ON THE NON-NEGATIVITY OF THE DIRICHLET ENERGY OF A WEIGHTED GRAPH

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ABSTRACT. In this short note, we address the question of when a weighted graph (with possibly negative weights) has non-negative Dirichlet energy.

Let G be a finite weighted graph, with vertices $V(G) = \{x_1, ..., x_n\}$, and weighting specified by its adjacency matrix $A \in \mathbb{R}^{n \times n}$. The Dirichlet energy for a weighted graph is defined

$$\mathcal{E}(f) := \sum_{i,j=1}^{n} A_{ij}(f(x_i) - f(x_j))^2,$$

where $f : V(G) \to \mathbb{R}$ is a function defined on the vertices of G.
The following elementary result is well-known:

Theorem 1. Suppose $A \in \mathbb{R}^{n \times n}$ is a non-negative matrix, i.e., the entries A_{ij} are non-negative real numbers, for each $i, j = 1, ..., n$. Then $\mathcal{E}(f) \geq 0$ for all $f : V(G) \to \mathbb{R}$.

It is natural to ask the following (apparently unknown) question:

Question. Given a finite weighted graph (G, A), where $A \in \mathbb{R}^{n \times n}$ is a real matrix, what conditions on A are necessary or sufficient for the inequality $\mathcal{E}(f) \geq 0$ to hold for all $f : V(G) \to \mathbb{R}$?

The main theorem of this note is to give an answer to this problem. To this end, let us recall some terminology arising from distance geometry:

Definition 2. Let $A = (A_{ij}) \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. We say that A is a Euclidean distance matrix if there is a vector $x = (x_1, ..., x_n) \in \mathbb{R}^n$ such that $A_{ij} = (x_i - x_j)^2$ for each $i, j = 1, ..., n$.
The set of all $n \times n$ Euclidean distance matrices forms a convex cone which we denote by $\mathbb{EDM}^n$. Recall that the Frobenius inner product of two matrices $A, B \in \mathbb{R}^{n \times n}$ is defined by

$$(A, B)_{\mathrm{F}} := \mathrm{tr}(AB^t).$$

This dual pairing allows us to define the dual EDM cone $\mathbb{EDM}^*$:

Definition 3. The dual EDM cone $\mathbb{EDM}^*$ is given by

$$\mathbb{EDM}^* := \{A \in \mathbb{R}^{n \times n} : (A, B)_{\mathrm{F}} \geq 0 \quad \forall B \in \mathbb{EDM}\}.$$

Theorem 4. Let (G, A) be a weighted finite graph. Then the Dirichlet energy $\mathcal{E}$ is non-negative if and only if A lies in the dual EDM cone.

Proof. If $V(G) = \{x_1, ..., x_n\}$ is the vertex set of some graph, then we may construct a Euclidean distance matrix $B(f)$ from a graph function $f : V(G) \to \mathbb{R}$ by setting $B(f)_{ij} = (f(x_i) - f(x_j))^2$. In particular, since

$$\mathrm{tr}(AB(f)) \quad = \quad \sum_{i,j=1}^{n} A_{ij} B(f)_{ij} \quad = \quad \sum_{i,j=1}^{n} A_{ij} (f(x_i) - f(x_j))^2,$$

we see that the Dirichlet energy $\mathcal{E}$ of a weighted graph (G, A) is non-negative if and only if $\mathrm{tr}(AB) \geq 0$ for all Euclidean distance matrices $B \in \mathbb{EDM}$. $\square$

Remark 5. It is natural to ask what the relation is (if any) between the EDM cone (and its dual) and the $\mathbb{PSD}$ cone, i.e., the cone of (symmetric) positive semi-definite matrices. Dattorro [2] has shown that

$$\mathbb{EDM}^n = \mathbb{S}^n_{\mathrm{H}} \cap \left((\mathbb{S}^n_{\mathrm{C}})^{\perp} - \mathbb{PSD}^n \right) \subset \mathbb{R}^{n \times n}_{\geq 0}.$$

Here, $\mathbb{S}^n_{\mathrm{H}}$ denotes the space of symmetric $n \times n$ hollow matrices, i.e., symmetric matrices with no non-zero entries on its diagonal; $\mathbb{S}^n_{\mathrm{C}}$ denotes the geometric centering subspace:[1]

$$\mathbb{S}^n_{\mathrm{C}} := \{A \in \mathbb{S}^n : A\mathbf{e} = 0\},$$

where $\mathbf{e} = (1, ..., 1)^t$. The orthogonal complement of $\mathbb{S}^n_{\mathrm{C}}$ is then

$$(\mathbb{S}^n_{\mathrm{C}})^{\perp} = \{u\mathbf{e}^t + \mathbf{e}u^t : u \in \mathbb{R}^n\}.$$

In particular, from standard properties of cones, we observe that

$$(\mathbb{EDM}^n)^* \quad = \quad \mathbb{S}^n_{\mathrm{D}} - \mathbb{S}^n_{\mathrm{C}} \cap \mathbb{PSD}^n,$$

where $\mathbb{S}^n_{\mathrm{D}}$ is the cone of diagonal matrices.

By appealing to the eigenvalue characterization of the dual EDM cone given in [1], we have the following corollary:

[1]It is more natural to refer to $\mathbb{S}^n_{\mathrm{C}}$ as the annihilator of $\mathbf{e} = (1, ..., 1)^t \in \mathbb{R}^n$.

Corollary 6. Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then $A \in \mathbb{EDM}_n^*$ if and only if

$$\lambda_1 \geq \sum_{k=2}^{n} r_k \lambda_k,$$

for all Perron-weights $0 \leq r_k \leq 1$.

Remark 7. The meaning of the non-standard terminology *Perron-weights* is the following: The well-known Schoenberg criterion [3] states that a symmetric hollow matrix[2] Σ is a Euclidean distance matrix if and only if it is negative semi-definite on the hyperplane $H = \{x \in \mathbb{R} : x^t \mathbf{e} = 0\}$, where $\mathbf{e} = (1, ..., 1)^t$. The Perron–Frobenius theorem asserts that the largest eigenvalue (the Perron root) of the EDM Σ is positive and occurs with eigenvector in the non-negative orthant $\mathbb{R}^n_{\geq 0}$. Therefore, if $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$ denote the eigenvalues of a non-trivial Euclidean distance matrix[3] Σ, then $\delta_1 > 0$ and $\delta_2, ..., \delta_n \leq 0$. We then define the Perron weights, for each $2 \leq k \leq n$, by

$$r_k := -\frac{\delta_k}{\delta_1} \in [0, 1].$$

References

[1] Broder, K., *An eigenvalue characterization of the dual EDM cone*, to appear in the Bulletin of the Australian Mathematical Society.

[2] Dattorro, Jon. Equality relating Euclidean distance cone to positive semidefinite cone. Linear Algebra Appl. 428 (2008), no. 11-12, 2597–2600. MR2416574

[3] Schoenberg., I. J., Remarks to Maurice Fréchet's article "Sur la définition axiomatique d'une classe déspace distanciés vectoriellement applicable sur l'espace de Hilbert". Annals of Mathematics, 36(3):724–732, July 1935.

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[2]That is, a matrix with non-zero entries on its diagonal.

[3]That is, a Euclidean distance matrix with $\delta_1 > 0$.