# CONE ANGLES AND LCTS

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Abstract. We compute the asymptotic expansion of the Hodge metric on the direct image of the relative canonical bundle for a Calabi–Yau fiber space with one-dimensional base.

Let $f : X^{n+1} \to Y$ be a Calabi–Yau fiber space, i.e., a surjective proper holomorphic map with connected fibers from a compact Kähler manifold $X$ onto a normal projective variety $Y$ whose generic fiber is Calabi–Yau. We assume that $\dim_{\mathbb{C}} X = n+1$ and $\dim_{\mathbb{C}} Y = 1$, so the relative dimension is $n$. Denote by $\mathrm{disc}(f)$ the discriminant locus of $f$, defined to be the set of all points $p \in Y$ such that the corresponding fiber $X_p := f^{-1}(p)$ is singular. Since the base is a curve, $\mathrm{disc}(f)$ consists of a finite number of points.

Such fiber spaces arise naturally in the study of the long-time behavior of the Kähler–Ricci flow and collapsed limits of Ricci-flat Kähler metrics. In general, it is known that on the base of such fiber spaces, there exists a twisted Kähler–Einstein metric $\omega_{\mathrm{can}}$ which satisfies

$$\mathrm{Ric}(\omega_{\mathrm{can}}) \ = \ -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}},$$

away from $\mathrm{disc}(f)$, where $\omega_{\mathrm{WP}}$ is the Weil–Petersson metric on $Y$ which measures the variation in the complex structure of the fibers of $f$. The Weil–Petersson metric was shown by Tian to be the curvature form of the Hodge metric $\omega_{\mathrm{H}}$ on the direct image of the relative canonical bundle $f_* \Omega^n_{X/Y}$.

In more detail, since the generic fiber of $f$ is Calabi–Yau[1], some multiple of the canonical bundle $K^{\ell}_{X_w}$ is holomorphically trivial. For simplicity, we assume that $K_{X_w}$ is holomorphically trivial. Let $\Omega_y$ be the holomorphic $(n,0)$–form which trivializes $K_{X_w}$. The Hodge metric $\omega_{\mathrm{H}}$ is then given by the fiber integral:

$$\omega_{\mathrm{H}} \ := \ (\sqrt{-1})^{n^2} \int_{X_w} \Omega_y \wedge \overline{\Omega}_y.$$

In this note, we compute the asymptotics of the Hodge metric and analyze its behavior near the discriminant locus. In particular, we prove the following theorem:

**Theorem.** Let $\alpha$ be the log canonical threshold of the singular fiber $X_p$. There exists a constant $C \geq 1$, a positive integer $N \in \mathbb{N}$ and a conical Kähler metric $\omega_{\mathrm{cone}}$ of cone angle $2\pi\alpha$ such that in any local holomorphic coordinate system centered at $p$,

[1] in the sense that $c_1(K_{X_w}) = 0$ in $H^2(X_w, \mathbb{R})$

$$C^{-1}(-\log|w|)^{N-1}\omega_{\text{cone}} \quad \leq \quad \omega_{\text{H}} \quad \leq \quad C(-\log|w|)^{N-1}\omega_{\text{cone}}.$$

That is, modulo some logarithmic poles, the Hodge metric is quasi-isometric to the conical metric:

$$\omega_{\text{cone}} \ := \ \sqrt{-1}\frac{1}{|w|^{2(1-\alpha)}}dw \wedge d\overline{w}.$$

The aforementioned twisted Kähler–Einstein metric $\omega_{\text{can}}$ is obtained from solving the complex Monge–Ampère equation

$$\omega_{\text{can}} = \omega_0 + \sqrt{-1}\partial\overline{\partial}\varphi = c_n \omega_{\text{H}} e^{\varphi},$$

for some dimensional constant $c_n > 0$ and some continuous plurisubharmonic function $\varphi$. Hence, the asymptotics of $\omega_{\text{H}}$ coincide with the asymptotics of $\omega_{\text{can}}$.
To simplify the notation, in what follows we will write

$$\Lambda_y \ := \ (\sqrt{-1})^{n^2}\Omega_y \wedge \overline{\Omega}_y.$$

*Proof.* The problem is local, so assume $f : X \to \Delta$ is a Calabi–Yau fiber space over the unit disk in $\mathbb{C}$ with singular fiber $X_0$ over the origin. Consider a log resolution of the pair $(X, X_0)$, i.e., a birational morphism

$$\pi : (\widetilde{X}, \widetilde{X}_0) \ \dashrightarrow \ (X, X_0)$$

such that $\widetilde{X}$ is smooth and

$$\widetilde{X}_0 \ := \ \pi^{-1}(X_0) \ = \ \sum_j a_j E_j$$

has simple normal crossings. Choose defining sections $\sigma_j$ for $E_j$ and take $h_j$ to be Hermitian metrics on the associated line bundles $\mathcal{O}(E_j)$. The ramification formula

$$K_{\widetilde{X}} \ = \ \pi^* K_X + \sum_j k_j E_j$$

allows us to write

$$\pi^*\Lambda_y \quad = \quad \prod_j |\sigma_j|_{h_j}^{2k_j} dV, \tag{1}$$

where $dV$ is a smooth non-degenerate volume form on $\widetilde{X}$. To compute the asymptotics of the Hodge metric near the discriminant locus, it suffices to compute

$$\int_{\widetilde{X}_w} \pi^*\Lambda_y \ = \ \int_{X_w} \Lambda_y, \tag{2}$$

as $w$ approaches 0 (since $\pi$ is an isomorphism away from the origin).
Fix a point $x_0 \in \widetilde{X}_0$ and let

$$J(x_0) \ := \ \{j \in \mathbb{N} : x_0 \in E_j\}$$

be the indexing set for those irreducible components of $\widetilde{X}_0$ which meet $x_0$ non-trivially.
Let $U_0$ be a coordinate chart centered $x_0$ with coordinates $z = (z_1, ..., z_n)$. In these coordinates, the composite map

$$\widetilde{f} \ := \ f \circ \pi \ : \ \widetilde{X} \longrightarrow \Delta$$

affords the following description:

$$\widetilde{f}(z) \quad = \quad \prod_{j \in J(x_0)} z_j^{a_j} \quad =: \quad w.$$

In the chart $U_0$, equations (1) and (2) read:

$$\int_{\widetilde{X}_w \cap U_0} \pi^* \Lambda_y \quad = \quad \int_{\widetilde{X}_w \cap U_0} h(z) \prod_{j \in J(x_0)} |z_j|^{2k_j} dz \wedge d\overline{z},$$

where $h$ is a smooth positive function on $U_0$. To simplify the notation, write $V_w := \widetilde{X}_w \cap U_0$ and take $h$ to be identically one. Observe that

$$\begin{aligned}
\widetilde{f}^* dw \ = \ d\left( \prod_{j \in J(x_0)} z_j^{a_j} \right) \quad &= \quad \sum_{k \in J(x_0)} a_k z_k^{a_k - 1} dz_k \prod_{j \in J(x_0) \setminus \{k\}} z_j^{a_j} \\
&= \quad \sum_{k \in J(x_0)} \frac{a_k}{z_k} dz_k \prod_{j \in J(x_0)} z_j^{a_j} \\
&= \quad \sum_{k \in J(x_0)} \frac{a_k}{z_k} dz_k \ \widetilde{f}^* w,
\end{aligned}$$

that is, we have the following relation on the logarithmic differentials:

$$\widetilde{f}^* \frac{dw}{w} \quad = \quad \sum_{j \in J(x_0)} a_j \frac{dz_j}{z_j}.$$

Fix some $j_0 \in J(x_0)$ to be determined later and write

$$dz \wedge d\overline{z} \quad = \quad dz_{j_0} \wedge dz^{\widehat{j_0}} \wedge d\overline{z}_{j_0} \wedge d\overline{z}^{\widehat{j_0}},$$

where $dz^{\widehat{j_0}} := dz_1 \wedge \cdots \wedge dz_{j_0 - 1} \wedge dz_{j_0+1} \wedge \cdots \wedge dz_n$.
Setting

$$\mu_0 \quad := \quad \frac{1}{\widetilde{f}^*(w)} \frac{z_{j_0}}{a_{j_0}} dz^{\widehat{j_0}}$$

we have

$$\int_{V_w} \prod_{j \in J(x_0)} |z_j|^{2k_j} dz \wedge d\overline{z} \quad = \quad \int_{V_w} \prod_{j \in J(x_0)} |z_j|^{2k_j} \widetilde{f}^* dw \wedge \mu_0 \wedge \overline{\widetilde{f}^* dw} \wedge \overline{\mu_0}.$$

The measure $\widetilde{f}^*dw \wedge \overline{\widetilde{f}^*dw}$ is pulled back from the base, and is therefore constant along the fibers. Hence, we need only compute

$$\int_{V_w} \prod_{j \in J(x_0)} |z_j|^{2k_j} \mu \wedge \overline{\mu}_0. \tag{3}$$

To this end, introduce the following change of coordinates: For $j \in J(x_0)$, set

$$z_j \ = \ e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j},$$

and for $k \notin J(x_0)$, set $z_k = r_k e^{\sqrt{-1}\vartheta_k}$. The differentials are

$$\begin{aligned} dz_j &= \frac{1}{a_j} e^{\sqrt{-1}\vartheta_j} e^{\rho_j/a_j} d\rho_j + \sqrt{-1} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j} d\vartheta_j, \\ dz_k &= e^{\sqrt{-1}\vartheta_k} dr_k + \sqrt{-1} r_k e^{\sqrt{-1}\vartheta_k} d\vartheta_k. \end{aligned}$$

In these coordinates, the unit disk $\Delta = \{|w| < 1\}$, with $w = \prod_{j \in J(x_0)} z_j^{a_j}$, is inscribed by the relations:

$$\rho_j \ \leq \ 0, \quad 0 \ \leq \ r_k \ \leq \ 1, \quad \sum_{j \in J(x_0)} \rho_j \ = \ \log|w|, \quad \sum_{j \in J(x_0)} a_j \vartheta_j \ = \ \arg(w).$$

Let $J(x_0)^* := J(x_0) - \{j_0\}$ and compute:

$$\begin{aligned} & dz^{\hat{j}_0} \\ = \quad & \bigwedge_{j \in J(x_0)^*} dz_j \bigwedge_{k \notin J(x_0)} dz_k \\ = \quad & \left( \bigwedge_{j \in J(x_0)^*} \frac{1}{a_j} e^{\sqrt{-1}\vartheta_j} e^{\rho_j/a_j} d\rho_j + \sqrt{-1} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j} d\vartheta_j \right) \wedge \left( \bigwedge_{k \notin J(x_0)} e^{\sqrt{-1}\vartheta_k} dr_k + \sqrt{-1} r_k e^{\sqrt{-1}\vartheta_k} d\vartheta_k \right) \\ = \quad & \bigwedge_{j \in J(x_0)^*, k \notin J(x_0)} \left( \frac{1}{a_j} e^{\sqrt{-1}\vartheta_j} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_k} d\rho_j \wedge dr_k + \sqrt{-1} \frac{1}{a_j} e^{\sqrt{-1}\vartheta_j} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_k} d\rho_j \wedge d\vartheta_k \right) \\ & + \bigwedge_{j \in J(x_0)^*, k \notin J(x_0)} \left( \sqrt{-1} e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j} e^{\sqrt{-1}\vartheta_k} d\vartheta_j \wedge dr_k - e^{\rho_j/a_j} e^{\sqrt{-1}\vartheta_j} d\vartheta_j \wedge d\vartheta_k \right) \\ =: \quad & \bigwedge_{j \in J(x_0)^*, k \notin J(x_0)} \left( A_{j,k} d\rho_j \wedge dr_k + B_{j,k} d\rho_j \wedge d\vartheta_k + C_{j,k} d\vartheta_j \wedge dr_k + D_{j,k} d\vartheta_j \wedge d\vartheta_k \right). \end{aligned}$$

Then

$$\begin{aligned}
dz^{\hat{j_0}} \wedge d\bar{z}^{\hat{j_0}} &= \bigwedge_{j\in J(x_0)^*, k\notin J(x_0)} (A_{j,k} d\rho_j \wedge dr_k + B_{j,k} d\rho_j \wedge d\vartheta_k + C_{j,k} d\vartheta_j \wedge dr_k + D_{j,k} d\vartheta_j \wedge d\vartheta_k) \\
&\quad \wedge \bigwedge_{j\in J(x_0)^*, k\notin J(x_0)} \left(\overline{A}_{j,k} d\rho_j \wedge dr_k + \overline{B}_{j,k} d\rho_j \wedge d\vartheta_k + \overline{C}_{j,k} d\vartheta_j \wedge dr_k + \overline{D}_{j,k} d\vartheta_j \wedge d\vartheta_k\right) \\
&= \bigwedge_{j\in J(x_0)^*, k\notin J(x_0)} A_{j,k}\overline{D}_{j,k} d\rho_j \wedge dr_k \wedge d\vartheta_j \wedge d\vartheta_k + D_{j,k}\overline{A}_{j,k} d\vartheta_j \wedge d\vartheta_k \wedge d\rho_j \wedge dr_k \\
&= \bigwedge_{j\in J(x_0)^*, k\notin J(x_0)} \frac{1}{a_j} e^{2\rho_j/a_j} e^{2\sqrt{-1}\vartheta_j} d\rho_j \wedge dr_k \wedge d\vartheta_j \wedge d\vartheta_k,
\end{aligned}$$

and subsequently,

$$\begin{aligned}
\mu_0 \wedge \overline{\mu}_0 &= \frac{1}{|w|^2} \frac{|z_{j_0}|^2}{a_{j_0}^2} \frac{1}{a_j} e^{2\rho_j/a_j} e^{2\sqrt{-1}\vartheta_j} d\rho_j \wedge dr_k \wedge d\vartheta_j \wedge d\vartheta_k \\
&= \frac{1}{|w|^2} \prod_{j\in J(x_0)} \frac{1}{a_j^2} e^{2\rho_j/a_j} d\rho_j \prod_{\ell\in J(x_0)^*} e^{2\sqrt{-1}\vartheta_\ell} d\vartheta_\ell \bigwedge_{k\notin J(x_0)} dr_k.
\end{aligned}$$

Equation (3) is then given by

$$\begin{aligned}
&\int_{V_w} \prod_{j\in J(x_0)} |z_j|^{2k_j} \mu_0 \wedge \overline{\mu}_0 \\
&= \frac{1}{|w|^2} \int_{V_w} \prod_{j\in J(x_0)} \frac{1}{a_j^2} e^{2k_j \frac{\rho_j}{a_j} + 2\rho_j/a_j} d\rho_j \bigwedge_{\ell\in J(x_0)^*} e^{2\sqrt{-1}\vartheta_\ell} d\vartheta_\ell \bigwedge_{k\notin J(x_0)} dr_k \\
&= \frac{1}{|w|^2} \int_{V_w} \prod_{j\in J(x_0)} \frac{1}{a_j^2} e^{2\left(\frac{k_j+1}{a_j}\right)\rho_j} d\rho_j \bigwedge_{\ell\in J(x_0)^*} e^{2\sqrt{-1}\vartheta_\ell} d\vartheta_\ell \bigwedge_{k\notin J(x_0)} dr_k \\
&= \frac{1}{|w|^{2\left(1-\frac{k_{j_0}+1}{a_{j_0}}\right)}} \int_{V_w} \prod_{j\in J(x_0)} \frac{1}{a_j^2} e^{2\left(\frac{k_j+1}{a_j} - \frac{k_{j_0}+1}{a_{j_0}}\right)\rho_j} d\rho_j \bigwedge_{\ell\in J(x_0)^*} e^{2\sqrt{-1}\vartheta_\ell} d\vartheta_\ell \bigwedge_{k\notin J(x_0)} dr_k \\
&\sim \frac{1}{|w|^{2\left(1-\frac{k_{j_0}+1}{a_{j_0}}\right)}} \int_{V_w} \prod_{j\in J(x_0)} e^{2\left(\frac{k_j+1}{a_j} - \frac{k_{j_0}+1}{a_{j_0}}\right)\rho_j} d\rho_j.
\end{aligned}$$

**Claim:** We claim that

$$\int_{V_w} \prod_{j\in J(x_0)} e^{2\left(\frac{k_j+1}{a_j} - \frac{k_{j_0}+1}{a_{j_0}}\right)\rho_j} d\rho_j \;\sim\; (-\log|w|)^{N_0-1},$$

where $N_0$ is the cardinality of the set $\Omega \;:=\; \{j \in J(x_0) : \frac{k_j+1}{a_j} = \frac{k_{j_0}+1}{a_{j_0}}\}$.

Set

$$\alpha_j \quad := \quad 2\left(\frac{k_j+1}{a_j} - \frac{k_{j_0}+1}{a_{j_0}}\right)$$

and assume that $1, ..., N_0 \in \mathcal{Q}$, i.e., $\alpha_1, ..., \alpha_{N_0} = 0$; while $N_0+1, ..., \ell \notin \mathcal{Q}$, i.e., $\alpha_{N_0+1}, ..., \alpha_\ell$ do not vanish.[2] It will be convenient to introduce the following notations: Set

$$S_a^b := \sum_{k=a}^{b} \rho_k, \qquad A := \log|w|.$$

Hence, we need to compute

$$\mathcal{I} \quad := \quad \int_A^0 \int_{A-\rho_\ell}^0 \int_{A-S_{\ell-1}^\ell}^0 \cdots \int_{A-S_2^\ell} e^{\alpha_{N_0+1}\rho_{N_0+1}+\cdots+\alpha_\ell\rho_\ell} d\rho_1 \cdots d\rho_\ell.$$

The integrand does not depend on the first $N_0$ variables, so we may write $\mathcal{I} := \mathcal{I}_1 \cdot \mathcal{I}_2$, where

$$\mathcal{I}_1 \quad := \quad \int_{A-S_{N_0+1}^\ell}^0 \cdots \int_{A-S_2^\ell}^0 d\rho_1 \cdots d\rho_{N_0},$$

and

$$\mathcal{I}_2 \quad := \quad \int_A^0 \int_{A-\rho_\ell}^0 \cdots \int_{A-S_{N_0+2}^\ell}^0 e^{\alpha_{N_0+1}\rho_{N_0+1}+\cdots+\alpha_\ell\rho_\ell} d\rho_{N_0+1} d\rho_{N_0+2} \cdots d\rho_\ell$$

Let us mention explicitly the following elementary observation:

$$S_a^b = \rho_a + S_{a+1}^b.$$

Calculate the three inner most integrals of $\mathcal{I}_1$ to illucidate the iterative nature of the computation:

$$\begin{aligned}
\int_{A-S_4^\ell}^0 \int_{A-S_3^\ell}^0 \int_{A-S_2^\ell}^0 d\rho_1 d\rho_2 d\rho_3 \quad &= \quad \int_{A-S_4^\ell}^0 \int_{A-S_3^\ell}^0 (S_2^\ell - A) d\rho_2 d\rho_3 \\
&= \quad \int_{A-S_4^\ell}^0 \int_{A-S_3^\ell}^0 (\rho_2 + S_3^\ell - A) d\rho_2 d\rho_3 \\
&= \quad \int_{A-S_4^\ell}^0 (S_3^\ell - A)^2 d\rho_3 \\
&= \quad \int_{A-S_4^\ell}^0 (\rho_3 + S_4^\ell - A)^2 d\rho_3 \\
&= \quad \frac{1}{2}(S_4^\ell - A)^3 \quad = \quad \frac{1}{2}(\rho_4 + S_5^\ell - A)^3.
\end{aligned}$$

The general formula for the iteration is

$$\int_{A-S_k^\ell}^0 \frac{1}{(k-2)!}(S_{k-1}^\ell - A)^{k-2} d\rho_{k-1} \quad = \quad \frac{1}{(k-1)!}(S_k^\ell - A)^{k-1}$$

[2]We can assume without loss of generality that $\alpha_{N_0+1}, ..., \alpha_\ell$ are positive.

Therefore,

$$\mathcal{I}_1 \quad = \quad \int_{A-S^{\ell}_{N_0+1}}^{0} \frac{1}{(N_0-1)!}(S^{\ell}_{N_0}-A)^{N_0-1}d\rho_{N_0} \quad = \quad \frac{1}{N_0!}(S^{\ell}_{N_0+1}-A)^{N_0},$$

which can be written explicitly as

$$\mathcal{I}_1 \quad = \quad \frac{1}{N_0!}\left(\sum_{k=N_0+1}^{\ell}\rho_k - \log|w|\right)^{N_0} \quad = \quad \frac{1}{N_0!}\sum_{r=1}^{N_0}\binom{N_0}{r}\left(\sum_{k=N_0+1}^{\ell}\rho_k\right)^{r}(-\log|w|)^{N_0-r}.$$

Now

$$\begin{aligned}
&\int_{A-S^{\ell}_{N_0+2}}^{0} e^{\alpha_{N_0+1}\rho_{N_0+1}+\cdots+\alpha_\ell\rho_\ell}\mathcal{I}_1(\rho_{N_0+1},...,\rho_\ell)d\rho_{N_0+1}\\
=\quad &\frac{1}{N_0!}e^{\alpha_{N_0+2}\rho_{N_0+2}+\cdots+\alpha_\ell\rho_\ell}\int_{A-S^{\ell}_{N_0+2}}^{0} e^{\alpha_{N_0+1}\rho_{N_0+1}}(\rho_{N_0+1}+S^{\ell}_{N_0+2}-A)^{N_0}d\rho_{N_0+1}.
\end{aligned}$$

Recall the following integration by parts fromula:

$$\int e^{ax}(x+b)^n dx \quad = \quad e^{ax}\sum_{k=0}^{n}(-1)^{n-k}\frac{n!}{k!a^{n-k+1}}(x+b)^k.$$

Hence,

$$\begin{aligned}
&\int_{A-S^{\ell}_{N_0+2}}^{0} e^{\alpha_{N_0+1}\rho_{N_0+1}}(\rho_{N_0+1}+S^{\ell}_{N_0+2}-A)^{N_0}d\rho_{N_0+1}\\
=\quad & e^{\alpha_{N_0+1}\rho_{N_0+1}}\sum_{k=0}^{N_0}(-1)^{N_0-k}\frac{N_0!}{k!\alpha_{N_0+1}^{N_0-k+1}}(\rho_{N_0+1}+S^{\ell}_{N_0+2}-A)^k\Bigg|_{A-S^{\ell}_{N_0+2}}^{0}\\
=\quad & \sum_{k=0}^{N_0}(-1)^{N_0-k}\frac{N_0!}{k!\alpha_{N_0+1}^{N_0-k+1}}(S^{\ell}_{N_0+2}-A)^k
\end{aligned}$$

and by inserting this into the above expression, we get

$$\begin{aligned}
&\int_{A-S^{\ell}_{N_0+2}}^{0} e^{\alpha_{N_0+1}\rho_{N_0+1}+\cdots+\alpha_\ell\rho_\ell}\mathcal{I}_1(\rho_{N_0+1},...,\rho_\ell)d\rho_{N_0+1}\\
=\quad & \sum_{k=0}^{N_0}(-1)^{N_0-k}\frac{1}{k!\alpha_{N_0+1}^{N_0-k+1}}(S^{\ell}_{N_0+2}-A)^k e^{\alpha_{N_0+2}\rho_{N_0+2}+\cdots+\alpha_\ell\rho_\ell}
\end{aligned}$$

The next integral to be evaluated is

$$\sum_{k=0}^{N_0}(-1)^{N_0-k}\frac{1}{k!\alpha_{N_0+1}^{N_0-k+1}}e^{\alpha_{N_0+3}\rho_{N_0+3}+\cdots+\alpha_\ell\rho_\ell}\int_{A-S^{\ell}_{N_0+3}}^{0}(S^{\ell}_{N_0+2}-A)^k e^{\alpha_{N_0+2}\rho_{N_0+2}}d\rho_{N_0+2}.$$

Applying the integration by parts formula again, we have

$$\begin{aligned}
&\int_{A-S^{\ell}_{N_0+3}}^{0} (S^{\ell}_{N_0+2} - A)^k e^{\alpha_{N_0+2}\rho_{N_0+2}} d\rho_{N_0+2} \\
= \; &\sum_{j=0}^{k} (-1)^{k-j} \frac{k!}{j! \alpha_{N_0+2}^{k-j+1}} (S^{\ell}_{N_0+3} - A)^j.
\end{aligned}$$

So the integral is

$$\sum_{k=0}^{N_0} \sum_{j=0}^{k} (-1)^{N_0-j} \frac{1}{j! \alpha_{N_0+1}^{N_0-k+1} \alpha_{N_0+2}^{k-j+1}} (S^{\ell}_{N_0+3} - A)^j.$$

The final stage of the integration computation is then

$$\begin{aligned}
&\sum_{i=1}^{\ell-N_0+1} \sum_{k_i=0}^{k_{i-1}} \frac{(-1)^{N_0-k_{\ell-N_0+1}}}{(k_\ell - N_0+1)!} \left( \prod_{t=1}^{\ell-N_0} \alpha_{N_0+t}^{1+k_{t-1}-k_{t-2}} \right)^{-1} \int_A^0 (\rho_\ell - A)^{k_\ell - N_0+1} \\
= \; &\sum_{i=1}^{\ell-N_0+1} \sum_{k_i=0}^{k_{i-1}} \frac{(-1)^{N_0-k_{\ell-N_0+1}}}{(k_\ell - N_0+1)!} \left( \prod_{t=1}^{\ell-N_0} \alpha_{N_0+t}^{1+k_{t-1}-k_{t-2}} \right)^{-1} \cdot \frac{1}{k_\ell+2-N_0} (-A)^{k_\ell+2-N_0} \\
= \; &\sum_{i=1}^{\ell-N_0+1} \sum_{k_i=0}^{k_{i-1}} \frac{(-1)^{N_0-k_{\ell-N_0+1}}}{(k_\ell - N_0+2)!} \left( \prod_{t=1}^{\ell-N_0} \alpha_{N_0+t}^{1+k_{t-1}-k_{t-2}} \right)^{-1} (-A)^{k_\ell+2-N_0}
\end{aligned}$$

Recall that we have $A = \log|w|$, so we can offer the more explicit expression:

$$\sum_{k_1=0}^{N_0} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\ell-N_0+1}=0}^{k_{\ell-N_0}} \frac{(-1)^{N_0-k_{\ell-N_0+1}}}{(k_\ell-N_0+1)!} \frac{1}{\alpha_{N_0+1}^{1-k_0-k_{-1}} \alpha_{N_0+2}^{1-k_1-k_{-1}} \cdots \alpha_\ell^{1+k_{\ell-N_0-1}-k_{\ell-N_0-2}}} (-\log|w|)^{k_\ell+2-N_0}$$

If $N_0 = 1$, we get

$$\sum_{k_1=0}^{1} \sum_{k_2=0}^{k_1} \cdots \sum_{k_{\ell-1}=0}^{k_{\ell-1}} \frac{(-1)^{1-k_\ell}}{k_\ell!} \frac{1}{\alpha_2^{1-k_0-k_{-1}} \alpha_3^{1-k_1-k_{-1}} \cdots \alpha_\ell^{1+k_{\ell-2}-k_{\ell-3}}} (-\log|w|)^{k_\ell+1}$$

**Example case 1.** Consider the case when $J(x_0) = 4$ and $N_0 = 2$. Then

$$\begin{aligned}
&\int_A^0 \int_{\log|w|-\rho_4}^0 \int_{\log|w|-S_3^4}^0 \int_{\log|w|-S_2^4}^0 e^{\alpha_3\rho_3+\alpha_4\rho_4} d\rho_1 d\rho_2 d\rho_3 d\rho_4 \\
= \; &\frac{1}{\alpha_3-\alpha_4} \left\{ \frac{1}{\alpha_3^3}|w|^{\alpha_3} - \frac{1}{\alpha_3^2}\log|w| - \frac{1}{2\alpha_3}(\log|w|)^2 - \frac{1}{\alpha_3^3} \right\} \\
&+ \frac{1}{\alpha_4-\alpha_3} \left\{ \frac{1}{\alpha_4^3}|w|^{\alpha_4} - \frac{1}{\alpha_4^2}\log|w| - \frac{1}{2\alpha_4}(\log|w|)^2 - \frac{1}{\alpha_4^3} \right\}
\end{aligned}$$

**Example case 2.** Consider the case when $J(x_0) = 5$ and $N_0 = 1$. Then

$$\int_A^0 \int_{A-\rho_5}^0 \int_{A-S_4^5}^0 \int_{A-S_3^5}^0 \int_{A-S_2^5}^0 e^{\alpha_2\rho_2+\alpha_3\rho_3+\alpha_4\rho_4+\alpha_5\rho_5} d\rho_1 \cdots d\rho_5$$

$\square$