DIVISORS AND LINE BUNDLES

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Abstract. We discuss the relation between Weil divisors, Cartier divisors, and line bundles.

This theory is standard, and can be found in

§1. Divisors and Line Bundles on Complex Manifolds

Let X be a complex manifold (not necessarily compact, but connected).

Definition 1.1. A *Weil divisor* on X is a formal integral combination of codimension-one irreducible closed subvarieties[1] of X:

$$\sum_{D_j \subset_{\text{codim-one}} X} a_j D_j,$$

where the $a_j \in \mathbb{Z}$ are all zero except a finite number of them.
In other words, a Weil divisor on X is a cycle of (pure) codimension one. The set of Weil divisors on X will be denoted by $\text{Div}(X)$. It is clear that $\text{Div}(X)$ is an additive group. If X is a curve, a Weil divisor is an integral combination of points on X. If all the coefficients of a Weil divisor D are non-negative, we say that D is *effective* and write $D \geq 0$.
Let now L be a holomorphic line bundle on X. Let s be a non-zero section of L. Given an irreducible closed codimension-one subvariety $D \subset X$, we introduce a valuation[2] $\text{ord}_D(s)$ as follows: Let $U \subset X$ be a sufficiently small open set over which L is trivial, i.e., $L|_U \simeq U \otimes \mathbb{C}$. The section $s|_U$ is a rational function on U
Given an irreducible closed subvariety $D \subset X$, write $D = \{s = 0\}$ locally near a fixed point $p \in D$, where s is a holomorphic function. For any holomorphic function f defined near p, we can define the *order of vanishing of f along D* to be the smallest integer λ such that $f = s^\lambda \cdot g$ in the stalk $\mathcal{O}_{X,p}$, where g is a non-vanishing holomorphic function near p, i.e., a unit in $\mathcal{O}_{X,p}$. Of course, this definition required a choice of the point p. To show that the order of vanishing is independent of the choice of $p \in D$, we need the following lemma which

[1] What we will from now on call an analytic hypersurface.

[2] Recall that if k is a field and G is a totally order abelian group, a *valuation* of k with values in G is a map $v : k\backslash\{0\} \to G$ such that for all $x, y \in k\backslash\{0\}$,

(i) $v(xy) = v(x) + v(y)$;
(ii) $v(x + y) \geq \min(v(x), v(y))$.

asserts that relatively prime elements in one stalk $\mathcal{O}_{X,p}$ remain relatively prime in nearby stalks:

Lemma 1.2. Let f and g be relatively prime elements of $\mathcal{O}_{\mathbb{C}^n,0}$. If $p \in \mathbb{C}^n$ is such that $|p| < \varepsilon$ for any $\varepsilon > 0$, then f and g are relatively prime elements of $\mathcal{O}_{\mathbb{C}^n,p}$.

The order of vanishing of a holomorphic function f along an analytic hypersurface $D \subset X$ is therefore independent of the point p used to originally define it, and we write $\text{ord}_D(f)$. In particular, a holomorphic function on X gives rise to an effective Weil divisor

$$\text{div}(f) := \sum \text{ord}_D(f) \cdot D.$$

It is clear that if f and g are two holomorphic functions, then

$$\text{ord}_D(f \cdot g) = \text{ord}_D(f) + \text{ord}_D(g),$$

and

$$\text{ord}_D(f + g) \geq \min\{\text{ord}_D(f), \text{ord}_D(g)\}.$$

In particular, ord_D defines a valuation. If f is meromorphic, written locally as the ratio $f = g/h$, then

$$\text{ord}_D(f) \ := \ \text{ord}_D(g) - \text{ord}_D(h).$$

The divisor associated to a meromorphic function will be effective only if the meromorphic function is, in fact, holomorphic.

Example 1.3. Let $f : \mathbb{C} \to \mathbb{C}$ be the function

$$f(z) := \frac{z}{z^3 - 1}.$$

The the analytic hypersurfaces of $\mathbb{C}$ are just points, and the divisor associated to f is given by

$$\text{div}(f) = 1 \cdot [0] - 3 \cdot [-1].$$

Remark 1.4. Every complex curve (i.e., a complex manifold of dimension one) has divisors, many of them! In higher dimensions, however, this is not the case, even for surfaces. The standard examples to mention are the Inoue surfaces – they have no curves on them, so they certainly do not have divisors. On the other hand, suppose X (holomorphically) embeds into some projective space $\mathbb{P}^n$. This gives us an easy way to get divisors on M: simply intersect M with hyperplanes in $\mathbb{P}^n$. The converse of this phenomenon will lead to the notion of ample divisors in the next section.

Definition. Let $(X, \mathcal{O}_X)$ be a complex manifold. For an open set $U \subset X$, we let $\mathcal{A}(U)$ denote the set of elements $f \in \mathcal{O}_X(U)$ such that $f_x \in \mathcal{O}_{X,x}$ is a not a zero-divisor for all $x \in U$. The sheaf $\mathcal{M}_X$ of meromorphic functions on X is the sheaf associated to the presheaf of ring of fractions

$$\mathcal{M}_X(U) := \mathcal{O}_X(U)[\mathcal{A}_X(U)]^{-1}.$$

The sheaf of units of $\mathcal{O}_X$ (respectively, $\mathcal{M}_X$) is denoted $\mathcal{O}_X^*$ (respectively, $\mathcal{M}_X^*$).

Definition. A *Cartier divisor* on X is a global section of the sheaf associated to the quotient presheaf $\mathcal{M}_X^*/\mathcal{O}_X^*$. The set of Cartier divisors on X is denoted $\mathrm{Div}(X)$.
In more detail, a Cartier divisor on X is specified by a collection of local data $\{(U_\alpha, f_\alpha)\}$, where (U_α) is an open cover of X, $f_\alpha \in \mathcal{M}_X^*(U_\alpha)$, and for any non-empty intersection $U_\alpha \cap U_\beta$, there are functions $g_{\alpha\beta} \in \mathcal{O}_X^*(U_\alpha \cap U_\beta)$ such that

$$f_\alpha = g_{\alpha\beta} f_\beta$$

on $U_\alpha \cap U_\beta$.

Definition. There is a (canonical) map $\mathrm{Div}(X) \to \mathrm{Div}_W(X)$ defined by sending a Cartier divisor D to the Weil divisor $\sum \mathrm{ord}_V(D) \cdot [V]$, where V is an analytic hypersurface in X.

Remark. In general, this map will fail to be injective or surjective. If X is normal, however, this map is injective, and is an isomorphism if X is smooth.

Definition. Let $(X, \mathcal{O}_X)$ be a complex manifold. The Picard group $\mathrm{Pic}(X)$ is the set of isomorphism classes of holomorphic line bundles on X. The group operation is given by the tensor product $(L_1, L_2) \mapsto L_1 \otimes L_2$, the inverse operation being $L \mapsto L^*$, the dual line bundle.

Proposition. The Picard group $\mathrm{Pic}(X)$ is isomorphic to $H^1(X, \mathcal{O}_X^*)$.

Proof. □

To describe the equivalence between holomorphic line bundles and divisors, we consider the exact sequence of sheaves

$$0 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathcal{M}_X^* \longrightarrow \mathcal{M}_X^*/\mathcal{O}_X^* \longrightarrow 0.$$

This gives rise to a long exact sequence on sheaf cohomology:

$$0 \to H^0(X, \mathcal{O}_X^*) \to H^0(X, \mathcal{M}_X^*) \to \mathrm{Div}(X) \xrightarrow{\delta} \mathrm{Pic}(X) \to H^1(X, \mathcal{M}_X^*) \to \cdots,$$

where δ is the connecting homomorphism.

Definition. A line bundle $L \to X$ is said to be *very ample* if the sections of L furnish an embedding of X into some projective space. A line bundle L is said to be *ample* if there is an integer $m > 0$ such that $L^{\otimes m}$ is very ample.

Let us give some details on how sections of line bundles give rise to embeddings in projective space. Let $s_0, ..., s_k$ be a basis for the vector space $H^0(X, L)$. Each section s_j is a map from X to L given by sending a point $x \in X$ into the fibre $L_x \ni s_j(x)$. The fibres are one-dimensional vector spaces, and locally $L|_U \simeq U \times \mathbb{C}$. So we can make sense of a map $f : X \longrightarrow \mathbb{P}^k$ by defining

$$f(x) := (s_0(x) : \cdots : s_k(x)) \in \mathbb{C}^{k+1},$$

at least locally. If we pick another trivializing of L, the transition functions are elements of $\mathbb{C}^*$, so the values of f differ only by a scalar multiple. Hence, so long as there are no $x \in X$ such that $s_j(x) = 0$ for all $0 \leq j \leq k$, we have an embedding of X into $\mathbb{P}^k$.

A Brief Reminder of Čech cohomology. Let X be a topological space with an open covering $(U_\alpha)_{\alpha \in A}$, where A has a fixed well-ordering. The intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ is denoted by $U_{\alpha_0, \cdots, \alpha_p}$. A p-simplex σ is an ordered collection of the open sets in the cover, i.e., $\sigma = (U_{\alpha_0}, ..., U_{\alpha_p})$, such that the support $|\sigma| := U_{\alpha_0, ..., \alpha_p}$ is non-empty. Define qth boundary of the p-simplex to be the $(p-1)$-simplex $\partial_j \sigma := (U_{\alpha_0}, ..., U_{\alpha_{q-1}}, U_{\alpha_{q+1}}, ..., U_{\alpha_p})$. The boundary of σ is then given by the usual formula

$$\partial \sigma := \sum_{j=0}^{p} (-1)^{j+1} \partial_j \sigma.$$

Let now $\mathscr{F}$ be a sheaf of abelian groups on X. A p–cochain of (U_α) with coefficients in $\mathscr{F}$ is a section $\mathscr{F}(|\sigma|) = \mathscr{F}(U_{\alpha_0, ..., \alpha_p})$ for some p–simplex σ. Introduce a complex of abelian groups $C^\bullet((U_\alpha), \mathscr{F})$ by setting, for each $p \geq 0$,

$$C^p((U_\alpha), \mathscr{F}) := \prod_{\alpha_0 < \cdots < \alpha_p} \mathscr{F}(U_{\alpha_0, ..., \alpha_p}).$$

To define the coboundary map $\delta_p : C^p \to C^{p+1}$, let $\mathrm{res}^{|\partial_j \sigma|}_{|\sigma|} : \mathscr{F}(|\partial_j \sigma|) \to \mathscr{F}(|\sigma|)$ denote the restriction map. Then set

$$(\delta_p f)(\sigma) := \sum_{j=0}^{p+1} (-1)^j \mathrm{res}^{|\partial_j \sigma|}_{|\sigma|} f(\partial_j \sigma).$$

It is easy to verify that $\delta_{p+1} \circ \delta_p = 0$. A p-cocycle is a p-cochain which lies in the kernel of δ_p. A 1-cocycle f satisfies, for every non-empty open set $U_{\alpha\beta\gamma} := U_\alpha \cap U_\beta \cap U_\gamma$,

$$f(U_\beta \cap U_\gamma)|_U - f(U_\alpha \cap U_\gamma)|_U + f(U_\alpha \cap U_\beta)|_U = 0.$$

Definition. Let X be a compact Kähler manifold. A line bundle $L \to X$ is said to be *geometrically positive* if it admits a Hermitian metric h whose curvature from $\Theta_h = -\sqrt{-1}\partial\overline{\partial}\log h$ is positive (as a $(1,1)$–form).

First Chern Class of a Line Bundle. Let X be a compact complex manifold. The exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \to 0$$

is an exact sequence of sheaves. This gives rise to a long exact sequence on cohomology

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \to \cdots .$$

In particular, we have a map $\delta : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ given by the connecting homomorphism on cohomology. For an equivalence class of holomorphic line bundles $[L]$ over X, the image $\delta([L]) \in H^2(X, \mathbb{Z})$ is called the *first Chern class of* L, written $c_1(L)$.

Remark. There is only one holomorphic line bundle one can associate to a given complex manifold X. For this reason, the line bundle is called the *canonical bundle of* X, denoted K_X. It is defined to be the top exterior power of the cotangent bundle, i.e., $K_X := \Lambda^{\dim X} T_X^*$. It is sometimes referred to as the *determinant line bundle*. The dual of the canonical bundle is called the *anti-canonical bundle*, additively denoted $-K_X$, or multiplicatively denoted K_X^{-1}.

Remark. A Hermitian metric on the canonical bundle $K_X \to X$ is equivalent to a volume form on X. In particular, if (X, ω) is Kähler, the Ricci form $\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial}\log(\omega^n)$ is the curvature form of a Hermitian metric on $-K_X$. In particular,

$$c_1(-K_X) = \left[-\frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log(\omega^n) \right],$$

i.e., (2π times) the first Chern class of the anti-canonical bundle is represented by the Ricci form.

Definition. Let X be a complex manifold. The *first Chern class of* X is defined to be the first Chern class of the anti-canonical bundle, i.e., $c_1(X) := c_1(-K_X)$.

Definition. A line bundle $L \to X$ is said to be *algebraically positive* (respectively, *algebraically negative*) if the cohomology class $c_1(L) \in H^2(X, \mathbb{Z})$ is represented by a positive form (respectively, negative form).[3]

Theorem. Let $L \to X$ be a holomorphic line bundle on X. Then L is geometrically positive if and only if L is algebraically positive.

Proof. □

[3]That is, there is a closed 2-form ω on X, which we can write locally as $\omega = \frac{\sqrt{-1}}{2}\sum_{i,j=1}^{n} \omega_{i\bar{j}}(z) dz_i \wedge d\bar{z}_j$, such that the matrix $(\omega_{i\bar{j}})$ is positive definite.

Definition. A Kähler manifold (X, ω) is said to be

(i) a *Fano manifold* if $c_1(X) > 0$ in $H^2(X, \mathbb{R})$;
(ii) a *Calabi–Yau manifold* if $c_1(X) = 0$ in $H^2(X, \mathbb{R})$;
(iii) of *general type* if $c_1(X) < 0$.

Examples.

(i) Fano manifolds of dimension one are curves of genus 0. In dimension two, Fano surfaces are called Del Pezzo surfaces.
(ii) Calabi–Yau manifolds of dimension one are curves of genus 1. In dimension two, according to the Kodaira classification of compact complex surfaces, the Calabi–Yau manifolds consist of:
  (a) Tori;
  (b) K3 surfaces (both algebraic and non-algebraic);
  (c) Primary and Secondary Kodaira surfaces.
(iii) One-dimensional manifolds of general type are curves of genus $g \geq 2$.

Remark. The condition of having a definite or trivial first Chern class is very restrictive. Most compact Kähler manifolds do not fit into one of these restrictive categories. Indeed, this is even seen at the level of surfaces by taking products: $\mathbb{S}^1 \times \mathbb{T}$, where $\mathbb{T} := \mathbb{C}/\Lambda$ is an elliptic curve.

Hermitian Vector Bundles.

Definition. Let $\pi : E \to M$ be a complex vector bundle over a smooth manifold. Each fiber $E_x := \pi^{-1}(x)$ is a complex vector space of rank k. A *Hermitian metric on* E is a smoothly varying family of Hermitian inner products on each fiber E_x.

Definition. Let $E \to M$ be a complex vector bundle over a smooth manifold. A *connection* on E is a linear map $\nabla : \Gamma(M, \Lambda^p T_M^* \otimes E) \longrightarrow \Gamma(M, \Lambda^{p+1} T_M^* \otimes E)$ satisfying the Leibniz rule

$$\nabla(u \wedge v) = du \wedge v + (-1)^p u \wedge \nabla v,$$

where u, v are E–valued forms of type p and q, respectively.

Recall that a frame for a vector bundle $E \to M$ over an open set $U \subset M$ is a set $\{\sigma_1, ..., \sigma_k\}$ of sections of E over U such that $\{\sigma_1(x), ..., \sigma_k(x)\}$ forms a basis for each E_x, with $x \in U$. Let us give a description of a connection in terms of frames. Let $\sigma_1, ..., \sigma_k$ be a frame for E, and let ∇ be a connection on E. The action of ∇ on a section σ_i can be expressed as

$$\nabla \sigma_i = \sum_{i,j} \vartheta_{ij} \sigma_j,$$

where ϑ_{ij} are 1–forms. The matrix (ϑ_{ij}) is called the *connection matrix* for ∇.