CONE ANGLES AND LCTS

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Abstract. We add details to the proof of [1, Theorem 8].

Let (X, ω) be a compact Kähler manifold of dimension n. Let $f : X \to \Delta^*$ be a holomorphic submersion over the punctured unit disk $\Delta^* = \Delta \backslash \{0\} \subset \mathbb{C}$. Let $K_{X/\Delta^*} := K_X \otimes f^* K_{\Delta^*}^{-1}$ denote the relative canonical bundle. We will assume $K_{X/\Delta^*} \simeq \mathcal{O}_X$, i.e., the fibers $X_s := f^{-1}(s)$ are Calabi–Yau for all $s \in \Delta^*$. A trivializing section $\Omega \in H^0(X, K_{X/\Delta^*})$ yields a family of trivializing sections $\Omega_s := \Omega|_{X_s}$ of K_{X_s}. In particular, we have a smooth family of volume forms $\mu_s := (\sqrt{-1})^{(n-1)^2} \Omega_s \wedge \overline{\Omega}_s$.

The following theorem, due to Yashan Zhang [1], shows that

$$f_* \mu = \int_{X_s} \mu_s$$

is of conical-type, modulo some logarithmic poles. That is, if we let ω_{cone} be the conical Kähler metric

$$\omega_{\text{cone}} = \sqrt{-1} \frac{ds \wedge d\overline{s}}{|s|^{2(1-\beta)}},$$

for some $\beta \in (0, 1] \cap \mathbb{Q}$, we show that

$$f_* \mu \sim (-\log |s|)^{N-1} \omega_{\text{cone}},$$

for some $N \in \mathbb{N}$.

Remark. In the course of the proof, we will see that the cone angle β of the metric ω_{cone} is given by the log canonical threshold of the central fiber X_0.

Theorem. Let X be a compact Kähler manifold of dimension n with μ as above. Let Δ denote the unit disk in $\mathbb{C}$ with coordinate s. Let $f : X \to \Delta$ be a family of Calabi–Yau manifolds with singular fiber $X_0 = f^{-1}(0)$.[1] There exists $\beta \in \mathbb{Q} \cap (0, 1]$, $N \in \mathbb{N}$, and $C \geq 1$ such that

$$C^{-1} (-\log |s|)^{N-1} \omega_{\text{cone}} \ \leq \ f_* \mu \ \leq \ C (-\log |s|)^{N-1} \omega_{\text{cone}}.$$

[1]That is, $f : X \to \Delta$ is a proper surjective holomorphic map of relative dimension $n-1$ with connected fibers whose generic fiber is Calabi–Yau.

In particular, since the canonical ω_{can} on the base solves the Monge–Ampère equation

$$\omega_{\text{can}}^{\kappa} = (\omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi)^{\kappa} = \frac{f_*\mu}{V_0\binom{n}{\kappa}} e^{\varphi} \sim f_*\mu,$$

this determines the asymptotics of the canonical metric when the base is a curve.

Proof. Let $\pi : \widetilde{X} \to X$ be a resolution of X_0 such that $\widetilde{X}_0 = \pi^{-1}(X_0)$ has simple normal crossings, i.e.,

$$\widetilde{X}_0 = \sum_j a_j E_j,$$

where E_j are irreducible divisors with defining sections σ_j. By adjunction, we know that

$$K_{\widetilde{X}} = \pi^* K_X + \sum_j k_j E_j.$$

In particular, we can write

$$\pi^*\mu = \prod_j |\sigma_j|^{2k_j} \widetilde{\mu},$$

where $\widetilde{\mu}$ is a smooth nondegenerate volume form on $\widetilde{X}$. We want to compute the asymptotics of $f_*\mu$ near the central fiber. It suffices to calculate the asymptotics of

$$\int_{\widetilde{X}_s} \pi^*\mu,$$

as s approaches 0, where $\widetilde{X}_s := \pi^{-1}(X_s)$.
Fix a point $x_0 \in \widetilde{X}_0$ and let $J(x_0) := \{j : x_0 \in E_j\}$. In a chart $U_0 \ni x_0$ with coordinates $(z_1, ..., z_n)$, the map $\widetilde{f} := f \circ \pi : \widetilde{X} \to \Delta$ can be expressed as

$$f(z_1, ..., z_n) \ = \ \prod_{j \in J(x_0)} z_j^{a_j} \ =: \ s.$$

Shrinking U_0 if necessary, write $\widetilde{\mu} = h(z) dz \wedge d\overline{z}$, where $dz = dz_1 \wedge \cdots \wedge dz_n$, and h is a smooth non-vanishing function on U_0. For simplicity, we will take $h \equiv 1$. Letting $V_s := \widetilde{X}_s \cap U_0$, we see that

$$\int_{V_s} \pi^*\mu \ = \ \int_{V_s} \prod_{j \in J(x_0)} |z_j|^{2k_j} dz \wedge d\overline{z}.$$

An elementary calculation gives

$$f^* \frac{ds}{s} \ = \ \sum_{j \in J(x_0)} \frac{a_j}{z_j} dz_j.$$

Fix $j_0 \in J(x_0)$ to be the index which minimizes the ratio $\frac{k_j+1}{a_j}$, and set

$$\xi_0 := \frac{(-1)^{j_0}}{s} \frac{z_{j_0}}{a_{j_0}} dz^{\widehat{j_0}},$$

where $dz^{\widehat{j_0}}$ is a short-hand for $dz_1 \wedge \cdots \wedge dz_{j_0-1} \wedge dz_{j_0+1} \wedge \cdots \wedge dz_n$. Then

$$\int_{V_s} \prod_{j\in J(x_0)} |z_j|^{2k_j} dz \wedge d\overline{z} \quad = \quad \int_{V_s} \prod_{j\in J(x_0)} |z_j|^{2k_j} f^*ds \wedge \xi_0 \wedge \overline{f^*ds} \wedge \overline{\xi_0},$$

and since $f^*ds \wedge \overline{f^*ds}$ is constant along the fibers, we simply need to calculate

$$\int_{V_s} \prod_{j\in J(x_0)} |z_j|^{2k_j} \xi_0 \wedge \overline{\xi_0}.$$

For $j \in J(x_0)$, set $z_j = e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j}$, and for $k \notin J(x_0)$, set $z_k = r_k e^{\sqrt{-1}\vartheta_k}$. For $j \in J(x_0)$,

$$dz_j \quad = \quad \frac{1}{a_j} e^{\sqrt{-1}\vartheta_j} e^{\rho_j/a_j} d\rho_j + \sqrt{-1} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j} d\vartheta_j,$$

and for $k \notin J(x_0)$,

$$dz_k \quad = \quad e^{\sqrt{-1}\vartheta_k} dr_k + \sqrt{-1} r_k e^{\sqrt{-1}\vartheta_k} d\vartheta_k.$$

Recall that the region of integration is the disk $\Delta = \{s \in \mathbb{C} : |s| < 1\}$. With $s = \prod_{j\in J(x_0)} z_j^{a_j}$, it follows that $|z_j| \leq 1$ for $j \in J(x_0)$. Thus, in these new coordinates,

$$\rho_j \leq 0, \qquad \textstyle\sum_{j\in J(x_0)} \rho_j = \log|s|,$$

$$0 \leq r_k \leq 1, \qquad \textstyle\sum_{j\in J(x_0)} a_j \vartheta_j = \arg(s).$$

Now

$$\begin{aligned}
& dz^{\hat{j_0}} \\
= \quad & \bigwedge_{j\in J(x_0)^*} dz_j \bigwedge_{k\notin J(x_0)} dz_k \\
= \quad & \left(\bigwedge_{j\in J(x_0)^*} \frac{1}{a_j} e^{\sqrt{-1}\vartheta_j} e^{\rho_j/a_j} d\rho_j + \sqrt{-1} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j} d\vartheta_j \right) \wedge \left(\bigwedge_{k\notin J(x_0)} e^{\sqrt{-1}\vartheta_k} dr_k + \sqrt{-1} r_k e^{\sqrt{-1}\vartheta_k} d\vartheta_k \right) \\
= \quad & \bigwedge_{j\in J(x_0)^*, k\notin J(x_0)} \left(\frac{1}{a_j} e^{\sqrt{-1}\vartheta_j} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_k} d\rho_j \wedge dr_k + \sqrt{-1} \frac{1}{a_j} e^{\sqrt{-1}\vartheta_j} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_k} d\rho_j \wedge d\vartheta_k \right) \\
& + \bigwedge_{j\in J(x_0)^*, k\notin J(x_0)} \left(\sqrt{-1} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j} e^{\sqrt{-1}\vartheta_k} d\vartheta_j \wedge dr_k - e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j} d\vartheta_j \wedge d\vartheta_k \right) \\
=: \quad & \bigwedge_{j\in J(x_0)^*, k\notin J(x_0)} \left(A_{j,k} d\rho_j \wedge dr_k + B_{j,k} d\rho_j \wedge d\vartheta_k + C_{j,k} d\vartheta_j \wedge dr_k + D_{j,k} d\vartheta_j \wedge d\vartheta_k \right).
\end{aligned}$$

Then

$$\begin{aligned} dz^{\hat{j_0}} \wedge d\bar{z}^{\hat{j_0}} &= \bigwedge_{j \in J(x_0)^*, k \notin J(x_0)} (A_{j,k} d\rho_j \wedge dr_k + B_{j,k} d\rho_j \wedge d\vartheta_k + C_{j,k} d\vartheta_j \wedge dr_k + D_{j,k} d\vartheta_j \wedge d\vartheta_k) \\ & \quad \wedge \bigwedge_{j \in J(x_0)^*, k \notin J(x_0)} \left(\overline{A}_{j,k} d\rho_j \wedge dr_k + \overline{B}_{j,k} d\rho_j \wedge d\vartheta_k + \overline{C}_{j,k} d\vartheta_j \wedge dr_k + \overline{D}_{j,k} d\vartheta_j \wedge d\vartheta_k \right) \\ &= \bigwedge_{j \in J(x_0)^*, k \notin J(x_0)} A_{j,k} \overline{D}_{j,k} d\rho_j \wedge dr_k \wedge d\vartheta_j \wedge d\vartheta_k + D_{j,k} \overline{A}_{j,k} d\vartheta_j \wedge d\vartheta_k \wedge d\rho_j \wedge dr_k \\ &= \bigwedge_{j \in J(x_0)^*, k \notin J(x_0)} \frac{1}{a_j} e^{2\rho_j / a_j} e^{2\sqrt{-1}\vartheta_j} d\rho_j \wedge dr_k \wedge d\vartheta_j \wedge d\vartheta_k, \end{aligned}$$

and subsequently,

$$\begin{aligned} \xi_0 \wedge \bar{\xi}_0 &= \frac{1}{|s|^2} \frac{|z_{j_0}|^2}{a_{j_0}^2} \frac{1}{a_j} e^{2\rho_j / a_j} e^{2\sqrt{-1}\vartheta_j} d\rho_j \wedge dr_k \wedge d\vartheta_j \wedge d\vartheta_k \\ &= \frac{1}{|s|^2} \prod_{j \in J(x_0)} \frac{1}{a_j^2} e^{2\rho_j / a_j} d\rho_j \prod_{\ell \in J(x_0)^*} e^{2\sqrt{-1}\vartheta_\ell} d\vartheta_\ell \bigwedge_{k \notin J(x_0)} dr_k. \end{aligned}$$

The main integral is then given by

$$\begin{aligned} & \int_{V_s} \prod_{j \in J(x_0)} |z_j|^{2k_j} \xi_0 \wedge \overline{\xi_0} \\ &= \frac{1}{|s|^2} \int_{V_s} \prod_{j \in J(x_0)} \frac{1}{a_j^2} e^{2k_j \frac{\rho_j}{a_j} + 2\rho_j / a_j} d\rho_j \bigwedge_{\ell \in J(x_0)^*} e^{2\sqrt{-1}\vartheta_\ell} d\vartheta_\ell \bigwedge_{k \notin J(x_0)} dr_k \\ &= \frac{1}{|s|^2} \int_{V_s} \prod_{j \in J(x_0)} \frac{1}{a_j^2} e^{2\left(\frac{k_j+1}{a_j}\right)\rho_j} d\rho_j \bigwedge_{\ell \in J(x_0)^*} e^{2\sqrt{-1}\vartheta_\ell} d\vartheta_\ell \bigwedge_{k \notin J(x_0)} dr_k \\ &= \frac{1}{|s|^{2\left(1 - \frac{k_{j_0}+1}{a_{j_0}}\right)}} \int_{V_s} \prod_{j \in J(x_0)} \frac{1}{a_j^2} e^{2\left(\frac{k_j+1}{a_j} - \frac{k_{j_0}+1}{a_{j_0}}\right)\rho_j} d\rho_j \bigwedge_{\ell \in J(x_0)^*} e^{2\sqrt{-1}\vartheta_\ell} d\vartheta_\ell \bigwedge_{k \notin J(x_0)} dr_k \\ &\sim \frac{1}{|s|^{2\left(1 - \frac{k_{j_0}+1}{a_{j_0}}\right)}} \int_{V_s} \prod_{j \in J(x_0)} e^{2\left(\frac{k_j+1}{a_j} - \frac{k_{j_0}+1}{a_{j_0}}\right)\rho_j} d\rho_j. \end{aligned}$$

We claim that

$$\int_{V_s} \prod_{j \in J(x_0)} e^{2\left(\frac{k_j+1}{a_j} - \frac{k_{j_0}+1}{a_{j_0}}\right)\rho_j} d\rho_j \ \sim \ (-\log|s|)^{N_0 - 1},$$

where N_0 is the cardinality of the set $\{j \in J(x_0) : \frac{k_j+1}{a_j} = \frac{k_{j_0}+1}{a_{j_0}}\}$. For simplicity, take $J(x_0) = \{1, 2, 3\}$, and $j_0 = 3$. We will consider three cases:

Case 1: $N_0 = 3$ and

$$\frac{k_1+1}{a_1} = \frac{k_2+1}{a_2} = \frac{k_3+1}{a_3}.$$

The region of integration R is described by $\rho_1+\rho_2+\rho_3 = \log|s|$ and $\rho_1, \rho_2, \rho_3 \le 0$. Since $\log|s| \le \rho_3 \le 0$, we can write

$$R = \{(\rho_1, \rho_2) \in \mathbb{R}^2 : \rho_1 \le 0,\ \rho_2 \le 0,\ \log|s| \ \le \ \rho_1+\rho_2 \ \le \ 0\},$$

which describes a triangle in the plane whose area is $\frac{1}{2}(\log|s|)^2 = \frac{1}{2}(-\log|s|)^2 \sim (-\log|s|)^2$.

Case 2: $N_0 = 2$, where we may assume that $\frac{k_1+1}{a_1} > \frac{k_3+1}{a_3}$ and $\frac{k_2+1}{a_2} = \frac{k_3+1}{a_3}$. Set $\alpha = 2\left(\frac{k_1+1}{a_1} - \frac{k_3+1}{a_3}\right) > 0$. Then

$$\begin{aligned}
\int_R e^{\alpha\rho_1} d\rho_1 d\rho_2 &= \int_{\log|s|}^0 \int_{\log|s|-\rho_1}^0 e^{\alpha\rho_1} d\rho_2 d\rho_1 \\
&= \int_{\log|s|}^0 (\rho_1 - \log|s|) e^{\alpha\rho_1} d\rho_1 \\
&= -\frac{1}{\alpha}\log|s| + \frac{1}{\alpha^2}(e^{\alpha\log|s|} - 1) \\
&= -\frac{1}{\alpha}\log|s| + \frac{1}{\alpha^2}(|s|^\alpha - 1) \\
&\sim -\log|s|
\end{aligned}$$

Case 3: $N_0 = 1$, where we may assume that $\frac{k_1+1}{a_1} > \frac{k_2+1}{a_2} > \frac{k_3+1}{a_3}$. Set $\alpha_i := \frac{k_i+1}{a_i} - \frac{k_3+1}{a_3} > 0$ for $i \in \{1,2\}$. We compute

$$\begin{aligned}
\int_R e^{\alpha_1\rho_1+\alpha_2\rho_2} d\rho_1 d\rho_2 &= \int_{\log|s|}^0 \int_{\log|s|-\rho_1}^0 e^{\alpha_1\rho_1+\alpha_2\rho_2} d\rho_2 d\rho_1 \\
&= \int_{\log|s|}^0 \left[\frac{1}{\alpha_2} e^{\alpha_1\rho_1+\alpha_2\rho_2}\right]_{\log|s|-\rho_1}^0 d\rho_1 \\
&= \int_{\log|s|}^0 \left[\frac{1}{\alpha_2} e^{\alpha_1\rho_1} - \frac{1}{\alpha_2} e^{\alpha_1\rho_1} e^{\alpha_2\log|s|-\alpha_2\rho_1}\right] d\rho_1 \\
&= \int_{\log|s|}^0 \frac{1}{\alpha_2} e^{\alpha_1\rho_1} d\rho_1 - \int_{\log|s|}^0 \frac{1}{\alpha_2} |s|^{\alpha_2} e^{(\alpha_1-\alpha_2)\rho_1} d\rho_1 \\
&= \left[\frac{1}{\alpha_1\alpha_2} e^{\alpha_1\rho_1}\right]_{\log|s|}^0 - \frac{1}{\alpha_2}|s|^{\alpha_2} \int_{\log|s|}^0 e^{(\alpha_1-\alpha_2)\rho_1} d\rho_1 \\
&= \frac{1}{\alpha_1\alpha_2} - \frac{1}{\alpha_1\alpha_2}|s|^{\alpha_1} - \frac{1}{\alpha_2}|s|^{\alpha_2}\left[\frac{1}{\alpha_1-\alpha_2} - |s|^{\alpha_1-\alpha_2}\right] \\
&= \frac{1}{\alpha_1\alpha_2} - \frac{1}{\alpha_1\alpha_2}|s|^{\alpha_1} - \frac{1}{\alpha_2(\alpha_1-\alpha_2)}|s|^{\alpha_2} + \frac{1}{\alpha_2}|s|^{\alpha_1},
\end{aligned}$$

where all of these terms are bounded.

Therefore,

$$\int_{V_s} \prod_{j\in J(x_0)} |z_j|^{2k_j} \xi_0 \wedge \overline{\xi_0} \quad \sim \quad |s|^{-2\left(1-\frac{k_{j_0}+1}{a_{j_0}}\right)} (-\log|s|)^{N_0-1},$$

and we set $\beta := \frac{k_{j_0}+1}{a_{j_0}}$, and let N be the maximal number such that there are N divisors E_j with $\frac{k_j+1}{a_j} = \beta$. Since X is compact, cover X by a finite number of charts U_α. Let ϕ_α be a partition of unity subordinate to (U_α). Then

$$\frac{\int_{\widetilde{X}_s} \pi^*\mu}{\sqrt{-1}ds \wedge d\overline{s}} = \sum_\alpha \frac{\int_{\widetilde{X}_s} \phi_\alpha \pi^*\mu}{\sqrt{-1}ds \wedge d\overline{s}} = \sum_\alpha \frac{\int_{\widetilde{X}_s \cap U_\alpha} \phi_\alpha \pi^*\mu}{\sqrt{-1}ds \wedge d\overline{s}},$$

completing the proof. ☐

References

[1] Zhang, Y., *Collapsing Limits of the Kähler–Ricci flow and the Continuity Method*, arXiv:1705.01434 v3, (2018).

We now compute the full expansion: That is, we compute the integrals

$$
\begin{aligned}
& \int_{V_s} \prod_{j\in J(x_0)} |z_j|^{2k_j} \xi_0 \wedge \overline{\xi_0} \\
= & \int_{V_s} |s|^{-2} \prod_{j\in J(x_0)} e^{2\left(\frac{k_j+1}{a_j}\right)\rho_j} d\rho_j \\
= & \int_{V_s} e^{-2\sum_{j\in J(x_0)}\rho_j} \prod_{j\in J(x_0)} e^{2\left(\frac{k_j+1}{a_j}\right)\rho_j} d\rho_j.
\end{aligned}
$$

where we omit the $1/a_j^2$ term. Let $\alpha_j := 2\left(\frac{k_j+1}{a_j} - 1\right)$. Then the integral simplifies to

$$
\int_{V_s} e^{\sum_{j\in J(x_0)} \alpha_j \rho_j} d\rho_j.
$$

Let $|J(x_0)| = \ell$, and recall that the region of integration is described by

$$
\begin{aligned}
V_s &= \left\{ (\rho_1, ..., \rho_\ell) \in (\mathbb{R}_{<0})^\ell : \sum_{j=1}^{\ell} \rho_j = \log|s| \right\} \\
&= \left\{ (\rho_1, ..., \rho_{\ell-1}) \in (\mathbb{R}_{<0})^{\ell-1} : \log|s| \;\leq\; \sum_{j=1}^{\ell-1} \rho_j \;\leq\; 0 \right\}.
\end{aligned}
$$

We claim that

$$
\int_{V_s} e^{\alpha_1\rho_1+\cdots+\alpha_\ell\rho_\ell} d\rho_1 \cdots d\rho_\ell \;=\; \frac{1}{\prod_{k=1}^{\ell} \alpha_k} + (-1)^\ell \sum_{k=1}^{\ell} \frac{1}{\alpha_k \prod_{1\leq j\neq k\leq \ell} A_{kj}} |s|^{\alpha_k}, \qquad A_{kj} := \alpha_k - \alpha_j.
$$

Proof. Proceed inductively on ℓ. For $\ell = 2$, we compute

$$\begin{aligned}
\int_{V_s} e^{\alpha_1\rho_1+\alpha_2\rho_2} d\rho_1 d\rho_2 &= \int_{\log|s|}^0 \int_{\log|s|-\rho_2}^0 e^{\alpha_1\rho_1+\alpha_2\rho_2} d\rho_1 d\rho_2 \\
&= \int_{\log|s|}^0 \left[\frac{1}{\alpha_1} e^{\alpha_1\rho_1+\alpha_2\rho_2}\right]_{\log|s|-\rho_2}^0 d\rho_2 \\
&= \int_{\log|s|}^0 \left[\frac{1}{\alpha_1} e^{\alpha_2\rho_2} - \frac{1}{\alpha_1} e^{\alpha_1(\log|s|-\rho_2)+\alpha_2\rho_2}\right] d\rho_2 \\
&= \int_{\log|s|}^0 \left[\frac{1}{\alpha_1} e^{\alpha_2\rho_2} - \frac{1}{\alpha_1} e^{\log|s|^{\alpha_1}+(\alpha_2-\alpha_1)\rho_2}\right] d\rho_2 \\
&= \int_{\log|s|}^0 \left[\frac{1}{\alpha_1} e^{\alpha_2\rho_2} - \frac{1}{\alpha_1}|s|^{\alpha_1} e^{(\alpha_2-\alpha_1)\rho_2}\right] d\rho_2 \\
&= \left[\frac{1}{\alpha_1\alpha_2} e^{\alpha_2\rho_2} - \frac{1}{\alpha_1(\alpha_2-\alpha_1)}|s|^{\alpha_1} e^{(\alpha_2-\alpha_1)\rho_2}\right]_{\log|s|}^0 \\
&= \left[\frac{1}{\alpha_1\alpha_2} - \frac{1}{\alpha_1(\alpha_2-\alpha_1)}|s|^{\alpha_1}\right] - \left[\frac{1}{\alpha_1\alpha_2}|s|^{\alpha_2} - \frac{1}{\alpha_1(\alpha_2-\alpha_1)}|s|^{\alpha_1}|s|^{\alpha_2-\alpha_1}\right] \\
&= \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1(\alpha_1-\alpha_2)}|s|^{\alpha_1} - \frac{1}{\alpha_1\alpha_2}|s|^{\alpha_2} + \frac{1}{\alpha_1(\alpha_2-\alpha_1)}|s|^{\alpha_2} \\
&= \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1(\alpha_1-\alpha_2)}|s|^{\alpha_1} + \frac{1}{\alpha_2(\alpha_2-\alpha_1)}|s|^{\alpha_2} \\
&= \frac{1}{\alpha_1\alpha_2} + \frac{1}{\alpha_1 A_{12}}|s|^{\alpha_1} + \frac{1}{\alpha_2 A_{21}}|s|^{\alpha_2}.
\end{aligned}$$

Suppose the claim is true for ℓ, i.e.,

$$\int_{V_s} e^{\alpha_1\rho_1+\cdots+\alpha_\ell\rho_\ell} d\rho_1\cdots d\rho_\ell = \frac{1}{\prod_{k=1}^{\ell}\alpha_k} + (-1)^\ell \sum_{k=1}^{\ell} \frac{1}{\alpha_k \prod_{1\le j\ne k\le \ell} A_{kj}}|s|^{\alpha_k}, \qquad A_{kj} := \alpha_k - \alpha_j.$$

We will show the result holds for $\ell+1$. To this end, $\square$

Proof. Evaluate the integral directly:

$$\begin{aligned}
&\int_{V_s} e^{\alpha_1\rho_1+\cdots+\alpha_\ell\rho_\ell} d\rho_1\cdots d\rho_\ell \\
= &\int_{\log|s|}^0 \int_{\log|s|-\rho_{\ell-1}}^0 \cdots \int_{\log|s|-(\rho_2+\cdots+\rho_\ell)}^0 e^{\alpha_1\rho_1+\cdots+\alpha_\ell\rho_\ell} d\rho_1\cdots d\rho_\ell \\
= &\int_{\log|s|}^0 \int_{\log|s|-\rho_{\ell-1}}^0 \cdots \int_{\log|s|-(\rho_3+\cdots+\rho_{\ell-1})}^0 \left[\frac{1}{\alpha_1} e^{\alpha_1\rho_1+\cdots+\alpha_\ell\rho_\ell}\right]_{\log|s|-(\rho_2+\cdots+\rho_\ell)}^0 d\rho_2\cdots d\rho_\ell \\
= &\int_{\log|s|}^0 \int_{\log|s|-\rho_{\ell-1}}^0 \cdots \int_{\log|s|-(\rho_3+\cdots+\rho_{\ell-1})}^0 \frac{1}{\alpha_1} e^{\alpha_2\rho_2+\cdots+\alpha_\ell\rho_\ell} d\rho_2\cdots d\rho_\ell \\
&-\int_{\log|s|}^0 \int_{\log|s|-\rho_{\ell-1}}^0 \cdots \int_{\log|s|-(\rho_3+\cdots+\rho_{\ell-1})}^0 \frac{1}{\alpha_1}|s|^{\alpha_1} e^{A_{21}\rho_2+A_{31}\rho_3+\cdots+A_{(\ell-1)1}\rho_{\ell-1}+\alpha_\ell\rho_\ell} d\rho_2\cdots d\rho_\ell.
\end{aligned}$$

Calculate:

$$\begin{aligned}
&\int_{\log|s|-(\rho_3+\cdots+\rho_{\ell-1})}^0 \frac{1}{\alpha_1} e^{\alpha_2\rho_2+\cdots+\alpha_\ell\rho_\ell} d\rho_2 \\
= &\left[\frac{1}{\alpha_1\alpha_2} e^{\alpha_3\rho_3+\cdots+\alpha_\ell\rho_\ell} e^{\alpha_2\rho_2}\right]_{\log|s|-(\rho_3+\cdots+\rho_{\ell-1})}^0 \\
= &\frac{1}{\alpha_1\alpha_2} e^{\alpha_3\rho_3+\cdots+\alpha_\ell\rho_\ell} - \frac{1}{\alpha_1\alpha_2} e^{A_{32}\rho_3+\cdots+A_{(\ell-1)2}\rho_{\ell-1}+\alpha_\ell\rho_\ell}|s|^{\alpha_2}
\end{aligned}$$

And:

$$\begin{aligned}
&\int_{\log|s|-(\rho_3+\cdots+\rho_{\ell-1})}^0 \frac{1}{\alpha_1}|s|^{\alpha_1} e^{A_{21}\rho_2+A_{31}\rho_3+\cdots+A_{(\ell-1)1}\rho_{\ell-1}+\alpha_\ell\rho_\ell} d\rho_2 \\
= &\frac{1}{\alpha_1 A_{21}}|s|^{\alpha_1} e^{A_{31}\rho_3+\cdots+A_{(\ell-1)1}\rho_{\ell-1}+\alpha_\ell\rho_\ell}\left(1-|s|^{A_{21}} e^{-A_{21}(\rho_3+\cdots+\rho_{\ell-1})}\right) \\
= &\frac{1}{\alpha_1 A_{21}}|s|^{\alpha_1} e^{A_{31}\rho_3+\cdots+A_{(\ell-1)1}\rho_{\ell-1}+\alpha_\ell\rho_\ell} - \frac{1}{\alpha_1 A_{21}}|s|^{\alpha_2} e^{A_{32}\rho_3+\cdots+A_{(\ell-1)2}\rho_{\ell-1}+\alpha_\ell\rho_\ell}
\end{aligned}$$

Combining the expressions gives:

$$\begin{aligned}
&\frac{1}{\alpha_1\alpha_2} e^{\alpha_3\rho_3+\cdots+\alpha_\ell+\rho_\ell} + \frac{1}{\alpha_1 A_{21}}|s|^{\alpha_1} e^{A_{31}\rho_3+\cdots+A_{(\ell-1)1}\rho_{\ell-1}+\alpha_\ell\rho_\ell} \\
&-\left(\frac{1}{\alpha_1\alpha_2}+\frac{1}{\alpha_1 A_{21}}\right)|s|^{\alpha_2} e^{A_{32}\rho_3+\cdots+A_{(\ell-1)2}\rho_{\ell-1}+\alpha_\ell\rho_\ell} \\
= &\frac{1}{\alpha_1\alpha_2} e^{\alpha_3\rho_3+\cdots+\alpha_\ell+\rho_\ell} + \frac{1}{\alpha_1 A_{21}}|s|^{\alpha_1} e^{A_{31}\rho_3+\cdots+A_{(\ell-1)1}\rho_{\ell-1}+\alpha_\ell\rho_\ell} \\
&-\frac{2\alpha_2-\alpha_1}{\alpha_1\alpha_2(\alpha_2-\alpha_1)}|s|^{\alpha_2} e^{A_{32}\rho_3+\cdots+A_{(\ell-1)2}\rho_{\ell-1}+\alpha_\ell\rho_\ell}
\end{aligned}$$

$\square$

For $\ell = 3$, we get

$$= \frac{1}{\alpha_1\alpha_2\alpha_3} - \frac{1}{\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}|s|^{\alpha_1} + \frac{1}{\alpha_2(\alpha_1-\alpha_2)(\alpha_2-\alpha_3)}|s|^{\alpha_2} - \frac{1}{\alpha_3(\alpha_1-\alpha_3)(\alpha_2-\alpha_3)}|s|^{\alpha_3},$$

and for $\ell = 4$, we get

$$= \frac{1}{\alpha_1\alpha_2\alpha_3\alpha_4} + \frac{1}{\alpha_1 A_{12}A_{13}A_{14}}|s|^{\alpha_1} - \frac{1}{\alpha_2 A_{12}A_{23}A_{24}}|s|^{\alpha_2} + \frac{1}{\alpha_3 A_{13}A_{23}A_{34}}|s|^{\alpha_3} - \frac{1}{\alpha_4 A_{14}A_{24}A_{34}}|s|^{\alpha_4}.$$