On the Existence of Kähler–Einstein Metrics.

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Contents

Preface

A complex manifold necessarily carries the structure of a Riemannian manifold. Every Riemannian manifold can be equipped with a Riemannian metric. It is not necessarily the case however, that this metric will be compatible with the complex structure. Kähler manifolds are those complex manifolds whose complex structure and Riemannian structure coalesce beautifully. Once we have one metric on a Kähler manifold, it is very easy to generate other metrics. We are therefore naturally led to restrict our concern to certain *canonical metrics.* Two most notable choices of canonical metrics that have been proposed are the Kähler–Einstein metrics and the metrics of constant scalar curvature.

A metric g is said to be *Kähler–Einstein* if its associated $(1,1)$–form ω is closed, i.e., $d\omega = 0$ (Kähler) and the metric is proportional to the Ricci tensor (Einstein),

$$R_{i\bar{j}} = \lambda g_{i\bar{j}},$$

where $\lambda \in \mathbb{R}$. The motivation for the consideration of such metrics comes from the *uniformisation theorem* from Riemann surface theory.

Indeed, in (complex) dimension 1, the Kähler–Einstein metrics are exactly those metrics of constant Gauss curvature. The uniformisation theorem asserts that

- if $\lambda > 0$, the manifold is the round sphere $(\mathbb{S}^2, g_{\text{round}})$.
- If $\lambda = 0$, then the manifold is a flat torus $(\mathbb{T}^2, g_{\text{flat}})$.
- If $\lambda < 0$, then the manifold is a compact hyperbolic surface $\mathbb{H}^2/\Gamma$, where $\Gamma \subseteq \text{PSL}_2(\mathbb{R})$ is a discrete subgroup which acts freely on $\mathbb{H}^2$.

We have the following analogue in higher dimensions[1].

- If $\lambda > 0$, then $c_1(M) > 0$ and M is a Fano manifold.
- If $\lambda = 0$, then $c_1(M) = 0$ and M is a Calabi–Yau manifold.
- If $\lambda < 0$, then $c_1(M) < 0$ and M is of general type.

In 1978, Shing-Tung Yau released his paper *On The Ricci Curvature of a Compact Kähler Manifold and the Complex Monge–Ampère Equation.* The paper details a proof of the Calabi–Conjecture and proves that any Kähler manifold with first Chern class $c_1(M) = 0$ admits a Ricci–flat metric. In the case of $c_1(M) < 0$, this problem was given an affirmative answer independently by S.T. Yau and T.

[1] Note that many complex manifolds do not have a definite first Chern class. More details of this may be found in [Au] or [AS].

Aubin. For $c_1(M) > 0$ however, the situation is substantially more difficult and there are obstructions to the existence of Kähler–Einstein metrics. In recent years, the existence of Kähler–Einstein metrics on manifolds with positive first Chern class have been linked with the algebro–geometric notion of stability due to Mumford.

The purpose of this exposition is to present the original proof of the Calabi–Yau theorem as well as the necessary background to understand the proof. We assume that the reader is familiar with at least a standard undergraduate knowledge of Riemannian geometry and complex analysis. Some understanding of sheaves and basic algebraic geometry may also prove useful but is not necessary.

Structure of the Exposition

The structure of this exposition is as follows.

Chapter 1 provides an introduction to almost–complex manifolds and complex geometry. The material pertaining to this chapter can be found in Chapter 0 of [GH], Chapter 9 of [KN], and Chapters 7–9 of [Mo]. This chapter may be skipped on first reading and used only to refer back to if necessary.

Chapter 2 introduces the main objects of this exposition, Kähler manifolds. This material can be found in Chapter 0 of [GH], Chapters 10–11 of [Mo], Chapter 1 of [Sz], and Chapter 4 of [Ba].

Chapter 3 details the Riemannian and Ricci curvature tensor of a Kähler manifold and introduces the Ricci–form. This material can be found in Chapter 0 of [GH], Chapters 12 and 17 of [Mo], Chapter 1 of [Sz], and Chapter 4 of [Ba].

In Chapter 4 we define the First Chern Class in terms of sheaf cohomology. Material for this chapter may be found in Chapter 1 of [GH].

In Chapter 5 we introduce the Calabi–Yau theorem and formulate the Calabi–conjecture as a second–order elliptic partial differential equation of Monge–Ampère type. The primary sources for this are the original paper [Ya76], Chapter 5 of [Ti], [We] and [Bl].

Chapter 6 details the proofs of the second order estimates that are found in the original paper. The primary reference for this material is [Ya76].

In Chapter 7 we provide a proof of the third order estimates. Our computation is based off the simplification due to [PSS]. A similar treatment may be found in [Sz].

Notation

$\mathbb{N}$	The *natual numbers.* We do not assume that 0 is contained in $\mathbb{N}$.
$\mathbb{Z}$	The *integers.*
$\mathbb{R}$	The *real numbers.*
$\mathbb{C}$	The *complex numbers.*
$\mathbb{R}_{>0}$	The *positive real numbers*, i.e., $(0, \infty)$.
$\oplus$	The *direct sum.*
$\otimes$	The *tensor product.*
A	Often denotes an indexing set.
$\overline{z}$	The *complex conjugate* of z, i.e., if $z = x + \sqrt{-1}y$, then $\overline{z} = x - \sqrt{-1}y$.
(x^j)	A *local coordinate*, $x = (x^j) : \mathscr{U} \longrightarrow \mathbb{R}^k$.
(z^j)	A *local holomorphic coordinate*, $z = (z^j) : \mathscr{U} \longrightarrow \mathbb{C}^k$.
$g^{i\overline{j}}$	The inverse of the Hermitian metric $g_{i\overline{j}}$.
$\det(g_{i\overline{j}})$	The determinant of the matrix $g_{i\overline{j}}$.
$\Gamma(E)$	The space of *smooth sections* of a vector bundle $\pi : E \longrightarrow M$.
T_pM	The *tangent space* of a manifold M at a point $p \in M$.
$T_p^{1,0}M$	The *holomorphic tangent space* of a manifold M at a point $p \in M$.
$T_p^{0,1}M$	The *anti–holomorphic tangent space* of a manifold M at a point $p \in M$.
TM	The *tangent bundle* of a manifold M.
$TM^{\mathbb{C}}$	The *complexified tangent bundle*, $TM^{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$.
$\Lambda^k(M)$	The *k–th exterior bundle* over a manifold M.
$\Omega^k(M)$	The space of *smooth k–forms* on a manifold M.
$\Omega^{p,q}(M)$	The space of *holomorphic (p,q)–forms* on a manifold M.
d	The *exterior derivative* $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$.
∂	The *holomorphic Dolbeault operator* $\partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M)$.
$\overline{\partial}$	The *anti–holomorphic Dolbeault operator* $\overline{\partial} : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M)$.
$\mathscr{X}(E)$	The space of *vector fields.*
Id_X	The identity map on some space X.
J	The *almost complex structure* $J : TM \longrightarrow TM$, $J^2 = -\mathrm{Id}$.
∇	A connection, often the Levi–Civita connection.
Γ_{ij}^k	The *Christoffel symbols* of a connection ∇.
$\overline{\nabla}$	The *Chern connection.*
$R_{i\overline{j}k\overline{\ell}}$	The *Riemannian curvature tensor.*

$R_{i\bar{j}}$	The *Ricci curvature tensor*.
ω	A Kähler form.
$Ric(\omega), \rho$	The *Ricci form* associated to the Kähler form ω.
$c_1(M)$	The *First Chern Class* of a manifold M.
ω^n	The nth wedge product of ω, i.e., $\omega^n = \underbrace{\omega \wedge \cdots \wedge \omega}_{\text{n times}}$.
$Vol(M)$	The *Volume* of a manifold M.
${}_M\mathcal{O}$	The *sheaf of holomorphic functions* on a manifold M.
${}_M\mathscr{M}$	The *sheaf of meromorphic functions* on a manifold M.
${}_M\mathcal{O}^*$	The *sheaf of non-vanishing holomorphic functions* on a manifold M.
${}_M\mathscr{M}^*$	The *sheaf of non-vanishing meromorphic functions* on a manifold M.
$\mathscr{S}(Y)$	The *space of sections of a sheaf* $\mathscr{S}$ over a space Y.
$\mathbb{P}^n$	The *complex projective space*.
$\mathscr{C}^\infty(M)$	The space of *smooth functions* on a manifold M.
$\mathscr{C}^\infty(X,Y)$	The space of *smooth functions* $f : X \longrightarrow Y$.
$\mathscr{C}^{k,\alpha}(M)$	The space of k–times differentiable *Hölder continuous* functions with Hölder exponent α.
$\mathcal{L}$	The *Lie derivative*.
Δ	The *Laplace operator* associated to the metric g.
$\widetilde{\Delta}$	The *Laplace operator* associated to the metric $\widetilde{g}$.
$Div(M)$	The space of *Divisors* on a manifold M.
$Pic(M)$	The *Picard Group*.
L^*	The dual of a line bundle $L \longrightarrow M$.
ω_φ	$\omega_\varphi = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$.
ω_t	$\omega_t = \omega + \sqrt{-1}\partial\overline{\partial}\varphi_t$.
Δ_t	The *Laplace operator* associated to the metric g_t.
$\exp(f)$	The exponential of an $\mathbb{R}$–valued function f, i.e., e^f.
S^k_{ij}	The tensor $S^k_{ij} = \widetilde{\Gamma}^k_{ij} - \Gamma^k_{ij}$.

CHAPTER 1

Geometric Preliminaries

In this chapter we collect the relevant complex differential geometry that will be used throughout our treatment. The reader may wish to skip this chapter upon first reading and refer back only when necessary.

1.1. First Definitions – Riemannian Geometry

Definition 1.1.1. A *manifold* of dimension n is a Haussdorff topological space M such that each point of M has a neighbourhood which is homeomorphic to an open subset of $\mathbb{R}^n$. A *chart* consists of an open set $U \subset M$ and a homeomorphism $\varphi : U \longrightarrow V$, where $V \subseteq \mathbb{R}^n$.

Charts allow us to define *coordinates* on a manifold. Indeed, for any point $p \in M$, there is a chart $\varphi : U \longrightarrow V$ which allows us to identify p with the corresponding point $\varphi(p) \in \mathbb{R}^n$.

Definition 1.1.2. An *atlas of class* $\mathscr{C}^k$ consists of a collection of charts $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ such that for any pair $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$, the transition functions

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \longrightarrow \varphi_\beta(U_\alpha \cap U_\beta) \tag{1}$$

are $\mathscr{C}^k$ in the usual sense, i.e., as a map $\mathbb{R}^n \longrightarrow \mathbb{R}^n$.

Definition 1.1.4. An atlas is said to be *smooth* if the transition functions (1) are $\mathscr{C}^k$ for any $k \in \mathbb{N}$.

Definition 1.1.5. A manifold with an atlas of class $\mathscr{C}^k$ will be referred to as a $\mathscr{C}^k$*–manifold*. A manifold with a smooth atlas will be referred to as a *smooth manifold* or *Riemannian manifold*.

Definition 1.1.6. A *Riemannian metric* g on a smooth manifold M is a symmetric positive definite $(2,0)$–tensor field. In local coordinates, the metric is given by $g = g_{ij} dx^i \otimes dx^j$ with $g_{ij} = g_{ji}$. Every smooth manifold can be equipped with a Riemannian metric, see p. 4 of [Au].

Definition 1.1.7. Let M and N be two $\mathscr{C}^k$ manifolds with atlases $(U_\alpha, \varphi_\alpha)_{\alpha \in A}$ and $(V_\beta, \psi_\beta)_{\beta \in B}$, respectively. A map $f : M \longrightarrow N$ is said to be $\mathscr{C}^\ell$ for any $\ell \leq k$ if the map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is $\mathscr{C}^\ell$ in the usual sense.

Definition 1.1.8. A *derivation* at $p \in M$ is an $\mathbb{R}$–linear map $\partial : \mathscr{C}^\infty(M) \longrightarrow \mathbb{R}$ suchthat

$$\partial(fg) = f(p)\partial(g) + \partial(f)g(p).$$

The vector space of derivations at a point $p \in M$ is called the *tangent space* of M at p and is denoted by T_pM.

Definiton 1.1.9. The set of all $\mathbb{R}$–linear maps $T_pM \longrightarrow \mathbb{R}$, i.e., the dual of T_pM, is called the *cotangent space* of M at p and is denoted by T_p^*M.

Definition 1.1.10. Let M be a smooth manifold. A *smooth vector bundle of rank* k is a surjective submersion $\pi : E \longrightarrow M$ such that for each $p \in M$, the fibres $E_p := \pi^{-1}(p)$ carry the structure of a k–dimensional vector space. Moreover, the following local trivialisation condition is satisfied. That is, for each $p \in M$, there is a neighbourhood U of p and a diffeomorphism $\varphi : \pi^{-1}(U) \longrightarrow U \times \mathbb{R}^k$ such that the diagram

$$\begin{tikzcd} \pi^{-1}(U) \arrow[r] \arrow[d, "\pi"'] & U \times \mathbb{R}^k \arrow[dl, "\text{projection}"] \\ U & \end{tikzcd}$$

commutes and the map $\varphi|_p : E_p \longrightarrow \{p\} \times \mathbb{R}^k$ is a linear isomorphism of vector spaces.

Definition 1.1.11. A *section* of a vector bundle $\pi : E \longrightarrow M$ is a map $s : U \longrightarrow E$ such that $\pi \circ s = \mathrm{Id}_U$, where $U \subset M$ is an open set. The space of sections of a vector bundle $\pi : E \longrightarrow M$ is denoted by $\Gamma(E)$.

Definition 1.1.12. The *tangent bundle* of a smooth manifold M is the vector bundle given by the disjoint union of tangent spaces

$$TM = \coprod_{p \in M} T_pM.$$

Smooth sections of TM, i.e., elements of $\Gamma(TM)$ are called *vector fields*.

The *cotangent bundle* of a smooth manifold M is vector bundle given by the disjoint union of the cotangent spaces

$$T^*M = \coprod_{p \in M} T_p^*M.$$

Smooth sections of T^*M, i.e., elements of $\Gamma(T^*M)$ are called *differential forms*.

Definition 1.1.13. A *differential* p*–form* η is a section of the pth exterior power of T^*M. In local coordinates we may write (in a neighbourhood of each point)

$$\eta \quad = \sum_{i_1 < i_2 < \cdots < i_p} f_{i_1 \cdots i_p} dx^{i_1} \wedge dx^{i_2} \wedge \cdots \wedge dx^{i_p},$$

where $\wedge$ denotes the *wedge product*. We will often write $\Omega^p(M)$ for the space of differential p–forms on M.

Proposition 1.1.14. Let M be a smooth manifold of dimension n. There exists a unique $\mathbb{R}$–linear map $d : \Omega^p(M) \longrightarrow \Omega^{p+1}(M)$ such that:

(i) The image of any p–form under d is a $(p+1)$–form.
(ii) On functions, i.e., elements of $\Omega^0(M)$, d coincides with usual differentiation.
(iii) For any $\eta \in \Omega^p(M)$, $\xi \in \Omega^q(M)$, we have

$$d(\eta \wedge \xi) = d\eta \wedge \xi + (-1)^p \xi \wedge d\xi.$$

(iv) $d^2 = 0$.

Proof. See p. 22 of [Mo]. □

The exterior derivative of η, in local coordinates, is given by

$$d\eta \quad = \sum_{i_1 < i_2 < \cdots < i_p} df_{i_1 \cdots i_p} \wedge dz^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

Definition 1.1.15. A *complex manifold* is a smooth manifold M such that each point $p \in M$ admits a neighbourhood U and a homeomorphism $\varphi : U \longrightarrow \mathbb{C}^n$ such that the transition function (1) are holomorphic.

In an analogous manner to what was seen in the previous section, a function $f : M \longrightarrow N$ between two complex manifolds is said to be *holomorphic* if $\psi_\alpha \circ f \circ \varphi_\beta^{-1}$ is holomorphic for each chart φ_β and ψ_α of M and N, respectively.

Definition 1.1.16. A *complex vector bundle* over a complex manifold M is a *smooth vector bundle* such that for each $p \in M$, the fibres $E_p := \pi^{-1}(p)$ carry the structure of a k–dimensional complex vector space.

Note that a *complex vector bundle* and *holomorphic vector bundle* are not the same object. Indeed, a *complex vector bundle* has smooth transition functions, while a *holomorphic vector bundle* is a complex vector bundle with holomorphic transition functions.

Examples of a holomorphic vector bundles include the tangent and cotangent bundle of a complex manifold. In the next section, we will provide an example of a complex vector bundle that is not holomorphic.

1.2. Almost Complex Geometry

The differences between smooth manifolds and complex manifolds is vast. For example, the Whitney embedding theorem asserts that every smooth manifold embeds into $\mathbb{R}^N$, for some N. As a consequence of Liouville's theorem however, no compact complex manifolds however that can embed into $\mathbb{C}^N$ for any N. One way to bridge this gap, is to introduce manifolds which carry an *almost–complex* structure.

The matrix $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ has the property that $J^2 = -\text{Id}_{\mathbb{R}^2}$. With this matrix, one may define complex scalar multiplication on $\mathbb{R}^2$ by setting

$$u + \sqrt{-1}v := u + Jv, \qquad u, v \in \mathbb{R}^2. \tag{2}$$

We may extend this J map to higher dimensions by defining

$$J = \begin{bmatrix} 0 & -\text{Id}_{\mathbb{R}^n} \\ \text{Id}_{\mathbb{R}^n} & 0 \end{bmatrix}.$$

This motivates the following definition.

Definition 1.3.1. An *almost complex structure* on a smooth manifold M (of even real dimension) is an endomorphism $J : TM \longrightarrow TM$ of the tangent bundle such that $J^2 = -\text{Id}_{TM}$.

It is worth making explicit that this endomorphism J equips each tangent space T_pM, $p \in M$ with a "complex scalar multiplication" as in (2).

Proposition 1.3.2. Let $\mathcal{J}(f)$ denote the Jacobian of a smooth function $f : \mathbb{C} \longrightarrow \mathbb{C}$. Then f is holomorphic if and only if $\mathcal{J}(f)$ commutes with the almost complex structure J.

Proof. An elementary computation reveals that commuting with J is equivalent to f satisfying the Cauchy–Riemann equations. $\square$

All manifolds considered throughout our exposition here will be assumed to be complex.

Let us also note that if $z^1, ..., z^n$ are holomorphic coordinates and $z^k = x^k + \sqrt{-1}y^k$, where $x^k, y^k \in \mathbb{R}$ for all $1 \leq k \leq n$, then

$$J\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial}{\partial y^k} \quad \text{and} \quad J\left(\frac{\partial}{\partial y^k}\right) = -\frac{\partial}{\partial x^k}.$$

We will denote by $TM^{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$ the *complexified tangent bundle*. A basis for which is given by the

$$\frac{\partial}{\partial z^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} - \sqrt{-1}\frac{\partial}{\partial y^k}\right), \quad \text{and} \quad \frac{\partial}{\partial \overline{z}^k} = \frac{1}{2}\left(\frac{\partial}{\partial x^k} + \sqrt{-1}\frac{\partial}{\partial y^k}\right),$$

where $1 \leq k \leq n$.

The endmorphism J extends by $\mathbb{C}$–linearity to an endomorphism of the complexified tangent bundle and splits $TM^{\mathbb{C}}$ into the sum of its eigenbundles $T^{1,0}M$ and $T^{0,1}M$. That is,

$$TM^{\mathbb{C}} \cong T^{1,0}M \oplus T^{0,1}M,$$

where $T^{1,0}M$ is the eigenbundle corresponding to the eigenvalue $\sqrt{-1}$ and $T^{0,1}M$ is the eigenbundle corresponding to the eigenvalue $-\sqrt{-1}$.

For each $p \in M$, the vector space $T_p^{1,0}M$ is called the *holomorphic tangent space* and is given by

$$T_p^{1,0}M = \operatorname{span}\left\{ \left.\frac{\partial}{\partial z^1}\right|_p, ..., \left.\frac{\partial}{\partial z^n}\right|_p \right\}.$$

Similarly, the vector space $T_p^{0,1}M$ is called the *anti–holomorphic tangent space* and is given by

$$T_p^{0,1}M = \operatorname{span}\left\{ \left.\frac{\partial}{\partial \overline{z}^1}\right|_p, ..., \left.\frac{\partial}{\partial \overline{z}^n}\right|_p \right\}.$$

An analogous decomposition may be obtained from the complexified cotangent bundle. The space of differential k–forms on M, $\Omega^k(M)$ can be written as

$$\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M),$$

where

$$\Omega^{p,q}(M) = \operatorname{span}\left\{ dz^{k_1} \wedge \cdots \wedge dz^{k_p} \wedge d\overline{z}^{k_1} \wedge \cdots \wedge d\overline{z}^{k_q} \right\}.$$

Note that the elements of $\Omega^{p,q}(M)$ are locally represented

$$\eta = \sum_{k=1}^{n} f_k dz^{k_1} \wedge \cdots \wedge dz^{k_p} \wedge d\overline{z}^{k_1} \wedge \cdots \wedge d\overline{z}^{k_q}.$$

Elements of $\Omega^{p,q}(M)$ are called *holomorphic* (p,q)*–forms.*

This decomposition of $\Omega^k(M)$ induces a decomposition of the exterior derivative. Indeed, the exterior derivative $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$ may be written as a sum of the *Dolbeault operators* $\partial : \Omega^{p,q}(M) \longrightarrow \Omega^{p+1,q}(M)$,

$$f \longmapsto \sum_{i,j,k} \frac{\partial}{\partial z^k} f_{i_1 \cdots i_p \overline{j}_1 \cdots \overline{j}_q} dz^k \wedge dz^{i_1} \wedge \cdots dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q},$$

and $\overline{\partial} : \Omega^{p,q}(M) \longrightarrow \Omega^{p,q+1}(M)$,

$$f \longmapsto \sum_{i,j,k} \frac{\partial}{\partial \overline{z}^k} f_{i_1 \cdots i_p \overline{j}_1 \cdots \overline{j}_q} d\overline{z}^k \wedge dz^{i_1} \wedge \cdots \wedge dz^{i_p} \wedge d\overline{z}^{j_1} \wedge \cdots \wedge d\overline{z}^{j_q}.$$

An elementary computation, which may be found on p. 24 of [GR], shows that $\partial^2 = 0$ and $\overline{\partial}^2 = 0$. By considering the boundary map given by $\overline{\partial}$, we obtain the *Dolbeault cohomology theory.* Details of this cohomology theory may be found in Chapter 1 of [GR].

Let us conclude this introductory chapter by stating the notorious $\partial\overline{\partial}$–lemma.

Lemma 1.3.3. (The $\partial\overline{\partial}$–lemma). Let $\eta \in \Omega^{1,1}(M)$ be a real $(1,1)$–form on a complex manifold M. Then η is closed if and only if for every point $p \in M$, there is a neighbourhood U such that the restriction of η to this neighbourhood is given by

$$\eta = \sqrt{-1}\partial\overline{\partial}u,$$

where $u \in \mathscr{C}^\infty(M, \mathbb{R})$.

Proof. See p. 148–149 of [GH] or p. 205 of [Bl]. $\square$

CHAPTER 2

Kähler Manifolds

Complex manifolds necessarily carry both a complex structure and a Riemannian structure. A natural question to raise is whether these two structures are compatible. The question is not trivial since, in general, the answer will be that they are not compatible. As we will see in this chapter however, the class of manifolds for which these two structures are compatible are the Kähler manifolds.

2.1. Hermitian Manifolds

In this section, we consider the parallels of Riemmanian geometry in the setting of complex manifolds.

Definition 2.1.1. Let $E \longrightarrow M$ be a complex vector bundle over an almost complex manifold M. A *Hermitian structure* H on E is a smooth field of Hermitian products on the fibres of E. That is, for any point $p \in M$, we have a map $H : E_p \times E_p \longrightarrow \mathbb{C}$ which satisfies the following properties.

(i) (Linearity). $H(u, v)$ is $\mathbb{C}$-linear with respect to u for every $v \in E_p$.

(ii) (Conjugate Symmetry). $H(u, v) = \overline{H(v, u)}$ for all $u, v \in E_p$.

(iii) (Positivity). $H(u, u) > 0$ for all nonzero $u \in E_p$.

(iv) (Smoothness). $H(u, v) \in \mathscr{C}^\infty(M)$ for each choice of $u, v \in E_p$.

We have the following easy lemma.

Lemma 2.1.2. Every complex vector bundle admits a Hermitian structure.

Proof. Let $\pi : E \longrightarrow M$ be a complex vector bundle of rank k. Locally trivialise (E, π) by $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ and consider a partition of unity $(f_\gamma)_{\gamma \in A}$ subordinate to $\{U_\alpha\}_{\alpha \in A}$. For each $\alpha \in A$ and $x \in U_\alpha$, set $\psi_\alpha := \varphi_\alpha|_{E_x}$. Take H_α to be the pullback of the standard metric on $\mathbb{C}^k$ via ψ_α. Setting $H = \sum_{\alpha \in A} f_\alpha H_\alpha$ yields a Hermitian structure on (E, π). □

Definition 2.1.3. A *Hermitian metric* on an almost complex manifold (M, J) is a Riemannian metric g which is compatible with the almost complex structure J; that is,

$$g(u, v) = g(Ju, Jv), \qquad \forall u, v \in TM. \tag{3}$$

Condition (3) may be formulated as requiring that J is an orthogonal transformation on each tangent space.

The extension by $\mathbb{C}$-linearity of g, also denoted by g, satisfies the following.

(i) $g(\overline{u}, \overline{v}) = \overline{g(u,v)}$ for all $u, v \in TM^{\mathbb{C}}$.

(ii) $g(u, \overline{u}) > 0$ for all nonzero $u \in TM^{\mathbb{C}}$.

(iii) $g(u, v) = 0$ for all $u, v \in T^{1,0}M$ and all $u, v \in T^{1,0}M$.

More precisely, in local holomorphic coordinates $z^1, ..., z^m$, a Hermitian metric g is determined by the components $g_{j\overline{k}}$, where

$$g_{j\overline{k}} = g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \overline{z}^j}\right).$$

Condition (3) may then be formulated as

$$g\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial z^j}\right) = g\left(\frac{\partial}{\partial \overline{z}^i}, \frac{\partial}{\partial \overline{z}^j}\right) = 0.$$

Hence, we may write the metric explicitly as

$$g = \sum_{i,j=1}^{m} g_{i\overline{j}} dz^i \otimes d\overline{z}^j.$$

As a consequence of the symmetry of g, we see that $\overline{g_{i\overline{j}}} = g_{j\overline{i}}$ and since g is positive, we note that for each point $p \in M$, $g_{i\overline{j}}(p)$ is a positive definite Hermitian matrix.

Definition 2.1.4. To every Hermitian metric g, we may associate a $(1,1)$-form ω given by

$$\omega(u, v) := g(Ju, v), \qquad u, v \in TM.$$

In local holomorphic coordinates, we may write

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{m} g_{i\overline{j}} dz^i \wedge d\overline{z}^j.$$

Notation. It is common in the literature to omit the $\frac{\sqrt{-1}}{2}$ factor and write $\omega = \sum_{i,j=1}^{m} g_{i\overline{j}} dz^i \wedge d\overline{z}^j$.

Definition 2.1.5. A complex manifold M whose tangent bundle carries a Hermitian metric is called a *Hermitian manifold*.

In light of Lemma 2.1.2, every complex manifold is Hermitian.

2.2. The Chern Connection

Let $E \longrightarrow M$ be a complex vector bundle over a complex manifold M, not necessarily compact.

Definition 2.2.1. A *connection* on E is a $\mathbb{C}$-linear operator $\nabla : \Gamma(E) \longrightarrow \Omega^1(E)$ which satisfies the Leibniz rule

$$\nabla(f\sigma) \quad = \quad df \otimes \sigma + f\nabla\sigma, \qquad\qquad f \in \mathscr{C}^\infty(M),\ \sigma \in \Gamma(E).$$

We may extend Definition 4.2.1 from $\Omega^1(E)$ to $\Omega^p(E)$ by setting

$$\nabla(\sigma \otimes \tau) \quad = \quad d\sigma \otimes \tau + (-1)^p \sigma \wedge \nabla\tau,$$

where it is understood that $\sigma \wedge \nabla\tau := \sum_{i=1}^n \sigma \wedge e_i^* \otimes \nabla_{e_i}\tau$.

Recall the decomposition

$$\Lambda^k M = \bigoplus_{p+q=k} \Lambda^{p,q} M,$$

induces the analogous decomposition on $\Omega^k M$; that is,

$$\Omega^k(M) \quad = \quad \bigoplus_{p+q=k} \Omega^{p,q}(M).$$

Therefore, given a connection ∇ on E, we may decompose ∇ into $\nabla^{1,0}$ and $\nabla^{0,1}$, as follows.

Let $\pi^{1,0} : \Lambda^1(E) \longrightarrow \Lambda^{1,0}(E)$ and $\pi^{0,1} : \Lambda^1(E) \longrightarrow \Lambda^{0,1}(E)$ denote the standard projections, then we set $\nabla^{1,0} := \pi^{1,0} \circ \nabla$ and $\nabla^{0,1} := \pi^{0,1} \circ \nabla$. This yields $\mathbb{C}$-linear maps $\nabla^{1,0} : \Omega^{p,q}(E) \longrightarrow \Omega^{p+1,q}(E)$ and $\nabla^{0,1} : \Omega^{p,q}(E) \longrightarrow \Omega^{p,q+1}(E)$ such that for $\sigma \in \Omega^{p,q}(E),\ \tau \in \Gamma(E)$,

$$\nabla^{1,0}(\sigma \otimes \tau) = \partial\sigma \otimes \tau + (-1)^{p+q} \sigma \wedge \nabla^{1,0}\tau,$$

and

$$\nabla^{0,1}(\sigma \otimes \tau) = \overline{\partial}\sigma \otimes \tau + (-1)^{p+q} \sigma \wedge \nabla^{0,1}\tau.$$

In the context of Riemannian geometry, we invite the reader to recall that a great deal of time is spent on determining the *natural* connection on which we endow a vector bundle. We inevitably end up discovering the Levi-Civita connection. Of course this arises from our desire for a torsion–free connection that is compatible with the metric on a Riemannian manifold.

We consider characteristics that we would like our connection on a Hermitian vector bundle to carry. Two obvious choices are:

1. Under the decomposition $TM = T^{1,0}M \oplus T^{0,1}M$, with $\nabla = \nabla^{1,0} + \nabla^{0,1}$, we say that the connection ∇ is *compatible with the complex structure* if $\nabla^{0,1} = \overline{\partial}$.
2. We say that ∇ is *compatible with the Hermitian metric* h if

$$g(u, v) = (\nabla u, v) + (u, \nabla v).$$

Proposition 2.2.2. For any Hermitian vector bundle E over a complex manifold M, there is a unique connection $\overline{\nabla}$ on E which is compatible with both the metric and the complex structure. We call this connection the *Chern connection*.

Proof. See p. 79 of [Mo]. $\square$

The above proposition provides us with a beautiful result, but it also burdens us with a deeper question. The tangent bundle on a complex manifold M now carries two *canonical* connections. If we view M as a Riemannian manifold, then the canonical choice of connection is the Levi–Civita connection. If we view M as a complex manifold however, then the canonical choice of connection is the Chern connection. It is not clear, a priori, whether these connections are equivalent, and in general, they are not equivalent. The rest of this chapter is devoted to the study of manifolds on which these two connections coincide.

2.3. Kähler Manifolds

Definition 2.3.1. A complex manifold M with a Hermitian metric g is said to be *Kähler* if the associated $(1,1)$–form ω is closed, i.e.,

$$d\omega = 0.$$

In such a case, we call ω the *Kähler form* and g the *Kähler metric*.

We caution the reader that it common in the literature to see ω being referred to as the Kähler metric without any explicit mention of g.

The following result provides the first motivation for the consideration of such manifolds. Let us recall that on a Riemannian manifold we have a notion of parallel transport, i.e., a way of transporting tangent vectors along smooth curves. The following proposition tells us that on a Kähler manifold, parallel transport commutes with the complex structure given by J.

Proposition 2.3.2. Let M be a complex manifold with a Hermitian metric g and denote by ∇ the Levi–Civita connection. Let ω be the $(1,1)$–form associated to g. Then $d\omega = 0$ if and only if $\nabla J = 0$, i.e., M is a Kähler manifold if and only if J is parallel with respect to the Levi–Civita connection.

Proof. Suppose that $\nabla J = 0$ and let us write[1]

$$d\omega(X_0, ..., X_p) = \sum_{k=0}^{p} (-1)^k (\nabla_{X_k}\omega)(X_0, ..., \widehat{X}_k, ..., X_p), \tag{4}$$

for all $X_0, ..., X_p \in \mathscr{X}(M)$. Given that $\omega = g(J\cdot, \cdot)$, it follows that $\nabla\omega = 0$. Then by (4) we see that $d\omega = 0$.

[1] See p. 53 of [Mo] for further details.

Conversely, suppose that $d\omega = 0$ and define $B(X, Y, Z) = g((\nabla_X J)Y, Z)$. The anti–commutativity of J and $\nabla_X J$ implies that $B(X, Y, JZ) = B(X, JY, Z)$. Moreover, since $(\nabla_{JX} J)Y = J(\nabla_X J)Y$, we see that $B(X, JY, Z) + B(JX, Y, Z) = 0$. Consequently, if $d\omega = 0$, then

$$B(X, Y, JZ) + B(Y, JZ, X) + B(JZ, X, Y) = 0$$

and

$$B(X, JY, Z) + B(JY, Z, X) + B(Z, X, JY) = 0.$$

Adding these two relations together, we see that $2B(X, Y, JZ) = 0$ and in particular, $\nabla J = 0$. $\square$

Our first example of a Kähler manifold is $\mathbb{C}^n$ equipped with the Euclidean metric. An elementary computation shows that the components of the metric are given by

$$g_{i\bar{j}} = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial \bar{z}^j}\right) = \frac{1}{2}\delta_{ij}.$$

The associated Kähler form is given by

$$\omega = \frac{\sqrt{-1}}{2} \sum_{i,j=1}^{m} dz^i \wedge d\bar{z}^j.$$

A less elementary example of a Kähler manifold is $\mathbb{P}^n$ equipped with the *Fubini-Study metric.*

Consider the standard projection $\pi : \mathbb{C}^{n+1}\backslash\{0\} \longrightarrow \mathbb{P}^n$ defined by

$$(z_0, ..., z_n) \longmapsto [z_0 : \cdots : z_n].$$

For an open set $V \subset \mathbb{P}^n$ we consider the lifting of V to $\mathbb{C}^{n+1}\backslash\{0\}$. Explicitly, we have a holomorphic map $z : V \longrightarrow \mathbb{C}^{n+1}\backslash\{0\}$ such that $\pi \circ z = \mathrm{Id}_V$. The form

$$\omega \quad = \quad \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|z\|$$

is a $(1,1)$-form on $\mathbb{P}^n$ and admits a well-defined lifting to $\mathbb{C}^{n+1}\backslash\{0\}$. To see this, consider another lifting $w : V \longrightarrow \mathbb{C}^{n+1}\backslash\{0\}$ given by $w = f \cdot z$, where $f \in {}_{\mathbb{P}^n}\mathcal{O}^*(V)$. Then

$$\begin{aligned}
\omega - \omega &= \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|z\| - \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\|w\| \\
&= \frac{\sqrt{-1}}{2\pi}\left[\partial\bar{\partial}\log\|z\| - \partial\bar{\partial}\log\|f \cdot z\|\right] \\
&= \frac{\sqrt{-1}}{2\pi}\left[\partial\bar{\partial}\log\|z\| - \partial\bar{\partial}(\log\|f\| + \log\|z\|)\right] \\
&= -\frac{\sqrt{-1}}{2\pi}\left[\partial\bar{\partial}\log f + \partial\bar{\partial}\log\overline{f}\right] = 0,
\end{aligned}$$

as claimed. Further, the metric is invariant under the action of $U(n+1)$. Indeed, write $\mathbb{P}^n$ as the quotient S^{2n+1}/S^1. Then the Fubini-Study metric is the unique metric on $\mathbb{P}^n$ such that the quotient $S^{2n+1} \longrightarrow \mathbb{P}^n$ is a Riemannian submersion[2].

Since ω is invariant under the action of $U(n+1)$, it suffices to show that it is positive for one point, say, $[1 : 0 : \cdots : 0]$. So lift the affine patch U_0 to $\mathbb{C}^{n+1}\backslash\{0\}$ via the lift $z = (1, z_1, ..., z_n)$. In U_0,

$$\begin{aligned}
\omega &= \frac{\sqrt{-1}}{2\pi}\partial\overline{\partial}\log\left(1+\sum_{j=1}^{n} z^j\overline{z}^j\right) \\
&= \frac{\sqrt{-1}}{2\pi}\partial\left(\frac{\sum_{j=1}^{n} z^j\, d\overline{z}^j}{1+\sum_{j=1}^{n} z^j\overline{z}^j}\right) \\
&= \frac{\sqrt{-1}}{2\pi}\left[\frac{\sum_{j=1}^{n} dz^j\wedge d\overline{z}^j}{1+\sum_{j=1}^{n} z^j\overline{z}^j} - \frac{\left(\sum_{j=1}^{n}\overline{z}^j\, dz^j\right)\wedge\left(\sum_{j=1}^{n} z^j\, d\overline{z}^j\right)}{\left(1+\sum_{j=1}^{n} z^j\overline{z}^j\right)}\right].
\end{aligned}$$

Evaluate ω at $[1 : 0 : \cdots : 0]$,

$$\omega = \frac{\sqrt{-1}}{2\pi}\sum_{j=1}^{n} dz^j \wedge d\overline{z}^j > 0;$$

so ω is indeed a Hermitian metric on $\mathbb{P}^n$.

We have seen that the Kähler condition $d\omega = 0$ could be expressed as the complex structure J being parallel with respect to the Levi–Civita connection. Another formulation of this condition, which will prove to be very useful in computations is that near every point of a Kähler manifold we have *holomorphic normal coordinates*.

Theorem 2.3.3. For a complex manifold M with Hermitian metric g and associated $(1,1)$–form ω, the following are equivalent to the Kähler condition $d\omega = 0$.

(i) In local holomorphic coordinates

$$\frac{\partial g_{i\overline{j}}}{\partial z^k} = \frac{\partial g_{k\overline{j}}}{\partial z^i}, \qquad \forall i, j, k.$$

(ii) At each point $p \in M$, we have holomorphic normal coordinates such that

$$g_{i\overline{j}}(p) = \delta_{ij}, \qquad \frac{\partial g_{i\overline{j}}}{\partial z^k}(p) = \frac{\partial g_{i\overline{j}}}{\partial \overline{z}^\ell}(p) = 0, \qquad \forall i, j, k, \ell.$$

Condition (ii) may also be expressed as the ability to choose coordinates such that the metric g approximates the Euclidean metric up to second order.

[2]Recall that a submersion $f : (M, g) \longrightarrow (N, h)$ of Riemannian manifolds is said to be a *Riemannian submersion* if for every $x \in M$, the restriction of $(f_*)_x$ to the g-orthogonal of the tangent space to the fibre $f^{-1}(f(x))$ is an isometry onto $T_{f(x)}N$.

PROOF. The fact that $d\omega = 0$ is equivalent to (i) follows from writing $d\omega = 0$ in local coordinates. We therefore need only concern ourselves with (ii). Indeed, via a linear change of coordinates, we may diagonalise the metric at a point, $g_{i\overline{j}}(p) = \delta_{ij}$. As a consequence, the Kähler form is given by

$$\omega = \frac{\sqrt{-1}}{2} g_{i\overline{j}} dz^i \wedge d\overline{z}^j = \frac{\sqrt{-1}}{2}\left(\delta_{ij} + a_{ijk}z^k + a_{ij\overline{k}}\overline{z}^k\right) dz^i \wedge d\overline{z}^j. \tag{5}$$

Since g is Hermitian, $g_{i\overline{j}} = \overline{g_{j\overline{i}}}$ and so $a_{ji\overline{k}} = \overline{a_{ijk}}$. Moreover, $d\omega(p) = 0$ implies that $a_{ijk} = a_{kji}$. Using the change of coordinates

$$z^i = \xi^i - \frac{1}{2}a_{kij}\xi^j\xi^k,$$

the linear terms in (5) vanish. Hence we may write

$$\begin{aligned}
\omega &= \frac{\sqrt{-1}}{2}\left(d\xi^i - a_{kij}\xi^j\xi^k\right) \wedge \left(d\overline{\xi}^i - \overline{a_{nim}}\overline{\xi}^m d\overline{\xi}^n\right) \\
&\quad + \frac{\sqrt{-1}}{2}\left(a_{ijk}\xi^k + a_{ij\overline{k}}\overline{\xi}^k\right) d\xi^i \wedge d\overline{\xi}^j + O(|z|^2) \\
&= \frac{\sqrt{-1}}{2}\delta_{ij} d\xi^i \wedge d\overline{\xi}^j + O(|z|^2).
\end{aligned}$$

□

The last formulation of the Kähler condition which we present here was illuded to in the previous section. Indeed, we saw that the tangent bundle of a complex manifold carried two canonical connections, the Levi–Civita connection and the Chern connection. We proceed to prove that these connections coincide exactly when the underlying manifold is Kähler. We need the following lemma which allows us to express the Dolbeault operator in terms of the Levi–Civita connection.

Lemma 2.3.4. Let $v \in \Gamma(TM)$, then we may write the Dolbeault operator $\overline{\partial}$ as

$$\overline{\partial}^{\nabla}(Y)(X) \quad := \quad \frac{1}{2}\left(\nabla_X Y + J\nabla_{JX}Y - J(\nabla_Y J)X\right),$$

where ∇ denotes the Levi-Civita connection.

PROOF. There are two things to prove. We first show that $\overline{\partial}^{\nabla}(Y)(X)$ satisfies the Leibniz rule. Then we show that $\overline{\partial}^{\nabla}(Y)$ vanishes for every holomorphic section Y. This last assertion shows that $\overline{\partial}^{\nabla} = \overline{\partial}$. For $f \in \mathscr{C}^{\infty}(M)$, $Y \in \Gamma(TM)$,

$$\begin{aligned}
\overline{\partial}^{\nabla}(fY)(X) &= \frac{1}{2}\left(\nabla_X(fY) + J\nabla_{JX}(fY) - J(\nabla_{fY}J)X\right) \\
&= \frac{1}{2}\left(f\nabla_X Y + (\partial_X f)Y + J(\nabla_{JX}Y)f + J(\partial_{JX}f)Y - fJ(\nabla_Y J)X\right) \\
&= \frac{1}{2}f\left[\nabla_X Y + J(\nabla_{JX}Y) - J(\nabla_Y J)X\right] + \frac{1}{2}\left[(\partial_X f)Y + (\partial_{JX}f)JY\right] \\
&= f \cdot \overline{\partial}^{\nabla}Y(X) + \overline{\partial}f(X)Y.
\end{aligned}$$

Let Y be a holomorphic section of the tangent bundle. Then $\mathcal{L}_Y J = 0$; writing

$$\begin{aligned}(\mathcal{L}_Y J)X &= [Y, JX] - J[Y, X]\\ &= (\nabla_Y J)X - \nabla_{JX}Y + J\nabla_X Y\\ &= J(\overline{\partial}^{\nabla} Y)(X) = 0,\end{aligned}$$

as required. □

Theorem 2.3.5. The Chern connection $\overline{\nabla}$ coincides with the Levi–Civita connection ∇ if and only if the underlying manifold M is Kähler.

PROOF. Suppose ∇ and $\overline{\nabla}$ coincide. By definition, J is parallel with respect to the Chern connection. By assumption, J is parallel with respect to the Levi–Civita connection. By Theorem 2.3.2, M must be Kähler.

Conversely, suppose that M is a Kähler manifold. Then $\nabla J = 0$, $\nabla g = 0$ and $\nabla\omega = 0$. By Lemma 2.3.4, we may write

$$\nabla_X^{0,1} = \frac{1}{2}\left(\nabla_X + \sqrt{-1}\nabla_{JX}\right) = \frac{1}{2}\left(\nabla_X + J\nabla_{JX}\right).$$

So $\nabla^{0,1} = \overline{\partial}$ and $\nabla = \overline{\nabla}$, as required. □

Before closing this chapter, we would like to mention a very simple cohomological obstruction that measures whether a compact complex manifold is Kähler.

Let us observe that since the Kähler form ω is a closed form, it represents a cohomology class, i.e., $[\omega] \in H^2(M, \mathbb{R}) \otimes \mathbb{C}$. The following lemma expresses the intimate relationship between the Kähler form ω and the volume form.

Lemma 2.3.6. The Kähler form ω on a compact Kähler manifold M of (complex) dimension n satisfies the relation

$$\frac{\omega^n}{n!} = \star(1).$$

PROOF. The statment is local in nature. Hence, for a fixed point $p \in M$, we diagonalise the Kähler metric $g_{i\bar{j}}(p) = \delta_{ij}$ and as a consequence, the Kähler form is given by

$$\omega = \frac{\sqrt{-1}}{2} dz^i \wedge d\overline{z}^i.$$

Since $dz^i \wedge d\overline{z}^j = -2\sqrt{-1}dx^i \wedge dy^i$, we see that $\omega = dx^i \wedge dy^i$. Consequently,

$$\omega^n = n! dx^1 \wedge dy^1 \wedge \cdots \wedge dx^n \wedge dy^n = n! \star (1),$$

as required. □

From Lemma 2.3.6, we may deduce the following which measures the obstruction of a compact complex manifold being Kähler.

Proposition 2.3.7. The Kähler form ω of a Kähler metric g on a compact Kähler manifold M represents a nontrivial cohomology class. The same is true for ω^k, $1 \leq k \leq n$.

Proof. By Lemma 2.3.6,

$$\int_M \omega^n = n! \int_M \star(1) = n!\mathrm{Vol}(M) > 0.$$

Moreover, if $\omega^k = d\eta$ for some $1 \leq k \leq n$, then

$$\begin{aligned} \int_M \omega^n = \int_M \omega^k \wedge \omega^{n-k} &= \int_M d\eta \wedge \omega^{n-k} \\ &\overset{\omega \text{ is closed}}{=} \int_M d(\eta \wedge \omega^{n-k}) \overset{\text{Stokes}}{=} 0. \end{aligned}$$

□

A corollary of Proposition 2.3.7 is that the even cohomology groups of a compact Kähler manifold must be nonzero. This immediately shows that the complex manifold $S^1 \times S^{2k-1}$ is not Kähler.

CHAPTER 3

Curvature Properties of Kähler Manifolds

The notion of curvature is central to geometry. In this chapter, we collect the relevant notions and results in the context of Kähler manifolds. Throughout this chapter, (M, ω) will denote a Kähler manifold, not necessarily compact, with Levi–Civita connection ∇.

3.1. Covariant Derivatives

Notation. Let us adopt the notation

$$\nabla_k := \nabla_{\frac{\partial}{\partial z^k}}, \quad \nabla_{\overline{k}} := \nabla_{\frac{\partial}{\partial \bar{z}^k}}$$

The Christoffel symbols Γ^k_{ij} which determine the connection, satisfy

$$\nabla_i \frac{\partial}{\partial z^j} = \Gamma^k_{ij} \frac{\partial}{\partial z^k}.$$

In this language, the Kähler condition may be formulated as the assertion that all mixed Christoffel symbols vanish. That is, the only nonzero Christoffel symbols are of the form Γ^k_{ij} and $\Gamma^{\overline{k}}_{\overline{ij}}$.

For reference, the absence of torsion in the Levi–Civita connection can be expressed by the following symmetry of the Christoffel symbols

$$\Gamma^k_{ij} = \Gamma^k_{ji}.$$

Moreover, for any tensor X, we have

$$\nabla_{\overline{k}} X = \overline{\nabla_k \overline{X}}.$$

We may also express the Christoffel symbols in terms of the metric $g_{i\bar{j}}$ as follows.

Lemma 3.1.1. The Christoffel symbols satisfy

$$\Gamma^k_{ij} = g^{k\bar{\ell}} \frac{\partial}{\partial z^i} g_{j\bar{\ell}}.$$

Proof. Expressing $\nabla g = 0$ in coordinates yields

$$0 = \nabla_i g_{j\bar{\ell}} = \frac{\partial}{\partial z^i} g_{j\bar{\ell}} - \Gamma^r_{ij} g_{r\bar{\ell}}.$$

This implies that

$$g^{k\bar{\ell}} \frac{\partial}{\partial z^i} g_{j\bar{\ell}} = g^{k\bar{\ell}} \Gamma^r_{ij} g_{r\bar{\ell}} = \Gamma^r_{ij} \delta_{kr} = \Gamma^k_{ij}.$$

$\square$

The failure of the covariant derivatives to commute is measured by the curvature tensor. This 4–tensor $R_{i\bar{j}k\bar{\ell}}$ satisfies

$$(\nabla_k \nabla_{\bar{\ell}} - \nabla_{\bar{\ell}} \nabla_k) \frac{\partial}{\partial z^i} = R_i{}^j{}_{k\bar{\ell}} \frac{\partial}{\partial z^j},$$

where $R_{i\bar{j}k\bar{\ell}} = g_{r\bar{j}} R_i{}^r{}_{k\bar{\ell}}$. In terms of the Christoffel symbols, we see that

$$R_i{}^j{}_{k\bar{\ell}} = -\frac{\partial}{\partial \bar{z}^\ell} \Gamma^j_{ki}.$$

Therefore, in terms of the metric, we have

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} + g^{r\bar{s}} \frac{\partial g_{i\bar{s}}}{\partial z^k} \frac{\partial g_{r\bar{j}}}{\partial \bar{z}^\ell}$$

In the previous chapter, we saw that we could choose coordinates such that at a point $p \in M$, we have

$$\frac{\partial g_{i\bar{s}}}{\partial z^k} = \frac{\partial g_{r\bar{j}}}{\partial \bar{z}^\ell} = 0.$$

Consequently, we may choose coordinates such that the curvature tensor may be written as

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell}.$$

From this, one may interpret the curvature tensor as a measure of the obstruction to finding holomorphic coordinates such that the metric may be approximated up to second order by the Euclidean metric.

Let us recall that the components of the Ricci tensor are given by

$$R_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}.$$

In local coordinates, we may express the Ricci tensor as

$$R_{i\bar{j}} = -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{\alpha\bar{\beta}}).$$

Indeed, we have the following elementary computation,

$$\begin{aligned} -\frac{\partial^2}{\partial z^i \partial \bar{z}^j} \log \det(g_{\alpha\bar{\beta}}) &= -\frac{\partial}{\partial \bar{z}^j} \left(g^{\alpha\bar{\beta}} \frac{\partial}{\partial z^i} g_{\alpha\bar{\beta}} \right) \\ &= -\frac{\partial}{\partial \bar{z}^j} \Gamma^\alpha_{i\alpha} \\ &= R_\alpha{}^\alpha{}_{i\bar{j}} = R_{i\bar{j}}. \end{aligned}$$

Definition 3.1.2. We define the *Ricci form* to be the $(1,1)$–form ρ determined by $\rho(X,Y) = \mathrm{Ric}(JX,Y)$. In local coordinates,

$$\rho = -\sqrt{-1}\frac{\partial^2}{\partial z^i \partial \overline{z}^j} \log\det(g_{\alpha\overline{\beta}}).$$

If ω is the Kähler form associated to the Kähler metric g, then we write $\mathrm{Ric}(\omega)$ for the Ricci form, i.e.,

$$\mathrm{Ric}(\omega) = -\sqrt{-1}\partial\overline{\partial} \log\det(g_{i\overline{j}}).$$

In light of the above expression for $\mathrm{Ric}(\omega)$, we see that $\mathrm{Ric}(\omega)$ is a closed $(1,1)$–form. Consequently, $\mathrm{Ric}(\omega)$ represents a cohomology class. Moreover, by multiplying the Ricci form by a factor of $\frac{1}{2\pi}$, this will be an integer cohomology class which is referred to as the first Chern class.

CHAPTER 4

The First Chern Class

A well established technique in both algebraic and analytic geometry in the study of manifolds is to study the line bundles over the manifold. From this viewpoint, we will define the first Chern class, a topological invariant which is central to both the Calabi–Yau theorem and Kähler geometry as a whole.

4.1. Divisors

Definition 4.1.1. An *analytic hypersurface* is an analytic subvariety of codimension one. More precisely, an analytic hypersurface is a set $V \subset M$ inside some manifold M, such that for each point $p \in V$, we may choose some neighbourhood in which V is given by the zero set of a single holomorphic function. This function is called a *locally defining function*.

We say that an analytic hypersurface is *irreducible* if it cannot be written as a union of two proper closed subsets.

Let us suppose that V is an analytic hypersurface and $p \in V$ is a point. Choose a neighbourhood U of p such that in U, V is given by the zero set of a holomorphic function f. Let g be a function which is holomorphic on some neighbourhood $\widetilde{U}$ of p such that $U \cap \widetilde{U} \neq \emptyset$. We define a function $\text{ord}_{V,p}$ which assigns to g to the largest integer α such that

$$g = f^{\alpha} h,$$

where h is a non–vanishing holomorphic function. An elementary argument using some commutative algebra, see p. 10 of [GH], shows that this function $\text{ord}_{V,p}$ is independent of p. We are therefore justified in omitting the subscript p and writing simply ord_V.

Let us note some elementary properties of this order function.

(i) For two holomorphic functions f, g,

$$\text{ord}_V(fg) = \text{ord}_V(f) + \text{ord}_V(g).$$

(ii) If f is a meromorphic function which admits the local representation $f = g/h$, then

$$\text{ord}_V(f) = \text{ord}_V(g) - \text{ord}_V(h).$$

(iii) A function f is holomorphic on V if $\text{ord}_V(f) \geq 0$.

Definition 4.1.2. Let M be a complex manifold. A (Cartier) *divisor* (f) associated to a meromorphic function f is an expression of the form

$$(f) \quad = \quad \sum_{V \subset M} \mathrm{ord}_V(f) \cdot V,$$

where the sum is taken over all analytic hypersurfaces in M.

The Cartier divisors lend themselves well to a sheaf-theoretic interpretation.

To see this, let us denote by ${}_M\mathcal{O}$ the sheaf of holomorphic function on M and by ${}_M\mathscr{M}$ the sheaf of meromorphic function on M. The sections of these sheaves over an open set $U \subset M$ will be denoted by ${}_M\mathcal{O}(U)$ and ${}_M\mathscr{M}(U)$, respectively. The sheaf of nonvanishing holomorphic functions on M is denoted by ${}_M\mathcal{O}^*$ and similarly, the sheaf of nonvanishing meromorphic functions on M is denoted by ${}_M\mathscr{M}^*$.

Now let (f) be a Cartier divisor, and choose an open cover $\{U_\alpha\}_{\alpha \in A}$ of M such that, for each $\alpha \in A$, every analytic hypersurface V appearing in (f) has a locally defining function $g_{V,\alpha} \in \mathcal{O}(U_\alpha)$. Setting

$$f_\alpha := \prod_{V \subset M} g_{V,\alpha}^{\mathrm{ord}_V(f)} \in {}_M\mathscr{M}^*(U_\alpha),$$

we obtain a global section of ${}_M\mathscr{M}^*/{}_M\mathcal{O}^*$. Conversely, a global section of ${}_M\mathscr{M}^*/{}_M\mathcal{O}^*$ is given by a collection $\{(U_\alpha, f_\alpha)\}_{\alpha \in A}$, where the U_α yield an open cover of X and the $f_\alpha \in {}_M\mathscr{M}^*(U_\alpha)$ such that

$$\frac{f_\alpha}{f_\beta} \in {}_M\mathcal{O}^*(U_\alpha \cap U_\beta), \qquad \forall \alpha, \beta \in A.$$

This implies that $\mathrm{ord}_V(f_\alpha) = \mathrm{ord}_V(f_\beta)$ for all $\alpha, \beta \in A$. So for any analytic hypersurface $V \subset M$, we choose α such that $V \cap U_\alpha \neq \emptyset$ and setting

$$(f) = \sum_{V \subset M} \mathrm{ord}_V(f_\alpha) \cdot V$$

yields a Cartier divisor.

Definition 4.1.3. The quotient sheaf ${}_M\mathscr{D} := {}_M\mathscr{M}^*/{}_M\mathcal{O}^*$ is called the *sheaf of Cartier divisors* and denote the space of global sections by $\mathrm{Div}(X)$.

4.2. Line Bundles

Let us recall that a holomorphic vector bundle of rank 1 is called a *holomorphic line bundle*. All line bundles considered here are assumed to be holomorphic.

The transition functions $g_{\alpha\beta}$ for a line bundle $\mathscr{L} \longrightarrow M$ are given by $g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*$. These can be shown to be independent of the choice of trivialisation and satisfy the cocycle condition

$$\begin{cases} g_{\alpha\beta} \cdot g_{\beta\alpha} & = 1, \\ g_{\alpha\beta} \cdot g_{\beta\gamma} \cdot g_{\gamma\alpha} & = 1. \end{cases} \tag{6}$$

Conversely, given a collection of Čech 1-cocyles $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$ satisfying the cocycle condition (6), we may construct a line bundle $\mathscr{L}$ whose transition functions are exactly $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$. Indeed, we set

$$\mathscr{L} = \coprod_{\alpha\in A} (U_\alpha \times \mathbb{C})/\sim,$$

where $\sim$ is the equivalence relation defined by making the identification $z \sim g_{\alpha\beta}(z)$.

We investigate the transformation behaviour of the transition functions $g_{\alpha\beta}$ under a change of local trivialization. Let $\{(U_\alpha, \psi_\alpha)\}_{\alpha\in A}$ and $\{(U_\alpha, \widetilde{\psi}_\alpha)\}_{\alpha\in A}$ be two distinct local trivialization of the line bundle $\mathscr{L}$ such that

$$\widetilde{\psi}_\alpha = f_\alpha \psi_\alpha, \qquad f_\alpha \in {}_M\mathcal{O}^*(U_\alpha),\ \forall \alpha \in A. \tag{7}$$

If $\{g_{\alpha\beta}\}$ and $\{\widetilde{g}_{\alpha\beta}\}$ are the transition functions relative to $\{\psi_\alpha\}$ and $\{\widetilde{\psi}_\alpha\}$ respectively, then (7) yields the relation

$$\widetilde{g}_{\alpha\beta} = \frac{f_\alpha}{f_\beta} g_{\alpha\beta}. \tag{8}$$

Since any local trivialization of $\mathscr{L}$ may be obtained in this way, (8) defines an equivalence class of line bundles. Let $\mathscr{L}, \widetilde{\mathscr{L}}$ be two lines bundles with transition functions $\{g_{\alpha\beta}\}$ and $\{\widetilde{g}_{\alpha\beta}\}$ respectively. Then $\mathscr{L} \otimes \widetilde{\mathscr{L}}$ has transition functions $\{g_{\alpha\beta} \cdot \widetilde{g}_{\alpha\beta}\}$ and the dual of $\mathscr{L}$ has transition functions $\{g_{\alpha\beta}^{-1}\}$.

Definition 4.2.1. The set of equivalence classes of line bundles over a complex manifold M forms a (multiplicative) group called the *Picard group*, denoted by $\mathrm{Pic}(M)$.

We interpret the Picard group using the language of sheaves. Let $\mathscr{L}$ be a line bundle over X with transition functions $\{g_{\alpha\beta}\}_{\alpha,\beta\in A}$. These functions $g_{\alpha\beta} \in {}_M\mathcal{O}^*(U_\alpha \cap U_\beta)$ represent a Čech 1-cochain on M, taking values in the multiplicative sheaf ${}_M\mathcal{O}^*$. The constraint (6) asserts that that $g_{\alpha\beta}$ form a Čech cocycle and (8) asserts that two such cocycles $\{g_{\alpha\beta}\}, \{\widetilde{g}_{\alpha\beta}^{-1}\}$ define the same bundle if and only their quotient $\{g_{\alpha\beta} \cdot \widetilde{g}_{\alpha\beta}^{-1}\}$ is a coboundary. Hence the Picard group $\mathrm{Pic}(M)$ is exactly $H^1(M, {}_M\mathcal{O}^*)$.

Let us now consider the exact exponential sequence of sheaves

$$0 \longrightarrow {}_M\mathbb{Z} \xrightarrow{2\pi\sqrt{-1}} {}_M\mathcal{O} \xrightarrow{\exp} {}_M\mathcal{O}^* \longrightarrow 1. \tag{9}$$

Combining the boundary maps associated to the exact sequence

$$1 \longrightarrow {}_M\mathcal{O}^* \longrightarrow {}_M\mathscr{M}^* \xrightarrow{\tau} {}_M\mathscr{D} \longrightarrow 0, \tag{10}$$

and (9) yields the sequence

$$\mathrm{Div}(M) \longrightarrow \mathrm{Pic}(M) \longrightarrow H^2(M, \mathbb{Z}),$$

from which we define a topological invariant.

Definition 4.2.2. For any $D \in \mathrm{Div}(M)$ we define the (first) *Chern class* of D to be the image of D in $H^2(M, \mathbb{Z})$. The Chern class of D is denoted by $c_1(D)$.

From how we have defined the Chern class, it follows that

(i) $c_1(L \otimes L') = c_1(L) + c_1(L')$,

(ii) $c_1(L^*) = -c_1(L)$.

Further, by considering a holomorphic map $f : M \longrightarrow N$ of complex manifolds, the diagram

$$\begin{tikzcd} H^1(N, {}_N\mathcal{O}^*) \arrow[r] \arrow[d, "f^*"'] & H^2(N, \mathbb{Z}) \arrow[d, "f^*"] \\ H^1(M, {}_M\mathcal{O}^*) \arrow[r] & H^2(M, \mathbb{Z}) \end{tikzcd}$$

commutes and for any line bundle L over N,

$$c_1(f^*L) = f^*c_1(L).$$

Before concluding this chapter, let us state one of the central theorems of Kähler geometry.

Theorem 4.2.3. Let (M, ω) be a compact Kähler manifold. Then $\frac{1}{2\pi}\mathrm{Ric}(\omega)$ represents the first Chern class of M.

PROOF. See p. 120 of [Mo]. $\square$

CHAPTER 5

PDE Formulation and Preliminaries

Let M be a compact Kähler manifold of (complex) dimension m whose Kähler form is given by

$$\omega = \sqrt{-1} \sum_{i,j=1}^{m} g_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Denote by $R_{i\bar{j}} = -\partial\overline{\partial} \log \det(g_{i\bar{j}})$ the Ricci tensor and by

$$\mathrm{Ric}(\omega) = \sqrt{-1} R_{i\bar{j}} dz^i \wedge d\bar{z}^j = -\sqrt{-1}\partial\overline{\partial} \log \det(g_{i\bar{j}})$$

the associated Ricci form. Recall that the Ricci form $\mathrm{Ric}(\omega)$ is a positive $(1,1)$–form which represents 2π times the first Chern class of M, i.e.,

$$\left[\frac{1}{2\pi}\mathrm{Ric}(\omega)\right] = c_1(TM) \in H^2(M, \mathbb{R}).$$

In 1957 E. Calabi conjectured that the converse was also true. That is, for a compact Kähler manifold (M, ω) as above, let η be a real $(1,1)$–form which represents $2\pi c_1(M)$. The Calabi conjecture asserts that there exists a Kähler form $\widetilde{\omega}$ on M such that ω and $\widetilde{\omega}$ are cohomologous and $\mathrm{Ric}(\widetilde{\omega}) = \eta$.

Despite the absence of a proof, Calabi did show that if such a metric exists, then it is unique, see p. 46 of [Ti]. The question of existence however remained open for almost two decades. In 1976, S–T. Yau proved that Calabi conjecture by formulating the conjecture as a non–linear elliptic partial differential equation of Monge–Ampère type.

Notation. Throughout, we will omit the 2π and also $\sqrt{-1}$. In particular, we will write $\omega + \sqrt{-1}\partial\overline{\partial}\varphi$ as simply $\omega + \partial\overline{\partial}\varphi$. We will also set $\omega_\varphi := \omega + \partial\overline{\partial}\varphi$ which is common throughout the literature.

5.1. Formulation of the Calabi Conjecture to PDE

Let η be a $(1,1)$–form which represents the first Chern class of M. Since $\mathrm{Ric}(\omega)$ also represents this cohomology class, the $\partial\overline{\partial}$–lemma yields the existence of a smooth function $f \in \mathscr{C}^\infty(M, \mathbb{R})$ such that

$$\eta - \mathrm{Ric}(\omega) = \partial\overline{\partial} f.$$

Let $\widetilde{\omega}$ be the Kähler metric for which $\text{Ric}(\widetilde{\omega}) = \eta$ and whose existence is asserted by the conjecture. By the $\partial\overline{\partial}$–lemma, we have a function $\varphi \in \mathscr{C}^\infty(M, \mathbb{R})$ such that $\widetilde{\omega} = \omega + \partial\overline{\partial}\varphi$. It then follows that the difference of the Ricci forms is given by

$$\text{Ric}(\widetilde{\omega}) - \text{Ric}(\omega) \quad = \quad \eta - \text{Ric}(\omega) = \partial\overline{\partial} f.$$

In local holomorphic coordinates $z^1, ..., z^n$, the above equation may be written

$$\partial\overline{\partial} \log \det \left(g_{i\overline{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j} \right) - \partial\overline{\partial} \log \det(g_{i\overline{j}}) = \partial\overline{\partial} f. \tag{11}$$

Such an f is made unique by imposing the *compatibility* constraint

$$\int_M \exp(f) \omega^n = \int_M \omega^n. \tag{12}$$

We caution the reader that the $\partial\overline{\partial}$–lemma only establishes existence locally. More precisely, for $U, V \subset M$ open sets, $U \cap V \neq \emptyset$, with local coordinates $z^1, ..., z^n$, and $w^1, ..., w^n$, respectively, it is not necessarily the case that $\log\det(g_{i\overline{j}})|_U$ and $\log\det(g_{i\overline{j}})|_V$ agree on $U \cap V$.

In an attempt to circumvent this, recall that for two volume forms Ω_1^n and Ω_2^n which induce the same orientation on M, there exists a function $\xi \in \mathscr{C}^\infty(M, \mathbb{R}_{>0})$ such that $\Omega_1^n = \xi \Omega_2^n$. In particular, if we consider the volume forms ω^n and $(\omega + \partial\overline{\partial}\varphi)^n$, then $(\omega + \partial\overline{\partial}\varphi)^n = \xi\omega^n$ for some $\xi \in \mathscr{C}^\infty(M, \mathbb{R}_{>0})$. Consequently,

$$\log \left(\frac{(\omega + \partial\overline{\partial}\varphi)^n}{\omega^n} \right) = \log \left(\frac{\xi \omega^n}{\omega^n} \right) = \log(\xi),$$

which is a well–defined function on all of M. From (11) we observe that

$$\partial\overline{\partial} \log \left(\frac{(\omega + \partial\overline{\partial}\varphi)^n}{\omega^n} \right) = \partial\overline{\partial} \log \left(\det \left(g_{i\overline{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j} \right) (\det(g_{i\overline{j}}))^{-1} \right) = \partial\overline{\partial} f$$

is globally defined. So for some constant k, the compactness of M together with the maximum principle implies that

$$\det \left(g_{i\overline{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \overline{z}^j} \right) (\det(g_{i\overline{j}}))^{-1} = \exp(f + k),$$

or in the more condensed notation,

$$(\omega + \partial\overline{\partial}\varphi)^n = \exp(f + k)\omega^n. \tag{13}$$

We make the claim that the constant $k = 0$. To see this, observe that

$$\begin{aligned}\int_M \exp(f+k)\omega^n &\overset{(13)}{=} \int_M (\omega+\partial\overline{\partial}\varphi)^n \\ &= \int_M \omega^n + \partial\overline{\partial}\varphi \wedge (\omega_\varphi^{n-1} + \omega_\varphi^{n-2}\wedge\omega + \cdots + \omega^{n-1}) \\ &= \int_M \omega^n + d\left(\overline{\partial}\varphi\wedge(\omega_\varphi^{n-1}+\cdots+\omega^{n-1})\right) \\ &\overset{\text{Stokes}}{=} \int_M \omega^n.\end{aligned} \tag{14}$$

We therefore see that Calabi's conjecture is equivalent to the assertion that there exists a solution $\varphi \in \mathscr{C}^\infty(M,\mathbb{R})$ of the complex Monge–Ampère equation

$$(\omega+\partial\overline{\partial}\varphi)^n = \exp(f)\omega^n. \tag{15}$$

In particular, (15) is an equation of volume forms associated to two metrics in the same cohomology class.

5.2. The Continuity Method

The approach that we adopt here to solve (15) is the *continuity method*. This is the approach that Yau takes in [Ya76] and it also seen in [Au]. Since the original publication, new approaches have been formulated including variational and viscosity type approaches, see [AS].

We introduce a one-parameter family of complex Monge–Ampère equations

$$(\omega+\partial\overline{\partial}\varphi_t)^n = \exp(tf)\omega^n \tag{MA$_t$}$$

where $t\in[0,1]$. The method of continuity enables us to bootstrap our way from a solution that can be obtained easily to the solution we desire. Indeed, for $t=0$, (MA$_0$) is simply $(\omega+\partial\overline{\partial}\varphi_0)^n = \omega^n$. Taking φ_0 to be constant provides a solution to (MA$_0$). While taking $t=1$ is the equation which we would like to solve.

Notation. Let us adopt the notation $\omega_t := \omega + \partial\overline{\partial}\varphi_t$ and $f_t := tf$.

Note that the compatibility constraint (12) is then expressed by the (non–linear) condition

$$\int_M (\exp(f_t)-1)\,\omega^n. \tag{16}$$

Consider the set

$$S := \left\{ t\in[0,1] \ : \ (\text{MA}_t) \text{ has a solution } \varphi_t \in \mathscr{C}^{k+1,\alpha}(M), \text{ and (16) holds} \right\}.$$

To show that $1 \in S$, it suffices to show that S is a non–empty connected set in $[0,1]$. From the preceeding paragraph, we see that $0 \in S$ and so S is non–empty. We now need only show that S is both open and closed.

Let us preface by recalling some facts from functional analysis.

Theorem 5.2.1. (Implicit Function Theorem). Let X, Y, Z denote Banach spaces. Let $T : X \times Y \longrightarrow Z$ be a $\mathscr{C}^k$–operator and consider the differential $\nabla_y T(x, y_0) \in \text{Hom}(Y, Z)$ of $y \mapsto T(x_0, y)$ evaluated at y_0. Suppose $\nabla_y T(x_0, y)$ is invertible at $(x_0, y_0) \in X \times Y$. Then there exist open neighbourhoods $U \subseteq X \times Y$ and $V \subseteq X \times Z$ of (x_0, y_0) and $(x_0, T(x_0, y_0))$ respectively, such that $(x, y) \longmapsto (x, T(x, y))$ is a $\mathscr{C}^k$–diffeomorphism of U onto V.

Proof. See p. 72 of [Au]. □

Theorem 5.2.2. (Arzela–Ascoli Theorem). Let $\Omega \subset \mathbb{R}^{2m}$ be a bounded set and consider a uniformly bounded sequence $(\varphi_j)_{j \in \mathbb{N}}$ in $\mathscr{C}^{k,\alpha}(\Omega)$. That is, we have the exists of a constant C, independent of $j \in \mathbb{N}$, such that $\|\varphi_j\|_{\mathscr{C}^{k,\alpha}} \leq C$. Then there exists a subsequence $(\varphi_{j_k})_{k \in \mathbb{N}}$ which converges to some $\varphi \in \mathscr{C}^{l,\beta}$, where $l + \beta < k + \alpha$.

Proof. See p. 29 of [Sz]. □

The implicit function theorem will be used to show that S is open. This will require a suitable choice of Banach spaces and an operator defined between them whose derivative is invertible. Given that we want a smooth solution of (15), a first candidate for a space to work in would be $\mathscr{C}^\infty(M, \mathbb{R})$. To our dismay however, $\mathscr{C}^\infty(M, \mathbb{R})$ does not carry the structure of a Frèchet space, let alone a Banach space. The next obvious candidate is a Sobolev space $W^{k,p}$, we know these to be Banach spaces. In what follows, we will see that the main technique that is employed in the proof of Calabi–Yau theorem is the maximum principle. This requires knowledge of the function at a particular point. The elements of a Sobolev space are equivalence classes of functions defined modulo a set of measure zero however[1]. The Sobolev spaces therefore no not capture this punctual[2] information of the function.

As illuded to throughout this chapter, an appropriate choice of Banach space are the Hölder spaces $\mathscr{C}^{k,\alpha}$. The Hölder spaces remedy the problems which arose in the consideration of $\mathscr{C}^\infty(M, \mathbb{R})$ and $W^{k,p}$. Moreover, we may obtain a smooth solution of (15) by first showing that the solution lies in $\mathscr{C}^{k,\alpha}$ and applying the Schauder estimate.

[1] More precisely, if $f, g \in W^{k,p}(\Omega)$, then the equality $f = g$ is interpretted as $f(z) = g(z)$ for all $z \in \Omega \backslash N$, where N is a set of (typically Lebesgue) measure zero.

[2] We are using the expression punctual to mean *at a point*.

Theorem 5.2.3. (Schauder estimate). Let (M, g) be a compact Riemannian manifold and L a second order elliptic operator on M. There exists a constant $C > 0$ such that, for all $k \in \mathbb{N}$ and $0 < \alpha < 1$,

$$\|u\|_{\mathscr{C}^{k+2,\alpha}(M)} \leq C\left(\|Lu\|_{\mathscr{C}^{k,\alpha}(M)} + \|u\|_{L^1(M)}\right).$$

PROOF. See p. 34 of [Sz]. □

5.3. Showing that S is open

To show that S is open, we need to show that given $s \in S$, we may choose some $\delta > 0$ such that $|t - s| < \delta$ implies that $t \in S$. In particular, if φ_s is a solution of (MA_s) and $|t - s| < \delta$, we need to show that we can solve (MA_t).

To this end, let $s \in S$ be given such that we have a solution to the equation $(\omega + \partial\overline{\partial}\varphi_s)^n = \exp(f_s)\omega^n$. We need to solve the equation $(\omega + \partial\overline{\partial}\varphi_t)^n = \exp(tf)\omega^n$, where $|t - s| < \delta$. We may write

$$(\omega + \partial\overline{\partial}\varphi_t)^n = (\omega + \partial\overline{\partial}\varphi_s - \partial\overline{\partial}\varphi_s + \partial\overline{\partial}\varphi_t)^n = (\omega_s + \partial\overline{\partial}(\varphi_t - \varphi_s))^n. \tag{17}$$

Further, since $(\omega + \partial\overline{\partial}\varphi_t)^n = \exp(f_t)\omega^n$,

$$(\omega + \partial\overline{\partial}\varphi_t)^n = \exp(f_t)\omega^n = \exp(f_t)\exp(-f_s)\omega_s^n = \exp(f_t - f_s)\omega_s^n. \tag{18}$$

Setting $\psi := \varphi_t - \varphi_s$, (17) and (18) allow us to formulate $(\omega + \partial\overline{\partial}\varphi_t)^n = \exp(f_t)\omega^n$ as

$$(\omega_s + \partial\overline{\partial}\psi)^n = \exp(f_t - f_s)\omega_s^n. \tag{19}$$

Equivalently, (19) may also be expressed as

$$\log\left(\frac{(\omega_s + \partial\overline{\partial}\psi)^n}{\omega_s^n}\right) = f_t - f_s. \tag{20}$$

Set

$$\mathscr{C}_{\star}^{k,\alpha}(M, \mathbb{R}) := \left\{u \in \mathscr{C}^{k,\alpha}(M, \mathbb{R}) : \int_M u\omega_s^n = 0\right\},$$

and

$$\mathscr{C}_{\star\star}^{k,\alpha}(M, \mathbb{R}) := \left\{u \in \mathscr{C}^{k,\alpha}(M, \mathbb{R}) : \int_M (\exp(u) - 1)\omega_s^n = 0\right\}.$$

Observe that since M is compact, the operator $u \longmapsto \int_M u\omega_s^n$ is bounded and $\mathscr{C}_{\star}^{k,\alpha}(M, \mathbb{R})$ is exactly the kernel of this operator. In particular, it is a closed subspace of a Banach space and is therefore Banach.

Lemma 5.3.1. The tangent space of $\mathscr{C}^{k,\alpha}_{\star\star}(M,\mathbb{R})$ at $v=0$ is $\mathscr{C}^{k,\alpha}_{\star}(M,\mathbb{R})$.

PROOF. Indeed, for $u=\xi v$,

$$0=\frac{d}{d\xi}\int_M(\exp(\xi v)-1)\omega_s^n\bigg|_{\xi=0}=\int_M v\exp(\xi v)\omega_s^n\bigg|_{\xi=0}=\int_M v\omega_s^n.$$

□

Let us now define a (non–linear) operator $\Phi:\mathscr{C}^{2,\frac{1}{2}}_{\star}(M,\mathbb{R})\longrightarrow\mathscr{C}^{0,\frac{1}{2}}_{\star\star}(M,\mathbb{R})$ by setting

$$\Phi(\psi):=\log\left(\frac{(\omega_s+\partial\overline{\partial}\psi)^n}{\omega_s^n}\right). \tag{21}$$

We saw previously that an expression of this form is globally defined. To see that this map is well-defined, we need to show that given $\psi\in\mathscr{C}^{2,\frac{1}{2}}_{\star}(M,\mathbb{R})$, $\Phi(\psi)$ lies in $\mathscr{C}^{0,\frac{1}{2}}_{\star\star}(M,\mathbb{R})$. Let us first show that $\Phi(\psi)$ satisfies the non-linear condition $\int_M(\exp(\Phi(\psi))-1)\omega_s^n=0$. The necessary computation is given by

$$\begin{aligned}\int_M(\exp(\Phi(\psi))-1)\omega_s^n &= \int_M\left(\frac{(\omega_s+\partial\overline{\partial}\psi)^n}{\omega_s^n}-1\right)\omega_s^n\\ &= \int_M(\omega_s+\partial\overline{\partial}\psi)^n-\int_M\omega_s^n\\ &\overset{(14)}{=} \int_M\omega_s^n+(\text{exact terms})-\int_M\omega_s^n\\ &\overset{\text{Stokes}}{=} \int_M\omega_s^n-\int_M\omega_s^n=0.\end{aligned}$$

To see that $\Phi(\psi)$ is Hölder continuous, choose $h>0$ and consider that

$$\begin{aligned}\frac{1}{h}\left(\Phi(\psi+h\xi)-\Phi(\psi)\right) &= \frac{1}{h}\log\left(\frac{(\omega_s+\partial\overline{\partial}(\psi+h\xi))^n}{\omega_s^n}\right)-\frac{1}{h}\log\left(\frac{(\omega_s+\partial\overline{\partial}\psi)^n}{\omega_s^n}\right)\\ &= \frac{1}{h}\log\left(\frac{(\omega_s+\partial\overline{\partial}\psi+h\partial\overline{\partial}\xi)^n}{(\omega_s+\partial\overline{\partial}\psi)^n}\right).\end{aligned} \tag{22}$$

Notation. Since $\omega_s+\partial\overline{\partial}\psi=\omega+\partial\overline{\partial}\varphi_s+\partial\overline{\partial}(\varphi_t-\varphi_s)=\omega+\partial\overline{\partial}\varphi_t$, we may write $\omega_t:=\omega_s+\partial\overline{\partial}\psi$.

Using the Taylor approximation of $\log(1+x)$, we may approximate (22) as

$$
\begin{aligned}
\frac{1}{h}\log\left(\frac{(\omega_t + h\partial\overline{\partial}\xi)^n}{\omega_t^n}\right) &\overset{(14)}{=} \frac{1}{h}\log\left(\frac{\omega_t^n + nh\partial\overline{\partial}\xi\wedge\omega_t^{n-1}}{\omega_t^n}\right)\\
&\sim \frac{1}{h}\log\left(1+\frac{nh\partial\overline{\partial}\xi\wedge\omega_t^{n-1}}{\omega_t^n}\right)\\
&\sim \frac{1}{h}\left[nh\frac{\partial\overline{\partial}\psi\wedge\omega_t^{n-1}}{\omega_t^n}\right] - \frac{1}{2h}\left[h\frac{\partial\overline{\partial}\psi\wedge\omega_t^{n-1}}{\omega_t^n}\right]^2\\
&\sim \frac{n\partial\overline{\partial}\xi\wedge\omega_t^{n-1}}{\omega_t^n}.
\end{aligned}
\tag{23}
$$

Notation. In the above, we have adopted the notation $\sim$ to mean modulo lower order terms involving h.

In local holomorphic coordinates, we may write

$$\omega^n = n!dz^1\wedge d\overline{z}^1\wedge\cdots\wedge dz^n\wedge d\overline{z}^n, \tag{24}$$

and

$$\omega^{n-1} = (n-1)!\sum_{i=1}^m (dz^1\wedge d\overline{z}^1)\wedge\cdots\wedge(\widehat{dz^i\wedge d\overline{z}^i})\wedge\cdots\wedge(dz^n\wedge d\overline{z}^n), \tag{25}$$

where $\widehat{\cdot}$ signifies the term which we omit.

Inserting (24) and (25) into (23), we see that

$$
\begin{aligned}
&\frac{n\partial\overline{\partial}\xi\wedge\omega_t^{n-1}}{\omega_t^n}\\
=\;&\frac{n\sum_{j=1}^m \xi_{i\overline{j}}dz^j\wedge d\overline{z}^j\wedge\Big((n-1)!\sum_{i=1}^m(dz^1\wedge d\overline{z}^1)\wedge\cdots\wedge(\widehat{dz^i\wedge d\overline{z}^i})\wedge\cdots\wedge(dz^n\wedge d\overline{z}^n)\Big)}{n!dz^1\wedge d\overline{z}^1\wedge\cdots\wedge dz^n\wedge d\overline{z}^n}\\
=\;&\frac{\sum_{i=1}^m \xi_{i\overline{j}}dz^1\wedge d\overline{z}^1\wedge\cdots\wedge dz^n\wedge d\overline{z}^n}{dz^1\wedge d\overline{z}^1\wedge\cdots\wedge dz^n\wedge d\overline{z}^n}\\
=\;&\Delta_t\xi,
\end{aligned}
$$

where Δ_t is the Laplace operator with respect to metric g_t whose Kähler form is ω_t.

Proposition 5.3.2. For the operator $\Phi : \mathscr{C}_\star^{2,\frac{1}{2}}(M,\mathbb{R}) \longrightarrow \mathscr{C}_{\star\star}^{0,\frac{1}{2}}(M,\mathbb{R})$ defined by (21), the linearisation of Φ at $\psi = 0$, i.e., the operator $\nabla\Phi|_{\psi=0} : \mathscr{C}_\star^{2,\frac{1}{2}}(M,\mathbb{R}) \longrightarrow \mathscr{C}_\star^{0,\frac{1}{2}}(M,\mathbb{R})$, is given by

$$\nabla\Phi|_{\psi=0}(u) = \Delta_t u,$$

and is invertible.

Proof. The computation preceeding the statement of Proposition 8.3.2 shows that the linearisation of Φ at $\psi = 0$ is the Laplace operator Δ_t. The content of the proposition therefore lies in showing that Δ_t is invertible on these Hölder spaces.

To see that $\nabla\Phi|_{\psi=0}$ is injective, suppose that $\Delta_t u = 0$, where $u \in \mathscr{C}_\star^{2,\frac{1}{2}}(M,\mathbb{R})$. Since M is compact, we may apply the maximum principle to conclude that u must be constant. Since u must also satisfy the normalisation condition $\int_M u\omega_s^n = 0$, it follows that u is identically zero.

Surjectivity is equivalent to the solvability of the inhomogeneous Laplace equation $\Delta_t u = f$, where $f \in \mathscr{C}_*^{0,\frac{1}{2}}(M,\mathbb{R})$. A proof of this may be found on p. 104 of [Au], see Theorem 4.7. □

Remark 5.3.3. The dilligent reader at this point may be concerned as to how we intend on using the implicit function theorem, Theorem 8.2.1, given that $\mathscr{C}_{\star\star}^{0,\frac{1}{2}}(M,\mathbb{R})$ is not a Banach space. We may however view $\mathscr{C}_{\star\star}^{0,\frac{1}{2}}(M,\mathbb{R})$ as a smooth graph over its tangent space. From Lemma 8.3.1, we know that the tangent space is $\mathscr{C}_\star^{0,\frac{1}{2}}(M,\mathbb{R})$ which is a Banach space and so there is no issue.

5.4. Showing that S is Closed

To show that S is closed in $[0,1]$, we need to show that given a sequence $(s^\nu)_{\nu\in\mathbb{N}} \subset S$ with $s^\nu \longrightarrow s$, then $s \in S$. For each $\nu \in \mathbb{N}$, the element $s^\nu \in S$ gives rise to a $\mathscr{C}^{k+1,\alpha}$–solution φ^ν of the Monge–Ampère equation

$$(\omega + \partial\overline{\partial}\varphi^\nu)^n = \exp(s^\nu f)\omega^n.$$

Hence, by passing to a subsequence if necessary, we must show that the limit φ of the sequence $(\varphi^\nu)_{\nu\in\mathbb{N}}$ is a solution to

$$(\omega + \partial\overline{\partial}\varphi)^n = \exp(sf)\omega^n.$$

Note that by $\varphi^\nu \longrightarrow \varphi$ we mean the convergence with respect to the norm topology on $\mathscr{C}^{k+1,\alpha}$.

The proof will proceed in the following manner. For each $\varphi^\nu \in \mathscr{C}^{k+1,\alpha}(M,\mathbb{R})$, the complex Monge–Ampere equation will take the form

$$\det\left(g_{i\overline{j}} + \frac{\partial^2\varphi^\nu}{\partial z^i\partial\overline{z}^j}\right)\det(g_{i\overline{j}})^{-1} \quad = \quad \exp(s^\nu f). \tag{26}$$

If we differentiate (26), we see that

$$\det\left(g_{i\overline{j}} + \frac{\partial^2\varphi^\nu}{\partial z^i\partial\overline{z}^j}\right)\sum_{i,j=1}^n\left(g_{i\overline{j}} + \frac{\partial^2\varphi^\nu}{\partial z^i\partial\overline{z}^j}\right)^{-1}\frac{\partial^2}{\partial z^i\partial\overline{z}^j}\left(\frac{\partial\varphi^\nu}{\partial z^k}\right) \tag{27}$$
$$= \frac{\partial}{\partial z^p}[\exp(s^\nu f)\det(g_{i\overline{j}})].$$

Set Lu to be the differential operator given by the left hand side of (27), where $u = \frac{\partial}{\partial z^k}\varphi^\nu$.

The central result of Yau's proof is the following.

Proposition 5.4.1. The differential operator

$$Lu = \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi^\nu}{\partial z^i \partial \bar{z}^j}\right) \sum_{i,j=1}^m \left(g_{i\bar{j}} + \frac{\partial^2 \varphi^\nu}{\partial z^i \partial \bar{z}^j}\right)^{-1} \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} \tag{28}$$

is an elliptic second–order differential operator with Hölder continuous coefficients (with Hölder exponent $0 \le \alpha \le 1$).

Recall that a differential operator of the form

$$Lu = \sum_{i,j=1}^m a_{ij}(z) \frac{\partial^2 u}{\partial z^i \partial \bar{z}^j} + \sum_{i=1}^m b_i(z) \frac{\partial u}{\partial z^i} + c(z) u \tag{29}$$

is said to be *elliptic* if the matrix (a_{ij}) is symmetric and positive–definite. Further, we say that L is *uniformly elliptic* if there is a constant $c > 0$ such that

$$\frac{1}{c} |\xi|^2 \le \sum_{i,j=1}^m a_{ij}(z) \xi_i \xi_j \le c |\xi|^2, \qquad \forall \xi \in \mathbb{R}^n \backslash \{0\},\ z \in \Omega.$$

Comparing equations (28) and (29), we observe that the $b_i \equiv 0$ and $c \equiv 0$ also. Hence to show that (28) is uniformly elliptic, we need to show that the eigenvalues of

$$a_{ij} \quad = \quad \det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi^\nu}{\partial z^i \partial \bar{z}^j}\right) \sum_{i,j=1}^n \left(g_{i\bar{j}} + \frac{\partial^2 \varphi_j}{\partial z^i \partial \bar{z}^j}\right)^{-1}$$

are bounded above and below.

This therefore requires an estimate on $\sup_M |\varphi_j|$, $\sup_M |\nabla \varphi_j|$ and $m + \Delta \varphi_j$. To show that the coefficients are Hölder continuous, we require an estimate on the (mixed) third derivatives of φ.

The Schauder estimate will then control the $\mathscr{C}^{2,\alpha}$–norm of $\frac{\partial}{\partial \bar{z}^p} \varphi^\nu$ and $\frac{\partial}{\partial z^p} \varphi^\nu$. In particular, the regularity of the coefficients a_{ij} is improved and the Schauder estimate will establish a $\mathscr{C}^{3,\alpha}$–estimate on $\frac{\partial}{\partial \bar{z}^p} \varphi^\nu$ and $\frac{\partial}{\partial z^p} \varphi^\nu$. Iterating this process, we see that $\varphi^\nu \in \mathscr{C}^{k+1,\alpha}$. Taking limits, we conclude that S is closed.

CHAPTER 6

Second Order Estimates

Throughout this chapter we assume that (M, ω) is a compact Kähler manifold of (complex) dimension m, with metric g and Kähler form ω. The notation of the previous chapter is maintained. The following notation convention will also be useful.

Notation. For a function $\varphi \in \mathscr{C}^\infty(M, \mathbb{Z})$, we write $\widetilde{g}$ for the metric whose components are given by

$$\widetilde{g}_{i\bar{j}} = g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j}.$$

Note that we have maintained the convention and omitted the $\sqrt{-1}$.

In this chapter we establish the second order estimates that are proved in Yau's original paper [Ya76].

The proof is often considered difficult relative to later proofs due to J. Kazdan [Ka] and T. Aubin [Au]. There is an elegance to the original proof that is arguably lost in the later simplifications however. One of the primary objectives of this exposition is to not simply present a complete proof of the Calabi–Yau theorem, but to understand the motivation for considerations that were made in the proof. After considering these motivations, the proof is quite simple. One may even crudely summarise the proof as an answer to the question:

How far can we get with the maximum principle?

6.1. Preliminary Remarks

For convenience, let us offer the statement of the maximum principle.

Theorem 6.1.1. (p. 96 of [Au]). Let $\Omega \subseteq \mathbb{R}^n$ be an open connected set. Denote by $L(u)$ the uniformly elliptic second–order differential operator in Ω given by

$$Lu = \sum_{i,j=1}^n a_{ij}(x) u_{ij}(x) + \sum_{i=1}^n b_i(x) u_i(x) + c(x) u,$$

where the coefficients are bounded and $c(x) \leq 0$. Suppose $u \in \mathscr{C}^2(\Omega)$ satisfies $Lu \geq 0$. If u attains its maximum $\sup_{x \in \Omega} u(x) \geq 0$ in Ω, then u is constant on Ω.

To apply the maximum principle, we need to choose some second–order elliptic differential operator L and some function $u \in \mathscr{C}^2(\Omega)$ such that $Lu \geq 0$. In particular, if we let p denote the point at which

u attains its maximum (the existence of p is ensured by the compactness fo M), then $Lu(p) \leq 0$. We therefore like to find constants C_1, C_2, independent of t, such that $Lu(p) \geq C_1 u(p) + C_2$. From this, we then see that

$$\sup_M u = u(p) \leq \frac{1}{C_1}(Lu(p) - C_2) \leq -\frac{C_2}{C_1}.$$

In what follows, we will see that the appropriate choices for L and u are $L = \tilde{\Delta}$ and $u = \exp(-\lambda\varphi)(m + \tilde{\Delta}\varphi)$ respectively, where λ is a real parameter to be determined.

From the estimate on $\sup_M |\Delta\varphi|$, we may use the Green's function associated to the Laplace operator Δ on M to control $\sup_M \varphi$. The Schauder estimate will then be used to control $\sup_M |\nabla\varphi|$ in terms of $-\inf_M \varphi$. Note that the estimates for $\sup_M \Delta\varphi$ and $\sup_M \varphi$ are also in terms of $-\inf_M \varphi$. The estimate on $-\inf_M \varphi$ uses a standard compactness argument.

6.2. Some Failed Attempts

The purpose of this section is to understand why certain more "obvious candidates" for L and u are not used in the proof. The reader who would prefer to see the proof without discussion of these considerations may wish to skip this section.

For reference, let us rewrite the Monge–Ampère equation

$$\det\left(g_{i\bar{j}} + \frac{\partial^2\varphi}{\partial z^i \partial \bar{z}^j}\right)\det(g_{i\bar{j}})^{-1} = \exp(f). \tag{30}$$

First Attempt. An obvious first choice would be to take $L = \Delta$. To test the suitability of this choice, we first note that

$$\Delta(\Delta\varphi) \quad = \quad \sum_{k,\ell=1}^{m} g^{k\bar{\ell}} \frac{\partial^2}{\partial z^k \partial \bar{z}^\ell}\left(\sum_{i,j=1}^{m} g^{i\bar{j}}\varphi_{i\bar{j}}\right). \tag{31}$$

Now choose local holomorphic coordinates such that at a point $p \in M$ we may write $g_{i\bar{j}} = \delta_{ij}$ and $\frac{\partial g_{i\bar{j}}}{\partial z^k}(p) = \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^\ell}(p) = 0$. Recall that

$$g^{i\bar{j}} g_{i\bar{j}} = \delta_{ij} \quad \Longrightarrow \quad g^{i\bar{j}}\frac{\partial g_{i\bar{j}}}{\partial \bar{z}^\ell} + \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^\ell} g_{i\bar{j}} = 0 \Longrightarrow \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^\ell} = -g^{i\bar{j}} g^{i\bar{j}} \frac{\partial g_{i\bar{j}}}{\partial \bar{z}^\ell} = 0. \tag{32}$$

Further, the Riemannian curvature tensor is (locally) given by

$$R_{i\bar{j}k\bar{\ell}} \quad = \quad -\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} = g_{i\bar{j}} \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} g_{i\bar{j}}. \tag{33}$$

We therefore compute,

$$\begin{aligned}
\Delta(\Delta\varphi) &= \sum_{k,\ell=1}^m g^{k\bar{\ell}} \frac{\partial^2}{\partial z^k \partial \bar{z}^\ell} \left(\sum_{i,j=1}^m g^{i\bar{j}} \varphi_{i\bar{j}} \right) \\
&= \sum_{k,\ell=1}^m g^{k\bar{\ell}} \frac{\partial}{\partial z^k} \left(\sum_{i,j=1}^m \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^\ell} \varphi_{i\bar{j}} + g^{i\bar{j}} \frac{\partial \varphi_{i\bar{j}}}{\partial \bar{z}^\ell} \right) \\
&= \sum_{k,\ell=1}^m g^{k\bar{\ell}} \left(\sum_{i,j=1}^m \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} \varphi_{i\bar{j}} + \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^\ell} \frac{\partial \varphi_{i\bar{j}}}{\partial z^k} + \frac{\partial g^{i\bar{j}}}{\partial z^k} \frac{\partial \varphi_{i\bar{j}}}{\partial \bar{z}^\ell} + g^{i\bar{j}} \frac{\partial^2 \varphi_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} \right) \\
&\overset{\text{(locally)}}{=} \sum_{i,j,k,\ell=1}^m \delta_{k\ell} \left(R_{i\bar{j}k\bar{\ell}} \varphi_{i\bar{j}} + \delta_{ij} \varphi_{i\bar{j}k\bar{\ell}} \right) \\
&= \sum_{i,k=1}^m R_{i\bar{i}k\bar{k}} \varphi_{i\bar{i}} + \sum_{i,k=1}^m \varphi_{i\bar{i}k\bar{k}}. \qquad (34)
\end{aligned}$$

The presence of the fourth derivative terms $\varphi_{i\bar{i}k\bar{k}}$ is concerning since they are difficult to control. We may however, attempt to exploit the Monge–Ampère equation in order to reduce the number of derivatives. More precisely, by taking two derivatives of (30), we may be able to isolate the fourth order terms and replace them with lower order terms.

To this end, observe that by differentiating both sides of (30), we have

$$\begin{aligned}
\frac{\partial}{\partial z^k} \sum_{i,j=1}^m \log\left(\det\left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right) \right) - \sum_{i,j=1}^m \frac{\partial f}{\partial z^k} \log\left(\det(g_{i\bar{j}}) \right) &= \frac{\partial f}{\partial z^k}, \\
\sum_{i,j=1}^m \left(g_{i\bar{j}} + \frac{\partial^2 \varphi}{\partial z^i \partial \bar{z}^j} \right)^{-1} \left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k} \right) - \sum_{i,j=1}^m g_{i\bar{j}}^{-1} \frac{\partial g_{i\bar{j}}}{\partial z^k} &= \frac{\partial f}{\partial z^k} \\
\sum_{i,j=1}^m \tilde{g}^{i\bar{j}} \left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \frac{\partial^3 \varphi}{\partial z^i \partial \bar{z}^j \partial z^k} \right) - \sum_{i,j=1}^m g^{i\bar{j}} \frac{\partial g_{i\bar{j}}}{\partial z^k} &= \frac{\partial f}{\partial z^k}. \qquad (35)
\end{aligned}$$

Before differentiation (35), let us observe that

$$\frac{\partial}{\partial \bar{z}^\ell} \tilde{g}^{i\bar{j}} = \frac{\partial}{\partial \bar{z}^\ell} (g_{i\bar{j}} + \varphi_{i\bar{j}})^{-1} = -\left(\frac{\partial g_{t\bar{n}}}{\partial \bar{z}^\ell} + \varphi_{i\bar{j}\ell} \right) \tilde{g}^{t\bar{j}} \tilde{g}^{i\bar{n}}. \qquad (36)$$

Using (36), the derivative of (35) is given by

$$\begin{aligned}
\frac{\partial^2 f}{\partial z^k \partial \bar{z}^\ell} &= -\sum_{i,j,t,n=1}^m \tilde{g}^{t\bar{j}} \tilde{g}^{i\bar{n}} \left(\frac{\partial g_{t\bar{n}}}{\partial \bar{z}^\ell} + \varphi_{t\bar{n}\bar{\ell}} \right) \left(\frac{\partial g_{i\bar{j}}}{\partial z^k} + \varphi_{i\bar{j}k} \right) + \sum_{i,j=1}^m \tilde{g}^{i\bar{j}} \left(\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} + \varphi_{i\bar{j}k\bar{\ell}} \right) \\
&+ \sum_{i,j,k,n=1}^m \tilde{g}^{t\bar{j}} g^{i\bar{n}} \frac{\partial g_{t\bar{n}}}{\partial \bar{z}^\ell} \frac{\partial g_{i\bar{j}}}{\partial z^k} - \sum_{i,j=1}^m g^{i\bar{j}} \frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell}. \qquad (37)
\end{aligned}$$

In local holomorphic coordinates, we write

$$\tilde{g}^{i\bar{j}} = (g_{i\bar{j}} + \varphi_{i\bar{j}})^{-1} = (\delta_{ij} + \delta_{ij}\varphi_{i\bar{i}})^{-1} = \delta_{ij}(1 + \varphi_{i\bar{i}})^{-1}.$$

Equation (37) therefore becomes

$$\begin{aligned}
\frac{\partial^2 f}{\partial z^k \partial \bar{z}^\ell} &= -\sum_{i,j,t,n=1}^{m} \delta_{tj}\delta_{i\bar{n}}(1+\varphi_{j\bar{j}})^{-1}(1+\varphi_{i\bar{i}})^{-1}\varphi_{t\bar{n}\bar{\ell}}\varphi_{i\bar{j}k} \\
&\quad + \sum_{i,j=1}^{m} \delta_{ij}(1+\varphi_{i\bar{i}})^{-1}\left(\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} + \varphi_{i\bar{j}k\bar{\ell}}\right) - \sum_{i,j=1}^{m} \delta_{ij}\frac{\partial^2 g_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} \\
&= -\sum_{i,j=1}^{m} \frac{\varphi_{j\bar{i}\bar{\ell}}\varphi_{i\bar{j}k}}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} + \sum_{i=1}^{m} \frac{1}{(1+\varphi_{i\bar{i}})}(\varphi_{i\bar{i}k\bar{\ell}} - R_{i\bar{i}k\bar{\ell}}) + \sum_{i=1}^{m} R_{i\bar{i}k\bar{\ell}}.
\end{aligned} \tag{38}$$

Taking the trace of both sides of (38) and using the fact that $\varphi_{j\bar{i}\bar{k}} = \overline{\varphi_{i\bar{j}k}}$, we write

$$\begin{aligned}
\Delta f &= -\sum_{i,j,k=1}^{m} \frac{|\varphi_{i\bar{j}k}|^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} - \sum_{i,k=1}^{m} \frac{1}{(1+\varphi_{i\bar{i}})} R_{i\bar{i}k\bar{k}} \\
&\quad + \sum_{i,k=1}^{m} \frac{\varphi_{i\bar{i}k\bar{k}}}{(1+\varphi_{i\bar{i}})} + \sum_{i,k=1}^{m} R_{i\bar{i}k\bar{k}}.
\end{aligned} \tag{39}$$

Comparing the terms of (34) and (39) containing fourth derivatives of φ, we see that there is a quotient of $(1+\varphi_{i\bar{i}})$ present in (39) that is not present in (34). Hence, the initial choice of $L = \Delta$ must be alterred to accomodate for this.

Attempt 2. Let us proceed as before. This time however, we will consider $\widetilde{\Delta}$, the Laplace operator with respect to the metric $\tilde{g}_{i\bar{j}}$.

We compute

$$
\begin{aligned}
\widetilde{\Delta}(\Delta\varphi) \quad &= \quad \sum_{k,\ell=1}^{m} \tilde{g}^{k\bar{\ell}} \frac{\partial^2}{\partial z^k \partial \bar{z}^\ell} \left(\sum_{i,j=1}^{m} g^{i\bar{j}} \varphi_{i\bar{j}} \right) \\
&= \quad \sum_{k,\ell=1}^{m} \tilde{g}^{k\bar{\ell}} \frac{\partial}{\partial z^k} \left(\sum_{i,j=1}^{m} \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^\ell} \varphi_{i\bar{j}} + g^{i\bar{j}} \frac{\partial \varphi_{i\bar{j}}}{\partial \bar{z}^\ell} \right) \\
&= \quad \sum_{k,\ell=1}^{m} \tilde{g}^{k\bar{\ell}} \left(\sum_{i,j=1}^{m} \frac{\partial^2 g^{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} \varphi_{i\bar{j}} + \frac{\partial g^{i\bar{j}}}{\partial \bar{z}^\ell} \frac{\partial \varphi_{i\bar{j}}}{\partial z^k} + \frac{\partial g^{i\bar{j}}}{\partial z^k} \frac{\partial \varphi_{i\bar{j}}}{\partial \bar{z}^\ell} + g^{i\bar{j}} \frac{\partial^2 \varphi_{i\bar{j}}}{\partial z^k \partial \bar{z}^\ell} \right) \\
&\overset{\text{(locally)}}{=} \quad \sum_{k,\ell=1}^{m} \delta_{k\ell} (1+\varphi_{k\bar{k}})^{-1} \left(\sum_{i,j=1}^{m} R_{i\bar{j}k\bar{\ell}} \varphi_{i\bar{j}} + \delta_{ij} \varphi_{i\bar{j}k\bar{\ell}} \right) \\
&= \quad \sum_{i,k=1}^{m} \frac{1}{(1+\varphi_{k\bar{k}})} R_{i\bar{i}k\bar{k}} \varphi_{i\bar{i}} + \sum_{i,k=1}^{m} \frac{\varphi_{i\bar{i}k\bar{k}}}{(1+\varphi_{k\bar{k}})}.
\end{aligned} \tag{40}
$$

Comparing (39) and (40), we see that the terms involving fourth derivatives of φ now coincide. Hence, by inserting (39) into (40),

$$
\begin{aligned}
\widetilde{\Delta}(\Delta\varphi) \quad &= \quad \sum_{i,k=1}^{m} \frac{1}{(1+\varphi_{k\bar{k}})} R_{i\bar{i}k\bar{k}} \varphi_{i\bar{i}} + \Delta f + \sum_{i,j,k=1}^{m} \frac{|\varphi_{i\bar{j}k}|^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} \\
&\quad + \sum_{i,k=1}^{m} \frac{1}{(1+\varphi_{i\bar{i}})} R_{i\bar{i}k\bar{k}} - \sum_{i,k=1}^{m} R_{i\bar{i}k\bar{k}} \\
&\geq \quad \Delta f + \sum_{i,j,k=1}^{m} \frac{|\varphi_{i\bar{j}k}|^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})} + \left(\inf_{i \neq k} R_{i\bar{i}k\bar{k}} \right) \left[\sum_{i,k=1}^{m} \frac{1+\varphi_{i\bar{i}}}{1+\varphi_{k\bar{k}}} - m^2 \right].
\end{aligned} \tag{41}
$$

Notation. Set $\kappa := \inf_{i \neq k} R_{i\bar{i}k\bar{k}}$.

The geometric–arithmetic mean inequality is given by

$$
\sum_{i=1}^{m} \frac{1}{1+\varphi_{i\bar{i}}} \quad \geq \quad \left(\frac{\sum_{i=1}^{m} (1+\varphi_{i\bar{i}})}{\prod_{i=1}^{m} 1+\varphi_{i\bar{i}}} \right)^{\frac{1}{m-1}}. \tag{42}
$$

Combining (30) and (42), we have

$$
\begin{aligned}
\sum_{i=1}^{m} \frac{1}{1+\varphi_{i\bar{i}}} \quad &\geq \quad \left(\sum_{i=1}^{m} (1+\varphi_{i\bar{i}}) \right)^{\frac{1}{m-1}} \left(\prod_{i=1}^{m} (1+\varphi_{i\bar{i}}) \right)^{-\frac{1}{m-1}} \\
&= \quad (m+\Delta\varphi)^{\frac{1}{m-1}} \exp(\tfrac{f}{m-1}).
\end{aligned} \tag{43}
$$

Hence, (41) becomes

$$\begin{aligned}
\widetilde{\Delta}(\Delta\varphi) &\geq \kappa \sum_{i,k=1}^{m} \frac{\varphi_{i\bar{i}}}{1+\varphi_{k\bar{k}}} + \Delta f + \underbrace{\sum_{i,j,k=1}^{m} \frac{|\varphi_{i\bar{j}k}|^2}{(1+\varphi_{i\bar{i}})(1+\varphi_{j\bar{j}})}}_{\geq 0} - \kappa \left(\sum_{i=1}^{m} \frac{1}{1+\varphi_{i\bar{i}}} \right) - m^2\kappa \\
&\geq \kappa\Delta\varphi \left(\sum_{k=1}^{m} \frac{1}{1+\varphi_{k\bar{k}}} \right) + \Delta f - \kappa \left(\sum_{i=1}^{m} \frac{1}{1+\varphi_{i\bar{i}}} \right) - m^2\kappa \\
&\geq \kappa\Delta\varphi \exp(\tfrac{f}{m-1}) + \Delta f - \kappa \exp(\tfrac{f}{m-1}) - m^2\kappa \qquad (44) \\
&=: C_1\Delta\varphi + C_2,
\end{aligned}$$

where $C_1 := \kappa\exp(\frac{f}{m-1})$ and $C_2 := \Delta f - \kappa\exp(\frac{f}{m-1}) - m^2\kappa$.

The constants C_1 and C_2 are independent of t as desired. Observe however that κ will not necessarily be positive, and consequently, we cannot ensure that C_1 is positive. To rectify this, we introduce in place of $\Delta\varphi$, the weighted Laplacian $u = \exp(-\lambda\varphi)(m + \Delta\varphi)$, where $\lambda \in \mathbb{R}$.

6.3. Main Proof

Third Attempt. Proceeding as before, we have

$$\begin{aligned}
\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi)) &= \sum_{i,j=1}^{m} \tilde{g}^{i\bar{j}} \frac{\partial^2}{\partial z^i \partial \bar{z}^j} (\exp(-\lambda\varphi)(m+\Delta\varphi)) \\
&= \sum_{i,j=1}^{m} \tilde{g}^{i\bar{j}} \frac{\partial}{\partial z^i} \left[-\lambda\varphi_{\bar{j}} \exp(-\lambda\varphi)(m+\Delta\varphi) + \exp(-\lambda\varphi)(\Delta\varphi)_{\bar{j}} \right] \\
&= \lambda^2 \exp(-\lambda\varphi) \left(\sum_{i,j=1}^{m} \tilde{g}^{i\bar{j}} \varphi_i \varphi_{\bar{j}} \right) (m+\Delta\varphi) \\
&\quad -\lambda\exp(-\lambda\varphi) \sum_{i,j=1}^{m} \tilde{g}^{i\bar{j}} \varphi_i (\Delta\varphi)_{\bar{j}} - \lambda\exp(-\lambda\varphi) \sum_{i,j=1}^{m} \tilde{g}^{i\bar{j}} (\Delta\varphi)_i \varphi_{\bar{j}} \\
&\quad -\lambda\exp(-\lambda\varphi)\widetilde{\Delta}\varphi(m+\Delta\varphi) + \exp(-\lambda\varphi)\widetilde{\Delta}(\Delta\varphi). \qquad (45)
\end{aligned}$$

Lemma 6.3.1. We have the following inequality,

$$\begin{aligned}
\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi)) &\geq -(m+\Delta\varphi)^{-1}\exp(-\lambda\varphi) \sum_{i,j=1}^{m} \tilde{g}^{i\bar{j}} (\Delta\varphi)_i (\Delta\varphi)_{\bar{j}} \\
&\quad -\lambda\exp(-\lambda\varphi)(m+\Delta\varphi)(\widetilde{\Delta}\varphi) + \exp(-\lambda\varphi)\widetilde{\Delta}(\Delta\varphi), \qquad (46)
\end{aligned}$$

c.f., Equation (2.13) of [Ya76].

Proof. For the purposes of clarity, we omit the summation and the exponential term $\exp(-\lambda\varphi)$ in the following computation. Hence we observe that

$$
\begin{aligned}
& \lambda^2\tilde{g}^{i\bar{j}}\varphi_i\varphi_{\bar{j}}(m+\Delta\varphi)-\lambda\tilde{g}^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}}-\lambda\tilde{g}^{i\bar{j}}(\Delta\varphi)_i\varphi_{\bar{j}}-\lambda(\tilde{\Delta}\varphi)(m+\Delta\varphi)+\tilde{\Delta}(\Delta\varphi) \\
= & \lambda^2\tilde{g}^{i\bar{j}}\varphi_i\varphi_{\bar{j}}\frac{(m+\Delta\varphi)^2}{(m+\Delta\varphi)}-\lambda\tilde{g}^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}}\frac{(m+\Delta\varphi)}{(m+\Delta\varphi)}-\lambda\tilde{g}^{i\bar{j}}(\Delta\varphi)_i\varphi_{\bar{j}}\frac{(m+\Delta\varphi)}{(m+\Delta\varphi)} \\
& -\lambda(\tilde{\Delta}\varphi)(m+\Delta\varphi)+\tilde{\Delta}(\Delta\varphi) \\
= & \frac{1}{m+\Delta\varphi}\left[\lambda^2\tilde{g}^{i\bar{j}}\varphi_i\varphi_{\bar{j}}(m+\Delta\varphi)^2-\lambda\tilde{g}^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}}(m+\Delta\varphi)-\lambda\tilde{g}^{i\bar{j}}(\Delta\varphi_i)\varphi_{\bar{j}}(m+\Delta\varphi)\right] \\
& -\lambda(\tilde{\Delta}\varphi)(m+\Delta\varphi)+\tilde{\Delta}(\Delta\varphi). \qquad (47)
\end{aligned}
$$

We now introduce the term $\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}}$ into (47),

$$
\begin{aligned}
& \lambda^2\tilde{g}^{i\bar{j}}\varphi_i\varphi_{\bar{j}}(m+\Delta\varphi)^2-\lambda\tilde{g}^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}}(m+\Delta\varphi)-\lambda\tilde{g}^{i\bar{j}}(\Delta\varphi_i)\varphi_{\bar{j}}(m+\Delta\varphi) \\
= & \lambda^2\tilde{g}^{i\bar{j}}\varphi_i\varphi_{\bar{j}}(m+\Delta\varphi)^2-\lambda\tilde{g}^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}}(m+\Delta\varphi)-\lambda\tilde{g}^{i\bar{j}}(\Delta\varphi_i)\varphi_{\bar{j}}(m+\Delta\varphi) \\
& +\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}}-\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} \\
= & \tilde{g}^{i\bar{j}}\left(\lambda^2(m+\Delta\varphi)^2\varphi_i\varphi_{\bar{j}}-\lambda\varphi_i(\Delta\varphi)_{\bar{j}}(m+\Delta\varphi)-\lambda(\Delta\varphi_i)\varphi_{\bar{j}}(m+\Delta\varphi)+(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}}\right) \\
& -\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} \\
= & \tilde{g}^{i\bar{j}}\left[(\lambda(m+\Delta\varphi)\varphi_i-(\Delta\varphi)_i)(\lambda(m+\Delta\varphi)\varphi_{\bar{j}}-(\Delta\varphi)_{\bar{j}})\right]-\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} \\
= & \tilde{g}^{i\bar{j}}\left|\lambda(m+\Delta\varphi)\nabla\varphi-\nabla(\Delta\varphi)\right|^2-\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}}. \qquad (48)
\end{aligned}
$$

Inserting computation (48) into (47), we see that

$$
\begin{aligned}
& \frac{1}{m+\Delta\varphi}\left[\lambda^2\tilde{g}^{i\bar{j}}\varphi_i\varphi_{\bar{j}}(m+\Delta\varphi)^2-\lambda\tilde{g}^{i\bar{j}}\varphi_i(\Delta\varphi)_{\bar{j}}(m+\Delta\varphi)-\lambda\tilde{g}^{i\bar{j}}(\Delta\varphi_i)\varphi_{\bar{j}}(m+\Delta\varphi)\right] \\
& -\lambda(\tilde{\Delta}\varphi)(m+\Delta\varphi)+\tilde{\Delta}(\Delta\varphi) \\
= & \frac{1}{m+\Delta\varphi}\left[\tilde{g}^{i\bar{j}}\left|\lambda(m+\Delta\varphi)\nabla\varphi-\nabla(\Delta\varphi)\right|^2-\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}}\right] \\
& -\lambda(\tilde{\Delta}\varphi)(m+\Delta\varphi)+\tilde{\Delta}(\Delta\varphi) \\
\geq & -\frac{1}{m+\Delta\varphi}\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}}-\lambda(\tilde{\Delta}\varphi)(m+\Delta\varphi)+\tilde{\Delta}(\Delta\varphi). \qquad (49)
\end{aligned}
$$

Therefore, from (45),

$$
\begin{aligned}
\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi)) \quad \geq \quad & -(m+\Delta\varphi)^{-1}\exp(-\lambda\varphi)\sum_{i,j=1}^{m}\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} \\
& -\lambda\exp(-\lambda\varphi)(m+\Delta\varphi)(\widetilde{\Delta}\varphi)+\exp(-\lambda\varphi)\widetilde{\Delta}(\Delta\varphi).
\end{aligned}
$$

□

In light of Lemma 6.3.1, to show that $\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi)) \geq 0$, we need to control $\sum_{i,j=1}^m \tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}}$ from above and $\widetilde{\Delta}(\Delta\varphi)$ from below.

Lemma 6.3.2. We have the following estimate

$$(m+\Delta\varphi)^{-1}\sum_{i,j=1}^m \tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} \quad \leq \quad \sum_{i,j,k=1}^m \frac{\varphi_{k\bar{i}\bar{j}}\varphi_{i\bar{k}j}}{(1+\varphi_{k\bar{k}})(1+\varphi_{i\bar{i}})}. \tag{50}$$

Proof. In local holomorphic coordinates, we may write

$$\sum_{i,j=1}^m \tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} \overset{\text{(locally)}}{=} \sum_{i,j=1}^m \delta_{ij}\frac{1}{1+\varphi_{i\bar{i}}}\varphi_{k\bar{k}i}\varphi_{k\bar{k}\bar{j}} = \sum_{i=1}^m \frac{|\varphi_{k\bar{k}i}|^2}{1+\varphi_{i\bar{i}}}. \tag{51}$$

Then we simply observe that

$$\begin{aligned}
& (m+\Delta\varphi)^{-1}\sum_{i=1}^m \frac{\left|\varphi_{k\bar{k}i}\right|^2}{(1+\varphi_{i\bar{i}})} \\
= \quad & (m+\Delta\varphi)^{-1}\sum_{i=1}^m \frac{1}{1+\varphi_{i\bar{i}}}\left|\sum_{k=1}^m \frac{\varphi_{k\bar{k}i}}{\sqrt{1+\varphi_{k\bar{k}}}}\sqrt{1+\varphi_{k\bar{k}}}\right|^2 \\
\leq \quad & (m+\Delta\varphi)^{-1}\left(\sum_{i,k=1}^m \frac{\varphi_{k\bar{k}i}\varphi_{\bar{k}k\bar{i}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})}\right)\sum_{k=1}^m(1+\varphi_{k\bar{k}}) \\
= \quad & \sum_{i,k=1}^m \frac{\varphi_{i\bar{k}k}\varphi_{k\bar{i}\bar{k}}}{(1+\varphi_{i\bar{i}})(1+\varphi_{k\bar{k}})} \\
\leq \quad & \sum_{i,j,k=1}^m \frac{\varphi_{k\bar{i}\bar{j}}\varphi_{i\bar{k}j}}{(1+\varphi_{k\bar{k}})(1+\varphi_{i\bar{i}})},
\end{aligned}$$

where in the last line we used the fact that $\varphi_{k\bar{k}i} = \varphi_{\bar{k}ki} = \varphi_{\bar{k}ik} = \varphi_{i\bar{k}k}$ and $\varphi_{\bar{k}k\bar{i}} = \varphi_{k\bar{k}\bar{i}} = \varphi_{k\bar{i}\bar{k}}$. □

Therefore, by combing Lemma 6.3.2 and (41), we see that

$$-(m+\Delta\varphi)^{-1}\sum_{i,j=1}^m \tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} + \widetilde{\Delta}(\Delta\varphi) \geq \Delta f + \kappa\left(\sum_{i,k=1}^m \frac{1+\varphi_{i\bar{i}}}{1+\varphi_{k\bar{k}}} - m^2\right). \tag{52}$$

Lemma 6.3.3. The expression $\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi))$ is bounded below by zero, i.e.,

$$\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi)) \quad \geq \quad 0. \tag{53}$$

Proof. By Lemma 6.3.1, we have

$$\begin{aligned}\exp(\lambda\varphi)\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi)) \quad &\geq \quad -(m+\Delta\varphi)^{-1}\sum_{i,j=1}^{m}\tilde{g}^{i\bar{j}}(\Delta\varphi)_i(\Delta\varphi)_{\bar{j}} \\ &\quad\;\; -\lambda(m+\Delta\varphi)(\widetilde{\Delta}\varphi)+\widetilde{\Delta}(\Delta\varphi).\end{aligned}$$

By Lemma 6.3.2, we have

$$\exp(\lambda\varphi)\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi)) \quad \geq \quad \Delta f+\kappa\left(\sum_{i,k=1}^{m}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{k\bar{k}}}-m^2\right)-\lambda(m+\Delta\varphi)(\widetilde{\Delta}\varphi).$$

In local holomorphic coordinates, we may write

$$\widetilde{\Delta}\varphi \quad \overset{\text{(locally)}}{=} \quad \sum_{i=1}^{m}\frac{\varphi_{i\bar{i}}}{1+\varphi_{i\bar{i}}}=\sum_{i=1}^{m}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{i\bar{i}}}-\sum_{i=1}^{m}\frac{1}{1+\varphi_{i\bar{i}}}=m-\sum_{i=1}^{m}\frac{1}{1+\varphi_{i\bar{i}}}.$$

Therefore,

$$\begin{aligned}&\exp(\lambda\varphi)\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi)) \\ \geq\quad & \Delta f-\kappa m^2+\kappa\sum_{i,k=1}^{m}\frac{1+\varphi_{i\bar{i}}}{1+\varphi_{k\bar{k}}}-\lambda(m+\Delta\varphi)\left(m-\sum_{i=1}^{m}\frac{1}{1+\varphi_{i\bar{i}}}\right) \\ =\quad & (\Delta f-\kappa m^2)+\kappa(m+\Delta\varphi)\left(\sum_{i=1}^{m}\frac{1}{1+\varphi_{i\bar{i}}}\right)+\lambda(m+\Delta\varphi)\left(\sum_{i=1}^{m}\frac{1}{1+\varphi_{i\bar{i}}}\right) \\ & -\lambda m(m+\Delta\varphi) \\ =\quad & (\Delta f-\kappa m^2)+(\kappa+\lambda)(m+\Delta\varphi)\left(\sum_{i=1}^{m}\frac{1}{1+\varphi_{i\bar{i}}}\right)-\lambda m(m+\Delta\varphi).\end{aligned}$$

Then from (43), we see that

$$\begin{aligned}&\exp(\lambda\varphi)\widetilde{\Delta}(\exp(-\lambda\varphi)(m+\Delta\varphi)) \\ \geq\quad & (\Delta f-\kappa m^2)+(\kappa+\lambda)(m+\Delta\varphi)(m+\Delta\varphi)^{\frac{1}{m-1}}\exp\left(\tfrac{f}{m-1}\right)-\lambda m(m+\Delta\varphi).\end{aligned} \tag{54}$$

Hence, by choosing $\lambda+\kappa>1$, we have the desired estimate. □

Applying the maximum principle, we see that at some point $p\in M$,

$$\left(\Delta f-\kappa m^2\right)-\lambda m(m+\Delta\varphi)+(\kappa+\lambda)(m+\Delta\varphi)^{1+\frac{1}{m-1}}(\exp(-f))^{\frac{1}{m-1}} \quad \leq \quad 0. \tag{55}$$

Further, we consider that if $y^{1+\frac{1}{m-1}} \leq ay + b$, then $y^{1+\frac{1}{m-1}} \leq 2ay$ or $y^{1+\frac{1}{m-1}} \leq 2b$. In particular, we may write (55) as

$$\begin{aligned}(m+\Delta\varphi)^{1+\frac{1}{m-1}} &\leq \frac{\exp(f)^{\frac{1}{m-1}}}{\kappa+\lambda}\left[\lambda m(m+\Delta\varphi) - \left(\Delta f - \kappa m^2\right)\right] \\ &= \underbrace{\frac{\lambda m \exp(f)^{\frac{1}{m-1}}}{\kappa+\lambda}}_{=:a}(m+\Delta\varphi) + \underbrace{\frac{\exp(f)^{\frac{1}{m-1}}}{\kappa+\lambda}\left(\kappa m^2 - \Delta f\right)}_{=:b}. \qquad (56)\end{aligned}$$

Hence, equation (56) shows that $(m+\Delta\varphi) \leq \Lambda_1$, where

$$\Lambda_1 := \max\left\{\frac{(2\lambda m)^{m-1}}{(\kappa+\lambda)^{m-1}}\exp\left(\sup_M f\right), \frac{2^{\frac{m-1}{m}}}{(\kappa+\lambda)^{\frac{m-1}{m}}}\exp\left(\sup_M f\right)\left(\kappa m^2 - \Delta f\right)^{\frac{m-1}{m}}\right\}. \qquad (57)$$

Given that $\exp(-\lambda\varphi)(m+\Delta\varphi)$ attains a maximum at p, we see that

$$\exp(-\lambda\varphi)(m+\Delta\varphi) \leq \exp\left(-\lambda\inf_M \varphi\right)(m+\Delta\varphi)(p). \qquad (58)$$

Since $m + \Delta\varphi = \sum_{i=1}^m (1+\varphi_{i\bar{i}}) = \sum_{i,j=1}^m g^{i\bar{j}}\tilde{g}_{i\bar{j}} > 0$, (58) implies that

$$0 < (m+\Delta\varphi) \leq \exp\left[\lambda\left(\varphi - \inf_M \varphi\right)\right]\Lambda_1. \qquad (59)$$

Lemma 6.3.4. Letting G denote the Green's function[1] for the Laplacian Δ, we have the following inequality

$$\sup_M \varphi \quad \leq \quad m \sup_{x\in M}\int_M [G(x,y)+K]\omega^n(y). \qquad (60)$$

Proof. Under the normalisation assumption $\int_M \varphi\omega^n = 0$, we see that

$$\begin{aligned}\varphi(x) &= \frac{1}{\mathrm{Vol}(M)}\int_M \varphi\omega^n - \int_M G(x,y)\Delta\varphi(y)\omega^n(y) \\ &= -\int_M [G(x,y)+K]\Delta\varphi\omega^n(y). \qquad (61)\end{aligned}$$

Taking the supremum of both sides of (61), we have

$$\sup_M \varphi \quad \leq \quad m \sup_{x\in M}\int_M [G(x,y)+K]\omega^n(y). \qquad (62)$$

□

[1] See p. 107 of [Au].

From this, we also obtain the L^1 and $\mathscr{C}^1$–estimates.

Corollary 6.3.5. We have the following estimate on the L^1–norm of φ,

$$\int_M |\varphi|\,\omega^n \quad\leq\quad 2m\mathrm{Vol}(M)\sup_{x\in M}\int_M [G(x,y)+K]\omega^n(y). \tag{63}$$

Proof. This follows easily from the supremum bound,

$$\begin{aligned}
\int_M |\varphi|\,\omega^n &= \int_M \left|\sup_M \varphi - \varphi - \sup_M \varphi\right| \omega^n \\
&= \int_M \left|\sup_M \varphi - \varphi\right| \omega^n + \int_M \left|\sup_M \varphi\right| \omega^n \\
&\leq 2\left(\sup_M \varphi\right)\mathrm{Vol}(M) - \int_M \varphi\omega^n \\
&= 2\left(\sup_M \varphi\right)\mathrm{Vol}(M) \\
&\overset{(62)}{\leq} 2m\mathrm{Vol}(M)\sup_{x\in M}\int_M [G(x,y)+K]\omega^n(y).
\end{aligned}$$

□

Corollary 6.3.6. We have the following estimate on the derivative of φ,

$$\sup_M |\nabla\varphi| \quad\leq\quad \Lambda_3\left[\exp\left(-\lambda \inf_M \varphi\right)+1\right]. \tag{64}$$

Proof. Using the Schauder estimate, Proposition 2.8 of [Sz], combined with (59), there exists a constant Λ_2 such that

$$\begin{aligned}
\sup_M |\nabla\varphi| &\leq \Lambda_2\left[\exp\left(-\lambda \inf_M \varphi\right) + \int_M |\varphi|\,\omega^n\right] \\
&\overset{(63)}{\leq} \Lambda_3\left[\exp\left(-\lambda \inf_M \varphi\right)+1\right].
\end{aligned} \tag{65}$$

□

All the remains now is to control $\inf_M \varphi$.

In place of λ we consider a new parameter $\mu > 0$ such that $\mu + \kappa \geq \frac{1}{2}\mu$. By (54), we see that

$$\begin{aligned}
\exp(\mu\varphi)\widetilde{\Delta}(\exp(-\mu\varphi)(m+\Delta\varphi)) &\geq (\Delta f - m^2\kappa) - \mu m(m+\Delta\varphi) \\
&\quad +(\mu+\kappa)(m+\Delta\varphi)\left(\sum_{i=1}^{m}\frac{1}{1+\varphi_{i\bar{i}}}\right) \\
&\overset{(54)}{\geq} (\Delta f - m^2\kappa) - \mu m(m+\Delta\varphi) \\
&\quad +\frac{1}{2}\mu(m+\Delta\varphi)\left[(m+\Delta\varphi)^{\frac{1}{m-1}}\exp\left(\tfrac{-f}{m-1}\right)\right] \\
&= (\Delta f - m^2\kappa) - \mu m(m+\Delta\varphi) \\
&\quad +\frac{1}{2}\mu(m+\Delta\varphi)^{\frac{m}{m-1}}\exp\left(\tfrac{-f}{m-1}\right).
\end{aligned}$$

Choose a constant Λ_4, depending on $\sup_M f$ and m, such that

$$\frac{1}{2}\mu\exp\left(-\tfrac{f}{m-1}\right)(m+\Delta\varphi)^{\frac{m}{m-1}} \geq 2\mu m(m+\Delta\varphi) - \mu\Lambda_4.$$

Then

$$\widetilde{\Delta}(\exp(-\mu\varphi)(m+\Delta\varphi)) \geq \exp(-\mu\varphi)\left[\Delta f - m^2\kappa - \mu\Lambda_4 + \mu m(m+\Delta\varphi)\right].$$

We then observe that

$$\begin{aligned}
&\exp(f)\widetilde{\Delta}(\exp(-\mu\varphi)(m+\Delta\varphi)) \\
\geq\ & \exp(-\mu\varphi)\left[\Delta f - m^2\kappa - \mu\Lambda_4\right] + \exp(-\mu\varphi)\mu m^2 + \exp(-\mu\varphi)\mu m\Delta\varphi \\
=\ & \left[\exp(-\mu\varphi)\left(\Delta f - m^2\kappa - \mu\Lambda_4\right) + m^2\mu\right] + \exp(-\mu\varphi)m\mu\Delta\varphi \\
\geq\ & \left[\exp(-\mu\varphi)\left(\Delta f - m^2\kappa - \mu\Lambda_4\right) + m^2\mu\exp\left(\inf_M f\right)\right] + m\mu\exp\left(\inf_M f\right)\exp(-\mu\varphi)\Delta\varphi. \qquad (66)
\end{aligned}$$

A simple computation shows that

$$\Delta(\exp(-\mu\varphi)) = -\mu(\Delta\varphi)\exp(-\mu\varphi) + \mu^2\left|\nabla\varphi\right|^2\exp(-\mu\varphi). \qquad (67)$$

Inserting (67) into (66), we see that

$$\begin{aligned}
\exp(f)\widetilde{\Delta}(\exp(-\mu\varphi)(m+\Delta\varphi)) &\geq \exp(-\mu\varphi)(\Delta f - m^2\kappa - \mu\Lambda_4) + m^2\mu\exp\left(\inf_M f\right) \\
&\quad +m\mu\exp\left(\inf_M f\right)\left[\mu^2\left|\nabla\varphi\right|^2\exp(-\mu\varphi) - \Delta(\exp(-\mu\varphi))\right] \\
&\geq -\Lambda_5\exp(-\mu\varphi) \\
&\quad +m\exp\left(\inf_M f\right)\left[\mu^2\left|\nabla\varphi\right|^2\exp(-\mu\varphi) - \Delta\exp(-\mu\varphi)\right]. \qquad (68)
\end{aligned}$$

Integrating (68) we see that

$$0 \quad \geq \quad -\Lambda_5 \int_M \exp(-\mu\varphi)\omega^n + m\mu^2 \exp\left(\inf_M f\right) \int_M \exp(-\mu\varphi) \left|\nabla\varphi\right|^2 \omega^n. \tag{69}$$

Hence we see that

$$\begin{aligned} \int_M \left|\nabla \exp\left(-\tfrac{1}{2}\mu\varphi\right)\right|^2 \omega^n &= \frac{1}{4}\mu^2 \int_M \exp\left(-\mu\varphi\right) \left|\nabla\varphi\right|^2 \omega^n \\ &\overset{(69)}{\leq} \frac{1}{4m}\Lambda_5 \exp\left(-\inf_M f\right) \int_M \exp(-\mu\varphi)\omega^n \end{aligned} \tag{70}$$

Lemma 6.3.7. Suppose that $\mu \in \mathbb{R}_{>0}$ satisfies $\mu + \kappa \geq \frac{1}{2}\mu$. Then (63) and (70) yield the existence of a constant Λ_3 (depending on μ, f and M) such that

$$\int_M \exp(-\mu\varphi)\omega^n \quad \leq \quad \Lambda_6. \tag{71}$$

Proof. We will proceed by contradiction and consider a sequence $(\varphi^\nu)_{\nu\geq 1}$ for which (63) and (70) are satisfied, and such that

$$\int_M \exp(-\mu\varphi^\nu)\omega^n \longrightarrow \infty,$$

as $\nu \to \infty$. We then normalise the sequence $(\exp(-\mu\varphi^\nu))$ and let $(\psi^\nu)_{\nu\geq 1}$ be the sequence given by

$$\exp(-\mu\psi^\nu) \quad = \quad \exp(-\mu\varphi^\nu) \left(\int_M \exp(-\mu\varphi^\nu)\omega^n\right)^{-1}. \tag{72}$$

Then (70) shows that the sequence $\int_M \left|\exp(-\frac{1}{2}\mu\psi^\nu)\right|^2 \omega^n$ is uniformly bounded from above. That is, we have a uniform estimate on the $W^{1,2}$–norm of $\exp(-\frac{1}{2}\mu\psi^\nu)$. Hence we may extract an L^2–convergent subsequence. Denote the L^2–limit function by ψ. We assume that this subsequence is the sequence itself.

Now let $A > 0$ and consider that

$$\mathrm{Vol}\left\{x \mid A \leq |\varphi|\right\} \quad \leq \quad \frac{1}{A}\int_M |\varphi|\,\omega^n.$$

We may write

$$\mathrm{Vol}\left\{x \mid A \leq \exp(-\tfrac{1}{2}\mu\psi^\nu)\right\} \quad = \quad \mathrm{Vol}\left\{x \;\middle|\; \tfrac{2}{\mu}\log A + \tfrac{1}{\mu}\log\int_M \exp(-\mu\varphi^\nu) \leq -\varphi^\nu\right\}.$$

Since $\int_M \exp(-\frac{1}{2}\mu\varphi^\nu)\omega^n$ grows unboundedly as $\nu \to \infty$, we may assert that for ν sufficiently large,

$$\begin{aligned} \mathrm{Vol}\left\{x \mid A \leq \exp(-\tfrac{1}{2}\mu\psi^\nu)\right\} &\leq \mathrm{Vol}\left\{x \;\middle|\; 0 < \tfrac{2}{\mu}\log A + \tfrac{1}{\mu}\log\int_M \exp(-\mu\varphi^\nu) \leq |\varphi^\nu|\right\} \\ &\leq \left(\tfrac{2}{\mu}\log A + \tfrac{1}{\mu}\int_M \exp(-\mu\varphi^\nu)\right)^{-1} \underbrace{\int_M |\varphi^\nu|\,\omega^n.}_{\text{uniformly bounded by (63)}} \end{aligned} \tag{73}$$

Hence, for all $A > 0$, (73) implies that

$$\mathrm{Vol}\left\{x \mid A \leq \exp(-\tfrac{1}{2}\mu\psi^{\nu})\right\} \longrightarrow 0, \tag{74}$$

as $\nu \to \infty$. Further, we see that

$$\begin{aligned}
\mathrm{Vol}\left\{x \mid A \leq \eta\right\} \quad &\leq \quad \mathrm{Vol}\left\{x \mid \tfrac{1}{2}A \leq \left|\eta - \exp(-\tfrac{1}{2}\mu\psi^{\nu})\right|\right\} + \mathrm{Vol}\left\{x \mid \tfrac{1}{2}A \leq \exp(-\tfrac{1}{2}\mu\psi^{\nu})\right\} \\
&\leq \quad \frac{2}{A}\int_M \left|\eta - \exp(-\tfrac{1}{2}\mu\psi^{\nu})\right|\omega^n + \mathrm{Vol}\left\{x \mid \tfrac{1}{2}A \leq \exp(-\tfrac{1}{2}\mu\psi^{\nu})\right\} \\
&\overset{\text{Hölder}}{\leq} \quad \frac{4}{A^2}\int_M \left|\eta - \exp(-\tfrac{1}{2}\mu\psi^{\nu})\right|^2\omega^n + \mathrm{Vol}\left\{x \mid \tfrac{1}{2}A \leq \exp(-\tfrac{1}{2}\mu\psi^{\nu})\right\}.
\end{aligned}$$

As $\nu \to \infty$, the first integral converges to zero and the second integral converges to zero by (74). We therefore see that η is zero almost everywhere. By (72) however, $\|\psi\|_{L^2} = 1$ which is of course a contradiction and this proves the result. $\square$

Lemma 6.3.8. There exists a constant Λ_8 (depending on M) such that

$$-\inf_M \varphi \quad \leq \quad \Lambda_4. \tag{75}$$

Proof. Let p be the point at which φ attains its infimum, i.e., $\varphi(p) = \inf_M \varphi$. From (64), we then see that for a constant $C > 0$,

$$\begin{aligned}
\varphi(z) \quad &\geq \quad \varphi(p) - \mathrm{dist}(z,p)\sup_M |\nabla\varphi| \\
&\geq \quad \inf_M \varphi - \mathrm{dist}(z,p)C\left[\exp\left(-\lambda\inf_M\varphi\right) + 1\right].
\end{aligned} \tag{76}$$

Now choose a geodesic ball $\mathscr{B}_r$ such that $\varphi \geq \frac{1}{2}\inf_M \varphi$ over this ball. From (76), we see that this is obtained by taking

$$r \quad \leq \quad \frac{1}{2}\left(\inf_M \varphi\right)\left(C\left[\exp\left(-\lambda\inf_M\varphi\right) + 1\right]\right)^{-1}.$$

Then, by (70), we see that

$$\begin{aligned}
\int_{\mathscr{B}_r} \exp(-\mu\varphi)\omega^n \quad &\geq \quad \frac{1}{\Lambda_5}\mu^2 m\exp\left(\inf_M f\right)\int_{\mathscr{B}_r}\exp(-\mu\varphi)\left|\nabla\varphi\right|^2\omega^n \\
&\geq \quad \underbrace{\frac{1}{\Lambda_5}\mu^2 m\exp\left(\inf_M f\right)}_{=:\Lambda_7}\exp(-\tfrac{1}{2}\mu\inf_M\varphi)\int_{\mathscr{B}_r}\left|\nabla\varphi\right|^2\omega^n \\
&\overset{(64)}{\geq} \quad \Lambda_7\exp(-\tfrac{1}{2}\mu\inf_M\varphi)r^{2m}.
\end{aligned} \tag{77}$$

Rearranging (77), we see that

$$-\inf_M \varphi \quad \geq \quad \log\left(\frac{2\mu}{\Lambda_7 r^{2m}} \int_{\mathscr{B}_r} \exp(-\lambda\varphi)\omega^n\right). \tag{78}$$

By Lemma 6.3.7, the integral in (78) is bounded and this controls the infimum. □

The estimate on $\sup_M \varphi$ then gives an estimate on $\sup_M |\varphi|$. Moreover, we also have control of $m+\Delta\varphi$ and $\sup_M |\nabla\varphi|$. An upper estimate on $1+\varphi_{k\bar{k}}$ is then given by noting that $(\delta_{ij}+\varphi_{ij})$ is a positive definite Hermitian matrix. Finally, a lower estimate on $1+\varphi_{k\bar{k}}$ is then given by $\prod_{k=1}^m (1+\varphi_{k\bar{k}}) = \exp(f)$.

CHAPTER 7

Third Order Estimates

The notation of the previous chapter is maintained. In this section we obtain an a priori estimate on mixed third derivatives $\partial_{i\bar{j}k}\varphi$ assuming that $f \in \mathscr{C}^3(M)$. The original proof is based on an estimate of the real Monge–Ampère equation that was obtained by Calabi, see [Ya76]. In recent years however, the technical computation seen in Appendix A of [Ya76] has been simplified and streamlined by Phong–Sesum–Sturm in [PSS]. The proof presented here is the original proof seen in [Ya76], but we follow the computation given in Phong–Sesum–Sturm. A variant of this presentation may also be found in Chapter 3.3 of [Sz].

Observe that since we have already estimated the metric $\widetilde{g}_{i\bar{j}}$, to obtain a third order estimates on φ it suffices to control the Christoffel symbols $\Gamma_{ij}^k = g^{k\bar{\ell}}\partial_i g_{j\bar{\ell}}$ of the Levi–Civita connection ∇. Let us recall however that the Christoffel symbols are not tensors, which makes computations difficult. Let $\widetilde{\Gamma}_{ij}^k = \tilde{g}^{k\bar{\ell}}\partial_i \tilde{g}_{j\bar{\ell}}$ denote the Christoffel symbols of the Levi–Civita connection of $\tilde{g}_{i\bar{j}}$. We introduce the tensor

$$S_{ij}^k \quad := \quad \widetilde{\Gamma}_{ij}^k - \Gamma_{ij}^k. \tag{79}$$

The norm of this tensor measured by g is given by

$$|S|^2 \quad = \quad \sum_{i,j,k,r,s,t=1}^{m} \tilde{g}^{i\bar{r}}\tilde{g}^{j\bar{s}}\tilde{g}^{k\bar{t}}\varphi_{i\bar{j}k}\varphi_{\bar{r}s\bar{t}}, \tag{80}$$

c.f., equation 3.1 of [Ya76]. By diagonalising the metric at a point and omitting the summation, we may write (80) as

$$|S|^2 \quad = \quad S_{ij}^k\overline{S_{ij}^k}. \tag{81}$$

The third order estimate is obtained from estimating $\widetilde{\Delta}\,|S|^2$ and applying the maximum principle. With $\widetilde{\nabla}$ denoting the Levi–Civita connection with respect to the metric $\widetilde{g}_{i\bar{j}}$, we compute

$$\begin{aligned}
\tilde{\Delta}\,|S|^2 &= \widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}}(S_{ij}^k \overline{S_{ij}^k}) \\
&= \widetilde{\nabla}_\alpha \left[(\widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k)\overline{S_{ij}^k} + S_{ij}^k(\widetilde{\nabla}_{\overline{\alpha}}\overline{S_{ij}^k})\right] \\
&= (\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k)\overline{S_{ij}^k} + (\widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k)(\widetilde{\nabla}_\alpha \overline{S_{ij}^k}) + (\widetilde{\nabla}_\alpha S_{ij}^k)(\widetilde{\nabla}_{\overline{\alpha}}\overline{S_{ij}^k}) + S_{ij}^k(\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}}\overline{S_{ij}^k}) \\
&= (\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k)\overline{S_{ij}^k} + \left|\widetilde{\nabla}_\alpha \overline{S_{ij}^k}\right|^2 + \left|\widetilde{\nabla}_\alpha S_{ij}^k\right|^2 + S_{ij}^k(\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}}\overline{S_{ij}^k}) \\
&\geq (\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k)\overline{S_{ij}^k} + S_{ij}^k(\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}}\overline{S_{ij}^k}) \\
&= (\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k)\overline{S_{ij}^k} + S_{ij}^k(\overline{\widetilde{\nabla}_{\overline{\alpha}} \widetilde{\nabla}_\alpha S_{ij}^k}). \qquad (82)
\end{aligned}$$

The failure of the covariant derivatives to commute is measured by the curvature tensor. We therefore see that

$$\widetilde{\nabla}_{\overline{\alpha}} \widetilde{\nabla}_\alpha S_{ij}^k = \widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k + \tilde{R}_i{}^\ell{}_{\alpha\overline{\alpha}} S_{\ell j}^k + \tilde{R}_j{}^\ell{}_{\alpha\overline{\alpha}} S_{i\ell}^k - \tilde{R}_\ell{}^k{}_{\alpha\overline{\alpha}} S_{ij}^\ell \qquad (83)$$

Let us write (83) in terms of the Ricci tensor, indeed, with $\tilde{R}_i^j = \tilde{g}^{j\bar{t}}\tilde{R}_{j\bar{t}}$, (83) becomes

$$\widetilde{\nabla}_{\overline{\alpha}} \widetilde{\nabla}_\alpha S_{ij}^k = \widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k + \tilde{R}_i^\ell S_{\ell j}^k + \tilde{R}_j^\ell S_{i\ell}^k - \tilde{R}_\ell^k S_{ij}^\ell. \qquad (84)$$

Now recall that the $\partial\overline{\partial}$–lemma established that the difference in the Ricci tensors $\mathrm{Ric}(\omega)$ and $\mathrm{Ric}(\widetilde{\omega})$ was given by

$$\mathrm{Ric}(\omega) - \mathrm{Ric}(\widetilde{\omega}) = -\partial\overline{\partial} f, \qquad f \in \mathscr{C}^\infty(M, \mathbb{R}). \qquad (85)$$

Consequently, $\mathrm{Ric}(\widetilde{\omega})$ is bounded. This together with the fact that the metrics $g_{i\bar{j}}$ and $\widetilde{g}_{i\bar{j}}$ are uniformly equivalent, (84) furnishes the estimate

$$\left|\widetilde{\nabla}_{\overline{\alpha}} \widetilde{\nabla}_\alpha S_{ij}^k\right| \leq \left|\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k\right| + K_1\,|S|\,, \qquad (86)$$

where $K_1 > 0$ is some constant. To compute $\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k$, we first recall the Bianchi identity

$$\widetilde{\nabla}_\alpha \tilde{R}_i{}^j{}_{k\overline{\alpha}} = \widetilde{\nabla}_k \tilde{R}_i{}^j{}_{\alpha\overline{\alpha}} = \widetilde{\nabla}_k \tilde{R}_i^j. \qquad (87)$$

We therefore compute

$$\begin{aligned}
\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k &= \widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} (\tilde{\Gamma}_{ij}^k - \Gamma_{ij}^k) \\
&= -\widetilde{\nabla}_\alpha \left(\tilde{R}_i{}^j{}_{k\overline{\alpha}} - R_i{}^j{}_{k\overline{\alpha}} \right) \\
&= -\widetilde{\nabla}_\alpha \tilde{R}_i{}^j{}_{k\overline{\alpha}} + \nabla_\alpha R_i{}^j{}_{k\overline{\alpha}} - \nabla_\alpha R_i{}^j{}_{k\overline{\alpha}} + \widetilde{\nabla}_\alpha R_i{}^j{}_{k\overline{\alpha}} \\
&= -\widetilde{\nabla}_\alpha \tilde{R}_i{}^j{}_{k\overline{\alpha}} + \nabla_\alpha R_i{}^j{}_{k\overline{\alpha}} + (\widetilde{\nabla}_\alpha - \nabla_\alpha) R_i{}^j{}_{k\overline{\alpha}} \\
&\overset{(87)}{=} -\widetilde{\nabla}_k \tilde{R}_i^j + \nabla_k R_i^j + (\widetilde{\nabla}_\alpha - \nabla_\alpha) R_i{}^j{}_{k\overline{\alpha}}.
\end{aligned}$$

By (85), the covariant derivatives $\widetilde{\nabla}_k \tilde{R}_i^j$ and $\nabla_k R_i^j$ are bounded. Further, by (79), we see that $\widetilde{\nabla}_\alpha - \nabla_\alpha$ is bounded by S. Then (86) becomes

$$\left| \widetilde{\nabla}_{\overline{\alpha}} \widetilde{\nabla}_\alpha S_{ij}^k \right| \leq K_2 + K_3 |S|. \tag{88}$$

Combining (88) with (82),

$$\begin{aligned}
\widetilde{\Delta} |S|^2 &\overset{(82)}{\geq} (\widetilde{\nabla}_\alpha \widetilde{\nabla}_{\overline{\alpha}} S_{ij}^k) \overline{S_{ij}^k} + S_{ij}^k (\overline{\widetilde{\nabla}_{\overline{\alpha}} \widetilde{\nabla}_\alpha S_{ij}^k}) \\
&\overset{(88)}{\geq} -(K_4 + K_5 |S|) |S| \\
&= -K_4 |S| - K_5 |S|^2.
\end{aligned} \tag{89}$$

Recall from the previous chapter that

$$\widetilde{\Delta}(\Delta\varphi) = \Delta f + \tilde{g}^{k\overline{j}} \tilde{g}^{i\overline{n}} \varphi_{k\overline{n}\overline{\ell}} \varphi_{i\overline{j}\ell} + \tilde{g}^{i\overline{j}} R_{i\overline{j}\ell\overline{\ell}} - R_{i\overline{i}\ell\overline{\ell}} + \tilde{g}^{k\overline{\ell}} R_{i\overline{j}k\overline{\ell}} \varphi_{i\overline{j}}$$

Diagonalising at a point and recalling the control of the Ricci tensor, we have constants $K_6, K_7 > 0$ such that

$$\widetilde{\Delta}(\Delta\varphi) \geq K_6 |S|^2 - K_7. \tag{90}$$

Now choose a constant C so that (89) and (90) furnish the estimate

$$\widetilde{\Delta}(|S|^2 + C\Delta\varphi) \geq |S|^2 - K_8 \tag{91}$$

If the function $|S|^2 + C\Delta\varphi$ achieves a maximum at a point q, then by (91), we see that $|S|^2(q) \leq K_8$. Therefore, we have

$$|S|^2 \leq |S|^2 + C\Delta\varphi \leq |S|^2(q) + C\Delta\varphi(q) \leq K_8 + C\Delta\varphi(q), \tag{92}$$

c.f., Equation (3.5) of [Ya76]. Since we have already controlled $\sup_M \Delta\varphi$, this yields the uniform estimate on $|S|^2$. In particular, we have an a priori estimate on the mixed partial derivatives $\partial_i \partial_{\bar{j}} \partial_k \varphi$, from which we have a bound on $\|\partial_i \partial_{\bar{j}} \varphi\|_{\mathscr{C}^{0,\alpha}}$.

The Schauder estimate then provides a $\mathscr{C}^{2,\alpha}$–bound on $\partial_p \varphi$ and $\partial_{\bar{p}} \varphi$. The coefficients of the elliptic operator L in the linearised Monge–Ampère equation then have improved regularity. In particular, we may iterate the Schauder estimate to obtain a $\mathscr{C}^{k+1,\alpha}$–bound on φ.

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