AN EIGENVALUE CHARACTERIZATION OF THE DUAL EDM CONE

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ABSTRACT. We show that the elements of the dual of the Euclidean distance matrix cone can be described via an inequality on a certain weighted sum of its eigenvalues.

Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix. Given a vector $v = (v_1, ..., v_n) \in \mathbb{R}^n \backslash \{0\}$ we can associate a Euclidean distance matrix $\Delta = \Delta^v$ (of embedding dimension one) by declaring the entries of Δ to be $\Delta_{ij} = (v_i - v_j)^2$. Let $(\cdot, \cdot)_{\mathrm{F}}$ denote the Frobenius pairing on $\mathbb{R}^{n \times n}$ given by $(A, B)_{\mathrm{F}} := \mathrm{tr}(AB^t)$. If $\mathbb{EDM}_n$ denotes the cone of Euclidean distance matrices in $\mathbb{R}^{n \times n}$, we denote by $\mathbb{EDM}_n^*$ the cone dual to $\mathbb{EDM}$, namely,

$$\mathbb{EDM}_n^* := \{A \in \mathbb{R}^{n \times n} : (A, B)_{\mathrm{F}} \geq 0 \quad \forall B \in \mathbb{EDM}_n\}.$$

The purpose of this note is to prove the following:

Theorem. Let $A \in \mathbb{R}^{n \times n}$ be a real symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Then $A \in \mathbb{EDM}_n^*$ if and only if

$$\lambda_1 \quad \geq \quad \sum_{k=2}^n r_k \lambda_k,$$

for all Perron-weights $0 \leq r_k \leq 1$.

Remark 1. Let us explain the meaning of the non-standard terminology *Perron-weights*. A symmetric matrix $\Sigma \in \mathbb{R}^{n \times n}$ is said to be a Euclidean distance matrix (EDM) if there is a vector $v = (v_1, ..., v_n) \in \mathbb{R}^n$ such that $\Sigma_{ij} = (v_i - v_j)^2$. The well-known Schoenberg criterion [4] states that a symmetric hollow matrix[1] Σ is a Euclidean distance matrix if and only if it is negative semi-definite on the hyperplane $H = \{x \in \mathbb{R} : x^t \mathbf{e} = 0\}$, where $\mathbf{e} = (1, ..., 1)^t$. Of course, a Euclidean distance matrix is, in particular, a non-negative matrix in the sense that each entry of the matrix is a non-negative real number. As a consequence, the Perron–Frobenius theorem asserts that the largest eigenvalue of the EDM Σ is positive and occurs with eigenvector in the non-negative orthant $\mathbb{R}^n_{\geq 0}$. This eigenvalue is often called the Perron

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[1]That is, a matrix with non-zero entries on its diagonal.

root of Σ, denote $r = r(\Sigma)$. Therefore, if $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$ denote the eigenvalues of a non-trivial Euclidean distance matrix[2] Σ, then $\delta_1 > 0$ and $\delta_2, ..., \delta_n \leq 0$. The Perron weights are then defined, for each $2 \leq k \leq n$ by

$$r_k := -\frac{\delta_k}{\delta_1} \in [0,1].$$

Remark 2. This theorem can be described in terms of zonahedra. Given a set of vectors $w_1, ..., w_k \in \mathbb{R}^d$, the zonahedron generated by $w_1, ..., w_k$ is the Minkowski sum of line segments connecting each of the points. That is, for $\vartheta_1, ..., \vartheta_k \in [0,1]$, the zonahedron generated by $w_1, ..., w_k$ is the set

$$Z(w_1, ..., w_k) := \left\{ \sum_{i=1}^k \vartheta_i w_i : \vartheta_i \in [0,1] \right\}.$$

The dual EDM cone, therefore, consists of those symmetric matrices whose largest eigenvalue lies on the right of the zonahedron formed by the remaining eigenvalues.

Proof of the main theorem. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of the symmetric matrix $A \in \mathbb{R}^{n \times n}$ and let $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$ denote the eigenvalues of a Euclidean distance matrix Δ. If $A = U^t \text{diag}(\lambda) U$ and $\Delta = V^t \text{diag}(\delta) V$ denote the eigenvalue decompositions for A and Δ, we see that

$$\begin{aligned} \text{tr}(A\Delta) = \text{tr}(U^t \text{diag}(\lambda) U V^t \text{diag}(\delta) V) &= \text{tr}(V U^t \text{diag}(\lambda) U V^t \text{diag}(\delta)) \\ &= \text{tr}(Q^t \text{diag}(\lambda) Q \text{diag}(\delta)) \\ &= \sum_{i,j} \lambda_i \delta_j Q_{ij}^2, \end{aligned}$$

where $Q = UV^t$ is orthogonal. The Hadamard square[3] of an orthogonal matrix is doubly stochastic (see, e.g., [3]). The class of $n \times n$ doubly stochastic matrices forms a convex polytope – the Birkhoff polytope $\mathcal{B}^n$. The minimum of $\text{tr}(A\Delta)$ is given by

$$\min_{S \in \mathcal{B}^n} \sum_{i,j=1}^n \lambda_i \delta_j S_{ij}.$$

This function is linear in S, and therefore, achieves its minimum on the boundary of the Birkhoff polytope. The Birkhoff–von Neumann theorem tells us that $\mathcal{B}^n$ is the convex hull of the set of permutation matrices, and moreover, the vertices of $\mathcal{B}^n$ are precisely the permutation matrices. Hence,

$$\min_{S \in \mathcal{B}^n} \sum_{i,j=1}^n \lambda_i \delta_j S_{ij} = \min_{\sigma \in S_n} \sum_{i=1}^n \lambda_i \delta_{\sigma(i)},$$

[2] That is, a Euclidean distance matrix with $\delta_1 > 0$.

[3] That is, the matrix $Q \circ Q$ with entries Q_{ij}^2

where S_n denotes the symmetric group on n letters. An elementary argument (by induction, for instance) shows that

$$\min_{\sigma \in S_n} \sum_{i=1}^n \lambda_i \delta_{\sigma(i)} \quad = \quad \sum_{i=1}^n \lambda_i \delta_i.$$

From the discussion in remark 1, this completes the proof.

Applications to graph theory. In [1], I addressed the problem of when a weighted finite graph (with possibly negative weights) has non-negative Dirichlet energy. I showed that the Dirichlet energy was non-negative if and only if the matrix describing the weighting was an element of the dual EDM cone. In the matrix is symmetric, the graph is said to be directed. The main theorem of this note has the following corollary:

Corollary. Let (G, A) be a directed weighted graph. Let $V(G) = \{x_1, ..., x_n\}$ denote the vertex set of G. The Dirichlet energy

$$\mathcal{E}(f) := \sum_{i,j=1}^n A_{ij}(f(x_i) - f(x_j))^2$$

is non-negative for all graph functions $f : V(G) \to \mathbb{R}$ if and only if the largest eigenvalue of A dominates the Perron-weighted average of the remaining eigenvalues.

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References

[1] Broder, K., *On the non-negativity of the Dirichlet energy of a weighted graph*, (submitted)

[2] Dattorro, Jon. Equality relating Euclidean distance cone to positive semidefinite cone. Linear Algebra Appl. 428 (2008), no. 11-12, 2597–2600. MR2416574

[3] Heinz, T. F., Topological Properties of Orthostochastic Matrices, Linear algebra and its applications, **20**, 265–269 (1978)

[4] Schoenberg., I. J., Remarks to Maurice Fréchet's article "Sur la définition axiomatique d'une classe déspace distanciés vectoriellement applicable sur l'espace de Hilbert". Annals of Mathematics, 36(3):724–732, July 1935.

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