one finds the following reflection concerning his version of the

Schwarz lemma: Ahlfors confesses that his generalization of the Schwarz lemma had "*more substance that I was aware of*", but "*without applications, my lemma would have been too lightweight for publication*". The applications that Ahlfors alludes to here were the following: First, a proof of Schottky's theorem with definite numical bounds: If $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function such that $f(\mathbb{D}) \cap \{0,1\} = \emptyset$, then

$$\log|f(z)| \ < \ \frac{1+\vartheta}{1-\vartheta}\left(7 + \log|f(0)|\right),$$

for all $|z| \leq \vartheta < 1$. The second was an improved lower bound on the Bloch constant:

$$\mathcal{B} \ \geq \ \frac{\sqrt{3}}{4}.$$

5. The Aftermath of the Ahlfors Schwarz Lemma

The Ahlfors incarnation again redirects the focus of the Schwarz lemma away from holomorphic maps, and away from the distance function; it is the curvature that is the primary object of focus.

5.1. **Reminder: Curvature.** Let (M^n, g) be a Riemannian manifold of dimension n. A *connection* (or *covariant derivative*) ∇ on M provides a coordinate-free tool for computing the directional derivative of a vector field on M.

A connection is said to be *metric* if $\nabla g = 0$, i.e., $w(g(u,v)) = g(\nabla_w u, v) + g(u, \nabla_w v)$ for all vector fields u, v, w. The *Levi-Civita connection* is the unique metric connection on TM which is torsion-free, i.e., $T(u,v) := \nabla_u v - \nabla_v u - [u,v] = 0$, where $[\cdot,\cdot]$ denotes the Lie bracket.

The obstruction to covariant derivatives commuting is measured by the *Riemannian curvature*:

$$R(u,v)w := \nabla_u \nabla_v w - \nabla_v \nabla_u w - \nabla_{[u,v]} w,$$

where ∇ is the Levi-Civita connection. Using the Riemannian metric g, we can write R as

$$R(u,v,w,z) := g(R(u,v)w, z),$$

rendering it scalar-valued. If $e_1, ..., e_n$ denote a local frame for TM near a point p, then we can write

$$R(u,v,w,z) = \sum R_{ijk\ell} u_i v_j w_k z_\ell,$$

where $R_{ijk\ell}$ are the components of R with respect to the frame.

If $u, v \in T_pM$ are unit vectors which form a basis for a two-dimensional subspace of T_pM, we define the *sectional curvature* by

$$\mathrm{Sec}(u,v) := R(u,u,v,v).$$

The metric trace of the curvature tensor yields the Ricci curvature

$$\mathrm{Ric}^g(u,v) = \sum_i R(u, e_i, v, e_i).$$

With respect to a frame, this reads

$$\mathrm{Ric}^g_{k\ell} = \sum_{i,j} g^{ij} R_{ki\ell j}.$$

5.2. **Harmonic maps.** Let $f : (M,g) \to (N,h)$ be a smooth map of Riemannian manifolds. The *energy* of f is defined

$$\mathcal{E}(f) := \int_M |df|^2 dV_g.$$

The Euler–Lagrange operator associated to $\mathcal{E}$ is the *tension field* of f, namely, $\tau(f) = \mathrm{div}(df)$. We say that f is *harmonic* if $\tau(f) = 0$. In [10], it was shown that if f is harmonic, then

$$\begin{aligned} \Delta|df|^2 \ &= \ |\nabla df|^2 + \langle \mathrm{Ric}_g(\nabla_v f), \nabla_v f\rangle \\ &\quad - \langle \mathrm{Riem}_h(\nabla_v f, \nabla_w f)\nabla_v f, \nabla_w f\rangle. \end{aligned} \tag{5.1}$$

In particular, if (M,g) is compact with $\mathrm{Ric}_g > 0$, and $\mathrm{Riem}_h \leq 0$, then f is constant.

5.3. **Curvature in the presence of a complex structure.** For harmonic maps $f : (M,g) \to (N,h)$ between Riemannian manifolds, we required a lower bound on the Ricci curvature of the source metric g and an upper bound on the sectional curvature of the target metric h. It is hard to do much better than this, in general. In the complex analytic setting, however, one can say a lot more:

5.4. **Complex manifolds.** A complex manifold is a smooth manifold M with local holomorphic coordinates in a neighborhood of each point. By a theorem of Newlander–Nirenberg, this is equivalent to existence of an endomorphism $J : TM \to TM$ satisfying $J^2 = -\mathrm{id}$ together with an additional integrability criterion: The relation $J^2 = -\mathrm{id}$ splits the (complexified) tangent bundle into a direct sum of eigenbundles $TM = T^{1,0}M \oplus T^{0,1}M$, corresponding to the eigenvalues $\pm\sqrt{-1}$. The complex structure J is said to be *integrable* if $T^{0,1}M$ is closed under the Lie bracket.

In contrast with the Riemannian theory, the Levi-Civita connection is not the natural connection on the tangent bundle of a complex manifold. Indeed, we say that a metric connection ∇ is Hermitian if $\nabla J = 0$. The preferred connection here is the *Chern connection*: the unique Hermitian connection whose torsion T satisfies $T(Ju, v) = T(u, Jv)$ for all $u, v \in TM$. A complex manifold, therefore, has two distinguished connections on its tangent bundle. The class of manifolds for which these two connections coincide are the well-known *Kähler manifolds*. Such manifolds exist in an almost-baffling abundance: Complex projective space $\mathbb{P}^n$, Euclidean space $\mathbb{C}^n$, the complex ball $\mathbb{B}^n$ all admit Kähler structures. Complex submanifolds of Kähler manifolds remain Kähler. Hence, all projective manifolds and Stein manifolds are Kähler.

5.5. **Curvature of Hermitian Metrics.** For a Hermitian manifold (M, g), the *holomorphic bisectional curvature* is defined

$$\mathrm{HBC}_g(u, v) := R(u, \overline{u}, v, \overline{v}) = \sum R_{i\overline{j}k\overline{\ell}} u_i \overline{u}_j v_k \overline{v}_\ell,$$

for unit vectors $u = (u_1, ..., u_n)$, $v = (v_1, ..., v_n) \in \mathbb{C}^n$, where R denotes the curvature of the Chern connection. The holomorphic bisectional curvature is the restriction of the sectional curvature to the 2–planes which are invariant under the complex structure. The restriction to the diagonal yields the *holomorphic sectional curvature*:

$$\mathrm{HSC}_g(v) := R(v, \overline{v}, v, \overline{v}) = \sum R_{i\overline{j}k\overline{\ell}} v_i \overline{v}_j v_k \overline{v}_\ell,$$

for a unit vector $v \in \mathbb{C}^n$.

The bisectional curvature is very restrictive on the geometry. For instance, a Kähler metric has positive bisectional curvature if and only if it is the Fubini–Study metric on $\mathbb{P}^n$. The holomorphic sectional curvature is comparatively much weaker, but does have important implications on the complex geometry: A complex manifold with negative holomorphic sectional curvature is Kobayashi hyperbolic.

5.6. **Higher-dimensional Schwarz Lemmas.** The first general result in higher-dimensions was obtained by Chern [9] and Lu [11], where they proved the following general Laplace formula (c.f., (5.1)):

5.7. **Theorem.** ([9], [11, Theorem 4.1]). Let $f : (M, g) \to (N, h)$ be a holomorphic map between Hermitian manifolds. Let $u = |\partial f|^2$. Then

$$\begin{aligned} \frac{1}{2}\Delta u \;\; = \;\; & |\nabla \partial f|^2 + g^{i\overline{j}} R^g_{i\overline{j}k\overline{\ell}} g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f^\alpha_p \overline{f^\beta_q} \qquad (5.2) \\ & - R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} \left(g^{i\overline{j}} f^\alpha_i \overline{f^\beta_j} \right) \left(g^{p\overline{q}} f^\gamma_p \overline{f^\gamma_q} \right). \end{aligned}$$

The right-hand side of formula (5.2) has three terms: $|\nabla \partial f|^2$ is the *second fundamental form of* f.[5] The second term is (more or less) the (second) Ricci curvature of g.[6]

The last term in formula (5.2) has caused the most confusion and is indeed the most mysterious. Lu [11] falsely identifies this term as the holomorphic sectional curvature. In the Kähler setting, Royden [13] showed that this term is controlled if one has a non-positive upper bound on the holomorphic sectional curvature of h. In general, however, Royden's technique cannot be used, and it is not clear whether the holomorphic sectional curvature gives suitable control.

[5] This terminology is justified by the fact that if f is a harmonic immersion between Riemannian manifolds, then $|\nabla \partial f|^2$ is the familiar second fundamental form.

[6] In the non-Kähler case, there are three distinct Ricci curvatures: $\mathrm{Ric}^{(1)} := g^{k\overline{\ell}} R_{i\overline{j}k\overline{\ell}}$, $\mathrm{Ric}^{(2)} := g^{i\overline{j}} R_{i\overline{j}k\overline{\ell}}$, and $\mathrm{Ric}^{(3)} := g^{i\overline{\ell}} R_{i\overline{j}k\overline{\ell}}$.

To deduce an estimate on $|\partial f|^2$ from (5.9), we require the maximum principle. If M is compact, then we have access to the maximum principle. In particular, applying (5.2), we discover:

5.8. **Theorem.** ([9, 11, 13]). Let $f : (M, g) \to (N, h)$ be a holomorphic map of rank r between Kähler manifolds. Assume $\text{Ric}_g \geq -C$ and $\text{HSC}_h \leq -\kappa$ for some constants $C > 0$ and $\kappa \geq 0$. If M is compact, then

$$|\partial f|^2 \leq \frac{2Cr}{(r+1)\kappa}.$$

If the source manifold is not compact, there is no guarentee that a maximum exists, and the maximum principle cannot be applied directly. One way to circument this in this in the non-compact case, is to consider manifolds with a certain exhaustion property: A manifold M^n has the K*–exhaustion property* if M is exhausted by a sequence of open submanifolds $M_1 \subset M_2 \subset \cdots \subset M$, with compact closures and such that: (i) for each $k \in \mathbb{N}$, there is a smooth function $v_k \geq 0$ on M_k with $\frac{1}{2}\Delta v_k \leq \frac{R}{n} + K \exp(v_k)$, for some fixed constant $K > 0$; (ii) if p_i is a divergent sequence[7] of points in M_k, then $v_k(p_i) \to \infty$. The unit ball $\mathbb{B}^n$ has the K–exhaustion property with $K = 2n(n+1)$. As a consequence, we can apply (5.2) if the source manifold is $\mathbb{B}^n$:

5.9. **Theorem.** ([9], [11]). Let $f : \mathbb{B}^n \to N$ be a holomorphic map, where $\mathbb{B}^n$ is equipped with the standard metric, and N is a Kähler manifold. If N is equipped with a Kähler metric of negative holomorphic sectional curvature, then f is distance-decreasing.

The K–exhaustion property is, of course, very restrictive. The breakthrough that was required for the Schwarz lemma in the non-compact case was made by Yau in [18]:

5.10. **Theorem.** ([18]). Let (M, g) be a complete Riemannian manifold. Assume $\text{Ric}_g \geq -C$ for some $C \in \mathbb{R}$. Let $f : M \to \mathbb{R}$ be a smooth function which is bounded above. Then for any $\varepsilon > 0$, there is a point $p \in M$ such that $|f(p) - \sup_M f| < \varepsilon$, $\|\text{grad}|_p f\| < \varepsilon$, and $\Delta|_p f < \varepsilon$.

With the maximum principle now available in this level of generality, we have:

5.11. **Theorem.** ([19]). Let $f : (M, g) \to (N, h)$ be a holomorphic map from a complete Kähler manifold wth $\text{Ric}_g \geq -A$, to a Hermitian manifold (N, h) with $\text{HBC}_h \leq B$. Then

$$|\partial f|^2 \leq \frac{A}{B}.$$

5.12. **The Aubin–Yau Inequality.** All forms of the Schwarz lemma so far have arisen (more or less) from applying the maximum principle to (5.9). If we additionally assume that f is biholomorphic, however, we have another Laplacian at our disposal; namely, the target metric Laplacian: $\Delta_h = \text{tr}_h \sqrt{-1}\partial\bar{\partial}$. The Schwarz lemma with the target metric Laplacian was first considered in [3, 20], and hence is referred to as the Aubin–Yau second-order estimate:

5.13. **Theorem.** ([3, 20]). Let $f : (M^n, g) \longrightarrow (N, h)$ be a holomorphic map, which is biholomorphic onto its image. Assume $\text{HBC}_g^{(1)} \geq -\kappa$ and $\text{Ric}_h^{(2)} \leq -C_1 h + C_2 (f^{-1})^* g$ for constants κ, C_1, C_2, with $C_1 > 0$. Then, if M is compact,

$$|\partial f|^2 \leq \frac{nC_2 + \kappa}{C_1}.$$

5.14. **A Unified Approach.** In [14], Rubinstein gave a unified treatment of the Chern–Lu and Aubin–Yau inequalities in the Kähler setting. The underlying philosophy being that these theorems are best understood via holomorphic maps, and not as abstract tensor calculations. This paradigm extends to the study of holomorphic maps between Hermitian manifolds and was made further transparent in [6] using the ideas in [17]. In effect, this goes back to the original formulation of the Schwarz lemma: the focus should be placed on the holomorphic map.

[7] An infinite sequence p_i in M_k is said to be *divergent* if every compact open set in M_k contains only a number of points in this sequence.

The Bochner formula details how to compute the complex Hessian of (the norm of) a section $\sigma \in H^0(\mathcal{E})$ of a holomorphic vector bundle $\mathcal{E}$:

$$\partial\overline{\partial}|\sigma|^2 = |\nabla\sigma|^2 - \langle R^{\mathcal{E}}\sigma, \sigma\rangle, \tag{5.3}$$

where $R^{\mathcal{E}}$ is the curvature of the Chern connection on $\mathcal{E}$. When $\sigma = \partial f$, and $\mathcal{E} = \mathcal{T}_M^* \otimes f^*\mathcal{T}_N$, we have

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}|\partial f|^2 &= \langle \nabla^{1,0}\partial f, \nabla^{1,0}\partial f\rangle \\ &\quad -\sqrt{-1}\langle R^{\mathcal{T}_M^*\otimes f^*\mathcal{T}_N}\partial f, \partial f\rangle.\end{aligned}$$

Since the curvature of the tensor product of bundles splits additively, we get opposing contributions to the curvature from the source and target metrics:

$$R^{\mathcal{T}_M^*\otimes f^*\mathcal{T}_N} = -R^{\mathcal{T}_M} \otimes \mathrm{id} + \mathrm{id} \otimes f^* R^{\mathcal{T}_N}. \tag{5.4}$$

Taking the trace of (5.4) with respect to g, we recover the Chern–Lu formula (5.9). As we indicated previously, we want to understand the target curvature term:

$$g^{i\overline{j}} g^{p\overline{q}} R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^\alpha \overline{f_j^\beta} f_p^\gamma \overline{f_q^\delta}. \tag{5.5}$$

Choose coordinates $(z_1, ..., z_n)$ centered at a point $p \in M$ and $(w_1, ..., w_n)$ at $f(p) \in N$ such that $g_{i\overline{j}} = \delta_{ij}$ and $h_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ at p and $f(p)$, respectively. If $f = (f^1, ..., f^n)$, then with $f_i^\alpha = \frac{\partial f^\alpha}{\partial z_i}$, the coordinates can be chosen such that $f_i^\alpha = \lambda_i \delta_i^\alpha$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = \cdots = 0$, and r is the rank of $\partial f = (f_i^\alpha)$. Hence, (5.5) reads

$$g^{i\overline{j}} g^{p\overline{q}} R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} f_i^\alpha \overline{f_j^\beta} f_p^\gamma \overline{f_q^\delta} = \sum_{\alpha,\gamma} R^h_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \lambda_\alpha^2 \lambda_\gamma^2. \tag{5.6}$$

This motivated Yang–Zheng [17] to consider the following: The *real bisectional curvature* RBC_g of a Hermitian metric g is the function $\mathrm{RBC}_g : \mathcal{F}_M \times \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}$,

$$\mathrm{RBC}_g(\vartheta, \lambda) := \frac{1}{|\lambda|^2} \sum_{\alpha,\gamma} R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \lambda_\alpha \lambda_\gamma.$$

Here, $\mathcal{F}_M$ denotes the unitary frame bundle, $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ denote the components of the Chern connection with respect to the frame ϑ, and $\lambda = (\lambda_1, ..., \lambda_n) \in \mathbb{R}^n \backslash \{0\}$.

If the metric is Kähler, the real bisectional curvature is comparable to the holomorphic sectional curvature. For a general Hermitian metric, however, the real bisectional curvature strictly dominates the holomorphic sectional curvature. [8]

It was observed in [6] that the real bisectional curvature does not give suitable control in the Aubin–Yau second-order estimate. Indeed, if $f : (M, g) \longrightarrow (N, h)$ is biholomorphic onto its image, taking the trace of (5.4), with respect to h, the corresponding curvature term we want to understand is

$$h^{\gamma\overline{\delta}} g^{i\overline{q}} g^{p\overline{j}} R^g_{k\overline{\ell}p\overline{q}} h_{\alpha\overline{\beta}} f_i^\alpha \overline{f_j^\beta} (f^{-1})_\gamma^k \overline{(f^{-1})_\delta^\ell}. \tag{5.7}$$

Again, choose coordinates at p and $f(p)$ such that $g_{i\overline{j}} = \delta_{ij}$, $h_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$, and $f_i^\alpha = \lambda_i \delta_i^\alpha$. Then (5.7) reads

$$\sum_{i,k} R^g_{i\overline{i}k\overline{k}} \lambda_i^2 \lambda_k^{-2}. \tag{5.8}$$

This is not controlled by the real bisectional curvature (what is the vector here?).

To understand both (5.6) and (5.8), introduce the matrix $\mathcal{R} \in \mathbb{R}^{n\times n}$ with entries $\mathcal{R}_{\alpha\gamma} := R_{\alpha\overline{\alpha}\gamma\overline{\gamma}}$. If $v = (v_1, ..., v_n) \in \mathbb{R}^n \backslash \{0\}$, then the real bisectional curvature can be written as the Rayleigh quotient

$$\mathrm{RBC}_g(v) := \frac{v^t \mathcal{R} v}{v^t v}.$$

From the scale invariance of the Rayleigh quotient, we may equivalently assume that $v \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. It is then immediate that the maximum of RBC_g is always achieved, and the maximum occurs precisely when v is the eigenvector of $\mathcal{R}$ corresponding to the largest eigenvalue.

Since the vector v arising in (5.6) is given by the vector of principal values

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = \cdots = 0,$$

of ∂f, where $r = \mathrm{rank}(\partial f)$, it is clear that we need only control the Rayleigh quotient over the cone

$$\Gamma := \{x \in \mathbb{R}^n_+ : x_1 \geq \cdots \geq x_n \geq 0\}.$$

For the Aubin–Yau inequality, we want to estimate (from below), the quantity (5.8). Let $\Gamma_+ := \{(x_1, ..., x_n) \in \mathbb{R}^n_+ : x_1 \geq x_2 \geq \cdots \geq x_n > 0\}$ denote the cone of ordered positive n-tuples. For a vector

[8] It is not strong enough, however, to dominate the Ricci curvatures.

$v \in \Gamma_+$, we denote by $u_v := v_\circ^{-1}$ the vector which inverts v with respect to the Hadamard product. That is, if $v = (v_1, ..., v_n) \in \Gamma_+$, then $u_v = (v_1^{-1}, ..., v_n^{-1})$. Then a bound on (5.8) translates to a bound on the generalized Rayleigh quotient

$$\frac{u_v^t \mathcal{R} v}{|u_v||v|}.$$

To distinguish this from the real bisectional curvature, we define the following:

5.15. **Definition.** Let (M, ω) be a Hermitian manifold. Define the *first Schwarz bisectional curvature*

$$\mathrm{SBC}_\omega^{(1)} : \mathcal{F}_M \times \Gamma_+ \to \mathbb{R}, \ \ \mathrm{SBC}_\omega^{(1)}(\vartheta, v) \ := \ \frac{u_v^t \mathcal{R} v}{|u_v||v|},$$

and the *second Schwarz bisectional curvature*

$$\mathrm{SBC}_\omega^{(2)} : \mathcal{F}_M \times \Gamma \to \mathbb{R}, \ \ \mathrm{SBC}_\omega^{(2)}(\vartheta, v) \ := \ \frac{v^t \mathcal{R} v}{|v|^2},$$

where $\mathcal{R}$ is the matrix with entries $\mathcal{R}_{\alpha\gamma} := R_{\alpha\overline{\alpha}\gamma\overline{\gamma}}$ with respect to the frame ϑ.

We now have the tools to give a unified treatment of (and extend) the various forms of the Schwarz lemmas:

5.16. **Theorem.** (Hermitian Chern–Lu). Let $f : (M^n, g) \longrightarrow (N, h)$ be a holomorphic map of rank r between Hermitian manifolds. Assume $\mathrm{Ric}_g^{(2)} \geq -C_1 g - C_2 f^* h$, and $\mathrm{SBC}_g^{(2)} \leq -\kappa$ for some constants such that $\kappa - C_2 > 0$. If M is compact, then

$$|\partial f|^2 \ \ \leq \ \ \frac{C_1 r}{\kappa - C_2}.$$

5.17. **Theorem.** (Hermitian Aubin–Yau). Let $f : (M^n, g) \longrightarrow (N, h)$ be a holomorphic map, which is biholomorphic onto its image. Assume $\mathrm{SBC}_g^{(1)} \geq -\kappa$ and $\mathrm{Ric}_h^{(2)} \leq -C_1 h + C_2 (f^{-1})^* g$ for constants κ, C_1, C_2, with $C_1 > 0$. Then, if M is compact,

$$|\partial f|^2 \ \ \leq \ \ \frac{n C_2 + \kappa}{C_1}.$$

The Schwarz bisectional curvatures yield an interesting comparison between the Chern–Lu and Aubin–Yau inequalities. Indeed, the Chern–Lu inequality requires an upper bound on $\mathrm{SBC}^{(2)}$, a Rayleigh quotient, which is well-known to give a variational characterization of the eigenvalues. The Aubin–Yau inequality requires a lower bound on $\mathrm{SBC}_\omega^{(1)}$, a generalized Rayleigh quotient, which is known to give a variational characterization of the singular values. Therefore, at least philosophically, the Chern–Lu inequality is to the Aubin–Yau inequality what the eigenvalue decomposition is to the singular value decomposition.

5.18. **The Chen–Cheng–Look Schwarz Lemma.** So far, the underlying idea is that a Schwarz lemma is what results from applying a Bochner formula to some elementary symmetric function associated with some holomorphic map. In particular, if one has a Bochner formula, one should get a Schwarz lemma.[9] The following Schwarz lemma of Chen–Cheng–Look [8] does not seem to fit into the Bochner paradigm:

5.19. **Theorem.** Let $f : (M, g) \longrightarrow (N, h)$ be a holomorphic map from a complete Kähler manifold (M, g) with $\mathrm{HSC}_g \geq \kappa_1$, into a Hermitian manifold (N, h) with $\mathrm{HSC}_h \leq \kappa_2 < 0$. If the sectional curvature of g is bounded from below, then

$$|\partial f|^2 \ \ \leq \ \ \frac{\kappa_1}{\kappa_2}.$$

The lower bound on the sectional curvature is, of course, automatic if M is compact, and appears because of the use of a comparison theorem.

Since the Bochner formula will typically yield a Ricci term, the Chen–Cheng–Look Schwarz lemma is, in a sense, the outlier. If one places holomorphic maps as the center of focus, however, we see that a lower bound on the holomorphic sectional appears in the source of the Aubin–Yau estimate, while an upper bound on the holomorphic sectional curvature appears on the target in the Chern–Lu inequality. This idea leads to the following 8 real dimensional family of Schwarz lemmas, which includes the Chen–Cheng–Look Schwarz lemma as a special case:

[9] This can be thought of as a 'dual' version of Bloch's principle.

5.20. **Theorem.** Let $f : (M^n, g) \longrightarrow (N, h)$ be a holomorphic map of rank r between Hermitian manifolds with $\text{SBC}_g^{(1)} \geq -\kappa_1$ and $\text{SBC}_h^{(2)} \leq -\kappa_2$, for some constants $\kappa_1, \kappa_2 \geq 0$. Assume there is a Hermitian metric μ on M such that, for constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$, with $C_3 > 0$, $\kappa_2 - C_2 > 0$, we have

$$-C_1\mu - C_2 f^* h \ \leq \ \text{Ric}_\mu^{(2)} \ \leq \ -C_3\mu + C_4 g.$$

Then, if M is compact,

$$|\partial f|^2 \ \leq \ \frac{C_1 nr(\kappa_1 + C_4)}{C_3(\kappa_2 - C_2)}.$$

Concluding Remarks. In the same manner that the Bloch principle predicts the Schwarz lemma from a vanishing or rigidity theorem, we have proposed that a Bochner principle will give rise to a Schwarz lemma. Given the number of possible Bochner formulas that exist, one would suspect an abundance of Schwarz lemmas to appear.

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