REMARKS ON THE QUADRATIC ORTHOGONAL BISECTIONAL CURVATURE

KYLE BRODER

Abstract. We give a few interpretations of the quadratic orthogonal bisectional curvature which elucidate its relationship to other curvatures in complex geometry.

1. Introduction

In [2], I discovered a useful algebraic formulation of the real bisectional curvature (as well as the holomorphic sectional curvature) that led to the notions of the first and second Schwarz bisectional curvatures. This algebraic formulation gives insight into other curvatures in complex geometry. Let us recall the following notion introduced in [11]:

Definition 1.1. Let (X, ω) be a Hermitian manifold. The *Quadratic Orthogonal Bisectional Curvature* (from now on, QOBC) is the function

$$\text{QOBC}_\omega : \mathcal{F}_X \times \mathbb{R}^n \backslash \{0\} \to \mathbb{R}, \qquad \text{QOBC}_\omega : (\vartheta, v) \mapsto \frac{1}{|v|_\omega^2} \sum_{\alpha,\gamma=1}^n R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} (v_\alpha - v_\gamma)^2,$$

where $R_{\alpha\overline{\alpha}\gamma\overline{\gamma}}$ denote the components of the Chern connection of ω with respect to the unitary frame ϑ (a section of the unitary frame bundle $\mathcal{F}_X$).

It is natural to compare the QOBC with other curvatures which arise in complex geometry. The most natural candidate is the real bisectional curvature [12]:

Definition 1.2. Let (X, ω) be a Hermitian manifold. The *Real Bisectional Curvature* is the function

$$\text{RBC}_\omega : \mathcal{F}_X \times \mathbb{R}^n \backslash \{0\} \to \mathbb{R}, \qquad \text{RBC}_\omega(\vartheta, v) = \frac{1}{|v|_\omega^2} \sum_{\alpha,\gamma=1}^n R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} v_\alpha v_\gamma.$$

In [2], it was shown that the real bisectional curvature was given by the Rayleigh quotient

$$\text{RBC}_\omega(v) \ = \ \frac{v^t \mathcal{R} v}{v^t v}$$

2010 *Mathematics Subject Classification.* 32Q25, 32Q20, 32Q05.

in each frame. Hence, $\text{RBC}_\omega \geq 0$ is equivalent to $\mathcal{R}$ being positive semi-definite in each frame. It would be interesting to obtain similar insight for the QOBC. In this note, we will give four interpretations of the QOBC:

(i) The first is the well-known interpretation of QOBC as the Weitzenböck curvature operator acting on $(1,1)$–forms. This is, in fact, where the QOBC first appeared (c.f., [1]).

(ii)–(iii) We will give two interpretations which are linear algebraic, a similar flavor to what was seen in [2]. These linear algebraic interpretations allow us to view $\text{QOBC}_\omega \geq 0$ and $\text{RBC}_\omega \geq 0$ as opposite conditions, in a suitable sense. This also gives an indication of the likelihood that $\text{QOBC}_\omega \geq 0 \implies \text{RBC}_\omega \geq 0$, and conversely.

(iv) The final interpretation is graph-theoretic: We realize that the QOBC is the Dirichlet energy of a certain weighted graph obtained from the frame. This observation is interesting in light of the first interpretation: We observe that the difference of the Riemannian and Hodge Laplacians, acting on $(1,1)$–forms yields the (discrete) Dirichlet energy of a graph weighted by the matrix $\mathcal{R}$.

2. Some Linear Algebra

For the real bisectional curvature, the class of semi-definite matrices played a key role. For the Schwarz bisectional curvatures, copositive and conegative matrices played a key role. For the quadratic orthogonal bisectional curvature (QOBC from here on) the following class of matrices will play a key role:

Definition 2.1. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is said to be a Euclidean distance matrix, written $A \in \mathbb{EDM}^n$, if there is a vector $v = (v_1, ..., v_n) \in \mathbb{R}^n$ such that $A_{ij} = (v_i - v_j)^2$. Oftentimes, we will omit the n, and simply write $A \in \mathbb{EDM}$.

Remark 2.2. For those readers familiar with Euclidean distance matrices, the above definition confines itself to Euclidean distance matrices of embedding dimension one.
Let us, moreover, make some obvious remarks:

(i) Euclidean distance matrices are non-negative, i.e., each element A_{ij} is non-negative.
(ii) Euclidean distance matrices are hollow, i.e., there are no non-zero entries along the main diagonal.

The space of (symmetric) hollow matrices is denoted $\mathcal{S}_{\text{H}}$.

Remark 2.3. The space $\mathbb{EDM}$ of Euclidean distance matrices forms a pointed closed convex cone within the space of symmetric matrices $\mathcal{S}$. The cone $\mathbb{EDM}$ is the intersection of an infinite number of half-spaces about the origin, and a finite number of hyperplanes through the origin. Since the EDM cone is confined to the symmetric hollow subspace $\mathcal{S}_{\text{H}}$, we see that $\mathbb{EDM}$ is not a full-dimensional cone, and moreover, has empty interior.

Remark 2.4. Within the space $\mathcal{S}$ of symmetric matrices, the EDM cone does not meet the cone of positive semi-definite matrices $\mathcal{S}_+$, except at the origin (their only vertex), i.e.,

$$\mathcal{S}_+^n \cap \mathbb{EDM}^n \;=\; \{0\}, \qquad \forall n \in \mathbb{N}.$$

Indeed, an old result of Schoenberg [10] states a matrix $A \in \mathcal{S}_{\mathrm{H}}^n$ is an element of the EDM cone if and only if A is negative semi-definite on the hyperplane $H = \{v \in \mathbb{R}^n : \mathbf{e}^t v = 0\}$, where $\mathbf{e} = (1, 1, ..., 1) \in \mathbb{R}^n$. Moreover, by the Perron–Frobenius theorem, since Euclidean distance matrices are non-negative, the largest eigenvalue (the Perron root) is non-negative, and the corresponding eigenvector lies in the non-negative orthant $\overline{\mathbb{R}_+^n} = \{(x_1, ..., x_n) \in \mathbb{R}^n : x_k \geq 0 \; \forall k\}$. Hence, a non-trivial Euclidean distance matrix is always indefinite.

Remark 2.5. The relation between the cones $\mathbb{EDM}$ and $\mathcal{S}_+$ was given by Dattorro in [6]. To describe Dattorro's result, we denote by $\mathcal{S}_{\mathrm{C}}$ the geometric center subspace

$$\mathcal{S}_{\mathrm{C}}^n := \{A \in \mathcal{S}^n : A\mathbf{e} = 0\}.$$

An arguably better name for $\mathcal{S}_{\mathrm{C}}$ would be the annihilator of $\mathbf{e} = (1, ..., 1)^t$.
The orthogonal complement of $\mathcal{S}_{\mathrm{C}}$ is denoted

$$\mathcal{S}_{\mathrm{C}}^{\perp} \;=\; \{u\mathbf{e}^t + \mathbf{e}^t u : u \in \mathbb{R}^n\}.$$

Theorem 2.6. (Dattorro [6]). The cone $\mathbb{EDM}$ of Euclidean distance matrices affords the description:

$$\mathbb{EDM} \;=\; \mathcal{S}_{\mathrm{H}} \cap (\mathcal{S}_{\mathrm{C}}^{\perp} - \mathcal{S}_+).$$

Remark 2.7. In contrast with $\mathcal{S}_+$, the EDM cone fails to be self-dual. We denote by $\mathbb{EDM}^*$ the dual EDM cone:

$$\mathbb{EDM}^* = \{A \in \mathbb{R}^{n\times n} : \langle A, B\rangle \geq 0 \; \forall B \in \mathbb{EDM}\}.$$

We can take $\langle \cdot, \cdot \rangle$ to be the Frobenius duality pairing

$$\mu_{\mathrm{F}} : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \to \mathbb{R}, \qquad \mu_{\mathrm{F}}(A, B) := \mathrm{tr}(AB^t).$$

Remark 2.8. It is in this sense that $\mathrm{QOBC}_\omega \geq 0$ is the "dual opposite"[1] of $\mathrm{RBC}_\omega \geq 0$. As we indicated previously, $\mathrm{RBC}_\omega \geq 0$ is equivalently to $\mathcal{R} \in \mathcal{PSD}$ (in each frame); while $\mathrm{QOBC}_\omega \geq 0$ is equivalent to $\mathcal{R} \in \mathcal{EDM}^*$ (in each frame). The positive semi-definite cone $\mathcal{PSD}$ is self-dual. Hence, in a fixed frame, the cone describing $\mathrm{RBC}_\omega \geq 0$ has dual cone which intersects the cone dual to the cone describing $\mathrm{QOBC}_\omega \geq 0$ trivially. That is,

$$\mathrm{RBC}_\omega \geq 0 \iff \mathcal{R} \in \mathcal{PSD}, \qquad \mathrm{QOBC}_\omega \geq 0 \iff \mathcal{R} \in \mathcal{EDM}^*,$$

and

$$\mathcal{PSD} \cap \mathcal{EDM} = \{0\}.$$

[1]Declare two cones $\mathcal{K}_1, \mathcal{K}_2$ to be dual opposing if $\mathcal{K}_1^* \cap \mathcal{K}_2^* = \{0\}$.

Of course, $\text{RBC}_\omega \geq 0$ and $\text{QOBC}_\omega \geq 0$ can still intersect non-trivially.

Example 2.9. Let us exhibit an example of a metric with $\text{QOBC}_\omega \geq 0$, but RBC_ω does not have a sign. In a small disk centered at the origin in $\mathbb{C}^2$, with coordinates (z, w), define the metric

$$g = \begin{pmatrix} 1 - |w|^2 & 0 \\ 0 & 1 - |z|^2 \end{pmatrix}.$$

At the origin, the metric is Euclidean, with vanishing 1–jets. Hence, the curvature tensor has non-zero components

$$R_{1\bar{1}1\bar{1}} \ = \ R_{2\bar{2}2\bar{2}} \ = \ 0, \qquad R_{1\bar{1}2\bar{2}} \ = \ R_{2\bar{2}1\bar{1}} \ = \ 1.$$

Hence,

$$\mathcal{R} \ = \ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

which is (entry-wise) non-negative, but is not positive semi-definite.

The following gives a useful criterion for checking whether a symmetric matrix lies in the dual EDM cone (c.f., [7, p. 434]):

Theorem 2.10. Let $\delta : \mathbb{R}^n \to \mathcal{S}^n_{\text{D}}$ be the operator which maps a vector $v \in \mathbb{R}^n$ to the diagonal matrix $\text{diag}(v)$. Then a real symmetric matrix $A \in \mathcal{S}^n$ lies in the dual EDM cone if and only if $\delta(A\mathbf{e}) - A \in \mathcal{PSD}$. In particular,

$$\text{QOBC}_\omega \geq 0 \quad \Longleftrightarrow \quad \delta(\mathcal{R}\mathbf{e}) - \mathcal{R} \in \mathcal{PSD}.$$

Example 2.11. Consider the matrix $\mathcal{R}$ in the case $n = 3$. Then

$$\mathcal{R} = \begin{pmatrix} \mathcal{R}_{11} & \mathcal{R}_{12} & \mathcal{R}_{13} \\ \mathcal{R}_{12} & \mathcal{R}_{22} & \mathcal{R}_{23} \\ \mathcal{R}_{13} & \mathcal{R}_{23} & \mathcal{R}_{33} \end{pmatrix}.$$

Compute

$$\delta(\mathcal{R}\mathbf{e}) - \mathcal{R} = \begin{pmatrix} \mathcal{R}_{12} + \mathcal{R}_{13} & -\mathcal{R}_{12} & -\mathcal{R}_{13} \\ -\mathcal{R}_{12} & R_{12} + \mathcal{R}_{23} & -\mathcal{R}_{23} \\ -\mathcal{R}_{13} & -\mathcal{R}_{23} & \mathcal{R}_{13} + \mathcal{R}_{23} \end{pmatrix}.$$

The eigenvalues are 0, and

$$\mathcal{R}_{12} + \mathcal{R}_{13} + \mathcal{R}_{23} \pm \sqrt{\mathcal{R}_{12}^2 - \mathcal{R}_{12}\mathcal{R}_{13} + \mathcal{R}_{13}^2 - \mathcal{R}_{12}\mathcal{R}_{23} - \mathcal{R}_{13}\mathcal{R}_{23} + \mathcal{R}_{23}^2}.$$

In particular, we have the following:

Proposition 2.12. Let (X, ω) be a Hermitian threefold. Then $\text{QOBC}_\omega \geq 0$ if and only if, in each frame, one of the following hold:

(i) $\mathcal{R}_{23} < 0$ and $\mathcal{R}_{23} + \mathcal{R}_{13} > 0$ and $\mathcal{R}_{13}\mathcal{R}_{23} + \mathcal{R}_{12}(\mathcal{R}_{13} + \mathcal{R}_{23}) \geq 0$;

(ii) $\mathcal{R}_{23} = 0$ and $\mathcal{R}_{13} \geq 0$ and $\mathcal{R}_{12} \geq 0$;

(iii) $\mathcal{R}_{23} > 0$ and $\mathcal{R}_{23} + \mathcal{R}_{13} > 0$ and $\mathcal{R}_{13}\mathcal{R}_{23} + \mathcal{R}_{12}(\mathcal{R}_{13} + \mathcal{R}_{23}) \geq 0$.

3. A Graph-Theoretic Interpretation

Let G be a finite weighted graph, with vertices $V(G) = \{x_1, ..., x_n\}$, and weighting specified by its adjacency matrix $A \in \mathbb{R}^{n \times n}$. The Dirichlet energy for a weighted graph is defined

$$\mathcal{E}(f) := \sum_{i,j=1}^{n} A_{ij}(f(x_i) - f(x_j))^2,$$

where $f : V(G) \to \mathbb{R}$ is a function defined on the vertices of G.

Theorem 3.1. ([3]). Let (G, A) be a weighted finite graph. Then the Dirichlet energy $\mathcal{E}$ is non-negative if and only if A lies in the dual EDM cone.

Remark 3.2. Let Δ_g, Δ denote the Bochner Laplace operator, and the Laplace–Beltrami opeartor, respectively, actively on real $(1,1)$–forms. Their difference $\Delta_g - \Delta$ is the Weitzenböck curvature, which, upon restricting to real $(1,1)$–forms, is precisely the QOBC. This observation goes back to Bishop–Goldberg [1]. Fix now a unitary frame $e_1, ..., e_n$ in $\mathcal{F}_X$, and let $\mathcal{R} = (\mathcal{R}_{\alpha\gamma})$ denote the matrix of certain components of the curvature, as before. We observe that

$$\Delta_g - \Delta \;\; = \;\; \text{QOBC}_\omega \;\; = \;\; \sum_{\alpha,\gamma} \mathcal{R}_{\alpha\gamma}(v_\alpha - v_\gamma)^2 \;\; = \;\; \mathcal{E}(f).$$

That is, the difference of the Bochner Laplacian and the Laplace–Beltrami operator is the Dirichlet energy of the discrete Laplacian on the frame.

References

[1] R. L. Bishop and S. I. Goldberg, On the second cohomology group of a Kaehler man- ifold of positive curvature, Proc. Amer. Math. Soc. 16 (1965), 119–122. MR0172221

[2] Broder, K., The Schwarz Lemma in Kähler and non-Kähler geometry, arXiv:2109.06331

[3] Broder, K., On the non-negativity of the Dirichlet energy of a weighted graph. (submitted)

[4] Broder, K., An eigenvalue characterization of the dual EDM cone, to appear in the Bull. of the Aust. Math. Soc.

[5] A. Chau and L.-F. Tam, Kähler C-spaces and quadratic bisectional curvature, J. Differential Geom. 94 (2013), no. 3, 409–468. MR3080488

[6] Dattorro, Jon. Equality relating Euclidean distance cone to positive semidefinite cone. Linear Algebra Appl. 428 (2008), no. 11-12, 2597–2600. MR2416574

[7] Dattorro, Jon. Convex optimization † Euclidean distance geometry. Available at the author's webpage: https://ccrma.stanford.edu/~dattorro/

[8] Heinz, T. F., Topological Properties of Orthostochastic Matrices, Linear algebra and its applications, **20**, 265–269 (1978)

[9] Q. Li, D. Wu, and F. Zheng, An example of compact Kähler manifold with non-negative quadratic bisectional curvature, Proc. Amer. Math. Soc. 141 (2013), no. 6, 2117–2126. MR3034437

[10] Schoenberg., I. J., Remarks to Maurice Fréchet's article "Sur la définition axiomatique d'une classe déspace distanciés vectoriellement applicable sur l'espace de Hilbert". Annals of Mathematics, 36(3):724–732, July 1935.

[11] D. Wu, S.-T. Yau, and F. Zheng, A degenerate Monge-Ampère equation and the boundary classes of Kähler cones, Math. Res. Lett. 16 (2009), no. 2, 365–374. MR2496750

[12] Yang, X., Zheng, F., On the real bisectional curvature for Hermitian manifolds, Trans. Amer. Math. Soc. **371** (2019), no. 4, 2703–2718

Mathematical Sciences Institute, Australian National University, Acton, ACT 2601, Australia

BICMR, Peking University, Beijing, 100871, People's republic of china

E-mail address: kyle.broder@anu.edu.au