Complex Manifolds of Hyperbolic and Non-Hyperbolic-Type

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Dedicated to the memory of
Alexander Isaev

Declaration

The present manuscript accounts for the research undertaken at the Mathematical Research Institute, the Australian National University, and the Beijing International Center for Mathematical Research, Peking University, between January 1st, 2019 and July 31st, 2022. Except where acknowledged in the customary manner, the material presented here is, to the best of the author's knowledge, original, and has not been submitted in whole or part for a degree in any university.

Kyle Broder
July, 2022

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Throughout my Ph.D., I have been lucky to meet and speak with several mathematicians.

I owe an incredible amount to Zhou Zhang, who shaped my early technique, knowledge, and research ability. It was upon his suggestion to establish the partial second-order estimate with $\varepsilon > 0$ arbitrarily small in [**47, 48**].

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I've been fortunate to have many communications with Franc Forstnerič concerning several topics in complex geometry. His Oka manifolds appear in parts of this thesis, but unfortunately, to no significant extent. The ideas in this thesis will hopefully be extended to the *positive case* which hopefully gives insight into a curvature characterization of Oka manifolds.

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Preface

The present manuscript gives an account of the research undertaken from January 1st, 2019, to July 31st, 2022, under the supervision of Ben Andrews and Gang Tian. Outside of the Ph.D. requirement, the purpose of the present thesis is to provide (at least the author) with a useful reference on the curvature aspects of Hermitian manifolds. At the time of writing this, many texts exist concerning the geometry of Kähler manifolds, but not many of them have considered the general Hermitian category. The best reference at present appears to be Zheng's Complex Differential Geometry [**324**], but the focus of this beautiful book is also not on Hermitian manifolds.

In many respects, Hermitian (non-Kähler) differential geometry remains in its infancy. The presence of torsion in the natural connections which reside on the tangent bundle of a non-Kähler Hermitian manifold renders the subject formidable to outsiders, and this is undoubtedly exacerbated by the absence of books on the subject.

The guiding narrative behind this thesis is to understand how the curvature of Hermitian metrics, which reside on a complex manifold, influences the complex geometry. The primary example of this type of investigation is in understanding when a sign on some curvature of a Hermitian metric force the manifold to be a known class (e.g., Kobayashi hyperbolic, Brody hyperbolic, Oka, homogeneous, etc.). This is the reason for titling the manuscript *Complex Manifolds of Hyperbolic and Non-Hyperbolic-Type*.

One of the central tools in studying hyperbolicity (and non-hyperbolicity) utilizing the curvature of Hermitian metrics is the Schwarz lemma (sometimes called the Yau–Schwarz lemma or Ahlfors–Schwarz lemma in this context). The new results established by the author primarily concern refinements and improvements on the Schwarz lemma (exhibited in §2.5 and §2.6), which appear in [**49**, **50**]. However, a better understanding of the Schwarz lemma (especially in the Hermitian category) requires an improved understanding of the holomorphic sectional curvature and Ricci curvature of a Hermitian metric. This, in turn, furnished some of the results in §2.3 and §2.4, which appear in [**57**, **58**, **59**, **60**, **52**, **51**, **53**].

Novel contributions. Throughout January 1st, 2019, and July 31st, 2022, the author, together with his collaborators, produced (or is currently in the process of preparing) [**47**, **48**, **49**, **50**, **51**, **52**, **53**, **54**, **55**, **56**, **57**, **58**, **59**, **60**, **61**]. The present version of the manuscript omits [**47**, **48**, **56**, **60**] because these results deviate from the central narrative.

Not all results of the remaining papers are included either; the main factor in this is time. For instance, the present manuscript only begins to glimpse the results appearing in [**57**, **58**].

Remarks on the structure of the manuscript. Throughout the manuscript, we have included a number of questions. These are not to be interpreted as exercises or difficult open problems. To the author's knowledge, the questions posed throughout the text are open and act as both a prompt to the author and the reader to address these gaps in the literature. Communication[1] of the answers to these questions is warmly encouraged, together with comments, feedback, and any suggested questions.

Remarks on exposition. Our exposition borrows from a number of places: For the geometry of smooth manifolds [**10**, **41**, **114**, **227**, **256**, **328**]. The theory of complex manifolds is borrowed from [**21**, **30**, **134**, **137**, **138**, **142**, **151**, **168**, **188**, **214**, **260**, **274**, **295**, **300**, **324**]. Sheaf theory is referenced from [**46**, **137**, **138**, **152**].

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Introduction

A complex manifold is a space that is locally modeled on $\mathbb{C}^n$ by maps that preserve the complex-analytic data. Such objects are significantly more rigid than their smooth, real counterparts. The most notable example of this is the failure of the holomorphic analog of the Whitney embedding theorem: If X is a compact complex manifold, then there is no holomorphic embedding $X \hookrightarrow \mathbb{C}^n$, for any $n \in \mathbb{N}$, unless X is a point. Indeed, if such an embedding exists, the coordinate functions on $\mathbb{C}^n$ restrict to holomorphic functions on a compact set. Hence, these functions must be constant by the maximum principle, and X must be a point.

One of the primary aims of complex geometry is to understand all types of complex manifolds. Complex manifolds lie at the intersection of many fields, and these different fields have approached the problem of classification from different perspectives. A very natural approach to studying the geometry of a complex manifold X is given by looking at the holomorphic functions $f : X \to \mathbb{C}$ which emanate from it. For instance, Stein manifolds [**137**], the manifolds which admit holomorphic embeddings into some $\mathbb{C}^N$ are intuitively described as the class of complex manifolds with an abundance of holomorphic functions [**188**].

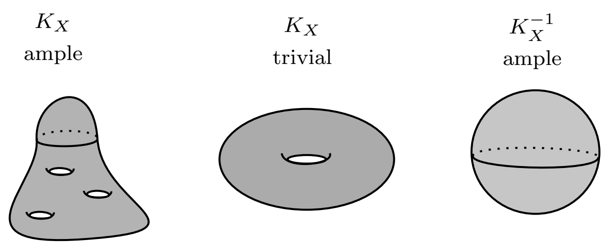
Of course, this method of investigation is useless if X is compact. In algebraic geometry, the natural way of rectifying this austerity of holomorphic functions is to realize holomorphic functions as holomorphic sections of the trivial line bundle $\mathbb{C} \to X$. Hence, one could instead study complex manifolds by considering holomorphic sections of line bundles $\mathcal{L} \to X$ instead of simply holomorphic functions. There is only one God-given line bundle intrinsic to any complex manifold, namely, the canonical bundle K_X – the top exterior power of the cotangent bundle[2]. If X is a complex manifold of (complex) dimension n, the sections of K_X are given by

$$f(z)dz^1 \wedge \cdots \wedge dz^n,$$

where $(z_1, ..., z_n)$ are local holomorphic coordinates, and f is a (locally-defined) holomorphic function on X.

[2]This is a less-famous, but still well-known, naturally occurring line bundle on a complex manifold. If X is a complex manifold with tangent bundle $T^{1,0}X$, then the projectivization $\mathbb{P}(T^{1,0}X)$ supports a hyperplane bundle $\mathcal{O}_{\mathbb{P}(T^{1,0}X)}(1)$.

One can now study the complex manifold X by looking at the 'amount of sections' of K_X. More precisely, a line bundle $\mathcal{L} \to X$ is said to be *ample* if the sections of $\mathcal{L}^{\otimes k}$ (for $k \in \mathbb{N}$ large) furnish a holomorphic embedding $\Phi : X \longrightarrow \mathbb{P}^{N_k}$. For compact Riemann surfaces (i.e., complex manifolds of (complex) dimension 1)[3], the ampleness of K_X (which in dimension 1, is just the cotangent bundle $\Omega_X^{1,0}$) is intimately related to the topological genus $g = \frac{1}{2}b_1(X)$. That is, if we let X be a compact Riemann surface of genus g, the canonical bundle K_X is ample for $g \geq 2$, trivial for $g = 1$, and for $g = 0$, the dual of the canonical bundle (namely, the anti-canonical bundle K_X^{-1}) is ample.



In complex analysis, holomorphic functions $X \to \mathbb{C}$ are replaced by holomorphic maps $\mathbb{C} \to X$ (i.e., entire curves). Or more generally, holomorphic maps $S \to X$ from a complex manifold *into* X. In this context, we do not think of $\mathbb{C}$ as a trivial line bundle but think of $\mathbb{C}$ as a Stein manifold. Then look at holomorphic maps $S \to X$ from a Stein manifold S into X.
This observation leads to a paradigm that splits into four distinct categories: The complex manifolds X for which (i) holomorphic functions are abundant $X \to \mathbb{C}$ (i.e., Stein manifolds); (ii) there are no holomorphic maps $\mathbb{C} \to X$ (i.e., Brody hyperbolic manifolds); (iii) there is an abundance of holomorphic maps $S \to X$ from a Stein manifold S (i.e., Oka manifolds); (iv) there are no holomorphic functions $X \to \mathbb{C}$. There are only three categories since class (iv) is too large to form any concrete structure.
Both classification schemes have their advantages and drawbacks. Both approaches have developed a large amount of machinery for studying such manifolds, and there has been extensive effort aiming to bridge these worlds. We also hope for a robust method of classification and techniques that can be used and applied to obtain and check examples. One of the most successful techniques of this type has been to attach smooth (and hence, more flexible) objects to the complex manifold, which preserves the (more rigid) complex-analytic structure—the primary example of this being a Hermitian metric. From the additional metric structure, one can look at its curvature and related invariants as a mode of inquiry into the underlying geometry.

[3]In algebraic geometry, these objects are sometimes called *curves*. This terminology, however, may be reserved for (compact) Riemann surfaces endowed with an embedding into some projective space.

The Newlander–Nirenberg theorem [220] states that the complex-analytic structure X of a complex manifold is encoded in an (integrable) endomorphism $J : TX \to TX$ satisfying $J^2 = -\mathrm{id}$ (as a morphism of bundles). Let g be a Hermitian metric on X in the sense that $g(Ju, Jv) = g(u, v)$ for all $u, v \in TX$. Highschool linear algebra tells us that the endomorphism J induces an embedding of the space of quadratic forms into the space of 2–forms via the map

$$g(\cdot, \cdot) \ \mapsto \ \omega_g(\cdot, \cdot) := g(J\cdot, \cdot).$$

Since an arbitrary Hermitian metric can be quite unwieldy, in general, it is common to afford oneself additional structure on the metric. The greatest success in this direction is our understanding of Hermitian metrics for which the 2–form ω_g is closed. These metrics were introduced by Schouten [246] and Kähler [177].

By Chern–Weil theory, the first Chern–Ricci curvature presents the first Chern class of the anti-canonical bundle. In particular, a sign on the Ricci curvature determines the ampleness of the anti-canonical bundle. Conversely, by Yau's solution of the Calabi conjecture [68, 318], the ampleness of the anti-canonical bundle yields the existence of a Kähler metric whose Ricci curvature is positive.

On the other hand, by the Ahlfors' Schwarz lemma [2], a complex manifold X with a Hermitian metric of negative (Chern) holomorphic sectional curvature is Brody hyperbolic. This observation goes back to Grauert–Reckziegel [136]. In particular, by Brody's theorem [62], if X is compact, then X is Kobayashi hyperbolic.

The converse was conjectured for many years:

Conjecture. A compact Kobayashi hyperbolic manifold admits a Hermitian metric of negative (Chern) holomorphic sectional curvature.

There is a large amount of evidence for the conjecture: Grauert–Reckziegel [136] produced a Hermitian metric of negative holomorphic sectional curvature in a neighborhood of a fiber of an analytic family of compact Riemann surfaces of genus $g \geq 2$ over a Riemann surface. This local (in a neighborhood of a fiber) construction was extended to all dimensions by Cowen [99]. The first global construction[4] was due to Cheung [94, 95]: The total space of a holomorphic submersion $f : X \to Y$ from a compact complex manifold X into a Hermitian manifold (Y, ω_h) with ${}^c\mathrm{HSC}_{\omega_h} \leq -\kappa < 0$ such that there is a smooth family of Hermitian metrics ω_y with ${}^c\mathrm{HSC}_{\omega_y} \leq -\kappa_y < 0$ on the fibers $f^{-1}(y)$, $y \in Y$, has a Hermitian metric of negative (Chern) holomorphic sectional curvature. See [79] for results in the positive holomorphic sectional curvature direction.

[4] There was related works prior due to Deschamps-Martin [109] and Schneider [243] who showed that the Kodaira surfaces of general type [186] have negative tangent bundle in the sense of Grauert.

Despite the growing evidence for this conjecture, Demailly [105] constructed a counterexample to *Conjecture* by producing a compact projective Kobayashi hyperbolic surface with a fiber sufficiently singular that it violates the following algebraic hyperbolicity criterion:

Theorem. Let (X, ω) be a compact Hermitian manifold with ${}^c\mathrm{HSC}_\omega \leq \kappa_0$ for some $\kappa_0 \in \mathbb{R}$. If $f : \mathcal{C} \to X$ is a non-constant holomorphic map from a compact Riemann surface $\mathcal{C}$ of genus g, then

$$2g - 2 \quad \geq \quad -\frac{\kappa_0}{2\pi} \deg_\omega(\mathcal{C}) - \sum_{p \in \mathcal{C}} (m_p - 1).$$

In particular, the negativity of the (Chern) holomorphic sectional curvature does not characterize (compact, or even projective) Kobayashi hyperbolic manifolds[5].

An important conjecture concerning the relationship between hyperbolicity in complex analysis (i.e., Kobayashi hyperbolicity) and hyperbolicity in algebraic geometry (i.e., ampleness of the canonical bundle) is the (Hermitian extension of the) Kobayashi conjecture[6]:

Conjecture. A compact Kobayashi hyperbolic manifold has ample canonical bundle.

It is hard to work with the Kobayashi hyperbolic assumption directly. It is simpler to work with the additional structure of a Hermitian metric with negative (Chern) holomorphic sectional curvature. This recovers the (Hermitian extension of the) Kobayashi–Yau conjecture:

Conjecture. A compact Hermitian manifold with a metric of negative (Chern) holomorphic sectional curvature has ample canonical bundle.

The above conjecture was verified for Kähler surfaces by Wong [302] and Campana [70] using the classification theory of Enriques and Kodaira. The first significant leap forward came from Heier–Lu–Wong [155] for projective threefolds. This was extended to projective manifolds of any dimension by (Wong–)Wu–Yau [303, 304]. The projective assumption was then dropped by Tosatti–Yang [286]:

Theorem. (Wu–Yau theorem). A compact Kähler manifold with a Kähler metric of negative holomorphic sectional curvature has ample canonical bundle.

[5]It remains open as to whether there is a curvature characterization of Kobayashi hyperbolic manifolds. The leading candidate is Demailly's *jet curvature* (see [105]).

[6]To the author's knowledge, Kobayashi only conjectured this for projective Kobayashi hyperbolic manifolds [184].

The Wu–Yau theorem hinges upon the Schwarz lemma. As a consequence, the bloodline of the present manuscript is developing the Schwarz lemma.

New results. Given the virgin nature of the differential geometry of Hermitian manifolds, many of the new results contained in the present manuscript are *foundational* in a sense. For instance, in [57], in a joint work with James Stanfield, the author proved the following relations among the t–Gauduchon–Ricci curvatures (see 2.3.24):

Theorem. Let (X, ω) be a Hermitian manifold. The Gauduchon–Ricci curvatures are given by

$$\begin{aligned}
{}^t\mathrm{Ric}^{(1)}_{\omega_g} &= {}^c\mathrm{Ric}^{(1)}_{\omega_g} + \frac{(t-1)}{2}(\partial\partial^*\omega_g + \bar{\partial}\bar{\partial}^*\omega_g), \\
{}^t\mathrm{Ric}^{(2)}_{\omega_g} &= t\,{}^c\mathrm{Ric}^{(2)}_{\omega_g} + (1-t)\,{}^c\mathrm{Ric}^{(1)}_{\omega_g} + \frac{(t-1)}{2}(\partial\partial^*\omega_g + \bar{\partial}\bar{\partial}^*\omega_g) + \frac{(1-t)^2}{4}\,{}^cT^{\diamondsuit} + \frac{(t-1)}{2}\,{}^cT^{\circ}, \\
{}^t\mathrm{Ric}^{(3)}_{\omega_g} &= t\,{}^c\mathrm{Ric}^{(3)}_{\omega_g} + \frac{(1-t)}{2}\left({}^c\mathrm{Ric}^{(1)}_{\omega_g} + {}^c\mathrm{Ric}^{(2)}_{\omega_g}\right) - \frac{(1-t)^2}{4}\,{}^cT^{\diamondsuit} + \frac{(1-t)^2}{4}\,{}^cT^{\heartsuit}, \\
{}^t\mathrm{Ric}^{(4)}_{\omega_g} &= t\,{}^c\mathrm{Ric}^{(4)}_{\omega_g} + \frac{(1-t)}{2}\left({}^c\mathrm{Ric}^{(1)}_{\omega_g} + {}^c\mathrm{Ric}^{(2)}_{\omega_g}\right) - \frac{(1-t)^2}{4}\,{}^cT^{\diamondsuit} + \frac{(1-t)^2}{4}\,\overline{{}^cT^{\heartsuit}}.
\end{aligned}$$

Liu–Yang [200] showed that the first Aeppli cohomology $c_1^{\mathrm{AC}}(K_X^{-1}) \in H_A^{1,1}(X)$ was represented by the first Lichnerowicz Ricci curvature (the first Ricci curvature of the $t = 0$ Gauduchon connection). The above theorem shows that the Aeppli cohomology class is represented by the first Ricci curvature of any t–Gauduchon connection. In particular, a number of the results which have only been established for the first Lichnerowicz Ricci curvature, readily extend to the first t–Gauduchon Ricci curvature for any $t \in \mathbb{R}$. For instance, we have (see 2.3.31):

Theorem. Let (X, ω) be a Hermitian manifold. The first Gauduchon–Ricci form ${}^t\mathrm{Ric}^{(1)}_{\omega}$ represents $c_1^{\mathrm{AC}}(K_X^{-1}) \in H_A^{1,1}(X)$ for all $t \in \mathbb{R}$. Moreover,

- (i) ${}^t\mathrm{Ric}^{(1)}_{\omega}$ is d–closed if and only if $\partial\bar{\partial}\bar{\partial}^*\omega = 0$.
- (ii) If $\bar{\partial}\partial^*\omega$, then ${}^t\mathrm{Ric}^{(1)}_{\omega}$ represents the $c_1(K_X^{-1}) \in H^2_{\mathrm{DR}}(X, \mathbb{R})$, i.e., $c_1(K_X^{-1}) = c_1^{\mathrm{AC}}(K_X^{-1})$.
- (iii) If ω is conformally balanced, then ${}^t\mathrm{Ric}^{(1)}_{\omega}$ represents $c_1(K_X^{-1}) \in H^{1,1}_{\bar{\partial}}(X)$ and also the first Bott–Chern class $c_1^{\mathrm{BC}}(K_X^{-1}) \in H^{1,1}_{\mathrm{BC}}(X)$.
- (iv) ${}^t\mathrm{Ric}^{(1)}_{\omega} = {}^s\mathrm{Ric}^{(1)}_{\omega}$ for $t \neq s$ if and only if ω is balanced.

Liu–Yang [200] (further refinements were made by Correa [98]) constructed first Lichnerowicz Ricci-flat metrics on Hopf manifolds $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. We extend their construction to produce first t–Gauduchon Ricci-flat metrics on Hopf manifolds $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ (see 2.3.36):

Theorem. Let (X, ω_0) be the Hopf manifold $X = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ endowed with the standard Boothby metric ω_0. The $(1,1)$–form

$$\omega \ := \ \omega_0 + \frac{4t(1-n)-4)}{n} {}^l\mathrm{Ric}_\omega^{(1)}$$

is a solution of the equation

$$^t\mathrm{Ric}_\omega^{(1)} \ = \ 0, \qquad \text{for all } t \in \mathbb{R} \backslash \{1\}.$$

Moreover, for all $t < 1$, the $(1,1)$–form ω is positive-definite and thus defines a first Gauduchon–Ricci-flat metric.

Remark. Observe that since $c_1^{\mathrm{BC}}(K_X^{-1}) \neq 0$ for $X = \mathbb{S}^{2n-1} \times \mathbb{S}^1$, there are no first Chern–Ricci-flat metrics on X.

For the t–Gauduchon holomorphic sectional curvature, we establish the following monotonicity theorem (see 2.5.27):

Theorem. Let (X, ω) be a Hermitian manifold. The t–Gauduchon altered holomorphic sectional curvature is given by

$$^t\widetilde{\mathrm{HSC}}_\omega \ = \ {}^c\widetilde{\mathrm{HSC}}_\omega - \frac{(t-1)^2}{4|v|_\omega^2} \sum_q \left({}^cT_{iq}^i \overline{{}^cT_{kq}^k} + {}^cT_{iq}^k \overline{{}^cT_{kq}^i} \right) v_i v_k. \tag{0.0.1}$$

In particular, ${}^t\widetilde{\mathrm{HSC}}_\omega \leq {}^c\widetilde{\mathrm{HSC}}_\omega$ for all $t \in \mathbb{R}$ and equality holds if and only if $t = 1$.

Since the altered holomorphic sectional curvature is comparable to the holomorphic sectional curvature, the following useful consequences of the above monotonicity result are easily obtained:

Corollary. Let (X, ω) be a Hermitian manifold.

(i) If ${}^c\mathrm{HSC}_\omega \leq 0$, then ${}^t\mathrm{HSC}_\omega \leq 0$ for all $t \in \mathbb{R}$.
(ii) If ${}^t\mathrm{HSC}_\omega > 0$ for some $t \in \mathbb{R}$, then ${}^c\mathrm{HSC}_\omega > 0$.

The understanding of the relations among the t–Gauduchon curvatures came from a desire to extend the known results for the Schwarz lemma, which held for the Chern connection, to the t–Gauduchon connection. The author, together with James Stanfield, recently established the following Schwarz lemma calculation in [58] (see 2.6.34):

Theorem. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Endow $T^{1,0}X$ with the s–Gauduchon connection ${}^s\nabla$ and endow $T^{1,0}Y$ with the t–Gauduchon connection ${}^t\nabla$. With respect to a local unitary frame such that $\partial f = f_i^\alpha = \lambda_i \delta_i^\alpha$, we have

$$\begin{aligned}
\Delta_{\omega_g} |\partial f|^2 \;\; = \;\; & |\nabla \partial f|^2 + \frac{s^2+2s-1}{2s(2s-1)} {}^s\mathrm{Ric}^{(2)}_{k\bar{k}} \lambda_k^2 + \frac{1-s}{4s(2s-1)} \left(2(1-s) {}^s\mathrm{Ric}^{(1)}_{k\bar{k}} - 2s({}^s\mathrm{Ric}^{(4)}_{k\bar{k}} + {}^s\mathrm{Ric}^{(3)}_{k\bar{k}}) \right) \lambda_k^2 \\
& + \frac{(s-1)^3}{8s^2(2s-1)} \left({}^sT_{ir}^i \overline{{}^sT_{kr}^k} + {}^sT_{kr}^k \overline{{}^sT_{ir}^i} \right) \lambda_k^2 + \frac{(s-1)(s^3+7s^2-5s+1)}{8s^3(2s-1)} {}^sT_{ir}^k \overline{{}^sT_{ir}^k} \lambda_k^2 \\
& + \frac{(1-s)(3s^3+7s^2-7s+1)}{8s^3(2s-1)} {}^sT_{kr}^i \overline{{}^sT_{kr}^i} \lambda_k^2 \\
& + \frac{t}{1-2t} \left({}^t\widetilde{R}_{\alpha\bar{\alpha}\beta\bar{\beta}} + {}^t\widetilde{R}_{\alpha\bar{\beta}\beta\bar{\alpha}} \right) \lambda_\alpha^2 \lambda_\beta^2 + \frac{1}{2t-1} {}^t\widetilde{R}_{\alpha\bar{\beta}\beta\bar{\alpha}} \lambda_\alpha^2 \lambda_\beta^2 \\
& + \frac{(1-t)^3}{8t^2(2t-1)} \left({}^t\widetilde{T}_{\alpha\gamma}^\alpha \overline{{}^t\widetilde{T}_{\beta\gamma}^\beta} + {}^t\widetilde{T}_{\beta\gamma}^\beta \overline{{}^t\widetilde{T}_{\alpha\gamma}^\alpha} \right) \lambda_\alpha^2 \lambda_\beta^2 \\
& + \frac{(t-1)^2(t+1)}{4t^2(2t-1)} {}^t\widetilde{T}_{\alpha\gamma}^\beta \overline{{}^t\widetilde{T}_{\alpha\gamma}^\beta} \lambda_\alpha^2 \lambda_\beta^2 + \frac{t-1}{t} {}^t\widetilde{T}_{\alpha\beta}^\gamma \overline{\widetilde{T}_{\alpha\beta}^\gamma} \lambda_\alpha^2 \lambda_\beta^2 \\
& + \left(\frac{1-t}{2t} - \frac{1-s}{2s} \right) \mathrm{Re} \left({}^sT_{ij}^k \overline{{}^t\widetilde{T}_{ij}^k} \right) \lambda_i \lambda_j \lambda_k.
\end{aligned}$$

Let us give the following specific instance of the above Schwarz lemma:

Theorem. (Bismut Schwarz Lemma). Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Assume (X, ω_g) is a compact balanced manifold with

$$3 {}^c\mathrm{Ric}^{(1)}_{\omega_g} - 3\sqrt{-1}\Lambda(\partial\bar{\partial}\omega_g) + {}^cT^\circ \;\; \geq \;\; -C_1\omega_g - C_2 f^*\omega_h,$$

for some constants $C_1, C_2 \in \mathbb{R}$. Assume $\left| {}^b\tilde{T} \right| \leq B$ and ${}^b\mathrm{RBC}_{\omega_h} + 2 {}^b\widetilde{\mathrm{RBC}}_{\omega_h} \leq \kappa_0$ for some constant $B, \kappa_0 \in \mathbb{R}$. Then

$$3\Delta_{\omega_g} |\partial f|^2 \;\; \geq \;\; -C_1 |\partial f|^2 - (C_2 + \kappa_0 + 2B) |\partial f|^4.$$

Hence, if $C_2 + \kappa_0 + 2B < 0$, then

$$|\partial f|^2 \;\; \leq \;\; -\frac{C_1}{C_2 + \kappa_0 + 2B}.$$

Even for the standard Schwarz lemma for the Chern connection, we include some novel developments. In [**49, 50**], the author introduced a refinement of the (Chern) real bisectional curvature ${}^c\mathrm{RBC}$ used by Yang–Zheng [**315**] to control the target curvature term appearing in the Schwarz lemma. The sharper curvature constraint we introduce is called the (Chern) *second Schwarz bisectional curvature* (see 2.6.19):

Theorem. Let $f : (X^n, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Assume

$$^c\mathrm{Ric}^{(2)}_{\omega_g} \geq -C_1\omega - C_2 f^*\eta$$

for constants $C_1, C_2 \in \mathbb{R}$. Assume that $^c\mathrm{SBC}^{(2)}_{\omega} \leq -\kappa_0$ for some constant $\kappa_0 \leq -C_2$. If X is compact, then

$$|\partial f|^2 \quad \leq \quad -\frac{C_1 r}{\kappa_0 + C_2},$$

where r is the rank of ∂f.

There are two primary issues in a further understanding of the Schwarz lemma:

(i) The first is how to relate the second Schwarz bisectional curvature to the holomorphic sectional curvature in the Hermitian non-Kähler-like category.
(ii) The second is how to relate the second Ricci curvature to the first Chern Ricci curvature.

The Hermitian extension of the Wu–Yau theorem capitalizes upon both of these problems. The assumption of the Hermitian Wu–Yau problem is negative (Chern) holomorphic sectional curvature, which is weaker than the (Chern) second Schwarz bisectional curvature. Moreover, the complex Monge–Ampère equation only gives control of the first Chern Ricci curvature, not the second Chern Ricci curvature which appears in the Schwarz lemma.

The main application of these Schwarz lemmas is the following version of the Wu–Yau theorem (see 3.1.15):

Theorem. Let (X, ω_g) be a compact Hermitian manifold with a Hermitian metric ω_h such that $^c\mathrm{SBC}^{(2)}_{\omega_h} \leq -\kappa_0 < 0$. If, for any $\varepsilon > 0$, the Hermitian metric

$$\omega_\varepsilon \ := \ (\varepsilon + \varepsilon_0)\omega_g - {}^c\mathrm{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar{\partial}u_\varepsilon$$

defined in (3.1.1) is Chern partially Kähler-like, then K_X is ample.

We say that a Hermitian metric is (Chern) *partially Kähler-like* if $^c\mathrm{Ric}^{(1)}_{\omega} = {}^c\mathrm{Ric}^{(2)}_{\omega}$. This class of manifolds is studied in [**57, 58**].

In the final sections of the present manuscript, we will consider a refinement of the Wu–Yau theorem, first established by Diverio–Trapani [**111**], which relaxed the negativity of the holomorphic sectional curvature to quasi-negativity (non-positive everywhere and negative at one point). One of the key results on which the theorem of Diverio–Trapani [**111**] hinges is the following:

Theorem. (Diverio–Trapani). Let (X^n, ω) be a compact Kähler manifold of (complex) dimension n. If the holomorphic sectional curvature of ω is quasi-negative, then

$$\int_X c_1(K_X)^n > 0.$$

We extended this result to the Hermitian category in a joint work with Kai Tang and Yashan Zhang [61]. The main theorem curiously does not require the curvature to have a sign. To state the main theorem, let us introduce the following terminology:

Definition. Let (X, ω) be a compact Kähler manifold. For positive constants $\delta_1, \delta_2 > 0$, we say that a Hermitian metric α on X is

(i) δ_1–*bounded* (relative to ω) if there is a smooth function $\psi : X \to \mathbb{R}$ such that

$$\alpha \ \leq \ \delta_1 \omega + \sqrt{-1}\partial\bar{\partial}\psi.$$

(ii) δ_2–*volume non-collapsed* on an open set $\mathcal{U} \subset X$ if $\alpha^n \geq \delta_2 \omega_0^n$.

A Hermitian metric α satisfying both (i) and (ii) is said to have (δ_1, δ_2)–*bounded geometry* (relative to ω and $\mathcal{U}$). The space of Hermitian metrics with (δ_1, δ_2)–bounded geometry (relative to ω_0 and $\mathcal{U}$) is denoted by $\mathcal{H}_{\delta_1,\delta_2}(\omega_0, \mathcal{U})$.

Let $\mathcal{F}_\omega$ be curvature function of the Hermitian metric[7] ω. For a positive constant $\delta > 0$, and a non-empty open set $\mathcal{U} \subset X$, we say that $\mathcal{F}_\omega$ is (ε, δ)–quasi-negative (relative to $\mathcal{U}$) if, there is a sufficiently small $\varepsilon > 0$ such that $\mathcal{F}_\omega \leq \varepsilon$ on X, and if $\mathcal{F}_\omega \leq -\delta$ on $\mathcal{U}$.

We now state the main theorem of [61] (see 3.3.6):

Theorem. Let (X^n, ω_0) be a compact Kähler manifold. Assume that there is a Hermitian metric $\eta \in \mathcal{H}_{\delta_1,\delta_2}(\omega_0, \mathcal{U})$ with (δ_1, δ_2)–bounded geometry. If the (Chern) second Schwarz bisectional curvature of η is (ε, δ)–quasi-negative (relative to $\omega_0, \mathcal{U}, \delta_1, \delta_2, \delta$), then

$$\int_X c_1(K_X)^n > 0.$$

In the final section of the manuscript, we discuss some of the results from [55] concerning the positive analog of the Wu–Yau theorem and other related questions. Hitchin's examples of Hodge metrics on Hirzebruch surfaces [161] show that a compact Kähler manifold with positive holomorphic sectional curvature need not support a Hermitian metric with positive first Chern Ricci curvature. We propose the following positive analog:

[7] For instance, $\mathcal{F}_\omega$ can be the scalar curvature, Ricci curvature, holomorphic sectional curvature, etc.

Conjecture. Let (X, ω) be a compact Kähler manifold with $\mathrm{Ric}_\omega > 0$. Then there is a Kähler metric $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ such that $\mathrm{HSC}_\omega > 0$.

Contents

CHAPTER 1

Foundational Theory

The geometry of complex manifolds lies at the intersection of a large number of subjects. As a consequence, complex geometry carries with it a vast and rich theory. The purpose of the present chapter is to exhibit this theory as it pertains to the developments in Chapters 2 and 3. We will start in §1.1 with reminders on the theory of smooth manifolds and then compare this with the theory of complex manifolds in §1.2. The algebro-geometric results concerning sheaves, their cohomology, divisors, line bundles, and characteristic classes is discuss in §1.3 and §1.4. Hermitian (and, in particular, Kähler metrics) are treated in §1.5. Hodge theory is discussed in §1.6. The chapter ends with the beautiful theory of compact complex surfaces due to Enriques and Kodaira in §1.7.

1.1. Smooth Manifolds

The concept of a manifold originates with Riemann's notion of a Riemann surface. Still, how we understand it today is very much in the style of twentieth-century mathematics. The idea is dialectically opposite to compactness: A compact space is controlled and well-behaved globally; a manifold is a space that is controlled and well-behaved locally. There is no control of a generic continuous function on a non-compact space, but the regularity of such a function, i.e., the extent to which one can understand its Taylor development, only concerns the function's behavior in a small neighborhood of a given point.

Definition 1.1.1. Let M be a connected paracompact Hausdorff topological space. For an arbitrary indexing set A, we assume M admits a covering $\mathcal{U} := (\mathcal{U}_\alpha)_{\alpha \in A}$ by connected open sets $\mathcal{U}_\alpha \subset M$ which are homeomorphic to balls $\mathbb{B}_\alpha := \{x_1^2 + \cdots + x_n^2 < 1\} \subset \mathbb{R}^n$. The pair $(\mathcal{U}_\alpha, \varphi_\alpha)$, where $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{B}_\alpha$ is a homeomorphism, is called a *chart* , and the set of charts $\mathscr{A} := \{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is said to be the *atlas* of the covering $\mathcal{U}$.

Convention 1.1.2. Unless otherwise stated, a topological space is assumed to be connected and paracompact. In particular, unless otherwise stated, there is a *partition of unity subordinate to any open cover* .

Remark 1.1.3. The charts permit one to locally identify a neighborhood of a point in M with a neighborhood of the origin in some Euclidean space $\mathbb{R}^n$. In particular, if $(x_1, ..., x_n)$

denote the coordinates on $\mathbb{R}^n$, these coordinates can be pulled back via the homeomorphism $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{B}_\alpha \subset \mathbb{R}^n$ to furnish a locally defined coordinate system on M, and hence, $\mathcal{U}_\alpha$ is sometimes called a *coordinate chart*. We say that the local coordinates are *centered at a point* $p \in M$ if $\varphi_\alpha(0) \in \mathbb{R}^n$.

On any overlap of $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, the composition

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_{\alpha\beta}) \longrightarrow \varphi_\beta(\mathcal{U}_{\alpha\beta})$$

defines a homeomorphism between open subsets of $\mathbb{R}^n$, which we call *transition maps*. These transition maps allow one to make sense of the regularity of M. Namely, if the transition maps are of class $\mathcal{C}^k$, for some $k \in \mathbb{N}$, we say that the atlas $\mathscr{A}$ is a $\mathcal{C}^k$*–atlas*[1]. If $\mathscr{A}$ is a $\mathcal{C}^k$–atlas for all $k \in \mathbb{N}$, we say $\mathscr{A}$ is a $\mathcal{C}^\infty$*–atlas* or a *smooth atlas*.

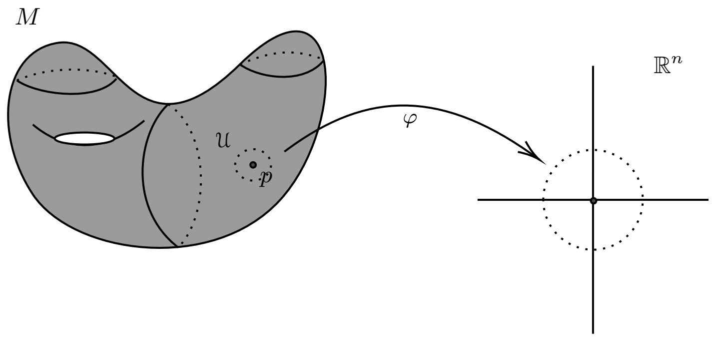
Remark 1.1.4. To remove any dependence on the specific choice of atlas, declare two $\mathcal{C}^k$–atlases $\mathscr{A}$ and $\mathscr{B}$ to be *equivalent* if their union is a $\mathcal{C}^k$–atlas. This defines an equivalence relation on the $\mathcal{C}^k$–atlases of M and ensures that the transition maps from the charts of one atlas to the charts of the other atlas have the same regularity as the regularity of the constituent transition maps for each atlas.

Definition 1.1.5. A $\mathcal{C}^k$*–manifold* is a connected Hausdorff topological space M endowed with an equivalence class of $\mathcal{C}^k$–atlases. The dimension of the balls to which the domains of the covering $\mathcal{U}$ are homeomorphic is called the (real) *dimension* of M, and is denoted $\dim_{\mathbb{R}} M$.

We will often indicate that a manifold M has real dimension n by writing M^n.

Remark 1.1.6. It is an elementary consequence of the rank-nullity theorem that the dimension of a *smooth* (or, at least, $\mathcal{C}^1$) manifold is well-defined. For *topological* manifolds (i.e., $\mathcal{C}^0$–manifolds) this remains true, but is non-trivial, requiring Brouwer's invariance of domain theorem (see, e.g., [**272**, §1.6.2]).

[1] Note that $\mathcal{C}^k$ is understood to mean k times continuously differentiable, for $k \in \mathbb{N}_0$. For $k = 0$, we identify $\mathcal{C}^k$ with continuity.



Example 1.1.7. The simplest example of a smooth manifold is $\mathbb{R}^n$. A smooth atlas is given by a single chart: $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$.

Example 1.1.8. The stereographic projection map defines a smooth atlas on the sphere

$$\mathbb{S}^n := \{(x_1, ..., x_{n+1}) \in \mathbb{R}^{n+1} : x_1^2 + \cdots + x_{n+1}^2 = 1\}.$$

Example 1.1.9. Let $(M^m, \mathscr{A})$ and $(N^n, \mathscr{B})$ be two smooth manifolds. The product $M \times N$ can be granted a smooth atlas $\mathscr{A} \times \mathscr{B} := \{(\alpha, \beta) : \alpha \in \mathscr{A}, \beta \in \mathscr{B}\}$. Here, $(\alpha, \beta)(x, y) := (\alpha(x), \beta(y)) \in \mathbb{R}^{m+n}$ for each $(x, y) \in \mathcal{U} \times \mathcal{V}$, where $\alpha : \mathcal{U} \to \mathbb{R}^m$ and $\beta : \mathcal{V} \to \mathbb{R}^n$.

Example 1.1.10. The torus $\mathbb{T}^2 := \mathbb{S}^1 \times \mathbb{S}^1$ is, therefore, a smooth manifold, with an atlas given by the product of the atlases on $\mathbb{S}^1$ specified in 1.1.8.

Cautionary Remark 1.1.11. One cannot replace the above definition of a manifold with the requirement that M is locally Euclidean. Indeed, the *double line* given by the quotient of $\mathbb{R} \times \{0, 1\}$ by the equivalence relation $(x, 0) \sim (x, 1)$ for all $x \in \mathbb{R}$ is locally Euclidean, but not Hausdorff.

Remark 1.1.12. Let us remark that a $\mathcal{C}^1$–manifold admits a unique real analytic $\mathcal{C}^\omega$–manifold structure. It is not the case that a $\mathcal{C}^0$–manifold admits a $\mathcal{C}^1$–manifold structure, however. Freedman's E_8–manifold is one such example (see [**125**] for details).

Definition 1.1.13. Let M be a manifold of class $\mathcal{C}^k$. A function $f : M \to \mathbb{R}$ is said to be of class $\mathcal{C}^\ell$, for some $\ell \leq k$, if the composite map $f \circ \varphi_\alpha^{-1}$ is of class $\mathcal{C}^\ell$ on the open set $\varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$.

Similarly, if M and N are two $\mathcal{C}^k$ manifolds, a map $f : M \to N$ is said to be of class $\mathcal{C}^\ell$, for some $\ell \leq k$, if the composite map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is of class $\mathcal{C}^\ell$ on the open set $\varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$. If M is a smooth manifold and f is $\mathcal{C}^k$ for all $k \in \mathbb{N}$, we say that f is a *smooth function*. These notions are clearly well-defined.

Convention 1.1.14. We will maintain the convention that the word *function* is used only for maps into a number space (e.g., $f : M \to \mathbb{R}$ or $f : M \to \mathbb{C}$). If the target space (codomain) is more general, we use only the term *map*. The space of smooth functions on a smooth manifold M is denoted by $\mathcal{C}^\infty(M)$.

It is common to view two smooth manifolds to be *equivalent* if there is a diffeomorphism between them:

Definition 1.1.15. Let $f : M \to N$ be a smooth map of smooth manifolds. If f is invertible with smooth inverse $f^{-1} : N \to M$, then f is said to be a *diffeomorphism*.

Remark 1.1.16. A smooth bijective map is not necessarily a diffeomorphism. Indeed, the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) := x^3$ is smooth, invertible, but does not admit a smooth inverse.

Remark 1.1.17. Let us remark that any smooth manifold homeomorphic to $\mathbb{R}^n$ is diffeomorphic to $\mathbb{R}^n$, unless $n = 4$ (see, e.g., [261]). On $\mathbb{R}^4$, there is an infinite number of *exotic smooth structures*[2]

Example 1.1.18. This equivalence of manifolds may identify two seemingly very different smooth manifolds: The connected sum[3] of the torus $\mathbb{T} \simeq \mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{RP}^2$ is diffeomorphic to the connected sum of three copies of $\mathbb{RP}^2$. Of course, there are the famous *exotic spheres* – smooth manifolds which are homeomorphic to $\mathbb{S}^n$ but not diffeomorphic to the standard $\mathbb{S}^n$ – discovered by Milnor [206].

Example 1.1.19. Let $\mathbb{P}^n$ denote the set of complex lines in $\mathbb{C}^{n+1}$, we call $\mathbb{P}^n$ the *complex projective space*. For $z \in \mathbb{C}^{n+1} \backslash \{0\}$, we denote by $[z]$ the complex line generated by z. For $0 \leq \alpha \leq n$, we let $\mathcal{U}_\alpha := \{[z] \in \mathbb{P}^n : z_\alpha \neq 0\}$. Each $\mathcal{U}_\alpha$ intersects the affine hyperplane $\{z_\alpha = 1\} \subset \mathbb{C}^{n+1}$ at exactly one point. We use this to define a coordinate map

$$\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{C}^n, \qquad \varphi_\alpha([z]) \ := \ \left(\frac{z_0}{z_\alpha}, ..., \frac{z_{\alpha-1}}{z_\alpha}, \frac{z_{\alpha+1}}{z_\alpha}, ..., \frac{z_n}{z_\alpha} \right).$$

[2] A smooth structure on a smooth manifold is said to be *exotic* if it is homeomorphic but not diffeomorphic to the standard smooth structure.

[3] Recall that a connected sum of two n–dimensional manifolds is a manifold formed by removing a disk inside each manifold and gluing together the resulting boundary disks. More precisely, let M and N be two oriented smooth manifolds of dimension n. Let $\Phi_M : \mathbb{B}^n \to M$ and $\Phi_N : \mathbb{B}^n \to N$ denote embeddings of the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$, such that Φ_M preserves orientation and Φ_N reverses the orientation. The *connected sum* $M \sharp N$ is the disjoint union

$$(M - \Phi_M(0)) \coprod (N - \Phi_2(0))$$

quotiented by the identification $\Phi_M(tu) = \Phi_N((1-t)u)$ for each $u \in \mathbb{S}^{n-1}$ and each $0 < t < 1$. The fact that this construction is well-defined forms the content of the disk theorem.

It is clear that φ_α is invertible. Moreover, the transition maps $\varphi_\alpha \circ \varphi_\beta^{-1}$ are defined by the composition of

$$\mathbb{C}^n \ni (w_1, ..., w_n) \mapsto \left(\frac{w_1}{w_\alpha}, ..., \frac{w_{\alpha-1}}{w_\alpha}, 1, \frac{w_{\alpha+1}}{w_\alpha}, ..., \frac{w_n}{w_\alpha}\right) \mapsto \left(\frac{w_1}{w_\alpha}, ..., \frac{w_{\alpha-1}}{w_\alpha}, \frac{w_{\alpha+1}}{w_\alpha}, ..., \frac{w_n}{w_\alpha}\right) \in \mathbb{C}^n.$$

Remark 1.1.20. The same discussion applies to the set of real lines in $\mathbb{R}^{n+1}$, producing the *real projective space* $\mathbb{RP}^n$. Similarly, the set of quaternionic lines in $\mathbb{H}^{n+1}$ yields the *quaternionic projective space* $\mathbb{HP}^n$.

An important class of smooth manifolds are those which support an additional level of algebraic structure, compatible with the smooth structure:

Definition 1.1.21. A group G is said to be a *Lie group* if G is a smooth manifold such that the composition map $\circ : G \times G \to G$ and the inversion map $\cdot^{-1} : G \to G$ are smooth.

Remark 1.1.22. Suppose one drops the requirement of connectedness and paracompactness. Any group can be viewed as a Lie group when given the discrete topology (and the discrete smooth structure).

Example 1.1.23. A theorem of Cartan (see, e.g., [**272**, Theorem 1.3.2]) asserts that any closed subgroup H of a Lie group G is a smooth (immersed) submanifold. In particular, closed subgroups of $\mathrm{GL}_n(\mathbb{C})$ such as the *real general linear group* $\mathrm{GL}_n(\mathbb{R})$, the *unitary group* $\mathrm{U}(n)$, the *special unitary group* $\mathrm{SU}(n)$, the *orthogonal group* $\mathrm{O}(n)$, and the *special orthogonal group* $\mathrm{SO}(n)$ are all Lie groups.

Definition 1.1.24. Let M be a smooth manifold. We say that M is a *homogeneous space* if there is a Lie group G which acts transitively on M.

Example 1.1.25. Let $\mathrm{U}(n+1)$ denote the unitary group, which acts transitively on $\mathbb{C}^{n+1}$. The action preserves complex subspaces of $\mathbb{C}^{n+1}$ and, in particular, maps complex lines to complex lines. The transitive action, therefore, descends to $\mathbb{P}^n$ showing that $\mathbb{P}^n$ is homogeneous.

Recall that if we have a curve $\mathcal{C} \subset \mathbb{R}^2$, given by the graph of a function $f : \mathbb{R} \to \mathbb{R}$, the derivative $f'(x)$ is a vector which lies in the one-dimensional vector space tangent to the curve $\mathcal{C}$ at $f(x)$. The higher-dimensional extension of this notion is called the tangent space:

Definition 1.1.26. Let M be a smooth manifold, and $p \in M$ be a point. A linear map $V : \mathcal{C}^\infty(M) \to \mathbb{R}$ is declared a *derivation at p* if it satisfies the following Leibniz rule:

$$V(fg) \quad = \quad f(p)V(g) + g(p)V(f).$$

Let T_pM denote the space of all derivations of $\mathcal{C}^\infty(M)$ at $p \in M$. We call T_pM the *tangent space* of M at p.

Remark 1.1.27. Clearly, T_pM forms a vector space with respect to the operators $(U + V)(f) = U(f) + V(f)$ and $(\lambda V)(f) = \lambda(V(f))$.

The most important fact concerning the space of derivations is that it is finite-dimensional and, in fact, isomorphic to $\mathbb{R}^n$ for each $p \in M$. Here n is the dimension of M. In fact, let $(x_1, ..., x_n)$ denote local coordinates on M centered at a point $p \in M$. Then a basis for T_pM is given by the derivations $\frac{\partial}{\partial x_1}|_p, ..., \frac{\partial}{\partial x_n}|_p$, where

$$\left.\frac{\partial}{\partial x_i}\right|_p f \quad := \quad \frac{\partial f}{\partial x_i}(p),$$

for each $1 \leq i \leq n$.

Definition 1.1.28. Let $f : M \to N$ be a smooth map between smooth manifolds M and N. For each point $p \in M$, there is an induced map $df_p : T_pM \to T_{f(p)}N$ called the *differential of f at p*.

Definition 1.1.29. Let $f : M^m \to N^n$ be a smooth map of smooth manifolds. We say that f is

(i) an *immersion* if df_p is injective for all $p \in M$.
(ii) a *submersion* if df_p is surjective for all $p \in M$.
(iii) an *embedding* if f is an immersion and is homeomorphic onto its image[4].

Remark 1.1.30. An immersion need not be injective: Take $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2$.

Definition 1.1.31. Let M be a smooth manifold. A (smooth) *submanifold* N of M is a smooth manifold N together with a smooth embedding $N \hookrightarrow M$.

Remark 1.1.32. Some authors relax the above definition, requiring a submanifold to merely be immersed. For us, a submanifold will always be an embedded submanifold unless otherwise stated. If this more general definition is used, we will emphasize its use by referring to the object as an *immersed submanifold*.

Theorem 1.1.33. (Whitney embedding theorem). Let M be a smooth manifold of real dimension $n \in \mathbb{N}$. Then there is a smooth embedding $M \hookrightarrow \mathbb{R}^{2n}$.

Example 1.1.34. Let $\mathbb{RP}^n$ denote the real projective space. The Whitney embedding theorem states that there is an embedding $\mathbb{RP}^n \hookrightarrow \mathbb{R}^{2n}$. However, the Boy's surface [44] shows that there is a (non-injective) immersion $\mathbb{RP}^2 \looparrowright \mathbb{R}^3$.

[4]Note that the image $f(M) \subset N$ is endowed with the subspace topology.

Theorem 1.1.35. (Submersion theorem). Let $f : M \to N$ be a submersion of smooth manifolds. The fibers $f^{-1}(p) \subset M$ (for $p \in N$) are smooth submanifolds of M.

Definition 1.1.36. Let $f : M \to N$ be a smooth map of smooth manifolds. A *critical point* for f is a point $p \in M$ such that df_p fails to have maximal rank. The corresponding value $f(p)$ is said to be a *critical value*. The set of critical values of f is called the *discriminant locus of* f.

Remark 1.1.37. The discriminant locus of a smooth submersion is empty.

The following class of submersions will play an important role:

Definition 1.1.38. Let $f : \mathcal{E} \to M$ be a smooth submersion whose fibers $\mathcal{E}_b := f^{-1}(b)$ (for $b \in M$) are all vector spaces isomorphic to $\mathbb{R}^k$. If, for any point $b \in M$, there is an open neighborhood $\mathcal{U} \ni b$ such that $f^{-1}(\mathcal{U}) \simeq_{\text{diffeo.}} \mathcal{U} \times \mathbb{R}^k$, and moreover, this map restricts to an isomorphism $f^{-1}(b) \simeq_{\text{iso.}} \{b\} \times \mathbb{R}^k$, then we say that f is a *smooth vector bundle of rank* k.

The manifold $\mathcal{E}$ is referred to as the *total space*, the manifold X is referred to as the *base space*, and f is called the *bundle projection*. The diffeomorphism (denote it by, say) $\varphi : f^{-1}(\mathcal{U}) \to \mathcal{U} \times \mathbb{R}^k$ is sometimes called a *bundle chart*.

Remark 1.1.39. The existence of neighborhoods around every point together with diffeomorphisms to products is referred to as the *local triviality condition* of a vector bundle. The condition asserts that vector bundles are locally modeled on products. The local triviality condition is essential if one wishes to obtain some degree of classification (the subject of K-theory). Suppose one relaxes the local triviality condition, considering only families of vector spaces parametrized by a manifold. In that case, one can take a disjoint union of vector spaces of the same rank, which can be arbitrarily bad. The local triviality condition forces enough structure to be amenable to homotopy theory and the construction of homotopy-theoretic invariants. For instance, every vector bundle over a contractible space is topologically trivial.

Example 1.1.40. The simplest method of producing a vector bundle of rank k is to take the product $M \times \mathbb{R}^k$ together with the projection onto the first factor $p : M \times \mathbb{R}^k \to M$. This is called the *trivial bundle*.

Remark 1.1.41. We can define vector bundles in more general categories. A *topological vector bundle* (of rank k) is a continuous map $f : \mathcal{E} \to \mathcal{B}$ such that $f^{-1}(b) \simeq \mathbb{R}^k$ and for every point $b \in \mathcal{B}$, there is an open neighborhood $\mathcal{U} \subseteq \mathcal{B}$ such that $f^{-1}(\mathcal{U}) \simeq_{\text{homeo.}} \mathcal{U} \times \mathbb{R}^k$. Further, we may apply these definitions to vector bundles with fibers being $\mathbb{C}^k$ in place of $\mathbb{R}^k$. We refer to (continuous or smooth) vector bundles as *complex vector bundles*. This is not to be confused with the holomorphic vector bundles we will encounter later.

Example 1.1.42. The most important example of a smooth vector bundle associated with a manifold M is the *tangent bundle* TM. The total space of the tangent bundle (which we abusively also denote by TM) is the disjoint union

$$TM = \coprod_{p \in M} T_pM \;=\; \{(p,v) : p \in M, v \in T_pM\}$$

of the tangent spaces T_pM. The bundle projection $f : TM \to M$ is defined $f(p,v) := p$.

Definition 1.1.43. Let $f : \mathcal{E} \to M$ be a smooth vector bundle of rank k. Fix a point $p \in M$. A *local frame* for $\mathcal{E}$ near p is a finite number of sections $\sigma_j : \mathcal{U} \to \mathcal{E}$ such that $\{\sigma_1|_p, ..., \sigma_k|_p\}$ furnishes a basis for the fiber $\mathcal{E}_p$.

Example 1.1.44. Let TM denote the tangent bundle of a smooth manifold M. For a point $p \in M$, let $(x_1, ..., x_n)$ denote local coordinates centered at p. Then $\partial_{x_1}, ..., \partial_{x_n}$ provides a local frame for TM near p.

Definition 1.1.45. We say that a smooth manifold M is *parallelizable* if the tangent bundle is trivial. That is, there exist (globally-defined) smooth vector fields $v_1, ..., v_n$ such that $v_1(p), ..., v_n(p)$ form a basis of T_pM for all $p \in M$.

Example 1.1.46. Every Lie group is parallelizable. Indeed, the group structure allows the tangent spaces at distinct points to be identified. The converse is false: $\mathbb{S}^7$ is paralellizable but not a Lie group[5]

Example 1.1.47. The hairy ball theorem states that $\mathbb{S}^2$ is not parallelizable. The only spheres which are Lie groups are $\mathbb{S}^1$ and $\mathbb{S}^3$, so these are parallelizable. The smooth Moufang loop structure on $\mathbb{S}^7$ implies that $\mathbb{S}^7$ is parallelizable. By the results of Adams, Bott, Hirzebruch, Kervaire, and Milnor, these are the only parallelizable spheres (see, e.g., [**238**] for a nice account).

Definition 1.1.48. Let $f : \mathcal{E} \to M$ be a smooth vector bundle. Let $\mathcal{U} \subset M$ be an open set. A smooth map $\sigma : \mathcal{U} \to \mathcal{E}$ is said to be a (smooth) *section* of $\mathcal{E}$ if $f \circ \sigma = \text{id}$. If $\mathcal{U} = M$, we say that $\sigma : M \to \mathcal{E}$ is a *global section*. The space of smooth sections of $\mathcal{E}$ over $\mathcal{U}$ is denoted $H^0(\mathcal{U}, \mathcal{E})$.

Example 1.1.49. The smooth sections of the tangent bundle TM are called *vector fields*. The space of (smooth) vector fields is denoted $\mathscr{X}(M)$.

Example 1.1.50. Let $f : \mathcal{E} \to M$ be a smooth vector bundle over a smooth manifold M. An important example of a global section of $\mathcal{E}$ is the *zero section*, the section $\sigma : M \to \mathcal{E}$ which assigns to each point $p \in M$, the origin in the fiber $\mathcal{E}_p = f^{-1}(p)$.

[5] It is worth remarking that $\mathbb{S}^7$ is not a Lie group only due to the failure of the multiplication to be associative. Because of this, $\mathbb{S}^7$ is a smooth Moufang loop (see, e.g., [**120**]).

Remark 1.1.51. There is a natural operation on $\mathscr{X}(M)$; namely, the Lie bracket:

$$[\cdot,\cdot] : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M), \qquad [X,Y](f) := XY(f) - YX(f),$$

for all $f \in \mathcal{C}^\infty(M)$.

Proliferation of Vector Bundles. The routine operations of direct sum, dual, tensor product, etc., which we know well in the context of vector spaces, extend easily to vector bundles:

Definition 1.1.52. Let $f : \mathcal{E} \to M$ be a smooth vector bundle of rank k. An embedded submanifold $\mathcal{F} \subseteq \mathcal{E}$ is said to be a *smooth subbundle* of $\mathcal{E}$ if, for each $p \in M$, the fiber $\mathcal{F}_p := \mathcal{F} \cap f^{-1}(p)$ is a linear subspace of $\mathcal{E}_p = f^{-1}(p)$, and moreover, the projection $f|_{\mathcal{F}} : \mathcal{F} \to M$ is a smooth vector bundle.

Definition 1.1.53. Let $f : \mathcal{E} \to M$ be a smooth vector bundle of rank k with bundle charts $(g_{\alpha\beta})$. The *dual vector bundle* $f^* : \mathcal{E}^* \to M$ is the vector bundle with fibers $\mathcal{E}_p^* = (\mathcal{E}_p)^* := \mathrm{Hom}_{\mathbb{R}}(\mathcal{E}_p, \mathbb{R})$, and bundle charts $(g_{\alpha\beta}^t)^{-1}$.

Example 1.1.54. The vector bundle dual to the tangent bundle TM is the *cotangent bundle* T^*M.

For a point $p \in M$, let $(x_1, ..., x_n)$ denote local coordinates centered at p. Then $dx_1, ..., dx_n$ provides a local frame for T^*M near p. This frame is dual to the frame $\partial_{x_1}, ..., \partial_{x_n}$ for T_pM in the sense that $dx_k(\partial_{x_\ell}) = \delta_k^\ell$ for all k, ℓ, and is thus sometimes called the *dual coframe.*

Tensor Products. The subject of linear algebra (in its traditional form) deals with vector spaces and natural maps, i.e., linear maps. In this respect, the content of inner products (e.g., the dot product) should not be treated in linear algebra; such objects reside within the more general subject of *multilinear algebra.* One of the miracles of the subject, however, is that there is a distinguished multilinear map, denoted by $\otimes$, which permits us to realize any multilinear map

$$\varphi : V_1 \times \cdots \times V_k \longrightarrow W$$

as a linear map

$$\Phi : V_1 \otimes \cdots \otimes V_k \longrightarrow W$$

from an auxiliary vector space $V_1 \otimes \cdots \otimes V_k$, independent of φ.

Definition 1.1.55. Let V and W be vector spaces. The *tensor product* $V \otimes W$ is a vector space together with a bilinear map

$$\otimes : V \times W \longrightarrow V \otimes W, \qquad \otimes : (v, w) \ \mapsto \ v \otimes w$$

such that, for any bilinear map $\varphi : V \times W \longrightarrow Z$, there is a unique linear map $\Phi : V \otimes W \longrightarrow Z$ such that $\varphi = \Phi \circ \otimes$.

Notation 1.1.56. We will write $V^{\otimes k}$ for the k–fold tensor product of V, i.e.,

$$V^{\otimes k} \ := \ V \otimes \cdots \otimes V \qquad (k\text{–times}).$$

Remark 1.1.57. Let $v_1, ..., v_n$ be a basis for V and let $w_1, ..., w_m$ be a basis for W. A basis for $V \otimes W$ is given by $\{v_k \otimes w_\ell\}_{1 \leq k \leq n, 1 \leq \ell \leq m}$. In particular,

$$\dim(V \otimes W) \ = \ \dim(V) \cdot \dim(W).$$

To get a more geometric understanding of the tensor product, let us make the following definition:

Definition 1.1.58. Let V be a vector space.

(i) A *covariant k–tensor* on V is a multilinear map

$$\varphi : V^{\otimes k} \longrightarrow \mathbb{R}.$$

The space of covariant k–tensors is denoted by $\mathcal{T}_k(V)$.

(ii) A *contravariant ℓ–tensor* on V is a multilinear map

$$\varphi : (V^*)^{\otimes \ell} \longrightarrow \mathbb{R}.$$

The space of contravariant ℓ–tensors is denoted by $\mathcal{T}^\ell(V)$.

(iii) A *(k, ℓ)–tensor* on V is a multilinear map

$$\varphi : (V^*)^{\otimes k} \otimes V^{\otimes \ell} \longrightarrow \mathbb{R}.$$

The space of (k, ℓ)–tensors is denoted by $\mathcal{T}_k^\ell(V)$.

Example 1.1.59. Let V and W be two vector spaces. We identify the space of endomorphisms $\mathrm{End}(V, W)$ with the tensor product $V^* \otimes W$ via the isomorphism

$$\Phi : V^* \otimes W \longrightarrow \mathrm{End}(V, W), \qquad \Phi(\alpha \otimes w) \ := \ (v \mapsto \alpha(v)w),$$

where $\alpha \in V^*$, $w \in W$, and $v \in V$. The space of endomorphism from a vector space V to itself is denoted by $\mathrm{End}(V)$, i.e., $\mathrm{End}(V) := \mathrm{End}(V, V)$. In particular,

$$\mathrm{End}(V) \ \simeq \ \mathcal{T}_1^1(V).$$

Example 1.1.60. Extending the above example, we can identify the space $\mathcal{T}_{\ell+1}^k(V)$ with the space of multilinear maps $\mathrm{Mult}((V^*)^\ell \times V^k, V)$ via the isomorphism

$$\Phi : \mathrm{Mult}((V^*)^\ell \times V^k, V) \longrightarrow \mathcal{T}_{\ell+1}^k(V),$$

specified by the formula

$$\Phi(A) \ := \ \Big((\alpha^0, \alpha^1, ..., \alpha^\ell, v_1, ..., v_k) \ \mapsto \ \alpha^0(A(\alpha^1, ..., \alpha^\ell, v_1, ..., v_k))\Big),$$

Remark 1.1.61. Let $e_1, ..., e_n$ be a basis for V and let $\varepsilon^1, ..., \varepsilon^n$ be the corresponding dual basis. Then a basis for $\mathcal{T}^k_\ell(V)$ is given by

$$e_{j_1} \otimes \cdots \otimes e_{j_\ell} \otimes \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}.$$

Hence, any (k, ℓ)–tensor $\xi \in \mathcal{T}^k_\ell(V)$, can be written as

$$\xi \;=\; \xi^{j_1 \cdots j_\ell}{}_{i_1 \cdots i_k} e_{j_1} \otimes \cdots \otimes e_{j_\ell} \otimes \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},$$

where $\xi^{j_1 \cdots j_\ell}{}_{i_1 \cdots i_k} = \xi(\varepsilon^{j_1}, ..., \varepsilon^{j_\ell}, e_{i_1}, ..., e_{i_k})$.

Example 1.1.62. Let us make explicit the important case when $V = T_pM$ is the tangent space to a smooth manifold M, at a point $p \in M$. If $(x_1, ..., x_n)$ are local coordinates on M centered at p, then a basis for T_pM is given by $\partial_{x_1}, ..., \partial_{x_n}$. The corresponding dual basis is given by $dx^1, ..., dx^n$. A (k, ℓ)–tensor on T_pM is then given in this basis by

$$\xi \;=\; \sum_{1 \leq i_1 \cdots i_k \leq n, 1 \leq j_1 \cdots j_n \leq n} \xi^{j_1 \cdots j_\ell}{}_{i_1 \cdots i_k} \partial_{x_{j_1}} \otimes \cdots \otimes \partial_{x_{j_\ell}} \otimes dx^{i_1} \otimes \cdots \otimes dx^{i_k}.$$

Extending these notions from vector spaces to vector bundles, we consider:

Example 1.1.63. Let M be a smooth manifold. The *bundle of* (k, ℓ)*–tensors* is the vector bundle $\mathcal{T}^\ell_k(M) \to M$ given by

$$\mathcal{T}^\ell_k(M) \;:=\; \coprod_{p \in M} \mathcal{T}^\ell_k(T_pM) \;=\; \coprod_{p \in M} \otimes^k T^*_pM \otimes^\ell T_pM.$$

Sections of $\mathcal{T}^\ell_k(M)$ are called (k, ℓ)*–tensors* or (k, ℓ)*–tensor fields*.

One of the most important examples of a tensor field are those $(2, 0)$–tensor fields that are positive-definite:

Definition 1.1.64. Let M be a smooth manifold. A *Riemannian metric* g on M is a smooth section of $\mathcal{T}^2_0(M)$ which is symmetric and positive-definite. In local coordinates $(x_1, ..., x_n)$, centered at a point $p \in M$, the Riemannian metric is written:

$$g = \sum_{i,j=1}^n g_{ij} dx^i \otimes dx^j,$$

where the *component functions* $g_{ij} := g(\partial_{x_i}, \partial_{x_j})$. A smooth manifold endowed with a Riemannian metric is referred to as a *Riemannian manifold*.

Example 1.1.65. The simplest Riemannian metric is the Euclidean metric $\delta_{\mathbb{R}^n}$ on $\mathbb{R}^n$ given in coordinates by

$$\delta_{\mathbb{R}^n} \;:=\; \sum_{k=1}^n dx_k \otimes dx_k.$$

Example 1.1.66. The *round metric* g_{round} on the sphere $\mathbb{S}^n$ is given in local coordinates $(x_1, ..., x_n)$ by

$$g_{\text{round}} := \sum_{i,j=1}^{n} \frac{4}{(1-|x|^2)^2} dx_i \otimes dx_j.$$

Since we assume that manifolds are paracompact, we have the following:

Proposition 1.1.67. Let M be a smooth manifold. Then M admits a Riemannian metric.

PROOF. Since a manifold is understood to be paracompact, there is a partition of unity (ρ_α), subordinate to any open cover $(\mathcal{U}_\alpha)$ of M. Cover M by coordinate charts $\varphi_\alpha : \mathcal{U}_\alpha \longrightarrow \mathbb{R}^n$ and pullback the Euclidean metric $\delta_{\mathbb{R}^n}$ on $\mathbb{R}^n$ via φ_α. The formula $g := \sum_\alpha \rho_\alpha \varphi_\alpha^* \delta_{\mathbb{R}^n}$ defines a Riemannian metric on M. □

A Riemannian metric g on a smooth manifold M endows M with several inherited structures, most notably, a length function:

Definition 1.1.68. Let (M, g) be a Riemannian manifold. Let $\alpha : [t_0, t_1] \to M$ be a smooth curve. Let $\dot{\alpha}(t) := \frac{d\alpha}{dt}$ and let g_t be the restriction of the Riemannian metric to $T_{\alpha(t)}M$. The *length* of α (with respect to g) is defined by

$$L_g(\alpha) := \int_{t_0}^{t_1} |\dot{\alpha}(t)|_{g_t} dt,$$

where $|\dot{\alpha}(t)|_{g_t} := \sqrt{g_t(\dot{\alpha}(t), \dot{\alpha}(t))}$.

The Riemannian metric, therefore, gives a functional on the space of $(\mathcal{C}^1)$ curves in M. The minimizers of this functional are called geodesics:

Definition 1.1.69. Let (M^n, g) be a Riemannian manifold. We say that a curve $\alpha : [t_0, t_1] \to M$ is a *geodesic* if α is a (local) minimum for the length functional L_g.

Definition 1.1.70. Let (M^n, g) be a Riemannian manifold and $p \in M$ a point. The *exponential map* $\exp_p : \mathcal{U} \to M$ is defined by sending $v_p \in T_pM$ to the endpoint $\alpha(1) \in M$ of the unique geodesic α with $\alpha(0) = p$ and $\dot{\alpha}(0) = v_p$.

Remark 1.1.71. The fact that the exponential map exists and is well-defined is well-known and can be found in [114, Chapter 3]. Moreover, the exponential map is always a local diffeomorphism.

Definition 1.1.72. Let (M^n, g) be a Riemannian manifold and $p \in M$ a point. Let $\{e_1, ..., e_n\}$ be a local orthonormal frame for T_pM. Define a chart $\varphi : \mathcal{U} \to \mathbb{R}^n$ on a neighborhood $\mathcal{U}$ of p by $\varphi^{-1}(x_1, ..., x_n) = \exp_p(x_j e_j)$. These coordinates are referred to as *Riemannian normal coordinates.*

The importance of Riemannian normal coordinates from the simple expression the metric takes when expressed in them, namely:

$$g_{ij}(p) = \delta_{ij}, \qquad \frac{\partial g_{ij}}{\partial x_k} \ = \ \frac{\partial g_{ik}}{\partial x_j} \ = \ 0.$$

By declaring the distance between $p, q \in M$ to be the infimum of the lengths of curves connecting p and q, we obtain a distance function:

Definition 1.1.73. Let (M, g) be a Riemannian manifold. The *Riemannian distance function* $d_g : M \times M \to \mathbb{R}$ is defined by

$$d_g(p, q) \ := \ \inf_{\alpha} L_g(\alpha),$$

where the infimum is over all $\mathcal{C}^1$ curves $\alpha : [t_0, t_1] \to M$ such that $\alpha(t_0) = p$ and $\alpha(t_1) = q$.

Definition 1.1.74. Let (M, g) be a Riemannian manifold. We say that g is *complete* if the metric space (M, d_g) is Cauchy complete, i.e., every Cauchy sequence (with respect to d_g) converges (with respect to d_g).

Remark 1.1.75. By the Hopf–Rinow theorem, a Riemannian manifold is complete in the above sense if and only if geodesics can be extended indefinitely (i.e., the exponential map is defined on the whole tangent bundle) (see [**10**, Theorem 11.5.1].

Definition 1.1.76. Let (M, g) be a Riemannian manifold. Let f be a diffeomorphism of a neighborhood of a point $p \in M$. We say that f is a *geodesic symmetry* if $f(p) = p$ and f reverses geodesics through p in the sense that

$$f(\gamma(t)) = \gamma(-t),$$

for any geodesic γ with $\gamma(0) = p$.

Definition 1.1.77. A Riemannian manifold (M, g) is understood to be *locally symmetric* if the geodesic symmetries are isometries. If, in addition, the geodesic symmetries extend to isometries on all of M, then (M, g) is called a *symmetric space*.

Example 1.1.78. Euclidean space, spheres, projective spaces, and hyperbolic spaces, endowed with their standard Riemannian metrics are symmetric spaces.

Example 1.1.79. Every compact Riemann surface of genus $g \geq 2$ with its metric of constant negative Gauss curvature is locally symmetric but not symmetric.

Although we will include the relevant theory in §2.1, we mention the following important fact:

Theorem 1.1.80. (Cartan–Ambrose–Hicks). A Riemannian manifold (M, g) is locally symmetric if and only if the curvature tensor is parallel with respect to the Levi-Civita connection.

Corollary 1.1.81. A simply connected complete locally symmetric space is a symmetric space.

Proposition 1.1.82. Every symmetric space is complete and homogeneous, with the isometry group acting transitively.

Definition 1.1.83. A simply connected symmetric space is said to be *irreducible* if it is not the product of two Riemannian symmetric spaces.

Tensor Contractions. Given a linear map $A : V \longrightarrow W$, we have a well-defined notion of the *trace of* A. Invariantly, if $\lambda_1, ..., \lambda_n$ are the eigenvalues of A, then $\mathrm{tr}(A) := \sum_{k=1}^n \lambda_k$. We can extend this notion to more general tensors. To this end, we observe that it is clear how to do this for tensors $\xi \in \mathcal{T}_1^1(V)$. Indeed, from 1.1.59, we can identify ξ with $\Phi(\xi) \in \mathrm{End}(V)$, and define $\mathrm{tr}(\xi) := \mathrm{tr}(\Phi(\xi))$. If $e_1, ..., e_n$ is a basis for V, with dual basis $\varepsilon^1, ..., \varepsilon^n$, write $\xi = \sum_{i,j=1}^n \xi^j{}_i e_j \otimes \varepsilon^i$. Then

$$\mathrm{tr}(\xi) \;\; = \;\; \sum_{k=1}^n \xi^k{}_k.$$

More generally, if $\xi = \xi^{j_1 \cdots j_\ell}{}_{i_1 \cdots i_k} e_{j_1} \otimes \cdots \otimes e_{j_\ell} \otimes \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k}$ is a (k, ℓ)–tensor, then the trace of ξ over the i_p, j_q indices is the $(k-1, \ell-1)$–tensor

$$\begin{aligned} \mathrm{tr}_{i_p j_q}(\xi) \;\; = \;\; & \xi^{j_1 \cdots j_{q-1} m j_{q+1} \cdots j_\ell}{}_{i_1 \cdots i_{p-1} m i_{p+1} \cdots i_k} e_{j_1} \otimes \cdots \otimes e_{j_{q-1}} \otimes e_{j_{q+1}} \otimes \cdots \otimes e_{j_\ell} \\ & \otimes \varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_{p-1}} \otimes \varepsilon^{i_{p+1}} \otimes \cdots \otimes \varepsilon^{i_k}. \end{aligned}$$

The above trace operation is often referred to as a *tensor contraction*. In the above case, the tensor ξ is said to be contracted over the indices i_p and j_q.

Remark 1.1.84. It is clear that the operation of tensor contraction is linear, and lowers the rank by 2, i.e., $\mathrm{tr} : \mathcal{T}_\ell^k(V) \longrightarrow \mathcal{T}_{\ell-1}^{k-1}(V)$.

Musical isomorphisms. The Riemannian metric yields the musical isomorphisms

$$\sharp : \mathcal{T}_k^\ell(M) \longrightarrow \mathcal{T}_{k-1}^{\ell+1}(M), \qquad \flat : \mathcal{T}_k^\ell(M) \longrightarrow \mathcal{T}_{k+1}^{\ell-1}(M)$$

defined as follows: Let $g = \sum_{i,j=1}^n g_{ij} dx_i \otimes dx_j$ denote the Riemannian metric in the local coordinates $(x_1, ..., x_n)$. Let

$$\xi = \xi^{i_1 \cdots i_k}{}_{j_1 \cdots j_\ell} dx^{i_1} \otimes \cdots dx^{i_k} \otimes \partial_{x_{j_1}} \otimes \cdots \otimes \partial_{x_{j_\ell}}$$

be a (k, ℓ)–tensor. Then

$$\xi^\sharp \;\; := \;\; (\xi^\sharp)^{i_1 \cdots i_k m}{}_{j_2 \cdots j_\ell} dx^{i_1} \otimes \cdots dx^{i_k} \otimes dx^m \otimes \partial_{x_{j_2}} \otimes \cdots \otimes \partial_{x_{j_\ell}},$$

where $(\xi^\sharp)^{i_1 \cdots i_k m}{}_{j_2 \cdots j_\ell} = g^{m j_1} \xi^{i_1 \cdots i_k}{}_{j_1 \cdots j_\ell}$. Similarly,

$$\xi^\flat \;\; := \;\; (\xi^\flat)^{i_1 \cdots i_{k-1}}{}_{j_1 \cdots j_\ell m} dx^{i_1} \otimes \cdots \otimes dx^{i_{k-1}} \otimes \partial_{x_{j_1}} \otimes \cdots \otimes \partial_{x_{j_\ell}} \otimes \partial_{x_m},$$

where $(\xi^\flat)^{i_1\cdots i_{k-1}}{}_{j_1\cdots j_\ell m} = g_{mi_k}\xi^{i_1\cdots i_k}{}_{j_1\cdots j_\ell}$.

Remark 1.1.85. This process of changing a tensor from type (k,ℓ) to type $(k+1,\ell-1)$, i.e., apply the isomorphism $\sharp$, is often referred to as *raising the index* of a tensor. Similarly, the process of changing a tensor from type (k,ℓ) to type $(k-1,\ell+1)$, i.e., apply the isomorphism $\flat$, is often referred to as *lowering the index* of a tensor.

Example 1.1.86. Let $R = R_{ijk}{}^p\varepsilon^i\otimes\varepsilon^j\otimes\varepsilon^k\otimes e_p$ be a $(3,1)$–tensor. Then

$$R^\flat \;=\; R_{ijk\ell}\varepsilon^i\otimes\varepsilon^j\otimes\varepsilon^k\otimes\varepsilon^\ell$$

is a $(4,0)$–tensor, where $R_{ijk\ell} := g_{p\ell}R_{ijk}{}^p$. On the other hand,

$$R^\sharp \;=\; R_{ij}{}^{qp}\varepsilon^i\otimes\varepsilon^j\otimes e_q\otimes e_p$$

is a $(2,2)$–tensor, where $R_{ij}{}^{qp} := g^{kq}R_{ijk}{}^p$.

Metric Contractions. The metric can be used to contract tensors. Indeed, with $g = g_{ij}dx^i\otimes dx^j$ and $\xi = \xi^{j_1\cdots j_\ell}{}_{i_1\cdots i_k}e_{j_1}\otimes\cdots\otimes e_{j_\ell}\otimes\varepsilon^{i_1}\otimes\cdots\otimes\varepsilon^{i_k}$ as before, we may contract ξ in two ways using the metric: the first is to contract ξ to a $(k-2,\ell)$–tensor:

$$g^{i_pi_q}\xi^{j_1\cdots j_\ell}{}_{i_1\cdots i_k}e_{j_1}\otimes\cdots\otimes e_{j_\ell}\otimes\varepsilon^{i_1}\otimes\cdots\otimes\varepsilon^{i_{p-1}}\otimes\varepsilon^{i_{p+1}}\otimes\cdots\otimes\varepsilon^{i_{q-1}}\otimes\varepsilon^{i_{q+1}}\otimes\cdots\otimes\varepsilon^{i_k}.$$

The second is to contract ξ to a $(k,\ell-2)$–tensor:

$$g_{j_pj_q}\xi^{j_1\cdots j_\ell}{}_{i_1\cdots i_k}e_{j_1}\otimes\cdots\otimes e_{j_{p-1}}\otimes e_{j_{p+1}}\otimes\cdots\otimes e_{j_{q-1}}\otimes e_{j_{q+1}}\otimes\cdots\otimes e_{j_\ell}\otimes\varepsilon^{i_1}\otimes\cdots\otimes\varepsilon^{i_k}.$$

Example 1.1.87. If $g = g_{ij}dx^i\otimes dx^j$ and $h = h_{ij}dx^i\otimes dx^j$ are two Riemannian metrics on a smooth manifold M, then

$$\mathrm{tr}_g(h) \;=\; g^{ij}h_{ij}$$

is a smooth function on M.

The notion of a Riemannian metric can be extended to any vector bundle:

Definition 1.1.88. Let $\mathcal{E}\to M$ be a smooth vector bundle over a smooth manifold M. A *bundle metric* g on $\mathcal{E}$ is a smooth family of positive-definite quadratic forms $g_p : \mathcal{E}_p\times\mathcal{E}_p\to\mathbb{R}$.

Remark 1.1.89. Repeating the argument in 1.1.67 shows that every smooth vector bundle admits a bundle metric.

Definition 1.1.90. Let (M,g) be a Riemannian manifold. Let ξ,ν be two smooth sections of the (k,ℓ)–tensor bundle $\mathscr{T}^\ell_k(M)$. Write $\xi = \xi^{i_1\cdots i_k}{}_{j_1\cdots j_\ell}dx^{i_1}\otimes\cdots dx^{i_k}\otimes\partial_{x_{j_1}}\otimes\cdots\otimes\partial_{x_{j_\ell}}$ and $\nu = \nu^{i_1\cdots i_k}{}_{j_1\cdots j_\ell}dx^{i_1}\otimes\cdots dx^{i_k}\otimes\partial_{x_{j_1}}\otimes\cdots\otimes\partial_{x_{j_\ell}}$. The metric g induces a bundle metric on $\mathscr{T}^\ell_k(M)$, given by

$$(\xi,\nu)_{\mathscr{T}^\ell_k(M)} \;=\; g^{a_1b_1}\cdots g^{a_kb_k}g_{i_1j_1}\cdots g_{i_\ell j_\ell}\xi_{a_1\cdots a_k}{}^{i_1\cdots i_\ell}\nu_{b_1\cdots b_k}{}^{j_1\cdots j_\ell}.$$

Example 1.1.91. If $f : (M,g) \longrightarrow (N,h)$ is a smooth of Riemannian manifolds. If $(x_1, ..., x_m)$ denote local coordinates centered at a point $p \in M$ and $(y_1, ..., y_n)$ denote local coordinates centered at a point $f(p) \in N$, let us write $g = g_{ij} dx^i \otimes dx^j$ and $h = h_{\gamma\delta} dy^\gamma \otimes dy^\delta$ for the metrics in these coordinates. Then f is locally written as $f = (f^1, ..., f^n)$, with each $f^\alpha = f^\alpha(x_1, ..., x_m)$. Set $f_k^\alpha := \frac{\partial f^\alpha}{\partial x_k}$, for each $1 \leq k \leq m$ and each $1 \leq \alpha \leq n$.

Observe that df is a section of $T^*M \otimes f^*TN$, where T^*M supports the metric g^{-1}, and f^*TN supports the metric f^*h. The pullback metric f^*h is

$$f^*h \ = \ (f^*h)_{k\ell} dx^k \otimes dx^\ell \ := \ h_{\gamma\delta} f_k^\gamma f_\ell^\delta dx^k \otimes dx^\ell.$$

and therefore,

$$|df|^2 \ = \ g^{k\ell} h_{\gamma\delta} f_k^\gamma f_\ell^\delta.$$

Observe that the above expression is equivalent to

$$|df|^2 = \mathrm{tr}_g(f^*h) \ = \ g^{k\ell}(f^*h)_{k\ell}.$$

Definition 1.1.92. Let V be a finite-dimensional vector space. The *tensor algebra* of V is defined

$$T(V) \ := \ \bigoplus_{k=0}^{\infty} T^k(V).$$

The multiplication on $T(V)$ is defined via the tensor product: $T^k V \otimes T^\ell V \to T^{k+\ell} V$, extended linearly to all of TV.

Definition 1.1.93. Let $T(V)$ denote the tensor algebra of a finite-dimensional vector space V. The quotient of $T(V)$ by the two-sided ideal generated by $v \otimes v$, for $v \in V$, defines the *exterior algebra* $\Lambda(V)$.

Definition 1.1.94. Let $\pi : T(V) \to \Lambda(V)$ denote the quotient map. We define the *wedge product* of two elements $\alpha \wedge \beta := \pi(A \otimes B)$, where $\pi(A) = \alpha$ and $\pi(B) = \beta$.

It is easy to check from the definition of $\Lambda(V)$, that $\wedge$ is well-defined, independent of the choice of the representatives of α and β.

Remark 1.1.95. The exterior algebra affords a grading:

$$\Lambda(V) \ = \ \bigoplus_{k=0}^{n} \Lambda^k(V),$$

where $\Lambda^k(V)$ is the *kth exterior power*. This forms a subspace $\Lambda^k(V) \subseteq \Lambda(V)$ spanned by the wedge of k elements of V. Further, if $\{e_1, ..., e_n\}$ is a basis for V, then a basis for $\Lambda^k(V)$ is given by

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

In particular, $\dim \Lambda^k(V) = \binom{n}{k}$.

Observe that if $\dim(V) = n$, then $\Lambda^n(V) = 1$ and is therefore isomorphic to $\mathbb{R}$. Moreover, $\Lambda^n(V) - \{0\}$ has two connected components.

Definition 1.1.96. An *orientation* of a vector space V is a choice of connected component of $\Lambda^n(V) - \{0\}$.

Definition 1.1.97. The *kth exterior power* $\Lambda^k(M)$ of the cotangent bundle T^*X of a smooth manifold M is the vector bundle $\Lambda^k(M) \to M$, where

$$\Lambda^k(M) \ := \ \Lambda^k(T^*M) \ := \ \coprod_{p\in M} \Lambda^k(T_p^*M).$$

Smooth sections of $\Lambda^k(M)$ are called *differential k–forms* or *k–form*. The space of smooth k–forms on M is denoted by Ω_M^k or $\Omega^k(M)$.

We can extend the definition of orientation of a vector space to define the orientation of a manifold as followings:

Definition 1.1.98. A smooth manifold M^n of dimension n is said to be *orientable* if the complement of the zero section $\Lambda_M^n - \{0\}$ has exactly two components. If $\Lambda_M^n - \{0\}$ is connected, then M is said to be *non-orientable*. If M is orientable, an *orientation* is defined to be a nowhere vanishing smooth section of $\Lambda_M^n - \{0\}$. A smooth manifold endowed with an orientation is said to be *oriented*.

Example 1.1.99. The Möbius strip – the surface given by gluing the ends of rectangle after performing a half twist is non-orientable.

Remark 1.1.100. If (M^n, g) is an oriented Riemannian manifold, then we get an induced metric on Λ_M^n given by the *Riemannian volume form*

$$dV_g := \sqrt{\det(g)} dV_{\mathbb{R}^n}.$$

The orientation is given by the choice of the square root of the determinant.

Theorem 1.1.101. Let M be a smooth manifold. There exists a unique linear map

$$d : \Omega^\bullet(M) \longrightarrow \Omega^\bullet(M)$$

such that

(i) $d : \Omega^k(M) \longrightarrow \Omega^{k+1}(M)$.
(ii) $d(f) = df$ (the ordinary differential) for $f \in \Omega^0(M)$.
(iii) (Leibniz rule). If $\sigma \in \Omega^k(M)$ and $\tau \in \Omega^\bullet(M)$, then

$$d(\sigma \wedge \tau) \ = \ (d\sigma) \wedge \tau + (-1)^k \sigma \wedge d\tau.$$

(iv) (Nilpotence). $d^2 = 0$.

Before proving the above theorem, we first establish the following lemma showing that for any exterior differentiation operator d, the value $(d\omega)(x)$ depends only on the behavior of ω in a small neighborhood of x. In particular, exterior differentiation operators are *local* in nature.

Lemma 1.1.102. Let d be an exterior differentiation operator, i.e., a linear map satisfying conditions (i)–(iv) above. Let ω be a differential form such that $\omega|_{\mathcal{U}} = 0$ for some open set $\mathcal{U} \subset M$. Then $(d\omega)|_{\mathcal{U}} = 0$. In particular, if ω and τ are differential forms such that $\omega|_{\mathcal{U}} = \tau|_{\mathcal{U}}$ for some open set $\mathcal{U} \subset M$, then $(d\omega)|_{\mathcal{U}} = (d\tau)|_{\mathcal{U}}$.

Proof. Suppose ω vanishes identically on the open set $\mathcal{U}$. Let $x_0 \in \mathcal{U}$. Take $f : M \to \mathbb{R}$ to be a smooth function such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \notin \mathcal{U}$. The differential form $f\omega$ then vanishes identically on M. Hence, by condition (iii), i.e., the Leibniz rule, we have

$$0 \;=\; d(f\omega) \;=\; (df) \wedge \omega + f d\omega.$$

Evaluating at x_0 shows that $(d\omega)(x_0) = 0$, and since this holds for all $x_0 \in \mathcal{U}$, we see that $(d\omega)|_{\mathcal{U}} = 0$. If $\omega|_{\mathcal{U}} = \tau|_{\mathcal{U}}$, then $(\omega - \tau)|_{\mathcal{U}} = 0$, and therefore,

$$0 \;=\; (d(\omega - \tau))|_{\mathcal{U}} \;=\; (d\omega - d\tau)\,|_{\mathcal{U}},$$

which implies that $(d\omega)|_{\mathcal{U}} = (d\tau)|_{\mathcal{U}}$. □

We are now ready to prove the theorem:

Proof of Uniqueness. Suppose an exterior differentiation operator d exists. Let us show that there is only one of them. To this end, let $x \in M$ be a point, contained in a chart $\mathcal{U} \subset M$, in which we have local coordinates $(x_1, ..., x_n)$. Let ω be a smooth k–form on M. Restricting ω to the coordinate chart $\mathcal{U}$ permits us to write

$$\omega|_{\mathcal{U}} \;=\; \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{1.1.1}$$

where $a_{i_1 \cdots i_k} \in \mathcal{C}^\infty(\mathcal{U}, \mathbb{R})$. Since the right-hand side of (1.1.1) is not a differential form on M, we cannot apply an exterior differentiation operator to it. To deal with this, let $\mathcal{U}_1$ be an open ball containing x such that the closure $\overline{\mathcal{U}}_1 \subset \mathcal{U}$ is contained in $\mathcal{U}$. Let $f \in \mathcal{C}^\infty(M, \mathbb{R})$ be a smooth function defined such that $f(x) = 1$ for all $x \in \mathcal{U}_1$, and $f(x) = 0$ for all $x \notin \mathcal{U}$. Then

$$\widetilde{\omega} \;:=\; \sum_{i_1 < \cdots < i_k} (f a_{i_1 \cdots i_k}) d(f x_{i_1}) \wedge \cdots \wedge d(f x_{i_k})$$

is a smooth k–form on M. Compute

$$\begin{aligned}
d\widetilde{\omega} &= \sum_{i_1<\cdots<i_k} d(fa_{i_1\cdots i_k}d(fx_{i_1})\wedge\cdots\wedge d(fx_{i_k}))\\
&= \sum_{i_1<\cdots<i_k} d(fa_{i_1\cdots i_k})\wedge d(fx_{i_1})\wedge\cdots\wedge d(fx_{i_k})\\
&\qquad + \sum_{i_1<\cdots<i_k} (fa_{i_1\cdots i_k})d(d(fx_{i_1})\wedge\cdots\wedge d(fx_{i_k}))\\
&= \sum_{i_1<\cdots<i_k} d(fa_{i_1\cdots i_k})\wedge d(fx_{i_1})\wedge\cdots\wedge d(fx_{i_k}),
\end{aligned}$$

where the first equality follows from linearity, the second from the Leibniz rule, and third equality from the Leibniz rule and nilpotence. From the lemma, $\widetilde{\omega}|_{\mathcal{U}_1} = \omega|_{\mathcal{U}_1}$ implies that $(d\widetilde{\omega})|_{\mathcal{U}_1} = (d\omega)|_{\mathcal{U}_1}$. Since f is identically 1 on $\mathcal{U}_1$, we see that

$$(d\omega)|_{\mathcal{U}_1} = \sum_{i_1<\cdots<i_k} \partial_{x_j}(a_{i_1\cdots i_k})dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Hence, if an exterior differentiation operator d exists, its value at x must be given by the above formula. Since the point x was arbitrary, this establishes uniqueness. □

Proof of Existence. Let $\mathcal{U}$ be a coordinate chart on M, in which we have local coordinates $(x_1, ..., x_n)$. We first define d locally on $\mathcal{U}$. To this end, let $\omega \in \Omega^k(\mathcal{U})$ be a smooth k–form given by

$$\omega := \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Define

$$d_{\mathcal{U}}\omega = \sum_{i_1<\cdots<i_k}\sum_{j=1}^{n} \partial_{x_j}(a_{i_1\cdots i_k})dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

We extend the definition of $d_{\mathcal{U}}$ to any differential form on $\mathcal{U}$ by forcing $d_{\mathcal{U}}$ to be linear. Properties (i) and (ii) are then immediate. It remains to verify the Liebniz rule and the nilpotence property. First note that any differential form is a sum of forms of the type $a_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}$. Since $d_{\mathcal{U}}$ is linear, and the wedge product is distributive, we need only verify (iii) and (iv) on forms of this type.

Let us verify (iii) for $d_{\mathcal{U}}$. Write $\sigma := a_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ and $\tau := b_{j_1\cdots j_\ell}dx^{j_1}\wedge\cdots\wedge dx^{j_\ell}$. Then

$$\sigma\wedge\tau = a_{i_1\cdots i_k}b_{j_1\cdots j_\ell}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_\ell},$$

and hence,

$$\begin{aligned}
d_{\mathcal{U}}(\sigma \wedge \tau) &= \sum_{r=1}^{n} \left[\partial_{x_r}(a_{i_1\cdots i_k}) b_{j_1\cdots j_\ell} + a_{i_1\cdots i_k}\partial_{x_r}(b_{j_1\cdots j_\ell})\right] dx^r \wedge dx^{i_1} \wedge \cdots \wedge dx^{j_\ell} \\
&= \left(\sum_{r=1}^{n} \partial_{x_r}(a_{i_1\cdots i_k}) dx^r \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}\right) \wedge \left(b_{j_1\cdots j_\ell} dx^{j_1} \wedge \cdots \wedge dx^{j_\ell}\right) \\
&\quad + (-1)^k (a_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}) \wedge \left(\sum_{r=1}^{n} \partial_{x_r}(b_{j_1\cdots j_\ell} dx^r \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_\ell}\right) \\
&= (d_{\mathcal{U}}\sigma) \wedge \tau + (-1)^k \mu \wedge (d_{\mathcal{U}}\tau).
\end{aligned}$$

This verifies property (iii) for $d_{\mathcal{U}}$. For property (iv), if $\sigma = a_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, then

$$d_{\mathcal{U}}\sigma = \sum_{r=1}^{n} \partial_{x_r}(a_{i_1\cdots i_k}) dx^r \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Applying the exterior derivative $d_{\mathcal{U}}$ again,

$$d_{\mathcal{U}}(d_{\mathcal{U}}\sigma) = \sum_{s=1}^{n}\sum_{r=1}^{n} \partial_{x_s}(\partial_{x_r}(a_{i_1\cdots i_k})) dx^s \wedge dx^r \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

The $r = s$ terms in the above expression vanish by the nilpotence of the wedge product, while the $r \neq s$ terms vanish by Clairaut's theorem.

This shows that $d_{\mathcal{U}}$ satisfies the properties of the exterior differentiation operator (i.e., properties (i)–(iv)). From the uniqueness proof given before, every linear operator satisfying properties (i)–(iv) is given by the formula specifying $d_{\mathcal{U}}$. In particular, if $\mathcal{U}_1$ is any open subset of $\mathcal{U}$, the coordinates on $\mathcal{U}$ restrict to coordinates on $\mathcal{U}_1$, and the formula for $d_{\mathcal{U}_1}$ coincides with the formula for $(d_{\mathcal{U}})|_{\mathcal{U}_1}$. Hence, we can define d globally by declaring $(d\omega)|_{\mathcal{U}} = d_{\mathcal{U}}(\omega|_{\mathcal{U}})$ for all differential forms ω, where $\mathcal{U}$ is any coordinate neighborhood. It remains to check that d is well-defined; but this is elementary, since for any pair of coordinate charts $\mathcal{U}$ and $\mathcal{V}$, we have

$$(d_{\mathcal{U}}(\omega|_{\mathcal{U}}))|_{\mathcal{U}\cap\mathcal{V}} = d_{\mathcal{U}\cap\mathcal{V}}(\omega|_{\mathcal{U}\cap\mathcal{V}}) = (d_{\mathcal{V}}(\omega|_{\mathcal{V}}))|_{\mathcal{U}\cap\mathcal{V}}.$$

It is clear that d has properties (i)–(iv), since $d_{\mathcal{U}}$ has these properties for all $\mathcal{U}$. □

Definition 1.1.103. Let M and N be smooth manifolds, and let $\varphi : M \to N$ be a smooth map. We define the *pullback of a k–form* as follows:

(i) If $f : N \to \mathbb{R}$ is a 0–form (i.e., a function), then $\varphi^*(f) = f \circ \varphi$.

(ii) If $\omega \in \Omega^k(N)$, then

$$(\varphi^*\omega)(x)(v_1, ..., v_k) = \omega(\varphi(x))(d\varphi(v_1), ..., d\varphi(v_k)),$$

where $v_1, ..., v_k \in T_xM$, and $x \in M$.

Remark 1.1.104. The following facts are straightforward:

(i) If ω is a smooth differential form, then $\varphi^*\omega$ is easily observed to be a smooth differential form.
(ii) The pullback of a k–form is a k–form.
(iii) The pullback is linear, and moreover, is compatible with the wedge product in the sense that

$$\varphi^*(\sigma \wedge \tau) \;=\; (\varphi^*\sigma) \wedge (\varphi^*\tau).$$

This can be formulated as stating that $\varphi^* : \Omega^k(N) \to \Omega^k(M)$ is an algebra homomorphism.

Theorem 1.1.105. Let $\varphi : M \to N$ be a smooth map between smooth manifolds. Then

$$d \circ \varphi^* \;=\; \varphi^* \circ d.$$

Proof. We first prove the statement on 0–forms. To this end, let $f : N \to \mathbb{R}$ be a smooth function. Then, for $v \in T_xM$, we compute

$$(d \circ \varphi^*)(f)(v) \;=\; d(f \circ \varphi)(v) \;=\; (df \circ d\varphi)(v) \;=\; \varphi^*(df)(v) \;=\; (\varphi^* \circ d)(f)(v).$$

Suppose now that $\omega \in \Omega^1(N)$ is a 1–form given by $\omega = df$. Then

$$(d \circ \varphi^*)(\omega) \;=\; d(\varphi^*(df)) \;=\; d(\varphi^* \circ d(f)) \;=\; d(d \circ \varphi^*(f)) \;=\; 0.$$

Similarly,

$$(\varphi^* \circ d)(\omega) \;=\; \varphi^*(d\omega) \;=\; \varphi^*(d^2 f) \;=\; \varphi^*(0) \;=\; 0.$$

From these two cases, and the fact that φ^* is an algebra homomorphism, the result is established in general by checking it locally on k–forms ω restricted to local coordinate neighborhoods

$$\omega|_{\mathfrak{U}} \;=\; \sum_{1<i_1<\cdots<i_k} a_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

This task is left to the reader. $\square$

Definition 1.1.106. Let $\alpha \in \Omega^p(M)$ be a smooth p–form. We will say that α is *closed* if $d\alpha = 0$, i.e., α lies in the kernel of d. If α lies in the image of d, i.e., if there exists a $(p-1)$–form β such that $\alpha = d\beta$, then we say that α is *exact*.

Example 1.1.107. We have all seen closed and exact forms, long ago, in our undergraduate courses on vector calculus (see, e.g., [54]). If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$, then the curl of $\mathbf{F}$ can be written as

$$\mathrm{curl}(\mathbf{F}) = (\star d(\mathbf{F}^\sharp))^\flat$$

(see § 1.6 for the definition of the Hodge $\star$–operator). We know that $\mathbf{F}$ is *irrotational* if $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$. Hence, from the invertibility of the Hodge $\star$–operator, irrotational vector fields correspond to closed 1–forms. On the other hand, we say that $\mathbf{F}$ is a *gradient field* if $\mathbf{F} = \nabla f$ for some smooth function f. Since $(\nabla f)^\sharp = df$, we see that gradient fields correspond (via the musical isomorphism) to exact 1–forms.

Remark 1.1.108. Since the exterior derivative has the nilpotent property $d^2 = d \circ d = 0$, every exact form is closed. The failure of the converse to hold is measured by the de Rham cohomology groups:

Definition 1.1.109. The cohomology of the complex $(\Omega^\bullet(M), d)$ defines the de Rham cohomology groups:

$$H^p_{\mathrm{DR}}(M, \mathbb{R}) \quad := \quad \frac{\{\alpha \in \Omega^p(M) : d\alpha = 0\}}{\{d\beta : \beta \in \Omega^{p-1}(M)\}}.$$

For $k \in \mathbb{N}_0$, we define the *kth Betti number*

$$b_k(M) := \dim_{\mathbb{R}} H^p_{\mathrm{DR}}(M, \mathbb{R}).$$

Remark 1.1.110. It is clear that the de Rham cohomology groups are diffeomorphism-invariant. That is, if M and N are diffeomorphic, then $H^p_{\mathrm{DR}}(M, \mathbb{R}) \simeq H^p_{\mathrm{DR}}(N, \mathbb{R})$ for all $p \in \mathbb{N}_0$. We will see toward the end of this section that the de Rham cohomology groups depend only on the topology of the underlying smooth manifold.

Example 1.1.111. We observe that $H^0(M, \mathbb{R})$ is the space of locally-constant functions, modulo constant functions. Hence, $b_0(M)$ measures the number of connected components of M.

Example 1.1.112. The vector space $H^1_{\mathrm{DR}}(M, \mathbb{R})$ measures the failure of a closed 1–form to be the exterior derivative of a function. In the classical language of vector calculus (with $M = \mathbb{R}^3$) this is equivalent to measuring the obstruction to every irrotational vector field (i.e., a vector field $\mathbf{F}$ with $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$) being a gradient field (i.e., $\mathbf{F} = \nabla f$, for some smooth function f). If M is simply connected, then $H^1_{\mathrm{DR}}(M, \mathbb{R}) = 0$, but the converse is not true in general. For instance, the Alexander Horned sphere [6] has $H^1_{\mathrm{DR}}(M, \mathbb{R}) = 0$, but is not simply connected (see, e.g., [54]).

Example 1.1.113. The vector space $H^2_{\mathrm{DR}}(M, \mathbb{R})$ also appears in vector calculus. This de Rham cohomology group measures the failure of an incompressible vector field (i.e., a vector field $\mathbf{F}$ with $\mathrm{div}(\mathbf{F}) = 0$) to be solenoidal (i.e., $\mathbf{F} = \mathrm{curl}(\mathbf{G})$ for some vector field $\mathbf{G}$).

Let us show that $H^k_{\mathrm{DR}}(\mathbb{R}^n, \mathbb{R}) = 0$ for all $k > 0$. Since $\mathbb{R}^n$ is diffeomorphic to the unit ball centered at the origin in $\mathbb{R}^n$, it suffices to show that $H^k_{\mathrm{DR}}(\mathbb{B}^n, \mathbb{R}) = 0$. For this, we need the following lemma:

Lemma 1.1.114. Let M be a smooth manifold. Then for each k, consider the maps

$$\Omega^{k-1}(M) \xrightarrow{d} \Omega^k(M) \xrightarrow{d} \Omega^{k+1}(M).$$

Suppose there exist linear maps

$$H_{k-1} : \Omega^k(M) \longrightarrow \Omega^{k-1}(M), \qquad H_k : \Omega^{k+1}(X) \longrightarrow \Omega^k(M)$$

such that

$$H_k \circ d + d \circ H_{k-1} \;=\; \mathrm{id}_k,$$

where id_k denotes the identity map on $\Omega^k(X)$. Then $H^k_{\mathrm{DR}}(M, \mathbb{R}) = 0$.

Proof. Let $\omega \in \Omega^k(M)$ be a closed k–form. Then

$$\omega \;=\; \mathrm{id}(\omega) \;=\; (H_k \circ d + d \circ H_{k-1})(\omega) \;=\; H_k(d\omega) + d(H_{k-1}(\omega)) \;=\; d(H_{k-1}(\omega)).$$

□

Definition 1.1.115. Let M be a smooth manifold. A sequence of linear maps

$$H_k : \Omega^{k+1}(M) \longrightarrow \Omega^k(M),$$

where $k \in \mathbb{N}_0$, satisfying

$$H_k \circ d + d \circ H_{k-1} = \mathrm{id}_k$$

for all k, is called a *homotopy operator*.

Theorem 1.1.116. (the Poincaré lemma). Let $\mathbb{B}^n \subset \mathbb{R}^n$ denote the unit ball centered at the origin in $\mathbb{R}^n$. Then for all $k > 0$,

$$H^k_{\mathrm{DR}}(\mathbb{B}^n, \mathbb{R}) = 0.$$

Proof. From the previous lemma, it suffices to construct a homotopy operator. For each k, the maps will be required to be linear, it suffices to define H_{k-1} on forms

$$\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

With ω defined as above, set

$$H_{k-1}(\omega)(x) \;:=\; \left(\int_0^1 t^{k-1} f(tx) dt\right) \sigma,$$

where

$$\sigma \;:=\; x_{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_k} - x_{i_2} dx^{i_1} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_k} + \cdots + (-1)^{k-1} x_{i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}.$$

Note that σ is precisely the $(k-1)$–form such that $d\sigma = k dx^{i_1} \wedge \cdots \wedge dx^{i_k}$. It suffices to show that

$$H_k \circ d + d \circ H_{k-1} = \mathrm{id}_k.$$

To this end, compute

$$
\begin{aligned}
(d\circ H_{k-1})(\omega)(x) &= d\left[\left(\int_0^1 t^{k-1}f(tx)dt\right)\sigma\right]\\
&= \sum_{j=1}^n \partial_{x_j}\left(\int_0^1 t^{k-1}f(tx)dt\right)dx^j\wedge\sigma+\left(\int_0^1 t^{k-1}f(tx)dt\right)d\sigma\\
&= \sum_{j=1}^n \left(\int_0^1 t^{k-1}\partial_{x_j}(f(tx))dt\right)dx^j\wedge\sigma+\left(\int_0^1 t^{k-1}f(tx)dt\right)d\sigma\\
&= \sum_{j=1}^n \left(\int_0^1 t^{k}f_{x_j}(tx)dt\right)dx^j\wedge\sigma+k\left(\int_0^1 t^{k-1}f(tx)dt\right)dx^{i_1}\wedge\cdots\wedge dx^{i_k}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(H_k\circ d)(\omega)(x) &= H_k\left(\sum_{j=1}^n f_{x_j}dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\\
&= \sum_{j=1}^n\left(\int_0^1 t^k f_{x_j}(tx)dt\right)(x_j dx^{i_1}\wedge\cdots\wedge dx^{i_k}-dx^j\wedge\sigma).
\end{aligned}
$$

Combining these expressions yields

$$
\begin{aligned}
&(d\circ H_{k-1}+H_k\circ d)(\omega)(x)\\
&= \left[k\left(\int_0^1 t^{k-1}f(tx)dt\right)+\sum_{j=1}^n\left(\int_0^1 t^k f_{x_j}(tx)x_j dt\right)\right]dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&= \left[\int_0^1\left(kt^{k-1}f(tx)+t^k\frac{d}{dt}f(tx)\right)dt\right]dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&= \left(\int_0^1\frac{d}{dt}(t^kf(tx))dt\right)dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&= \Big[t^kf(tx)\Big]_0^1 dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
&= f(x)dx^{i_1}\wedge\cdots\wedge dx^{i_k} \;=\; \omega(x),
\end{aligned}
$$

for all $x\in\mathbb{B}^n$. □

Remark 1.1.117. The homotopy operator constructed in the proof of the Poincaré lemma are not plucked out of thin air. To illuminate their definition, let us observe that for a vector v in vector space V, we can define a map

$$
i_v:\Lambda^k(V^*)\longrightarrow\Lambda^{k-1}(V^*),\qquad i_v(\omega)(u_1,...,u_{k-1}) \;=\; \omega(v,u_1,...,u_{k-1}).
$$

The map $i.:V\otimes\Lambda^k(V^*)\to\Lambda^{k-1}(V^*)$ is bilinear.

Definition 1.1.118. The bilinear map $i. : V \otimes \Lambda^k(V^*) \longrightarrow \Lambda^{k-1}(V^*)$ defined by

$$i_v : \Lambda^k(V^*) \longrightarrow \Lambda^{k-1}(V^*), \qquad i_v(\omega)(u_1, ..., u_{k-1}) \ = \ \omega(v, u_1, ..., u_{k-1})$$

is called the *interior product*.

Remark 1.1.119. The $(k-1)$th homotopy operator H_{k-1} is given by applying i_x to ω and averaging over the line through the origin in the direction of x.

Remark 1.1.120. The above theorem is, in fact, a special case of a more general result: Let $\mathcal{U}$ be a smooth manifold. Suppose there exists a smooth map $\Psi : \mathcal{U} \times (-\varepsilon, 1+\varepsilon) \to \mathcal{U}$, where $\Psi(u, 1) = u$ for all $u \in \mathcal{U}$, and $\Psi(u, 0) = u_0$ for all $u \in \mathcal{U}$, and some $u_0 \in \mathcal{U}$. Then $H^k_{\text{DR}}(\mathcal{U}) = 0$ for all $k \in \mathbb{N}$.

The map Ψ is a smooth homotopy. The theorem asserted here states that if $\mathcal{U}$ is smoothly homotopic to a point, then the cohomology of $\mathcal{U}$ is the cohomology of a point. In the theorem we proved above, the smooth homotopy is given by

$$\Psi(x, t) \ := \ tx, \qquad t \in (-\varepsilon, 1+\varepsilon), \ x \in \mathbb{B}^n.$$

Remark 1.1.121. The proof of the Poincaré lemma given above works equally well for domains which are *star-shaped*, i.e., there is a point $x_0 \in \mathcal{U}$ such that the line segment joining x_0 to any other point in $\mathcal{U}$ is contained in $\mathcal{U}$.

In general, the de Rham cohomology groups are difficult to calculate, even for very simple manifolds. Knowing that certain de Rham cohomology groups vanish, however, can be very fruitful. De Rham's theorem asserts that the de Rham cohomology groups are isomorphic to the singular cohomology groups. As a consequence, one can work with singular cohomology for the purposes of calculations, and then appeal to de Rham's theorem to deduce the vanishing of the de Rham cohomology groups. In more detail, let us make the following definition:

Definition 1.1.122. Define the standard p–simplex by

$$\Delta_p \ := \ \left\{ (x_0, ..., x_p) \in \mathbb{R}^{p+1} : \sum_{i=0}^p x_i = 1, \ x_i \geq 0 \right\}.$$

We orient Δ_p with respect to the normal $\mathbf{e} = (1, 1, ..., 1) \in \mathbb{R}^{p+1}$. Let $\Delta_p^{(i)} : \Delta_{p-1} \to \Delta_p$ denote the ith face of Δ_p defined by $(x_0, ..., x_{p-1}) \mapsto (x_0, ..., x_{i-1}, 0, x_i, ..., x_{p-1})$.

Definition 1.1.123. Let M be a smooth manifold. Recall that a *singular p–chain* is a formal linear combination $\sum_k a_k f_k$, where $f_k : \Delta_p \to M$ are maps from the standard p–simplex

$$\Delta_p := \left\{ (x_1, ..., x_p) \in \mathbb{R}^p : \sum_{i=1}^p x_i = 1 \right\}$$

into M. The singular p–chain is said to be *integral* if the coefficients a_k are integers.

Definition 1.1.124. We will declare a singular p–chain to be *piecewise smooth* if the maps f_k extend to $\mathcal{C}^\infty$ maps in a neighborhood of Δ_p into M. The space of piecewise smooth integral p–chains is denoted by $\mathcal{C}_p^{\mathrm{ps}}(M,\mathbb{Z})$.

Let $\delta : \mathcal{C}_p^{\mathrm{ps}}(M,\mathbb{Z}) \to \mathcal{C}_{p-1}^{\mathrm{ps}}(M,\mathbb{Z})$ be the boundary morphism defined by

$$\delta(\gamma) \ := \ \sum_{i=1}^{p}(-1)^i \gamma \circ \Delta_p^{(i)}.$$

Observe that this endows $\mathcal{C}_\bullet^{\mathrm{ps}}(M,\mathbb{Z})$ with the structure of a complex, since:

Proposition 1.1.125. The boundary morphism $\delta : \mathcal{C}_p^{\mathrm{ps}}(M,\mathbb{Z}) \to \mathcal{C}_{p-1}^{\mathrm{ps}}(M,\mathbb{Z})$ is nilpotent in the sense that $\delta^2 = 0$.

PROOF. We first compute

$$\begin{aligned}
\Delta_n^{(i)} \circ \Delta_{n-1}^{(j)}(x_0,...,x_{n-1}) &= \Delta_n^{(i)}(x_0,...,x_{j-1},0,x_j,...,x_{n-1}) \\
&= (x_0,...,x_{j-1},0,x_j,...,x_{i-2},0,x_{i-1},...,x_{n-2}) \\
&= \Delta_n^{(j)}(x_0,...,x_{i-2},0,x_{i-1},...,x_{n-1}) \\
&= \Delta_n^{(j)} \circ \Delta_{n-1}^{(i-1)}(x_0,...,x_{n-1}).
\end{aligned}$$

By the linearity of δ, we need only show that $\delta^2 = 0$ for the standard simplices. To this end, we have

$$\begin{aligned}
\delta_p \circ \delta_{p+1}(\Delta_{p+1}) &= \delta_p\left(\sum_{i=0}^{p+1}(-1)^i \Delta_{p+1}^{(i)}\right) \\
&= \sum_{i=0}^{p+1}(-1)^i \delta_p(\Delta_{p+1}^{(i)}) \\
&= \sum_{i=0}^{p+1}(-1)^i \sum_{j=0}^{p}(-1)^j \Delta_{p+1}^{(j)} \circ \Delta_p^{(i)} \\
&= \sum_{0\le j<i\le p+1}(-1)^{i+j}\Delta_{p+1}^{(i)} \circ \Delta_p^{(j)} + \sum_{0\le i\le j\le p}(-1)^{i+j}\Delta_{p+1}^{(i)} \circ \Delta_p^{(j)} \\
&= \sum_{0\le j<i\le p+1}(-1)^{i+j}\Delta_{p+1}^{(j)} \circ \Delta_p^{(i-1)} + \sum_{0\le i\le j\le n}(-1)^{i+j}\Delta_{p+1}^{(i)} \circ \Delta_p^{(j)}.
\end{aligned}$$

Since, after reindexing the second summation, the second summation is merely the negative of the first summation, the claim follows. □

Let δ denote the boundary morphism associated to the complex $\mathcal{C}_\bullet(M,\mathbb{Z})$ of integral singular p–chains. It is clear that the image of a piecewise smooth integral chain under δ will also be a piecewise smooth integral chain. Hence, $\mathcal{C}^{\mathrm{ps}}_\bullet(M,\mathbb{Z})$ forms a subcomplex of $\mathcal{C}_\bullet(M,\mathbb{Z})$. We can therefore define the homology associated to the complex $(\mathcal{C}^{\mathrm{ps}}_\bullet(M,\mathbb{Z}),\delta)$:

Definition 1.1.126. Let $\mathcal{Z}^{\mathrm{ps}}_p(M,\mathbb{Z}) := \ker(\delta : \mathcal{C}^{\mathrm{ps}}_p(M,\mathbb{Z}) \to \mathcal{C}^{\mathrm{ps}}_{p-1}(M,\mathbb{Z}))$. The homology groups of the complex $(\mathcal{C}^{\mathrm{ps}}_\bullet(M,\mathbb{Z}),\delta)$ are denoted

$$H^{\mathrm{ps}}_p(M,\mathbb{Z}) \;\; := \;\; \frac{\mathcal{Z}^{\mathrm{ps}}_p(M,\mathbb{Z})}{\delta\mathcal{C}^{\mathrm{ps}}_{p+1}(M,\mathbb{Z})}.$$

Observe that since the inclusion map $\mathcal{C}^{\mathrm{ps}}_\bullet(M,\mathbb{Z}) \to \mathcal{C}_\bullet(M,\mathbb{Z})$ induces an isomorphism[6]

$$H^{\mathrm{ps}}_p(M,\mathbb{Z}) \;\simeq\; H_p(M,\mathbb{Z}),$$

every homology class in $H_p(M,\mathbb{Z})$ can be represented by a piecewise smooth p–cycle. Moreover, if a piecewise smooth p–cycle σ is homologous to 0 (in the sense of singular homology), then there is a piecewise smooth $(p+1)$–chain τ with $\sigma = \delta\tau$.

The relation between the homology groups $H^{\mathrm{ps}}_p(M,\mathbb{Z})$ and the de Rham cohomology groups $H^p_{\mathrm{DR}}(M,\mathbb{R})$ is given by integrating forms on chains. We therefore need to make sense of integration of p–forms on a smooth manifold.

Definition 1.1.127. Let M be a topological space. A *boundary chart* centered at a point $x \in M$ is a continuous map $\varphi : \mathcal{U} \to \mathcal{V}$ from an open set $\mathcal{U} \subseteq M$ to a (relatively) open subset $\mathcal{V}$ of $\mathbb{R}^{n-1} \times \{x_n \geq 0\}$ with $\varphi(x) \in \mathbb{R}^{n-1} \times \{0\}$.

Definition 1.1.128. Let M be a topological space. A *smooth boundary atlas* $\mathscr{A}$ is a collection of maps $\varphi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$ each of which is either a chart or a boundary chart for M such that M is covered by the open sets $\mathcal{U}_\alpha$ and such that $\varphi_\alpha \circ \varphi_\beta^{-1}$ is a smooth between open subsets of $\mathbb{R}^{n-1} \times \{x_n \geq 0\}$ for each α and β.

Let M be a smooth manifold with boundary. The *boundary of* M is the subset $\partial M \subset M$ consisting of those points $x \in M$ for which there is a boundary chart about x.

Remark 1.1.129. As in the case of smooth atlases, two smooth boundary atlases are said to be *equivalent* if their union is again a smooth boundary atlas.

Definition 1.1.130. A *smooth manifold with boundary* is a paracompact Hausdorff topological space M endowed with an equivalence class of smooth boundary atlases.

The existence proof of partitions of unity on manifolds (without boundary) readily extends to the case of manifolds with boundary. The only distinction is that we include functions with support $\mathbb{B}(r) \times [0,r)$, but these can be constructed from smooth compactly supported functions we're already familiar with [10]:

[6]The details can be found in the beautiful set of notes by Viaclovsky [292].

Proposition 1.1.131. Let M be a smooth manifold with boundary. Then there exists a partition of unity on M subordinate to any boundary atlas for M.

Since we want to understand integration on manifolds with boundary, we need to understand orientation. If M is a manifold with boundary, we say that M is *oriented* if M supports a smooth boundary atlas whose transition maps are orientation-preserving. From [**10**, Proposition 14.2.1], we have:

Proposition 1.1.132. Let M be an oriented smooth manifold with boundary. Then the boundary ∂M is an oriented smooth manifold of dimension $\dim_{\mathbb{R}} M - 1$.

The orientation on ∂M given by the above Proposition is referred to as the *induced orientation.*

Let us now make sense of integrating differential forms on manifolds with boundary: Let M be a compact oriented smooth manifold with boundary. Let $(\rho_\alpha)_{\alpha \in A}$ be a partition of unity subordinate to an oriented boundary atlas for M such that for each α there is an oriented chart $\varphi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$ such that $\text{supp}(\rho_\alpha) \subset \mathcal{U}_\alpha$. For a smooth n–form $\eta \in \Omega_M^n$, write $\eta = \sum_\alpha \rho_\alpha \eta$ as a sum of n–forms, supported in charts. The integral $\int_M \omega$ is then understood to be the sum of the integrals in each chart upon inserting the coordinate tangent vectors of that chart into ω, i.e.,

$$\int_M \eta \quad := \quad \sum_{\alpha \in A} \int_{\mathcal{V}_\alpha} ((\varphi_\alpha^{-1})_* (\rho_\alpha \eta))(e_1, ..., e_n) dx_1 \wedge \cdots \wedge dx_n.$$

It is not hard to check that the above definition is well-defined, independent of the choice of partition of unity (see, e.g., [**10**, §14.4]).

We now state the higher-dimensional incarnation of the fundamental theorem of calculus [**10**, Proposition 14.5.1]:

Theorem 1.1.133. (Stokes' theorem). Let M be a compact oriented smooth manifold with boundary ∂M. For any $\omega \in \Omega^{n-1}(M)$,

$$\int_M d\omega \quad = \quad \int_{\partial M} \omega,$$

where the integral on the right-hand side is understood to be with respect to the induced orientation on ∂M, integrating the pullback $\iota^* \omega$ under the inclusion map $\iota : \partial M \hookrightarrow M$.

Remark 1.1.134. An immediate corollary: If M is compact, without boundary, the integral of the exterior derivative of any $(n-1)$–form vanishes identically.

Let $\omega \in \Omega^p(M)$ be a smooth p–form on M. Let $\sigma = \sum_k a_k f_k$ be a piecewise smooth p–chain. Integration provides the pairing:

$$\langle \omega, \sigma \rangle \;\; := \;\; \int_\sigma \omega \;=\; \sum_k a_k \int_{\Delta_p} f_k^* \omega.$$

We will construct a map $H^\bullet_{\mathrm{DR}}(M) \to H^\bullet_{\mathrm{sing}}(M)$. To this end, let $\omega \in \Omega^p(M)$ be a smooth closed p–form (i.e., $d\omega = 0$), and σ the boundary of a piecewise smooth $(p+1)$–chain τ (i.e., $\sigma = \delta\tau$). By Stokes' theorem:

$$\int_\sigma \omega \;\; = \;\; \int_{\delta\tau} \omega \;\; = \;\; \int_\tau d\omega \;\; = \;\; 0.$$

Let $\eta = \omega + d\varphi$, for some $\varphi \in \Omega^{p-1}(M)$. Then, for σ a piecewise smooth p–chain (not necessarily a boundary), a further application of Stokes' theorem yields:

$$\int_\sigma \omega - \int_\sigma \eta \;\; = \;\; -\int_\sigma d\varphi \;\; = \;\; 0.$$

The de Rham theorem states that this map is not only well-defined, but an isomorphism:

Theorem 1.1.135. (de Rham theorem). The map

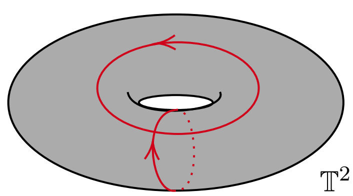
$$H^\bullet_{\mathrm{DR}}(M, \mathbb{R}) \longrightarrow H^\bullet_{\mathrm{sing}}(M, \mathbb{R})$$

is an isomorphism.

Example 1.1.136. The Betti numbers of the (real) n–torus $\mathbb{T}^n \simeq \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n times) are

$$b_p(\mathbb{T}^n) \;\; = \;\; \dim \Lambda^p(\mathbb{R}^n) \;\; = \;\; \binom{n}{p},$$

for each $0 \leq p \leq n$.



Since the bilinear form $H^p_{\mathrm{DR}}(M, \mathbb{R}) \times H^{n-p}_{\mathrm{DR}}(M, \mathbb{R}) \longrightarrow \mathbb{R}$, defined by

$$(\omega, \eta) \;\; \mapsto \;\; \int_M \omega \wedge \eta$$

is non-degenerate, we see that

Theorem 1.1.137. (Poincaré duality). Let M be a compact oriented smooth manifold of dimension n. Then for all $p \geq 0$,

$$H^p_{\mathrm{DR}}(M, \mathbb{R}) \;\; \simeq \;\; H^{n-p}_{\mathrm{DR}}(M, \mathbb{R}).$$

Since $H_p(M,\mathbb{R}) \simeq H^p_{\mathrm{DR}}(M,\mathbb{R})$, we have

Corollary 1.1.138. Let M be a compact oriented smooth manifold of dimension n. Then for all $p \geq 0$,

$$H_p(M,\mathbb{R}) \;\simeq\; H^{n-p}_{\mathrm{DR}}(M,\mathbb{R}).$$

1.2. Complex Manifolds

A smooth manifold, as we defined in the previous section, is a Hausdorff space locally modeled on $\mathbb{R}^n$; moreover, the model is required to have $\mathcal{C}^\infty$ regularity. The regularity requirement is meaningful since it is defined exclusively in terms of the regularity of functions on open subsets of $\mathbb{R}^n$. The same will be true of complex manifolds – Hausdorff spaces locally modeled on $\mathbb{C}^n$, with the regularity of the model required to be holomorphic. We begin by reminding the reader of the meaning of holomorphy for functions of several complex variables.

Let $\mathcal{U}$ be a connected open subset of $\mathbb{C}^n$. Write $z = (z_1, ..., z_n)$ for the coordinates on $\mathbb{C}^n$ (and hence, on $\mathcal{U}$).

Definition 1.2.1. A function $f : \mathcal{U} \to \mathbb{C}$ is said to be *k–differentiable* (for $k = \mathbb{R}$ or $\mathbb{C}$) at a point $z \in \mathcal{U}$ if

$$f(z+\varepsilon) \;=\; f(z) + df(\varepsilon) + o(\varepsilon),$$

where df is a k–linear function, and $o(\varepsilon)/|\varepsilon| \to 0$ as $\varepsilon \to 0$. We refer to df as the *differential of f*.

If $f : \mathcal{U} \to \mathbb{C}$ is $\mathbb{R}$–differentiable, then we can write the differential in real coordinates as

$$df \;=\; \sum_{\nu=1}^{n} \left(\frac{\partial f}{\partial x_\nu} dx_\nu + \frac{\partial f}{\partial y_\nu} dy_\nu \right).$$

Setting $z_\nu := x_\nu + \sqrt{-1} y_\nu$ and $\overline{z}_\nu := x_\nu - \sqrt{-1} y_\nu$, we may write

$$df \;=\; \sum_{\nu=1}^{n} \left(\frac{\partial f}{\partial z_\nu} dz_\nu + \frac{\partial f}{\partial \overline{z}_\nu} d\overline{z}_\nu \right),$$

where, for each $\nu = 1, ..., n$, we have

$$\frac{\partial f}{\partial z_\nu} \;:=\; \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu} - \sqrt{-1}\frac{\partial f}{\partial y_\nu} \right) \qquad \text{and} \qquad \frac{\partial f}{\partial \overline{z}_\nu} \;:=\; \frac{1}{2}\left(\frac{\partial f}{\partial x_\nu} + \sqrt{-1}\frac{\partial f}{\partial y_\nu} \right).$$

We write

$$\partial f \;:=\; \sum_{\nu=1}^{n} \frac{\partial f}{\partial z_\nu} dz_\nu, \qquad \bar\partial f \;:=\; \sum_{\nu=1}^{n} \frac{\partial f}{\partial \overline{z}_\nu} d\overline{z}_\nu.$$

Observe that since $df(\sqrt{-1}w) = \sqrt{-1}\partial f(w) - \sqrt{-1}\bar\partial f(w)$, comparing this with $\sqrt{-1}df(w) = \sqrt{-1}\partial f(w) + \sqrt{-1}\bar\partial f(w)$, we see that df is complex differentiable only if $\bar\partial f$ vanishes identically:

Theorem 1.2.2. Let $f : \mathcal{U} \to \mathbb{C}$ be a function which is $\mathbb{R}$–differentiable at a point $z \in \mathcal{U}$. Then f is $\mathbb{C}$–differentiable if and only if the Cauchy–Riemann equations

$$\bar\partial f \;=\; 0$$

hold.

Remark 1.2.3. Observe that the Cauchy–Riemann condition is equivalent to the system $2n$ real equations:

$$\frac{\partial}{\partial x_\nu}\mathrm{Re}(f) \;=\; \frac{\partial}{\partial y_\nu}\mathrm{Im}(f), \qquad \frac{\partial}{\partial y_\nu}\mathrm{Re}(f) \;=\; -\frac{\partial}{\partial x_\nu}\mathrm{Im}(f),$$

where $\nu = 1, ..., n$. For $n > 1$, the system is overdetermined, which is one of the principal differences between the function theory of several complex variables and the function theory of a single complex variable.

Definition 1.2.4. A function $f : \mathcal{U} \to \mathbb{C}$ is said to be *holomorphic at a point* $p \in \mathcal{U}$ if it is $\mathbb{C}$–differentiable in some neighborhood of p.

Remark 1.2.5. The definition of holomorphic extends readily to maps $f : \mathcal{U} \subseteq \mathbb{C}^n \to \mathbb{C}^m$. Indeed, we can write such a map locally as $f(z) = (f^1(z), ..., f^m(z))$, where $z = (z_1, ..., z_n)$. Then f is said to be *holomorphic* if each of the component functions f^α are holomorphic functions for $1 \leq \alpha \leq m$.

Remark 1.2.6. There is a subtlety concerning the definition of holomorphy and $\mathbb{C}$–differentiability on closed sets, as the example on [**254**, p. 14] illustrates.

Remark 1.2.7. Recall that the real and imaginary parts of a holomorphic function $f : \mathcal{U} \subseteq \mathbb{C} \to \mathbb{C}$ are harmonic. In the higher-dimensional setting, we have the following: Let $f : \mathcal{U} \subseteq \mathbb{C}^n \to \mathbb{C}$ be a holomorphic function. Set

$$u \;:=\; \mathrm{Re}(f) \;=\; \frac{1}{2}(f + \overline{f}), \qquad v \;:=\; \mathrm{Im}(f) \;=\; \frac{1}{2}(f - \overline{f}).$$

Then

$$\frac{\partial u}{\partial \overline{z}_\nu} \;=\; \frac{1}{2}\left(\frac{\partial f}{\partial \overline{z}_\nu} + \overline{\frac{\partial f}{\partial z_\nu}}\right) \;=\; \frac{1}{2}\overline{\frac{\partial f}{\partial z_\nu}}.$$

Since partial derivatives of holomorphic functions are holomorphic (see, e.g., [**254**, p. 20]), we see that for any $\mu, \nu = 1, ..., n$,

$$\frac{\partial^2 u}{\partial z_\mu \partial \overline{z}_\nu} \;=\; \frac{1}{2}\frac{\partial}{\partial z_\mu}\overline{\frac{\partial f}{\partial z_\nu}} \;=\; 0. \tag{1.2.1}$$

This motivates the following definition:

Definition 1.2.8. Let $\mathcal{U} \subseteq \mathbb{C}^n$ be a connected open set. We say that a $\mathcal{C}^2$–function $u : \mathcal{U} \to \mathbb{R}$ is *pluriharmonic* if

$$\partial\bar{\partial}u \;=\; 0.$$

Example 1.2.9. The above discussion shows that the real part of a holomorphic function is pluriharmonic. The argument extends immediately to show that the imaginary part of a holomorphic function is also pluriharmonic. By expanding (1.2.1) in real coordinates and setting $\mu = \nu$, it readily follows that pluriharmonic functions are harmonic.

The following natural extensions of subharmonic functions will play an important role:

Definition 1.2.10. Let $\mathcal{U} \subset \mathbb{C}^n$ be a connected open set. We say that a $\mathcal{C}^2$–function $\varphi : \mathcal{U} \to \mathbb{R}$ is *plurisubharmonic* (respectively, *strictly plurisubharmonic*) at a point $p \in \mathcal{U}$ if the complex Hessian

$$\left(\frac{\partial^2 \varphi}{\partial z_i \partial \overline{z}_j} \right)$$

is positive-semi-definite (respectively, positive-definite) at p. We say that φ is plurisubharmonic on $\mathcal{U}$ if it is plurisubharmonic for every point $p \in \mathcal{U}$ (and similarly, for strict plurisubharmonicity).

Remark 1.2.11. We emphasize that plurisubharmonicity is the familiar convexity suitably relaxed such that the notion is invariant under biholomorphism.

Remark 1.2.12. It is sometimes more advantageous to consider the more general definition of plurisubharmonic functions: An upper semi-continuous function $f : \mathcal{U} \subseteq \mathbb{C}^n \to \mathbb{R} \cup \{-\infty\}$ is said to be plurisubharmonic if the restriction of f to any complex line in $\mathcal{U}$ is a subharmonic function.

Example 1.2.13.

(i) If $f : \mathcal{U} \to \mathbb{C}$ is a holomorphic function, then $\log |f|^2$ is plurisubharmonic. If f vanishes, we understand $\log |f|^2$ to be plurisubharmonic in the sense of 1.2.11.

(ii) The pointwise limit of any decreasing sequence of plurisubharmonic functions is plurisubharmonic.

(iii) If $u_1, ..., u_p \in \mathrm{PSH}(\mathcal{U})$ and $\chi : \mathbb{R}^p \to \mathbb{R}$ is a convex function such that $\chi(t_1, ..., t_p)$ is an increasing function in each t_j, then $\chi(u_1, ..., u_p) \in \mathrm{PSH}(\mathcal{U})$. In particular, if f is holomorphic, then by taking $u = \log |f|$ and $\chi(t) = e^{\varepsilon t}$ for $\varepsilon > 0$, the function $|f|^\varepsilon$ is plurisubharmonic.

Definition 1.2.14. Let X be a smooth manifold of (real) dimension $\dim_{\mathbb{R}}(X)$. We say that X is a *complex manifold* if X admits an atlas whose transition maps are holomorphic. The *complex dimension* of X is defined to be $\dim_{\mathbb{C}} X := \frac{1}{2} \dim_{\mathbb{R}} X$.

Definition 1.2.15. Let X be a complex manifold. A smooth submanifold $Y \subseteq X$ is said to be a *complex submanifold* of (complex dimension k) if for all $y \in Y$, there is an open neighborhood $\mathcal{U} \subset X$ and a holomorphic chart $\varphi : \mathcal{U} \to \mathbb{C}^n$ such that $\varphi(Y \cap \mathcal{U}) = \varphi(\mathcal{U}) \cap \mathbb{C}^k$.

Example 1.2.16. The simplest example of a complex manifold is complex Euclidean space $\mathbb{C}^n$. The real dimension of $\mathbb{C}^n$ is $\dim_{\mathbb{R}} \mathbb{C}^n = 2n$, and the complex dimension is $\dim_{\mathbb{C}} \mathbb{C}^n = n$.

Remark 1.2.17. On $\mathbb{R}^2$, there are precisely two distinct complex structures: the canonical structure on $\mathbb{C}$ and that of the unit disk in $\mathbb{C}$. For $n > 1$, however, $\mathbb{R}^{2n}$ affords an infinite number of different complex structures. As discovered by Calabi–Eckmann [69], there is even a complex structure on $\mathbb{R}^{2n}$ which does not admit any non-constant holomorphic functions.

Example 1.2.18. From 1.1.19, observe that the transition maps for complex projective space $\mathbb{P}^n$ are holomorphic. In particular, $\mathbb{P}^n$ is a compact complex manifold of (complex) dimension n.

Remark 1.2.19. There is only one complex structure on $\mathbb{P}^1$ and $\mathbb{P}^2$. That is, any complex manifold diffeomorphic to $\mathbb{P}^n$ is biholomorphic to $\mathbb{P}^n$ if $n = 1$ or $n = 2$. The uniqueness of the complex structure on $\mathbb{P}^n$ for $n = 3$ is intimately related to the existence of a complex structure on $\mathbb{S}^6$. Indeed, it is an old observation of Hirzebruch [159] that if $\mathbb{S}^6$ supports a complex structure (i.e., is diffeomorphic to a complex manifold), then the blow-up of $\mathbb{S}^6$ at one point is diffeomorphic to an exotic[7] $\mathbb{P}^3$. Let us note that the only spheres which can be complex manifolds are $\mathbb{S}^2$ (with the $\mathbb{P}^1$ complex structure) and $\mathbb{S}^6$. We know that $\mathbb{S}^2$ is a complex manifold, but it remains open as to whether $\mathbb{S}^6$ supports a holomorphic atlas.

Terminology 1.2.20. A complex manifold of (complex) dimension 1 is referred to as a *Riemann surface*. This is unfortunate, since a *complex surface* will always be understood to be a complex manifold of (complex) dimension 2. A Riemann surface that is projective (i.e., admits a holomorphic embedding into some $\mathbb{P}^n$) is typically referred to as a *curve*.

Remark 1.2.21. Complex manifolds are significantly more rigid than their real smooth counterparts. The most notable example is the failure of the holomorphic analog of the Whitney embedding theorem. Indeed, no compact complex manifold supports a holomorphic embedding to $\mathbb{C}^n$ for any $n \in \mathbb{N}$. Suppose otherwise, and let $X \hookrightarrow \mathbb{C}^n$ be a holomorphic embedding of a compact complex manifold X. The coordinates on $\mathbb{C}^n$ restrict to X yielding bounded (in modulus) holomorphic functions, which will attain a local maximum. By the maximum principle, such functions must be constant.

Those complex manifolds which do admit a holomorphic embedding into some $\mathbb{C}^n$ form the important class:

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[7]It is clear that this exotic $\mathbb{P}^3$ is not Kähler: From the vanishing of the fourth Betti number, $\mathbb{S}^6$ is not balanced. Alessandrini–Bassanelli showed that the balanced condition is preserved under bimeromorphic map (in particular, under blow-ups). Hence, the exotic $\mathbb{P}^3$ does not admit a balanced metric, let alone a Kähler metric.

Definition 1.2.22. A complex manifold S is said to be *Stein* if there is a holomorphic embedding $S \hookrightarrow \mathbb{C}^n$ for some $n \in \mathbb{N}$.

Example 1.2.23. A Riemann surface is Stein if and only if it is not compact. An open set $\mathcal{D} \subset \mathbb{C}^n$ is Stein if and only if it is holomorphically convex, or equivalently, by the Cartan–Thullen theorem, a domain of holomorphy. By Cartan's theorem B, Stein manifolds may be characterized by the vanishing of the higher sheaf cohomology $H^q(X, \mathscr{S}) = 0$ $(q \geq 1)$ for any coherent analytic sheaf $\mathscr{S}$. We invite the reader to consult [**137, 254**] for the proofs of these statements and further details.

Remark 1.2.24. Stein manifolds X capture the idea of admitting an abundance of holomorphic functions $X \to \mathbb{C}$ (see, e.g., [**188**]). It is natural to ask whether there is any rich structure to the class of manifolds for which there are no holomorphic functions $X \to \mathbb{C}$. This class of manifolds is easily seen to be too vast: Every compact complex manifold resides within this class by 1.2.20. One does obtain a rich class of complex manifolds; however, if one considers the question of the existence of holomorphic maps $\mathbb{C} \to X$:

Definition 1.2.25. A complex manifold X is said to be *Brody hyperbolic* if there are no entire curves, i.e., non-constant holomorphic maps $\mathbb{C} \to X$.

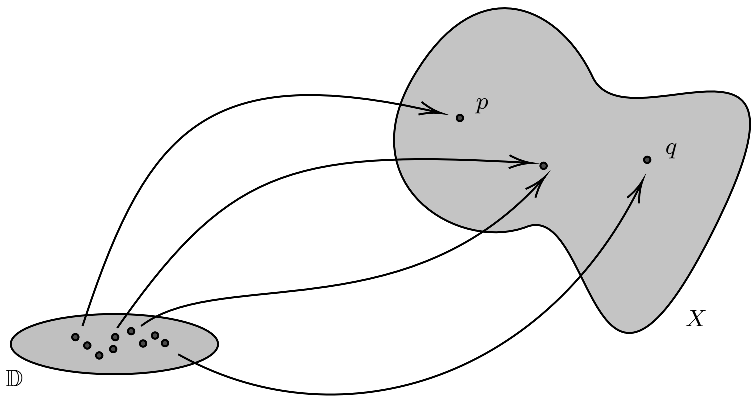
Example 1.2.26. The unit disk $\mathbb{D} \subset \mathbb{C}$ is Brody hyperbolic by Liouville's theorem.

A significant class of Brody hyperbolic manifolds is given by the following class of manifolds introduced by Kobayashi:

Definition 1.2.27. (Kobayashi pseudodistance). Let X be a complex manifold. The *Kobayashi pseudodistance* d_X on X is defined, for $p, q \in X$

$$d_X(p,q) \quad := \quad \inf \sum_{j=1}^m d_\rho(s_j, t_j),$$

where the infimum is taken over all $m \in \mathbb{N}$, all pairs of points $(s_j, t_j) \in \mathbb{D} \times \mathbb{D}$, and all collections of holomorphic maps $f_j : \mathbb{D} \to X$, where $j = 1, ..., m$, such that $f_1(s_1) = p$, $f_m(t_m) = q$, and $f_j(t_j) = f_{j+1}(s_{j+1})$ for $j = 1, ..., m-1$.



Remark 1.2.28. It is straightforward to verify that d_X defines a pseudo-distance in the sense that d_X is symmetric and satisfies the triangle inequality. In general, however, d_X may degenerate in the sense that $d_X(p, q) = 0$ for $p \neq q$. Indeed, this is the case for the Kobayashi pseudodistance on the complex line $\mathbb{C}$, or projective line $\mathbb{P}^1$. On the other hand, the Kobayashi pseudodistance on the disk $\mathbb{D}$ coincides with the Poincaré distance function, i.e., $d_{\mathbb{D}} = d_\rho$. Hence, in this case, the Kobayashi pseudo-distance is non-degenerate and defines an honest distance function.

Remark 1.2.29. The Kobayashi pseudodistance has the important property that it does not increase under holomorphic maps, i.e., for any holomorphic map $f : X \to Y$, we have

$$d_Y(f(p), f(q)) \leq d_X(p, q).$$

This property, together with the fact that the Kobayashi pseudodistance on $\mathbb{C}$ is identically zero, i.e., $d_{\mathbb{C}} \equiv 0$, we have

Proposition 1.2.30. Let X be a complex manifold for which any two points $p, q \in X$ are connected by an entire curve $\mathbb{C} \to X$. Then the Kobayashi pseudodistance is identically zero, $d_X \equiv 0$.

Remark 1.2.31. More generally, the above proposition can be formulated with the assumption that any two points $p, q \in X$ are connected by a (finite) chain of entire curves $\mathbb{C} \to X$.

Definition 1.2.32. A complex manifold X is said to be *Kobayashi hyperbolic* if the Kobayashi pseudodistance d_X is an honest distance function, in the sense that d_X is non-degenerate.

Example 1.2.33. Since a complex manifold is Kobayashi hyperbolic if and only if its universal cover is Kobayashi hyperbolic [184], the uniformization theorem tells us that a compact

Riemann surface is Kobayashi hyperbolic if and only if the genus is $g \geq 2$. In particular, the projective line $\mathbb{P}^1$ and elliptic curves $\mathbb{T} = \mathbb{C}/\Lambda$, are not Kobayashi hyperbolic.

Observe that an immediate consequence of the distance-decreasing under holomorphic maps property of the Kobayashi pseudo-distance is the following:

Proposition 1.2.34.

(i) Closed complex submanifolds of Kobayashi hyperbolic manifolds are Kobayashi hyperbolic.
(ii) If X is Kobayashi hyperbolic, then X is Brody hyperbolic.

The second claim follows from the fact that holomorphic maps are distance-decreasing for the Kobayashi pseudo-distance, and the Kobayashi pseudodistance of $\mathbb{C}$ vanishes identically. If X happens to be compact, however, then the two notions coincide [**62, 324**]:

Theorem 1.2.35. (Brody's theorem). Let X be a compact complex manifold. Then X is Kobayashi hyperbolic if and only if X is Brody hyperbolic.

Proof. Suppose there is a non-constant entire curve $\mathbb{C} \to X$. From the distance-decreasing property of the Kobayashi pseudo-distance, we see that $d_{f^*X} \leq d_{\mathbb{C}} = 0$, and therefore, X is not Kobayashi hyperbolic.
Conversely, suppose that X is not Kobayashi hyperbolic. Then there is a sequence of holomorphic maps $f_\nu : \mathbb{D} \to X$ such that $|f_\nu'(0)|_\omega^2 \to \infty$. To see this, we observe that if this is not the case, then there is a constant $C > 0$ such that $|f'(0)|_\omega^2 \leq C$ for all holomorphic maps $f : \mathbb{D} \to X$. Since $\mathbb{D}$ is homogeneous, $|f'(z)|_\omega^2 \leq C$ for any holomorphic map $f : \mathbb{D} \to X$ and any $z \in \mathbb{D}$. Hence,

$$d_\rho(0, z) \ = \ \frac{1}{2} \ln \left(\frac{1 + |z|}{1 - |z|} \right) \ \geq \ |z| \ \geq \ C^{-1} d_\omega(f(0), f(z)),$$

where d_ρ is the Poincaré distance on $\mathbb{D}$ and d_ω is the distance function associated to the Hermitian metric ω. In particular, the length of any chain of holomorphic disks from p to q in X is at least $C^{-1} d_\omega(p, q) > 0$, which violates the assumption that X is not Kobayashi hyperbolic.
Given the sequence $f_\nu : \mathbb{D} \to X$ with $|f_\nu'(0)|_\omega^2 \to \infty$, we apply the Brody reparametrization lemma to obtain holomorphic maps $g_\nu = f_\nu \circ \psi_\nu : \mathbb{D}(R_\nu) \to X$ such that

$$|g_\nu'(t)|_\omega^2 \ \leq \ \frac{1}{1 - |t|^2/R_\nu^2},$$

with $R_\nu \to \infty$ and $|g_\nu'(0)|_\omega^2 = 1$. Since X is compact, the derivatives are uniformly bounded, by Montel's theorem we can extract a convergent subsequence $g_{\nu_i} : \mathbb{D}(R_{\nu_i}) \to X$. Since the radii $R_{\nu_i} \to \infty$, the limit will be a non-constant entire curve $g : \mathbb{C} \to X$ with $|g'(t)|_\omega^2 \leq 1$. □

Example 1.2.36. The above theorem fails in the non-compact case. The standard example is the domain $\mathcal{D} \subset \mathbb{C}^2$ given by

$$\mathcal{D} = \{(z,w) \in \mathbb{C}^2 : |z| < 1, \ |zw| < 1\} - \{(0,w) : |w| \geq 1\}.$$

It is clear that $\mathcal{D}$ is not Kobayashi hyperbolic since the distance between the origin and any point $(0,w) \in \mathcal{D}$ is zero. In any case, however, there are no entire curves $\mathbb{C} \to \mathcal{D}$, rendering $\mathcal{D}$ Brody hyperbolic.

Example 1.2.37. More examples of Brody hyperbolic manifolds which are not Kobayashi hyperbolic were constructed[8] by Eisenman–Taylor [**184**, p. 130] and Campbell–Howard–Ochiai [**72**]. An example of a pseudoconvex (hence, Stein) domain in $\mathbb{C}^2$ which is Brody hyperbolic, but not Kobayashi hyperbolic, was given by Barth [**20**].

Vector Bundles. Just as we did in the smooth case, we can define holomorphic immersions, submersions, and embeddings by requiring that f is holomorphic in the definition. In particular, we have the following important class of holomorphic submersions:

Definition 1.2.38. Let $f : \mathcal{E} \to X$ be a holomorphic submersion between complex manifolds. We say that f is a *holomorphic vector bundle* of rank k if the fibers support a complex vector space structure $\mathcal{E}_x := f^{-1}(x) \simeq \mathbb{C}^k$, for all $x \in X$, and in a neighborhood of every point $x \in X$, there is an open neighborhood $\mathcal{U}$ such that $f^{-1}(\mathcal{U}) \simeq_{\text{bihol.}} \mathcal{U} \times \mathbb{C}^k$. Moreover, the biholomorphism between $f^{-1}(\mathcal{U})$ and $\mathcal{U} \times \mathbb{C}^k$ restricts the isomorphism of vector spaces $\mathcal{E}_x \simeq \mathbb{C}^k$ for all $x \in \mathcal{U}$.

Cautionary Remark 1.2.39. Note that a holomorphic vector bundle is far from a complex vector bundle. A complex vector bundle is merely a smooth vector bundle with fibers isomorphic to $\mathbb{C}^k$. On the other hand, a holomorphic vector bundle has the fibers varying holomorphically.

Example 1.2.40. Let X be a complex manifold. The tangent bundle[9] $T^{1,0}X$ is a holomorphic vector bundle. The holomorphic sections of $T^{1,0}X$ are called *holomorphic vector fields*. If $(z_1, ..., z_n)$ denote local holomorphic coordinates on X, a holomorphic vector field ξ is locally given by

$$\xi \ = \ \sum_\alpha \xi^\alpha(z) \frac{\partial}{\partial z_\alpha},$$

where ξ^α are locally defined holomorphic functions.

[8] In effect, the construction amounts to working on a hyperbolic manifold, which is not Kobayashi hyperbolic but contains only a few entire curves. Then removing small enough pieces of these curves such that the resulting domain becomes Brody hyperbolic, without disturbing the absence of Kobayashi hyperbolicity.

[9] This notation is explained in 1.2.62.

Example 1.2.41. The cotangent bundle T^*X of a complex manifold X is a holomorphic vector bundle. The holomorphic sections of T^*X are denoted by $\Omega_X^{1,0}$ and are called *holomorphic* $(1,0)$*–forms.* In local holomorphic coordinates $(z_1, ..., z_n)$, a holomorphic $(1,0)$–form η is given by

$$\eta \ = \ \sum_\alpha \eta_\alpha(z) dz^\alpha.$$

Example 1.2.42. A very important role in complex geometry is played by *line bundles* – holomorphic vector bundles of rank 1. Let X be a complex manifold of (complex) dimension n. The most important line bundle is given by the canonical bundle $K_X := \Lambda_X^{n,0}$. The local holomorphic sections of K_X are given by

$$\omega \ = \ f(z) dz^1 \wedge \cdots \wedge dz^n,$$

where f is a locally defined holomorphic function.

Example 1.2.43. Let V be an $(n+1)$–dimensional vector space and write $\mathbb{P}^n(V)$ for the vector space of lines $\ell \subset V$ through the origin. Let ℓ_x denote the line corresponding to a point $x \in \mathbb{P}^n$. Let $\mathcal{E} \subset \mathbb{P}^n \times V$ denote the set of points (x, v_x), where $v_x \in \ell_x$. On V, introduce the coordinates $(x_0, ..., x_n)$. Let $\mathcal{U}_\alpha \subset V$ be the open set given by $\mathcal{U}_\alpha := \{x_\alpha \neq 0\}$. Then $\mathcal{E}|_{\mathcal{U}_\alpha}$ corresponds to the set of points $\xi = (t_1 : \cdots : t_n ; y_0 : \cdots : y_n)$, where $y_i = t_i y_\alpha$ and $t_i = x_i / x_\alpha$. The map $\xi \mapsto ((t_1, ..., t_n), y_\alpha)$ defines an isomorphism $\mathcal{E}|_{\mathcal{U}_\alpha} \simeq \mathcal{U}_\alpha \simeq \mathbb{C}$.

Definition 1.2.44. The *tautological bundle* $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the line bundle over $\mathbb{P}^n$ with fiber $\mathcal{O}_{\mathbb{P}^n}(-1)_x \simeq \ell_x$. From 1.2.42, we see that $\mathcal{O}_{\mathbb{P}^n}(-1)$ is a vector bundle of rank 1, i.e., a (holomorphic) *line bundle.*

Example 1.2.45. The *hyperplane bundle* $\mathcal{O}_{\mathbb{P}^n}(1) \to \mathbb{P}^n$ is the line bundle corresponding to a hyperplane in $\mathbb{P}^n$. In more detail, let $D \subset \mathbb{P}^n$ be the hyperplane corresponding to, say, $x_0 = 0$. In the open set $\mathcal{U}_\alpha = \{x_\alpha \neq 0\}$, the local equation for this hyperplane is x_0/x_α. Associated to D, therefore, is the line bundle whose transition maps are $g_{\alpha\beta} = x_\alpha / x_\beta$. From 1.2.42, the transition maps of the tautological bundle $\mathcal{O}_{\mathbb{P}^n}(-1)$ are given by $h_{\alpha\beta} = x_\beta / x_\alpha$. Hence, $\mathcal{O}_{\mathbb{P}^n}(1)$ is the line bundle dual to $\mathcal{O}_{\mathbb{P}^n}(-1)$.

In contrast with $\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$, there are many global sections of $\mathcal{O}_{\mathbb{P}^n}(1) \to \mathbb{P}^n$. We can identify $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1))$ with the space of homogenous polynomials of degree 1 in $(n+1)$ variables.

Associated Projective Bundle. Let $\mathcal{E} \to X$ be a holomorphic vector bundle of rank k over a complex manifold X. We can associate to $\mathcal{E}$ another holomorphic vector bundle $\mathbb{P}(\mathcal{E})$, the *projectivized bundle associated to* $\mathcal{E}$, whose fiber over $x \in X$ is the complex projective space $\mathbb{P}(\mathcal{E}_x^*) \simeq \mathbb{P}^{k-1}$, where $\mathcal{E}^*$ is the dual bundle.

Observe that since $\mathbb{P}(\mathcal{E})$ can be identified with a quotient of the unit sphere bundle of $\mathcal{E}$ (for any Hermitian metric on $\mathcal{E}$), it is clear that $\mathbb{P}(\mathcal{E})$ is compact.

This construction will be used extensively in the present manuscript since it gives a means of associating to a vector bundle $\mathcal{E}$ of any rank, a line bundle: Indeed, if $\mathcal{E} \to X$ is a holomorphic vector bundle of rank k, then $\mathbb{P}(\mathcal{E}) \to X$ is a holomorphic vector bundle of rank $k-1$. On $\mathbb{P}(\mathcal{E})$, we have the tautological line bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1) \to \mathbb{P}(\mathcal{E})$, whose fiber at a point $(p, [v]) \in \mathbb{P}(\mathcal{E})$ is the line spanned (over $\mathbb{C}$) by $v \in \mathcal{E}_p^* \backslash \{0\}$. The dual of the tautological bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(-1)$ is the hyperplane bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. The hyperplane bundle over a projectivized bundle will play a central role in discussions of positivity notions for vector bundles of rank $k > 1$.

Blow-ups. Let X be a complex manifold. An important mechanism for generating new complex manifolds $\widetilde{X}$ which are bimeromorphic but not biholomorphic, is given by blow-ups:

Definition 1.2.46. Let X be a complex manifold and $p \in X$ a point. The *blow-up* of X at p is a bimeromorphic map $f : \text{Bl}_p(X) \longrightarrow X$ given by adjoining $\widetilde{\mathcal{U}} := \{(w, \ell) : w \in \ell\} \subset \mathcal{U} \times \mathbb{P}^{n-1}$ to $X \backslash \{p\}$ via the map $\widetilde{\mathcal{U}} \backslash \{z = 0\} \simeq \mathcal{U} \backslash \{p\}$ defined by $(z, \ell) \mapsto z$. This defines a bimeromorphic map $\text{Bl}_p(X) \longrightarrow X$ which extends to a morphism $f : \text{Bl}_p(X) \to X$. The preimage $\mathcal{E} := f^{-1}(p)$ is isomorphic to $\mathbb{P}(T_p^{1,0}X) \simeq \mathbb{P}^{n-1}$, and is called the *exceptional divisor* of the blow-up.

Remark 1.2.47. The total space $\text{Bl}_p(X)$ of the blow-up is diffeomorphic to the connected sum $X \sharp \overline{\mathbb{P}^{n-1}}$ of X with $\mathbb{P}^{n-1}$ endowed with the reverse orientation. A detailed proof of this assertion is given in [**168**, Proposition 2.5.8]. The key points here, however, are the following: The blow-up of $\mathbb{C}^n$ at $0 \in \mathbb{C}^n$ is biholomorphic to the total space of the tautological bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$, the exceptional divisor being the zero section of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. For a complex line bundle $\mathcal{L}$, the dual of a complex line bundle $\mathcal{L}$ is isomorphic (as a complex line bundle) to the conjugate bundle $\overline{\mathcal{L}}$. For any $x \in \mathbb{P}^n$, projecting away from x defines a line bundle $\mathbb{P}^n \backslash \{x\} \to \mathbb{P}^{n-1}$ which is isomorphic (as a complex line bundle) to the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$. Indeed, if we abusively write $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ and $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$ for the total space of the tautological bundle and hyperplane bundle over $\mathbb{P}^{n-1}$, respectively, then (i) implies

$$\mathbb{C}^n \backslash \{0\} \simeq \mathcal{O}_{\mathbb{P}^{n-1}}(-1) - E.$$

Statement (ii) then asserts that $\mathcal{O}_{\mathbb{P}^{n-1}}(-1) \simeq \overline{\mathcal{O}_{\mathbb{P}^{n-1}}(1)}$ as smooth manifolds, with the diffeomorphism mapping the zero section of $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$ to the zero section of $\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. From (iii), we identify $\overline{\mathcal{O}_{\mathbb{P}^{n-1}}(1)} \simeq \overline{\mathbb{P}^n \backslash \{x\}}$, and hence, the one-point compactification is diffeomorphic to $\overline{\mathbb{P}^n}$. The geometric picture is that under the identification in (i), real rays directed into the origin are transformed into rays going out of x in $\mathbb{P}^n$.

Hermitian Vector Bundles.

Definition 1.2.48. Let $\mathcal{E} \to X$ be a holomorphic vector bundle. A *Hermitian metric* h on $\mathcal{E}$ is a family of Hermitian inner products $h_p : \mathcal{E}_p \times \mathcal{E}_p \to \mathbb{C}$ smoothly parametrized by $p \in X$.

Notation 1.2.49. If $\xi = \{\xi^1, ..., \xi^r\}$ is a local frame for $\mathcal{E}$, defined in an open neighborhood of $x \in X$, we write

$$h_{ij}(x) \ := \ h(\xi^i(x), \xi^j(x))$$

for the components of the metric. We often omit the variable argument and write h_{ij}.

Definition 1.2.50. A frame $\xi = \{\xi^1, ..., \xi^r\}$ is said to be *unitary* (with respect to h) if

$$h_{ij} = h(\xi^i, \xi^j) = \delta_{ij}.$$

Unitary frames always exist locally: take any given frame and apply the Gram–Schmidt procedure.

Definition 1.2.51. A holomorphic vector bundle endowed with a Hermitian metric is referred to as a *Hermitian vector bundle*.

Example 1.2.52. Let (X, ω) be a complex manifold of (complex) dimension n with canonical bundle $K_X = \Lambda_X^{n,0}$. A Hermitian metric h on K_X is given by a volume form (i.e., non-vanishing section of $\Lambda_X^{n,n}$).

Almost Complex Structures. We are primarily interested in how the metric and the complex structure interact and how this interaction determines the geometry of the manifold. The existence of local holomorphic coordinates to define a complex manifold is not very amenable to this study. It will therefore be important to formulate the complex manifold structure differently:

Definition 1.2.53. Let M be a smooth manifold. We say that M supports an *almost complex structure* if there is a smooth section $J \in H^0(M, \text{End}(TM))$ satisfying $J^2 = -\text{id}$ as a morphism of bundles. A smooth manifold supporting an almost complex structure J is said to be an *almost complex manifold*.

Remark 1.2.54. Since an almost complex structure is an endomorphism of TM, it is locally described in a coordinate frame:

$$J = J_k^\ell dx^k \otimes \partial_{x_\ell}.$$

The condition $J^2 = -\text{id}$ then reads

$$J_i^k J_k^\ell \ = \ -\delta_i^\ell.$$

Example 1.2.55. Let $e_1 = (1, 0)$ and $e_2 = (0, 1)$ denote the standard basis on $\mathbb{R}^2$. An almost complex structure on $\mathbb{R}^2$ is given by

$$J := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

More generally, if $\mathrm{id}_{\mathbb{R}^n}$ denotes the identity matrix on $\mathbb{R}^n$, an almost complex structure on $\mathbb{R}^{2n}$ is given by

$$J := \begin{pmatrix} 0 & \mathrm{id}_{\mathbb{R}^n} \\ -\mathrm{id}_{\mathbb{R}^n} & 0 \end{pmatrix}.$$

Example 1.2.56. Consider the unit sphere $\mathbb{S}^2$ in $\mathbb{R}^3$. On $\mathbb{R}^3$, we have a multiplication – the cross product $\times$ – inherited from identifying $\mathbb{R}^3$ with the imaginary quaternions. We identify any point $p \in \mathbb{S}^2$ with the corresponding vector (also denoted by p) in $\mathbb{R}^3$. An almost complex structure $J_p : T_p\mathbb{S}^2 \to T_p\mathbb{S}^2$ is then given by

$$J_p(v) := p \times v.$$

Definition 1.2.57. Let $f : (X, J_X) \longrightarrow (Y, J_Y)$ be a smooth map between almost complex manifolds. We say that f is holomorphic if the differential $f_* : TX \longrightarrow TY$ is compatible with the complex structure in the sense that

$$f_* \circ J_X = J_Y \circ f_*.$$

Example 1.2.58. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a smooth map, and let (x, y) denote the coordinates on $\mathbb{R}^2$. We can write $f(x, y) = [u(x, y), v(x, y)]^t$. The differential is then

$$f_* = \begin{pmatrix} u_x & v_x \\ u_y & v_y \end{pmatrix}.$$

If we endow $\mathbb{R}^2$ with the almost complex structure defined in 1.2.54, then $f_* \circ J_{\mathbb{R}^2} = J_{\mathbb{R}^2} \circ f_*$ is equivalent to the *Cauchy–Riemann equations*: $u_x = v_y$ and $v_x = -u_y$.

Example 1.2.59. Any complex manifold supports an almost complex structure. Indeed, let $p \in X$ be a point contained in holomorphic charts $\varphi : \mathcal{U} \to \mathbb{C}^n$ and $\psi : \mathcal{V} \to \mathbb{C}^n$. Let $J_{\mathcal{U}} := \varphi_*^{-1} \circ J_{\mathbb{C}^n} \circ \varphi_*$ and $J_{\mathcal{V}} := \psi_*^{-1} \circ J_{\mathbb{C}^n} \circ \psi_*$. Let $\sigma : \varphi(\mathcal{U} \cap \mathcal{V}) \to \psi(\mathcal{U} \cap \mathcal{V})$ be the transition map. Then

$$\begin{aligned} J_{\mathcal{V}} = \psi_*^{-1} \circ J_{\mathbb{C}^n} \circ \psi_* &= \psi_*^{-1} \circ J_{\mathbb{C}^n} \circ \psi_* \\ &= \psi_*^{-1} \circ J_{\mathbb{C}^n} \circ (\sigma \circ \varphi)_* \\ &= \psi_*^{-1} \circ J_{\mathbb{C}^n} \circ \sigma_* \circ \varphi_* \\ &= \psi_*^{-1} \circ \sigma_* \circ J_{\mathbb{C}^n} \circ \varphi_* \\ &= \varphi_*^{-1} \circ J_{\mathbb{C}^n} \circ \varphi_* = J_{\mathcal{U}}. \end{aligned}$$

Hence, J is independent of the chart and yields a well-defined tensor on X.

Remark 1.2.60. It is easy to see that any even-dimensional smooth manifold M admits a map $J_p : T_pM \to T_pM$ satisfying $J_p^2 = -\text{id}_p$ for each $p \in M$. Conversely, any almost complex manifold must be of even dimension. This certainly does not guarantee that such maps glue together to yield a section of the endomorphism bundle $\text{End}(TM)$. Further, since the problem of the existence of an almost complex structure can be formulated in terms of the existence of a section of a vector bundle, characteristic classes give obstructions to finding an almost complex structure. This will be discussed later.

Example 1.2.61. An old theorem of Kirchoff [**180**] states that if $\mathbb{S}^n$ supports an almost complex structure, then $\mathbb{S}^{n+1}$ is parallelizable. From 1.1.47, we see that the only spheres to admit almost complex structures are $\mathbb{S}^2$ and $\mathbb{S}^6$.

Remark 1.2.62. Let V be a real vector space of even (real) dimension. Let $J : V \to V$ be a linear map satisfying $J^2 = -\text{id}$. Denote by the same symbol J, the complex-linear extension of J to the complexification $V \otimes_{\mathbb{R}} \mathbb{C}$. From $J^2 = -\text{id}$, the eigenvalues of J are $\pm\sqrt{-1}$ with corresponding eigenspaces

$$V^{1,0} \;:=\; \{v - \sqrt{-1}Jv : v \in V\}, \qquad V^{0,1} \;:=\; \{v + \sqrt{-1}Jv : v \in V\}.$$

In particular, we have a splitting of the complexification

$$V \otimes_{\mathbb{R}} \mathbb{C} \;=\; V^{1,0} \oplus V^{0,1}, \qquad V^{1,0} \;=\; \overline{V^{0,1}}.$$

The reader can easily show that this splitting structure is, in fact, equivalent to the data of an almost complex structure on the vector space V. That is, if V is a (real) even-dimensional vector space with $\mathcal{L}$ a complex subspace of $V \otimes_{\mathbb{R}} \mathbb{C}$, with the properties $\mathcal{L} \cap \overline{\mathcal{L}} = \{0\}$ and $\mathcal{L} \oplus \overline{\mathcal{L}} = V \otimes_{\mathbb{R}} \mathbb{C}$, then there is a unique complex linear map $J : V \to V$ such that $\mathcal{L}$ and $\overline{\mathcal{L}}$ are the eigenspaces of (the $\mathbb{C}$–linear extension of) J corresponding to $\sqrt{-1}$ and $-\sqrt{-1}$, respectively.

Definition 1.2.63. Let (X, J) be an almost complex manifold. We say that a vector $u \in T^{1,0}X$ (respectively, $v \in T^{0,1}X$) is a $(1,0)$–*tangent vector* (respectively, a $(0,1)$–*tangent vector*). In terms of tangent vectors on TX, a $(1,0)$–tangent vector and $(0,1)$–tangent vector is given by

$$u - \sqrt{-1}Ju \in T^{1,0}X, \qquad v + \sqrt{-1}Jv \in T^{0,1}X,$$

respectively.

We have seen that a complex manifold, i.e., a smooth manifold supporting a holomorphic atlas, is an almost complex manifold. The converse is not true and is most notably obstructed by the integrability of the tangent distribution $T^{0,1}X \subset T^{\mathbb{C}}X$:

Definition 1.2.64. Let (X, J) be an almost complex manifold. The almost complex structure J is said to be *integrable* if the tangent distribution $T^{0,1}X$ is integrable.

Proposition 1.2.65. Let X be a complex manifold. Then the natural almost complex structure J on X is integrable.

PROOF. For any point $p \in X$, we have a holomorphic chart $\varphi : \mathcal{U} \to \mathbb{C}^n$ centered at p. We write $\varphi = (\varphi_1, ..., \varphi_n)$ and set $\varphi_k = z_k = x_k + \sqrt{-1} y_k$. Let $\{\mathbf{e}_1, ..., \mathbf{e}_{2n}\}$ denote the standard basis on $\mathbb{R}^{2n}$. Then by definition

$$\frac{\partial}{\partial x_k} \;=\; \varphi_*^{-1}(\mathbf{e}_k), \qquad \frac{\partial}{\partial y_k} \;=\; \varphi_*^{-1}(\mathbf{e}_{n+k}).$$

Let $J_{\mathbb{C}^n}$ denote the standard almost complex structure on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$. Then $J_{\mathbb{C}^n}(\mathbf{e}_k) = \mathbf{e}_{n+k}$, and since the chart φ is holomorphic, we have

$$J\left(\frac{\partial}{\partial x_k}\right) \;=\; \frac{\partial}{\partial y_k}. \tag{1.2.2}$$

Introduce the notation

$$\frac{\partial}{\partial z_k} \;:=\; \frac{1}{2}\left(\frac{\partial}{\partial x_k} - \sqrt{-1}\frac{\partial}{\partial y_k}\right), \qquad \frac{\partial}{\partial \overline{z}_k} \;:=\; \frac{1}{2}\left(\frac{\partial}{\partial x_k} + \sqrt{-1}\frac{\partial}{\partial y_k}\right).$$

From (1.2.2), we see that $\frac{\partial}{\partial z_k}$ and $\frac{\partial}{\partial \overline{z}_k}$ define local sections of $T^{1,0}X$ and $T^{0,1}X$, respectively, forming a local frame at each point of $\mathcal{U}$. Let now $u = \sum_k u_k \frac{\partial}{\partial \overline{z}_k}$ and $v = \sum_k v_k \frac{\partial}{\partial \overline{z}_k}$ denote local sections of $T^{0,1}X$. The Lie bracket is computed to be

$$[u, v] \;=\; \sum_{k,\ell=1}^{n} u_k \frac{\partial v_\ell}{\partial \overline{z}_k}\frac{\partial}{\partial \overline{z}_\ell} - \sum_{k,\ell=1}^{n} v_k \frac{\partial u_\ell}{\partial \overline{z}_k}\frac{\partial}{\partial \overline{z}_\ell},$$

which is undoubtedly a local section of $T^{0,1}X$. $\square$

The converse is the celebrated Newlander–Nirenberg theorem [220]:

Theorem 1.2.66. (Newlander–Nirenberg). Suppose (X, J) is an almost complex manifold with J an integrable almost complex structure. Then (X, J) is a complex manifold.

Corollary 1.2.67. An almost complex structure J is integrable if and only if the *Nijenhuis tensor* of J:

$$\mathcal{N}^J(u, v) \;:=\; [u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv]$$

vanishes identically.

PROOF. The statement is effectively immediate from 1.2.65. Indeed, for vector fields u, v, an elementary calculation shows that

$$N^J(u, v) \;=\; [u + \sqrt{-1}Ju, v + \sqrt{-1}Jv].$$

Hence, $T^{0,1}X$ is integrable if and only if $N^J(u,v) - \sqrt{-1}JN^J(u,v) \in T^{0,1}X$ if and only if $N^J(u,v) = 0$. $\square$

Remark 1.2.68. Observe if X is an oriented smooth manifold of (real) dimension 2. The Nijenhuis vanishes identically; hence, any almost complex structure on X is integrable. Indeed, for any vector field u (defined locally near a point $p \in X$), the tangent space T_pX is spanned by u and Ju. Hence, we need only compute the Nijenhuis tensor on u, Ju. In this case, we see that

$$\begin{aligned} N_J(u,Ju) &= [u,Ju] + J[Ju,Ju] + J[u,J^2u] - [Ju,J^2u] \\ &= [u,Ju] + [Ju,u] \; = \; 0. \end{aligned}$$

Example 1.2.69. The above remark implies that any almost complex structure on $\mathbb{S}^2$ is integrable.

Remark 1.2.70. We have seen that in the presence of an almost complex structure, the complexified tangent bundle affords the splitting

$$T^{\mathbb{C}}X \;\simeq\; T^{1,0}X \oplus T^{0,1}X.$$

The complexified cotangent bundle affords a similar splitting

$$T^*X \otimes \mathbb{C} \;\simeq\; \Lambda^{1,0}X \oplus \Lambda^{0,1}X.$$

Here, $\Lambda^{1,0}_X$ and $\Lambda^{0,1}_X$ denote the bundles of $(1,0)$–forms and $(0,1)$–forms, respectively. A 1–form α is a $(1,0)$–form if $\alpha(Jv) = \sqrt{-1}\alpha(v)$, or a $(0,1)$–form if $\alpha(Jv) = -\sqrt{-1}\alpha(v)$. Let $\Lambda^{*,0}X := \Lambda_{\mathbb{C}}(\Lambda^{1,0}X)$ and $\Lambda^{0,*}X := \Lambda_{\mathbb{C}}(\Lambda^{0,1}X)$ denote the full exterior algebra over $\mathbb{C}$. We then have the following isomorphism of complex vector bundles:

$$\Lambda_{\mathbb{C}}(T^*X \otimes \mathbb{C}) \;\simeq\; \Lambda^{*,0}X \otimes \Lambda^{0,*}X.$$

Let $\Lambda^{p,0}X$ denote the bundle of complex-valued p–forms α such that

$$\alpha(Jv_1, ..., v_p) = \sqrt{-1}\alpha(v_1, ..., v_p),$$

and similarly, denote by $\Lambda^{0,q}$ the bundle of complex-valued q–forms η such that

$$\eta(Jv_1, ..., v_q) = -\sqrt{-1}\eta(v_1, ..., v_q).$$

This yields gradings:

$$\Lambda^{*,0} \;=\; \bigoplus_{p=0}^{n} \Lambda^{p,0}X, \qquad \Lambda^{0,*} \;=\; \bigoplus_{q=0}^{n} \Lambda^{0,q}X.$$

Set $\Lambda^{p,q}(X) := \Lambda^{p,0}(X) \otimes \Lambda^{0,q}(X)$.

Definition 1.2.71. Let X be a complex manifold. For $0 \leq p, q \leq n = \dim_{\mathbb{C}} X$, we refer to the sections of the bundle $\Lambda^{p,q}(X)$ as (p,q)*–forms*, denoting the space of (p,q)–forms by $\Omega^{p,q}(X)$.

Remark 1.2.72. In a (holomorphic) local coordinate system $(z_1, ..., z_n)$, a (p,q)–form α is given by

$$\alpha = \sum_{1 \leq i_1 < \cdots < i_p \leq n} \sum_{1 \leq j_1 < \cdots < j_q \leq n} f_{i_1 \cdots i_p j_1 \cdots j_q} dz_{i_1} \wedge \cdots \wedge dz_{i_p} \wedge d\bar{z}_{j_1} \wedge \cdots \wedge d\bar{z}_{j_q},$$

where the locally-defined functions $f_{i_1 \cdots i_p j_1 \cdots j_q}$ are smooth. If these functions are holomorphic, we refer to α as a *holomorphic* (p,q)*–form.*

Definition 1.2.73. If a (p,q)–form $\alpha \in \Omega^{p,q}(X)$ is invariant under conjugation, then we say that α is a *real* (p,q)*–form.* The space of real (p,q)–forms is denoted by $\Omega^{p,q}_{\mathbb{R}}(X)$.

Definition 1.2.74. Let $\alpha \in \Omega^{p,p}_X$ be a real (p,p)–form. We say that α is *positive* if

$$(-\sqrt{-1})^p \alpha(v_1, \overline{v}_1, ..., v_p, \overline{v}_p) > 0$$

for any set of linearly indepedent (over $\mathbb{C}$) vectors $v_1, ..., v_p$.

Remark 1.2.75. If $\alpha \in \Omega^k(X)$ is a smooth k–form on X, then $\alpha = \sum_{p+q=k} \alpha^{p,q}$, where $\alpha^{p,q} \in \Omega^{p,q}(X)$ is the (p,q)–part. Moreover, if α is a real k–form of type (p,q), then an elementary argument shows that $p = q$ and α is J–invariant. In particular, if ω is a real $(1,1)$–form, then

$$\omega(Ju, Jv) = \omega(u, v)$$

for all $u, v \in TX$.

The splitting of forms into forms of type (p,q) comes equipped with two natural operators which arise from the splitting of the exterior derivative:

Definition 1.2.76. Let X^n be a complex manifold, with $d : \Omega^k_{\mathbb{C}}(X) \to \Omega^{k+1}_{\mathbb{C}}(X)$ denoting the exterior derivative acting on complex-valued forms. We define the *Dolbeault operator* $\partial : \Omega^{p,q}(X) \to \Omega^{p+1,q}(X)$ and $\overline{\partial} : \Omega^{p,q}(X) \to \Omega^{p,q+1}(X)$ by the formulae

$$\partial\alpha := \pi^{p+1,q}(d\alpha), \qquad \overline{\partial}\alpha := \pi^{p,q+1}(d\alpha),$$

where $\pi^{p+1,q} : \Omega^{k+1}_{\mathbb{C}}(X) \to \Omega^{p+1,q}(X)$ and $\pi^{p,q+1} : \Omega^{k+1}_{\mathbb{C}}(M) \to \Omega^{p,q+1}(X)$ denote the natural projection maps.

Terminology 1.2.77. Let X be a complex manifold with Dolbeault operators ∂ and $\bar{\partial}$. We refer to the second-order differential operator $\partial\bar{\partial}$ (or $\sqrt{-1}\partial\bar{\partial}$) as the *complex Hessian.* If X

supports a Hermitian metric ω_g (see 1.5.1), then we define the *complex Laplacian* Δ_{ω_g} to be the trace (with respect to ω_g) of the complex Hessian:

$$\Delta_{\omega_g} f \;:=\; \mathrm{tr}_{\omega_g}(\sqrt{-1}\partial\bar{\partial} f) \;=\; g^{i\bar{j}}\partial_i \partial_{\bar{j}} f,$$

where $f \in \mathcal{C}^\infty(X)$.

The Dolbeault operators, like the exterior derivative, are nilpotent:

Proposition 1.2.78. Let X be a complex manifold. Then

$$\partial^2 = 0 \qquad \bar{\partial}^2 = 0, \qquad \text{and} \qquad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

The proof of the above proposition is a straightforward extension of the proof that $d^2 = 0$ together with the fact that $d = \partial + \bar{\partial}$.

Definition 1.2.79. A (p, q)–form α is said to be $\bar{\partial}$*–closed* if $\bar{\partial}\alpha = 0$. We say that α is $\bar{\partial}$*–exact* if there is a $(p, q-1)$–form β such that $\alpha = \bar{\partial}\beta$.

The analogous definitions can be made for ∂. Moreover, immediate from $\bar{\partial}^2 = 0$ is the fact that $\bar{\partial}$–exact forms are $\bar{\partial}$–closed.

Definition 1.2.80. Let X be a complex manifold. The *Dolbeault cohomology groups* of X are defined

$$H^{p,q}_{\bar{\partial}}(X) \quad := \quad \frac{\{\alpha \in \Omega^{p,q}(X) : \overline{\partial}\alpha = 0\}}{\{\overline{\partial}\beta : \beta \in \Omega^{p,q-1}(X)\}}.$$

We define the *Hodge numbers* $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}_{\bar{\partial}}(X)$.

Notation 1.2.81. We will often omit the subscript $\bar{\partial}$ and simply write $H^{p,q}_{\bar{\partial}}$ for the Dolbeault cohomology groups.

We want to establish the Dolbeault analog of the Poincaré lemma; that is, we will show that the Dolbeault cohomology groups of the polydisk $\mathbb{D}^n$ vanish.

Theorem 1.2.82. (Cauchy–Pompeiu formula [151]). Let $\mathcal{D} \subset \mathbb{C}$ be a bounded domain with a smooth boundary. Let $\mathcal{U}$ be an open neighborhood of the closure of $\mathcal{D}$. If f is an $\mathbb{R}$–differentiable function on $\mathcal{U}$, then for all $z \in \mathcal{D}$, we have

$$f(z) \quad = \quad \frac{1}{2\pi\sqrt{-1}} \int_{\partial\mathcal{D}} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi\sqrt{-1}} \int_{\mathcal{D}} \frac{\partial f}{\partial\bar{\zeta}} \frac{d\zeta \wedge d\bar{\zeta}}{\zeta - z}.$$

Proof. Fix a point $p \in \mathcal{D}$ and let $\mathbb{D}(r) \subset \mathcal{D}$ be a sufficiently small disk centered at p such that $\overline{\mathbb{D}(r)} \subset \mathcal{D}$. Write Γ_r for the boundary of $\overline{\mathbb{D}(r)}$ and set $\mathcal{D}_r := \mathcal{D} - \mathbb{D}(r)$. We observe that at any $\zeta \in \mathcal{D}_r$, we have

$$d\left(\frac{f(\zeta)}{\zeta - z} d\zeta\right) \quad = \quad \frac{\partial f}{\partial\bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

Stokes' theorem then implies that

$$\int_{\mathcal{D}_r} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} = \int_{\partial \mathcal{D}_r} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_{\partial \mathcal{D}} \frac{f(\zeta)}{\zeta - z} d\zeta - \int_{\Gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta. \tag{1.2.3}$$

Let $\zeta = z + re^{\sqrt{-1}\vartheta}$. Then

$$\int_{\Gamma_r} \frac{f(\zeta)}{\zeta - z} d\zeta = \int_0^{2\pi} f(z + re^{\sqrt{-1}\vartheta})\sqrt{-1} d\vartheta.$$

We estimate

$$\begin{aligned} \left| \int_0^{2\pi} f(z + re^{\sqrt{-1}\vartheta})\sqrt{-1} d\vartheta - 2\pi\sqrt{-1} f(z) \right| &\leq \int_0^{2\pi} \left| f(z + re^{\sqrt{-1}\vartheta}) - f(z) \right| d\vartheta \\ &\leq 2\pi r \max_{0 \leq \vartheta \leq 2\pi} \left| f(z + re^{\sqrt{-1}\vartheta}) \right|. \end{aligned}$$

Letting $r \to 0$, we see that the right-hand side of (1.2.3) converges to

$$\int_{\partial \mathcal{D}} \frac{f(\zeta)}{\zeta - z} d\zeta - 2\pi\sqrt{-1} f(z).$$

On the other hand, if we compute the left-hand side of (1.2.3) directly, we see that

$$\int_{\mathcal{D}_r} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} = \int_{\mathcal{D}} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} - \int_{\mathbb{D}(r)} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}.$$

As before, set $\zeta = z + re^{i\vartheta}$. Then $d\zeta = e^{i\vartheta} dr + rie^{i\vartheta} d\vartheta$ and $d\bar{\zeta} = e^{-i\vartheta} dr - rie^{-i\vartheta} d\vartheta$. Hence

$$\frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z} = 2ie^{-i\vartheta} d\vartheta \wedge dr,$$

which is bounded as $r \to 0$. Since $\frac{\partial f}{\partial \bar{\zeta}}$ is continuous, the above argument shows that

$$\int_{\mathbb{D}(r)} \frac{\partial f}{\partial \bar{\zeta}} \frac{d\bar{\zeta} \wedge d\zeta}{\zeta - z}$$

converges to 0 as $r \to 0$. This proves the desired statement. □

Remark 1.2.83. The Cauchy–Pompeiu formula does not even scratch the surface of the fascinatingly rich subject concerning integral representations of holomorphic functions. The reader may wish to consult the [**237**, **254**, **294**] for more on this subject.

Proposition 1.2.84. Let $\mathcal{D} \subset \mathbb{C}$ be a bounded domain with a smooth boundary. Let $\mathcal{U}$ be an open neighborhood of the closure of $\mathcal{D}$. If f is an $\mathbb{R}$–differentiable function on $\mathcal{U}$, then there is an $\mathbb{R}$–differentiable function $u : \mathcal{U} \to \mathbb{C}$ such that

$$f = \frac{\partial u}{\partial \bar{z}}.$$

PROOF. We set

$$u(z) \;:=\; \frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{D}}\frac{f(\zeta)}{\zeta - z}d\zeta\wedge d\bar{\zeta}.$$

Fix a point $p \in \mathcal{D}$ and let $\mathbb{D}(r) \subset \mathcal{D}$ be a sufficiently small disk such that $\overline{\mathbb{D}(r)} \subset \mathcal{D}$. Set $\mathcal{D}_r := \mathcal{D} - \mathbb{D}(r)$. For any $\zeta \in \mathcal{D}_r$, we have

$$d\log|\zeta - z|^2 \;=\; \frac{d\zeta}{\zeta - z} + \frac{d\bar{\zeta}}{\bar{\zeta}-\bar{z}}.$$

The same argument as in the proof of 1.2.81 shows that

$$\int_{\partial\mathcal{D}} f(\zeta)\log|\zeta - z|^2 d\bar{\zeta} \;=\; \int_{\mathcal{D}}\frac{\partial f}{\partial \zeta}\log|\zeta - z|^2 d\zeta\wedge d\bar{\zeta} + \int_{\mathcal{D}} f(\zeta)\frac{d\zeta\wedge d\bar{\zeta}}{\zeta - z}.$$

In particular, differentiating u, we see that

$$\begin{aligned}
\frac{\partial u}{\partial \bar{z}} &= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\mathcal{D}} f(\zeta)\left(\frac{\partial}{\partial\bar{z}}\log|\zeta - z|^2\right)d\bar{\zeta} - \frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{D}}\frac{\partial f}{\partial\zeta}\left(\frac{\partial}{\partial\bar{z}}\log|\zeta - z|^2\right)d\bar{\zeta} \\
&= \frac{1}{2\pi\sqrt{-1}}\int_{\partial\mathcal{D}}\frac{f(\zeta)}{\bar{\zeta}-\bar{z}}d\bar{\zeta} - \frac{1}{2\pi\sqrt{-1}}\int_{\mathcal{D}}\frac{\partial f}{\partial\zeta}\frac{d\zeta\wedge d\bar{\zeta}}{\zeta - z} \\
&= f(z).
\end{aligned}$$

□

Theorem 1.2.85. (Dolbeault lemma). Let ω be a smooth $\bar{\partial}$–closed (p,q)–form on a polydisk $\mathcal{U}$ in $\mathbb{C}^n$. Then there exists a smooth $(p, q-1)$–form α on $\mathcal{U}$ such that $\omega = \bar{\partial}\alpha$.

PROOF. We give the standard proof, which appears in [**151, 295**]: We first reduce to the case when $p = 0$. For multi-indices I, J with $|I| = p$ and $|J| = q$, we can locally write the (p,q)–form α as

$$\alpha \;=\; \sum_{I,J}\alpha_{IJ}dz_I\wedge d\overline{z}_J.$$

Introduce the $(0,q)$–forms $\alpha_I := \sum_J \alpha_{IJ}d\overline{z}_J$. Since

$$0 \;=\; \bar{\partial}\alpha \;=\; \sum_{I,J}\bar{\partial}\alpha_{IJ}dz_I\wedge d\overline{z}_J,$$

the $(0,q)$–forms α_I are $\bar{\partial}$–closed. If the Dolbeault lemma holds for $(0,q)$–forms, then locally we can find $(0, q-1)$–forms β_I such that $\alpha_I = \bar{\partial}\beta_I$. Hence, $\alpha = (-1)^p\bar{\partial}\left(\sum_I dz_I\wedge\beta_I\right)$, proving Dolbeault lemma, in general. It suffices, therefore, to prove the Dolbeault lemma for $(0,q)$–forms, which we can locally write as

$$\alpha \;=\; \sum_J \alpha_J d\overline{z}_J.$$

We will proceed by induction on the largest $k \in \mathbb{Z}$ such that there exists a multi-index J containing k and such that $\alpha_J \neq 0$; necessarily, $k \geq q$. If $k = q$, then

$$\alpha = f d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_q,$$

for some smooth function f. The condition that α is $\bar{\partial}$–closed is then equivalent to f being holomorphic in the variables $z_{q+1}, ..., z_n$. From 1.2.83, there is a smooth function g, holomorphic in the variables $z_{q+1}, ..., z_n$, such that

$$\frac{\partial g}{\partial \overline{z}_q} \ = \ f.$$

Therefore, if we set $\beta := (-1)^{q-1} g d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_{q-1}$, we see that

$$\bar{\partial}\beta = f d\overline{z}_1 \wedge \cdots \wedge d\overline{z}_q \ = \ \alpha.$$

This proves the result for $k = q$. The induction hypothesis is then to assume that $k \geq q+1$. Let us write

$$\alpha = \gamma + \delta \wedge d\overline{z}_k,$$

where only the coordinates $z_1, ..., z_{k-1}$ appear in the local expressions for γ and δ.
Let $\delta := \sum_J \delta_J d\overline{z}_J$, where $|J| = q-1$ and the entries of the multi-index J are contained in $\{1, ..., k-1\}$. The condition $\bar{\partial}\alpha = 0$ implies that the functions δ_J are holomorphic in the variables $z_{k+1}, ..., z_n$. Hence, by 1.2.83, we can find smooth functions η_J, holomorphic in the variables $z_1, ..., z_{k-1}$, such that

$$\delta_J \ = \ \frac{\partial \eta_J}{\partial \overline{z}_k}.$$

We can therefore write $\alpha = \gamma' + \bar{\partial}\beta$, where only the $z_1, ..., z_{k-1}$ coordinates appear in the local expression for γ'. Since $\bar{\partial}\alpha = 0$, it follows that $\bar{\partial}\gamma' = 0$. The induction hypothesis implies that $\gamma' = \bar{\partial}\beta'$, and hence, $\alpha = \bar{\partial}(\beta' + \beta)$, proving the claim. □

Remark 1.2.86. The Dolbeault cohomology groups are less intimately related to the underlying topology of the manifold in comparison with the de Rham cohomology groups (which are entirely topological). For instance, we will see in the next section that there are simply connected complex manifolds with $h^{0,1} \neq 0$.

Remark 1.2.87. However, there is a relationship between the Dolbeault cohomology groups and the topology of the underlying manifold. This is given by the Frölicher spectral sequence, which we will not discuss here since it will take us too far afield.

1.3. Sheaves and their Cohomology

Despite the many illustrious properties of vector bundles, the category of vector bundles does not have good exactness[10] properties. We will need to consider more general objects, namely, sheaves.

Definition 1.3.1. Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups is specified by the following data:

(i) for every open subset $\mathcal{U} \subseteq X$, there is an abelian group $\mathcal{F}(\mathcal{U})$.
(ii) for any inclusion of open sets $\mathcal{V} \subseteq \mathcal{U} \subseteq X$, there is a morphism of groups

$$\varrho_{\mathcal{V}}^{\mathcal{U}} : \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{F}(\mathcal{V})$$

called a *restriction map.*

Further, we demand that $\mathcal{F}(\emptyset) = 0$; for all open sets, we have $\rho_{\mathcal{U}}^{\mathcal{U}} = \mathrm{id}$; if $\mathcal{W} \subseteq \mathcal{V} \subseteq \mathcal{U}$, then $\varrho_{\mathcal{U}}^{\mathcal{W}} = \varrho_{\mathcal{V}}^{\mathcal{W}} \circ \varrho_{\mathcal{V}}^{\mathcal{U}}$.

Remark 1.3.2. For the reader familiar with the categorical jargon, the above definition specifies that a presheaf of abelian groups is merely a contravariant functor from the category of open subsets of a topological space to the category of abelian groups.

Moreover, one can consider more general presheaves, not just presheaves of abelian groups. For instance, with appropriate modifications, one can define presheaves of sets, rings, $\mathcal{O}_X$–modules. We invite the reader to consult [**46, 152, 173**] for more details.

Terminology 1.3.3. Let $\mathcal{U} \subseteq X$ be an open subset of a topological space X. Let $\mathcal{F}$ be a presheaf of abelian groups on X. The elements of $\mathcal{F}(\mathcal{U})$ are called the *sections of* $\mathcal{F}$ *over* $\mathcal{U}$.

Example 1.3.4. Vector bundles form a specific class of presheaves (locally free sheaves). We discuss vector bundles in greater detail in 1.3.13.

Example 1.3.5. Let X be a complex manifold. The most important example of a presheaf is the *presheaf of holomorphic functions* $\mathcal{O}_X$. The presheaf $\mathcal{O}_X$ assigns to an open set $\mathcal{U} \subset X$ the abelian group $\mathcal{O}_X(\mathcal{U}) := \{f : \mathcal{U} \to \mathbb{C} : f \text{ is holomorphic}\}$ of holomorphic functions on $\mathcal{U}$. More precisely, the elements are *germs of holomorphic functions* – two holomorphic functions $f : \mathcal{U} \to \mathbb{C}$ define the same element in $\mathcal{O}_X(\mathcal{U})$ if and only if they coincide on some non-empty open subset of $\mathcal{U}$.

Example 1.3.6. Let X be a smooth manifold. Other important examples of presheaves are the presheaf $\mathcal{C}_X$ of (germs of) continuous functions[11], the presheaf $\mathcal{C}_X^{\infty}$ of (germs of) smooth functions, and the presheaf Ω_X^p of (germs of) smooth (or holomorphic) p–forms on X. If G is

[10] More precisely, vector bundles do not form an abelian category.

[11] This, of course, does not require the topological space X to be a smooth manifold.

an abelian group, the *constant presheaf* is the presheaf which assigns to each open set $\mathcal{U} \subseteq X$ the abelian group G.

Definition 1.3.7. Let $\mathcal{F}, \mathcal{G}$ be two presheaves over a topological space X. A *morpshism of presheaves* $f : \mathcal{F} \longrightarrow \mathcal{G}$ is a collection of morphisms

$$f_{\mathcal{U}} : \mathcal{F}(\mathcal{U}) \longrightarrow \mathcal{G}(\mathcal{U}),$$

for each open set $\mathcal{U}$, such that for any inclusion of open sets $\mathcal{V} \subset \mathcal{U}$,

$$\varrho^{\mathcal{V}}_{\mathcal{U}} \circ f_{\mathcal{U}} \;=\; f_{\mathcal{V}} \circ \varrho^{\mathcal{U}}_{\mathcal{V}}.$$

Definition 1.3.8. A *sheaf of abelian groups* on a topological space X is a presheaf $\mathcal{F}$ such that if $\{\mathcal{U}_\alpha\}$ is an open cover of an open set $\mathcal{U} \subset X$, then

(i) (uniqueness). $\varrho^{\mathcal{U}}_{\mathcal{U}_\alpha}(\sigma) = 0$ for all α implies that $\sigma = 0$.

(ii) (existence). if there are sections $\sigma_\alpha \in \mathcal{F}(\mathcal{U}_\alpha)$ such that, for all pairs α, β, they coincide on overlaps:

$$\varrho^{\mathcal{U}_\alpha}_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}(\sigma_\alpha) \;=\; \varrho^{\mathcal{U}_\beta}_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}(\sigma_\beta),$$

then there exists a section $\sigma \in \mathcal{F}(\mathcal{U})$ such that $\varrho^{\mathcal{U}}_{\mathcal{U}_\alpha}(\sigma) = \sigma_\alpha$ for each α.

Example 1.3.9. The reader can easily verify that the presheaves $\mathcal{O}_X$, $\mathcal{C}_X$, $\mathcal{C}^\infty_X$, and Ω^p_X are also sheaves. The constant presheaf, however, is not a sheaf: Take X to be a two-point set $X = \{a, b\}$ endowed with the discrete topology, for instance.

Remark 1.3.10. As in 1.3.2, the definition of sheaf can be extended to more general algebraic objects beyond abelian groups.

We are well-acquainted with vector bundles. These turn out to be a special class of sheaves:

Definition 1.3.11. A sheaf $\mathcal{F}$ on a topological space X is said to be *locally free* of rank k if $\mathcal{F}$ is locally isomorphic to $\mathcal{O}^{\oplus k}_X$.

Remark 1.3.12. There are several remarks which need to be made concerning the above definition: The first is that implicit in the above definition is that $\mathcal{O}^{\oplus k}_X$ (the direct sum of k–copies of $\mathcal{O}_X$) is a sheaf. This is undoubtedly true since the direct sum of sheaves is easily seen to be a sheaf.

The second, more important remark is on the meaning of $\mathcal{O}_X$ if X is not a complex manifold. In the above definition, we understand $\mathcal{O}_X$ not as the sheaf of (germs of) holomorphic functions but as the *structure sheaf* of X. That is, $\mathcal{O}_X$ is a sheaf of (germs of) functions such that if X is a topological space with no extra structure, then $\mathcal{O}_X$ is $\mathcal{C}_X$, the sheaf of (germs of) continuous functions. If X is a smooth manifold, then $\mathcal{O}_X = \mathcal{C}^\infty_X$, the sheaf of (germs of) smooth functions.

Example 1.3.13. If $\mathcal{E} \to X$ is a vector bundle, then we can associate to it a locally free sheaf $\mathcal{E}_X$ given by the sections of $\mathcal{E}$. The map which takes a vector bundle $\mathcal{E}$ to the associated locally free sheaf $\mathcal{E}_X$ defines a bijection (in fact, an equivalence of categories) between vector bundles and sheaves of locally free sheaves (see [**295**, Lemma 4.8] for a complete proof).

Definition 1.3.14. Let $\mathcal{F}$ be a presheaf of abelian groups on a topological space X. For any $x \in X$, we set

$$\mathcal{F}_x \ := \ \varinjlim_{\mathcal{U}} \mathcal{F}(\mathcal{U}),$$

where the right-hand side is the direct limit over all open neighborhoods $\mathcal{U}$ of x. The group $\mathcal{F}_x$ is called the *stalk* of $\mathcal{F}$ at x. Moreover, the image of $\sigma \in \mathcal{F}(\mathcal{U})$, where $x \in \mathcal{U}$, is called the *germ* of σ at x, and is denoted σ_x.

Example 1.3.15. Let X be a complex manifold and $\mathcal{O}_X$ denote the presheaf of holomorphic functions on X. For any $x \in X$, the stalk $\mathcal{O}_{X,x}$ is identified with the group of (local) power series convergent in some neighborhood of $x \in X$.

Definition 1.3.16. A *morphism of sheaves* $f : \mathcal{F} \to \mathcal{G}$ is a map such that $f(\mathcal{F}_x) \subset \mathcal{G}_x$ for all $x \in X$ and the restriction $f_x : \mathcal{F}_x \to \mathcal{G}_x$ of f to stalks is a morphism of groups for all $x \in X$.

Remark 1.3.17. A morphism of sheaves induces a morphism of presheaves in the obvious way. On the other hand, if $f : \mathcal{F} \to \mathcal{G}$ is a morphism of presheaves, there is an induced morphism on stalks $f_x : \mathcal{F}_x = \lim_{x\in\mathcal{U}} \mathcal{F}(\mathcal{U}) \to \lim_{x\in\mathcal{U}} \mathcal{G}(\mathcal{U}) = \mathcal{G}_x$.

The distinction between morphisms of presheaves and morphisms of sheaves, however, is critical. We first recall (see, e.g., [**152**, §2.1], [**173**, §2.2]):

Proposition 1.3.18. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. Then f is injective as a morphism of sheaves (i.e., $f_x : \mathcal{F}_x \to \mathcal{G}_x$ is injective for all $x \in X$) if and only if f is injective as a morphism of presheaves (i.e., $f_{\mathcal{U}} : \mathcal{F}(\mathcal{U}) \to \mathcal{G}(\mathcal{U})$ for all open sets $\mathcal{U} \subseteq X$).

Remark 1.3.19. The corresponding statement with injectivity replaced by surjectivity is false. For instance, let $\mathcal{Z}_d^1$ denote the sheaf of closed smooth 1–forms on a domain $\mathcal{D} \subseteq \mathbb{R}^n$. The Poincaré lemma asserts that the exterior derivative $d : \mathcal{C}_{\mathcal{D}}^\infty \to \mathcal{Z}_d^1$ is a surjection at the level of stalks, and is therefore, a surjection at the level of sheaves. If we choose $\mathcal{D}$ such that $H^1_{\mathrm{DR}}(\mathcal{D}, \mathbb{R}) \neq 0$, however, e.g., take $\mathcal{D}$ to be a non-simply connected domain[12] in $\mathbb{R}^2$, the exterior derivative will fail to be a surjection on global sections and hence, fail to be a surjection of presheaves.

[12]Note that, in general, we have $\pi_1(X) = 0 \implies H^1_{\mathrm{DR}}(X, \mathbb{R}) = 0$, and hence, $H^1_{\mathrm{DR}}(X, \mathbb{R}) \neq 0 \implies \pi_1(X) \neq 0$. For domains in $\mathbb{R}^2$, however, we have $\pi_1 = 0 \iff H^1_{\mathrm{DR}} = 0$ (see, e.g., [**54**]). This is certainly false in general, as illustrated by the Alexander Horned sphere [**6, 54**].

Definition 1.3.20. Let $\mathcal{G}$ be a sheaf on a topological space X. A sheaf (or presheaf) $\mathcal{F}$ is said to be a *subsheaf* (or *subpresheaf*) of $\mathcal{G}$ if $\mathcal{F}_x = \mathcal{F} \cap \mathcal{G}_x$ is a subgroup of $\mathcal{G}_x$ for all $x \in X$.

Of course, for (pre)sheaves of more general algebraic objects – rings, modules – sub(pre)sheaves are defined with the obvious modifications.

Example 1.3.21. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The presheaf which assigns to each open set $\mathcal{U} \subseteq X$ the group $(\mathcal{K}\text{er} f)(\mathcal{U}) := \ker(f(\mathcal{U})) \subseteq \mathcal{F}(\mathcal{U})$ is a subsheaf of $\mathcal{F}$. The assertion that $\mathcal{K}\text{er} f$ is, in fact, a sheaf and not just a presheaf, requires proof (see, e.g., [**46**, **152**, **173**]).

Example 1.3.22. Let $f : \mathcal{F} \to \mathcal{G}$ be a morphism of sheaves. The presheaf which assigns to each open set $\mathcal{U} \subseteq X$ the group $(\mathcal{I}\text{m} f)(\mathcal{U}) := \text{im}(f(\mathcal{U})) \subseteq \mathcal{G}(\mathcal{U})$ is called the *presheaf image*, and is a subpresheaf of $\mathcal{G}$, but not a subsheaf in general.

Definition 1.3.23. We say that a sequence of morphism of sheaves (or presheaves)

$$0 \xrightarrow{\alpha_0} \mathcal{F}_1 \xrightarrow{\alpha_1} \mathcal{F}_2 \xrightarrow{\alpha_2} \mathcal{F}_3 \to \cdots \to \mathcal{F}_n \xrightarrow{\alpha_n} 0$$

is *exact* if $\ker(\alpha_k) = \text{im}(\alpha_{k-1})$ for each $1 \leq k \leq n$.

For instance, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is injective if and only if the sequence $0 \to \mathcal{F} \xrightarrow{\varphi} \mathcal{G}$ is exact. Similarly, a morphism $\varphi : \mathcal{F} \to \mathcal{G}$ is surjective if and only if the sequence $\mathcal{F} \xrightarrow{\varphi} \mathcal{G} \to 0$ is exact.

Remark 1.3.24. Let us emphasize that although both the category of presheaves of abelian groups and the category of sheaves of abelian groups are abelian categories, the notions of cokernel are distinct: A short exact sequence of sheaves is not necessarily a short exact sequence of presheaves. On the other hand, a short exact sequence of presheaves $0 \to \mathcal{F}_1 \to \mathcal{F}_2 \to \mathcal{F}_3 \to 0$, with $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ sheaves, is an exact sequence of sheaves. We will see that this discrepancy is measured by sheaf cohomology.

Sheafification.

Theorem 1.3.25. Let $\mathcal{F}$ be a presheaf (of abelian groups, rings, modules) on X. There is a unique sheaf $\widetilde{\mathcal{F}}$ on X together with a morphism (of presheaves)

$$f : \mathcal{F} \to \widetilde{\mathcal{F}}$$

such that for every morphism (of presheaves) $\varphi : \mathcal{F} \to \mathcal{G}$, where $\mathcal{G}$ is a sheaf, there is a unique morphism (of sheaves) $\psi : \widetilde{\mathcal{F}} \to \mathcal{G}$ such that $\varphi = \psi \circ f$. We call $\widetilde{\mathcal{F}}$ the *sheaf associated to the presheaf* $\mathcal{F}$ or the *sheafification* of $\mathcal{F}$.

There are two ways in which a presheaf $\mathcal{F}$ may fail to be a sheaf:

(i) There is a non-zero section that vanishes on each open set in a covering; or

(ii) there are local sections that do not glue together to yield a global section.

The construction of the sheafification $\widetilde{\mathcal{F}}$ from the presheaf $\mathcal{F}$ is therefore given by removing the sections in (i) and restricting to the sections which satisfy (ii). More precisely, we let $\mathcal{F}_0$ denote the presheaf which assigns to an open set $\mathcal{U}$ the group

$$\mathcal{F}_0(\mathcal{U}) \;:=\; \{\sigma \in \mathcal{F}(\mathcal{U}) : \exists \mathcal{V} \text{ such that } \sigma|_V = 0 \;\forall V \in \mathcal{V}\},$$

where $\mathcal{V}$ belongs to the set of coverings by open subsets of $\mathcal{U}$. The quotient presheaf

$$\mathcal{F}_1(\mathcal{U}) \;:=\; \mathcal{F}(\mathcal{U})/\mathcal{F}_0(\mathcal{U})$$

will satisfying the uniqueness sheaf axiom. To build from $\mathcal{F}_1$ a presheaf which also satisfies the existence axiom (and is therefore a sheaf), we introduce the following groups: For any covering $\mathcal{V}$ by open subsets of $\mathcal{U}$, we set

$$\mathcal{A}_{\mathcal{V}}(\mathcal{U}) \;:=\; \{(\sigma_V)_{V\in\mathcal{V}} : \sigma_V \in \mathcal{F}_1(V) \text{ and } \sigma_V|_{W\cap V} = \sigma_W|_{W\cap V},\; \forall V, W \in \mathcal{V}\}.$$

We will define the sheafification $\widetilde{\mathcal{F}}$ as a certain direct limit of the groups $\mathcal{A}_{\mathcal{V}}$. Let us, therefore, make the following definitions:

Definition 1.3.26. Let $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in A}$ and $\mathcal{V} = \{\mathcal{V}_\beta\}_{\beta\in B}$ be two locally finite open coverings of X. Declare $\mathcal{V}$ to be a *refinement* of $\mathcal{U}$ if for each $\beta \in B$, there is some $\alpha \in A$, such that $\mathcal{V}_\beta \subset \mathcal{U}_\alpha$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, we write $\mathcal{V} \prec \mathcal{U}$.

Remark 1.3.27. The relation $\mathcal{V} \prec \mathcal{U}$ defined a preordering between coverings of X. This relation is, moreover, filtered since, if $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha\in A}$ and $\mathcal{V} = \{\mathcal{V}_\beta\}_{\beta\in B}$ are two coverings, then $\mathcal{W} := \{(\mathcal{U}_\alpha \cap \mathcal{V}_\beta)\}_{(\alpha,\beta)\in A\times B}$ is a covering such that $\mathcal{W} \prec \mathcal{U}$ and $\mathcal{W} \prec \mathcal{V}$.

Two coverings $\mathcal{U}$ and $\mathcal{V}$ are *equivalent* if $\mathcal{U} \prec \mathcal{V}$ and $\mathcal{V} \prec \mathcal{U}$. We can therefore speak of the set of classes of coverings for this equivalence relation, which is an ordered filtered set.[13]

In light of these remarks, we observe that if $\tilde{\mathcal{V}}$ is finer than $\mathcal{V}$, and if $\sigma : \tilde{\mathcal{V}} \to \mathcal{V}$ denotes the refinement map such that $V \subset \sigma(V)$ for all $\tilde{V} \in \mathcal{V}$, then we have the obvious restriction map $\rho(\sigma)^{\tilde{\mathcal{V}}}_{\mathcal{V}} : \mathcal{A}_{\mathcal{V}} \to \mathcal{A}_{\tilde{\mathcal{V}}}$. From the definition of $\mathcal{A}_{\mathcal{V}}$, the restriction maps are independent of the choice of refinement map σ. Let $\mathcal{R}$ be the directed set of coverings of $\mathcal{U}$. We set

$$\widetilde{\mathcal{F}}(\mathcal{U}) \;:=\; \lim_{\mathcal{V}\in\mathcal{R}} \mathcal{A}_{\mathcal{V}}.$$

Explicitly, this direct limit is the group consisting of the $(\sigma_{\mathcal{V}})_{\mathcal{V}\in\mathcal{R}}$ satisfying the property that there exists a covering $\mathcal{V}$ such that $\sigma_{\tilde{\mathcal{V}}} = \rho^{\tilde{\mathcal{V}}}_{\mathcal{V}}(\sigma_{\mathcal{V}})$ for $\tilde{\mathcal{V}}$ finer than $\mathcal{V}$, quotiented by the subgroup consisting of the $(\sigma_{\mathcal{V}})_{\mathcal{V}\in\mathcal{R}}$ such that for some $\mathcal{V}$, we have $\sigma_{\tilde{\mathcal{V}}} = 0$ for everying covering $\tilde{\mathcal{V}}$ which is finer than $\mathcal{V}$. It is clear from the construction that $\widetilde{\mathcal{F}}$ satisfies the second sheaf

[13] Note that we cannot speak of the set of all coverings of X since the indexing sets can be arbitrary.

axiom and is thus a sheaf.

Observe that we have a natural map $f : \mathcal{F} \to \widetilde{\mathcal{F}}$ given by restriction and passage to the quotient. Finally, given a morphism of presheaves $\varphi : \mathcal{F} \to \mathcal{G}$, there is an associated morphism $\tilde{\varphi} : \widetilde{\mathcal{F}} \to \widetilde{\mathcal{G}}$, which satisfies $\tilde{\varphi} \circ f = g \circ \varphi$, where $g : \mathcal{G} \to \widetilde{\mathcal{G}}$ is the sheafification of $\mathcal{G}$. If $\mathcal{G}$ is a sheaf, then g is an isomorphism. Hence, there is a morphism χ such that $\varphi = \chi \circ f$. The uniqueness of χ is clear.

Definition 1.3.28. Let $\mathcal{F}$ be a subsheaf of a sheaf $\mathcal{G}$ of abelian groups on a topological space X. The *quotient presheaf* $\mathcal{G}/\mathcal{F}$ is the presheaf which assigns to each open set $\mathcal{U} \subseteq X$, the quotient group $\mathcal{G}(\mathcal{U})/\mathcal{F}(\mathcal{U})$. The *quotient sheaf* is the sheaf associated with the quotient presheaf.

The sheaf of Meromorphic functions. Our intuition from our training in complex analysis tells us that a meromorphic function is specified by the ratio of holomorphic functions defined locally in a neighborhood of each point. That is, we intuit that for any point $p \in X$ in a complex manifold X, a meromorphic function is to be specified by the ratio f/g in some open neighborhood $\mathcal{U}$ of $p \in X$, where $f, g \in \mathcal{O}_X(\mathcal{U})$. The subtlety in defining meromorphic functions was pointed out by Kleiman [**182**]. Let $\mathcal{M}_X$ denote the sheaf of meromorphic functions (whatever this is) on a complex manifold X. The following statements are *false*:

(i) The sheaf $\mathcal{M}_X$ is the sheaf associated with the presheaf of total fraction rings[14] $\mathcal{U} \mapsto H^0(\mathcal{U}, \mathcal{O}_X)_{\text{tot}}$.
(ii) The stalks $\mathcal{M}_{X,x}$ are equal to the total fraction rings $(\mathcal{O}_{X,x})_{\text{tot}}$.
(iii) If X is a scheme, $\mathcal{U} = \text{Spec}(R) \subseteq X$ is an affine open set, then $H^0(\mathcal{U}, \mathcal{M}_X) = R_{\text{tot}}$. That is, the presheaf in statement (i) is a sheaf if $\mathcal{U}$ ranges only over affine subsets.[15]

Counterexamples to each of the these statements (i)–(iii) are given in [**182**]. One of the key points is that a non-zero divisor may restrict to a zero divisor on a smaller open subset. In light of this, following the language of [**138**, p. 119], we define:

[14]Recall that if S is a subset of a commutative ring R, then S is said to be *multiplicatively closed* if the multiplicative identity element of R is contained in S and $x \cdot y \in S$ for all $x, y \in S$. The *total ring of fractions* $S^{-1}R$ is then defined (as a set) by $S^{-1}R := \{r/s : r \in R, s \in S\}$, with addition defined by $(r/s) + (q/t) = (rt + qs)/st$ and multiplication defined by $(r/s) \cdot (q/t) = rq/st$. It is easy to check that this definition is well-defined.

[15]We omit any extensive discussion of the meaning of a scheme, affine open sets, etc. These notions can be found in any standard book on algebraic geometry, most notably [**152**]. Let us mention, however, that for the reader not so familiar with algebraic geometry, one can replace scheme with complex manifold and (in the GAGA string of analogies, see, e.g., [**219**]) smooth affine varieties are the algebraic analog of Stein manifolds. For the rich subtleties concerning this analog between Steins and affines, see [**218**].

Definition 1.3.29. Let X be a complex manifold. The sheaf $\mathcal{A}_X$ of *active holomorphic germs* to be the sheaf associated to the following presheaf: For any open subset $\mathcal{U} \subset X$, we define $\mathcal{A}_X(\mathcal{U})$ to be the ring of germs of holomorphic functions which do not restrict to a zero-divisor on any open subset $\mathcal{V} \subseteq \mathcal{U}$.

Let $\mathcal{A}_{X,x}$ denote the stalk of the sheaf of active holomorphic germs at $x \in X$. Then $\mathcal{A}_{X,x}$ forms a multiplicatively closed set in $\mathcal{O}_{X,x}$. We may, therefore, give a definition for the sheaf of meromorphic functions $\mathcal{M}_X$:

Definition 1.3.30. Let X be a complex manifold. For each $x \in X$, we set

$$\mathcal{M}_{X,x} \ := \ \{f_x/g_x : f_x \in \mathcal{O}_{X,x}, g_x \in \mathcal{A}_{X,x}\},$$

and hence, the *sheaf of meromorphic functions* $\mathcal{M}_X$ is defined

$$\mathcal{M}_X \ := \ \bigcup_{x \in X} \mathcal{M}_{X,x}.$$

Remark 1.3.31. The quotient of two sheaves, in general, is not a sheaf. For instance, let $X = \mathbb{R}^2 - \{0\}$ (endowed with its standard topology). Let $\mathcal{C}_X$ denote the sheaf of germs of continuous $\mathbb{R}$–valued functions on X. Let $2\pi\mathbb{Z}$ be the subsheaf of (germs of) locally constant functions with values in integer multiples of 2π. The angle function $\vartheta : X \to \mathbb{R}$ is locally well-defined as a section of $\mathcal{C}_X$, but it is not well-defined globally on X. It gives a well-defined section of the quotient sheaf $\mathcal{Q} := \mathcal{C}_X/2\pi\mathbb{Z}$, however. Note that we may identify $\mathcal{Q}$ with the sheaf of (germs of) continuous functions $f : X \to \mathbb{S}^1$. To see that $\mathcal{C}_X/2\pi\mathbb{Z}$ is not a sheaf, we observe that the existence sheaf axiom is violated: Cover $\mathbb{S}^1$ by the open sets $\mathcal{U}_1 := \{-\varepsilon < \vartheta < \varepsilon\}$ and $\mathcal{U}_2 := \{0 < \vartheta < 2\pi - \varepsilon\}$, where ϑ is the angle measured relative to the first coordinate axis. On $\mathcal{U}_1, \mathcal{U}_2$, the angle function is a well-defined section of $\mathcal{Q}(\mathcal{U}_1)$ and $\mathcal{Q}(\mathcal{U}_2)$ and coincides on the overlap $\mathcal{U}_1 \cap \mathcal{U}_2$. But there is no global section $\mathcal{Q}(X)$ which restricts to the angle function on each of these open sets.[16]

The Cohomology of Sheaves. Let X denote the complex plane $\mathbb{C}$. On X, we consider the sheaf of locally constant $\mathbb{Z}$–valued functions, which we denote by $\mathbb{Z}$. Further, we denote by $\mathcal{O}_X^*$ the sheaf of non-vanishing holomorphic functions. Since we can locally write a non-vanishing holomorphic function as the exponential of a holomorphic function, we have the following exact sequence of sheaves:

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\ \exp\ } \mathcal{O}_X^* \longrightarrow 0.$$

However, this will fail to produce an exact sequence on global sections since there is no globally-defined logarithm.

[16]This example is beautifully illustrated in M. C. Escher's 1961 lithograph *Waterfall*.

Sheaf cohomology measures the obstruction to an exact sequence of sheaves giving rise to an exact sequence on sections. More precisely, let

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{G} \longrightarrow 0$$

be an exact sequence of sheaves of abelian groups on a topological space X. The corresponding sequence at the level of global sections is left-exact in the sense that

$$0 \longrightarrow \mathcal{E}(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X)$$

is an exact sequence of abelian groups. In general, however, the sequence

$$0 \longrightarrow \mathcal{E}(X) \longrightarrow \mathcal{F}(X) \longrightarrow \mathcal{G}(X) \longrightarrow 0$$

will fail to be exact. The obstruction to an exact sequence of sheaves yielding an exact sequence on the space of sections is measured by the sheaf cohomology groups.

Some Homological Algebra.

Definition 1.3.32. Let R be a commutative ring. A sequence

$$A^0 \xrightarrow{d_0} A^1 \xrightarrow{d_1} A^2 \to \cdots \to A^k \xrightarrow{d_k} A^{k+1} \xrightarrow{d_{k+1}} \cdots$$

of R–modules and R–morphisms is called a *complex* (of R–modules) if, for all $k \in \mathbb{N}_0$, we have $d_{k+1} \circ d_k = 0$. We denote such a complex by $A^\bullet := (A^k, d_k)_{k \in \mathbb{N}_0}$ and refer to the sequence of R–morphisms d_k as the *coboundary map* of $A^\bullet$.

Definition 1.3.33. Let R be a commutative ring. Let $A^\bullet := (A^k, d_k)_{k \in \mathbb{N}_0}$ and $B^\bullet := (B^k, \delta_k)_{k \in \mathbb{N}_0}$ be two complexes (of R–modules). A *morphism of complexes* (of R–modules) is a sequence φ_k of R–morphisms $\varphi_k : B^k \to A^k$ which are compatible with the constituent coboundary maps in the sense that

$$d_k \circ \varphi_k \;=\; \varphi_{k+1} \circ \delta^k, \qquad \forall k \in \mathbb{N}_0.$$

Remark 1.3.34. Note that with morphisms of complexes defined as above, the category of complexes (of R–modules) form an abelian category.

Definition 1.3.35. Let R be a commutative ring. Let

$$A^\bullet \xrightarrow{\varphi_\bullet} B^\bullet \xrightarrow{\psi_\bullet} C^\bullet$$

be a sequence of complexes (of R–modules). We say that this sequence is *exact* (as a sequence of complexes of R–modules) if

$$A^k \xrightarrow{\varphi_k} B^k \xrightarrow{\psi_k} C^k$$

is exact (as a sequence of R–modules) for all $k \in \mathbb{N}_0$.

Let $A^\bullet := (A^k, d_k)_{k\in\mathbb{N}_0}$ be a complex of R–modules. We introduce the R–modules

$$\mathcal{Z}^k(A^\bullet) \ := \ \ker(d_k), \qquad \mathcal{B}^k(A^\bullet) \ := \ \operatorname{Im}(d_{k-1}),$$

of k*–cocycles* and k*–coboundaries*, respectively.

Definition 1.3.36. Let $A^\bullet := (A^k, d_k)_{k\in\mathbb{N}_0}$ be a complex of R–modules. The *kth cohomology module* of $A^\bullet$ is defined by

$$H^k(A^\bullet) \ := \ \mathcal{Z}^k(A^\bullet)/\mathcal{B}^k(A^\bullet), \qquad \forall\ k \in \mathbb{N}_0,$$

where $\mathcal{B}^0(A^\bullet) := \{0\}$.

Let $\varphi_k : B^k \to A^k$ be a morphism of complexes (of R–modules). Then $\varphi_k(\mathcal{Z}^k(B^\bullet)) \subseteq \mathcal{Z}^k(A^\bullet)$ and $\varphi_k(\mathcal{B}^k(B^\bullet)) \subseteq \mathcal{B}^k(A^\bullet)$. Hence, a morphism of complexes $\varphi_\bullet$ induces morphisms

$$H^k(B^\bullet) \to H^k(A^\bullet), \qquad \forall\ k \in \mathbb{N}_0$$

on the corresponding cohomology modules.

Lemma 1.3.37. ([137, p. 29]). Let R be a commutative ring. Let

$$0 \to A^\bullet \xrightarrow{\varphi_\bullet} B^\bullet \xrightarrow{\psi_\bullet} C^\bullet \to 0$$

be a sequence of complexes (of R–modules)[17]. For each $k \in \mathbb{N}_0$, there exists a natural morphism

$$\delta_k : H^k(C^\bullet) \longrightarrow H^{k+1}(A^\bullet)$$

which depends functorially on $\varphi_\bullet$ and $\psi_\bullet$ so that the long sequence of cohomology modules

$$0 \to H^0(A^\bullet) \to \cdots \to H^k(A^\bullet) \to H^k(B^\bullet) \to H^k(C^\bullet) \xrightarrow{\delta_k} H^{k+1}(A^\bullet) \to \cdots$$

is exact.

We want to apply this theory to sheaves. Hence, we need to build complexes. This is achieved by introducing resolutions:

Definition 1.3.38. Let $\mathcal{R}$ be a sheaf of rings on a topological space X. Let $\mathcal{G}$ be a sheaf of $\mathcal{R}$–modules on X. An (*injective*) $\mathcal{R}$*–resolution* of sheaves of $\mathcal{R}$–modules is a long exact sequence

$$0 \to \mathcal{G} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \cdots \to \mathcal{F}^k \to \cdots$$

of sheaves of $\mathcal{R}$–modules.

[17] Here, 0 denotes the zero complex.

Example 1.3.39. Let M be a smooth manifold of (real) dimension n. Denote the sheaf associated with the constant presheaf by $\mathbb{R}$. By the Poincaré lemma, a resolution of $\mathbb{R}$ is given by

$$0 \longrightarrow \mathbb{R} \to \Omega^0_M \xrightarrow{d} \Omega^1_M \xrightarrow{d} \cdots \xrightarrow{d} \Omega^n_M \longrightarrow 0.$$

From an injective $\mathcal{R}$–resolution

$$0 \longrightarrow \mathcal{G} \longrightarrow \mathcal{F}^0 \xrightarrow{\varphi_1} \mathcal{F}^1 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_k} \mathcal{F}^k \xrightarrow{\varphi_{k+1}} \cdots \tag{1.3.1}$$

of a sheaf $\mathcal{G}$ of $\mathcal{R}$–modules, we build a complex (of $\mathcal{R}(X)$–modules) at the level of sections:

$$\mathcal{F}^0(X) \xrightarrow{\varphi_1^*} \mathcal{F}^1(X) \xrightarrow{\varphi_2^*} \cdots \xrightarrow{\varphi_k^*} \mathcal{F}^k(X) \xrightarrow{\varphi_{k+1}^*} \cdots .$$

Definition 1.3.40. Let $\mathcal{G}$ be a sheaf of $\mathcal{R}$–modules on a topological space X. Let $\mathcal{F}^\bullet(\mathcal{G})$ denote the injective resolution (1.3.1). We define the *kth cohomology module* (relative to $\mathcal{F}^\bullet(\mathcal{G})$) by

$$H^k(X, \mathcal{F}^\bullet(\mathcal{G})) \;:=\; \ker(\varphi_k^*)/\mathrm{Im}(\varphi_{k-1}^*), \qquad \forall\ k \in \mathbb{N},$$

and $H^0(X, \mathcal{F}^\bullet(\mathcal{G})) := \mathcal{G}(X)$.

Definition 1.3.41. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. Let $\{\mathcal{U}_\alpha\}_\alpha$ be a locally finite open covering of X. A *partition of unity* of the sheaf $\mathcal{F}$ subordinate to the covering $\{\mathcal{U}_\alpha\}_\alpha$ is a collection of morphisms of sheaves $\gamma_\alpha : \mathcal{F} \longrightarrow \mathcal{F}$ such that

(i) γ_α is the zero morphism on an open neighborhood of the complement of $\mathcal{U}_\alpha$ in X.
(ii) $\sum_\alpha \gamma_\alpha = \mathrm{id}$, where $\mathrm{id} : \mathcal{F} \longrightarrow \mathcal{F}$ is the identity morphism.

A sheaf is said to be *fine* if it admits a partition of unity subordinate to some open cover of X.

Example 1.3.42. The sheaf of smooth p–forms Ω^p_M on a smooth manifold M is a fine sheaf. Any sheaf of $\mathcal{C}^\infty_M$–modules (in particular, any locally free sheaf on M) is a fine sheaf.

Definition 1.3.43. Let $\mathcal{E}$ be a sheaf of abelian groups on a topological space X. An exact sequence of sheaves on X of the form

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_0 \xrightarrow{d_0} \mathcal{F}_1 \xrightarrow{d_1} \mathcal{F}_2 \longrightarrow \cdots$$

is called a *fine resolution* if, for each $k \in \mathbb{N}_0$, the sheaves $\mathcal{F}_k$ are fine.

Example 1.3.44. The resolution in 1.3.39 is a fine resolution of the constant sheaf $\mathbb{R}$.

Fine resolutions play a central role in sheaf cohomology in light of the following theorem [**151**, p. 178]:

Lemma 1.3.45. Let $\mathcal{G}$ be a sheaf (of abelian groups, rings, modules) on a topological space X. There is a canonical fine resolution $(\mathcal{F}^\bullet(\mathcal{G})$:

$$0 \longrightarrow \mathcal{G} \xrightarrow{\alpha} \mathcal{F}^0 \xrightarrow{d_0} \mathcal{F}^2 \xrightarrow{d_1} \cdots$$

such that for any morphism of sheaves $\varphi : \mathcal{G} \to \mathcal{H}$, there is a commutative diagram between the canonical resolutions of $\mathcal{G}$ and $\mathcal{H}$.

The above result has the important consequence that on a paracompact topological space, the sheaf cohomology groups, computed with respect to a fine resolution, are independent of the choice of fine resolution [**151**, p. 180]:

Theorem 1.3.46. Let $\mathcal{G}$ be a sheaf (of abelian groups, rings, etc.) on a paracompact topological space X. The sheaf cohomology groups $H^k(X, \mathcal{F}^\bullet(\mathcal{G}))$ are independent of the choice of fine resolution $\mathcal{F}^\bullet(\mathcal{G})$.

In particular, the following definition is well-defined:

Definition 1.3.47. Let $\mathcal{G}$ be a sheaf (of abelian groups, rings, etc.) on a paracompact topological space X. The *sheaf cohomology groups* of $\mathcal{G}$ are defined

$$H^k(X, \mathcal{G}) \ := \ H^k(X, \mathcal{F}^\bullet(\mathcal{G})), \qquad \forall\ k \in \mathbb{N},$$

for some fine resolution $\mathcal{F}^\bullet(\mathcal{G})$, and $H^0(X, \mathcal{G}) := \mathcal{G}(X)$.

Theorem 1.3.48. (Dolbeault theorem). Let X be a complex manifold. Denote by Ω_X^p the sheaf of holomorphic p–forms on X. Then there is an isomorphism

$$H_{\bar{\partial}}^{p,q}(X) \ \simeq \ H^q(X, \Omega_X^p).$$

Proof. Let $\mathcal{Z}_X^{p,q}$ denote the sheaf of smooth $\bar{\partial}$–closed (p,q)–forms on X. This is a subsheaf of the sheaf $\mathscr{A}_X^{p,q}$ of smooth (p,q)–forms on X. From the Dolbeault lemma, the sequence of sheaves

$$0 \longrightarrow \mathcal{Z}_X^{p,q} \longrightarrow \mathscr{A}_X^{p,q} \xrightarrow{\bar{\partial}} \mathcal{Z}_X^{p,q+1} \longrightarrow 0$$

is exact for each $p, q \in \mathbb{N}_0$. Since the sheaf $\mathscr{A}_X^{p,q}$ is fine, the long exact sequence on cohomology implies that

$$H^k(X, \mathcal{Z}_X^{p,q+1}) \ \simeq \ H^{k+1}(X, \mathcal{Z}_X^{p,q}),$$

for each $k > 0$, and the sequence

$$H^0(X, \mathscr{A}_X^{p,q}) \xrightarrow{\bar{\partial}} H^0(X, \mathcal{Z}_X^{p,q+1}) \longrightarrow H^1(X, \mathcal{Z}_X^{p,q}) \longrightarrow 0$$

is an exact sequence of groups. In particular, $H^1(X, \mathcal{Z}_X^{p,q-1})$ is isomorphic to the quotient of $H^0(X, \mathcal{Z}_X^{p,q})$ by the image of $H^0(X, \mathscr{A}_X^{p,q-1})$ under $\bar{\partial}$. From $\mathcal{Z}_X^{p,0} = \Omega_X^p$, we see that

$$H^q(X, \Omega_X^p) \ = \ H^q(X, \mathcal{Z}_X^{p,0}) \ \simeq \ H^{q-1}(X, \mathcal{Z}_X^{p,1}) \ \simeq \ H^{q-2}(X, \mathcal{Z}_X^{p,2}) \ \simeq \ \cdots \ \simeq \ H^1(X, \mathcal{Z}_X^{p,q-1}).$$

From the previous remark, we see that

$$H^1(X, \mathcal{Z}_X^{p,q-1}) \simeq H^0(X, \mathcal{Z}_X^{p,q})/\bar{\partial}(H^0(X, \mathscr{A}_X^{p,q-1})) = H^{p,q}(X).$$

$\square$

A Brief Reminder of Čech cohomology. Let X be a topological space, on which we have a sheaf $\mathcal{F}$ of abelian groups. Let $\mathcal{U} := \{\mathcal{U}_\alpha\}_{\alpha \in A}$ be a locally finite open covering of X.

Definition 1.3.49. For $p \in \mathbb{N}_0$, a *p–cochain of $\mathcal{U}$ with values in $\mathcal{F}$* is a function σ which assigns to each $(p+1)$–tuple $(\alpha_0, ..., \alpha_p)$ of A, a section $\sigma_{\alpha_0 \cdots \alpha_p} \in \mathcal{F}(\mathcal{U}_{\alpha_0} \cap \cdots \cap \mathcal{U}_{\alpha_p})$.

The p–cochains form an abelian group, which we denote by

$$\mathcal{C}^p(\mathcal{U}, \mathcal{F}) := \prod_{\alpha_0 \neq \alpha_1 \neq \cdots \neq \alpha_p} \mathcal{F}(\mathcal{U}_{\alpha_0} \cap \cdots \cap \mathcal{U}_{\alpha_p}).$$

For $p = 0$, we have $\mathcal{C}^0(\mathcal{U}, \mathcal{F}) := \prod_\alpha \mathcal{F}(\mathcal{U}_\alpha)$.

Definition 1.3.50. The *coboundary operator* $\delta : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \to \mathcal{C}^{p+1}(\mathcal{U}, \mathcal{F})$, is defined by the formula

$$(\delta\sigma)_{\alpha_0,...,\alpha_{p+1}} := \sum_{k=0}^{p+1} (-1)^k \sigma_{\alpha_0,...,\widehat{\alpha_k},...,\alpha_{p+1}} \Big|_{\mathcal{U}_{\alpha_0} \cap \cdots \cap \mathcal{U}_{\alpha_p}}.$$

Example 1.3.51. If $\sigma = \{\sigma_\alpha\} \in \mathcal{C}^0(\mathcal{U}, \mathcal{F})$, then

$$(\delta\sigma)_{\alpha\beta} = \sigma_\beta - \sigma_\alpha.$$

If $\sigma = \{\sigma_{\alpha\beta}\} \in \mathcal{C}^1(\mathcal{U}, \mathcal{F})$, then

$$(\delta\sigma)_{\alpha\beta\gamma} = \sigma_{\alpha\beta} + \sigma_{\beta\gamma} - \sigma_{\alpha\gamma}.$$

It is straightforward to show that $\delta \circ \delta = 0$. Hence, the coboundary operator δ endows $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ with the structure of a complex. Hence, we can speak of the cohomology of $(\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F}), \delta)$:

Definition 1.3.52. Let $\mathcal{U}$ be a locally finite open covering of a topological space X. Let $\mathcal{F}$ be a sheaf (of abelian groups, rings, modules) on X. The *Čech cohomology groups* of X with values in $\mathcal{F}$ (relative to $\mathcal{U}$) are

$$\check{H}^p(\mathcal{U}, \mathcal{F}) := \frac{\{\sigma \in \mathcal{C}^p(\mathcal{U}, \mathcal{F}) : \delta\sigma = 0\}}{\{d\tau : \tau \in \mathcal{C}^{p-1}(\mathcal{U}, \mathcal{F})\}}.$$

The construction of these cohomology groups hinges on the choice of locally finite open covering $\mathcal{U}$ of X. To remove this dependence, we consider:

Remark 1.3.53. If $\mathcal{V} \prec \mathcal{U}$, with $\mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in A}$ and $\mathcal{V} = \{\mathcal{V}_\beta\}_{\beta \in B}$, there is a map $f : A \to B$ such that $\mathcal{V}_\beta \subseteq \mathcal{U}_{f(\beta)}$. This furnishes a morphism

$$\rho_f : \mathcal{C}^p(\mathcal{U}, \mathcal{F}) \longrightarrow \mathcal{C}^p(\mathcal{V}, \mathcal{F}), \qquad (\rho_f \sigma)_{\beta_0 \cdots \beta_p} := \sigma_{f(\beta_0) \cdots f(\beta_p)}|_{\mathcal{U}_{\beta_0} \cap \cdots \cap \mathcal{U}_{\beta_p}}.$$

Given two such maps $f : A \to B$ and $g : A \to B$, it can be checked that the induced maps ρ_f and ρ_g are chain homotopic[18]. In particular, since $\delta \circ \rho_f = \rho_f \circ \delta$, we have induced morphisms on cohomology

$$\rho : \check{H}^p(\mathcal{U}, \mathcal{F}) \to \check{H}^p(\mathcal{V}, \mathcal{F})$$

which are well-defined, independent of the choice of f.

Definition 1.3.54. Let X be a topological space, endowed with a sheaf $\mathcal{F}$ of abelian groups. We denote by $H^p(X, \mathcal{F})$ the inductive limit of groups $H^p(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}$ runs over all filtered orderings of classes of coverings of X, with respect to the morphisms ρ.

In other words, an element of $H^p(X, \mathcal{F})$ is a pair $(\mathcal{U}, \gamma)$ with $\gamma \in H^q(\mathcal{U}, \mathcal{F})$ subject to the following identification: We identify $(\mathcal{U}, \gamma)$ and $(\mathcal{V}, \eta)$ whenever there exists a covering $\mathcal{W}$ such that $\mathcal{W} \prec \mathcal{U}$, $\mathcal{W} \prec \mathcal{V}$, and $\rho(\mathcal{W}, \mathcal{U})(\gamma) = \rho(\mathcal{W}, \mathcal{V})(\eta)$ in $H^p(\mathcal{W}, \mathcal{F})$.

Theorem 1.3.55. Let X be a topological space on which we have a simplicial complex Σ_X. Then there is an isomorphism

$$H^p(\Sigma_X, \mathbb{Z}) \simeq \check{H}^p(X, \mathbb{Z}),$$

where the left-hand side is the simplicial cohomology of Σ_X, and the right-hand side is the Čech cohomology of the constant sheaf $\mathbb{Z}$ on X.

Theorem 1.3.56. (Serre duality). Let X be a compact complex manifold of (complex) dimension n. Let $\mathcal{E} \to X$ be a holomorphic vector bundle. There is a conjugate linear isomorphism

$$H^k(X, \Omega_X^p(\mathcal{E})) \longrightarrow H^{n-k}(X, \Omega_X^{n-p}(\mathcal{E}^*)).$$

Corollary 1.3.57. Let X be a compact complex manifold of (complex) dimension n. Then

(i) $b_k(X) = b_{2n-k}(X)$ for all $k \in \{0, ..., 2n\}$.
(ii) $h^{p,q}(X) = h^{n-q,n-p}(X)$, for all $p, q \in \{0, ..., n\}$.

[18] For details, see [249, Proposition 3.3] and [116, Chapter IV, §3].

1.4. Divisors, Line Bundles, and Characteristic Classes

One of the essential features of the theory of complex-analytic functions is that the inspiring beauty of such functions demands a high compensation in the form of their rigidity. This strips the study of holomorphic functions on a compact manifold of any interest: they are all constant. There are two natural ways to recover from this:

(i) Allow (reasonable) singularities to be introduced in the holomorphic map, i.e., we may consider meromorphic maps.
(ii) Allow the target space of the holomorphic map to be a non-trivial line bundle (noting that a holomorphic function $X \to \mathbb{C}$ is a section of the trivial bundle $\mathbb{C} \to X$).

These two routes turn out to be intimately related. Let us first recall:

Definition 1.4.1. Let $\mathcal{D} \subseteq \mathbb{C}^n$ be a connected open set. A subset $\mathcal{A} \subset \mathcal{D}$ is said to be *analytic at* $p \in \mathcal{D}$ if there is an open neighborhood $\mathcal{U} \subset \mathcal{D}$ of p and holomorphic functions $f_1, ..., f_k \in \mathcal{O}(\mathcal{U})$ such that

$$\mathcal{A} \cap \mathcal{U} \ = \ V(f_1, ..., f_k) \ := \ \{x \in \mathcal{U} : f_i(x) = 0 \ \forall i = 1, ..., k\}.$$

We say that $\mathcal{A}$ is an *analytic subvariety* of $\mathcal{D}$ if $\mathcal{A}$ is analytic at every point $p \in \mathcal{D}$.

Remark 1.4.2. By Oka's coherence theorem [**138**] the number of analytic functions $f_1, ..., f_k$ which locally describe an analytic subvariety is always finite. Further, it is clear that the above definition easily extends to define analytic subvarieties of complex manifolds.

Definition 1.4.3. Let $\mathcal{A}$ be an analytic subvariety of a complex manifold X. We say that $\mathcal{A}$ is *irreducible* if $\mathcal{A}$ cannot be written as the union of two analytic subvarieties of X.

It is a standard fact (see, e.g., [**148**]) that an analytic subvariety can be written as a union of its irreducible components.

Definition 1.4.4. Let $\mathcal{A}$ be an analytic subvariety of a complex manifold X. Let $p \in \mathcal{A}$ be a point with $\mathcal{A}$ locally described (in some open neighborhood $\mathcal{U}$ of p) by the holomorphic functions $f_1, ..., f_k$. We say that p is a *smooth point* of $\mathcal{A}$ if the Jacobian of the local defining functions $\mathcal{J}(f_1, ..., f_k)$ has maximal rank k. The set of smooth points of an analytic subvariety $\mathcal{A}$ is denoted $\mathcal{A}^\circ$. The *dimension* of an analytic subvariety $\mathcal{A}$ is defined to be the dimension of $\mathcal{A}^\circ$.

Remark 1.4.5. From [**142**, p. 21], an analytic subvariety $\mathcal{A}$ is irreducible if and only if $\mathcal{A}^\circ$ is connected.

Let Σ be a compact Riemann surface, and let f be a meromorphic function[19] on Σ. Let $V(f) := \{x \in \Sigma : f(x) = 0\}$ denote the *vanishing locus* of f, and $P(f) := \{x \in \Sigma : 1/f(x) =$

[19] That is, a section of the sheaf of (germs of) meromorphic functions on Σ.

$0\}$ denote the *polar locus* of f. Since Σ is compact, both $V(f)$ and $P(f)$ are locally finite sets in Σ. The points of $V(f)$ and $P(f)$ contain more data than merely set-theoretic information: the points are associated with integers – the multiplicity of f. This leads to the notion of a divisor:

Definition 1.4.6. Let X be a complex manifold. A *divisor* $\mathcal{D}$ on X is a (locally finite) formal $\mathbb{Z}$–linear combination of codimension-one analytic subvarieties

$$\mathcal{D} \ := \ \sum_{p \in X} m_p \cdot \mathcal{A}_p.$$

The set of divisors on a complex manifold X form a group $\mathrm{Div}(X)$ with respect to addition.

Example 1.4.7. Let f be a meromorphic function on a compact Riemann surface Σ. For a point $p \in \Sigma$, let $\mathrm{ord}_p(f)$ denote the order of vanishing (if $\mathrm{ord}_p(f) > 0$) or the order of the pole (if $\mathrm{ord}_p(f) < 0$). Then

$$\mathrm{div}(f) \ := \ \sum_{p \in \Sigma} \mathrm{ord}_p(f) \cdot p$$

is a divisor on Σ.

Definition 1.4.8. A divisor $\mathcal{D}$ on a compact complex manifold X is said to be a *principal divisor* if $\mathcal{D} = \mathrm{div}(f)$ for some meromorphic function f on X.

If f, g are two meromorphic functions on X, then it is easy to show that

$$\mathrm{div}(f) + \mathrm{div}(g) = \mathrm{div}(fg).$$

In particular, the space of principal divisors forms a subgroup of $\mathrm{Div}(X)$ which we denote by $\mathrm{PDiv}(X)$.

Remark 1.4.9. There is a natural group homomorphism attached to the group of divisors $\mathrm{Div}(X)$ – the *degree function*:

$$\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}, \qquad \deg\left(\sum_{p \in X} m_p \cdot p\right) \ := \ \sum_{p \in X} m_p.$$

The kernel of deg is the subgroup $\mathrm{Div}_0(X)$ of divisors of degree zero.

Remark 1.4.10. Since the number of poles of a meromorphic function must be equal to the number of zeroes (counted with multiplicity) on a compact Riemann surface (see, e.g., [**207**]), we see that

$$\deg(\mathrm{div}(f)) \ = \ 0$$

for any meromorphic function f. In particular, $\mathrm{PDiv}(\Sigma)$ is a subgroup of $\mathrm{Div}_0(\Sigma)$.

Example 1.4.11. Let f be a meromorphic function on $\mathbb{P}^1$. Restricting f to the affine part $\mathbb{C} \simeq \mathbb{P}^1 - \{+\infty\}$, and letting z denote the affine coordinate on $\mathbb{C}$, write

$$f(z) \ = \ a_0 \prod_{k=1}^{n} (z - w_k)^{m_k},$$

where $m_k \in \mathbb{Z}$ and $a_0, w_k \in \mathbb{C}$. Then

$$\operatorname{div}(f) \ = \ \sum_{k=1}^{n} m_k \cdot w_k - \left(\sum_{k=1}^{n} m_k \right) \cdot \infty.$$

Example 1.4.12. Let ϑ denote the standard theta function. This defines an entire function which has simple zeroes at the points $(1/2) + (\tau/2) + \ell$, for all lattice points $\ell \in \mathbb{Z} + \tau\mathbb{Z}$. Then

$$\operatorname{div}(\vartheta) \ = \ \sum_{m,n \in \mathbb{Z}} 1 \cdot (1/2) + (\tau/2) + m + n\tau.$$

Let $\mathcal{M}_X^*$ denote the multiplicative sheaf of meromorphic functions on X. Write $\mathcal{O}_X^*$ for the multiplicative sheaf of holomorphic functions on X. A global section of the quotient $\mathcal{M}_X^*/\mathcal{O}_X^*$ is specified by an open cover $(\mathcal{U}_\alpha)_\alpha$ of X together with meromorphic functions $f_\alpha \in \mathcal{M}_X(\mathcal{U}_\alpha)$ such that

$$\frac{f_\alpha}{f_\beta} \in \mathcal{O}_X^*(\mathcal{U}_\alpha \cap \mathcal{U}_\beta).$$

For any analytic subvariety of codimension one $\mathcal{A} \subset X$, we have $\operatorname{ord}_{\mathcal{A}}(f_\alpha) = \operatorname{ord}_{\mathcal{A}}(f_\beta)$. Hence, we can associate to the global section of $\mathcal{M}_X^*/\mathcal{O}_X^*$, the divisor

$$D \ := \ \sum_{\mathcal{A}} \operatorname{ord}_{\mathcal{A}}(f_\alpha) \cdot \mathcal{A},$$

where, for each $\mathcal{A}$, we choose α such that $\mathcal{A} \cap \mathcal{U}_\alpha \neq 0$.

On the other hand, if we have a divisor $D = \sum_k m_k \cdot \mathcal{A}_k$, then may cover X by open sets $\mathcal{U}_\alpha$ such that $\mathcal{A}_k \cap \mathcal{U}_\alpha$ is the vanishing locus of an analytic function $f_{k\alpha} \in \mathcal{O}_X(\mathcal{U}_\alpha)$. If we set

$$f_\alpha \ := \ \prod_k f_{k\alpha}^{m_k} \in \mathcal{M}_X^*(\mathcal{U}_\alpha)$$

we obtain a global section of $\mathcal{M}_X^*/\mathcal{O}_X^*$:

Theorem 1.4.13. The map which sends a meromorphic function to its principal divisor defines an isomorphism of groups

$$\operatorname{Div}(X) \ \simeq \ H^0(X, \mathcal{M}_X^*/\mathcal{O}_X^*).$$

Remark 1.4.14. Since any point on a Riemann surface Σ is an irreducible analytic subvariety of codimension one, $\operatorname{Div}(\Sigma)$ is always large. This phenomenon does not persist for complex manifolds of higher dimensions, however. For instance, a sufficiently generic complex torus

$\mathbb{C}^{n>1}/\Lambda$ has no analytic subvarieties of positive dimension. If X is a projective manifold, however, i.e., X embeds into some $\mathbb{P}^n$, then intersecting X with hyperplanes can be used to generate a bountiful number of divisors. We will see later that projective manifolds can be characterized (amongst compact complex manifolds) by the existence of a large number of divisors.

Definition 1.4.15. Let $\mathcal{D} = \sum c_\alpha \mathcal{V}_\alpha$ be a divisor in a complex manifold X. We say that $\mathcal{D}$ is *effective* if the coefficients $c_\alpha \geq 0$.

Definition 1.4.16. Let X be a complex manifold. Two divisors $\mathcal{D}_1, \mathcal{D}_2 \subset X$ are said to be *linearly equivalent* if their difference is a principal divisor

$$\mathcal{D}_1 \;=\; \mathcal{D}_2 + (f).$$

The set of divisors linearly equivalent to a divisor $\mathcal{D}$ is called the *linear system* associated to $\mathcal{D}$, denoted by $|\mathcal{D}|$.

Line Bundles. Let $\mathcal{L} \longrightarrow X$ be a (holomorphic) line bundle over a complex manifold. Cover X by open sets $\mathcal{U}_\alpha$ such that $\mathcal{L}|_{\mathcal{U}_\alpha} \simeq \mathcal{U}_\alpha \times \mathbb{C}$. Denote the local trivializations by $\varphi_\alpha : \mathcal{L}|_{\mathcal{U}_\alpha} \longrightarrow \mathcal{U}_\alpha \times \mathbb{C}$. Over any overlap $\mathcal{U}_\alpha \cap \mathcal{U}_\beta$, we have transition maps $\varphi_{\alpha\beta} : \mathcal{U}_\alpha \cap \mathcal{U}_\beta \longrightarrow \mathbb{C}^*$ for $\mathcal{L}$ given by $\varphi_{\alpha\beta} = \varphi_\alpha \circ \varphi_\beta^{-1}$. Observe that the following Čech 1–cocycle condition holds:

$$\varphi_{\alpha\beta}\varphi_{\beta\alpha} \;=\; 1, \qquad \varphi_{\alpha\beta}\varphi_{\beta\gamma}\varphi_{\gamma\alpha} \;=\; 1. \tag{1.4.1}$$

Indeed, (1.4.1) implies that the Čech 1–cochain $\Phi_{\alpha\beta}$ defined by transition maps $\{\varphi_{\alpha\beta} \in \mathcal{O}_X^*(\mathcal{U}_\alpha \cap \mathcal{U}_\beta)\}$ satisfies $\delta(\Phi_{\alpha\beta}) = 0$.

If consider another local trivialization for $\mathcal{L}$, given by $\psi_\alpha : \mathcal{L}|_{\mathcal{U}_\alpha} \longrightarrow \mathcal{U}_\alpha \times \mathbb{C}$, then we can find holomorphic functions $f_\alpha \in \mathcal{O}_X^*(\mathcal{U}_\alpha)$ such that $\psi_\alpha \;=\; f_\alpha \varphi_\alpha$. Hence, the transition maps are

$$\psi_{\alpha\beta} \;=\; \psi_\alpha \circ \psi_\beta^{-1} \;=\; \frac{f_\alpha}{f_\beta}\varphi_{\alpha\beta}, \tag{1.4.2}$$

and therefore, two collections of transition maps $\{\psi_{\alpha\beta}\}$ and $\{\varphi_{\alpha\beta}\}$ define the same line bundle if and only if there are holomorphic functions $f_\alpha \in \mathcal{O}_X^*(\mathcal{U}_\alpha)$ satisfying (1.4.2). In other words, the Čech 1–cocycles $\{\psi_{\alpha\beta}\}$ and $\{\varphi_{\alpha\beta}\}$ define the same line bundle if and only if their difference[20] $\{\psi_{\alpha\beta}\varphi_{\alpha\beta}^{-1}\}$ is a Čech 1–coboundary:

Theorem 1.4.17. The map which sends a line bundle $\mathcal{L} \longrightarrow X$ to the Čech 1–cocycle given by its transition maps defines an isomorphism of groups:

$$\mathrm{Pic}(X) \;\simeq\; H^1(X, \mathcal{O}_X^*).$$

[20]Since the sheaves are multiplicative, difference is understood to mean ratio.

Example 1.4.18. For any $n \in \mathbb{N}$,

$$\mathrm{Pic}(\mathbb{P}^n) \simeq \mathbb{Z},$$

the generator given by the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

Remark 1.4.19. Let us briefly remark that the group $H^2(X, \mathcal{O}_X^*)$ is also of particular importance. Let X be a compact complex manifold (or, more generally, a complex space). The set of isomorphism classes of holomorphic $\mathbb{P}^k$–bundles over X is denoted by $\mathrm{Proj}_k(X)$. We have an exact sequence of sheaves on X:

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \mathrm{GL}_k(\mathcal{O}_X) \longrightarrow \mathrm{PGL}_k(\mathcal{O}_X) \longrightarrow 1,$$

from which we obtain an exact sequence

$$H^1(X, \mathrm{GL}_k(\mathcal{O}_X)) \xrightarrow{\mathbb{P}} H^1(X, \mathrm{PGL}_k(\mathcal{O}_X)) \xrightarrow{\delta_k} H^2(X, \mathcal{O}_X^*)).$$

Identifying $H^1(X, \mathrm{GL}_k(\mathcal{O}_X))$ with the set of isomorphism classes of holomorphic rank k vector bundles on X and $H^1(X, \mathrm{PGL}_k(\mathcal{O}_X))$ with $\mathrm{Proj}_{k-1}(X)$, we may write

$$\mathrm{Vect}_k(X) \xrightarrow{\mathbb{P}} \mathrm{Proj}_{k-1}(X) \xrightarrow{\delta_k} H^2(X, \mathcal{O}_X^*).$$

A projective bundle of the form $\mathbb{P}(\mathcal{E})$, for some vector bundle $\mathcal{E}$, is said to be *insignificant*. Hence, δ_k is the obstruction to $\mathcal{P} \in \mathrm{Proj}_{k-1}(X)$ coming from a vector bundle $\mathcal{E}$, i.e., to $\mathcal{P}$ being insignificant. Let

$$\mathrm{Proj}(X) := \coprod_{k \in \mathbb{N}} \mathrm{Proj}_k(X).$$

The composition law $\otimes$ endows $\mathrm{Proj}(X)$ with a natural monoid structure, which, on insignificant bundles, is given by $\mathbb{P}(\mathcal{E}) \otimes \mathbb{P}(\mathcal{F}) = \mathbb{P}(\mathcal{E} \otimes \mathcal{F})$. Moreover, the involution $\mathbb{P}(\mathcal{E}) \to \mathbb{P}(\mathcal{E}^*)$ on insignificant bundles extends to an involution on $\mathrm{Proj}(X)$.

An equivalence relation on $\mathrm{Proj}(X)$ is then defined by declaring that

$$\mathcal{P} \sim \mathcal{Q} \iff \mathcal{P} \otimes \mathbb{P}(\mathcal{E}) \simeq \mathcal{Q} \otimes \mathbb{P}(\mathcal{F}).$$

Definition 1.4.20. The *Brauer group* (in the sense of Grothendieck) of X is the quotient

$$\mathrm{Br}(X) := \mathrm{Proj}(X)/\sim.$$

Correspondence between divisors and line bundles. Let us briefly describe the correspondence between divisors and line bundles. Let X be a complex manifold and $\mathcal{D} \subset X$ a divisor. Relative to an open cover $\{\mathcal{U}_\alpha\}_\alpha$ of X, let $f_\alpha \in \mathcal{M}^*(\mathcal{U}_\alpha)$ be the local defining functions for $\mathcal{D}$. We construct the *line bundle associated to* $\mathcal{D}$, which we denote by $\mathcal{O}_X(\mathcal{D})$, by declaring the transition maps to be $g_{\alpha\beta} := f_\alpha / f_\beta$. It is clear that the map $\mathcal{D} \mapsto \mathcal{O}_X(\mathcal{D})$ is well-defined, and moreover, defines a morphism of groups $\mathrm{Div}(X) \to \mathrm{Pic}(X)$.

Example 1.4.21. An easy argument (see, e.g., [**142**, p. 134] for details) shows that $\mathcal{O}_X(\mathcal{D})$ is trivial if and only if $\mathcal{D}$ is a principal divisor.
Let us also mention that if $H = \mathbb{P}^{n-1}$ denotes the hyperplane divisor in $\mathbb{P}^n$, then $\mathcal{O}_{\mathbb{P}^n}(H)$ is the hyperplane bundle $\mathcal{O}_{\mathbb{P}^n}(1)$.

Chern classes. Let X be a complex manifold. The exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

is an exact sequence of sheaves, which fails to be exact at the level of global sections. We, therefore, have the corresponding long exact sequence on cohomology:

$$\cdots \longrightarrow H^1(X, \mathcal{O}_X) \longrightarrow \mathrm{Pic}(X) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow \cdots, \tag{1.4.3}$$

where δ is the coboundary map.

Definition 1.4.22. Let $\mathcal{L} \longrightarrow X$ be a (holomorphic) line bundle. The *first Chern class of* $c_1(\mathcal{L})$ of $\mathcal{L}$ is the cohomology $c_1(\mathcal{L})$ class $c_1(\mathcal{L}) \in H^2(X, \mathbb{Z})$ given by $c_1(\mathcal{L}) := \delta([\mathcal{L}])$, where $[\mathcal{L}]$ is the holomorphic equivalence class of $\mathcal{L}$ in $\mathrm{Pic}(X)$.

Remark 1.4.23. Since $\delta : \mathrm{Pic}(X) \longrightarrow H^2(X, \mathbb{Z})$ is a morphism of groups, if $\mathcal{L}^*$ is the line bundle dual to $\mathcal{L}$, then $\mathcal{L}^* \otimes \mathcal{L} \simeq \mathcal{O}_X$, implies that

$$c_1(\mathcal{L}^*) + c_1(\mathcal{L}) = \delta([\mathcal{L}^*]) + \delta([\mathcal{L}]) = \delta([\mathcal{L}^* \otimes \mathcal{L}]) = \delta([\mathcal{O}_X]) = 0 \in H^2(X, \mathbb{Z}).$$

In particular, the first Chern class reverses sign under inversion of line bundles in $\mathrm{Pic}(X)$:

$$c_1(\mathcal{L}^*) \quad = \quad -c_1(\mathcal{L}).$$

Similarly, the first Chern class is additive:

$$c_1(\mathcal{L} \otimes \mathcal{A}) \;=\; \delta([\mathcal{L} \otimes \mathcal{A}]) \;=\; \delta([\mathcal{L}]) + \delta([\mathcal{A}]) \;=\; c_1(\mathcal{L}) + c_1(\mathcal{A}).$$

Finally, if $f : X \longrightarrow Y$ is a holomorphic map between complex manifolds, and $\mathcal{L} \longrightarrow Y$ is a line bundle, then the first Chern class commutes with pullback:

$$c_1(f^*\mathcal{L}) \;=\; f^*c_1(\mathcal{L}).$$

Example 1.4.24. Let us consider the sequence (1.4.3) with $X = \mathbb{P}^n$. Since

$$H^1(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \;=\; H^2(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}) \;=\; 0,$$

it follows that $\delta : \mathrm{Pic}(\mathbb{P}^n) \longrightarrow H^2(\mathbb{P}^n, \mathbb{Z}) \simeq \mathbb{Z}$ is an isomorphism. In particular, every line bundle on $\mathbb{P}^n$ is determined by its first Chern class. In other words, every divisor on $\mathbb{P}^n$ is linearly equivalent to some multiple of the hyperplane divisor $H = \mathbb{P}^{n-1} \subset \mathbb{P}^n$.

Example 1.4.25. If Σ is a compact Riemann surface of genus g, then $c_1(K_\Sigma) = -\chi(\Sigma) = 2 - 2g$. In particular, the first Chern class of a compact Riemann surface is homotopy-invariant.

Intersection Theory. Let us now discuss the pairing between line bundles and curves in a complex manifold X. Here, a *curve* in X is understood to mean a (not necessarily irreducible) analytic subvariety of dimension 1. In the following definition, $\Theta^{(\mathcal{L},h)}$ denotes the curvature form of the Hermitian metric h on $\mathcal{L}$ (see §2.2 for the relevant theory).

Definition 1.4.26. Let $\mathcal{C}$ be a curve in a complex manifold X. Let $\mathcal{L} \to X$ be a holomorphic line bundle over X. We define the *intersection pairing*

$$\mathcal{L} \cdot \mathcal{C} \ := \ \int_{\mathcal{C}^\circ} \Theta^{(\mathcal{L},h)},$$

where h is a Hermitian metric on $\mathcal{L}$ and $\mathcal{C}^\circ$ denotes the set of smooth points of $\mathcal{C}$.

Remark 1.4.27. By Stokes' theorem (see [**142**, p. 33] for the proof of Stokes' theorem if $\mathcal{C}$ is not smooth), the definition is well-defined, independent of the choice of Hermitian metric.

The above definition extends to a pairing between divisors $\mathcal{D} \subset X$ and curves $\mathcal{C} \subset X$. Indeed, let $\mathcal{O}_X(\mathcal{D})$ be the line bundle associated to the divisor $\mathcal{D}$. Then we set

$$\mathcal{D} \cdot \mathcal{C} \ := \ \mathcal{O}_X(\mathcal{D}) \cdot \mathcal{C}.$$

In fact, we can still further extend the definition to define a pairing between a curve $\mathcal{C}$ and any cohomology class $[\alpha] \in H^{1,1}(X, \mathbb{R})$ by setting

$$[\alpha] \cdot \mathcal{C} \ := \ \int_{\mathcal{C}} \alpha.$$

Definition 1.4.28. Let $\mathcal{L} \to X$ be a holomorphic line bundle over a complex manifold X of (complex) dimension n. We define the *top intersection number* of $\mathcal{L}$ to be

$$c_1(\mathcal{L})^n \ := \ \int_X \left(\Theta^{(\mathcal{L},h)}\right)^n.$$

Remark 1.4.29. We may sometimes

$$\int_X c_1(\mathcal{L})^n,$$

which is understood to be the top intersection number of $\mathcal{L}$ in the above sense.

Nakai Moishezon Criterion. Let $\mathcal{L} \to X$ be a holomorphic line bundle over a compact complex manifold X. If $\mathcal{D}$ is an ample divisor[21] in X, then

$$\int_Y c_1(\mathcal{D})^k > 0$$

for any subvariety $Y \subset X$ of dimension k, where $0 < k \leq n = \dim_{\mathbb{C}} X$. The Nakai–Moishezon criterion asserts that if X is projective, the converse is true [**324**, p. 205]:

[21]That is, if the associated line bundle $\mathcal{O}_X(\mathcal{D})$ is an ample line bundle in the sense of 3.4.

Theorem 1.4.30. (Nakai–Moishezon criterion). Let X^n be a projective manifold which supports a divisor $\mathcal{D}$. Then $\mathcal{D}$ is ample if and only if for any $0 < k \leq n$, and any irreducible subvariety $Y \subset X$ of dimension k, we have

$$\int_Y c_1(\mathcal{D})^k > 0.$$

1.5. Hermitian and Kähler Manifolds

From now on, unless otherwise stated, all complex structures are assumed to be integrable.

Definition 1.5.1. Let X be a complex manifold with an underlying complex structure J. A Riemannian metric g on X is said to be *Hermitian* (or more precisely, *J–Hermitian*) if

$$g(Ju, Jv) = g(u, v),$$

for all $u, v \in TX$.

Remark 1.5.2. That is, the complex structure J is orthogonal with respect to the Hermitian metric g. If (g_{ij}) denote the components of the Riemannian metric g, then the Hermitian condition translates to

$$g_{ij} J_k^i J_\ell^j = g_{k\ell}.$$

Definition 1.5.3. A complex manifold X endowed with a Hermitian metric g is referred to as a *Hermitian manifold*. If the underlying almost complex structure is not integrable, then we say (X, g) is an *almost Hermitian manifold*.

Proposition 1.5.4. Let X be a complex manifold with a complex structure J. Then X admits a Hermitian metric.

Proof. A manifold (by our definition) is paracompact, and hence, admits a Riemannian metric. Call this metric g. If J denotes the complex structure on X, then the prescription

$$h(u, v) \ := \ g(u, v) + g(Ju, Jv)$$

defines a Hermitian metric on X. □

Suppose (M^{2n}, g) supports an almost complex structure $J : T^{\mathbb{R}}M \longrightarrow T^{\mathbb{R}}M$ compatible with g in the sense that (by writing J for the complex-linear extension of J)

$$g(Ju, Jv) \ = \ g(u, v), \tag{1.5.1}$$

for all $u, v \in T^{\mathbb{C}}M$. Let $\{x_1, ..., x_n, x_{n+1}, ..., x_{2n}\}$ denote smooth (real-valued) local coordinates on M. Introduce the notation $I := i + n$ to apply the Einstein summation convention. Assume the almost complex structure acts according to

$$J\left(\frac{\partial}{\partial x_i}\right) \ = \ \frac{\partial}{\partial x_I}, \qquad J\left(\frac{\partial}{\partial x_I}\right) \ = \ -\frac{\partial}{\partial x_i}.$$

In these coordinates, the Riemannian metric reads

$$g \ = \ g_{ik} dx_i \otimes dx_k + g_{iK} dx_i \otimes dx_K + g_{Ik} dx_I \otimes dx_k + g_{IK} dx_I \otimes dx_K.$$

From (1.5.1), we see that

$$g_{ik} \ = \ g_{IK}, \qquad g_{iK} = g_{Ki} \ = \ -g_{kI} \ = \ -g_{Ik}.$$

Introduce the following complex coordinates $\{z_i, \bar{z}_i\}$ on M, given by

$$z_i \ := \ x_i + \sqrt{-1}x_I, \qquad \bar{z}_i \ := \ x_i - \sqrt{-1}x_I.$$

Then $dz_i = dx_i + \sqrt{-1}dx_I$ and $d\bar{z}_i = dx_i - \sqrt{-1}dx_I$, and hence,

$$\frac{\partial}{\partial z_i} \ := \ \frac{1}{2}\left(\frac{\partial}{\partial x_i} - \sqrt{-1}\frac{\partial}{\partial x_I}\right), \qquad \frac{\partial}{\partial \bar{z}_i} \ := \ \frac{1}{2}\left(\frac{\partial}{\partial x_i} + \sqrt{-1}\frac{\partial}{\partial x_I}\right).$$

Let us write $h : T^{\mathbb{C}}M \times T^{\mathbb{C}}M \longrightarrow \mathbb{C}$ for the Hermitian form coming from the complexification of g, i.e.,

$$h(u,v) := g(u,v),$$

for $u, v \in T^{\mathbb{C}}M$. We observe that

$$h_{ij} \ = \ h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial z_j}\right) \ = \ 0 \ = \ h\left(\frac{\partial}{\partial \bar{z}_i}, \frac{\partial}{\partial \bar{z}_j}\right) \ = \ h_{\bar{i}\bar{j}},$$

moreover,

$$h_{i\bar{j}} \ := \ h\left(\frac{\partial}{\partial z_i}, \frac{\partial}{\partial \bar{z}_j}\right) \ = \ g_{ij} + \sqrt{-1}g_{iJ}.$$

In particular, we see that $\mathrm{Re}(h) = g$ and $\mathrm{Im}(h) = -\omega$, where ω is the $(1,1)$–form

$$\omega(u,v) \ := \ g(Ju,v).$$

Cautionary Remark 1.5.5. Let h be a Hermitian metric with underlying Riemannian metric g and $(1,1)$–form ω. We will often express this data by simply writing ω; when we wish to specify the additional data given by g, we write ω_g. It is a ubiquitous tradition in complex geometry to refer to the $(1,1)$–form ω_g as the *Hermitian metric*, and this tradition will be maintained here. Let us also caution the reader that ω_g appears under a number of names in the literature, most commonly referred to as the *fundamental* 2*–form, fundamental* $(1,1)$*–form, associated* $(1,1)$*–form*, or *Kähler form*[22]. We will also at times adopt the notation $\{\cdot,\cdot\}_h$ in place of $h(\cdot,\cdot)$.

In general, a Hermitian metric can be a rather ungodly object. As a consequence, great success has been achieved by considering classes of Hermitian metrics which support some additional structure, the most famous of which are the following:

Definition 1.5.6. Let (X, ω) be a Hermitian manifold. The Hermitian metric ω is said to be a *Kähler metric* if the associated $(1,1)$–form is closed

$$d\omega = 0. \tag{1.5.2}$$

A Hermitian manifold (X, ω) is said to be *Kähler* if ω is a Kähler metric. A complex manifold X is said to be *Kähler* if it supports a Kähler metric.

[22]This name is maintained even if the Hermitian metric is not a Kähler metric.

Remark 1.5.7. It is clear from writing $d = \partial + \bar{\partial}$ and decomposing the *Kähler condition* (1.5.2) into types, that (1.5.2) is equivalent to $\partial\omega = 0$ or $\bar{\partial}\omega = 0$.

Proposition 1.5.8. Let (X, ω_g) be a Kähler manifold. Then in local coordinates $(z_1, ..., z_n)$, the metric $g = \sum_{i,j=1}^n g_{i\bar{j}} dz_i \otimes d\bar{z}_j$ has the following symmetry:

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \qquad \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_k} = \frac{\partial g_{i\bar{k}}}{\partial \bar{z}_j}.$$

Proof. Let $\omega = \sqrt{-1}\sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. Then

$$0 = d\omega = \sqrt{-1}\sum_{i,j,k} \frac{\partial g_{i\bar{j}}}{\partial z_k} dz_k \wedge dz_i \wedge d\bar{z}_j + \sqrt{-1}\sum_{i,j,k} \frac{\partial g_{i\bar{j}}}{\partial \bar{z}_k} d\bar{z}_k \wedge dz_i \wedge d\bar{z}_j,$$

from which the statement readily follows. □

Remark 1.5.9. We will see in Chapter 2 that the above proposition states precisely that the torsion of the Chern connection of a Kähler metric vanishes (and conversely).

Given the restrictive nature of the Kähler condition (1.5.2), it is surprising that Kähler metrics exist at all. One of the remarkable features of the subject, however, is that they happen to exist in abundance:

Example 1.5.10. Let $(z_1, ..., z_n)$ denote the standard coordinates on $\mathbb{C}^n$. The Euclidean metric

$$\omega_{\mathbb{C}^n} := \sqrt{-1}\sum_{k=1}^n dz_k \wedge d\bar{z}_k$$

is certainly closed, and thus $(\mathbb{C}^n, \omega_{\mathbb{C}^n})$ is a Kähler manifold.

Example 1.5.11. Let $[z_0 : z_1 : \cdots : z_n]$ be homogeneous coordinates on $\mathbb{P}^n$. Let $\mathcal{U}_0 = \{z_0 = 1\} \simeq \mathbb{C}^n$. The Fubini–Study metric ω_{FS} affords the following description in the open affine chart $\mathcal{U}_0$:

$$\omega_{\text{FS}} = \sqrt{-1}\partial\bar{\partial}\log(1 + |z|^2),$$

where $|z|^2 = \sum_{i=1}^n |z_i|^2$.

To see that this metric is globally defined, let $\mathcal{U}_1 = \{w_1 = 1\} \simeq \mathbb{C}^n$ be another open affine chart on $\mathbb{P}^n$. On the overlap $\mathcal{U}_0 \cap \mathcal{U}_1$, we have $z_1 = \frac{1}{w_0}$ and $z_i = \frac{w_i}{w_0}$ for all $i \geq 2$. Hence,

$$\begin{aligned}
\omega_{\text{FS}} &= \sqrt{-1}\partial\bar{\partial}\log(1 + |z|^2) \\
&= \sqrt{-1}\partial\bar{\partial}\log\left(1 + \frac{1}{|w_0|^2} + \sum_{i=2}^n \left|\frac{w_i}{w_0}\right|^2\right) \\
&= \sqrt{-1}\partial\bar{\partial}\log(1 + |w|^2) - \sqrt{-1}\partial\bar{\partial}\log|w_0|^2 \\
&= \sqrt{-1}\partial\bar{\partial}\log(1 + |w|^2).
\end{aligned}$$

Example 1.5.12. The Bergman metric

$$\omega_{\mathbb{B}} := \sqrt{-1}\partial\bar{\partial}\log(1-|z|^2)$$

on the ball $\mathbb{B}^n \subset \mathbb{C}^n$ is certainly a Kähler metric.

Example 1.5.13. Any Riemann surface is Kähler. This is immediate from the fact that a $(2,2)$–form $d\omega$ vanishes identically on a Riemann surface.

Example 1.5.14. Let $X := \mathbb{S}^{2n-1} \times \mathbb{S}^1$ denote the *Hopf manifold*. Then X is diffeomorphic to the quotient of $\mathbb{C}^n \backslash \{0\}$ by the cyclic group generated by $z \mapsto \frac{1}{2}z$. The *Boothby metric* (or *standard metric*) on X is the Hermitian metric induced by the Euclidean metric on $\mathbb{C}^n \backslash \{0\}$:

$$\omega_0 \ := \ \sqrt{-1}\sum_{i,j}\frac{4}{|z|^2}dz_i \wedge d\overline{z}_j.$$

It is easy to show that ω_0 is Hermitian but not Kähler.

Example 1.5.15. The Kähler condition is preserved under holomorphic immersions. More precisely, let $f : Y \to (X,\omega)$ be a holomorphic immersion with ω a Kähler metric. The pullback of ω to Y is given by $(f^*\omega)(u,v) = \omega(df(u), df(v))$. Since f is holomorphic, the complex structure is preserved, and since f is an immersion $f^*\omega$ will be non-degenerate.

An important specific case of this is the following:

Example 1.5.16. The Kähler property is preserved under restriction to complex submanifolds. In particular, all projective manifolds are Kähler (being complex submanifolds of $\mathbb{P}^n$), and all Stein manifolds are Kähler (being complex submanifolds of $\mathbb{C}^n$).

Remark 1.5.17. Not all compact Kähler manifolds are projective, however. Indeed, a sufficiently generic complex torus is not projective.

Historically, although non-Kähler metrics are quite easy to produce (for instance, deforming a Kähler metric ω within its conformal class $\omega \mapsto e^{2u}\omega$ will violate the Kähler condition, in general), finding explicit complex manifolds which do not support Kähler metrics are not so easy to find.

We will see in Chapter 1.7 that for compact complex surfaces, the Kähler condition is determined entirely by the topology of the underlying manifold:

Theorem 1.5.18. A compact complex surface is Kähler if and only if its first Betti number b_1 is even.

The above theorem can be used to exhibit the first non-Kähler complex manifold:

Example 1.5.19. The Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$, discovered by Hopf [163], has $b_1 = 1$, and thus, does not support a Kähler metric.

There are now many known examples of non-Kähler manifolds, which we will progressively encounter. To describe another readily-available obstruction to the existence of a Kähler metric on a complex manifold, we make the following important definition:

Definition 1.5.20. A cohomology class in $H^2_{\text{DR}}(X,\mathbb{R})$ is called a *Kähler class* if it can be represented by a Kähler metric. The set of all Kähler classes on X is denoted by $\mathcal{K}(X)$, and is called the *Kähler cone*.

Proposition 1.5.21. Let (X,ω) be a compact Kähler manifold. Then the cohomology class $[\omega] \in H^2_{\text{DR}}(X,\mathbb{R})$ is non-trivial. In particular, if $b_2(X) = 0$, then X does not support a Kähler structure.

PROOF. Suppose $[\omega] = 0$ in $H^2_{\text{DR}}(X,\mathbb{R})$. Then $\omega = d\alpha$ for some $\alpha \in \Lambda^1(X)$, and we may write

$$\omega^n = d\alpha \wedge \omega^{n-1}.$$

Observe that

$$d(\alpha \wedge \omega^{n-1}) \ = \ d\alpha \wedge \omega^{n-1} - \alpha \wedge d\omega^{n-1} \ = \ d\alpha \wedge \omega^{n-1} - \alpha \wedge (n-1)(d\omega) \wedge \omega^{n-2}.$$

Since ω is closed, we see that

$$d(\alpha \wedge \omega^{n-1}) \ = \ d\alpha \wedge \omega^{n-1} \ = \ \omega^n.$$

By Stokes' theorem,

$$\int_X \omega^n \ = \ \int_X d(\alpha \wedge \omega^{n-1}) \ = \ 0,$$

since X is compact without boundary. □

Remark 1.5.22. The above argument was communicated to me by Ramiro Lafuente. If one has access to the $\partial\overline{\partial}$–lemma, then we can also argue as follows[23]: If $[\omega] = 0$ in $H^2_{\text{DR}}(X,\mathbb{R})$, then $\omega = d\alpha$ for some 1–form α. By the $\partial\overline{\partial}$–lemma, we can write $\omega = \sqrt{-1}\partial\overline{\partial}\varphi$ for some smooth strictly plurisubharmonic function $\varphi : X \to \mathbb{R}$. By compactness, the maximum principle shows that this is not possible.

Remark 1.5.23. The above proposition is certainly false if X is not compact – take the cohomology class represented by the Euclidean metric on $\mathbb{C}^n$.

An immediately corollary of 1.5.21:

Corollary 1.5.24. No Kähler structure exists on a complex manifold homeomorphic to $\mathbb{S}^6$.

One beautifully transparent instance of the contrast between the real and complex-analytic categories is the Wirtinger theorem – an elementary (but striking) consequence of the interplay between the real and imaginary parts of a Hermitian metric [**254**, p. 101]:

[23] This was the original argument I had in mind.

Theorem 1.5.25. (Wirtinger's theorem). Let Y be a k–dimensional complex submanifold of a Hermitian manifold (X, ω). Then

$$\mathrm{vol}(Y) \;=\; \frac{1}{k!}\int_Y \omega^k.$$

Proof. Let X be a complex manifold with a Hermitian metric h. In a neighborhood $\mathcal{U}$ of a fixed point, let $\{\varphi^1, ..., \varphi^n\}$ be a holomorphic unitary frame for the tangent bundle $\Omega^{1,0}_X$. In $\mathcal{U}$, write

$$h = \sum_k \varphi^k \otimes \bar{\varphi}^k.$$

Decompose the Hermitian metric into real and imaginary parts $h = g - \sqrt{-1}\omega$, where g is the Riemannian metric and ω is real $(1,1)$–form. Write $\varphi^k = \alpha^k + \sqrt{-1}\beta^k$ for the decomposition of the $(1,0)$–forms into real and imaginary parts. Then

$$g \;=\; \mathrm{Re}(h) \;=\; \sum_k (\alpha^k \otimes \alpha^k + \beta^k \otimes \beta^k), \quad \text{ and } \quad \omega \;=\; \sum_k \alpha^k \wedge \beta^k.$$

The Riemannian volume element dV_g is then computed to be

$$dV_g = \alpha^1 \wedge \beta^1 \wedge \cdots \wedge \alpha^n \wedge \beta^n.$$

On the other hand, the top exterior power of ω is

$$\omega^n \;=\; n!\, \alpha^1 \wedge \beta^1 \wedge \cdots \wedge \alpha^n \wedge \beta^n \;=\; n!\, dV_g.$$

Now for any complex submanifold $Y \subset X$ of dimension d. The $(1,1)$–form associated with the metric on Y (induced by restriction) is given by restricting ω to Y. Applying the above to the induced metric on Y proves the theorem. $\square$

Remark 1.5.26. The principle that the volume of a complex submanifold $Y \subset X$ is computed by integrating a differential form defined on all of X, in contrast to the computation of arc length, surface area, etc., in the Riemannian setting, is of fundamental importance.

Remark 1.5.27. It is worth emphasizing that the Kähler condition is not used in the proof of Wirtinger's theorem. The Kähler condition does yield further mileage, however. Indeed, the Wirtinger theorem states that for Kähler manifolds, a suitable normalization of the exterior power of the Kähler form defines a calibration.

To remind the reader, let (M^n, g) be a Riemannian manifold. We understand an *oriented tangent k–plane* $\mathcal{V}$ to be a k–dimensional subspace of some tangent space T_pM, endowed with an orientation. Since $\mathcal{V}$ supports an orientation, the restriction of the Riemannian metric g to $\mathcal{V}$ induces a volume form (i.e., a k–form) on $\mathcal{V}$, which we write as $\mathrm{vol}_{\mathcal{V}}$.

Definition 1.5.28. Let (M, g) be a Riemannian manifold. A closed k–form α on M is said to be a *calibration* on M if, for any oriented k–plane $\mathcal{V}$, we have

$$\alpha|_{\mathcal{V}} \ \leq \ \mathrm{vol}_{\mathcal{V}}.$$

A Riemannian manifold with a calibration is said to be a *calibrated manifold*.

Example 1.5.29. Let (X, ω) be a Kähler manifold. The (normalized) exterior powers

$$\alpha_k \ := \ \frac{1}{k!}\,\omega^k$$

of the Kähler form define calibrations.

Let N^k be an oriented submanifold of a calibrated manifold (M, g, α). Then each tangent space T_pN, $p \in N$, is an oriented tangent k–plane.

Definition 1.5.30. We say that N is a *calibrated submanifold* (with respect to the calibration α) if

$$\alpha|_{T_pN} \ = \ \mathrm{vol}_{T_pN},$$

for all $p \in N$.

Example 1.5.31. The Wirtinger theorem asserts that complex submanifolds of Kähler manifolds are calibrated submanifolds (with respect to the calibrations defined in 1.5.29).

The theory of calibrated geometry gives a useful context for understanding several results in Kähler geometry. One such example that we will make use of later is the following:

Theorem 1.5.32. Let N be a compact submanifold of a calibrated manifold (M, g, α). Then N is volume-minimizing within its homology class.

PROOF. Let $\dim_{\mathbb{R}} N = k$, and write $[N] \in H_k(M, \mathbb{R})$ for the corresponding homology class. Since the calibration is, by definition, closed, α represents a de Rham cohomology class $[\alpha] \in H^k_{\mathrm{DR}}(M, \mathbb{R})$. Write $\cdot : H^k_{\mathrm{DR}}(M, \mathbb{R}) \times H_k(M, \mathbb{R}) \to \mathbb{R}$ for the dual pairing. Then, since N is a calibrated submanifold,

$$[\alpha] \cdot [N] \ \ = \ \ \int_{p \in N} \alpha\,|_{T_pN} \ = \ \int_{p \in N} \mathrm{vol}_{T_pN} \ = \ \mathrm{Vol}(N).$$

Let $\widetilde{N}$ be a compact k–dimensional submanifold of M, homologous to N. Then, since α is a calibration,

$$[\alpha] \cdot [N] \ = \ [\alpha] \cdot [\widetilde{N}] \ \ = \ \ \int_{p \in \widetilde{N}} \alpha|_{T_p\widetilde{N}} \ \leq \ \int_{p \in \widetilde{N}} \mathrm{vol}_{T_p\widetilde{N}} \ = \ \mathrm{vol}(\widetilde{N}).$$

□

Corollary 1.5.33. Let (X, ω) be a Kähler manifold. Complex submanifolds of X are volume-minimizing within their homology classes.

Balanced and Pluriclosed Metrics. There are several ways to relax the Kähler condition. One wants to form classes of Hermitian metrics, which adequately uniformize the wilderness of Hermitian manifolds. In this respect, two natural classes of metrics readily emerge:

Definition 1.5.34. Let (X,ω) be a Hermitian manifold of (complex) dimension n. We say that ω is

(i) a *balanced metric* if $d\omega^{n-1}=0$.
(ii) a *pluriclosed metric* if $\partial\bar{\partial}\omega=0$.

Remark 1.5.35. It is worth noting that the balanced condition is the only non-trivial d–closed condition one can place on a power of the metric ω. Indeed, it easy to see that $d\omega^k=0$ for $1\leq k\leq n-2$ implies that $d\omega=0$.

Example 1.5.36. It is clear that for complex surfaces, the balanced condition is equivalent to the Kähler condition. On the other hand, we will see momentarily that pluriclosed metrics always exist on compact complex surfaces. To see this, we make the following definition:

Definition 1.5.37. Let (X,ω) be a Hermitian manifold of (complex) dimension n. We say that ω is a *Gauduchon metric* if

$$\partial\bar{\partial}\omega^{n-1}=0.$$

The following theorem of Gauduchon [131] shows that Gauduchon metrics always exist:

Theorem 1.5.38. Let (X,ω) be a compact Hermitian manifold. Then there is a smooth function $u:X\to\mathbb{R}$ such that $\omega_u:=e^{2u}\omega$ is a Gauduchon metric. Moreover, the Gauduchon metric is unique in its conformal class.

Corollary 1.5.39. Any compact complex surface admits a pluriclosed metric.

Remark 1.5.40. There is a curious duality between balanced and pluriclosed metrics. We will see in Chapter 2 that a straightforward argument shows that a Hermitian metric which is simultaneously balanced and pluriclosed is Kähler. On the other hand, we have the following conjecture:

Conjecture 1.5.41. (Fino–Vezzoni [121]). Let X be a compact complex manifold. If X supports a balanced metric and a pluriclosed metric, then X admits a Kähler metric.

There is a growing amount of evidence for this conjecture:

Example 1.5.42. Michelsohn [205] showed that all twistor spaces admit balanced metrics. By an older result of Hitchin [162], twistor spaces never admit Kähler metrics, with two exceptions $\mathbb{P}^3$ (the twistor space of $\mathbb{S}^4$), and the flag space (the twistor space of $\mathbb{P}^2$). Verbitsky [291] showed that the twistor space of a compact anti-self-dual Riemannian manifold of dimension 4 which admits a pluriclosed metric must be Kähler.

Example 1.5.43. A very interesting example is given by k–copies $X := \sharp_k(\mathbb{S}^3 \times \mathbb{S}^3)$ of the connected sum of $\mathbb{S}^3 \times \mathbb{S}^3$. It is clear that X is not Kähler. Further, since the Aeppli cohomology group $H_A^{1,1}(X) = 0$ vanishes, any pluriclosed metric ω on X is of the form $\omega = \partial\varphi + \overline{\partial\varphi}$, for some $\varphi \in \Omega_X^{1,0}$. Suppose, in addition, that α is a balanced metric on X. Then

$$0 \ < \ \int_X \omega \wedge \alpha^{n-1} \ \ = \ \ \int_X (\partial\bar{\varphi} + \bar{\partial}\varphi) \wedge \alpha^{n-1} \ = \ 0.$$

Fu–Li–Yau [**126**] showed (via conifold transitions) that for $k \geq 2$, X_k admits balanced metrics. It is known that X_k does not support pluriclosed metrics. At present, the following question remains open:

Question 1.5.44. Does $\mathbb{S}^3 \times \mathbb{S}^3$ admit a balanced metric?

The relationship between balanced and pluriclosed metrics is particularly curious given their vastly different nature. It was elucidated in the seminal work of Michelsohn [**205**] that balanced metrics are, in a very precise sense, dual to Kähler manifolds. Pluriclosed metrics are more complex-analytic in nature, however, since the operator $\partial\bar{\partial}$ is a primarily complex-analytic operator.

Recall that in 1.5.15, we saw that the Kähler condition was preserved by holomorphic immersions. For balanced manifolds, we have the following dual statement:

Proposition 1.5.45. Let $f : X \to Y$ be a holomorphic submersion. If X is balanced, then Y is balanced.

Definition 1.5.46. A complex manifold M^n is said to be *homologically balanced* if every closed de Rham current of dimension $2n-2$ whose $(n-1, n-1)$–component is positive and non-zero represents a non-zero class in $H_{2n-2}(M, \mathbb{R})$.

Theorem 1.5.47. A compact complex manifold M admits a balanced metric if and only if it is homologically balanced.

Remark 1.5.48. Observe, therefore, that the existence of a balanced structure imposes a non-trivial constraint on the complex manifold, i.e., there is an obstruction: On a compact complex balanced manifold, no complex hypersurface can be homologous to 0. Of course, on a Kähler manifold, no complex subvariety can be homologous to zero.

Example 1.5.49. Note that a complex manifold can certainly admit compact hypersurfaces that are homologous to zero: Consider the Calabi–Eckmann complex manifold $\mathbb{S}^{2k+1} \times \mathbb{S}^1$ for $k > 0$. The complex structure is given by identifying $\mathbb{S}^{2k+1} \times \mathbb{S}^1$ as $\mathbb{C}^{k+1} - \{0\}$ modulo the action of scaling by 2. The image in $\mathbb{S}^{2k+1} \times \mathbb{S}^1$ of a complex dimension k complex linear subspace of $\mathbb{C}^{k+1}$ is a compact complex hypersurface homologous to 0. In particular, no Calabi–Eckmann manifolds support balanced metrics.

One surprising feature of balanced manifolds is their behavior under bimeromorphic maps. Note that the following example (*Hironaka's example*) illustrates that Kähler manifolds are not closed under bimeromorphic map:

Example 1.5.50. (Hiroanka). The following example of Hironaka [157], however, shows that the same statement cannot be made for a general modification: Consider the projective space $\mathbb{P}^3$ with coordinates (x, y, z), and in it, the curve $\mathcal{C}$ given by the equation $y^2 = x^2 + x^3$, $z = 0$. In a little ball near zero, blow up one branch of $\mathcal{C}$ first, then the other; outside of the origin, blow up $\mathcal{C} - \{0\}$. Then glue together to obtain the compact complex manifold X with holomorphic map $f : X \to \mathbb{P}^3$. In particular, the Kähler property is not a bimeromorphic invariant.

Remark 1.5.51. We note that Hironaka's example necessarily occurs in dimension at least 3. Indeed, for compact complex surfaces, Kodaira showed that the existence of a Kähler structure is preserved under bimeromorphism. He shows that the blow-up of a compact Kähler surface at one point remains Kähler. Every compact complex surface is obtained by blowing up one of the minimal models. Therefore, it suffices to check that blowing up points on a non-Kähler surface does not yield a Kähler surface. This can be done explicitly.

Remark 1.5.52. We caution the reader that Hironaka's example does not assert that blow-ups of Kähler manifolds are not necessarily Kähler. Indeed, this is always true. Hironaka's example illustrates that a non-Kähler complex manifold may blow up to a Kähler manifold.

The fact that the class of (compact) balanced manifolds is large enough to be closed under bimeromorphic maps is due to Alessandrini–Basanelli [5]:

Theorem 1.5.53. Let $f : M \longrightarrow N$ be a bimeromorphic map. If M supports a balanced metric, then N supports a balanced metric.

The above theorem yields the existence of balanced metrics on some important classes of complex manifolds:

Definition 1.5.54. Let X be a compact complex manifold. We say that X is

(i) *Moishezon* if X is bimeromorphic to a smooth projective variety.
(ii) in the *Fujiki class* $\mathcal{C}$ if X is bimeromorphic to a compact Kähler manifold.

Example 1.5.55. Hironaka's example (Example 1.5.50) provides an example of a Moishezon non-Kähler manifold.

Example 1.5.56. Any compact complex manifold X which supports a big line bundle $\mathcal{L} \to X$ is Moishezon. Indeed, if $\mathcal{L}$ is big, the sections of $\mathcal{L}$ furnish a bimeromorphic map $\Phi : X \dashrightarrow Y \subset \mathbb{P}^N$ with $\dim_{\mathbb{C}} X = \dim_{\mathbb{C}} Y$.

Theorem 1.5.57. Let X be a compact complex manifold in the Fujiki class $\mathcal{C}$.

(i) (Chiose). If X admits a pluriclosed metric, then X is Kähler.
(ii) (Biswas–McKay). If X does not contain any rational curves, then X is Kähler.

Proof. The proof of (i) is given in [96]. The proof of statement (ii) makes use of a number results and is not present in the existing literature. Suppose X is in the Fujiki class $\mathcal{C}$ with no rational curves. By [29, Corollary 8] every bimeromorphic map between compact complex manifolds with no rational curves is a biholomorphic map. $\square$

Corollary 1.5.58. Let X be a compact Moishezon manifold.

(i) If X admits a pluriclosed metric, then X is projective.
(ii) If X does not contain any rational curves, then X is projective with K_X nef.

Proof. The first statement follows from Chiose's theorem [96] together with Moishezon's theorem [210] – a Moishezon Kähler manifold is projective. The second statement follows from [77, Theorem 3.1] and Mori's theorem [213]. $\square$

Remark 1.5.59. Since complex maniflds in the Fujiki class $\mathcal{C}$ admit balanced metrics, Chiose's theorem provides evidence for the Fino–Vezzoni conjecture 1.5.41.

Further directions. Recall that in 1.2.18 we discussed an old observation of Hirzebruch, linking the existence of an integrable complex structure on $\mathbb{S}^6$ to an exotic complex structure on $\mathbb{P}^3$. If it exists, we showed that this exotic $\mathbb{P}^3$ structure could not support a balanced metric. Can such a structure support a pluriclosed metric?

1.6. Harmonic Theory

Let V be a real vector space endowed with a scalar product $(\cdot,\cdot)$. There is an induced scalar product on the pth exterior power $\Lambda^p(V)$ given by (the bilinear extension of)

$$(v_1 \wedge \cdots \wedge v_p, w_1 \wedge \cdots \wedge w_p) \quad = \quad \det((v_i, w_j)).$$

If $e_1, ..., e_n$ is an orthonormal basis for $(V, (\cdot,\cdot))$, then an orthonormal basis for $\Lambda^p(V)$ is given by

$$\{e_{i_1} \wedge \cdots \wedge e_{i_p} : 1 \leq i_1 < i_2 < \cdots < i_p \leq n\}.$$

An orientation for a vector space V is given by declaring a given basis as 'positive'. Any basis obtained from this distinguished 'positive basis' via an invertible matrix with a positive determinant is similarly declared 'positive'.

Definition 1.6.1. If $(V, (\cdot,\cdot))$ is endowed with an orientation, define the *Hodge $\star$–operator* to be the linear extension of the map

$$\star : \Lambda^p(V) \to \Lambda^{n-p}(V), \qquad \star(e_{i_1} \wedge \cdots \wedge e_{i_p}) = e_{j_1} \wedge \cdots \wedge e_{j_{n-p}},$$

where the indices $j_1, ..., j_{n-p}$ are those such that $e_{i_1}, ..., e_{i_p}, e_{j_1}, ..., e_{j_{n-p}}$ is a positive basis for V.

Remark 1.6.2. The Hodge $\star$–operator does not depend on the choice of positive orthonormal basis of V. Given a square matrix $A \in \mathbb{R}^{n \times n}$, and vectors $v_1, ..., v_p \in V$, we see that

$$\star(Av_1 \wedge \cdots \wedge Av_p) \quad = \quad (\det A) \star (v_1 \wedge \cdots \wedge v_p).$$

Since any two positive orthonormal bases are related by a matrix with determinant 1, this verifies the claim.

Example 1.6.3. The image of the 0–form $1 \in \mathcal{C}^\infty(M)$ under the Hodge $\star$–operator yields the volume form of g:

$$\star(1) = \sqrt{\det(g_{ij})} dx_1 \wedge \cdots \wedge dx_n.$$

Remark 1.6.4. Let $\mathcal{E} \to M$ be a smooth vector bundle over a Riemannian manifold (M^n, g). The space of smooth sections of $\mathcal{E}$ are denoted by $H^0(\mathcal{E})$. We denote the space of compactly supported sections by $H^0_0(\mathcal{E})$. If $\mathcal{E}$ is a Hermitian vector bundle, endowed with a bundle metric $h_{\mathcal{E}}$, we denote by $L^2(\mathcal{E})$ the Hilbert space of square-integrable sections $\sigma \in H^0(\mathcal{E})$.

Definition 1.6.5. Let $\mathcal{E}$ and $\mathcal{F}$ be two vector bundles over M. A map $D : H^0(\mathcal{E}) \to H^0(\mathcal{F})$ is said to be *linear differential operator of order k* if it is of the form

$$D(u) \quad = \quad \sum_{|\alpha| \leq k} A_\alpha \partial^\alpha u,$$

where $A_\alpha \in \text{Hom}(\mathcal{E}, \mathcal{F})$ is a bundle morphism, and α is a multi-index.

Example 1.6.6. A general differential operator of second-order is of the form

$$L(f) \;=\; \sum_{i,j=1}^{n} a_{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} + \sum_{k=1}^{n} b_k \frac{\partial f}{\partial x_k} + cf,$$

where $f, a_{ij}, b_k, c : \mathcal{U} \to \mathbb{R}$ are functions on an open set $\mathcal{U} \subseteq \mathbb{R}^n$.

Definition 1.6.7. The second-order differential operator L is *elliptic* if the matrix (a_{ij}) is positive-definite. We say that L is *uniformly elliptic* if

$$\lambda |v|^2 \;\leq\; \sum_{i,j=1}^{n} a_{ij}(x) v_i v_j \;\leq\; \Lambda |v|^2$$

for all $x \in \mathcal{U}$, all vectors v, and some constants $\lambda, \Lambda > 0$.

Although we typically assume the coefficients of the operator L are smooth, in constructing solutions it is typically easier to first obtain weak solutions: A weak solution is defined in terms of the formal adjoint L^* of L, i.e., the operator

$$L^*(f) \;=\; \sum_{i,j=1}^{n} \frac{\partial^2}{\partial x_i \partial x_j}(a_{ij} f) - \sum_{k=1}^{n} \frac{\partial}{\partial x_k}(b_k f) + cf.$$

A locally integrable function $f : \mathcal{U} \to \mathbb{R}$ is said to be a *weak solution* of the equation $L(f) = g$ if

$$\int_{\mathcal{U}} f L^*(\varphi) d\mu \;=\; \int_{\mathcal{U}} g \varphi d\mu,$$

for all compactly supported smooth functions φ on $\mathcal{U}$.

Definition 1.6.8. Let $\omega, \eta \in \Omega_0^p(M)$ be compactly supported smooth p–forms on M. Define the L^2–scalar product $(\cdot,\cdot)_{L^2} : \Omega_0^p(M) \times \Omega_0^p(M) \to \mathbb{R}$ by the formula

$$(\omega, \eta) := \int_M (\omega, \eta) \star (1) \;=\; \int_M \omega \wedge \star\eta.$$

Definition 1.6.9. Let (M^n, g) be an oriented Riemannian manifold. Let $(\mathcal{E}, h_{\mathcal{E}})$ and $(\mathcal{F}, h_{\mathcal{F}})$ be two complex vector bundles over M (endowed with bundles metrics). Let $P : H^0(M, \mathcal{E}) \to H^0(M, \mathcal{F})$ be a differential operator. We say that a differential operator $Q : H^0(M, \mathcal{F}) \to H^0(M, \mathcal{E})$ is the *formal adjoint of* P if

$$\int_M h_{\mathcal{F}}(P\alpha, \beta) dV_g \;=\; \int_M h_{\mathcal{E}}(\alpha, Q\beta) dV_g,$$

for all compactly supported smooth sections $\alpha \in H_0^0(M, \mathcal{E})$ and $\beta \in H_0^0(M, \mathcal{F})$.

Remark 1.6.10. The formal adjoint is unique: Given a differential operator $P : H^0(\mathcal{E}) \to H^0(\mathcal{F})$, suppose that $Q_1 \neq Q_2$ are formal adjoints of P. If $Q := Q_1 - Q_2$, then for all $\alpha \in H_0^0(\mathcal{E})$ and $\beta \in H_0^0(\mathcal{F})$, we see that

$$\int_M h_{\mathcal{E}}(\alpha, Q\beta) dV_g \;=\; 0. \tag{1.6.1}$$

Assume there is a section $\sigma \in H^0(\mathcal{F})$ such that the restriction of $Q(\sigma)$ to the fiber over a point a point $p \in M$ is non-zero. Let ρ be a smooth bump function that is identically 1 in a neighborhood of p and vanishes identically outside of a compact set. Since Q is a differential operator, the value of $Q(\sigma)_x$ depends only on the germ of σ at x. Hence, $Q(f\sigma)$ is compactly supported with $Q(f\sigma)_x = R(\sigma)_x \neq 0$. With $\alpha = Q(f\sigma) \in H^0_0(\mathcal{E})$ and $\beta = f\sigma \in H^0_0(\mathcal{F})$, from (1.6.1), we have

$$0 \;=\; \int_M h_{\mathcal{E}}(Q(f\sigma), Q(f\sigma))dV_g \;=\; \int_M |Q(f\sigma)|^2 dV_g.$$

Hence, $Q(f\sigma) \equiv 0$, contradicting $Q(f\sigma)_x \neq 0$.

Remark 1.6.11. Consider the exterior derivative $d : \mathcal{C}^\infty(M) \to \Omega^1(M)$ acting on functions. Choose local coordinates $(x_1, ..., x_n)$ in an open set $\mathcal{U} \subset M$, and suppose that $f \in \mathcal{C}^\infty_0(\mathcal{U})$ and $\alpha = \alpha_j dx_j \in \Omega^1_0(\mathcal{U})$. Let $dV_g = \rho(x)dx = \rho(x)dx_1 \wedge \cdots \wedge dx_n$ denote the volume form associated to the metric, in these coordinates. Integrating by parts, we observe that

$$\begin{aligned}\int_M \langle df, \alpha\rangle dV_g \;&=\; \int_{\mathcal{U}} g^{ij}\partial_i f \alpha_j \rho dx \\ &=\; -\int_{\mathcal{U}} f\partial_i(g^{ij}\alpha_j\rho)dx \;=\; -\int_{\mathcal{U}} f\rho^{-1}\partial_i(g^{ij}\alpha_j\rho)dV_g.\end{aligned}$$

Hence,

$$d^*\alpha \;=\; -\rho^{-1}\partial_i(\rho g^{ij}\alpha_j). \tag{1.6.2}$$

Theorem 1.6.12. Let $d : \Omega^p(M) \to \Omega^{p+1}(M)$ denote the exterior derivative. The formal adjoint $d^* : \Omega^p(M) \to \Omega^{p-1}(M)$ is defined by the formula

$$d^* \;=\; (-1)^{n(p+1)+1} \star d \star.$$

PROOF. Let $\alpha \in \Omega^{p-1}(M)$ and $\beta \in \Omega^p(M)$. We calculate

$$d(\alpha \wedge \star\beta) \;=\; d\alpha \wedge \star\beta + (-1)^{p-1}\alpha \wedge d \star \beta.$$

Since β is p–form, $d \star \beta$ is an $(n-p+1)$–form, we see that

$$\star \star d \star \beta \;=\; (-1)^{(n-p+1)(n-(n-p+1))} d \star \beta \;=\; (-1)^{(p-1)(n-p+1)} d \star \beta.$$

Hence,

$$\begin{aligned} d\alpha \wedge \star\beta + (-1)^{p-1}\alpha \wedge d \star \beta \;&=\; d\alpha \wedge \star\beta + (-1)^{p-1}(-1)^{(p-1)(n-p+1)}\alpha \wedge \star \star d \star \beta \\ &=\; d\alpha \wedge \star\beta + (-1)^{(p-1)(n+2-p)}\alpha \wedge \star \star d \star \beta \\ &=\; d\alpha \wedge \star\beta - (-1)^{n(p+1)+1}\alpha \wedge \star \star d \star \beta \\ &=\; \pm \star ((d\alpha, \beta) - (-1)^{n(p+1)+1}(\alpha, \star d \star \beta)).\end{aligned}$$

Integrating the formula

$$d(\alpha \wedge \star\beta) = \pm \star ((d\alpha, \beta) - (-1)^{n(p+1)+1}(\alpha, \star d \star \beta)),$$

and applying Stokes' theorem completes the proof. $\square$

Definition 1.6.13. The *Laplace–Beltrami operator* (or *Hodge Laplacian*) is defined

$$\Delta_d \;:=\; dd^* + d^*d.$$

The forms $\eta \in \Omega^p(M)$ which are annihilated by the Laplace–Beltrami operator, i.e., $\Delta\eta = 0$, are called *harmonic p–forms*. The space of harmonic p–forms is denoted by $\mathcal{H}^p(M)$.

Example 1.6.14. On functions, $\Delta = d^*d$. Hence, from (1.6.2), we have the following local coordinate expression:

$$\Delta f \;=\; -\frac{1}{\sqrt{\det(g_{ij})}}\partial_i \left(g^{ij}\sqrt{\det(g_{ij})}\partial_j f \right).$$

Example 1.6.15. The Laplace–Beltrami operator is elliptic and, moreover, self-adjoint with respect to the above L^2–pairing, i.e.,

$$(\Delta_d \alpha, \beta) \;=\; (\alpha, \Delta_d \beta),$$

for all $\alpha, \beta \in \Omega^p$.

Lemma 1.6.16. Let $\alpha \in \Omega^p(M)$. Then α is harmonic if and only if $d\alpha = 0$ and $d^*\alpha = 0$.

Proof. The if statement is obvious. For the non-trivial part of the lemma, write

$$\begin{aligned}(\Delta\alpha, \alpha) \;=\; ((dd^* + d^*d)\alpha, \alpha) \;&=\; (dd^*\alpha, \alpha) + (d^*d\alpha, \alpha)\\ &=\; \|d^*\alpha\|^2_{L^2} + \|d\alpha\|^2_{L^2}.\end{aligned}$$

$\square$

Theorem 1.6.17. Let (M^n, g) be a compact orientable Riemannian manifold. Then every cohomology class in $H^p_{\mathrm{DR}}(M, \mathbb{R})$ can be represented by a unique harmonic p–form.

Proof of Uniqueness. Uniqueness is straightforward: Let $\omega, \widehat{\omega} \in \Omega^p(M)$ be two cohomologous harmonic p–forms on M. If $p = 0$, there is nothing to prove, so assume $p > 0$. Then $\widehat{\omega} = \omega + d\alpha$ for some $(p-1)$–form α. Compute

$$\|\widehat{\omega} - \omega\|^2 \;=\; (\widehat{\omega} - \omega, \widehat{\omega} - \omega) \;=\; (\widehat{\omega} - \omega, d\alpha) \;=\; (\delta(\widehat{\omega} - \omega), \alpha).$$

Since $\widehat{\omega}$ and ω are harmonic, $\delta(\widehat{\omega} - \omega) = 0$ and uniqueness follows. $\square$

Sketch of Existence. The proof of existence is more delicate but extends the classical variational approach of solving Laplace's equation (for functions) via the Dirichlet principle. That is, fix a cohomology class $[\widehat{\omega}] \in H^p_{\mathrm{DR}}(M, \mathbb{R})$, and consider the Dirichlet energy

$$\mathcal{E}(\omega) \ := \ \frac{1}{2} \int_M \|\omega\|^2 dV_g,$$

where ω ranges over all p–forms in $[\widehat{\omega}]$. The key point is to show that the infimum of $\mathcal{E}$ is *achieved* by a *smooth* form ξ. The resulting minimum will then be harmonic: ξ must satisfy the Euler–Lagrange equations for $\mathcal{E}$, i.e., for all $\alpha \in \Omega^{p-1}(M)$,

$$0 \ = \ \frac{d}{d\varepsilon}\bigg|_{\varepsilon=0} (\xi + \varepsilon d\alpha, \xi + \varepsilon d\alpha) \ = \ 2(\xi, d\alpha) \ = \ 2(\delta\xi, \alpha).$$

□

To make this sketch more precise, we start by reminding the reader of the Sobolev space $H^s(\mathbb{R}^n)$ of tempered distributions f on $\mathbb{R}^n$ such that the Fourier transform satisfies

$$\|f\|^2_{H^s(\mathbb{R}^n)} \ := \ \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 (1 + |\xi|^2)^s |d\xi|^n \ < \ \infty.$$

Equivalently, $H^s(\mathbb{R}^n)$ is the space of functions $f \in L^2(\mathbb{R}^n)$ which admit s weak derivatives in L^2, and

$$\|f\|^2_{H^s(\mathbb{R}^n)} \ \sim \ \sum_{|\alpha| \le s} \|\partial_\alpha f\|^2_{L^2(\mathbb{R}^n)}.$$

Remark 1.6.18. These definitions coincide when $s \in \mathbb{N}_0$, but the first definition is more general: it can be defined for any $s \in \mathbb{R}$.

Remark 1.6.19. Let $\mathcal{E} \to M$ be a smooth vector bundle over a compact manifold M. We denote by $\mathcal{C}^k(M, \mathcal{E})$ the space of sections of $\mathcal{E}$ of regularity $\mathcal{C}^k$. In any local trivialization of $\mathcal{E}$ and any coordinate chart of M, the coefficients of the section are $\mathcal{C}^k$. We similarly define $H^s(M, \mathcal{E})$ to be the space of sections of $\mathcal{E}$ whose coefficients have regularity in the Sobolev space H^s.

If M is covered by a finite number of charts $(\mathcal{U}_\alpha)$ with trivializations of $\mathcal{E}|_{\mathcal{U}_\alpha}$ given by a basis of sections $\sigma_{\alpha,k}$, we may choose a partition of unity (ρ_α) subordinate to $(\mathcal{U}_\alpha)$. Then a section u of $\mathcal{E}$ can be written as

$$u = \sum_{\alpha,k} \rho_\alpha u_{\alpha,k} \sigma_{\alpha,k},$$

where $\rho_\alpha u_{\alpha,k}$ is a function with compact support in $\mathcal{U}_\alpha \subset \mathbb{R}^n$. As a consequence, we define

$$\|u\|_{\mathcal{C}^k} \ := \ \sup_{\alpha,k} \|\rho_\alpha u_{\alpha,k}\|_{\mathcal{C}^k(\mathbb{R}^n)}, \qquad \|u\|^2_{H^s} \ := \ \sum_{\alpha,k} \|\rho_\alpha u_{\alpha,k}\|^2_{H^s(\mathbb{R}^n)}.$$

Up to equivalence of norms, the result is independent of the choice of coordinate charts and trivializations of $\mathcal{E}$.

Example 1.6.20. If M is a torus $\mathbb{T}^n$, the regularity can be seen at the level of Fourier series: We see that $f \in H^s(\mathbb{T}^n)$ if and only if

$$\|f\|_{H^s(\mathbb{T}^n)} \;=\; \sum_{\xi \in \mathbb{Z}^n} (1+|\xi|^2)^s |\hat{f}(\xi)|^2 \;<\; \infty.$$

From the inverse formula $f(x) = \sum_\xi \hat{f}(\xi) e^{\sqrt{-1}\langle \xi, x\rangle}$ and the Cauchy–Schwartz inequality, we have for $s > \frac{n}{2}$:

$$|f(x)| \;\leq\; \sum_{\xi \in \mathbb{Z}^n} |\hat{f}(\xi)| \;\leq\; \|f\|_{H^s(\mathbb{T}^n)} \left(\sum_\xi (1+|\xi|^2)^{-s} \right)^{\frac{1}{2}} \;<\; \infty.$$

In particular, we see that there is a continuous inclusion $H^s(\mathbb{T}^n) \hookrightarrow \mathcal{C}^0(\mathbb{T}^n)$ if $s > \frac{n}{2}$. It similarly follows that $H^s(\mathbb{T}^n) \subset \mathcal{C}^k(\mathbb{T}^n)$ if $s > k + \frac{n}{2}$.
Using the Fourier transform, we see that the same results are true on $\mathbb{R}^n$, therefore:

Lemma 1.6.21. (Sobolev). Let M be a compact manifold and $k \in \mathbb{N}$. There is a continuous and compact injection

$$H^s \hookrightarrow \mathcal{C}^k, \qquad \text{if } s > k + \frac{n}{2}.$$

The fact that the inclusion is compact follows from the following lemma:

Lemma 1.6.22. (Rellich). If M is a compact manifold, then the inclusion

$$H^s \hookrightarrow H^t \qquad \text{for } s > t$$

is compact.

The above lemma is obvious on the torus; hence, the general case easily follows by using the Fourier transform.

Let $P : H^0(\mathcal{E}) \to H^0(\mathcal{F})$ be a differential operator of order d. By looking at P in local coordinates, it is clear that P induces continuous operators

$$P : H^{s+d}(M, \mathcal{E}) \longrightarrow H^s(M, \mathcal{F}).$$

In general, a *weak solution* of the equation $Pu = v$ is an L^2 section u of $\mathcal{E}$ such that for any $\varphi \in \mathcal{C}_0^\infty(M, \mathcal{F})$, we have

$$(u, P^* \varphi)_{L^2} \;=\; (v, \varphi)_{L^2}.$$

Lemma 1.6.23. (Local elliptic estimate). Let $P : H^0(\mathcal{E}) \to H^0(\mathcal{F})$ be an elliptic operator.Fix a ball $\mathbb{B}$ in a chart with local coordinates (x_i) and a smaller ball $\mathbb{B}(\frac{1}{2})$. Suppose that $u \in L^2(\mathbb{B}, \mathcal{E})$ and $Pu \in H^s(\mathbb{B}, \mathcal{F})$, then $u \in H^{s+d}(\mathbb{B}(\frac{1}{2}), \mathcal{E})$ and

$$\|u\|_{H^{s+d}(\mathbb{B}(\frac{1}{2})} \leq C\left(\|Pu\|_{H^s(\mathbb{B})} + \|u\|_{L^2(\mathbb{B})}\right).$$

Remark 1.6.24. We will not prove the above lemma, but let us mention that there are effectively two approaches to its proof: The first meth is to approximate the operator locally on small balls by an operator with constant coefficients on $\mathbb{R}^n$ or $\mathbb{T}^n$, where an explicit inverse is available using the Fourier transform. These inverses are then glued together to get an approximate inverse for P, which will give what is needed on u. The details of this approach can be found in [**297**]. A modern approach is via microlocal analysis: one inverts the operator "microlocally", i.e., fiber by fiber on each cotangent space, using the theory of pseudo-differential operators. The details of this approach can be found in [**104**].

Corollary 1.6.25. (Global elliptic estimate). Let $p : H^0(\mathcal{E}) \to H^0(\mathcal{F})$ be an elliptic operator. If $u \in L^2(M, \mathcal{E})$ and $Pu \in H^s(M, \mathcal{F})$, then $u \in H^{s+d}(M, \mathcal{E})$ and

$$\|u\|_{H^{s+d}} \leq C(\|Pu\|_{H^s} + \|u\|_{L^2}).$$

From the elliptic estimate and the fact that $\bigcap_s H^s = \mathcal{C}^\infty$, we have:

Corollary 1.6.26. If P is an elliptic differential operator and $Pu = 0$, then u is smooth. More generally, $Pu \in \mathcal{C}^\infty \implies u \in \mathcal{C}^\infty$.

Hodge theory. Let $\mathcal{H}^p$ be the space of harmonic p–forms.

Theorem 1.6.27. Let (M^n, g) be a compact oriented Riemannian manifold. Then the space $\mathcal{H}^p$ of harmonic p–forms is finite-dimensional and there is a decomposition

$$\Omega^p_M \simeq \mathcal{H}^p \oplus \Delta(\Omega^p_M)$$

which is orthogonal for the L^2 inner product.

Remark 1.6.28. It is clear that $\ker(\Delta)$ is orthogonal to $\mathrm{Im}(\Delta)$, since Δ is formally self-adjoint. Moreover, the general theory of unbounded operators gives (almost immediately) that $L^2(M, \Omega^p_M)$, the space of p–forms with L^2–regularity decomposes as

$$L^2(M, \Omega^p_M) \simeq \mathcal{H}^p \oplus \overline{\mathrm{Im}(\Delta)}.$$

The non-trivial components of the Hodge theorem are the finite-dimensionality of $\mathcal{H}^p$, closedness of $\mathrm{Im}(\Delta)$, and the fact that smooth forms in the L^2 image of Δ are images of smooth forms.

The proof of 1.6.27 is a consequence of the following theorem on elliptic operators:

Theorem 1.6.29. Suppose (M^n, g) is a compact oriented Riemannian manifold. Let $P : H^0(\mathcal{E}) \to H^0(\mathcal{F})$ be an elliptic operator with $\mathrm{rank}(\mathcal{E}) = \mathrm{rank}(\mathcal{F})$. Then

(i) $\ker(P)$ is finite-dimensional.

(ii) There is an L^2 orthogonal direct sum decomposition

$$H^0(\mathcal{F}) \;\simeq\; \ker(P^*) \oplus P(H^0(\mathcal{E})).$$

Proof. Let $\Sigma := \ker(P)$ and consider the identity map $\mathrm{id} : (\Sigma, \|\cdot\|_{L^2}) \to (\Sigma, \|\cdot\|_{H^{s+d}})$. From the elliptic estimate, this map is continuous. From the Rellich compactness lemma, the identity map $\mathrm{id} : (\Sigma, \|\cdot\|_{H^{s+d}}) \to (\Sigma, \|\cdot\|_{L^2})$ is compact. The composition

$$(\Sigma, \|\cdot\|_{L^2}) \longrightarrow (\Sigma, \|\cdot\|_{H^{s+d}}) \longrightarrow (\Sigma, \|\cdot\|_{L^2})$$

is a compact operator. Hence, the closed unit ball in Σ is compact, and Σ is finite-dimensional. For the second statement, let us first show that

$$H^s(M, \mathcal{F}) \simeq \ker(P^*) \oplus \mathrm{Im}(P). \tag{1.6.3}$$

Assume that for any $\varepsilon > 0$ there is an L^2–orthonormal sequence $(v_1, ..., v_N)$ in H^{s+d} such that

$$\|u\|_{L^2} \;\;\leq\;\; \varepsilon\|u\|_{H^{s+d}} + \left(\sum_{k=1}^{N} |(v_k, u)|^2\right)^{\frac{1}{2}}.$$

Combining this with the elliptic estimate, we see that

$$(1 - C\varepsilon)\|u\|_{H^{s+d}} \;\;\leq\;\; C\|Pu\|_{H^s} + C\left(\sum_{k=1}^{N} |(v_k, u)|^2\right)^{\frac{1}{2}}.$$

Choose $\varepsilon = (2C)^{-1}$ and let Λ be the subspace of sections in $H^{s+d}(M, \mathcal{E})$ which are orthogonal to the (v_k). Then for any $u \in \Lambda$, we have

$$2\|u\|_{H^{s+d}} \;\;\leq\;\; C\|Pu\|_{H^s}.$$

It follows that $P(\Lambda)$ is closed in $H^s(M, \mathcal{F})$. Since $\mathrm{Im}(P)$ is the sum of $P(\Lambda)$ and the image of the finite-dimensional space generated by the (v_k), it follows that $\mathrm{Im}(P)$ is closed in $H^s(M, \mathcal{F})$. The statement for the Sobolev spaces H^s is sufficient to imply the decomposition on $\mathcal{C}^\infty$. Indeed, if $v \in H^0(M, \mathcal{F})$ is a smooth section which is L^2–orthogonal to $\ker(P^*)$, by fixing any $s \geq 0$ and applying (1.6.3) in H^s, we can find $u \in H^{s+d}(M, \mathcal{E})$ such that $Pu = v$. By 1.6.26, u is smooth.

It remains to prove the claim: To this end, let (v_k) be an L^2–orthonormal basis of L^2. Proceed by contradiction and suppose the claim is not true. Then there is a sequence $(u_N) \in$

$H^{s+d}(M, \mathcal{E})$ such that $\|u_N\|_{L^2} = 1$ and

$$\varepsilon\|u_N\|_{H^{s+d}} + \left(\sum_{k=1}^{N} |(v_k, u_N)|^2\right)^{\frac{1}{2}} \;<\; 1.$$

This second condition implies that (u_N) is bounded in $H^{s+d}(\mathcal{E})$, and therefore admits a weakly convergent subsequence in $H^{s+d}(\mathcal{E})$, with limit satisfying

$$\varepsilon\|u\|_{H^{s+d}} + \|u\|_{H^0} \;\leq\; 1.$$

By the compact inclusion $H^{s+d} \subset L^2$, this subsequence converges strongly in $L^2(\mathcal{E})$. From the first condition, this implies that the limit u satisfies $\|u\|_0 = 1$, which is a contradiction. $\square$

Corollary 1.6.30. Let M be a compact, oriented, Riemannian manifold. The de Rham cohomology groups $H^p_{\mathrm{DR}}(M, \mathbb{R})$ are finite-dimensional.

Proof. Proceed by contradiction, and assume that $H^p_{\mathrm{DR}}(M, \mathbb{R})$ is not finite-dimensional. Then there exists an orthonormal sequence of harmonic forms $(\omega_k)_{k\in\mathbb{N}}$ in $H^p_{\mathrm{DR}}(M, \mathbb{R})$. By the Rellich compactness lemma, there is an L^2–convergent subsequence, which converges to some $\omega \in H^{1,2}_p(M)$. Since the ω_k are orthonormal, however, $(\omega_k, \omega_\ell) = \delta_{k\ell}$ for any $k, \ell \in \mathbb{N}$. In particular, if $k \neq \ell$, then

$$\|\omega_k - \omega_\ell\|^2 \;=\; (\omega_k - \omega_\ell, \omega_k - \omega_\ell) \;=\; \|\omega_k\|^2 + \|\omega_\ell\|^2 \;\geq\; 1,$$

and the sequence (ω_k) is not Cauchy with respect to the L^2–norm. $\square$

Remark 1.6.31. Note that the orthogonality of $\ker(\Delta)$ and $\mathrm{Im}(\Delta)$ is immediate from the fact that Δ is self-adjoint. Moreover, the general theory of unbounded operators gives $L^2(M, \Omega^p) = H^p \oplus \overline{\mathrm{im}(\Delta)}$. The non-trivial part of the Hodge theorem is the finite-dimensionality of H^p, the fact that the image of Δ is closed, and the fact that smooth forms in the L^2 image of Δ are images of smooth forms.

Hodge Theory for Complex Manifolds. We now want to extend these results to the complex-analytic category. To this end, let (X, ω) be a Hermitian manifold. Recall that the exterior derivative d splits into a sum of Dolbeault operators $\partial : \Omega^{p,q}(X) \longrightarrow \Omega^{p+1,q}(X)$ and $\bar{\partial} : \Omega^{p,q}(X) \longrightarrow \Omega^{p,q+1}(X)$. The inner product $(\cdot, \cdot)$ on $\Omega^\bullet(X)$ extends to a Hermitian inner product on $\Omega^{p,q}$ by the demanding the formulae:

$$(a\alpha + b\beta, \gamma) \;\;=\;\; a(\alpha, \gamma) + b(\beta, \gamma), \qquad\quad (\alpha, a\gamma + b\varepsilon) \;=\; \overline{a}(\alpha, \gamma) + \bar{b}(\alpha, \varepsilon),$$

where $a, b \in \mathbb{C}$, $\alpha, \beta, \varepsilon, \delta \in \Omega^\bullet(X)$.

The $\mathbb{C}$–linear extension of the Hodge $\star$–operator to complex-valued forms (which we abusively write as $\star$) satisfies

$$\star : \Omega^{p,q}(X) \longrightarrow \Omega^{n-q,n-p}(X), \qquad \alpha \wedge \star\bar{\beta} \ = \ (\alpha, \beta)\frac{\omega^n}{n!},$$

where $\alpha, \beta \in \Omega^{p,q}(X)$. Observe that

$$\overline{\star\alpha} \ = \ \star(\overline{\alpha}), \qquad \star(\star\alpha) \ = \ (-1)^{p+q}\alpha, \qquad (\star\alpha, \star\beta) \ = \ (\alpha, \beta).$$

Example 1.6.32. Let ω be the $(1,1)$–form of a Hermitian metric. Then

$$\star\omega^k \ = \ \frac{k!}{(n-k)!}\omega^{n-k}.$$

The following lemma is straightforward:

Lemma 1.6.33. The formal adjoints ∂^* and $\bar{\partial}^*$ of the Dolbeault operators ∂ and $\bar{\partial}$ are specified by the formulae

$$\partial^* \ = \ -\star\bar{\partial}\star, \qquad \bar{\partial}^* \ = \ -\star\partial\star.$$

Remark 1.6.34. We will see in Chapter 2 that an important role is played by $\tau = -\sqrt{-1}\bar{\partial}^*\omega$, where ω is the $(1,1)$–form of a Hermitian metric. These $(1,0)$–forms τ are referred to as the torsion 1–forms of the Chern connection.

Definition 1.6.35. The *Dolbeault Laplace operators* are defined

$$\Delta_\partial \ := \ \partial\partial^* + \partial^*\partial, \qquad \Delta_{\bar{\partial}} \ := \ \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}.$$

Definition 1.6.36. A (p,q)–form η is said to be *$\bar{\partial}$–harmonic* if $\Delta_{\bar{\partial}}\eta = 0$. The space of $\overline{\partial}$–harmonic (p,q)–forms is denoted $\mathcal{H}^{p,q}(X)$.

It is straightfoward to see that a form $\eta \in \Omega^{p,q}(X)$ is $\overline{\partial}$–harmonic if and only if $\overline{\partial}\eta = 0$ and $\overline{\partial}^*\eta = 0$.

Definition 1.6.37. Let (X, ω) be a Kähler manifold. The *Lefschetz operator* is defined to be the linear map

$$L : \Omega^k(X) \to \Omega^{k+2}(X), \qquad L(\alpha) := \omega \wedge \alpha.$$

The adjoint of the Lefschetz operator is denoted $\Lambda : \Omega^k(X) \to \Omega^{k-2}(X)$.

Recall that if M is a smooth manifold of (real) dimension $2n$, then M has the homotopy type of a CW-complex of (real) dimension $\leq 2n$. The following theorem asserts that Stein manifolds have half the topology than one would expect:

Theorem 1.6.38. Let $S \subset \mathbb{C}^n$ be a Stein manifold of dimension m. Then S has the homotopy type of a CW-complex of real dimension $\leq m$. As a consequence

$$H^k(S,\mathbb{Z}) \;=\; H_k(S,\mathbb{Z}) \;=\; 0, \qquad \forall k > m.$$

Theorem 1.6.39. (Lefschetz hyperplane theorem [236]). Let X be a smooth projective variety of dimension n. Let $Y \subset X$ be a hyperplane section. Then the maps

$$r_k : H^k(X,\mathbb{Z}) \longrightarrow H^k(Y,\mathbb{Z})$$

induced by the restriction $X \to Y$ is an isomorphism for $k \leq n-2$ and injective for $k = n-1$.

Theorem 1.6.40. Let (X,ω) be a Kähler manifold. Then we can find local holomorphic coordinates $(z_1, ..., z_n)$ such that

$$g_{i\bar{j}} \;=\; \delta_{ij} + O(|z|^2).$$

That is, a Kähler metric affords (holomorphic) coordinates in which the coefficients coincide with those of the standard Euclidean metric up to second-order.

Proof. For $(z_1, ..., z_n)$ local holomorphic coordinates centered at some point $p \in X$, make a linear change of coordinates if necessary to ensure that $dz_1, ..., dz_n$ yields a local unitary frame for Ω_X^1 near p. In this frame, write

$$\omega \;=\; \sqrt{-1}\sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\overline{z}_j,$$

where $g_{i\bar{j}} = \delta_{ij} + O(|z|)$. to first order, the Taylor development of $g_{i\bar{j}}$ reads:

$$g_{i\bar{j}} \;=\; \delta_{ij} + \sum_{k=1}^n (A_{i\bar{j}k} z_k + B_{i\bar{j}\bar{k}} \overline{z}_k) + O(|z|^2).$$

Since the metric is Hermitian, $B_{i\bar{j}\bar{k}} = \overline{A_{j\bar{i}k}}$. The Kähler condition $\partial_k g_{i\bar{j}} = \partial_i g_{k\bar{j}}$ implies $A_{i\bar{j}k} = A_{k\bar{j}i}$. For each $1 \leq j \leq n$, introduce the holomorphic functions

$$\zeta_j \;:=\; z_j + \frac{1}{2}\sum_{i,k=1}^n A_{i\bar{j}k} z_i z_k.$$

Compute

$$\begin{aligned} d\zeta_j \;&=\; dz_j + \frac{1}{2}\sum_{i,k=1}^n A_{i\bar{j}k}(z_i dz_k + z_k dz_i) \\ &=\; dz_j + \frac{1}{2}\sum_{i,k=1}^n (A_{i\bar{j}k} + A_{k\bar{j}i}) z_k dz_k \;=\; dz_j + \sum_{i,k=1}^n A_{i\bar{j}k} z_k dz_i. \end{aligned}$$

Hence,

$$\begin{aligned}\sqrt{-1}\sum_{j=1}^n d\zeta_j \wedge d\bar{\zeta}_j &= \sqrt{-1}\sum_{j=1}^n dz_j \wedge d\bar{z}_j + \sqrt{-1}\sum_{i,j,k=1}^n A_{i\bar{j}k} z_k dz_i \wedge d\bar{z}_j \\ &\qquad + \sqrt{-1}\sum_{i,j,k=1}^n \overline{A_{i\bar{j}k}}\bar{z}_k dz_j \wedge d\bar{z}_i + O(|z|^2).\end{aligned}$$

Further,

$$\begin{aligned}\sqrt{-1}\sum_{i,j,k=1}^n \overline{A_{i\bar{j}k}}\bar{z}_k dz_j \wedge d\bar{z}_i &= \sqrt{-1}\sum_{i,j,k=1}^n \overline{A_{k\bar{j}i}}\bar{z}_k dz_j \wedge d\bar{z}_i \\ &= \sum_{i,j,k=1}^n B_{i\bar{j}k}\bar{z}_k dz_j \wedge d\bar{z}_i.\end{aligned}$$

Coalescing the above, we have

$$\sqrt{-1}\sum_{j=1}^n d\zeta_j \wedge d\bar{\zeta}_j = \sqrt{-1}\sum_{i,j=1}^n \left(\delta_{ij} + \sum_{k=1}^n A_{i\bar{j}k} z_k + B_{i\bar{j}k}\bar{z}_k\right) dz_i \wedge d\bar{z}_j + O(|z|^2),$$

as required. □

Remark 1.6.41. The coordinates in the above theorem are sometimes referred to as *normal coordinates*. This is unfortunate since the Riemannian normal coordinates (i.e., geodesic normal coordinates) are, in general, not holomorphic (even in the Kähler category). Indeed, let (X^n, ω) be a Hermitian manifold, and $p \in X$ a point. Suppose the exponential map $\exp_p : T_pX \to X$ is holomorphic. Pass to the universal cover $\widetilde{X}$ to get a holomorphic map $\mathbb{C}^n \to \widetilde{X}$. If $\widetilde{X}$ is a bounded domain in $\mathbb{C}^n$ (for instance, if X is Kobayashi hyperbolic), we are gifted a bounded entire function, which is, of course, constant. It is clear, however, that if the exponential map is holomorphic in a neighborhood of every point, then the metric coincides up to second-order with the Euclidean metric and is, therefore, Kähler.

Theorem 1.6.42. (Kähler identities). Let (X, ω) be a Kähler manifold. Then

$$[\overline{\partial}^*, L] = \sqrt{-1}\partial, \quad [\partial^*, L] = -\sqrt{-1}\overline{\partial}, \quad [\Lambda, \overline{\partial}] = -\sqrt{-1}\partial^*, \quad [\Lambda, \partial] = \sqrt{-1}\overline{\partial}^*.$$

Proof. The Lefschetz operator L, and its adjoint Λ, are both real (i.e., invariant under conjugation). Hence, the second identity follows from the first, and the fourth identity follows from the third. Assuming the first identity holds, we prove the third identity: Let $\alpha, \beta \in \Omega^{p,q}(X)$ be smooth (p,q)–forms. Then with respect to the L^2–scalar product, we have

$$([\Lambda, \overline{\partial}]\alpha, \beta) = (\alpha, [\overline{\partial}^*, L]\beta) = (\alpha, \sqrt{-1}\partial\beta) = (-\sqrt{-1}\partial^*\alpha, \beta).$$

It suffices, therefore, to prove the first identity. The key point is that the operator L uses the coefficients of the metric only up to order zero, while $\sqrt{-1}\partial$ and $\overline{\partial}^* = -\star\partial\star$ use only the

1–jets of the metric coefficients. Therefore, it suffices to prove the identity for the Euclidean metric on $\mathbb{C}^n$. To this end, let $\omega = \sqrt{-1}\sum_{i=1}^n dz_i \wedge d\bar{z}_i$ denote the standard Euclidean metric on some open subset $\mathcal{U} \subset \mathbb{C}^n$. Let $\alpha \in \Omega_0^p(\mathcal{U})$ be a compactly supported smooth p–form on $\mathcal{U}$. We have

$$\begin{aligned}
[\overline{\partial}^*, L]\alpha &= -\sum_{k=1}^n \partial_{\overline{k}} \lrcorner \partial_k(\omega \wedge \alpha) + \omega \wedge \sum_{k=1}^n \partial_{\overline{k}} \lrcorner \partial_k \alpha \\
&= -\sum_{k=1}^n \partial_{\overline{k}} \lrcorner (\omega \wedge \partial_k \alpha) + \omega \wedge \sum_{k=1}^n \partial_{\overline{k}} \lrcorner \partial_k \alpha \\
&= \sum_{k=1}^n \big(-(\partial_{\overline{k}} \lrcorner \omega) \wedge \partial_k \alpha - \omega \wedge \partial_{\overline{k}} \lrcorner \partial_k \alpha\big) + \omega \wedge \partial_{\overline{k}} \lrcorner \partial_k \alpha \\
&= -\sum_{k=1}^n (\partial_{\overline{k}} \lrcorner \omega) \wedge \partial_k \alpha \;=\; \sum_{k=1}^n \sqrt{-1} dz_k \wedge \partial_k \alpha \;=\; \sqrt{-1}\partial\alpha.
\end{aligned}$$

□

Theorem 1.6.43. Suppose (X, ω) is a Kähler manifold. Then

$$\frac{1}{2}\Delta_d \;=\; \Delta_\partial \;=\; \Delta_{\overline{\partial}}.$$

Proof. Use the Kähler identities $\partial^* = -\sqrt{-1}[\Lambda, \partial]$ and $\partial^* = \sqrt{-1}[\Lambda, \overline{\partial}]$

$$\begin{aligned}
\Delta_d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) \\
&= \partial\partial^* + \partial\overline{\partial}^* + \overline{\partial}\partial^* + \overline{\partial}\overline{\partial}^* \\
&= -\sqrt{-1}\partial[\Lambda, \overline{\partial}] - \sqrt{-1}\partial[\Lambda, \partial] + \sqrt{-1}\overline{\partial}[\Lambda, \overline{\partial}] - \sqrt{-1}\overline{\partial}[\Lambda, \partial] \\
&= -\sqrt{-1}\partial[\Lambda, \overline{\partial}] - \sqrt{-1}\partial\Lambda\partial + \sqrt{-1}\overline{\partial}\Lambda\overline{\partial} - \sqrt{-1}\overline{\partial}[\Lambda, \partial] \\
&= \sqrt{-1}\partial\overline{\partial}\Lambda - \sqrt{-1}\partial\Lambda\overline{\partial} - \sqrt{-1}\partial\Lambda\partial + \sqrt{-1}\overline{\partial}\Lambda\overline{\partial} - \sqrt{-1}\overline{\partial}\Lambda\partial + \sqrt{-1}\overline{\partial}\partial\Lambda \\
&= -\sqrt{-1}\partial\Lambda\overline{\partial} - \sqrt{-1}\partial\Lambda\partial + \sqrt{-1}\overline{\partial}\Lambda\overline{\partial} - \sqrt{-1}\overline{\partial}\Lambda\partial \\
&= \sqrt{-1}(\Lambda\overline{\partial} - \overline{\partial}\Lambda)\partial + \sqrt{-1}\partial(\Lambda\overline{\partial} - \overline{\partial}\Lambda) \\
&= 2\Delta_{\overline{\partial}}.
\end{aligned}$$

In particular, since Δ_d is real, and $\overline{\Delta_{\overline{\partial}}} = \Delta_\partial$, this completes the proof. □

Since Δ_∂ and $\Delta_{\overline{\partial}}$ preserve the type decomposition of a form, the Laplace–Beltrami operator inherits this property:

Corollary 1.6.44. Let (X, ω) be a Kähler manifold. If $\alpha \in \Omega^{p,q}(X)$, then $\Delta_d\alpha \in \Omega^{p,q}(X)$.

Remark 1.6.45. More can be said: Let (X, ω) be a Kähler manifold. Write $\alpha^{p,q}$ for the (p, q)–part of α. The above corollary tells us that α is harmonic if and only if each $\alpha^{p,q}$ is harmonic. Indeed, the corollary implies

$$\Delta_d \alpha \;=\; \sum_{p+q=k} \Delta_d \alpha^{p,q}.$$

The converse if given by

$$\overline{\Delta_\partial \overline{\alpha}} \;=\; \overline{\Delta_\partial} \alpha \;=\; \Delta_{\overline{\partial}} \alpha \;=\; \Delta_\partial \alpha \;=\; 0.$$

In the compact Kähler case, we have now established the following well-known theorem:

Theorem 1.6.46. (Hodge decomposition theorem). Let (X, ω) be a compact Kähler manifold. Then we have the Hodge decomposition

$$H^k(X, \mathbb{C}) \;=\; \bigoplus_{p+q=k} H^{p,q}(M),$$

with the Hodge duality $H^{p,q}(X) = \overline{H^{q,p}(X)}$.

Immediate from the above theorem is the following corollary:

Corollary 1.6.47. The odd Betti numbers of a compact Kähler manifold must be even.

This generalizes the well-known fact that the first Betti number of a compact Riemann surface is always even, equal to twice its (topological) genus.

Example 1.6.48. Since the first Betti number of the Hopf surface $X := \mathbb{S}^3 \times \mathbb{S}^1$ is $b_1(X) = 1$, we see that X is not Kähler. Similarly, the Inoue surfaces (compact complex surfaces with universal cover $\mathbb{C} \times \mathbb{H}$) have $b_1(X) = 1$ and are thus not Kähler. In the next section, we will discuss the compact complex surfaces of class VII. These are defined by $\kappa = -\infty$ and $b_1 = 1$.

Remark 1.6.49. The Hodge decomposition theorem extends to compact complex manifolds in the Fujiki class $\mathcal{C}$ (in particular, for Moishezon manifolds).

Remark 1.6.50. Observe that, a priori, there is only an isomorphism between $H^k(M, \mathbb{C})$ and $\bigoplus_{p+q=k} H^{p,q}(M)$. The presence of a $\partial\overline{\partial}$–lemma on compact Kähler manifolds, however, grants a *canonical* isomorphism.

Remark 1.6.51. It was a long-standing problem whether equality of the Laplacians could characterize Kähler manifolds. An straightforward calculation (going back to Gauduchon [**132**]) shows that

$$(\Delta_{\bar{\partial}} - \Delta_\partial) f \;=\; -(\delta\omega) df,$$

for any smooth function $f \in \mathcal{C}^\infty(X)$. In particular, we see that the equality of the Laplacians is equivalent to the Hermitian metric being balanced. It is a theorem of Ogawa [**224**] that the equality of Laplacians on functions and 1–forms implies Kähler.

Let (X, ω_g) be a compact Kähler manifold. If $\{\cdot, \cdot\}$ denotes the Hermitian form induced by ω_g, and $dV_g = \frac{1}{n!}\omega^n$ is the volume form, we can define the formal adjoint $\partial_g^* : \Omega^{1,0}(M) \to \mathcal{C}^\infty(M)$ of ∂ by

$$\int_M \langle \alpha, \partial f \rangle dV_g \quad = \quad \int_M \langle \partial_g^* \alpha, f \rangle dV_g,$$

where $f \in \mathcal{C}^\infty(M)$ and $\alpha \in \Omega^{1,0}(M)$.

Lemma 1.6.52. Let (X, ω) be a compact Kähler manifold, and let $u : X \to \mathbb{R}$ be a smooth function with vanishing average:

$$\int_M u dV_g = 0.$$

Then there exists a smooth function $f : X \to \mathbb{R}$ such that $\Delta f = u$ on X.

Proof sketch. Let H^1 denote the completion of $\mathcal{C}^\infty(X)$ with respect to the norm

$$\|f\|_{H^1}^2 \;:=\; \int_X (|\nabla f|^2 + |f|^2) dV_g.$$

One approach to prove this result is to minimize the functional

$$\mathcal{E}(f) \;=\; \int_X \left(\frac{1}{2} |\nabla f|^2 + uf \right) dV_g,$$

over all $f \in H^1$ such that $\int_X f dV_g = 0$. The Poincaré inequality gives constants $\varepsilon, C > 0$ such that

$$\mathcal{E}(f) \;\geq\; \varepsilon \|f\|_{H^1} - C$$

for all such f. Hence, a minimizing sequence is bounded H^1, and a subsequence converges weakly in H^1 to some F. The lower semi-continuity of the H^1–norm implies that F is a minimizer of $\mathcal{E}$, and the weak convergence ensures that $\int_X F dV_g = 0$. Since the average of F vanishes, computing the variation of $\mathcal{E}$ at F, we find that F is a weak solution to $\Delta F = u$. Since Δ is elliptic, weak solutions of the Poisson equation are smooth. $\square$

Lemma 1.6.53. ($\partial\overline{\partial}$–lemma). Let (X, ω) be a compact Kähler manifold. Let ω, η be two cohomologous real $(1,1)$–forms on X. Then there is a smooth function $f : X \to \mathbb{R}$ such that

$$\eta = \omega + \sqrt{-1}\partial\overline{\partial} f.$$

Proof. Since $[\omega] = [\eta] \in H^2_{\mathrm{DR}}(X, \mathbb{R})$, there is a real 1–form α such that $\eta = \omega + d\alpha$. Decompose $\alpha = \alpha^{1,0} + \alpha^{0,1}$ into types, noting that $\overline{\alpha^{1,0}} = \alpha^{0,1}$, since α is real. Since ω, η are of $(1,1)$–type, we see that

$$\eta \;=\; \omega + \partial\alpha^{0,1} + \overline{\partial}\alpha^{1,0}, \qquad \overline{\partial}\alpha^{0,1} = 0, \qquad \partial\alpha^{1,0} = 0.$$

The divergence theorem shows that $\partial^*\alpha$ has average zero. Hence, by the existence theorem for Poisson's equation, there is a function f such that

$$\partial_g^*\alpha^{1,0} \;=\; \Delta_g f \;=\; -\partial_g^*\partial f.$$

Therefore, $\partial(\alpha^{1,0}+\partial f)=0$ and $\partial_g^*(\alpha^{1,0}+\partial f)=0$, i.e., $\alpha^{1,0}+\partial f$ is ∂–harmonic. The $\overline{\partial}$–Laplacian coincides with the ∂–Laplacian if the metric g is Kähler. Hence, $\alpha^{1,0}+\partial f$ is also $\overline{\partial}$–harmonic. Since $\partial(\alpha^{1,0}+\partial f)=\partial_g^*(\alpha^{1,0}+\partial f)=0$, it follows that $g^{k\overline{\ell}}\nabla_k\nabla_{\overline{\ell}}(\alpha_i^{1,0}+\partial_i f)=0$. Integrating by parts shows that $\overline{\partial}(\alpha^{1,0}+\partial f)=0$. Hence,

$$\eta \;=\; \omega+\partial\alpha^{0,1}+\overline{\partial}\alpha^{1,0} \;\;=\;\; \omega-\partial\overline{\partial}f-\overline{\partial}\partial f \;=\; \sqrt{-1}\partial\overline{\partial}\mathrm{Im}(f),$$

where $\mathrm{Im}(f)$ denotes the imaginary part of f. □

Definition 1.6.54. Let X be a compact complex manifold. We say that X is a $\partial\bar{\partial}$*–manifold* if for any d–exact, ∂–closed, and $\bar{\partial}$–closed form is $\partial\bar{\partial}$–exact.

Proposition 1.6.55. ([**11, 103**]). Let $f:\widetilde{X}\to X$ be a bimeromorphic modification between compact complex manifolds. If $\widetilde{X}$ is a $\partial\bar{\partial}$–manifold, then X is also a $\partial\bar{\partial}$–manifold. In particular, Moishezon manifolds and manifolds in the Fujiki class $\mathcal{C}$ are $\partial\bar{\partial}$–manifolds.

Remark 1.6.56. Ceballos–Otal–Ugarte–Villacampa [**78**] have produced an example of a compact complex manifold with the symmetry of the Hodge diamond $h^{p,q}=h^{q,p}$, but is not a $\partial\overline{\partial}$–manifold.

1.7. The Enriques–Kodaira Classification of Complex Surfaces

This section describes the classification of compact complex surfaces due to Enriques and Kodaira. By Riemann's uniformization theorem, the geometry of compact Riemann surfaces Σ is determined, to a large extent, by its (topological) genus $g := \frac{1}{2}b_1(\Sigma)$. This leads to the well-known trichotomy:

(i) $g = 0 \iff \Sigma \simeq \mathbb{P}^1$;
(ii) $g = 1 \iff \Sigma \simeq \mathbb{C}/\Lambda$, where $\Lambda \simeq \mathbb{Z} + \tau\mathbb{Z}$ is a lattice of maximal rank, $\mathrm{Im}(\tau) > 0$;
(iii) $g \geq 2 \iff \Sigma \simeq \mathbb{D}/\Gamma$, where Γ is a discrete group of automorphisms acting freely.

This trichotomy incarnates in several ways:

(a) The existence of metrics of constant (Gauss) curvature $K \equiv c$: $g = 0 \iff c > 0$; $g = 1 \iff c = 0$; $g \geq 2 \iff c < 0$.
(b) The fundamental group: $g = 0 \iff \pi_1(\Sigma) = 0$; $g = 1 \iff \pi_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$; $g \geq 2 \iff \pi_1(\Sigma)$ is large, highly non-commutative, and grows with the genus.
(c) The set of rational points: $\mathbb{P}^1$ has a large number of rational points; by Falting's theorem, the space of rational points on $\mathbb{T}^2$ is finitely generated; Falting's theorem states that there are only a finite number of rational points on $\Sigma_{g \geq 2}$.

Remark 1.7.1. If the genus is fixed, there are still many curves within each class: In case (i), there is only one object in the class: The complex structure on $\mathbb{P}^1$ is unique. For case (ii), there is a one-parameter family, parametrized by the j–invariant. For curves in class (iii), there is a (coarse) moduli space $\mathcal{M}_g$ of dimension $3g - 3$.

The remarkable fact that the geometry of compact Riemann surfaces is determined primarily by its genus is not maintained by compact complex manifolds of higher dimensions. Further, the problem of understanding complex manifolds up to biholomorphism is tremendously formidable. In the non-compact case, Poincaré's discovery that the bidisk $\mathbb{D} \times \mathbb{D}$ and unit ball $\mathbb{B}^2 \subset \mathbb{C}^2$ are not biholomorphic already hints at the complexity of the problem. Serre's example (see, e.g., [**152**, p. 440]) further illustrates the dire nature of the quest for biholomorphic classification. In place of biholomorphic classification, one can relax the identification to bimeromorphic classification.

For complex manifolds of dimension ≥ 2, there is no unique smooth representative in a bimeromorphic isomorphism class. This is due to the existence of blow-ups. We need to understand and determine canonical representatives of a bimeromorphic isomorphism class to solve the classification problem. Let X be a compact complex surface, and $p \in X$ is a point. The blow-up $\varphi : \widetilde{X} \longrightarrow X$ of X at $p \in X$ is a bimeromorphic map with an exceptional divisor $\mathcal{E} = \pi^{-1}(p) \subset \widetilde{X}$. The exceptional divisor is a rational curve with self-intersection -1.

Definition 1.7.2. Let X be a compact complex surface. A curve $\mathcal{C}$ in X is said to be a $(-k)$–*curve*, for $k \in \mathbb{N}$, if the intersection number

$$\mathcal{C} \cdot \mathcal{C} \;=\; -k.$$

We saw from the above discussion that the exceptional divisor of the blow-up (at a point) of a complex surface yields a (-1)–curve. The following theorem of Castelnuovo (see, e.g., [**21**]) asserts that this is the only way in which (-1)–curves arise:

Theorem 1.7.3. Let X be a compact complex surface. Let $\mathcal{C} \subset X$ be a (-1)–curve. Then there exists a smooth surface Y and a bimeromorphism $\varphi : X \longrightarrow Y$ such that $X = \mathrm{Bl}_p(Y)$ and $\mathcal{C} = \varphi^{-1}(p)$ is the exceptional divisor of φ.

In particular, (-1)–curves can be *blown down*. Blow-downs are paid for by the second Betti number in the sense that if $\varphi : \mathrm{Bl}_p(X) \longrightarrow X$ is the blow up of X at a point $p \in X$, then (see, e.g., [**21**, p 28])

$$H^2(\mathrm{Bl}_p(X), \mathbb{Z}) \;\simeq\; H^2(X, \mathbb{Z}) \oplus \mathbb{Z}e,$$

where $e := c_1(\mathcal{O}_{\mathrm{Bl}_p(X)}(\mathcal{E}))$ is the first Chern class of the line bundle associated to the exceptional divisor $\mathcal{E}$.

Hence, blowing down (-1)–curves terminates after a finite number of steps. We, therefore, have the following candidate for a canonical representative of a bimeromorphic isomorphism class:

Definition 1.7.4. Let X be a compact complex surface. We say that X is a *minimal model* if X does not contain any (-1)–curves.

We can define minimal models in all dimensions, but we require an alternative description. To this end, let us define:

Definition 1.7.5. Let $\mathcal{L} \to (X, \omega)$ be a holomorphic line bundle over a compact Hermitian manifold (X, ω). We say that $\mathcal{L}$ is *nef* (or *numerically effective*) if for every $\varepsilon > 0$, there is a smooth Hermitian metric h_ε on $\mathcal{L}$ such that $\Theta^{(\mathcal{L}, h_\varepsilon)} \geq -\varepsilon\omega$.

Here, $\Theta^{(\mathcal{L}, h)}$ denotes the curvature form of a Hermitian metric h on $\mathcal{L}$ (see §2.2 for details).

Remark 1.7.6. Let $\mathcal{L} \to X$ be a nef line bundle over a compact complex manifold. In general, it is not possible to extract a smooth limit h_0 such that $\Theta^{(\mathcal{L}, h_0)} \geq 0$. That is, a nef line bundle is not, in general, semi-positive[24] (of course, the converse is certainly true). The first example showing that a nef line bundle was not necessarily semi-positive was constructed

[24]A holomorphic line bundle $\mathcal{L} \to X$ is said to be *semi-positive* if there is a Hermitian metric h such that $\Theta^{(\mathcal{L}, h)} \geq 0$.

by Demailly–Peternell–Schneider [108]: Let $\mathcal{C}$ be an elliptic curve and $\mathcal{E}$ a vector bundle given by a non-split extension

$$0 \to \mathcal{O} \to \mathcal{E} \to \mathcal{O} \to 0.$$

Then $\mathcal{E}$ is nef, but it is easy to show that $\mathcal{E}$ cannot be semi-positive. Otherwise, the curvature of $\mathcal{E}$ would vanish, and $\mathcal{E}$ would be Hermitian-flat. In particular, the exact sequence would split.

Remark 1.7.7. Let $\mathcal{L} \to X$ be a holomorphic line bundle over a projective manifold. Then $\mathcal{L}$ is nef if and only if

$$\mathcal{L} \cdot \mathcal{C} \ := \ \int_{\mathcal{C}} c_1(\mathcal{L}) \ \geq \ 0$$

for every closed curve $\mathcal{C} \subset X$. The only if direction is clear; for the only if, we invite the reader to consult [106, p. 50].

Definition 1.7.8. Let X be a compact complex manifold. We say that X is *minimal* if the canonical bundle K_X is nef.

Example 1.7.9. A complex manifold with K_X semi-positive is certainly minimal. A compact Kähler manifold with K_X holomorphically torsion (i.e., $K_X^{\otimes \ell} \simeq \mathcal{O}_X$ for some $\ell \in \mathbb{N}$) is said to be *Calabi–Yau*. Hence, Calabi–Yau manifolds are certainly minimal. The Kähler assumption is not required here: A compact Hermitian manifold X with K_X holomorphically torsion is said to be *non-Kähler Calabi–Yau*. There are many non-Kähler Calabi–Yau manifolds (hence, many non-Kähler compact minimal complex manifold), see [283].

Definition 1.7.10. Let $\varphi : X \longrightarrow Y$ be a meromorphic map between compact complex manifolds. We say that φ is a *modification* if there is a proper analytic subvariety $S \subset Y$ such that $\varphi : X \backslash f^{-1}(S) \longrightarrow Y \backslash S$ is a biholomorphism.

Example 1.7.11. If $\Sigma \subset Y$ is a complex submanifold of codimension at least 2, then the blow-up $\varphi : X \longrightarrow Y$ of the compact complex manifold Y along Σ yields a modification. Conversely, if $\varphi : X \longrightarrow Y$ is a modification, by applying the embedded resolution of singularities to the graph of φ, we obtain a compact complex manifold Z together with two holomorphic maps $p_1 : Z \to X$ and $p_2 : Z \to Y$ such that p_1^{-1} and p_2^{-1} are the composition of finitely many blow-ups along compact complex submanifolds. For surfaces, this yields the following:

Proposition 1.7.12. Let $\varphi : X \longrightarrow Y$ be a modification between compact complex surfaces. Then φ is the composition of a finite number of blow-ups of points and blow-downs.

Remark 1.7.13. In particular, if we wish to understand the properties of compact complex surfaces that are invariant under modification, it suffices to understand invariants of blow-ups of points and blow-downs. Let us first observe that the (topology) genus $g = \frac{1}{2}b_1$ is one such invariant: Let $\varphi : \widetilde{X} \to X$ be the blow-up of X at a point $p \in X$. Then $\widetilde{X}$ is diffeomorphic to the connected sum $X\sharp\overline{\mathbb{P}^2}$. Hence,

$$g(\widetilde{X}) \;=\; \frac{1}{2}b_1(\widetilde{X}) \;=\; \frac{1}{2}b_1(X) \;=\; g(X).$$

Since a blow-down contracts a (-1)–curve, and this does not contribute to $b_1(\widetilde{X})$, it follows that the genus is invariant under modification.

To build further invariants, we observe that the (topological) genus g can be identified with the dimension $h^0(K_X)$ of the space of sections of the canonical bundle K_X. In light of this, we make the following definition:

Definition 1.7.14. Let X be a compact complex manifold. The *mth plurigenus* $p_m = p_m(X)$ is defined

$$p_m \;:=\; \dim H^0(X, K_X^{\otimes m}).$$

The *Kodaira dimension* κ_X is then defined as a measure of the growth of plurigenera:

$$\kappa_X \quad:=\quad \limsup_{m\to\infty} \frac{\log(p_m)}{\log(m)}.$$

If $p_m = 0$ for all $m \geq 0$, then $\kappa_X := -\infty$, and we say that X has *negative Kodaira dimension*.

Theorem 1.7.15. The plurigenera, and hence, the Kodaira dimension of a compact complex surface, are invariant under modification.

Proof. Let $\varphi : \widetilde{X} \longrightarrow X$ be the blow up of X at a point $p \in X$. We will show that

$$p_m(\widetilde{X}) = H^0(\widetilde{X}, K_{\widetilde{X}}^{\otimes m}) \simeq H^0(X, K_X^{\otimes m}) = p_m(X). \tag{1.7.1}$$

Let $\mathcal{E}$ be the exceptional divisor of φ. Then the canonical bundles are related by

$$K_{\widetilde{X}} = \pi^* K_X + \mathcal{E}.$$

Let $\mathcal{D}$ be an effective divisor in the linear system $|K_{\widetilde{X}}^{\otimes m}|$, we claim that the map

$$|K_{\widetilde{X}}^{\otimes m}| \ni \mathcal{D} \;\mapsto\; \mathcal{D} - m\mathcal{E} \in |mK_{\widetilde{X}} - m\mathcal{E}| = |\pi^* K_X|$$

yields an isomorphism of linear systems (and hence, proves (1.7.1)). It suffices to show that $\mathcal{D} - m\mathcal{E}$ is an effective divisor. To this end, for $\mathcal{D} \in |mK_{\widetilde{X}}|$, write $\mathcal{D} = \mathcal{D}_0 + k\mathcal{E}$, for $k \geq 0$ and $\mathcal{D}_0$ an effective divisor such that $\mathcal{E}$ is not one of its irreducible components. Then

$$\begin{aligned} 0 \;\leq\; \mathcal{D}_0 \cdot \mathcal{E} \;&=\; (\mathcal{D} - k\mathcal{E}) \cdot \mathcal{E} \\ &=\; \mathcal{D} \cdot \mathcal{E} - k\mathcal{E} \cdot \mathcal{E} \;=\; mK_{\widetilde{X}} \cdot \mathcal{E} + k \;=\; -m + k. \end{aligned}$$

Hence, $\mathcal{D} - m\mathcal{E}$ is an effective divisor, and the map $\mathcal{D} \mapsto \mathcal{D} - m\mathcal{E}$ yields the desired isomorphism of linear systems. $\square$

The above result is true in higher dimensions, but we need only consider the case of surfaces.

Example 1.7.16. Let Σ_g be a compact Riemann surface of genus g. Then $\kappa(\Sigma_g) = -\infty \iff g = 0$; $\kappa(\Sigma_g) = 0 \iff g = 1$; $\kappa(\Sigma_g) = 1 \iff g \geq 2$.

Example 1.7.17. Since the canonical bundle of $\mathbb{P}^n$ is $K_{\mathbb{P}^n} \simeq \mathcal{O}_{\mathbb{P}^n}(-n-1)$, we see that $p_m(\mathbb{P}^n) = 0$ for all $m \geq 0$. In particular, the Kodaira dimension of $\mathbb{P}^n$ is negative.

Example 1.7.18. Let $X_d \subset \mathbb{P}^n$ be a smooth hypersurface of degree d. Let $\mathcal{O}_{X_d}(1) := \mathcal{O}_{\mathbb{P}^n}(1)|_{X_d}$ be the restriction of the hyperplane bundle to X_d. The adjunction formula[25] implies that

$$K_{X_d} \simeq \mathcal{O}_{X_d}(d-n-1).$$

Hence,

(i) if $d \leq n$, the plurigenera vanish and $\kappa_{X_d} = -\infty$.
(ii) if $d = n+1$, the canonical bundle is holomorphically trivial $K_{X_d} \simeq \mathcal{O}_{X_d}$. Hence, the plurigenera $p_m(X) = 1$ for all $m \geq 0$, and $\kappa_{X_d} = 0$.
(iii) if $d \geq n+2$, then K_{X_d} is ample, and we will see in a moment that this implies $\kappa_{X_d} = \dim_{\mathbb{C}} X_d$.

Example 1.7.19. It is easy to see that the Kodaira dimension splits additively on the product of complex manifolds, i.e., $\kappa_{X \times Y} = \kappa_X + \kappa_Y$ (see, e.g., [**289**, p. 69]). Let Σ_g be a compact Riemann surface of genus g. Then

(i) $\kappa(\mathbb{P}^1 \times \Sigma_g) = -\infty$.
(ii) $\kappa(\Sigma_1 \times \Sigma_1) = 0$.
(iii) $\kappa(\Sigma_1 \times \Sigma_{g \geq 2}) = 1$.
(iv) $\kappa(\Sigma_{g\geq 2} \times \Sigma_{g \geq 2}) = 2$.

Remark 1.7.20. Suppose $\kappa_X \geq 0$. Let $\sigma_0, ..., \sigma_{N_m}$ be a basis for the vector space $H^0(X, K_X^{\otimes m})$. We define a meromorphic map

$$\Phi_m : X \longrightarrow \mathbb{P}^{N_m}, \qquad \Phi_m(x) := [\sigma_0(x) : \cdots : \sigma_{N_m}(x)] \in \mathbb{P}^{N_m}.$$

We call Φ_m the *mth pluricanonical map*. The Φ_m will not be holomorphic, in general, since the *base locus*

$$\text{Bs}(X) := \bigcap_{k=0}^{N_m} \sigma_k^{-1}(0)$$

[25]Let X be smooth projective manifold. Let $Y \subset X$ be a smooth hypersurface. Then the canonical bundles are related by

$$K_Y \simeq (K_X \otimes \mathcal{O}_X(Y))|_Y.$$

will be non-empty, in general. Further, since a change of basis of $H^0(X, K_X^{\otimes m})$ is specified by a unitary matrix, the pluricanonical maps are well-defined.

Proposition 1.7.21. Let X be a compact complex manifold. The Kodaira dimension κ_X is the maximal rank of the pluricanonical maps

$$\kappa_X = \max_{m \in \mathbb{N}} \operatorname{rank}(\Phi_m).$$

In particular, if $\kappa_X \geq 0$, then κ_X is a non-negative integer $0 \leq \kappa_X \leq \dim_{\mathbb{C}}(X)$.

Remark 1.7.22. The same construction holds for any holomorphic line bundle, not just the canonical bundle. In this greater level of generality, the Kodaira dimension is referred to as the *Iitaka dimension*.

Returning to the classification problem, we note that for complex surfaces of negative Kodaira dimension, minimal models are not necessarily unique: For instance, $\mathbb{P}^2$ is bimeromorphic to $\mathbb{P}^1 \times \mathbb{P}^1$, and both are minimal. For surfaces of non-negative Kodaira dimension, however, we have [**324**, p. 134]:

Theorem 1.7.23. Let X be a compact complex surface with $\kappa_X \geq 0$. Then there exists a unique minimal model bimeromorphic to X.

Definition 1.7.24. Let X be a compact complex manifold. If $\kappa_X = \dim_{\mathbb{C}}(X)$, we say that X is *of general type*.

Example 1.7.25. A compact Riemann surface Σ is of general type if and only if $g \geq 2$. Since the Kodaira dimension splits additively on the product of complex manifolds, i.e., $\kappa_{X \times Y} = \kappa_X + \kappa_Y$ (see, e.g., [**289**, p. 69]), the product of two manifolds of general type will also be of general type. If X is a compact complex manifold with K_X ample (i.e., X is *canonically polarized*), then by definition, the sections of a suitably high power of K_X furnish an embedding $\Phi : X \to \mathbb{P}^N$, for some $N \in \mathbb{N}$. In particular, $\dim_{\mathbb{C}} \Phi(X) = \dim_{\mathbb{C}} X$, and thus, X is of general type.

The Kodaira dimension will provide the first stratification of the landscape of compact complex surfaces. We start with the surfaces of negative Kodaira dimension $\kappa_X = -\infty$, and systematically work through the classification, concluding with surfaces of general type.

Kähler Surfaces of Negative Kodaira Dimension. We saw previously that $\mathbb{P}^n$ has negative Kodaira dimension. From 1.7.15, we see that the following class of complex manifolds have negative Kodaira dimension:

Definition 1.7.26. Let X be a compact complex manifold. We say that X is *rational* if there is a modification $\varphi : X \longrightarrow \mathbb{P}^n$.

Example 1.7.27. Obviously, $\mathbb{P}^2$ is rational, and any surface obtained from blowing up points on $\mathbb{P}^2$ is rational. Observe that since $\mathbb{C} \times \mathbb{C} \subset \mathbb{P}^1 \times \mathbb{P}^1$ is isomorphic to $\mathbb{C}^2 \subset \mathbb{P}^2$, it follows that $\mathbb{P}^1 \times \mathbb{P}^1$ is rational.

To describe the remaining rational surfaces, we recall that if $\mathcal{E} \to \mathbb{P}^1$ is a holomorphic vector bundle of rank k, then

$$\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(n_k).$$

In particular, any rank 2 holomorphic vector bundle over $\mathbb{P}^1$ is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1}(n_2)$ for some pair of integers $n_1, n_2 \in \mathbb{Z}$. The projectivization of this vector bundle is invariant under the twisting by a line bundle $\mathcal{L}$, i.e.,

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1}(n_2) \otimes \mathcal{L}) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1}(n_2)).$$

Hence, by twisting $\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1}(n_2)$ with $\mathcal{O}_{\mathbb{P}^1}(-n_2)$, and letting $n := n_1 - n_2$, we may write

$$\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n_1) \oplus \mathcal{O}_{\mathbb{P}^1}(n_2)) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}).$$

Definition 1.7.28. The *nth Hirzebruch surface* $\mathcal{F}_n$ is the $\mathbb{P}^1$–bundle over $\mathbb{P}^1$ whose total space is isomorphic to $\mathcal{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$.

Example 1.7.29. We observe that $\mathcal{F}_0 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, $\mathcal{F}_1$ is the blow up of $\mathbb{P}^2$ at one point, $\mathcal{F}_2$ is the blow up of the Fermat conic $z_0^2 + z_1^2 + z_2^2 = 0$ in $\mathbb{P}^3$, blown up at the node.

Remark 1.7.30. Let $\mathcal{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}) \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1})$ denote the nth Hirzebruch surface. Let $\mathcal{C}$ be the zero section of the line bundle $\mathcal{O}_{\mathbb{P}^1}(-n)$. The zero section $\mathcal{C}$ is a rational curve with self-intersection $\mathcal{C}^2 = -n$. This is the only curve in $\mathcal{F}_n$ with negative self-intersection. In particular, $\mathcal{F}_1$ is not minimal (which we know, since $\mathcal{F}_1$ is the blow-up of $\mathbb{P}^2$ at one point), but all other $\mathcal{F}_n$ are minimal.

Remark 1.7.31. The diffeomorphism-type of Hirzebruch surfaces is completely understood. If $n \in \mathbb{N}$ is even, then the Hirzebruch surface $\mathcal{F}_n$ is diffeomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. If n is odd, then $\mathcal{F}_n$ is diffeomorphic to $\mathbb{P}^2 \sharp \overline{\mathbb{P}^2}$. From developments in Gauge theory (see, e.g., [**21**, Chapter IX]), if $\mathbb{S}^2 \times \mathbb{S}^2$ or $\mathbb{P}^2 \sharp \overline{\mathbb{P}^2}$ is endowed with a complex structure, then it is biholomorphic to a Hirzebruch surface $\mathcal{F}_n$.

Proposition 1.7.32. The Hirzebruch surfaces $\mathcal{F}_n$ are rational and hence, have negative Kodaira dimension.

An important property of the Hirzebruch surfaces $\mathcal{F}_n$ are that they are rational surfaces which (for $n \geq 2$) do not have positive first Chern class. More precisely, let us first recall the following result which appears implicitly in [**161**]:

Lemma 1.7.33. Let X be a compact complex surface. Let $\varphi : \widetilde{X} \longrightarrow X$ be the blow-up of X at a point $p \in X$ with exceptional divisor $\mathcal{E}$. Then

(i) $c_1(K_{\widetilde{X}}^{-1}) = \varphi^* c_1(K_X^{-1}) - [\mathcal{E}]$.
(ii) If $\mathcal{C}$ is a non-singular curve passing through p, then $\mathcal{C}$ lifts to a non-singular curve $\widetilde{\mathcal{C}}$ with self-intersection
$$\widetilde{\mathcal{C}}^2 \;=\; \mathcal{C}^2 - 1.$$
(iii) If a point $p \in \mathcal{C}$ has multiplicity m, then the cohomology classes of $\widetilde{\mathcal{C}}$ and $\mathcal{C}$ are related by
$$[\widetilde{\mathcal{C}}] \;=\; \varphi^*[\mathcal{C}] - m[\mathcal{E}].$$

The above lemma is used to prove the following (see [**161**]):

Proposition 1.7.34. Let X be a compact complex surface. Then $c_1(K_X^{-1}) > 0$ if and only if $X \simeq \mathbb{P}^1 \times \mathbb{P}^1$ or is obtained from $\mathbb{P}^2$ by blowing up $k \leq 8$ distinct points in general position[26].

This gives the very important corollary:

Corollary 1.7.35. Let $\mathcal{F}_n$ denote the nth Hirzebruch surface. Then $c_1(K_{\mathcal{F}_n}^{-1}) > 0$ if and only if $n = 0$ or $n = 1$.

We have now seen all the (minimal) Kähler surfaces of negative Kodaira dimension:

Theorem 1.7.36. Let X be a minimal compact Kähler surface with $\kappa_X = -\infty$. Then X is bimeromorphic to $\mathbb{P}^2$ or a Hirzebruch surface $\mathcal{F}_n$ for $n \in \mathbb{N}\backslash\{1\}$.

A proof of the above theorem is exhibited in [**21**, p. 250]. A significant consequence of the above theorem is the following rationality criterion due to Castelnuovo:

Theorem 1.7.37. (Castelnuovo's criterion). An algebraic surface X is rational if and only if $p_2 = q = 0$.

Proof. Since q and p_2 are birational invariants, it suffices to assume that X is a smooth minimal surface. Moreover, since $p_2(\mathbb{P}^2) = q(\mathbb{P}^2) = 0$, we need only show that X is rational if $p_2 = q = 0$. To this end, since $p_2 = 0$, we see that $p_g = 0$. Therefore, the holomorphic Euler characteristic[27] $\chi(\mathcal{O}_X) = 1 - q + p_g = 1$. By Riemann–Roch, this implies that
$$h^0(K_X^{-1}) \;\;=\;\; h^0(K_X^{\otimes 2}) + h^0(K_X^{-1}) \;\geq\; K_X \cdot K_X + 1.$$

[26] That is, no three points are collinear, no six lie on a conic, and no eight of them lie on a cubic with one of them a double point.

[27] Let X be a complex manifold of (complex) dimension n. The *holomorphic Euler characteristic* $\chi(\mathcal{O}_X)$ is defined to be the alternating sum
$$\chi(\mathcal{O}_X) \;:=\; \sum_{k=0}^n (-1)^k h^{0,k}(X).$$

If $K_X \cdot K_X \geq 0$, then $h^0(K_X^{-1}) \geq 1$. Since $p_g = 0$, it follows that K_X cannot be trivial, and in particular, K_X cannot be nef. On the other hand, if $K_X \cdot K_X < 0$, then K_X cannot be nef. The previous theorem, therefore, implies that X is biholomorphic to $\mathbb{P}^2$ or one of the Hirzebruch surfaces. As a consequence, X must be rational. □

There are a number of relaxations on the rationality condition. Two important relaxations are the following:

Definition 1.7.38. A complex manifold X is said to be *unirational* if there exists a dominant meromorphic map $\mathbb{P}^N \longrightarrow X$.

Example 1.7.39. It is clear that if X is rational, then X is unirational. For algebraic surfaces, the notions coincide. Indeed, if $\mathbb{P}^N \longrightarrow X$ is a dominant meromorphic map, then we may assume that X is smooth by resolving singularities if necessary. Then $q(X) \leq q(\mathbb{P}^N) = 0$ and $p_2(X) \leq p_2(\mathbb{P}^N) = 0$. Hence, by 1.7.37, X is rational.

Remark 1.7.40. The implication rational $\implies$ unirational is strict for manifolds of dimension greater than two. Artin and Mumford [**14**] constructed examples, making use of the Brauer group. Iskovskikh–Manin [**170**] produced examples of smooth quartic surfaces $X \subset \mathbb{P}^4$ which are unirational but not rational. Their technique exploits the birational automorphism group of a rational variety is large; their examples have a finite birational automorphism group.

The second relaxation of the rationality criterion is the following:

Definition 1.7.41. A complex manifold X is said to be *rationally connected* if any two points of X lie in the image of a rational curve.

Example 1.7.42. By [**187**, Theorem 0.1], every projective Fano manifold (i.e., a projective manifold X with K_X^{-1} ample) is rationally connected. In particular, $\mathbb{P}^n$ is rationally connected, and any smooth hypersurface $X_d \subset \mathbb{P}^n$ of degree $d \leq n$ is rationally connected.

The property of being rationally connected is preserved under modification:

Theorem 1.7.43. Let $\varphi : \widetilde{X} \longrightarrow X$ be a modification of X. If X is rationally connected, then $\widetilde{X}$ is rationally connected.

Example 1.7.44. The above theorem implies that any rational variety is rationally connected. In particular, the Hirzebruch surfaces $\mathcal{F}_n$ are rationally connected. In fact, for compact algebraic surfaces, the properties of rationality, unirationality, and rationally connectedness all coincide. The proof of this fact relies upon the following theorem (see, e.g., [**102**]):

Theorem 1.7.45. Let X be a rationally connected projective manifold. Then

$$H^0(X, \Omega_X^p) \ = \ 0$$

for all $p > 0$. In particular, $H_{\bar{\partial}}^{p,0}(X) = 0$ for all $p > 0$.

An important (but not immediate, see, e.g., [102]) consequence of this is the following corollary:

Corollary 1.7.46. Let X be a rationally connected projective manifold. Then X is simply connected.

Remark 1.7.47. In a similar manner to 1.7.40, the notions of rationality and rationally connectedness diverge in dimensions > 2. Clemens–Griffiths [97] showed, by considering intermediate Jacobians, that smooth cubic hypersurfaces in $\mathbb{P}^4$ are rationally connected but not rational.

Remark 1.7.48. We know that for projective manifolds,

$$\text{rational} \implies \text{unirational} \implies \text{rationally connected.}$$

In dimensions ≤ 2, the reverse implications hold. The examples of Artin–Mumford [14], Iskovskikh–Manin [170] show that the first implication is strict, while the examples of Clemens–Griffiths [97] show that the implication rational $\implies$ rationally connected is strict:

$$\begin{tikzcd} \text{Rational} \arrow[r, Rightarrow] & \text{Unirational} \arrow[l, bend right, dotted, "\substack{\text{Artin–Mumford \&} \\ \text{Iskovskikh–Manin}}"'] \arrow[r, Rightarrow] & \text{Rationally connected} \arrow[ll, bend left, dotted, "\text{Clemens–Griffiths}"] \end{tikzcd}$$

At present, the following question remains open:

Question. Do there exist rationally connected varieties which are not unirational?

Remark 1.7.49. The main stumbling block concerning the above question is unirationality – there are no robust techniques for showing that a variety is unirational. For instance, is a hypersurface of $X_n \subset \mathbb{P}^n$ of degree $n \geq 5$ unirational?

Campana and Kollár–Miyaoka–Mori [187] gave a useful construction, which measures the failure of a variety being rationally connected:

Definition 1.7.50. For a variety X, the *maximal rationally connected fibration* (MRC fibration) associates to X a (birational isomorphism class of a) variety Z and a rational map $\Phi : X \to Z$ with the following properties:

(i) the fibers $\Phi^{-1}(z)$ are rationally connected; and conversely,

(ii) almost all the rational curves in X lie in the fibers of Φ: for a very general point $z \in Z$, any rational curve in X meeting $\Phi^{-1}(z)$ lies in $\Phi^{-1}(z)$.

The variety Z and morphism Φ are unique to birational isomorphism and are called the *MRC quotient* and *MRC fibration* of X, respectively.

Remark 1.7.51. The MRC quotient and MRC fibration measure the failure of X to be rationally connected: If X is rationally connected, then Z is a point; on the other hand, if X is not uniruled, then $Z = X$.

Complex surfaces of vanishing Kodaira dimension. Let X be a compact complex surface with $\kappa_X = 0$. Then the plurigenera $p_1, p_2 \in \{0, 1\}$, with at least one being non-zero. We observe that

$$p_1(X) \; = \; \dim H^0(X, K_X) \; = \; \dim H^0(X, \Omega_X^2) \; = \; \dim H_{\bar{\partial}}^{2,0}(X) \; = \; h^{2,0}(X).$$

Hence, either $h^{2,0}(X) = 0$ or $h^{2,0}(X) = 1$. Moreover, the vanishing of the Kodaira dimension implies that the canonical bundle has vanishing self-intersection, and therefore, $c_1^2 = 0$. Noether's formula now tells us that

$$\begin{aligned} c_1^2 + c_2 \; = \; 12\chi \quad &\Longrightarrow \quad c_2 \; = \; 12\chi \\ &\Longrightarrow \quad b_0 - b_1 + b_2 - b_3 + b_4 \; = \; 12(h^{0,0} - h^{0,1} + h^{0,2}) \\ &\Longrightarrow \quad -2b_1 + b_2 \; = \; 10 + 12(h^{0,2} - h^{0,1}), \end{aligned}$$

where the last line follows from Poincaré duality. Assume X is Kähler, then Hodge theory implies that $b_1 = 2h^{0,1}$. Inserting this into the above formula, we see that

$$8h^{0,1} + b_2 \; = \; 10 + 12h^{0,2}.$$

We now consider the two cases constrained by the vanishing of the Kodaira dimension:

$$h^{2,0} = h^{0,2} = 0 \quad \Longrightarrow \quad 8h^{0,1} + b_2 = 10 \quad \Longrightarrow \quad h^{0,1} = 0 \text{ or } h^{0,1} = 1.$$

If $h^{0,1} = 0$, then $b_2 = 10$, while if $h^{0,1} = 1$, then $b_2 = 2$. Similarly,

$$h^{2,0} = h^{0,2} = 1 \quad \Longrightarrow \quad 20h^{0,1} + b_2 = 22 \quad \Longrightarrow \quad h^{0,1} = 0 \text{ or } h^{0,1} = 1.$$

If $h^{0,1} = 0$, then $b_2 = 22$, while if $h^{0,1} = 1$, then $b_2 = 2$. It follows that the only possibilities for compact Kähler surfaces with $\kappa = 0$ are given by:

Proposition 1.7.52. Let X be a compact Kähler surface. Then X is one of the following four types:

(i) $h^{2,0} = 1$, $h^{1,0} = 0$, and $b_2 = 22$.
(ii) $h^{2,0} = h^{1,0} = 0$ and $b_2 = 10$.
(iii) $h^{2,0} = 0$, $h^{1,0} = 1$, and $b_2 = 2$.
(iv) $h^{2,0} = h^{1,0} = 1$, and $b_2 = 2$.

The class of complex surfaces of type (i) in 1.7.52 were given the name K3 surfaces by André Weil: "*Dans la seconde partie de mon rapport, il s'agit des variétés kählériennes dites K3, ainsi nommées en l'honneur de Kummer, Kähler, Kodaira et de la belle montagne K2 au Cachemire.*"[28] [**299**, p. 546]:

Definition 1.7.53. A compact complex surface is said to be a *K3 surface* if $h^{0,1} = 0$ and the canonical bundle $K_X \simeq \mathcal{O}_X$ is holomorphically trivial.

Since $K_X \simeq \mathcal{O}_X$, it is clear that $h^{2,0} = 1$, hence K3 surfaces provide examples of surfaces of type (i) in 1.7.52.

Example 1.7.54. The simplest example of a K3 surface is a degree 4 hypersurface in $\mathbb{P}^3$. By adjunction, the canonical bundle is holomorphically trivial.

Remark 1.7.55. Siu showed that every K3 surface is Kähler [**257**], which was a significant gap in the general classification theory. Moreover, there is only one diffeomorphism-type for K3 surfaces (i.e., all K3 surfaces are diffeomorphic).

The class of complex surfaces of type (ii) in 1.7.52 were discovered by Enriques:

Definition 1.7.56. A compact complex surface X is called an *Enriques surface* if $h^{0,1} = 0$, the canonical bundle K_X is not holomorphically trivial, but $K_X^{\otimes 2} \simeq \mathcal{O}_X$ is holomorphically trivial.

In other words, Enriques surfaces have a holomorphically torsion canonical bundle. Observe that it is clear from the definition that Enriques surfaces are examples of type (ii) in 1.7.52: if $K_X^{\otimes 2} \simeq \mathcal{O}_X$, then $p_1 = h^{0,2} = 0$.

To describe the surfaces of class (iii) in 1.7.52, we recall the following definition:

Definition 1.7.57. Let $f : X \to Y$ be a surjective holomorphic map between complex manifolds. We say that f is a *fibration* or a *fiber space* if the fibers of f are connected.

Remark 1.7.58. The term fibration in the above sense is standard in both algebraic and complex geometry. However, it is not to be confused with the use of (Serre) fibration in homotopy theory. See the example given by Francesco Polizzi here [**235**].

[28] In the second part of my report, we deal with the Kähler varieties known as K3, names in honor of Kummer, Kähler, Kodaira and of the beautiful mountain K2 in Kashmir.

The following theorem of Fischer–Grauert [**122**] gives an important characterization of fibrations that are holomorphically locally trivial:

Theorem 1.7.59. (Fischer–Grauert). Let $f : X \to Y$ be a holomorphic fibration between complex manifolds. Then f is a fiber bundle if and only if all the fibers of f are biholomorphic.

We now describe the surfaces of class (iii) in 1.7.52:

Definition 1.7.60. A *bielliptic surface* is a compact complex surface X with $b_1 = 2$ which is the total space of a locally trivial holomorphic fibration $f : X \to \mathbb{T}$ over an elliptic curve with fiber an elliptic curve $\mathbb{T}$.

Remark 1.7.61. The assumption that the holomorphic fibration is locally trivial is superfluous in the above definition. Indeed, since elliptic curves are parametrized by the j–invariant, if $f : X \to \mathbb{T}$ is a holomorphic fibration with elliptic curves for fibers, then the j–invariant defines a holomorphic map $j : \mathbb{T} \to \mathbb{C}$. Since $\mathbb{T}$ is compact, j is constant, and all fibers are biholomorphic. By a theorem of Fischer–Grauert [**122**], the fibration must be locally trivial. Note, however, that if the fibration is permitted to have singular fibers, then the fibration is not necessarily locally trivial.

The surfaces of type (iv) in 1.7.52 are simply the complex tori: $\mathbb{C}^2/\Lambda$, where Λ is a lattice in $\mathbb{C}^2$ of maximal rank.

We may now rephrase 1.7.52 more qualitatively:

Theorem 1.7.62. Let X be a compact Kähler surface $\kappa_X = 0$. Then X is either a K3 surface, an Enriques surface, a bielliptic surface, or a torus.

Assume now that X is a compact non-Kähler complex surface with $\kappa_X = 0$. From Noether's formula, we have

$$b_2 - 2b_1 \;=\; 10 + 12(h^{0,2} - h^{0,1}).$$

For non-Kähler complex surfaces, $b_1 = 1 + 2h^{0,1}$, therefore

$$b_2 = 12 + 12h^{0,2} - 8h^{0,1} \quad \Longrightarrow \quad 2h^{0,1} \;\leq\; 3 + 3h^{0,2}.$$

The two cases constrained by the vanishing of the Kodaira dimension imply that

$$h^{0,2} = 0 \quad \Longrightarrow \quad h^{0,1} \leq 1, \qquad \text{or} \qquad h^{0,2} = 1 \quad \Longrightarrow \quad h^{0,1} \;\leq\; 3.$$

Definition 1.7.63. Let X be a compact complex surface. We say that X is a

(i) *primary Kodaira surface* if $b_1(X) = 3$ and X is the total space of an elliptic fibration $f : X \to \mathbb{T}$ over an elliptic curve.

(ii) *secondary Kodaira surface* if X is not a primary Kodaira surface but there is an unramified covering $p : \tilde{X} \to X$ with $\widetilde{X}$ a primary Kodaira surface.

Surfaces of General Type. Recall that the Kodaira dimension is either $-\infty$ or a non-negative integer which is at most the (complex) dimension of the manifold. The complex manifolds of general type are those for which the Kodaira dimension is maximal:

Definition 1.7.64. A compact complex manifold X is said to be of *general type* if $\kappa(X) = \dim(X)$, or equivalently, if K_X is big.

In particular, if X is of general type, the Iitaka map $\Phi : X \longrightarrow \mathbb{P}^N$ has maximal rank with zero-dimensional fibers. That is, if X is of general type, X is bimeromorphic to a projective variety $\Phi(X) \subseteq \mathbb{P}^N$. We, therefore, have the following:

Proposition 1.7.65. Let X be a compact complex manifold of general type. Then X is Moishezon and therefore balanced. If, in addition, X is pluriclosed, then X is projective.

Proof. From the preceding discussion, X is Moishezon. The balanced condition is preserved under bimeromorphism [**5**], so X supports a balanced metric. By [**96**], if a Moishezon manifold admits a pluriclosed metric, it is Kähler. Finally, if X is both Kähler and Moishezon, then X is projective by [**210**, Chapter 1, Theorem 11]. $\square$

Example 1.7.66. Let Σ be a compact Riemann surface. If the genus of Σ is ≥ 2, then $\kappa(\Sigma) = 1$, and Σ is of general type. More generally, if X is a compact complex manifold with K_X ample, then $\kappa(X) = \dim(X)$, thus, of general type.

Remark 1.7.67. Let X be a compact complex manifold. We say that X is *polarized* if there exists a positive holomorphic line bundle $\mathcal{L} \to X$. In this case, we say that $(X, \mathcal{L})$ is a *polarized manifold* or X is *polarized by* $\mathcal{L} \to X$. Note that this already implies the manifold X is Kähler: If $\mathcal{L} \to X$ is positive, then there is a Hermitian metric h with positive[29] curvature form $\Theta^{(\mathcal{L},h)} \in \Omega_X^{1,1}$. Since $\Theta^{(\mathcal{L},h)}$ is locally $\partial\bar{\partial}$–exact, it is certainly closed, and gives a Kähler metric on X. Conversely, given a Kähler metric ω with integral Kähler class $[\omega] \in H^2(X, \mathbb{Z})$, we may find a Hermitian line bundle $(\mathcal{L}, h) \to X$ with such that $\Theta^{(\mathcal{L},h)} = \omega$. Hence, we may define a *polarization* to be a Kähler class[30].

Definition 1.7.68. We say that X is *canonically polarized* if X is polarized by the canonical bundle K_X.

Remark 1.7.69. By the Aubin–Yau [**15, 318**] solution of the Calabi conjecture, if X is compact Kähler with K_X ample, there is a unique Kähler–Einstein metric on X with $\text{Ric}_\omega = -\omega$. Hence, canonically polarized manifolds have a canonical polarization given by the Kähler class of the Kähler–Einstein metric.

[29] Here, positive is understood in the sense of $(1,1)$–forms.

[30] Oftentimes, when a polarization is defined in this manner, the Kähler class is not assumed to be integral.

Remark 1.7.70. 1.7.66 shows that if X is canonically polarized, then X is of general type. The ampleness of the canonical bundle is not equivalent to the manifold being of general type, however. Some examples of big non-ample line bundles are discussed in [**189**, p. 140].

On the other hand, we have the following important theorem of Kodaira [**186**]:

Theorem 1.7.71. (Kodaira). Let X be a minimal surface of general type, then its canonical bundle K_X ample if and only if there are no (-2)–curves.

Let us now discuss an important example of a surface of general type:

Definition 1.7.72. A compact complex surface X is said to be a *Kodaira fibration surface* if X is the total space of a holomorphic fibration $f : X \to \Sigma$ over a compact Riemann surface Σ such that f is a holomorphic submersion and the fibers are not biholomorphic[31].

Class VII Surfaces. A fundamental theorem in complex geometry states that a compact complex surface is Kähler if and only if $b_1(X)$ is even. This follows indirectly from Siu's theorem [**257**] and directly from the theorem of Buchdahl [**66**]. The non-Kähler compact complex surfaces have minimal models X_{min} which belong to one of the following three classes:

(i) Primary and secondary Kodaira surfaces.
(ii) Non-Kähler properly elliptic surfaces.
(iii) Minimal class VII surfaces.

Let us describe these three classes in more detail:

Definition 1.7.73. A *primary Kodaira surface* is a topologically non-trivial, locally trivial principal elliptic fiber bundle over an elliptic base.

Remark 1.7.74. From [**21**, p. 197], the invariants of a primary Kodaira surface are $b_1 = 3$, $b_2 = 4$, $e = 0$, $h^{0,1} = 2$, $h^{0,2} = 1$, and $K_X \simeq \mathcal{O}_X$.

In some cases, a primary Kodaira surface admits a finite group of automorphisms that acts freely. The smooth quotients subsequently obtained are called secondary Kodaira surfaces:

Definition 1.7.75. A *secondary Kodaira surface* is a compact complex surface X which admits an unramified covering $\widehat{X} \to X$ such that $\widehat{X}$ is a primary Kodaira surface.

Definition 1.7.76. A compact complex surface X is said to be of *Class VII* if $\kappa(X) = -\infty$ and $b_1(X) = 1$.

[31] More precisely, the associated Kodaira–Spencer map $\delta_p : T_p\Sigma \to H^1(X_p, T^{1,0}X_p)$ is injective at each point $p \in \Sigma$. The right language for this condition is that the family is *effectively parametrized*. Note that, in general, this condition is stronger than a family having *maximal variation* (in the sense of Viehweg [**293**]).

Remark 1.7.77. Since class VII surfaces have $\kappa = -\infty$, all plurigenera $p_m = 0$. Given that $b_1 = 1$, we know ([21, Theorem 2.7, p. 139]) that $b^+ = 2p_m = 0$. In particular, class VII surfaces are interesting from the point of view of differential topology: they form a class of (real) 4–manifolds with $b_1 = 1$ and negative-definite intersection form.

Definition 1.7.78. A compact complex surface is said to be a *Hopf surface* if its univeral cover is biholomorphic to $\mathbb{C}^2 - \{0\}$.

Remark 1.7.79. The original Hopf surface, defined by Hopf in [163], was the quotient of $\mathbb{C}^2 - \{0\}$ by the infinite cyclic group generated by the homothety $(z_1, z_2) \;\mapsto\; \left(\frac{1}{2}z_1, \frac{1}{2}z_2\right)$. This surface H is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$ and has $b_1(H) = 1$. In particular, H does not support a Kähler metric. On the other hand, the surface H is an elliptic fiber bundle over $\mathbb{P}^1$ and is homogeneous. Hopf's construction immediately generalizes to the case where $\mathbb{C}^2 - \{0\}$ is quotiented by an infinite cyclic group generated by particular automorphisms:

Definition 1.7.80. A *primary Hopf surface* is the quotient of the punctured plane $\mathbb{C}^2 - \{0\}$ by an infinite cyclic group H which acts properly discontinuously by holomorphic transformations:

$$(z_1, z_2) \;\mapsto\; (\alpha_1 z_1, \alpha_2 z_2), \tag{1.7.2}$$

where $0 < |\alpha_1| \le |\alpha_2| < 1$.

Proposition 1.7.81. ([21, p. 226]). Let H_α be the primary Hopf surface given by the quotient of $\mathbb{C}^2 - \{0\}$ by the cyclic group generated by (1.7.2). Then

(i) H_α is diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$.
(ii) $h^{1,0} = h^{2,0} = h^{0,2} = h^{1,1} = 0$ and $h^{0,1} = 1$.
(iii) H_α contains two elliptic curves.

It follows immediately from (i) or (ii) of 1.7.81 that a primary Hopf surface does not support a Kähler metric. It follows immediately from (iii) that a primary Hopf surface is not Kobayashi hyperbolic.

Proposition 1.7.82. A compact complex surface X is a primary Hopf surface if and only if one of the following equivalent conditions hold:

(i) X is homeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$.
(ii) $b_2(X) = 0$ and $\pi_1(X) \simeq \mathbb{Z}$.

Remark 1.7.83. There are many Hopf surfaces which are not of the type H_α or even diffeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$ (see page [21, p. 227]).

Proposition 1.7.84. Let X be a Hopf surface. Then there exists a finite unramified covering $Y \to X$ such that Y is a primary Hopf surface.

Definition 1.7.85. An *Inoue surface* is a class VII surface which is the free quotient of $\mathbb{C} \times \mathbb{H}$ by a properly discontinuous affine action.

By [**36**, **73**, **192**, **273**], we can characterize Inoue surfaces as follows:

Theorem 1.7.86. A class VII surface X is an Inoue surface if and only if $b_2 = 0$ and X has no holomorphic curves.

Proposition 1.7.87. ([**119**, p. 3]). An Inoue surface X admits a non-singular holomorphic foliation $\mathcal{F}$ whose leaves are the images of $\mathbb{C} \times \{z_2\}$, for every $z_2 \in \mathbb{H}$, under the quotient map $\mathbb{C} \times \mathbb{H} \to X$.

Theorem 1.7.88. (Bogomolov's theorem [**35**, **36**, **273**, **192**]). A class VII surface with $b_2 = 0$ is biholomorphic to either a Hopf surface or an Inoue surface.

There are many examples of class VII_0 surfaces with positive second Betti number. By the results of Kato [**174**, **175**, **176**], and Dloussky [**112**], these surfaces contain a global spherical shell:

Definition 1.7.89. Let X be a compact complex surface. A *global spherical shell* is an open submanifold Σ which is biholomorphic to a standard neighborhood $\mathcal{U}$ of $\mathbb{S}^3$ in $\mathbb{C}^2$ which does not separate X in the sense that $X - \mathcal{U}$ is connected.

Definition 1.7.90. A *Kato surface* (or *GSS surface*) is a class VII_0 surface with $b_2 > 0$ containing a global spherical shell.

Kato surfaces are constructed as follows: Let $\mathbb{B}(r)$ denote the open ball of radius $r > 0$ in $\mathbb{C}^2$. We consider a (finite) sequence of blow-ups $\varphi : \widetilde{\mathbb{B}} \to \mathbb{B}$, where $\mathbb{B} = \mathbb{B}(1)$ is the unit ball in $\mathbb{C}^2$. The sequence of blow-ups is subject to the following constraint: The first blow-up is given by blowing up the origin, and the next blow-up is given by blowing up a point in the exceptional divisor. Let $\sigma : \overline{\mathbb{B}} \to \widetilde{\mathbb{B}}$ be a holomorphic (up to the boundary) embedding which maps the origin to a point belonging to the last exceptional divisor created by φ. Set $W := \widetilde{\mathbb{B}} - \sigma(\overline{\mathbb{B}})$. We can glue the two boundary components of $\partial W = \partial\widetilde{\mathbb{B}} \cup \sigma(\partial\mathbb{B})$ using the real analytic CR diffeomorphism $\sigma \circ \varphi : \partial\widetilde{\mathbb{B}} \to \sigma(\partial\mathbb{B})$. The result is a minimal compact complex surface S with $b_1(S) = 1$ and b_2 equal to the number of blow-ups in φ.

Kato surfaces are well understood. For instance, we have the following theorem of Kato:

Theorem 1.7.91. (Kato). Every Kato surface contains b_2 rational curves and is a global deformation (a degeneration) of a 1–parameter family of blown up primary Hopf surfaces.

Corollary 1.7.92. All Kato surfaces with fixed second Betti number $b_2 > 0$ are deformation equivalent. Moreover, they are all diffeomorphic to $(\mathbb{S}^3 \times \mathbb{S}^1)\sharp b_2\overline{\mathbb{P}^2}$.

The biholomorphic classification of global spherical shell surfaces is understood. The following conjecture of Nakamura [217], therefore, would, in principle, solve the classification problem for class VII surfaces:

Conjecture 1.7.93. (Global spherical shell conjecture). Every class VII_0 surface with $b_2 > 0$ has a global spherical shell.

A global spherical shell is difficult to work with, especially given that it is a non-compact object. One can show that a surface X of class VII_0 contains at most $b_2(X)$ rational curves. Further, if X admits a global spherical shell, there are precisely $b_2(X)$ rational curves on X. Kato conjectured that the converse should also be true. The following theorem of Dloussky–Oeljeklaus–Toma [113] reduces the problem to the existence of sufficiently many rational curves:

Theorem 1.7.94. If X is a compact complex surface of class VII_0 with $b_2(X) > 0$ rational curves, then X admits global spherical shells.

Remark 1.7.95. It is customary to divide Kato surfaces into three classes:

(i) Enoki surfaces [117] – Kato surfaces which support a non-trivial divisor $\mathcal{D}$ with $H^0(X, \mathcal{O}_X(\mathcal{D})) \neq 0$ and $\mathcal{D} \cdot \mathcal{D} = 0$.

(ii) Kato surfaces of intermediate type [112, 64, 12] – A Kato surface that contains a cycle of rational curves with branches.

(iii) Inoue–Hirzebruch surfaces [169] – Kato surfaces with no meromorphic functions.

In [64], Brunella refers to the Kato surfaces of class (i) as *parabolic Kato surfaces*, and the Kato surfaces of class (ii) and (iii) as *hyperbolic Kato surfaces*.

Further directions. In light of 1.7.65, to produce examples of complex manifolds which support a big line bundle but no ample line bundles, one can look at Moishezon manifolds that do not support pluriclosed metrics. Further, it would be curious to explore the relationship between 1.7.65 and the Fino–Vezzoni conjecture [121]. Perhaps there is a useful algebraic formulation of the Fino–Vezzoni conjecture.

The role of geometric flows in the classification of compact complex surfaces is explored by Streets and Tian in [263, 264, 265, 262]. The developments in §2.4 and §2.5 of the present manuscript may be of some utility in better understanding the flow and hence, understanding the classification problem.

CHAPTER 2

Curvature

In the previous chapter, we became acquainted with the main objects we will be interested in – complex manifolds. The present chapter begins to get at the heart of the matter. Given the rigid nature of complex manifolds, an effective method for studying these objects has been to attach to them smooth (more flexible) objects that preserve the underlying complex structure in a suitable sense. The most notable of these smooth objects is a Hermitian metric. The curvature is a measure of the convexity of this Hermitian matrix-valued function. The use of curvature in studying complex manifolds is fruitful since its theory is robust and well-developed.

In the present chapter, we remind the reader of the basic theory of connections, first on a Riemannian manifold and then on a Hermitian manifold. In contrast to the Riemannian setting, the connections associated with a Hermitian manifold will generally have torsion; we will discover many distinguished connections, not just Levi-Civita. In the last section of this chapter, we discuss one of the primary research focuses of the author – the *Schwarz lemma.*

2.1. The Curvature of a Riemannian metric

Let $\mathcal{C}$ be a curve in $\mathbb{R}^n$ given by a parametrization $\alpha : (a,b) \to \mathbb{R}^n$, $\alpha(t) = (x^1(t), ..., x^n(t))$. Let $V(t) = V(\alpha(t))$ be a vector field defined along $\mathcal{C}$, i.e., to each $t \in (a,b)$, we have a vector

$$V(t) = \sum_i a^i(t) \left(\frac{\partial}{\partial x^i} \right)_{\alpha(t)} \in T_{\alpha(t)}\mathbb{R}^n.$$

We want to understand how to compute the derivative of V along the curve $\mathcal{C}$. Of course, in general, neither $V(t)$ nor its "derivative" need to be tangent to the curve. Since $\mathcal{C}$ sits inside $\mathbb{R}^n$, however, we can exploit the natural *parallelism* which $\mathbb{R}^n$ possess (i.e., the natural isomorphism between $T_p\mathbb{R}^n$ and $T_q\mathbb{R}^n$, for distinct $p, q \in \mathbb{R}^n$). Making use of this, we can identify $V(t_0 + \Delta t) \in T_{\alpha(t_0+\Delta t)}\mathbb{R}^n$ with a vector in $T_{\alpha(t_0)}\mathbb{R}^n$, using this parallelism. We can then make sense of the difference

$$V(t_0 + \Delta t) - V(t_0)$$

by using the vector space structure on $T_{\alpha(t_0)}\mathbb{R}^n$. Further, we may define the difference quotient

$$\frac{V(t_0+\Delta t)-V(t_0)}{\Delta t} = \sum_i \frac{a^i(t_0+\Delta t)-a^i(t_0)}{\Delta t}\left(\frac{\partial}{\partial x^i}\right)_{\alpha(t_0)}.$$

The equality is due to the fact that if we write vectors in terms of the basis $\partial/\partial x^1, ..., \partial/\partial x^n$, which defines a field of *parallel* frames on $\mathbb{R}^n$, then vectors at distinct points are parallel if and only if they have the same components. Permitting $\Delta t \to 0$ yields the definition of the derivative

$$\frac{dV}{dt} = \lim_{\Delta t\to 0}\frac{V(t_0+\Delta t)-V(t_0)}{\Delta t}.$$

For a general smooth manifold M, however, there are no canonical means of identifying the tangent spaces T_pM and T_qM for distinct points $p, q \in M$. The object which allows us to identify tangent spaces T_pM and T_qM for distinct $p, q \in M$ is given by a *connection*:

Definition 2.1.1. Let $\mathcal{E} \to M$ be a smooth vector bundle over a smooth manifold M. An $\mathbb{R}$–linear first-order differential operator $\nabla : \mathscr{X}(M) \times H^0(\mathcal{E}) \to H^0(\mathcal{E})$ is called a *connection*[1] if, for all $f, g \in \mathcal{C}^\infty(X, \mathbb{R})$, $u, v \in \mathscr{X}(X)$, $\sigma \in H^0(\mathcal{E})$,

(i) it is $\mathcal{C}^\infty$–linear in the first variable:

$$\nabla_{fu+gv}(\sigma) = f\nabla_u\sigma + g\nabla_v\sigma,$$

(ii) it satisfies the Leibniz rule:

$$\nabla_u(f\sigma) = u(f)\sigma + f\nabla_u\sigma.$$

A connection on the tangent bundle TM is called an *affine connection.*

Remark 2.1.2. Property (i) ensures that the value of $\nabla_u\sigma$ at a point $p \in X$ depends only on the value of u at p. Similarly, property (ii) ensures that the value of $\nabla_u\sigma$ at $p \in X$ depends only on the value of σ in a neighborhood of p.

Let $\sigma \in H^0(\mathcal{E})$ be a smooth section of $\mathcal{E}$. With respect to a local frame $\{e_\alpha\}$ of $\mathcal{E}$, we write $\sigma = \sigma^\alpha e_\alpha$. Let $\gamma : [t_0, t_1] \longrightarrow X$ be a smooth curve in X, whose image we denote by $\mathcal{C}$. By setting $\sigma(t) := \gamma^*\sigma$, we define a section *along* $\mathcal{C}$. Let us write $\dot{\gamma}(t) := \frac{d}{dt}\gamma(t)$, which in a local coordinate frame ∂_{x_i}, we may write as $\dot{\gamma}(t) = \dot{\gamma}^i(t)\partial_{x_i}$. Then

$$\begin{aligned}\nabla_{\dot{\gamma}(t)}\sigma(t) &= \nabla_{\dot{\gamma}(t)}\sigma^\alpha(\gamma(t))e_\alpha(\gamma(t)) \\ &= \dot{\sigma}^\alpha(\gamma(t))e_\alpha(\gamma(t)) + \dot{\gamma}^i(t)\sigma^\alpha(t)\Gamma^j_{ik}(\gamma(t))e_j(\gamma(t)).\end{aligned}$$

[1] Or more precisely, the covariant derivative associated to a connection.

Hence, $\nabla_{\dot{\gamma}(t)}\sigma = 0$ represents a linear system of first-order ODEs for the coefficients $\sigma^1(t), ..., \sigma^n(t)$ of $\sigma(t)$. Hence, for any given initial value $\sigma(0) \in \mathcal{E}_{\gamma(t_0)}$, there is a unique of

$$\nabla_{\dot{\gamma}(t)}\sigma(t) \;\;=\;\; 0. \tag{2.1.1}$$

Definition 2.1.3. The solution $\sigma(t)$ of (2.1.1) is said to be the *parallel transport* of $\sigma(0)$ along the curve $\mathcal{C}$.

A choice of connection on $\mathcal{E}$, therefore, provides a means of identifying distinct fibers. In light of this discussion, we make the following definition:

Definition 2.1.4. Let $\mathcal{E} \to M$ be a smooth vector bundle over a smooth manifold M. Let ∇ be a connection on $\mathcal{E}$. We say that a section $\sigma \in H^0(\mathcal{E})$ is *parallel* (with respect to ∇) if

$$\nabla\sigma = 0.$$

A section being parallel with respect to a connection is understood to mean that the connection preserves the section. For instance, we have the following:

Definition 2.1.5. Let $(\mathcal{E}, h) \to M$ be a smooth vector bundle with bundle metric h over a smooth manifold M. We say that a connection ∇ on $\mathcal{E}$ is *compatible with the metric* h (or a *metric connection*) if

$$\nabla h \;\;=\;\; 0.$$

That is, for any vector field $u \in \mathscr{X}(M)$ and sections $\sigma, \tau \in H^0(\mathcal{E})$ we have

$$uh(\sigma, \tau) \;\;=\;\; h(\nabla_u\sigma, \tau) + h(\sigma, \nabla_u\tau).$$

Remark 2.1.6. If ∇ is a metric connection on TM, then the operation of parallel transport acts by isometries of the metric.

Remark 2.1.7. In the above discussion, we obtained a notion of parallel transport from (the covariant derivative associated to) a connection. From a notion of parallel transport (i.e., an identification of the fibers of a vector bundle along curves), one can produce a covariant derivative as follows: Let $v \in T_pX$, and let $\gamma : [t_0, t_1] \longrightarrow X$ be a smooth curve with $\gamma(0) = p$ and $\dot{\gamma}(0) = v$. For $\sigma \in H^0(\mathcal{E})$, we define

$$\nabla_v\sigma \;\;:=\;\; \lim_{t\to 0}\frac{P_{\gamma,t}(\sigma(\gamma(t))) - \sigma(\gamma(0))}{t},$$

where $P_{\gamma,t} : \mathcal{E}_{\gamma(t)} \longrightarrow \mathcal{E}_{\gamma(0)}$ is the identification by parallel transport along γ.
To see that the notions of parallel transport and covariant derivative are equivalent, let $\{e_\alpha(t)\}$ be a frame of parallel sections of $\mathcal{E}$ along γ, i.e.,

$$\nabla_{\dot{\gamma}(t)}e_\alpha(t) \;\;=\;\; 0, \qquad \alpha = 1, ..., n.$$

Any section $\sigma \in H^0(\mathcal{E})$ along γ can be written (locally) as $\sigma(t) = \sigma^\alpha(t) e_\alpha(t)$. Hence, with $v = \dot{\gamma}(0)$, we have

$$\nabla_v \sigma(t) \;=\; \dot{\sigma}^\alpha(t) e_\alpha(t).$$

Therefore,

$$(\nabla_v \sigma)(\gamma(0)) \;=\; \lim_{t \to 0} \frac{\sigma^\alpha(t) - \sigma^\alpha(0)}{t} e_\alpha(0) \;=\; \lim_{t \to 0} \frac{P_{c,t}(\sigma(t)) - \sigma(0)}{t},$$

allowing us to define a connection if we have a parallel transport operator.

Torsion and Curvature. From an affine connection, we can build two invariants: its torsion, and its curvature. The torsion is a measure of the failure of the connection to parallel transport sections without '*slipping*'; this is a measure of the failure of the covariant derivative to commute up to first-order. The curvature is a measure of the failure of the covariant derivative action to commute up to second-order.[2]

Definition 2.1.8. Let ∇ be a connection on the tangent bundle TM of a smooth manifold M. The *torsion* $T = T^\nabla$ of ∇ is the $(2,1)$–tensor defined by

$$T(u,v) \;:= \nabla_u v - \nabla_v u - [u,v].$$

In Riemannian geometry, there is a strong aversion to the torsion of a connection. This is because a Riemannian manifold supports a distinguished metric connection on its tangent bundle whose torsion vanishes:

Theorem 2.1.9. Let (M,g) be a Riemannian manifold. There exists a unique metric connection ∇^{LC} – the *Levi-Civita connection* – whose torsion vanishes.

We invite the reader to consult [**227**, p. 53] for a proof of the above theorem.

From an affine connection ∇, we can build the torsion tensor T^∇. This expression is meaningless on a general smooth vector bundle, but the following $(3,1)$–tensor can be defined on any smooth vector bundle with a connection:

Definition 2.1.10. Let $\mathcal{E} \to M$ be a smooth vector bundle over a smooth manifold. Let ∇ be a connection on $\mathcal{E}$. The *curvature* $\Theta = \Theta^\nabla$ of ∇ is the $\mathrm{End}(\mathcal{E})$–valued 2–form defined by

$$\Theta(\xi,\eta)\sigma \quad := \quad \nabla_\xi \nabla_\eta \sigma - \nabla_\eta \nabla_\xi \sigma - \nabla_{[\xi,\eta]}\sigma, \tag{2.1.2}$$

where $\xi, \eta \in \mathscr{X}(M)$ and $\sigma \in H^0(\mathcal{E})$. If ∇ is the Levi-Civita connection on TM, then the curvature tensor is referred to as the *Riemannian curvature tensor*.

[2]See also the notion of Cartan displacement [**154**].

Notation 2.1.11. The curvature of a connection on an arbitrary vector bundle will typically be denoted by Θ. The curvature of an affine connection will typically be denoted by R.

Let $\mathcal{E} \to M$ be a smooth vector bundle over a smooth manifold M. Endow $\mathcal{E}$ with a bundle metric h. Using the bundle metric, the curvature $\Theta = \Theta^{\nabla}$ of a connection ∇ on $\mathcal{E}$ can be considered scalar-valued by setting

$$\Theta(\xi, \eta, \sigma, \tau) \quad := \quad h(\Theta(\xi, \eta)\sigma, \tau),$$

where $\xi, \eta \in \mathscr{X}(M)$ and $\sigma, \tau \in H^0(\mathcal{E})$.

If we let $\{e_i\}$ be a smooth local frame for $\mathcal{E}$. We write $\Theta_{ijk\ell} \; := \; \Theta(e_i, e_j, e_k, e_\ell)$ for the components of Θ with respect to this frame. The *connection coefficients* Γ_{ij}^k are the functions given by

$$\nabla_{e_i} e_j \;=\; \Gamma_{ij}^k e_k.$$

From (2.1.2), we see that

$$\begin{aligned} R(e_i, e_j)e_k &= \nabla_{e_i}\nabla_{e_j} e_k - \nabla_j \nabla_{e_i} e_k - \nabla_{[e_i, e_j]} e_k \\ &= \nabla_{e_i}\left(\Gamma_{jk}^\ell e_\ell\right) - \nabla_{e_j}\left(\Gamma_{ik}^\ell e_\ell\right) - \Gamma_{[i,j]k}^p e_p \\ &= \left(\partial_i \Gamma_{jk}^\ell\right) e_\ell + \Gamma_{jk}^\ell \Gamma_{i\ell}^p e_p - \left(\partial_j \Gamma_{ik}^\ell\right) e_\ell - \Gamma_{ik}^\ell \Gamma_{j\ell}^p e_p - \Gamma_{[i,j]k}^p e_p. \end{aligned}$$

Let now ∇ denote the Levi-Civita connection on TM, take $(x_1, ..., x_n)$ to be (smooth) local coordinates near a point $p \in M$, then $\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}$ yields a (smooth) local frame for TM near p. Then

$$R_{ijk}{}^{\ell} \;=\; \frac{\partial \Gamma_{jk}^\ell}{\partial x_i} - \frac{\partial \Gamma_{ik}^\ell}{\partial x_j} + \sum_p (\Gamma_{jk}^\ell \Gamma_{i\ell}^p - \Gamma_{ik}^\ell \Gamma_{j\ell}^p).$$

Since

$$\Gamma_{ij}^k \;=\; \frac{1}{2} g^{k\ell} \left(\partial_i g_{j\ell} + \partial_j g_{i\ell} - \partial_\ell g_{ij}\right),$$

we see that in a local coordinate frame, the Riemannian curvature tensor reads:

$$R_{ijk}{}^{\ell} \;=\; \frac{1}{2}\left(\frac{\partial^2 g_{j\ell}}{\partial x_i \partial x_k} + \frac{\partial^2 g_{ik}}{\partial x_j \partial x_\ell} - \frac{\partial^2 g_{i\ell}}{\partial x_j \partial x_k} - \frac{\partial^2 g_{jk}}{\partial x_i \partial x_\ell}\right) + g_{\ell p}(\Gamma_{jk}^m \Gamma_{im}^p - \Gamma_{ik}^m \Gamma_{jm}^p).$$

From (2.1.2), we have the following immediate symmetries:

Proposition 2.1.12. Let (M, g) be a Riemannian manifold. The Riemannian curvature tensor R has the following symmetries:

(i) $R_{ijk\ell} + R_{jik\ell} = 0$.
(ii) $R_{ijk\ell} + R_{jki\ell} + R_{kij\ell} = 0$.

(iii) $R_{ijk\ell} + R_{ij\ell k} = 0$.
(iv) $R_{ijk\ell} = R_{k\ell ij}$.

From the symmetries of the Riemannian curvature tensor, we can define a symmetric bilinear form:

Definition 2.1.13. Let (M, g) be a Riemannian manifold. Let R denote the Riemannian curvature tensor. The *Riemannian curvature operator* is the symmetric bilinear form $\mathfrak{R} : \Lambda^2(M) \longrightarrow \Lambda^2(M)$ defined by

$$\mathfrak{R}(u \wedge v, z \wedge w) \ := \ R(u, v, z, w).$$

Constraints on the curvature operator are very restrictive. For instance, we have:

Example 2.1.14. Böhm–Wilking [38] showed that manifolds with positive curvature operator $\mathfrak{R} > 0$ are diffeomorphic to spherical space forms. A compact simply connected Riemannian manifold with nonnegative curvature operator $\mathfrak{R} \geq 0$ is isometric to a Riemannian product of spheres with metrics of nonnegative curvature operator, $\mathbb{P}^n$ endowed with a Kähler metric with nonnegative curvature operator (when acting on real $(1,1)$–forms), and compact irreducible Riemannian symmetric spaces with their natural metrics of nonnegative curvature operator.

Given the restrictive nature of the curvature operator, we want to build weaker invariants from the curvature.

Remark 2.1.15. An element of $\Lambda^2(T_pM)$ is said to be *decomposable* if it can be written as $u \wedge v$ for some $u, v \in T_pM$. Let $\Sigma \subset \Lambda^2(T_pM)$ denote the set of all decomposable elements. It is easy to show that Σ forms a cone, and its projectivization $\mathbb{P}(\Sigma)$ is exactly the Grassmannian of 2–planes in T_pM. The restriction of the curvature operator $\mathcal{R}$ to the set of decomposable elements defines the following:

Definition 2.1.16. Let (M, g) be a Riemannian manifold. The *sectional curvature* of g is defined

$$\text{Sec}_g(u \wedge v) \ = \ \frac{\mathfrak{R}(u \wedge v, u \wedge v)}{|u \wedge v|^2} \ = \ \frac{R(u, v, v, u)}{|u|_g^2 |v|_g^2 - (u, v)^2}.$$

The sectional curvature measures the extent to which the exponential map distorts distances.

Remark 2.1.17. It is easy to show that $\text{Sec}_g(u, v)$ is independent of the choice of tangent vectors u, v, and depends only on the two-plane $\Pi \subset TM$ spanned by u, v. In particular, the sectional curvature descends to a function on the Grassmannian of two–planes in TM:

$$\text{Sec}_g : \text{Gr}_2(TM) \longrightarrow \mathbb{R}.$$

An important feature of the sectional curvature is that it determines the curvature tensor completely:

Theorem 2.1.18. Let (M, g) be a Riemannian manifold. The sectional curvature completely determines the curvature tensor R. In particular,

$$\begin{aligned} R_{ijk\ell} \;&=\; \frac{1}{3}\mathrm{Sec}_g\left(\frac{(e_i+e_k)\wedge(e_j+e_\ell)}{2}\right) + \frac{1}{3}\mathrm{Sec}_g\left(\frac{(e_i-e_k)\wedge(e_j-e_\ell)}{2}\right) \\ &\quad -\frac{1}{3}\mathrm{Sec}_g\left(\frac{(e_j+e_k)\wedge(e_i+e_\ell)}{2}\right) - \frac{1}{3}\mathrm{Sec}_g\left(\frac{(e_j-e_k)\wedge(e_i-e_\ell)}{2}\right) \\ &\quad -\frac{1}{6}\mathrm{Sec}_g(e_j\wedge e_\ell) - \frac{1}{6}\mathrm{Sec}_g(e_i\wedge e_k) + \frac{1}{6}\mathrm{Sec}_g(e_i\wedge e_\ell) + \frac{1}{6}\mathrm{Sec}_g(e_j\wedge e_k). \end{aligned}$$

Example 2.1.19. The model spaces $\mathbb{R}^n, \mathbb{S}^n$, and $\mathbb{H}^n$ (endowed with their standard metrics) have isometry groups that act transitively on the orthonormal frames and therefore act transitively on the 2–planes in the tangent bundle. It follows that each of these Riemannian manifolds has *constant sectional curvature*, i.e., the sectional curvatures are the same for all planes at all points.

Example 2.1.20. The classical examples of smooth manifolds which support metrics of positive sectional curvature are the spheres $\mathbb{S}^n$, the complex projective space $\mathbb{P}^n$, the quaternionic projective space $\mathbb{HP}^n$, and the Cayley plane $\mathrm{Ca}\mathbb{P}^2$. These are precisely the simply connected rank one symmetric spaces. Other examples of compact smooth manifolds with metrics of positive curvature are known in (real) dimensions 6 (there are two), 7 (there is an infinite number), 12 (there is one), 13 (there is an infinite number), and 24 (there is one).

Remark 2.1.21. All constructions of positively curved manifolds essentially rely on quotient or metric projections. In particular, one needs a good source of non-negatively curved manifolds (e.g., Lie groups) in order to produce manifolds with positive curvature.

Although weaker than constraints on the curvature operator, constraints on the sectional curvature are still very restrictive (albeit, there are not many known obstructions). Let us mention some of the important structural results concerning the sectional curvature [**114**, p. 200]:

Theorem 2.1.22. (Bonnet–Myers). Let (M, g) be a complete n–dimensional Riemannian manifold with $\mathrm{Sec}_g \geq 1$. Then

$$\mathrm{diam}(M, g) \;\leq\; \mathrm{diam}(\mathbb{S}^n, g_{\mathrm{round}}) \;=\; \pi.$$

In particular, the fundamental group $\pi_1(M)$ is finite.

In other words, the Bonnet–Myers theorem asserts that a manifold curved as much as the round (unit) sphere has a diameter bounded by the round (unit) sphere.

Remark 2.1.23. It is worth remarking that, at present, there is no obstruction to any finite group being the fundamental group of a manifold with a (complete) Riemannian metric of positive sectional curvature. Completeness is essential since Gromov's H-principle [**146**] asserts that any open manifold has a (possibly non-complete) metric of positive sectional curvature.

The second main structure theorem for the sectional curvature is due to Synge [**114**, p. 203]:

Theorem 2.1.24. (Synge). Let (M^n, g) be a compact n–dimensional Riemannian manifold with $\mathrm{Sec}_g > 0$.

(i) If n is even, then $\pi_1(M) = 0$ or $\pi_1(M) = \mathbb{Z}_2$.
(ii) If n is odd, then M is orientable.

In particular, $\mathbb{RP}^n \times \mathbb{RP}^n$ does not admit a metric with positive sectional curvature.

Essentially the same argument used to prove Synge's theorem yields the following [**124**]:

Theorem 2.1.25. (Frankel). Let $(M^n, g$ be an n–dimensional Riemannian manifold with $\mathrm{Sec}_g > 0$. Let U and V be two closed totally geodesic submanifolds. If $\dim(U)+\dim(V) \geq n$, then $U \cap V \neq \emptyset$.

Negative (or non-positive) sectional curvature also places severe constraints on the topology of the manifold [**114**, p. 149]:

Theorem 2.1.26. (Cartan–Hadamard). Let (M^n, g) be a complete Riemannian manifold with $\mathrm{Sec}_g \leq 0$. Then the universal cover is diffeomorphic to $\mathbb{R}^n$.

An easy consequence of the Cartan–Hadamard theorem is the following:

Corollary 2.1.27. A complete Riemannian manifold (M, g) with $\mathrm{Sec}_g \leq 0$ is a $K(\Gamma, 1)$–space (i.e., an Eilenberg–Maclane space).

If the sectional curvature is strictly negative, we have [**114**, § 12.3]:

Theorem 2.1.28. (Priessmann). Let (M, g) be a compact Riemannian manifold with $\mathrm{Sec}_g < 0$. Then any non-trivial abelian subgroup of $\pi_1(M)$ is cyclic.

Priessmann's theorem implies, in particular, that M cannot be homeomorphic to a product manifold; otherwise, $\pi_1(M)$ would contain $\mathbb{Z} \oplus \mathbb{Z}$ as a subgroup. From [**227**, Exercise 6.7.20] (see also [**114**, p. 263]), Priessmann's theorem can be generalized to Any solvable subgroup of the fundamental group of a compact negatively curved manifold must be cyclic.

Remark 2.1.29. The compactness assumption is necessary, as shown by the main theorem in [**9**].

Remark 2.1.30. In the late 50s, Chern conjectured that the same phenomenon occurs for compact Riemannian manifolds with positive sectional curvature. At the time, one only knew the compact rank one symmetric spaces. The conjecture was shown to be false, there are examples of Riemannian manifolds (M, g) with $\mathrm{Sec}_g > 0$ having fundamental group $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_3 \oplus \mathbb{Z}_3$.

The Bonnet–Myers theorem asserts that positive sectional curvature bounds the number of generators on the fundamental group. The following result of Gromov achieves this for the Betti numbers (in any coefficient field):

Theorem 2.1.31. (Gromov). Let (M^n, g) be a compact Riemannian manifold with $\mathrm{Sec}_g \geq 0$. Then there is a universal constant $C = C(n) > 0$ such that

$$b_k(M, F) \;\leq\; c(n),$$

for all k and any field of coefficients F. Moreover, the fundamental group has a generating set with at most $c(n)$ elements.

Besides the results stated here for Riemannian manifolds with positive sectional curvature, there are no other known obstructions. There are several important conjectures, most notably the following conjecture of Hopf:

Conjecture 2.1.32. (Hopf). There is no metric of positive sectional curvature on $\mathbb{S}^2 \times \mathbb{S}^2$.

More generally, one can ask whether the product of two simply connected manifolds with positive sectional curvature supports a metric with positive sectional curvature. The example of $\mathbb{RP}^n \times \mathbb{RP}^n$ shows that we cannot entirely drop the simply connectedness assumption.

Theorem 2.1.33. (Soul theorem). Let (M, g) be a complete non-compact Riemannian manifold with $\mathrm{Sec}_g \geq 0$. Then there exists a compact totally convex submanifold $S \subset M$ such that M is diffeomorphic to the (total space of the) normal bundle of S. Moreover, S supports a Riemannian metric with nonnegative sectional curvature, and if $\mathrm{Sec}_g > 0$, then S is a point.

Remark 2.1.34. The Soul theorem [82] gives us a structure (or reduction) theorem for complete noncompact Riemannian manifolds with nonnegative sectional curvature: They are all total spaces of a vector bundle over a compact manifold with nonnegative sectional curvature (up to diffeomorphism). It gives a complete classification of complete noncompact Riemannian manifolds with positive sectional curvature: the manifold is diffeomorphic to $\mathbb{R}^n$.

The converse problem is the following:

Question 2.1.35. Let (M, g) be a compact Riemannian manifold with $\mathrm{Sec}_g \geq 0$. Let $\mathcal{E} \to M$ be a smooth vector bundle over M. Does there exist a Riemannian metric $\hat{g}$ on the total space of $\mathcal{E}$ with $\mathrm{Sec}_{\hat{g}} > 0$?

It is known that the answer to the above problem is negative, in general. All known examples have infinite fundamental group. No counterexample is known for finite (in particular, trivial) fundamental group. The first examples were given by Özaydin–Walschap [**226**].

Remark 2.1.36. There are a number of variants that bridge the gap between the sectional curvature and the curvature operator. One such instance is the notion of *strongly positive curvature*, coined in the paper of Grove–Verdiani–Ziller [**147**], though the concept stems from earlier work of Thorpe [**275, 276**]. Observe that any 4–form $\alpha \in \Omega^4_M$ induces a symmetric operator $\hat{\alpha} : \Omega^2_M \to \Omega^2_M$ by $g(\alpha(\xi), \zeta) = g(\alpha, \xi \wedge \zeta)$. The quadratic form associated to $\hat{\alpha}$ vanishes on $\sigma \in \mathrm{Gr}_2(TM)$. Hence, the sectional curvature can be written as

$$\mathrm{Sec}_g(\sigma) \;=\; g(\mathfrak{R}(\sigma), \sigma) \;=\; g((\mathfrak{R} + \alpha)(\sigma), \sigma).$$

This observation, which is referred to as *Thorpe's trick*, implies that if there exists $\alpha \in \Omega^4_M$ such that the *modified curvature operator* $\mathfrak{R} + \hat{\alpha}$ is positive-definite, then $\mathrm{Sec}_g > 0$. A Riemannian manifold (M, g) is said to have *strongly positive curvature* if there is a 4–form $\alpha \in \Omega^4_M$ such that $\mathfrak{R} + \hat{\alpha}$ is positive-definite at all points of M. For more in this direction, see [**25**].

The Ricci Curvature. From the curvature tensor, we can produce less restrictive curvatures by taking averages:

Definition 2.1.37. Let (M^n, g) be a Riemannian manifold. The (Riemannian) *Ricci curvature* Ric_g is the $(2,0)$–tensor given by the (metric) trace of the Riemannian curvature tensor:

$$\mathrm{Ric}_g(u, v) \;:=\; \sum_{k=1}^{n} R(u, e_k, v, e_k), \qquad u, v \in TM,$$

where $\{e_k\}$ is a local frame for TM.

We write

$$\mathrm{Ric}_g(e_i, e_k) \;=\; \mathrm{Ric}_{ik} = \sum_{j,\ell=1}^{n} g^{j\ell} R_{ijk\ell}$$

for the components of the Ricci curvature with respect to the frame $\{e_k\}$.

Remark 2.1.38. The above definition for the Ricci curvature can apply more generally to the curvature of any affine connection.

The (Riemannian) Ricci curvature measures the extent to which volumes along geodesics are distorted under the exponential map. Indeed, let $(x_1, ..., x_n)$ denote geodesic normal coordinates defined on a neighborhood of a point in M. In these coordinates, the Riemannian volume form is given by

$$dV_g \;=\; \sqrt{\det(g_{ij})}dx_1 \wedge \cdots \wedge dx_n.$$

Computing the Taylor expansion of dV_g, we see that

$$dV_g \;=\; \left[1 - \frac{1}{6}\mathrm{Ric}_{jk}x_j x_k + O(|x|^3)\right] dx_1 \wedge \cdots \wedge dx_n.$$

In particular, if $\mathrm{Ric}_g > 0$, the volume along geodesics decreases, while they it increases if $\mathrm{Ric}_g < 0$.

The Ricci curvature is significantly weaker than the sectional curvature. The most striking example of this is the following well-known theorem of Gao [129], Gao–Yau [130] (in dimension 3), and Lohkamp [202] (in all dimensions):

Theorem 2.1.39. (Gao–Yau, Lohkamp). Let M be a smooth manifold of dimension $n \geq 3$. Then there exists a complete Riemannian metric g on M with

$$-c_0 \;\leq\; \mathrm{Ric}_g \;\leq\; -c_1,$$

for positive constants $c_0, c_1 > 0$.

In particular, there are no obstructions to a Riemannian manifold of dimension ≥ 3 admitting a *complete* Riemannian metric with negative Ricci curvature. Of course, for (compact) surfaces, the Gauss–Bonnet formula restricts the existence of metrics of negative curvature to surfaces of genus $g \geq 2$.

Remark 2.1.40. It should be emphasized that the main content of the above theorem is that the metric with negative Ricci curvature is complete. Indeed, by Gromov's H-principle [146], any smooth open manifold admits a (possibly non-complete) Riemannian metric with sectional curvature pinched between two negative constants.

Remark 2.1.41. Gromoll–Meyer [144] constructed examples of Riemannian manifolds with $\mathrm{Ric} \geq 0$ but do not admit metrics with $\mathrm{Sec} \geq 0$. These examples all have finite homotopy type. Abresch–Gromoll [1] showed that a complete Riemannian manifold with $\mathrm{Ric} > 0$ has finite homotopy type under some growth assumptions on the diameter. These assumptions on the diameter are necessary, as the examples of Sha–Yang [253] illustrate.

The (both compact and non-compact) examples of Sha–Yang [253] of simply connected Riemannian manifolds of dimension ≥ 7 with $\mathrm{Ric} > 0$, which do not support metrics with $\mathrm{Sec} \geq 0$ also show that Gromov's Betti number theorem does not hold with $\mathrm{Sec} > 0$ replaced with Ric > 0.

Let us mention some important structure theorems for the Ricci curvature. By the theorem of Gao and Lohkhamp, there are no structure theorems for Riemannian metrics with negative Ricci curvature. We have the following extension of the Bonnet–Myers' theorem, which was originally stated for the sectional curvature [**114**, p. 200]:

Theorem 2.1.42. (Bonnet–Myers). Let (M^n, g) be a complete Riemannian manifold of dimension n. Suppose $\mathrm{Ric}_g \geq (n-1)k > 0$. Then

$$\mathrm{diam}(M, g) \ \leq \ \frac{\pi}{\sqrt{k}}. \tag{2.1.3}$$

In particular, a compact Riemannian manifold with positive Ricci curvature has finite fundamental group.

Remark 2.1.43. Cheng [**91**] extended the Bonnet–Myers theorem, showing that equality in (2.1.3) is achieved if and only if (M, g) is isometric to the round sphere of constant curvature k.

An important problem within Riemannian geometry is the search of metrics of constant Ricci curvature:

Definition 2.1.44. Let (M, g) be a Riemannian manifold. A Riemannian metric g is said to be *Einstein* with *Einstein constant* λ if

$$\mathrm{Ric}_g = \lambda g.$$

Example 2.1.45. Besides metrics of constant curvature, Einstein metrics have been difficult to find, in general. Jensen [**172**] showed that the sphere $\mathbb{S}^{4k+3}$, $k > 1$, have an $\mathrm{Sp}(k+1)$–homogeneous Einstein metric. Bourguignon–Karchar [**43**] showed that $\mathbb{S}^{15}$ supports a Spin(9)–homogeneous Einstein metric. Ziller [**327**] showed that these were the only homogeneous Einstein metrics on spheres. Böhm [**37**] constructed infinite sequences of non-isometric Einstein metrics of positive scalar curvature on $\mathbb{S}^k$ for $k = 5, 6, 7, 8, 9$. Böhm's metrics are of cohomogeneity one and were the first inhomogeneous and non-classical Einstein metrics to be found on even-dimensional spheres. Boyer–Galicki–Kollár [**45**] showed that on $\mathbb{S}^5$ there are (at least) 68 inequivalent families of (Sasaki–)Einstein metrics; and all 28 oriented diffeomorphism classes on $\mathbb{S}^7$ admit inequivalent families of (Sasaki–)Einstein metrics.

Scalar curvature. Just as we obtained the Ricci curvature tensor from taking a trace of the curvature tensor, the scalar curvature is given by taking a trace of the Ricci curvature:

Definition 2.1.46. Let (M^n, g) be a Riemannian manifold. The (Riemannian) *scalar curvature* is the (metric) trace of the Ricci curvature

$$\mathrm{Scal}_g \;:=\; \mathrm{tr}_g \mathrm{Ric}_g \;=\; \sum_{i,j=1}^{n} R(e_i, e_j, e_j, e_i),$$

where $\{e_i\}$ is a local frame for TM.

The (Riemannian) scalar curvature measures the extent to which volumes of balls are distorted under the exponential map. Indeed, if we look at the distortion of the volume of a small ball $\mathbb{B}^M(\varepsilon) \subset M$ of radius $\varepsilon > 0$, compared with the volume of the corresponding ball $\mathbb{B}^{\mathbb{R}^n}(\varepsilon)$ in $\mathbb{R}^n$, we see that

$$\mathrm{vol}(\mathbb{B}^M(\varepsilon)) \;=\; \left[1 - \frac{\mathrm{Scal}_g}{6(n+2)}\varepsilon^2 + O(\varepsilon^4)\right] \mathrm{vol}(\mathbb{B}^{\mathbb{R}^n}(\varepsilon)).$$

Remark 2.1.47. We observe that an immediate consequence of 2.1.39, every Riemannian manifold admits a complete Riemannian metric with negative scalar curvature.

On the other hand, like all the curvatures we have exhibited thus far, the existence of metrics with positive scalar curvature is obstructed:

Theorem 2.1.48. (Lichnerowicz). Let $\mathcal{D}$ be the (Riemannian) Dirac operator on a spin manifold M. Then

$$\mathcal{D}^2 \;=\; \nabla^*\nabla + \frac{1}{4}\mathrm{Scal}_g,$$

where ∇ is the covariant derivative on the spinor bundle induced by the Levi-Civita connection.

Recall that a Dirac operator is a self-adjoint elliptic first-order differential operator which acts on sections of the spinor bundle. If M is a spin manifold of dimension n, there is a version of the Dirac operator which commutes with the action of the Clifford algebra $\mathcal{C}_n$. In particular, its kernel is a (graded) $\mathcal{C}_n$–module which represents an element $\alpha(M)$ in the real K–theory group $\mathrm{KO}_n = \mathrm{KO}^{-n}(\mathrm{pt})$ (see [**160**, **197**, **240**] for more details).

Theorem 2.1.49. (Lichnerowicz, Hitchin). If M^n is a compact spin manifold for which $\alpha(M) \neq 0$ in KO_n, then M does not admit a metric of positive scalar curvature.

Example 2.1.50. Let

$$X \;:=\; \{z_0^4 + z_1^4 + z_2^4 + z_3^4 = 0\} \subseteq \mathbb{P}^3$$

denote the Fermat hypersurface of degree 4 in $\mathbb{P}^3$. Then, from [**160**], we know that X is spin and has $\widehat{A}(X) = 2$. Therefore, by 2.1.49, X does not support a metric of positive scalar curvature.

Theorem 2.1.51. (Gromov–Lawson [145]). Let (M, g) be a compact Riemannian manifold with $\text{Sec}_g \leq 0$. Then there is no metric on M with positive scalar curvature.

Although we saw that the pinching of the sectional curvature and the Einstein condition was very restrictive, the corresponding conditions are much less restrictive for the scalar curvature. Indeed, the so-called Yamabe problem asks whether a metric of constant scalar curvature exists in each conformal class. More precisely, we have:

The Yamabe problem. Let M be a smooth manifold. For any Riemannian metric g on M does there exist a conformally related metric $g_u := e^{2u}g$ such that $\text{Scal}_{g_u} \equiv c$ for some constant $c \in \mathbb{R}$?

The Yamabe problem was solved positively by Yamabe [308], Trudinger [287] on compact Riemannian manifolds:

Theorem 2.1.52. Let M be a compact smooth manifold. For any Riemannian metric g on M, there exists a conformally related $g_u := e^{2u}g$ such that $\text{Scal}_{g_u} \equiv c$ for some constant $c \in \mathbb{R}$. Moreover, the constant scalar curvature metric is unique in the conformal class.

Remark 2.1.53. The Yamabe problem for complete noncompact Riemannian manifolds was posed by Kazdan [178]. A counterexample in the noncompact case was given by Zhiren [325]. For more discussion on this, see [16].

There is a result of Kazdan–Warner [179] which classifies the functions which can be realized as the scalar curvature of a Riemannian metric on compact Riemannian manifolds of dimension $n \geq 3$.

Theorem 2.1.54. (Kazdan–Warner). Let (M^n, g) be a compact Riemannian manifold of (real) dimension $n \geq 3$. The scalar curvature Scal_g is a function of three possible types:

(i) There is no restriction on the possible scalar curvatures – every function can be realized as the scalar curvature of some metric.
(ii) A function is the scalar curvature of a metric if and only if it is negative somewhere.
(iii) A function is the scalar curvature of a metric if and only if it is identically zero or negative somewhere.

From this, on a compact manifold, any negative function is the scalar curvature of some metric – In particular, spheres $\mathbb{S}^n$ support metrics of negative scalar curvature. Some manifolds, however, do not admit metrics of strictly positive scalar curvature – like the torus $\mathbb{T}^n$.

Remark 2.1.55. There is a similar story for the Ricci curvature of a Riemannian manifold, which remains a very active area of research – the so-called *prescribed Ricci curvature problem.*

2.2. Generalities on Hermitian Connections

Let (M^{2n}, g) be a smooth Riemannian manifold of (real) dimension $2n$. Let $T^{\mathbb{R}}M$ denote the tangent bundle of M. Suppose M supports an almost complex structure $J : T^{\mathbb{R}}M \longrightarrow T^{\mathbb{R}}M$ compatible with g in the sense that

$$g(Ju, Jv) \;\; = \;\; g(u, v) \tag{2.2.1}$$

for all tangent vectors $u, v \in T^{\mathbb{R}}M$. We can extend g and J complex-linearly to the complexified tangent bundle $T^{\mathbb{C}}M$, abusively denoting the $\mathbb{C}$–linear extensions by the same symbols, such that (2.2.1) holds for all $u, v \in T^{\mathbb{C}}M$. As we did in §1.5, we write $h : T^{\mathbb{C}}M \times T^{\mathbb{C}}M \to \mathbb{C}$ for the Hermitian form given by the (complexification of) g, i.e.,

$$h(u, v) \;:=\; g(u, \overline{v}), \qquad \forall u, v \in T^{\mathbb{C}}M.$$

If R denotes the Riemannian curvature tensor of the Riemannian metric g (defined on $T^{\mathbb{R}}M$), then we can similarly extend R complex-linearly to a complex-linear quadrilinear form

$$R : T^{\mathbb{C}}M \times T^{\mathbb{C}}M \times T^{\mathbb{C}}M \times T^{\mathbb{C}}M \longrightarrow \mathbb{C}$$

retaining all the symmetries of the usual Riemannian curvature tensor. If $\{e_i\}$ defines a local frame for $T^{1,0}M$, and $\{\overline{e}_i\}$ defines a local frame for $T^{0,1}M$, then we write

$$R_{i\overline{j}k\overline{\ell}} \;:=\; R(e_i, \overline{e}_j, e_k, \overline{e}_\ell).$$

The Levi-Civita connection will not be the most appropriate for a general Hermitian manifold. To make this transparent, let us make the following definition:

Definition 2.2.1. Let (M, g, J) be an almost complex manifold. An affine connection ∇ on $T^{\mathbb{R}}M$ is said to be *almost Hermitian* if

$$\nabla g \;=\; \nabla J \;=\; 0.$$

If the almost complex structure J is integrable, we say that ∇ is a *Hermitian connection*.

The following shows how restrictive the (almost) Hermitian requirement on ∇^{LC} is:

Theorem 2.2.2. Let (M, g, J) be an almost complex manifold. Then the Levi-Civita connection ∇ is almost Hermitian if and only if (g, J) is Kähler.

Proof. Fix a point $p \in M$ and extend $u, v \in T_p^{1,0}M$ to vector fields which are ∇–parallel at p. The Nijenhuis tensor is then

$$\begin{aligned} N^J(u, v) &= [u, v] + J[Ju, v] + J[u, Jv] - [Ju, Jv] \\ &= J(\nabla_u J)v - J(\nabla_v J)u - (\nabla_{Ju} J)v + (\nabla_{Jv} J)v. \end{aligned}$$

Hence, if $\nabla J = 0$, the Nijenhuis tensor vanishes $N^J = 0$. By 1.2.66, the almost complex structure J is integrable. Let $\omega_g(\cdot,\cdot) := g(J\cdot,\cdot)$. Then for vector fields u, v, w, we have

$$\begin{aligned} d\omega_g(u,v,w) \;=\; \nabla_w g(Ju,v) &= g(\nabla_w(Ju),v) + g(Ju,\nabla_w v) \\ &= g((\nabla_w J)u,v) + g(J(\nabla_w u),v) + g(Ju,\nabla_w v) \\ &= -g(Jv,\nabla_w u) + g(Ju,\nabla_w v) \\ &= -\omega_g(v,\nabla_w u) + \omega_g(u,\nabla_w v) \;=\; 0. \end{aligned}$$

The converse is straightforward (see, e.g., [**214**]). ☐

Remark 2.2.3. In particular, if the Levi-Civita connection is compatible with an almost Hermitian structure, then the almost Hermitian structure must have an integrable complex structure and a compatible symplectic structure. Given the restrictive nature of the compatibility of the Levi-Civita connection with the Hermitian structure, the Levi-Civita connection is not the most natural connection on the tangent bundle of a complex manifold. To describe the replacement for the Levi-Civita connection, we will need to further explore the discrepancy between complex vector bundles and holomorphic vector bundles.

We recall that in 1.2.38, we saw that a holomorphic vector bundle is very far from a complex vector bundle. The obstruction to a complex vector bundle having fibers that vary holomorphically can be encoded in a first-order differential operator, which satisfies an integrability condition[3].

Definition 2.2.4. Let $\mathcal{E} \to X$ be a complex vector bundle over an almost complex manifold X. A first-order $\mathbb{C}$–linear differential operator

$$\bar\partial^{\mathcal{E}} : H^0(\mathcal{E}) \longrightarrow \Omega_X^{0,1} \otimes H^0(\mathcal{E})$$

is said to be a *CR–operator* (i.e., a *Cauchy–Riemann operator* or *Dolbeault operator*) if it satisfies the following variant of the Leibniz rule:

$$\bar\partial^{\mathcal{E}}(f\sigma) \;=\; \bar\partial f \otimes \sigma + f\bar\partial^{\mathcal{E}}\sigma,$$

where $f \in \mathcal{C}^\infty(X,\mathbb{C})$, $\sigma \in H^0(\mathcal{E})$ is a smooth section, and $\bar\partial : \mathcal{C}^\infty(X,\mathbb{C}) \longrightarrow \Omega_X^{0,1}$ is the standard Dolbeault operator acting on functions (given in 1.2.75).

Notation 2.2.5. We will denote by $\mathrm{Diff}_{\mathrm{CR}}(\mathcal{E})$ the space of CR–operators on the complex vector bundle $\mathcal{E}$.

Definition 2.2.6. Let $\mathcal{E} \to X$ be a complex vector bundle endowed with a CR–operator $\bar\partial^{\mathcal{E}}$ over an almost complex manifold X. A smooth section $\sigma \in H^0(\mathcal{E})$ is said to be *quasi-holomorphic* (relative to $\bar\partial^{\mathcal{E}}$) if $\bar\partial^{\mathcal{E}}\sigma = 0$.

[3] This is, in effect, a linear version of the Newlander–Nirenberg theorem (1.2.65).

Example 2.2.7. The Dolbeault operator $\bar{\partial} : \mathcal{C}^\infty(X, \mathbb{C}) \to \Omega^{0,1}_X$ acting on (complex-valued) functions on an almost complex manifold X defines a CR–operator on the trivial bundle $X \times \mathbb{C} \to X$.

The exterior derivative defines a flat connection on the trivial bundle, and the Dolbeault operator $\bar{\partial}$ is the $(0,1)$–part. In particular, we construct a CR–operator from the $(0,1)$–part of a connection. This special case turns out to illustrate most of the story for CR–operators:

Proposition 2.2.8. Let $\mathcal{E} \to X$ be a complex vector bundle over an almost complex manifold (X, J). For any complex-linear connection ∇ on $\mathcal{E}$, we obtain a CR–operator $\bar{\partial}^{\mathcal{E}}$ via the prescription:

$$\bar{\partial}^{\mathcal{E}} \; := \; \nabla^{0,1}.$$

Proof. Let ∇ be a $\mathbb{C}$–linear connection on $\mathcal{E}$. For any vector field v on X and smooth section $\sigma \in H^0(\mathcal{E})$, we define

$$\nabla^{1,0}_v \sigma \; := \; \frac{1}{2}(\nabla_v \sigma - \sqrt{-1}\nabla_{Jv}\sigma), \qquad \nabla^{0,1}_v \sigma \; := \; \frac{1}{2}(\nabla_v \sigma + \sqrt{-1}\nabla_{Jv}\sigma).$$

Then $\nabla = \nabla^{1,0} + \nabla^{0,1}$ and $\nabla^{0,1}$ is a first-order $\mathbb{C}$–linear differential operator satisfying

$$\nabla^{0,1}_v(f\sigma) \;\; = \;\; (\bar{\partial} f)(v) \otimes \sigma + f\nabla^{0,1}_v \sigma.$$

□

From 2.2.8 we obtain a map from the space of $\mathbb{C}$–linear connections on $\mathcal{E}$ to the space of CR–operators $\mathrm{Diff}_{\mathrm{CR}}(\mathcal{E})$. In general, this map will be far from injective. However, we may recover injectivity of this map by placing an additional compatibility requirement on the connection:

Proposition 2.2.9. Let $(\mathcal{E}, h) \to (X, J)$ be an almost Hermitian vector bundle over an almost complex manifold. Let ∇ be a complex-linear connection compatible with h in the sense that $\nabla h = 0$. Then there is a unique CR–operator $\bar{\partial}^{\mathcal{E}}$ such that

$$\nabla^{0,1} \; = \; \bar{\partial}^{\mathcal{E}}.$$

Proof. To construct ∇, let $\bar{\partial}^{\mathcal{E}^*}$ denote the CR–operator induced on the (complex) dual bundle $\mathcal{E}^*$, specified by the formula

$$\{\bar{\partial}^{\mathcal{E}^*}\alpha, \sigma\} \;\; = \;\; \bar{\partial}\{\alpha, \sigma\} - \{\sigma, \bar{\partial}^{\mathcal{E}}\sigma\},$$

where $\{\cdot, \cdot\}$ denotes the dual pairing between $\mathcal{E}$ and $\mathcal{E}^*$, and $\alpha \in H^0(\mathcal{E}^*)$, $\sigma \in H^0(\mathcal{E})$, are smooth sections. Let $\mathcal{H}$ denote the Hermitian duality (i.e., the musical isomorphism coming from the Hermitian metric) from $\mathcal{E}$ to $\mathcal{E}^*$ defined by

$$\{\xi_1, \mathcal{H}(\xi_2)\} \; = \; h(\xi_1, \xi_2).$$

Define

$$\nabla\sigma \;:=\; \mathcal{H}^{-1}(\bar{\partial}^{\mathcal{E}*}\mathcal{H}(\sigma)) + \bar{\partial}^{\mathcal{E}}\sigma.$$

This connection is uniquely determined by the Hermitian metric and $\bar{\partial}^{\mathcal{E}}$. Indeed, if two $\mathbb{C}$–linear connections have the same $(0,1)$–part, their difference A is an $\mathrm{End}(\mathcal{E})$–valued $(1,0)$–form. In particular,

$$A(Jv) = \sqrt{-1}A(v) \tag{2.2.2}$$

for all v. On the other hand, if both connections are compatible with the metric, then $A(v)$ is skew-Hermitian with respect to this Hermitian metric. This holds in conjunction with (2.2.2) if and only if $A = 0$. □

The CR–operator measures the failure of a complex vector bundle to be holomorphic. To state this precisely, define:

Definition 2.2.10. Let $\mathcal{E} \to (X, J)$ be a complex vector bundle over an almost complex manifold X. We say that a CR–operator $\bar{\partial}^{\mathcal{E}}$ is *integrable* if

$$\bar{\partial}^{\mathcal{E}} \circ \bar{\partial}^{\mathcal{E}} \;=\; 0.$$

Theorem 2.2.11. (Koszul–Malgrange). Let $\mathcal{E} \to X$ be a complex vector bundle over a complex manifold. Then $\mathcal{E}$ is a holomorphic vector bundle if and only if $\mathcal{E}$ supports an integrable CR–operator $\bar{\partial}^{\mathcal{E}}$. Moreover, the holomorphic vector bundle structure on $\mathcal{E}$ is uniquely determined by $\bar{\partial}^{\mathcal{E}}$.

The proof of the Koszul–Malgrange theorem is a linear version of the argument given to prove the Newlander–Nirenberg theorem (1.2.65). The crux of the Newlander–Nirenberg theorem is that if an almost complex structure J satisfies an integrability condition (i.e., $N^J \equiv 0$), then there is a coordinate system in which J is locally constant, equal to the complex structure on $\mathbb{C}^n$. The Koszul–Malgrange theorem is proven by first completing $\bar{\partial}^{\mathcal{E}}$ to a Hermitian connection ∇ (relative to some auxiliary Hermitian metric) such that $\nabla^{0,1} = \bar{\partial}^{\mathcal{E}}$. The condition that $\bar{\partial}^{\mathcal{E}} \circ \bar{\partial}^{\mathcal{E}} = 0$ translates to an integrability criterion on $\nabla^{0,1}$, specified by the Cartan structure equations. The crux of the proof is then in showing that one can find a suitable gauge transformation such that for any point $x \in X$, there is a local frame for $\mathcal{E}$, given by quasi-holomorphic sections.

Lemma 2.2.12. Let $(\mathcal{E}, \bar{\partial}^{\mathcal{E}}) \longrightarrow X$ be a complex vector bundle of rank r. Then $\mathcal{E}$ is a holomorphic vector bundle if and only if, for each $x \in X$, there is an open neighborhood $\mathcal{U}$ containing x and r quasi-holomorphic sections which yield a frame for $\mathcal{E}$ over $\mathcal{U}$.

Proof. The only if direction is clear: If $\varphi : \pi^{-1}(\mathcal{U}) \longrightarrow \mathcal{U} \times \mathbb{C}^r$ is a holomorphic trivialization of $\mathcal{E}$, then the pseudo-holomorphic frame, in the neighborhood $\mathcal{U}$ of x, is given by $\sigma_k(x) := \varphi^{-1}(x, e_k)$, where e_k is the kth standard basis vector of $\mathbb{C}^r$. Conversely, let $\{\sigma_k\}$ and $\{\tau_k\}$ be two pseudo-holomorphic frames defined on open neighborhoods $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ of two points x and $\tilde{x}$, respectively. If $\mathcal{U}$ and $\widetilde{\mathcal{U}}$ intersect non-trivially, then there are bundle transition maps (A_i^j) such that $\sigma_i = A_i^j \tau_j$. From

$$0 \;=\; \bar{\partial}^{\mathcal{E}} \sigma_i \;=\; (\bar{\partial} A_i^j)\tau_j + A_i^j \bar{\partial}^{\mathcal{E}} \tau_j \;=\; (\bar{\partial} A_i^j)\tau_j,$$

we see that the bundle transition maps are holomorphic. □

Proof of Koszul–Malgrange theorem. Let $\bar{\partial}^{\mathcal{E}}$ be an integrable CR–structure on the complex vector bundle $\mathcal{E}$. Let ∇ be the Hermitian connection given by completing $\bar{\partial}^{\mathcal{E}}$ according to 2.2.9. Fix a local frame for $\mathcal{E}$, with respect to which we write ϑ for the connection matrix. Write $\lambda := \vartheta^{0,1}$ for the $(0,1)$–part of ϑ. The integrability of $\bar{\partial}^{\mathcal{E}} = \nabla^{0,1}$ then translates to

$$0 \;=\; \nabla^{0,1}(\nabla^{0,1} e_i) \;=\; \nabla^{0,1}(\lambda_i^j e_j) \;=\; (\bar{\partial}\lambda_i^j) e_j + \lambda_i^j \wedge \lambda_j^k e_k \;=\; (\bar{\partial}\lambda_i^k + \lambda_i^j \wedge \lambda_j^k) e_k.$$

The proof is complete if we can find a gauge transformation $A = (A_i^j)$, defined on a possibly smaller neighborhood $\widetilde{\mathcal{U}} \subseteq \mathcal{U}$ of $x \in X$ such that

$$\bar{\partial} A_i^j + A_i^k \lambda_k^j \;=\; 0. \tag{2.2.3}$$

Indeed, suppose such a gauge transformation exists. Define sections over $\widetilde{\mathcal{U}}$ by the formula $\sigma_i := A_i^j e_j$. Then

$$\bar{\partial}^{\mathcal{E}} \sigma_i \;=\; (\bar{\partial} A_i^j) e_j + A_i^j \bar{\partial}^{\mathcal{E}} e_j \;=\; (\bar{\partial} A_i^j) e_j + A_i^j (\lambda_j^k e_k) \;=\; (\bar{\partial} A_i^j + A_i^k \lambda_k^j) e_k.$$

Then (2.2.3) implies that the σ_i are quasi-holomorphic, and 2.2.12 completes the proof. For the existence of the gauge transformation, we invite the reader to consult [**214**, Lemma 3.3]. □

Remark 2.2.13. In light of 2.2.11, it is natural to refer to an integrable Cauchy–Riemann operator as a *holomorphic structure*.

From 2.2.11, a holomorphic vector bundle supports a canonical first-order differential operator. We may therefore encode the compatibility of a connection ∇ on a holomorphic vector bundle $\mathcal{E} \to X$ by demanding that the $(0,1)$–part of the connection $\nabla^{0,1}$ coincides with the CR–operator $\bar{\partial}^{\mathcal{E}}$. We therefore make the following (non-standard) definition:

Definition 2.2.14. Let $\mathcal{E} \to X$ be a holomorphic vector bundle over a complex manifold X. Denote by $\bar{\partial}^{\mathcal{E}}$ the integrable CR–operator on $\mathcal{E}$. A complex-linear connection ∇ on $\mathcal{E}$ is said to be *complex-analytic* (or *compatible with the holomorphic structure*) if

$$\nabla^{0,1} \;=\; \bar{\partial}^{\mathcal{E}}.$$

Question 2.2.15. Is the Einstein–Weyl connection complex-analytic?

Remark 2.2.16. 2.2.14 is non-standard in that we do not require the connection to be compatible with a Hermitian structure. Complex-analytic connections (as defined in 2.2.14) are independent of any metric structure. The complex-analyticity of a connection on a Hermitian vector bundle $(\mathcal{E}, h) \to X$ is not to be confused with a connection being *Hermitian* in the sense that $\nabla h = 0$. If $\mathcal{E} = T^{1,0}X$, this coincides with 2.2.1.

If one requires a complex-linear connection on a Hermitian vector bundle $(\mathcal{E}, h) \to X$ to be both complex-analytic and Hermitian, then we discover the following uniqueness theorem:

Theorem 2.2.17. Let $(\mathcal{E}, h) \to X$ be a Hermitian vector bundle over a complex manifold X. There is a unique complex-analytic Hermitian connection ∇ on $\mathcal{E}$.

Definition 2.2.18. Let (X, ω_g) be a Hermitian manifold. The unique complex-analytic Hermitian connection on $T^{1,0}X$ given by 2.2.17 is called the *Chern connection.*

Remark 2.2.19. The term *Chern connection* is sometimes extended to refer to the unique complex-analytic Hermitian connection on an arbitrary holomorphic vector bundle.

To prove 2.2.17, we will make use of the Cartan theory of connections, which grew to prominence because of the work of Chern.

Cartan–Chern theory of connections. Let $\mathcal{E} \to X$ be a holomorphic vector bundle, endowed with a complex-linear connection ∇, over a complex manifold. By linearity, to understand how ∇ acts on an arbitrary (locally-defined) smooth section of $\mathcal{E}$, it suffices to understand the action of ∇ on a local frame $\{e_\alpha\}$ for $\mathcal{E}$.

Definition 2.2.20. Let ∇ be a complex-linear connection on a holomorphic vector bundle $\mathcal{E} \to X$. The *connection matrix* for ∇ (relative to a local frame $\{e_\alpha\}$ for $\mathcal{E}$) is the matrix $\vartheta = (\vartheta_i^j)$ of 1–forms given by

$$\nabla e_i \ = \ \sum_j \vartheta_i^j e_j.$$

If we identify $e = (e_1, ..., e_r)^t$ with a column vector (where r denotes the rank of $\mathcal{E}$), then in matrix notation we have $\nabla e = \vartheta e$.

Remark 2.2.21. If ∇ is a complex-analytic connection, the connection 1–form ϑ is a $(1,0)$–form. For a general Hermitian connection, however, ϑ will have a non-zero $(0,1)$–part.

Let σ be a smooth section of $\mathcal{E}$. With respect to the frame $\{e_\alpha\}$, write $\sigma = \sum_\alpha \sigma^\alpha e_\alpha$. The connection ∇ then acts by

$$\nabla \sigma \ = \ \sum_\gamma \left(d\sigma^\gamma + \sigma_\alpha \vartheta_\gamma^\alpha \right) e_\gamma, \tag{2.2.4}$$

where d is the exterior derivative acting on functions.

Remark 2.2.22. The connection ∇ on $\mathcal{E}$ is, a priori, only defined on sections of $\mathcal{E}$. We can extend ∇ to a complex-linear map (which we abusively denote by the same symbol) $\nabla : \Omega^p_X(\mathcal{E}) \longrightarrow \Omega^{p+1}_X(\mathcal{E})$ by forcing the Leibniz rule

$$\nabla(\sigma \otimes \alpha) \;\; := \;\; (\nabla\sigma) \wedge \alpha + \sigma \otimes d\alpha,$$

where $\sigma \in H^0(\mathcal{E})$ and $\alpha \in \Omega^p_X$.

Definition 2.2.23. Let ∇ be a complex-linear connection on a holomorphic vector bundle $\mathcal{E} \to X$. The *curvature matrix* for ∇ (relative to a local frame $\{e_\alpha\}$ for $\mathcal{E}$) is the matrix $\Theta = (\Theta^\gamma_\alpha)$ of 2–forms given by

$$\nabla^2 e_\alpha \;:=\; \nabla(\nabla e_\alpha) \;=\; \sum_\gamma \Theta^\gamma_\alpha e_\gamma.$$

Remark 2.2.24. In the special case that $\mathcal{E}$ is a holomorphic line bundle, the connection matrix ϑ of a connection ∇ on $\mathcal{E}$ is given by a (scalar-valued) 1–form $\vartheta \in \Omega^1_X$. Similarly, the curvature matrix Θ is given by a (scalar-valued) 2–form $\Theta \in \Omega^2_X$ in this case.

Proposition 2.2.25. Let ∇ be a complex-linear connection on a holomorphic vector bundle $\mathcal{E} \to X$ over a complex manifold X. Let $\Theta \in \Omega^2_X \otimes \mathrm{End}(\mathcal{E})$ denote the curvature form ∇. If ∇ is complex-analytic, then the $(0,2)$–part of Θ vanishes.

Proof. Since $\nabla^{0,1} = \bar\partial^{\mathcal{E}}$ and $\bar\partial^{\mathcal{E}} \circ \bar\partial^{\mathcal{E}} = 0$, we see that

$$\begin{aligned}
\Theta \;=\; \nabla \circ \nabla \;&=\; (\nabla^{1,0} + \nabla^{0,1}) \circ (\nabla^{1,0} + \nabla^{0,1}) \\
&=\; \nabla^{1,0} \circ \nabla^{1,0} + \nabla^{1,0} \circ \nabla^{0,1} + \nabla^{0,1} \circ \nabla^{1,0} + \nabla^{0,1} \circ \nabla^{0,1} \\
&=\; \nabla^{1,0} \circ \nabla^{1,0} + \nabla^{1,0} \circ \bar\partial^{\mathcal{E}} + \bar\partial^{\mathcal{E}} \circ \nabla^{1,0} + \bar\partial^{\mathcal{E}} \circ \bar\partial^{\mathcal{E}} \\
&=\; \underbrace{\nabla^{1,0} \circ \nabla^{1,0}}_{=\,\Theta^{2,0}} \;+\; \underbrace{\nabla^{1,0} \circ \bar\partial^{\mathcal{E}} + \bar\partial^{\mathcal{E}} \circ \nabla^{1,0}}_{=\,\Theta^{1,1}}.
\end{aligned}$$

□

On the other hand, for Hermitian (not necessarily complex-analytic) connections we have:

Proposition 2.2.26. Let ∇ be a complex-linear connection on a Hermitian vector bundle $(\mathcal{E}, h) \to X$ over a complex manifold X. Let $\Theta \in \Omega^{1,1}_X \otimes \mathrm{End}(\mathcal{E})$ denote the curvature form ∇. If ∇ is Hermitian, then the connection 1–form $\vartheta \in \Omega^1_X \otimes \mathrm{End}(\mathcal{E})$ and curvature 2–form $\Theta \in \Omega^2_X \otimes \mathrm{End}(\mathcal{E})$ are skew-Hermitian in the sense that

$$\bar\vartheta^t \;=\; -\vartheta, \quad \text{and} \quad \bar\Theta^t \;=\; -\Theta.$$

Hence, if ∇ is Hermitian and complex-analytic, we see that $\Theta^{2,0} = -(\bar\Theta^{0,2})^t = 0$. Therefore, we have the following:

Corollary 2.2.27. Let ∇ be the unique complex-analytic and Hermitian connection on a Hermitian vector bundle $(\mathcal{E}, h) \to X$ over a complex manifold X. The connection 1–form ϑ is a skew-Hermitian $\mathrm{End}(\mathcal{E})$–valued $(1,0)$–form and the curvature 2–form Θ is a skew-Hermitian $\mathrm{End}(\mathcal{E})$–valued $(1,1)$–form.

We can now give a proof of 2.2.17:

Proof of 2.2.17. Let us first show that the complex-analytic Hermitian connection ∇ is unique. To this end, let $\{e_\alpha\}$ be a local holomorphic frame with respect to which we define the connection 1–form ϑ. Write $h_{\alpha\overline{\gamma}} := h(e_\alpha, e_\gamma)$ for the components of the Hermitian metric relative to the frame e. Since ∇ is both complex-analytic and Hermitian,

$$dh_{\alpha\overline{\gamma}} \;=\; dh(e_\alpha, e_\gamma) \;=\; \sum_\nu \left(\vartheta^\nu_\alpha h_{\nu\overline{\gamma}} + \overline{\vartheta^\nu_\gamma} h_{\alpha\overline{\nu}}\right).$$

Decomposing the above equation into its $(1,0)$ and $(0,1)$–parts shows that $\partial h = \vartheta h$. Hence, $\vartheta = (\partial h) \cdot h^{-1}$, from which it follows that the Hermitian metric and the complex-analytic structure $\bar{\partial}$ uniquely determine ϑ (and hence, determine ∇).
Existence now follows readily, since we may define ∇ to be the connection with connection matrix $\vartheta = (\partial h) \cdot h^{-1}$. Since both ∂h and h^{-1} are independent of the choice of local frame, the connection is subsequently well-defined. The fact that this connection is complex-analytic and Hermitian is straightforward. $\square$

Cartan structure equation. Let $\{e_\alpha\}$ be a local frame for a holomorphic vector bundle $\mathcal{E} \to X$ over a complex manifold X. Let ϑ and Θ denote the connection and curvature matrices of a connection ∇ on $\mathcal{E}$. Observe that

$$\sum_\gamma \Theta^\gamma_\alpha e_\gamma \;=\; \nabla(\nabla e_\alpha) \;=\; \nabla\left(\sum_\gamma \vartheta^\gamma_\alpha e_\gamma\right) \;=\; \sum_{\gamma,\nu} \left(d\vartheta^\gamma_\alpha + \vartheta^\nu_\alpha \wedge \vartheta^\gamma_\nu\right) e_\gamma.$$

Hence, we recover the structure equation of Cartan:

$$\Theta^\gamma_\alpha \;=\; d\vartheta^\gamma_\alpha + \vartheta^\nu_\alpha \wedge \vartheta^\gamma_\nu.$$

In matrix notation:

$$\Theta \;=\; d\vartheta + \vartheta \wedge \vartheta. \tag{2.2.5}$$

Example 2.2.28. Let $\mathcal{L} \to X$ be a holomorphic line bundle over a complex manifold X. Let ∇ be the unique complex-analytic Hermitian connection on $\mathcal{L}$ given by 2.2.17. The proof of 2.2.17 shows that the connection matrix is given by $\vartheta = (\partial h) \cdot h^{-1} = \partial \log(h)$. Hence, by (2.2.5), the curvature $\Theta^{(\mathcal{L},h)}$ of the complex-analytic Hermitian connection on a Hermitian line bundle is given by

$$\Theta^{(\mathcal{L},h)} \;=\; d\vartheta + \vartheta \wedge \vartheta \;=\; d\vartheta \;=\; \bar{\partial}\partial \log(h) \;=\; -\partial\bar{\partial}\log(h).$$

Remark 2.2.29. Since a Hermitian metric h on a holomorphic line bundle $\mathcal{L} \to X$ is given by a non-negative function, we often write $h = e^{-\varphi}$, where φ is a smooth function on X. We observe that the curvature form of the complex-analytic Hermitian connection associated to h is then

$$\Theta^{(\mathcal{L},h)} \;=\; -\partial\bar{\partial}\log(e^{-\varphi}) \;=\; \partial\bar{\partial}\varphi.$$

In particular, the line bundle $\mathcal{L}$ is positive if and only if the function φ is plurisubharmonic. Moreover, if $\Theta^{(\mathcal{L},h)}$ is positive, the complex manifold X supports a Kähler metric given locally by $\partial\bar{\partial}\varphi$.

Let us mention some important (albeit elementary to prove) properties of the curvature:

Proposition 2.2.30. Let $(\mathcal{E}, h_{\mathcal{E}}) \to X$ and $(\mathcal{F}, h_{\mathcal{F}}) \to X$ be two Hermitian vector bundles over a complex manifold X. Then the curvature form

(i) splits additively under tensor product

$$\Theta^{(\mathcal{E}\otimes\mathcal{F}, h_{\mathcal{E}}\otimes h_{\mathcal{F}})} \;=\; \Theta^{(\mathcal{E},h_{\mathcal{E}})} \otimes \mathrm{id} + \mathrm{id} \otimes \Theta^{(\mathcal{F},h_{\mathcal{F}})}.$$

(ii) is natural in the sense that it commutes with pullback

$$\Theta^{(f^*\mathcal{E}, f^*h_{\mathcal{E}})} \;=\; f^*\Theta^{(\mathcal{E},h_{\mathcal{E}})},$$

where $f : Y \to X$ is a holomorphic map.

Remark 2.2.31. The properties of the curvature form indicate that it defines a natural group homomorphism from the space of isometric isomorphism classes of Hermitian line bundles. It is difficult to make this more precise, however, since the target space of this group homomorphism is not clear.

Frame-dependence. The connection matrix (as defined in 2.2.20) is not invariant under a change of a frame. Indeed, let $\{e_\alpha\}$ and $\{f_\alpha\}$ be two local frames for the holomorphic vector bundle $\mathcal{E} \to X$. Let $A = (A^\gamma_\alpha)$ be the $\mathrm{GL}(\mathbb{C})$–valued map incarnating the change of frame:

$$e_\alpha \;=\; \sum_\gamma A^\gamma_\alpha f_\gamma.$$

From (2.2.4), we see that

$$\nabla e_\alpha \;=\; \sum_{\gamma,\nu} (dA^\gamma_\alpha + A^\nu_\alpha \vartheta^\gamma_\nu)\, f_\gamma \tag{2.2.6}$$

Let $\vartheta(e)$ and $\vartheta(f)$ denote the connection matrices for ∇ relative to the local frames $\{e_\alpha\}$ and $\{f_\alpha\}$, respectively. From (2.2.6), we see that

$$\sum_\gamma \vartheta(e)^\gamma_\alpha e_\gamma \;=\; \sum_{\gamma,\nu} \vartheta(e)^\gamma_\alpha A^\nu_\gamma f_\nu \;=\; \sum_{\gamma,\nu} (dA^\gamma_\alpha + A^\nu_\alpha \vartheta(f)^\gamma_\nu) f_\gamma.$$

Hence, in (the more transparent) matrix notation,

$$\vartheta(e) \;=\; dA \cdot A^{-1} + A \cdot \vartheta(f) \cdot A^{-1}.$$

Similarly, the curvature matrix Θ is not invariant under a change of frame, but transforms by the adjoint action

$$\Theta(e) \;=\; A \cdot \Theta(f) \cdot A^{-1}.$$

Curvature of Subbundles and Quotients. Let $(\mathcal{E}, h) \longrightarrow X$ be a Hermitian holomorphic vector bundle. Let $\mathcal{F} \subset \mathcal{E}$ be a holomorphic subbundle and let $\mathcal{Q} := \mathcal{E}/\mathcal{F}$ denote the quotient. Let $\nabla^{\mathcal{E}}, \nabla^{\mathcal{F}}$, and $\nabla^{\mathcal{Q}}$ denote the connections on $\mathcal{E}$, $\mathcal{F}$, and $\mathcal{Q}$, respectively.

Definition 2.2.32. The *second fundamental form* (of $\mathcal{F}$ in $\mathcal{E}$) is the map

$$\Pi : H^0(\mathcal{F}) \longrightarrow \Omega^1_X(\mathcal{Q}), \qquad \Pi := \nabla^{\mathcal{E}}|_{\mathcal{F}} - \nabla^{\mathcal{F}}.$$

Work in a local unitary frame $\xi = \{\xi^1, ..., \xi^r\}$ for $\mathcal{E}$ such that $\{\xi^1, ..., \xi^p\}$ yields a local frame for $\mathcal{F}$. With respect to this frame, the connection matrix $\vartheta^{\mathcal{E}}$ can be written as

$$\vartheta^{\mathcal{E}} \;=\; \begin{pmatrix} \vartheta^{\mathcal{F}} & \overline{\Pi}^t \\ \Pi & \vartheta^{\mathcal{Q}} \end{pmatrix}.$$

Then

$$\Theta^{\mathcal{E}} \;=\; d\vartheta^{\mathcal{E}} - \vartheta^{\mathcal{E}} \wedge \vartheta^{\mathcal{E}} \;=\; \begin{pmatrix} d\vartheta^{\mathcal{F}} - \vartheta^{\mathcal{F}} \wedge \vartheta^{\mathcal{F}} & \cdot \\ \cdot & d\vartheta^{\mathcal{Q}} - \vartheta^{\mathcal{Q}} \wedge \vartheta^{\mathcal{Q}} \end{pmatrix},$$

and in particular,

$$\Theta^{\mathcal{F}} \;=\; \Theta^{\mathcal{E}}|_{\mathcal{F}} + \overline{\Pi}^t \wedge \Pi, \qquad \Theta^{\mathcal{Q}} \;=\; \Theta^{\mathcal{E}}|_{\mathcal{Q}} + \Pi \wedge \overline{\Pi}^t.$$

The second fundamental form is a $\text{Hom}(\mathcal{F}, \mathcal{Q})$–valued $(1,0)$–form. Hence, we may write

$$\Pi \;=\; \sum_{1 \leq j \leq p, p < \lambda \leq r} \Pi^{\alpha}_{\lambda j} dz_{\alpha} \otimes \xi_{\lambda} \otimes \varepsilon_j.$$

Hence,

$$\Pi \wedge \overline{\Pi}^t \;=\; \sum \Pi^{\alpha}_{ik} \overline{\Pi^{\beta}_{jk}} dz_{\alpha} \wedge d\overline{z}_{\beta} \otimes \xi_i \otimes \varepsilon_j,$$

and so

$$(\Pi \wedge \overline{\Pi}^t)(\partial_{\alpha}, \partial_{\overline{\beta}}) \;=\; \sum \Pi^{\alpha}_{ik} \overline{\Pi^{\beta}_{jk}} \xi_i \otimes \varepsilon_j \;=\; \Pi^{\alpha} \overline{\Pi^{\alpha}}^t \;\geq\; 0.$$

In summary, this implies the following (whose importance cannot be overstated):

Subbundle Decreasing and Quotient Increasing Property. If $\mathcal{F}$ is a holomorphic subbundle of the Hermitian vector bundle $(\mathcal{E}, h)$, and $\mathcal{Q} := \mathcal{E}/\mathcal{F}$ is the quotient, then

$$\Theta^{\mathcal{F}} \;\leq\; \Theta^{\mathcal{E}}|_{\mathcal{F}}, \qquad \Theta^{\mathcal{Q}} \;\geq\; \Theta^{\mathcal{E}}|_{\mathcal{Q}}. \tag{2.2.7}$$

Chern–Weil theory. The fact that the connection form and curvature form of a connection ∇ on a holomorphic vector bundle $\mathcal{E} \to X$ yield differential forms on X paves the way for a cohomological study of vector bundles; or, conversely, this bridge offers a differential-geometric approach to cohomology.

Theorem 2.2.33. Let $(\mathcal{L}, h) \longrightarrow X$ be a Hermitian line bundle on a complex manifold X. Let $\Theta^{(\mathcal{L},h)}$ be the curvature form of the Hermitian metric h on $\mathcal{L}$. Then

$$c_1(\mathcal{L}) \;=\; \left[\frac{\sqrt{-1}}{2\pi}\Theta^{(\mathcal{L},h)}\right] \in H^2_{\mathrm{DR}}(X, \mathbb{R}).$$

In particular, the cohomology class represented by $\Theta^{(\mathcal{L},h)}$ is independent of the choice of Hermitian metric.

Proof. Let ∇ be a complex-linear connection on $\mathcal{L}$. With respect to a local frame defined on an open set $\mathcal{U}_\alpha$, denote by ϑ_α and Θ_α the connection and curvature forms of ∇. Since $\mathcal{L}$ is a line bundle, $\Theta_\alpha = d\vartheta_\alpha$ for all α, by the Cartan structure equation. On any overlap $\mathcal{U}_\alpha \cap \mathcal{U}_\gamma$, we have

$$\vartheta_\alpha \;=\; A_{\alpha\gamma} \cdot \vartheta_\gamma \cdot A_{\alpha\gamma}^{-1} + dA_{\alpha\gamma} \cdot A_{\alpha\gamma}^{-1}.$$

Hence,

$$\begin{aligned}
\vartheta_\gamma - \vartheta_\alpha &= \vartheta_\gamma - A_{\alpha\gamma}^{-1} \cdot \vartheta_\alpha \cdot A_{\alpha\gamma} \\
&= \vartheta_\gamma - A_{\alpha\gamma}^{-1}\left(A_{\alpha\gamma} \cdot \vartheta_\gamma \cdot A_{\alpha\gamma}^{-1} + dA_{\alpha\gamma} \cdot A_{\alpha\gamma}^{-1}\right) A_{\alpha\gamma} \\
&= -A_{\alpha\gamma}^{-1} \cdot dA_{\alpha\gamma} \;=\; -d\log(A_{\alpha\gamma}),
\end{aligned}$$

that is, the connection forms differ by a d–exact form. In particular, $\Theta_\gamma = \Theta_\alpha = d\vartheta_\alpha$, and the connection form Θ is globally well-defined. Further, $\sqrt{-1}\Theta$ represents a cohomology class in $H^2_{\mathrm{DR}}(X, \mathbb{R})$. To see that the cohomology class represented by $\sqrt{-1}\Theta$ coincides with the first Chern class $c_1(\mathcal{L}) \in H^2_{\mathrm{DR}}(X, \mathbb{R})$, recall: Let $A_{\alpha\gamma}$ denote the bundle charts relative to an open cover $\{\mathcal{U}_\alpha\}$ of X. We can choose a refinement of $\{\mathcal{U}_\alpha\}$ if necessary such that each $\mathcal{U}_\alpha$ is simply connected. The Čech cocycle (relative to the covering $\{\mathcal{U}_\alpha\}$) representing $c_1(\mathcal{L})$ is then

$$\frac{1}{2\pi\sqrt{-1}}\left(\log(A_{\alpha\beta} + \log(A_{\beta\gamma}) - \log(A_{\alpha\gamma})\right).$$

From the considerations in §1.3, the results do not depend on the choice of covering. Let now $\mathcal{Z}^k_d$ denote the sheaf of d–closed k–forms and write Ω^k_X for the sheaf of smooth k–forms on X. We have exact sequences of sheaves

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0_X \longrightarrow \mathcal{Z}^1_d \longrightarrow 0, \qquad 0 \longrightarrow \mathcal{Z}^1_d \longrightarrow \Omega^1_X \longrightarrow \mathcal{Z}^2_d \longrightarrow 0.$$

On sheaf cohomology, these exact sequences furnish boundary morphisms

$$\delta_1 : H^0(X, \mathcal{Z}^2_d)/dH^0(X, \Omega^1_X) \longrightarrow H^1(X, \mathcal{Z}^1_d), \qquad \delta_2 : H^1(X, \mathcal{Z}^1_d) \longrightarrow H^2(X, \mathbb{R}).$$

We will compute $\delta_2\delta_1(\Theta)$. To this end, relative to the covering $\{\mathcal{U}_\alpha\}$, we can write $\Theta_\alpha = d\vartheta_\alpha$. The image of Θ under δ_1 is then the Čech 1–cocycle represented by $\{\vartheta_\gamma - \vartheta_\alpha\}_{\alpha,\gamma} \in \mathcal{Z}^1(X, \mathcal{Z}^1_d)$. The image of this Čech 1–cocycle under δ_2 is then the Čech 2–cocycle represented by

$$\{-\log(A_{\alpha\beta}) - \log(A_{\beta\gamma}) + \log(A_{\alpha\gamma})\}_{\alpha,\beta,\gamma}.$$

Of course, this is precisely, $-2\pi\sqrt{-1}c_1(\mathcal{L})$. □

Remark 2.2.34. We emphasize that defining the first Chern class of a holomorphic line bundle as the cohomology class represented by $\frac{\sqrt{-1}}{2\pi}\Theta^{(\mathcal{L},h)}$ is not equivalent to the definition given in 1.4.22. Indeed, although $\left[\frac{\sqrt{-1}}{2\pi}\Theta^{(\mathcal{L},h)}\right]$ is an integral cohomology class, it is an integral cohomology class within $H^2_{\mathrm{DR}}(X, \mathbb{R})$, since it is defined via forms; in this process, torsion is lost. To see this very explicitly, let X be an Enriques surface. Then K_X is not holomorphically trivial, but $K_X^{\otimes 2} \simeq \mathcal{O}_X$ is. In particular, K_X represents a torsion element in $\mathrm{Pic}(X)$. The first Chern class of K_X (according to 1.4.22) satisfies $c_1(K_X) \neq 0$ and $c_1(K_X^{\otimes 2}) = 2c_1(K_X) = 0$. On the other hand, if h is a Hermitian metric on K_X, then $h \otimes h$ is a Hermitian metric on $K_X^{\otimes 2} \simeq \mathcal{O}_X$, and we may assume that $\Theta^{h\otimes h} \equiv 0$. But $\Theta^{h\otimes h} = \Theta^h \otimes \mathrm{id} + \mathrm{id} \otimes \Theta^h$, which implies that $\Theta^h \equiv 0$, and hence, $\left[\frac{\sqrt{-1}}{2\pi}\Theta^{(K_X,h)}\right] = 0$. This illustrates that an integral class in $H^2_{\mathrm{DR}}(X, \mathbb{R})$ is different from a cohomology class in $H^2(X, \mathbb{Z})$.

Combining 2.2.33 with the properties of the curvature form 2.2.30, we see that:

Proposition 2.2.35. *Let $\mathcal{L} \to X$ and $\mathcal{V} \to X$ be two holomorphic line bundles over a complex manifold X. Then*

(i) $c_1(\mathcal{L} \otimes \mathcal{V}) = c_1(\mathcal{L}) \otimes \mathrm{id} + \mathrm{id} \otimes c_1(\mathcal{V})$.
(ii) $c_1(\mathcal{L}^*) = -c_1(\mathcal{L})$*, where $\mathcal{L}^*$ denotes the dual line bundle.*
(iii) $c_1(f^*\mathcal{L}) = f^*c_1(\mathcal{L})$*, where $f : Y \to X$ is a holomorphic map.*

Remark 2.2.36. The above properties hold for the Chern class defined more generally as the image of the boundary morphism $c_1 : \mathrm{Pic}(X) \to H^2(X, \mathbb{Z})$ as defined in 1.4.22 But these properties do not follow from the properties of the curvature form, as we indiciated in 2.2.34.

There exist variants of the first Chern class which play a more suitable role in some contexts. These are defined not as cohomology classes in de Rham or Dolbeault cohomology, but within the following cohomology theories:

Definition 2.2.37. Let X be a complex manifold.

(i) The *Bott–Chern cohomology groups* are defined

$$H^{p,q}_{\mathrm{BC}}(X) \quad := \quad \frac{\{\alpha \in \Omega^{p,q}_X : d\alpha = 0\}}{\{\sqrt{-1}\partial\bar{\partial}\beta : \beta \in \Omega^{p-1,q-1}_X\}}.$$

(ii) The *Aeppli cohomology groups* are defined

$$H^{p,q}_{\mathrm{A}}(X) \quad := \quad \frac{\{\alpha \in \Omega^{p,q}_X : \partial\bar{\partial}\alpha = 0\}}{\{\partial\gamma + \bar{\partial}\gamma : \gamma \in \Omega^{p-1,q-1}_X\}}.$$

If $\alpha \in \Omega^{p,q}_X$ is a d–closed (p,q)–form, we write $[\alpha]_{\mathrm{BC}} \in H^{p,q}_{\mathrm{BC}}(X)$ (respectively, $[\alpha]_{\mathrm{A}} \in H^{p,q}_{\mathrm{A}}(X)$) for the corresponding Bott–Chern class (respectively, Aeppli cohomology class).

Definition 2.2.38. Let $(\mathcal{L}, h) \to X$ be a Hermitian line bundle over a complex manifold X. The *first Bott–Chern class* $c_1^{\mathrm{BC}}(\mathcal{L})$ (respectively, *first Aeppli–Chern class* $c_1^{\mathrm{AC}}(\mathcal{L})$) is the cohomology class in $H^{1,1}_{\mathrm{BC}}(X)$ (respectively, $H^{1,1}_{A}(X)$) represented by the curvature form $\Theta^{(\mathcal{L},h)} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(h)$.

Remark 2.2.39. The above definition is well-defined, independent of the specific choice of Hermitian metric. Indeed, if $\tilde{h}$ is another Hermitian metric on $\mathcal{L} \to X$, then the difference of the curvature forms

$$\Theta^{(\mathcal{L},\tilde{h})} - \Theta^{(\mathcal{L},h)} \quad = \quad \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\frac{\tilde{h}}{h}\right)$$

is globally $\partial\bar{\partial}$–exact.

An immediate consequence of the definition is that

$$c_1^{\mathrm{BC}}(\mathcal{L}) = 0 \quad \Longrightarrow \quad c_1(\mathcal{L}) = 0 \quad \Longrightarrow \quad c_1^{\mathrm{AC}}(\mathcal{L}) = 0.$$

If X supports the $\partial\bar{\partial}$–lemma (e.g., if X is Kähler, 1.6.53), then the converse implications hold [200]:

Proposition 2.2.40. Let $\mathcal{L} \to X$ be a holomorphic line bundle over a complex manifold X. If the $\partial\bar{\partial}$–lemma holds on X, then

$$c_1^{\mathrm{BC}}(\mathcal{L}) = 0 \quad \Longleftrightarrow \quad c_1(\mathcal{L}) = 0 \quad \Longleftrightarrow \quad c_1^{\mathrm{AC}}(\mathcal{L}) = 0.$$

Proof. Suppose $c_1^{\mathrm{AC}}(\mathcal{L}) = 0$. It suffices to show that $c_1^{\mathrm{BC}}(\mathcal{L}) = 0$. Then there is a Hermitian metric h on $\mathcal{L}$ such that

$$\Theta^{(\mathcal{L},h)} = \partial\alpha + \overline{\partial\beta},$$

where $\alpha, \beta \in \Omega^{0,1}_X$. Differentiating this equation, $\partial\alpha$ is $\bar{\partial}$–closed and $\overline{\partial\beta}$ is ∂–closed. Since the $\partial\bar{\partial}$–lemma holds, we can find smooth functions u and v such that $\partial\alpha = \partial\bar{\partial}u$ and $\partial\beta = \partial\bar{\partial}v$. Then

$$\Theta^{(\mathcal{L},h)} \quad = \quad \partial\bar{\partial}(u - \overline{v}),$$

proving the claim. □

Example 2.2.41. Let $X = \mathbb{S}^3 \times \mathbb{S}^1$ be the Hopf surface. From the Kunneth formula, the second Betti number $b_2(X) = 0$, and therefore, $c_1(\mathcal{L}) = c_1^{\mathrm{AC}}(\mathcal{L}) = 0$ for any holomorphic line bundle $\mathcal{L} \to X$. We will see in 2.3.33, however, that $c_1^{\mathrm{BC}}(K_X^{-1}) \neq 0$.

Higher Chern Classes. In light of 2.2.33, we can construct higher Chern classes by considering certain variants of the curvature. The starting observation comes from the fact that the curvature of a Hermitian metric on a (holomorphic) vector bundle $\mathcal{E} \longrightarrow X$ is an $\mathrm{End}(\mathcal{E})$–valued $(1,1)$–form. The $(1,1)$–part of the curvature is invariant under a change of frame, but the endomorphism component transforms via the adjoint action of $\mathrm{GL}_k(\mathbb{C})$:

$$\Theta \ \longmapsto \ A \cdot \Theta \cdot A^{-1}.$$

In light of this, let us denote by $M_n(\mathbb{C}) \simeq \mathbb{C}^{n^2}$ the space of $n \times n$ matrices with complex entries.

Definition 2.2.42. A k–homogeneous polynomial $f : M_n(\mathbb{C}) \longrightarrow \mathbb{C}$ is said to be *invariant* if

$$f(M) \ = \ f(AMA^{-1})$$

for all $M \in M_n(\mathbb{C})$ and all $A \in \mathrm{GL}_n(\mathbb{C})$.

Example 2.2.43. The most important examples for our purposes are the elementary symmetric polynomials $P^k(A)$ of the eigenvalues of a matrix $A \in M_n(\mathbb{C})$, defined by

$$\det(A + t\mathrm{Id}) \ = \ \sum_{k=0}^{n} P^{n-k}(A) \cdot t^k.$$

For instance, $P^1(A) = \mathrm{tr}(A)$ and $P^n(A) = \det(A)$.

Lemma 2.2.44. For any invariant polynomial P of degree k, the form $P(\Theta_\alpha)$ is closed, and the cohomology class it represents is independent of the choice of connection.

From [142], we have the following important theorem:

Proposition 2.2.45. Let $\mathcal{P} = \bigoplus_{k \geq 0} \mathcal{P}^k$ denote the graded algebra of invariant polynomials. There is a well-defined morphism of algebras

$$\mathrm{W} : \mathcal{P} \longrightarrow \bigoplus_{k \geq 0} H^{2k}_{\mathrm{DR}}(X, \mathbb{R}), \qquad \mathrm{W}(P) \ := \ [P(\Theta)].$$

The morphism W is called the *Weil morphism*.

Definition 2.2.46. Let $(\mathcal{E}, h) \to X$ be a Hermitian holomorphic vector bundle over a complex manifold X. Let Θ denote the curvature form of h. The *kth Chern class* of $\mathcal{E}$ is

$$c_k(\mathcal{E}) \ := \ \left[P^k \left(\frac{\sqrt{-1}}{2\pi} \Theta^{(\mathcal{E},h)} \right) \right] \in H^{2k}_{\mathrm{DR}}(X, \mathbb{R})$$

is called the *kth Chern class* of $\mathcal{E}$.

Example 2.2.47. Let $(\mathcal{E}, h) \to X$ be a Hermitian vector bundle over a complex surface X. Let $\Theta = \Theta^{(\mathcal{E},h)} \in \Omega^{1,1}_X \otimes \mathrm{End}(\mathcal{E})$ denote the curvature form of the complex-analytic Hermitian connection on $(\mathcal{E}, h)$. Then the two Chern classes

$$\begin{aligned} c_1(\mathcal{E}) &= \left[\frac{\sqrt{-1}}{2\pi} \mathrm{tr}(\Theta) \right] \\ c_2(\mathcal{E}) &= \left[\frac{\mathrm{tr}(\Theta \wedge \bar{\Theta}) - \mathrm{tr}(\Theta)^2}{8\pi^2} \right]. \end{aligned}$$

In particular, for $\alpha \in \mathbb{R}$, the cohomology class $c_1^2(\mathcal{E}) - \alpha c_2(\mathcal{E}) \in H^4_{\mathrm{DR}}(X, \mathbb{R})$ is represented by

$$\left(\frac{\sqrt{-1}}{2\pi} \mathrm{tr}(\Theta) \right)^2 - \alpha \left(\frac{\mathrm{tr}(\Theta \wedge \bar{\Theta}) - \mathrm{tr}(\Theta)^2}{8\pi^2} \right) \ = \ \frac{(\alpha - 2)\mathrm{tr}(\Theta)^2 - \alpha \mathrm{tr}(\Theta \wedge \bar{\Theta})}{8\pi^2}.$$

The torsion of Hermitian connections on the tangent bundle. Let (X, ω_g) be a Hermitian manifold. The torsion T of a Hermitian connection ∇ is defined by the standard formula

$$T(u, v) \ := \ \nabla_u v - \nabla_v u - [u, v],$$

where $u, v \in \mathscr{X}^{1,0}(X)$. For the Chern connection, the Christoffel symbols Γ^k_{ij} are given by the formula

$$\Gamma^k_{ij} \ := \ g^{k\bar{\ell}} \partial_i g_{j\bar{\ell}}.$$

The components of the (Chern) torsion ${}^cT$ are then specified by

$${}^cT^k_{ij} \ := \ \Gamma^k_{ij} - \Gamma^k_{ji} \ = \ g^{k\bar{\ell}} (\partial_i g_{j\bar{\ell}} - \partial_j g_{i\bar{\ell}})$$

Write $\omega_g = \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}} dz_i \wedge d\bar{z}_j$. Then

$$\partial \omega_g \ = \ \sqrt{-1} \sum_{i,j,k=1}^n \frac{\partial g_{i\bar{j}}}{\partial z_k} dz_k \wedge dz_i \wedge d\bar{z}_j,$$

and it is clear that $\partial \omega_g = 0$ if and only if ${}^cT = 0$.

Definition 2.2.48. The (Chern) *torsion* 1*–form* of the Chern connection is the $(1,0)$–form τ defined by

$$\tau \ := \ \sum_{k=1}^n \tau_k dz_k \ := \ \sum_{k=1}^n {}^cT^k_{ik} dz_k.$$

Remark 2.2.49. If Λ denotes the formal adjoint of the Lefschetz operator, then we can easily see that $\tau = \Lambda(\partial\omega)$.

The vanishing of the torsion 1–form τ (which amounts to a vanishing of the trace of the Chern torsion) is one of the two natural constraints one can place on the torsion of the Chern connection. The other natural condition being that the torsion is analytic. We will discuss this latter condition in the next section. From the (Chern) torsion $(1,0)$–form τ, we define the following real 1–form:

Definition 2.2.50. Let (X,ω) be a Hermitian manifold of (complex) dimension n. The *Lee form* is defined

$$\theta := \frac{2}{n-1}\mathrm{Re}(\tau),$$

where τ is the torsion 1–form.

The (Chern) Lee form θ and the (Chern) torsion $(1,0)$–form τ are important objects in part due to the following formulae:

Proposition 2.2.51. Let (X,ω) be a Hermitian manifold of (complex) dimension n. Then

$$d\omega^{n-1} = (n-1)\theta \wedge \omega^{n-1}.$$

In particular, the $(2,1)$–part of this equation yields $\partial\omega^{n-1} = \tau \wedge \omega^{n-1}$.

Proof. By the Leibniz rule for the exterior derivative, we have

$$\begin{aligned} d\omega^{n-1} &= (n-1)d\omega \wedge \omega^{n-2} \\ &= (n-1)\partial\omega \wedge \omega^{n-2} + (n-1)\bar{\partial}\omega \wedge \omega^{n-2} \\ &= (n-1)L(\Lambda(\partial\omega)) \wedge \omega^{n-2} + (n-1)L(\Lambda(\bar{\partial}\omega)) \wedge \omega^{n-2} \\ &= (n-1)\Lambda(\partial\omega) \wedge \omega^{n-1} + (n-1)\Lambda(\bar{\partial}\omega) \wedge \omega^{n-1} \\ &= (n-1)\tau \wedge \omega^{n-1} + (n-1)\overline{\tau} \wedge \omega^{n-1} \\ &= (n-1)(\tau + \bar{\tau}) \wedge \omega^{n-1} \ = \ \theta \wedge \omega^{n-1}. \end{aligned}$$

□

2.2.51 has the following important corollary:

Corollary 2.2.52. Let (X,ω) be a Hermitian manifold. If the torsion $(1,0)$–form τ vanishes, or X is compact with τ analytic, then ω is balanced. In particular, a compact Hermitian surface with analytic torsion $(1,0)$–form is Kähler.

Proof. It is clear from 2.2.51 that if $\tau = 0$, then $d\omega^{n-1} = 0$. Hence, assume X is compact with holomorphic torsion $(1,0)$–form τ. We will show that

$$\tau \ = \ -\sqrt{-1}\bar{\partial}^* \omega.$$

To this end, let $\alpha \in \Omega_X^{1,0}$ and denote by $\{\cdot, \cdot\}$ the inner product coming from ω, extended to $\Omega_X^{p,q}$. Then

$$\{\alpha, \bar{\partial}^* \omega\} \frac{\omega^n}{n!} \ = \ \alpha \wedge \bar{\tau} \wedge \frac{\omega^{n-1}}{(n-1)!} \ = \ \langle \alpha \wedge \bar{\tau}, \omega \rangle \frac{\omega^n}{n!} \ = \ \langle \alpha, \sqrt{-1}\tau \rangle \frac{\omega^n}{n!}.$$

Suppose now that X is balanced and τ is holomorphic. Then $\bar{\partial}\tau = -\sqrt{-1}\bar{\partial}\bar{\partial}^*\omega = 0$. If $(\cdot, \cdot)$ denotes the L^2–pairing, then

$$\|\bar{\partial}^* \omega\|_{L^2(X)}^2 \ = \ (\bar{\partial}\bar{\partial}^* \omega, \omega) \ = \ 0.$$

Hence, $\tau = -\sqrt{-1}\bar{\partial}^*\omega = 0$ and the metric is balanced. □

The Gauduchon Connections. If (X, ω) is a Hermitian manifold, then by forgetting the complex structure, (X, ω) can be identified with an even-dimensional Riemannian manifold. In particular, $T^{\mathbb{R}}X$ supports the Levi-Civita connection ∇^{LC} (compatible with the Riemannian metric underlying ω). We can complex-linearly extend ∇^{LC} to $T^{\mathbb{C}}X$, yielding a complex-linear connection $\nabla^{\text{LC}} \otimes \mathbb{C}$ on $T^{\mathbb{C}}X$ compatible with the Hermitian metric ω. The space of complex-linear connections compatible with ω can be identified with $\Omega_X^2 \otimes T^{\mathbb{C}}X$, i.e., the sections of the $\Lambda_X^2 \otimes T^{\mathbb{C}}X$. The connections in $\Omega_X^2 \otimes T^{\mathbb{C}}X$, which also preserve the complex structure, form an affine subbundle. Therefore define a connection ${}^l\nabla$ at any point $x \in X$ to be the orthogonal projection of the origin into the fiber of this affine subbundle at the point x.

In other words, with orthogonal projection understood in the above sense, we make the following definition:

Definition 2.2.53. Let (X, ω) be a Hermitian manifold. The *Lichnerowicz connection* ${}^l\nabla$ is the Hermitian connection on $T^{1,0}X$ given by the orthogonal projection of the complexified Levi-Civita connection of the underlying Riemannian metric.

Remark 2.2.54. The Lichnerowicz connection was first considered by Lichnerowicz in [**196**], where it is referred to as the *first canonical connection*.

In [**133**], Gauduchon introduced the following one-parameter family of Hermitian connections:

Definition 2.2.55. Let (X, ω) be a Hermitian manifold. The *t–Gauduchon connection* is the Hermitian connection ${}^t\nabla$ on $T^{1,0}X$ defined by

$${}^t\nabla \ \ := \ \ t^c\nabla + (1-t)^l\nabla,$$

where ${}^c\nabla$ and ${}^l\nabla$ are the Chern and Lichnerowicz connections on $T^{1,0}X$, respectively.

The Gauduchon connections capture many of the distinguished connections which have been discovered over the years:

Example 2.2.56.

(i) For $t = -1$, the t–Gauduchon connection coincides with the *Strominger–Bismut connection* ${}^b\nabla$ – the unique Hermitian connection with totally skew-symmetric torsion [**266**, **31**].

(ii) For $t = \frac{1}{2}$, the t–Gauduchon connection coincides with Libermann's *conformal connection* [**194**, **195**] – the unique Hermitian connection on $T^{1,0}X$ whose torsion satisfies the Bianchi identity (c.f., 2.2.67).

(iii) For $t = \frac{1}{3}$, the t–Gauduchon connection coincides with the *minimal connection* ${}^{\min}\nabla$ – the unique Hermitian connection whose torsion has minimal norm [**133**].

Remark 2.2.57. The Strominger–Bismut connection first appeared in the 1986 paper of Strominger [**266**] (referred to in [**266**] as the *H–connection*) with motivations coming from heterotic string theory and was later independently discovered by Bismut [**31**] (who proved existence and uniqueness). It was later independently discovered by Bismut [**31**], where it was used in his study of local index theorems. The Strominger–Bismut connection plays an important role in complex-geometric flows, most notably the *pluriclosed flow* [**264**, **265**]:

$$\frac{\partial \omega_t}{\partial t} \ = \ -{}^b\text{Ric}_{\omega_t}^{(1)},$$

where ${}^b\text{Ric}_{\omega_t}^{(1)}$ is the $(1,1)$–part of the first Bismut–Ricci curvature.

Remark 2.2.58. Libermann's conformal connection can be equivalently characterized as the unique Gauduchon connection for which the corresponding Dirac operator is conformally covariant[4].

Lemma 2.2.59. (Gauduchon symmetries). Let (X, ω_g) be a Hermitian manifold. Let $T = {}^tT$ be the torsion of the t–Gauduchon connection ${}^t\nabla$. In any unitary frame, we have

[4]Recall that a differential operator P_g (where g is a Riemannian metric) is conformally covariant if, under a conformal rescaling $g \mapsto e^{2u}g =: g_u$, the operator transforms like

$$P_{g_u}(\varphi) \quad = \quad e^{-\frac{n+2}{2}u} P_g(e^{\frac{n-2}{2}u}\varphi),$$

for all smooth functions $\varphi : X \to \mathbb{R}$. The most notable example is the *conformal Laplacian* (sometimes called the *Yamabe operator*):

$$\mathcal{L}_g \ := \ -\Delta_g + \frac{n-2}{4(n-1)}\text{Scal}_g,$$

where Scal_g is the Riemannian scalar curvature.

(i) $T_{ij}^k = t\,{}^cT_{ij}^k$, where ${}^cT$ denotes the torsion of the Chern connection.

(ii) $T_{\bar{i}j}^k = (1-t)\,{}^lT_{\bar{i}j}^k$, where ${}^lT$ denotes the torsion of the Lichnerowicz connection.

(iii) $T_{\bar{i}j}^k = -T_{\bar{k}j}^i$.

(iv) $T_{\bar{i}j}^k = \frac{1-t}{2t}\overline{T_{ik}^j}$.

Proof. Let $\{e_i\}$ be a local unitary frame. Since the $(1,1)$–part of the Chern torsion, and the $(2,0)$–part of the Lichnerowicz torsion vanish,

$$ {}^cT^{1,1} \;=\; {}^lT^{2,0} \;=\; 0, $$

assertions (i) and (ii) follow immediately from the definition ${}^t\nabla = t\,{}^c\nabla + (1-t)\,{}^l\nabla$. For (iii), we note that from (ii),

$$ T_{\bar{i}j}^k \;=\; \frac{1-t}{2}\,{}^bT_{\bar{i}j}^k, \qquad \text{and} \qquad T_{ij}^k \;=\; -t\,{}^bT_{ij}^k. $$

Since the torsion of the Bismut connection ${}^b\nabla$ is totally skew-symmetric,

$$ \begin{aligned} 2tT_{\bar{i}j}^k \;&=\; 2tg(T(\overline{e_i}, e_j), \overline{e_k}) \;=\; t(1-t)g({}^bT(\overline{e_i}, e_j), \overline{e_k}) \\ &=\; t(t-1)g({}^bT(\overline{e_i}, \overline{e_k}), e_j) \;=\; (1-t)\overline{T_{ik}^j}. \end{aligned} $$

Similarly, for (iv), from the fact that ${}^bT$ is totally skew-symmetric,

$$ T_{\bar{i}j}^k \;=\; \frac{1-t}{2}\,{}^bT_{\bar{i}j}^k \;=\; -\frac{1-t}{2}\,{}^bT_{\bar{k}j}^i \;=\; -T_{\bar{k}j}^i, $$

as required. □

Structure Equations for the complexified Levi-Civita Connection. Let (X, ω_g) be a Hermitian manifold. Let ∇^{LC} denote the Levi-Civita connection on $T^{\mathbb{R}}X$, compatible with the underlying Riemannian metric g. Let $\{e_\alpha\}$ be a local unitary frame for $T^{1,0}X$. Then a unitary frame for $T^{\mathbb{C}}X \simeq T^{1,0}X \oplus T^{0,1}X$ is given by $\{e_\alpha, \bar{e}_\alpha\}$. Denote by $\{\varphi^\alpha, \bar{\varphi}^\alpha\}$ the dual coframe. The action of ∇^{LC} on a member of this unitary frame is specified by the connection forms $\Phi, \Psi \in \Omega_X^1 \otimes \mathrm{End}(TX)$,

$$ \nabla^{\mathrm{LC}} e_\alpha \;=\; \Phi_\alpha^\gamma e_\gamma + \bar{\Psi}_\alpha^\gamma \bar{e}_\gamma, \qquad \nabla^{\mathrm{LC}} \overline{e}_\alpha \;=\; \Psi_\alpha^\gamma e_\gamma + \bar{\Phi}_\alpha^\gamma \bar{e}_\alpha. $$

The corresponding matrices for the connection and curvature of ∇^{LC} are therefore

$$ \vartheta^{\mathrm{LC}} \;=\; \begin{pmatrix} \Phi & \bar{\Psi} \\ \Psi & \bar{\Phi} \end{pmatrix}, \qquad \Theta^{\mathrm{LC}} \;=\; \begin{pmatrix} \Lambda & \bar{\Upsilon} \\ \Upsilon & \bar{\Lambda} \end{pmatrix}, $$

where

$$ \Lambda \;=\; d\Phi - \Phi \wedge \Phi - \bar{\Psi} \wedge \Psi, \tag{2.2.8} $$

$$ \Upsilon \;=\; d\Psi - \Psi \wedge \Phi - \bar{\Phi} \wedge \Psi. \tag{2.2.9} $$

Further,

$$d\varphi \quad = \quad -\Phi^{\mathrm{T}} \wedge \varphi - \Psi^{\mathrm{T}} \wedge \bar{\varphi}.$$

Let $\{f_\alpha, \bar{f}_\alpha\}$ be another unitary frame for $T^{\mathbb{C}}X$, given by transforming the frame $\{e_\alpha, \bar{e}_\alpha\}$ by a unitary matrix A. It is straightforward to show that

$$\Phi(f) \quad = \quad A\Phi(e)A^{-1} + dA \cdot A^{-1},$$

While the other terms transform like:

$$\Psi(f) = \bar{A}\Psi(e)A^{-1}, \qquad \Lambda(f) = A\Lambda(e)A^{-1}, \qquad \Upsilon(f) = \bar{A}\Upsilon(e)A^{-1}.$$

Let ${}^c\vartheta \in \Omega^{1,0}_X \otimes \mathrm{End}(T^{1,0}X)$ denote the connection $(1,0)$–form for the Chern connection ${}^c\nabla$. Similarly, denote the curvature $(1,1)$–form for the Chern connection by ${}^c\Theta \in \Omega^{1,1}_X \otimes \mathrm{End}(T^{1,0}X)$. Define

$$\gamma \ := \ \Phi - {}^c\vartheta,$$

and note that γ transforms like $\gamma(f) = A\gamma A^{-1}$. We can express γ in terms of the Chern torsion ${}^cT$. Indeed, from [**309**, Lemma 2], we see that

$$\gamma^\beta_\alpha \quad = \quad \Phi^\beta_\alpha - {}^c\vartheta^\beta_\alpha \ = \ {}^cT^\beta_{\alpha\delta}\varphi^\delta - \overline{{}^cT^\alpha_{\beta\delta}}\bar{\varphi}^\delta.$$

Expressing (2.2.8) and (2.2.9) in terms of γ, we see that

$$\Lambda^\beta_\alpha - {}^c\Theta^\beta_\alpha \quad = \quad d\gamma^\beta_\alpha - \gamma^\delta_\alpha \wedge \gamma^\beta_\delta - \gamma^\delta_\alpha \wedge {}^c\vartheta^\beta_\delta - {}^c\vartheta^\delta_\alpha \wedge \gamma^\beta_\delta - \bar{\Psi}^\delta_\alpha \wedge \Psi^\beta_\delta.$$

The Lichnerowicz connection ${}^l\nabla$ is the restriction of the complexified Levi-Civita connection ∇^{LC} to $T^{1,0}X$, and therefore,

$${}^l\nabla e_\alpha \ = \ \Phi^\gamma_\alpha e_\gamma.$$

Denote by ${}^t\nabla$ the t–Gauduchon connection ${}^t\nabla := t{}^c\nabla + (1-t){}^l\nabla$. Write ${}^t\vartheta$ and ${}^t\Theta$ for the connection 1–form and curvature 2–form. Then

$${}^t\vartheta^\beta_\alpha \quad = \quad t{}^c\vartheta^\beta_\alpha + (1-t){}^l\vartheta^\beta_\alpha \ = \ {}^c\vartheta^\beta_\alpha + (1-t)\gamma^\beta_\alpha.$$

The Cartan structure equation then implies that

$${}^t\Theta^\beta_\alpha \quad = \quad d{}^t\vartheta^\beta_\alpha - {}^t\vartheta^\delta_\alpha \wedge {}^t\vartheta^\beta_\delta.$$

Hence,

$${}^t\Theta^\beta_\alpha - {}^c\Theta^\beta_\alpha \quad = \quad (1-t)d\gamma^\beta_\alpha - (1-t)\gamma^\delta_\alpha \wedge {}^c\vartheta^\beta_\delta - (1-t){}^c\vartheta^\delta_\alpha \wedge \gamma^\beta_\delta - (1-t)^2\gamma^\delta_\alpha \wedge \gamma^\beta_\delta,$$

and moreover,

$${}^t\Theta^\beta_\alpha - \Lambda^\beta_\alpha \quad = \quad -td\gamma^\beta_\alpha + t\gamma^\delta_\alpha \wedge {}^c\vartheta^\beta_\delta + t{}^c\vartheta^\delta_\alpha \wedge \gamma^\beta_\delta + t(2-t)\gamma^\delta_\alpha \wedge \gamma^\beta_\delta + \bar{\Psi}^\delta_\alpha \wedge \Psi^\beta_\delta.$$

The $(1,1)$–part of the t–Gauduchon curvature tensor is given by ${}^tR_{k\bar{\ell}i\bar{j}} = {}^t\Theta_i^j(e_k,\bar{e}_\ell)$ and the $(1,1)$–part of the complexified Riemannian curvature tensor is given by $R_{k\bar{\ell}i\bar{j}} = \Lambda_i^j(e_k,\bar{e}_\ell)$. Hence,

$$ {}^tR_{k\bar{\ell}i\bar{j}} \;=\; R_{k\bar{\ell}i\bar{j}} + t\left({}^cT^j_{ik,\bar{\ell}} + \overline{{}^cT^i_{j\ell,\bar{k}}}\right) + t(t-2)\left({}^cT^r_{ik}\overline{{}^cT^r_{j\ell}} - {}^cT^j_{rk}\overline{{}^cT^i_{r\ell}}\right) - \overline{{}^cT^k_{rj}}\,{}^cT^\ell_{ir}, $$

where the comma indicates covariant derivatives with respect to the Chern connection ${}^c\nabla$.

From the Bianchi identity 2.2.67, we have ${}^cT^j_{ik,\bar{\ell}} = {}^cR_{k\bar{\ell}i\bar{j}} - {}^cR_{i\bar{\ell}k\bar{j}}$. Hence, we recover the following (see [89, Lemma 2.2], [127, Proposition 4.2] also):

Theorem 2.2.60. Let (X,ω_g) be a Hermitian manifold. The $(1,1)$–part of the t–Gauduchon curvature tensor ${}^tR$ is related to the Chern curvature tensor ${}^cR$ and the $(1,1)$–part of the complexified Riemannian curvature tensor R in any local unitary frame by

$$ \begin{aligned} {}^tR_{k\bar{\ell}i\bar{j}} \;=\; & R_{k\bar{\ell}i\bar{j}} + t\left(2\,{}^cR_{k\bar{\ell}i\bar{j}} - {}^cR_{i\bar{\ell}k\bar{j}} - {}^cR_{k\bar{j}i\bar{\ell}}\right) \\ & + t(t-2)\left({}^cT^r_{ik}\overline{{}^cT^r_{j\ell}} - {}^cT^j_{rk}\overline{{}^cT^i_{r\ell}}\right) - \overline{{}^cT^k_{rj}}\,{}^cT^\ell_{ir}, \end{aligned} $$

where the comma indicates covariant derivatives with respect to the Chern connection ${}^c\nabla$.

Remark 2.2.61. Note that the special case of $t=0$ for the above theorem was computed by Liu–Yang in [200, Corollary 3.1] and [199, Proposition 4.2]. In [199, 200], the authors denote the Lichnerowicz curvature tensor by $\mathfrak{R}$ (see also [98, 201].

The curvature of the t–Gauduchon curvature tensor ${}^tR$ expressed exclusively in terms of the Chern curvature tensor ${}^cR$ is given by the following theorem:

Theorem 2.2.62. Let (X,ω_g) be a Hermitian manifold. The $(1,1)$–part of the t–Gauduchon curvature tensor ${}^tR$ is related to the Chern curvature tensor ${}^cR$ in any local unitary frame by

$$ {}^tR_{k\bar{\ell}j\bar{i}} \;=\; {}^cR_{k\bar{\ell}j\bar{i}} + \frac{1-t}{2}\left({}^cR_{j\bar{\ell}k\bar{i}} + {}^cR_{k\bar{i}j\bar{\ell}} - 2\,{}^cR_{k\bar{\ell}j\bar{i}}\right) + \left(\frac{t-1}{2}\right)^2\left({}^cT^r_{kj}\overline{{}^cT^r_{\ell i}} - {}^cT^i_{kr}\overline{{}^cT^j_{\ell r}}\right). $$

Corollary 2.2.63. Let (X,ω_g) be a Hermitian manifold. The $(1,1)$–part of the Strominger–Bismut curvature tensor ${}^bR$ is related to the Chern curvature tensor ${}^cR$ in any local unitary frame by

$$ {}^bR_{k\bar{\ell}j\bar{i}} \;=\; {}^cR_{j\bar{\ell}k\bar{i}} + {}^cR_{k\bar{i}j\bar{\ell}} - {}^cR_{k\bar{\ell}j\bar{i}} + \left({}^cT^r_{kj}\overline{{}^cT^r_{\ell i}} - {}^cT^i_{kr}\overline{{}^cT^j_{\ell r}}\right). $$

Corollary 2.2.64. Let (X,ω_g) be a Hermitian manifold. The $(1,1)$–part of the Lichnerowicz curvature tensor ${}^lR$ is related to the Chern curvature tensor ${}^cR$ in any local unitary frame by

$$ {}^lR_{k\bar{\ell}j\bar{i}} \;=\; \frac{1}{2}{}^cR_{j\bar{\ell}k\bar{i}} + {}^cR_{k\bar{i}j\bar{\ell}} + \frac{1}{4}\left({}^cT^r_{kj}\overline{{}^cT^r_{\ell i}} - {}^cT^i_{kr}\overline{{}^cT^j_{\ell r}}\right), $$

and related to the $(1,1)$–part of the complexified Riemannian curvature tensor R by

$$ {}^{l}R_{k\bar{\ell}i\bar{j}} \;\; = \;\; R_{k\bar{\ell}i\bar{j}} + \overline{{}^{c}T^{k}_{jr}}\,{}^{c}T^{\ell}_{ir}, $$

If we choose local coordinates , the t–Gauduchon curvature tensor is given by [19]:

Corollary 2.2.65. Let (X,ω_g) be a Hermitian manifold. Choose coordinates $(z_1,...,z_n)$ near a point $p \in X$ such that the metric g is Euclidean at p and the Christoffel symbols of the Levi-Civita connection vanish at p. In these coordinates, the t–Gauduchon curvature tensor ${}^{t}R$ is given by

$$ \begin{aligned} {}^{t}R_{i\bar{j}k\bar{\ell}} \;\; &= \;\; \frac{(1-t)}{2}\left(\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \overline{z}_j} - \frac{\partial^2 g_{k\bar{j}}}{\partial z_i \partial \overline{z}_\ell} - \frac{\partial^2 g_{i\bar{\ell}}}{\partial z_k \partial \overline{z}_j}\right) - \frac{(1+t)}{2}\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \overline{z}_j} \\ &\quad + \frac{(1-t)^2}{4}\sum_q {}^{c}T^{\ell}_{iq}\overline{{}^{c}T^{k}_{qj}} + \frac{t^2}{4}\sum_q {}^{c}T^{q}_{ik}\overline{{}^{c}T^{q}_{j\ell}}, \end{aligned} $$

where ${}^{c}T$ is the torsion of the Chern connection.

For $t = 1$, we recover the standard local expression for the Chern curvature tensor:

$$ {}^{c}R_{i\bar{j}k\bar{\ell}} \;\; = \;\; -\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \overline{z}_j} + \sum_{p,q=1}^{n} g^{p\bar{q}} \frac{\partial g_{k\bar{q}}}{\partial z_i}\frac{\partial g_{p\bar{\ell}}}{\partial \overline{z}_j}. \tag{2.2.10} $$

Of course, a direct proof is equally elementary to deduce: Locally, write ${}^{c}R_{i\bar{j}k}{}^{p} = -\frac{\partial \Gamma^{p}_{ik}}{\partial \overline{z}_j}$. Then

$$ \begin{aligned} {}^{c}R_{i\bar{j}k\bar{\ell}} \;\; = \;\; -g_{p\bar{\ell}}\partial_{\bar{j}}\Gamma^{p}_{ik} \;\; &= \;\; -g_{p\bar{\ell}}\partial_{\bar{j}}\left(g^{p\bar{q}}\partial_i g_{k\bar{q}}\right) \\ &= \;\; -g_{p\bar{\ell}}(\partial_{\bar{j}}g^{p\bar{q}})(\partial_i g_{k\bar{q}}) - g_{p\bar{\ell}}g^{p\bar{q}}\partial_{\bar{j}}\partial_i g_{k\bar{q}}. \end{aligned} $$

Differentiating the identify $g^{p\bar{q}}g_{p\bar{\ell}} = \delta^{q}_{\ell}$ yields (2.2.10).

Remark 2.2.66. From (2.2.10) we observe the following conjugate symmetry of the Chern curvature tensor:

$$ \overline{{}^{c}R_{i\bar{j}k\bar{\ell}}} \;\; = \;\; {}^{c}R_{j\bar{i}\ell\bar{k}}. \tag{2.2.11} $$

For a general Hermitian metric, the Chern curvature tensor violently fails to have the symmetries of the Riemannian curvature. Indeed, we have the following:

Proposition 2.2.67. Let ∇ denote the Chern connection on a Hermitian manifold (X,ω). If T and R respectively denote the torsion and curvature of the Chern connection, then with respect to any local frame, we have

(i) $R_{k\bar{j}i\bar{\ell}} - R_{i\bar{j}k\bar{\ell}} = \nabla_{\bar{j}}T^{\ell}_{ik}$.
(ii) $T^{k}_{\{\bar{j}\ell,i\}} = R_{\{i\bar{j}\ell\}\bar{k}} + T^{r}_{\{i\bar{j}}T^{k}_{\ell\}r} + T^{\bar{r}}_{\{i\bar{j}}T^{k}_{\ell\}\bar{r}}$,

where $\{i,j,k\}$ denotes cyclic summation.

PROOF. We compute directly:

$$R_{k\bar{j}i\bar{\ell}} - R_{i\bar{j}k\bar{\ell}} \;=\; -g_{p\bar{\ell}}\nabla_{\bar{j}}\Gamma^p_{ki} + g_{p\bar{\ell}}\nabla_{\bar{j}}\Gamma^p_{ik} \;=\; -g_{p\bar{\ell}}\nabla_{\bar{j}}(\Gamma^p_{ki} - \Gamma^p_{ik}) \;=\; \nabla_{\bar{j}}T^{\ell}_{ik}.$$

This proves the first statement. For the second statement, see [**185**, p. 135]. □

Of course, if the metric is Kähler, the torsion of the Chern connection vanishes, and the Chern curvature tensor has the symmetries of the Riemannian curvature tensor.
Unsurprisingly, there has been a growing interest in the study of Hermitian metrics whose curvature tensors satisfy particular symmetries in recent years. The following class of Hermitian metrics is of particular interest [**309**]:

Definition 2.2.68. A Hermitian metric is said to be (t–Gauduchon) *Kähler-like* if its t–Gauduchon curvature tensor has the symmetries of the Chern curvature tensor of a Kähler metric.

Example 2.2.69. Let

$$\mathbb{G}(k) := \left\{ \begin{pmatrix} 1 & z_1 & z_3 \\ 0 & 1 & z_2 \\ 0 & 0 & 1 \end{pmatrix} : z_1, z_2, z_3 \in k \right\}.$$

The *Iwasawa threefold* is the quotient $X := \mathbb{G}(\mathbb{C})/\mathbb{G}(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$. From [**205**], X is a non-Kähler balanced manifold. The holomorphic 1–forms dx, dy, and $dz - xdy$ are left-invariant by the group $\mathbb{G}(\mathbb{Z} + \sqrt{-1}\mathbb{Z})$, yielding a global frame for the cotangent bundle. In particular, X admits a Hermitian metric which is Chern-flat, and the metric is (Chern) Kähler-like (albeit for trivial reasons).

2.3. On the Curvature of a Hermitian Metric

Granted the theory of Hermitian (in particular, Gauduchon) connections developed in the previous section, we want to extend the discussion of §2.1 to the Hermitian category.

The Ricci curvatures. In contrast with the Riemannian curvature tensor, the presence of torsion in the Chern connection produces distinct Ricci curvatures by taking various traces:

Definition 2.3.1. Let (X, ω_g) be a Hermitian manifold. Let ${}^tR$ denote the t–Gauduchon curvature tensor. The *t–Gauduchon–Ricci forms* of ω_g are defined in a local frame $\{e_k\}$ by

$$\begin{aligned} {}^t\mathrm{Ric}^{(1)}_{\omega_g} &:= \sqrt{-1}\,{}^t\mathrm{Ric}^{(1)}_{k\bar{\ell}} e_k \wedge \bar{e}_\ell \ := \ \sqrt{-1} g^{i\bar{j}} \left({}^tR_{k\bar{\ell}i\bar{j}} \right) e_k \wedge \bar{e}_\ell, \\ {}^t\mathrm{Ric}^{(2)}_{\omega_g} &:= \sqrt{-1}\,{}^t\mathrm{Ric}^{(2)}_{k\bar{\ell}} e_k \wedge \bar{e}_\ell \ := \ \sqrt{-1} g^{i\bar{j}} \left({}^tR_{i\bar{j}k\bar{\ell}} \right) e_k \wedge \bar{e}_\ell, \\ {}^t\mathrm{Ric}^{(3)}_{\omega_g} &:= \sqrt{-1}\,{}^t\mathrm{Ric}^{(3)}_{k\bar{\ell}} e_k \wedge \bar{e}_\ell \ := \ \sqrt{-1} g^{i\bar{j}} \left({}^tR_{k\bar{j}i\bar{\ell}} \right) e_k \wedge \bar{e}_\ell, \\ {}^t\mathrm{Ric}^{(4)}_{\omega_g} &:= \sqrt{-1}\,{}^t\mathrm{Ric}^{(4)}_{k\bar{\ell}} e_k \wedge \bar{e}_\ell \ := \ \sqrt{-1} g^{i\bar{j}} \left({}^tR_{i\bar{\ell}k\bar{j}} \right) e_k \wedge \bar{e}_\ell. \end{aligned}$$

Remark 2.3.2. We refer to ${}^t\mathrm{Ric}^{(k)}_{\omega_g}$ as the kth t–Gauduchon–Ricci curvature (or kth t–Gauduchon–Ricci form). We also note that the above definition only considers the $(1,1)$–part of the 'Ricci curvatures' obtained by taking various traces of the t–Gauduchon curvature tensor. In general, the Gauduchon–Ricci curvatures will not be of type $(1,1)$, but for many applications (e.g., the Schwarz lemma), only the $(1,1)$–part of the Ricci curvatures appear.

Let us consider first, the Ricci curvatures of the Chern connection ${}^c\nabla$. Because of the conjugate symmetry (2.2.11) of the Chern curvature tensor, we have the following:

Proposition 2.3.3. Let (X, ω_g) be a Hermitian manifold.

(i) The first and second Chern–Ricci curvatures ${}^c\mathrm{Ric}^{(1)}_{\omega_g}$ and ${}^c\mathrm{Ric}^{(2)}_{\omega_g}$ are real.

(ii) The third and fourth Chern–Ricci curvatures are conjugate to each other:

$$\overline{{}^c\mathrm{Ric}^{(3)}_{\omega_g}} \ = \ {}^c\mathrm{Ric}^{(4)}_{\omega_g}.$$

Proof. We omit the superscript c for the moment and write

$$\begin{aligned} \overline{\mathrm{Ric}^{(1)}_{i\bar{j}}} &= \overline{g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}} \ = \ g^{\ell\bar{k}} R_{j\bar{i}\ell\bar{k}} \ = \ \mathrm{Ric}^{(1)}_{j\bar{i}}, \\ \overline{\mathrm{Ric}^{(2)}_{k\bar{\ell}}} &= \overline{g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}} \ = \ g^{j\bar{i}} R_{j\bar{i}\ell\bar{k}} \ = \ \mathrm{Ric}^{(2)}_{\ell\bar{k}}. \end{aligned}$$

This proves the first statement. For the second statement, we similarly argue:

$$\mathrm{Ric}^{(4)}_{k\bar{\ell}} \ := \ g^{i\bar{j}} R_{k\bar{j}i\bar{\ell}} \ = \ \overline{g^{j\bar{i}} R_{j\bar{k}\ell\bar{i}}} \ = \ \overline{\mathrm{Ric}^{(3)}_{\ell\bar{k}}}.$$

$\square$

Proposition 2.3.4. Let (X, ω_g) be a Hermitian manifold. The first Chern–Ricci form is locally $\partial\bar{\partial}$–exact, with

$$^c\mathrm{Ric}^{(1)}_{\omega_g} \;=\; -\sqrt{-1}\partial\overline{\partial}\log(\omega_g^n).$$

In particular, $^c\mathrm{Ric}^{(1)}_{\omega_g}$ represents the first Bott–Chern class $c_1^{\mathrm{BC}}(K_X^{-1})$ of the anti-canonical bundle K_X^{-1}.

Proof. Write

$$\begin{aligned}
^c\mathrm{Ric}^{(1)}_{i\bar{j}} \;&=\; -\sum_{k,\ell=1}^{n} g^{k\bar{\ell}}\left(\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \overline{z}_j} - \sum_{p,q=1}^{n} g^{p\bar{q}}\frac{\partial g_{k\bar{q}}}{\partial z_i}\frac{\partial g_{p\bar{\ell}}}{\partial \overline{z}_j}\right) \\
&=\; -\sum_{k,\ell=1}^{n} g^{k\bar{\ell}}\left(\frac{\partial}{\partial z_i}\left(\frac{\partial g_{k\bar{\ell}}}{\partial \overline{z}_j}\right) + \sum_{p,q=1}^{n} g_{k\bar{q}}\frac{\partial g^{p\bar{q}}}{\partial z_i}\frac{\partial g_{p\bar{\ell}}}{\partial \overline{z}_j}\right) \\
&=\; -\sum_{k,\ell=1}^{n}\left[g^{k\bar{\ell}}\frac{\partial}{\partial z_i}\left(\frac{\partial g_{k\bar{\ell}}}{\partial \overline{z}_j}\right) + \frac{\partial g^{k\bar{\ell}}}{\partial z_i}\frac{\partial g_{k\bar{\ell}}}{\partial \overline{z}_j}\right] \\
&=\; -\frac{\partial}{\partial z_i}\left(\sum_{k,\ell=1} g^{k\bar{\ell}}\frac{\partial g_{k\bar{\ell}}}{\partial \overline{z}_j}\right) \\
&=\; -\frac{\partial}{\partial z_i}\left(\frac{\partial}{\partial \overline{z}_j}\log\det(g)\right),
\end{aligned}$$

where the last equality follows from the formula

$$\frac{\partial}{\partial \overline{z}_j}\log\det(g) \;=\; \sum_{k,\ell=1}^{n} g^{k\bar{\ell}}\frac{\partial g_{k\bar{\ell}}}{\partial \overline{z}_j},$$

which can be deduced from Cramer's rule. The final assertion follows from the fact that the volume form ω_g^n defines a Hermitian metric on the canonical bundle K_X. The curvature form of this Hermitian metric is then $\sqrt{-1}\partial\bar{\partial}\log(\omega_g^n)$. The curvature of the induced metric on the dual bundle is then $-\sqrt{-1}\partial\bar{\partial}\log(\omega_g^n) = {}^c\mathrm{Ric}^{(1)}_{\omega_g}$. $\square$

Remark 2.3.5. One can also realize the closedness of the Ricci form of a Kähler metric ω as a consequence of the second Bianchi identity:

$$
\begin{aligned}
d\mathrm{Ric}(\omega) &= \sqrt{-1}\sum_{\alpha,i,j=1}^{n}\left(\nabla_\alpha \mathrm{Ric}_{i\bar{j}} dz_\alpha\wedge dz_i\wedge d\bar{z}_j + \nabla_{\bar{\alpha}}\mathrm{Ric}_{i\bar{j}} d\bar{z}_\alpha\wedge dz_i\wedge d\bar{z}_j\right)\\
&= \sqrt{-1}\sum_{\alpha,i,j=1}^{n}\left(\nabla_i \mathrm{Ric}_{\alpha\bar{j}} dz_\alpha\wedge dz_i\wedge d\bar{z}_j + \nabla_{\bar{j}}\mathrm{Ric}_{i\bar{\alpha}} d\bar{z}_\alpha\wedge dz_i\wedge d\bar{z}_j\right)\\
&= -\sqrt{-1}\sum_{\alpha,i,j=1}^{n}\left(\nabla_i \mathrm{Ric}_{\alpha\bar{j}} dz_i\wedge dz_\alpha\wedge d\bar{z}_j + \nabla_{\bar{j}}\mathrm{Ric}_{i\bar{\alpha}} d\bar{z}_j\wedge dz_i\wedge d\bar{z}_\alpha\right)\\
&= -\sqrt{-1}\sum_{\alpha,i,j=1}^{n}\left(\nabla_\alpha \mathrm{Ric}_{i\bar{j}} dz_\alpha\wedge dz_i\wedge d\bar{z}_j + \nabla_{\bar{\alpha}}\mathrm{Ric}_{i\bar{j}} d\bar{z}_\alpha\wedge dz_i\wedge d\bar{z}_j\right).
\end{aligned}
$$

Corollary 2.3.6. Let (X,ω_g) be a Hermitian $\partial\bar{\partial}$–manifold. Then the first Chern–Ricci form ${}^c\mathrm{Ric}_{\omega_g}^{(1)}$ represents the first Chern class $c_1(K_X^{-1})$ in $H^2_{\mathrm{DR}}(X,\mathbb{R})$.

The Calabi Conjecture. It is natural to ask whether any presentative of $c_1^{\mathrm{BC}}(X)$ is given by the first Chern–Ricci curvature of a Hermitian metric. For Kähler metrics, this is the famous Calabi conjecture:

Conjecture 2.3.7. (Calabi). Let (X,ω) be a compact Kähler manifold. Given any representative α of $c_1(K_X^{-1})$, there is a smooth function $\varphi\in\mathcal{C}^\infty(X,\mathbb{R})$ such that $\omega_\varphi := \omega+\sqrt{-1}\partial\bar{\partial}\varphi > 0$ and $\mathrm{Ric}_{\omega_\varphi} = \alpha$.

The Calabi conjecture implies that if X is a compact Kähler manifold with definite or trivial first Chern class, then X admits a Kähler metric with correspondingly definite or trivial Ricci curvature. Calabi [**68**] showed that the problem reduces to a complex Monge–Ampère equation. Indeed, let $\alpha = \lambda[\omega]$ for some $\lambda\in\mathbb{R}$. Then $\mathrm{Ric}_{\omega_\varphi} = \lambda\omega_\varphi$ and

$$
\begin{aligned}
\mathrm{Ric}_{\omega_\varphi} = \lambda\omega_\varphi \quad &\iff \quad \mathrm{Ric}_{\omega_\varphi} - \mathrm{Ric}_\omega = \lambda\omega_\varphi - \lambda\omega + \lambda\omega - \mathrm{Ric}_\omega\\
&\iff \quad \mathrm{Ric}_{\omega_\varphi} - \mathrm{Ric}_\omega = \lambda\sqrt{-1}\partial\bar{\partial}\varphi - \sqrt{-1}\partial\bar{\partial}f\\
&\iff \quad -\sqrt{-1}\partial\bar{\partial}\log(\omega_\varphi^n) + \sqrt{-1}\partial\bar{\partial}\log(\omega^n) = \sqrt{-1}\partial\bar{\partial}(\lambda\varphi - f)\\
&\iff \quad \omega_\varphi^n = e^{f-\lambda\varphi}\omega^n.
\end{aligned}
$$

Hence, the resolution of the Calabi conjecture is equivalent to the solvability of the complex Monge–Ampère equation

$$\omega_\varphi^n = e^{f-\lambda\varphi}\omega^n,$$

where $f: X\to\mathbb{R}$ is a smooth function subject to the normalization condition

$$\int_X e^{f-\lambda\varphi}\omega^n = \int_X \omega_\varphi^n.$$

For compact Kähler manifolds with $c_1(K_X^{-1}) > 0$, there are obstructions to the existence of Kähler–Einstein metrics. The first obstruction was observed by Matsushima [**204**], noticing that it was necessary for the Lie group of holomorphic automorphisms to be reductive, i.e., the complexification of a real compact Lie group. Later obstructions were found by Futaki [**128**] who introduced a Lie algebra character for the Lie algebra of holomorphic vector fields, which necessarily vanished for the existence of Kähler–Einstein metrics on Fano manifolds. This led to a folklore conjecture that the only obstructions were to arise from the Lie algebra of holomorphic vectors. An example of Mukai (see, e.g., [**279**]), however, provides an example of a Fano manifold with no non-trivial holomorphic vector fields but does not admit a Kähler–Einstein metric. In [**279**], Tian proposed the notion of K–stability which he showed to be a necessary condition for existence, and conjectured that it was also sufficient. This so-called Yau–Tian–Donaldson conjecture (YTD conjecture) was proved in 2012 by Tian [**280**] and Chen–Donaldson–Sun [**87**]. We will discuss the Calabi conjecture and its resolution in Chapter 3.3.

The Second Chern–Ricci Curvature. We have seen that the first Chern–Ricci curvature of a Hermitian metric is intimately related to the (anti-)canonical bundle K_X and has a cohomological nature. The second Chern–Ricci curvature is not a cohomological object, in general, and is much more mysterious. We will see that in the subsequent sections that the second Chern–Ricci curvature naturally appears in the Schwarz lemma, but is unfortunately, not within the jurisdiction of a complex Monge–Ampère equation. The second Chern–Ricci curvature, however, does support the subbundle decreasing property:

Proposition 2.3.8. Let (X, ω) be a Hermitian manifold. Let $f : Y \hookrightarrow X$ be an embedded complex submanifold. Then

$$ {}^c\mathrm{Ric}^{(2)}_{f^*\omega} \leq {}^c\mathrm{Ric}^{(2)}_{\omega}. $$

PROOF. Since the curvature of the Chern connection satisfies the subbundle decreasing property, and the trace commutes with restriction, the result is immediate. □

Example 2.3.9. A Stein manifold supports a Hermitian metric (induced by the ambient Euclidean metric) of non-positive second Chern–Ricci curvature.

Remark 2.3.10. In general, the first and second Chern–Ricci curvatures are not comparable. For instance, the standard metric on the Hopf manifold $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ has ${}^c\mathrm{Ric}^{(1)}_{\omega_0} \geq 0$ and ${}^c\mathrm{Ric}^{(2)}_{\omega_0} > 0$.

Question 2.3.11. Is there an example of a compact Hermitian manifold such that the first Chern–Ricci curvature does not satisfy the subbundle decreasing property?

Of course, the second Ricci curvature can be defined for any complex-linear connection on a Hermitian vector bundle $(\mathcal{E}, h) \to X$. If we consider the Hermitian complex-analytic connection on $(\mathcal{E}, h)$ with curvature form $\Theta^{(\mathcal{E},h)}$, the second Ricci curvature is defined by $\text{tr}_\omega \Theta^{(\mathcal{E},h)}$. From [**314**, Theorem 3.7], we have:

Theorem 2.3.12. (Yang). Let $(\mathcal{E}, h) \to X$ be a Hermitian vector bundle with quasi-positive second Ricci curvature. Let $\mathcal{L} \to X$ be a pseudo-effective line bundle. Then

$$H^0(X, \mathcal{E}^* \otimes \mathcal{L}^*) \ = \ 0.$$

Theorem 2.3.13. (Yang). Let $(\mathcal{E}, h) \to X$ be a Hermitian vector bundle over a compact complex manifold X. Suppose there is a smooth Hermitian metric ω on X such that $\text{tr}_\omega \Theta^{(\mathcal{E},h)}$ is quasi-positive. Then

(i) any invertible subsheaf $\mathcal{L}$ of $\mathcal{O}_X(\otimes^k \mathcal{E}^*)$, for $k \geq 1$, is not pseudo-effective.
(ii) $\det(\mathcal{E}^*)$ is not pseudo-effective.

Remark 2.3.14. The key point in the above theorem is that the line bundle $\mathcal{L}$ in statement (i) is only required to be a subsheaf, not a subbundle. In the case that it is a subbundle, this result was shown by Campana–Demailly–Paun [**71**].

Before proving the main theorem (due to Yang) concerning the second Chern–Ricci curvature, we mention the following result of Boucksom–Demailly–Paun–Peternell [**42**]:

Theorem 2.3.15. (BDPP). Let X be a projective manifold such that K_X is not pseudo-effective. Then X is uniruled.

The above theorem has a number of important consequences. For instance, we have:

Corollary 2.3.16. Let X be a projective Kobayashi hyperbolic manifold. Then the canonical bundle K_X is pseudo-effective.

We will consider a number of more general statements relating the hyperbolicity of a complex manifold to the positivity of the canonical bundle in Chapter 3. For the moment, we state one of the main structure theorems for the second Chern–Ricci curvature:

Theorem 2.3.17. (Yang). Let X be a compact Kähler manifold. Suppose X admits a Hermitian metric with quasi-positive second Chern–Ricci curvature. Then X is projective and rationally connected; in particular, X is simply connected.

PROOF. From 2.3.12, taking $\mathcal{E} = T^{1,0}X$, we see that $H^0(X, (\Omega_X^{1,0})^{\otimes k}) = 0$. Since, for $1 \leq p \leq \dim_{\mathbb{C}}(X)$, the pth exterior power $\Lambda^p(\Omega_X^{1,0})$ is a sum of $(\Omega_X^{1,0})^{\otimes k}$ for suitably large k, we have

$$H^{p,0}_{\bar{\partial}}(X) \ \simeq \ H^0(X, \Omega_X^p) \ = \ 0.$$

In particular, $h^{2,0}(X) = 0$ and by Kodaira's projectivity criterion, X is projective. To show that X is rationally connected, we observe that statement (ii) of 2.3.13 implies that K_X is not pseudo-effective. Hence, by 2.3.15, X is uniruled. Let $\varphi : X \dashrightarrow Z$ denote the MRC fibration. By resolving the singularities if necessary, we may assume that Z is smooth and φ is a proper morphism. There are two cases: (i) Z is a point, or (ii) Z is a positive-dimensional variety which is not uniruled. Proceed by contradiction and suppose that X is not rationally connected. Then Z is a positive-dimensional variety which is not uniruled. Hence, by 2.3.15, the canonical bundle K_Z is pseudo-effective. Since K_Z is a direct summand of the vector bundle $(\Omega_Z^{1,0})^{\otimes k}$ for some large k, and $\mathcal{O}_Z((\Omega_Z^{1,0})^{\otimes k})$ is a subsheaf of $\mathcal{O}_X(\Omega_X^{1,0})^{\otimes k})$, we see that K_Z is a pseudo-effective invertible subsheaf of $\mathcal{O}_X((\Omega_X^{1,0})^{\otimes k})$. This contradicts statement (i) in 2.3.13. □

Corollary 2.3.18. Let X be a compact Kähler manifold. If X supports a Hermitian metric with quasi-positive first Chern–Ricci curvature, then X is projective and rationally connected; in particular, X is simply connected.

PROOF. Let ω be a Hermitian metric with quasi-positive first Chern–Ricci curvature. By Yau's theorem [318], there is a Kähler metric $\widetilde{\omega}$ such that $\text{Ric}_{\widetilde{\omega}} = {}^c\text{Ric}_{\omega}^{(1)}$. Since the Chern–Ricci curvatures of a Kähler metric all coincide, the result follows. □

From [298, Theorem 6.2], we have the following:

Corollary 2.3.19. Let $f : X \to Y$ be a holomorphic submersion from between compact Kähler manifolds. If X supports a Hermitian metric with quasi-positive second Chern–Ricci curvature, then $b_1(Y) = b_1(X) = 0$.

PROOF. From 2.3.17, we see that $h^{1,0}(X) = h^{0,1}(X) = 0$. From [298, Theorem 6.2], we know that a holomorphic submersion between compact Kähler manifolds satisfies $b_1(Y) \leq b_1(X)$. □

The Gauduchon–Ricci curvatures.

Notation 2.3.20. Building from [200, p. 17], we introduce the following notation:

$$T^{\diamondsuit}_{i\bar{j}} := g^{p\bar{q}} g_{k\bar{\ell}} T^k_{ip} \overline{T^{\ell}_{jq}}, \qquad T^{\circ}_{i\bar{j}} := g^{p\bar{q}} g^{s\bar{r}} g_{k\bar{j}} g_{i\bar{\ell}} T^k_{sp} \overline{T^{\ell}_{rq}}, \qquad T^{\heartsuit}_{i\bar{\ell}} := g^{p\bar{r}} g_{k\bar{r}} T^{\ell}_{iq} \overline{T^k_{qr}}.$$

In particular, if $g_{i\bar{j}} = \delta_{ij}$, then

$$T^{\diamondsuit}_{i\bar{j}} = T^k_{iq} \overline{T^k_{jq}}, \qquad T^{\circ}_{i\bar{j}} = T^j_{sp} \overline{T^i_{sp}}, \qquad T^{\heartsuit}_{i\bar{j}} = T^j_{iq} \overline{T^k_{qk}}.$$

Note that $T^{\circ}_{i\bar{j}} = (T \circ \overline{T})_{i\bar{j}}$ in the notation of [200].

Lemma 2.3.21. ([200, Lemma 3.4]). Let (X, ω_g) be a Hermitian manifold. For any point $p \in X$, there are local holomorphic coordinates $(z_1, ..., z_n)$ centered at p, such that

$$g_{i\bar{j}}(p) \;=\; \delta_{ij} \qquad \text{and} \qquad \Gamma^k_{ij}(p) \;=\; 0.$$

In particular, at p, we have

$$\Gamma^k_{\bar{j}i}(p) \;=\; \frac{\partial g_{i\bar{k}}}{\partial \bar{z}_j}(p) \;=\; -\frac{\partial g_{i\bar{j}}}{\partial \bar{z}_k}(p).$$

For the remaining Chern–Ricci curvatures, we have the following [200, Theorem 4.1]:

Proposition 2.3.22. Let (X, ω) be a Hermitian manifold. The Chern–Ricci curvatures are related by:

$$\begin{aligned}
{}^c\mathrm{Ric}^{(2)}_\omega &= {}^c\mathrm{Ric}^{(1)}_\omega - \sqrt{-1}\Lambda(\partial\bar{\partial}\omega) - (\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega) + {}^cT^\diamondsuit, \\
{}^c\mathrm{Ric}^{(3)}_\omega &= {}^c\mathrm{Ric}^{(1)}_\omega - \partial\partial^*\omega, \\
{}^c\mathrm{Ric}^{(4)}_\omega &= {}^c\mathrm{Ric}^{(1)}_\omega - \bar{\partial}\bar{\partial}^*\omega,
\end{aligned}$$

where ${}^cT^\diamondsuit$ is the $(1,1)$–form with components $T^\diamondsuit_{i\bar{j}} \;:=\; g^{p\bar{q}} g_{k\bar{\ell}} {}^cT^k_{ip} \overline{{}^cT^\ell_{jq}}$.

Proof. We fix a point $x_0 \in X$ and work in the coordinates of 2.3.21. Then

$$\begin{aligned}
{}^c\mathrm{Ric}^{(1)}_{i\bar{j}} &= -\sum_p \frac{\partial^2 g_{p\bar{p}}}{\partial z_i \partial \bar{z}_j} + \frac{1}{4} {}^cT^\diamondsuit_{i\bar{j}}, \\
{}^c\mathrm{Ric}^{(2)}_{i\bar{j}} &= -\sum_p \frac{\partial^2 g_{i\bar{j}}}{\partial z_p \partial \bar{z}_p} + \frac{1}{4} {}^cT^\diamondsuit_{i\bar{j}}, \\
{}^c\mathrm{Ric}^{(3)}_{i\bar{j}} &= -\sum_p \frac{\partial^2 g_{p\bar{j}}}{\partial z_i \partial \bar{z}_p} - \frac{1}{4} {}^cT^\diamondsuit_{i\bar{j}} \\
{}^c\mathrm{Ric}^{(4)}_{i\bar{j}} &= -\sum_p \frac{\partial^2 g_{i\bar{p}}}{\partial z_p \partial \bar{z}_j} - \frac{1}{4} {}^cT^\diamondsuit_{i\bar{j}}.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
(\partial\partial^*\omega)_{i\bar{j}} &= \sum_p \left(\frac{\partial^2 g_{p\bar{j}}}{\partial z_i \partial \bar{z}_p} - \frac{\partial^2 g_{p\bar{p}}}{\partial z_i \partial \bar{z}_j} \right) + \frac{1}{2} {}^cT^\diamondsuit_{i\bar{j}}, \\
(\bar{\partial}\bar{\partial}^*\omega)_{i\bar{j}} &= \sum_p \left(\frac{\partial^2 g_{i\bar{p}}}{\partial z_p \partial \bar{z}_j} - \frac{\partial^2 g_{p\bar{p}}}{\partial z_i \partial \bar{z}_j} \right) + \frac{1}{2} {}^cT^\diamondsuit_{i\bar{j}}.
\end{aligned}$$

We immediately see the relationship between the first Chern–Ricci curvature and the third and fourth Chern–Ricci curvatures. To prove the relationship between the first and second

Chern–Ricci curvature, we note that a straightforward calculation gives

$$\Lambda(\partial\bar\partial\omega)_{i\bar j} \;=\; \sum_p \left(\frac{\partial^2 g_{i\bar j}}{\partial z_p \partial \bar z_p} + \frac{\partial^2 g_{p\bar p}}{\partial z_i \partial \bar z_j} - \frac{\partial^2 g_{i\bar p}}{\partial z_p \partial \bar z_j} - \frac{\partial^2 g_{p\bar j}}{\partial z_i \partial \bar z_p} \right).$$

Combining these expressions, we deduce the relationship between the first and second Chern–Ricci curvatures. □

Since $\partial^*\omega = \bar\partial^*\omega = 0$ if ω is balanced, we see that

Corollary 2.3.23. Let (X,ω) be a balanced manifold. Then the Chern–Ricci curvatures are related by

$$\begin{aligned} {}^c\mathrm{Ric}^{(2)}_\omega &= {}^c\mathrm{Ric}^{(1)}_\omega - \sqrt{-1}\Lambda(\partial\bar\partial\omega) + {}^cT^\diamondsuit, \\ {}^c\mathrm{Ric}^{(3)}_\omega &= {}^c\mathrm{Ric}^{(4)}_\omega \;=\; {}^c\mathrm{Ric}^{(1)}_\omega. \end{aligned}$$

Proposition 2.3.24. Let (X,ω) be a Hermitian manifold. The Gauduchon–Ricci curvatures are given by

$$\begin{aligned} {}^t\mathrm{Ric}^{(1)}_{\omega_g} &= {}^c\mathrm{Ric}^{(1)}_{\omega_g} + \frac{(t-1)}{2}(\partial\partial^*\omega_g + \bar\partial\bar\partial^*\omega_g), \\ {}^t\mathrm{Ric}^{(2)}_{\omega_g} &= t\,{}^c\mathrm{Ric}^{(2)}_{\omega_g} + (1-t){}^c\mathrm{Ric}^{(1)}_{\omega_g} + \frac{(t-1)}{2}(\partial\partial^*\omega_g + \bar\partial\bar\partial^*\omega_g) + \frac{(1-t)^2}{4}T^\diamondsuit + \frac{(t-1)}{2}T^\circ, \\ {}^t\mathrm{Ric}^{(3)}_{\omega_g} &= t\,{}^c\mathrm{Ric}^{(3)}_{\omega_g} + \frac{(1-t)}{2}\left({}^c\mathrm{Ric}^{(1)}_{\omega_g} + {}^c\mathrm{Ric}^{(2)}_{\omega_g}\right) - \frac{(1-t)^2}{4}T^\diamondsuit + \frac{(1-t)^2}{4}T^\heartsuit, \\ {}^t\mathrm{Ric}^{(4)}_{\omega_g} &= t\,{}^c\mathrm{Ric}^{(4)}_{\omega_g} + \frac{(1-t)}{2}\left({}^c\mathrm{Ric}^{(1)}_{\omega_g} + {}^c\mathrm{Ric}^{(2)}_{\omega_g}\right) - \frac{(1-t)^2}{4}T^\diamondsuit + \frac{(1-t)^2}{4}\overline{T^\heartsuit}. \end{aligned}$$

PROOF. Choose coordinates such that $g_{i\bar j}(p) = \delta_{ij}$ and $\Gamma^k_{ij}(p) = 0$. In these coordinates we have

$$\begin{aligned} {}^c\mathrm{Ric}^{(1)}_{i\bar j} &= -\frac{\partial^2 g_{kk}}{\partial z_i \partial \bar z_j} + \frac{1}{4}T^\diamondsuit_{i\bar j}, \\ (\partial\partial^*\omega)_{i\bar j} &= \frac{\partial^2 g_{k\bar j}}{\partial z_i \partial \bar z_k} - \frac{\partial^2 g_{k\bar k}}{\partial z_i \partial \bar z_j} + \frac{1}{2}T^\diamondsuit_{i\bar j}, \quad (\bar\partial\bar\partial^*\omega)_{i\bar j} = \frac{\partial^2 g_{i\bar k}}{\partial z_k \partial \bar z_j} - \frac{\partial^2 g_{k\bar k}}{\partial z_i \partial \bar z_j} + \frac{1}{2}T^\diamondsuit_{i\bar j}. \end{aligned}$$

From the above curvature formula for the Gauduchon connection, we have

$$\begin{aligned} {}^t\mathrm{Ric}^{(1)}_{i\bar j} &= \left(\frac{1-t}{2}\right)\left[-\frac{\partial^2 g_{k\bar j}}{\partial z_i \partial \bar z_k} - \frac{\partial^2 g_{i\bar k}}{\partial z_k \partial \bar z_j} + \frac{\partial^2 g_{k\bar k}}{\partial z_i \partial \bar z_j}\right] - \left(\frac{1+t}{2}\right)\frac{\partial^2 g_{k\bar k}}{\partial z_i \partial \bar z_j} \\ &\quad - \frac{(1-t)^2}{4}T^\diamondsuit_{i\bar j} + \frac{t^2}{4}T^\diamondsuit_{i\bar j} \\ &= \left(\frac{t-1}{2}\right)(\partial\partial^*\omega + \bar\partial\bar\partial^*\omega) + \left(\frac{1-t}{2}\right)\left(2\,{}^c\mathrm{Ric}^{(1)}_{i\bar j} + \frac{1}{2}T^\diamondsuit_{i\bar j}\right) + t\,{}^c\mathrm{Ric}^{(1)}_{i\bar j} + \left(\frac{t-1}{4}\right)T^\diamondsuit_{i\bar j} \\ &= {}^c\mathrm{Ric}^{(1)}_{i\bar j} + \frac{(t-1)}{2}(\partial\partial^*\omega + \bar\partial\bar\partial^*\omega). \end{aligned}$$

Similarly, we compute

$$\begin{aligned}
{}^t\mathrm{Ric}^{(2)}_{k\bar{\ell}} &= \left(\frac{1-t}{2}\right)\left[\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_i} - \frac{\partial^2 g_{k\bar{i}}}{\partial z_i \partial \bar{z}_\ell} - \frac{\partial^2 g_{i\bar{\ell}}}{\partial z_k \partial \bar{z}_i}\right] - \left(\frac{1+t}{2}\right)\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_i} \\
&\qquad + \left(\frac{1-t}{2}\right)^2 \sum_p T^\ell_{ip}\overline{T^k_{qi}} + \frac{t^2}{4}\sum_p T^p_{ik}\overline{T^p_{i\ell}} \\
&= \left(\frac{1-t}{2}\right)\left[\left(-{}^c\mathrm{Ric}^{(2)}_{k\bar{\ell}} + \frac{1}{4}T^\diamondsuit_{k\bar{\ell}}\right) - \left(-{}^c\mathrm{Ric}^{(4)}_{k\bar{\ell}} - \frac{1}{4}T^\diamondsuit_{k\bar{\ell}}\right) - \left(-{}^c\mathrm{Ric}^{(3)}_{k\bar{\ell}} - \frac{1}{4}T^\diamondsuit_{k\bar{\ell}}\right)\right] \\
&\qquad - \left(\frac{1+t}{2}\right)\left(-{}^c\mathrm{Ric}^{(2)}_{k\bar{\ell}} + \frac{1}{4}T^\diamondsuit_{k\bar{\ell}}\right) - \left(\frac{1-t}{2}\right)T^\circ_{k\bar{\ell}} + \frac{t^2}{4}T^\diamondsuit_{k\bar{\ell}} \\
&= t^c\mathrm{Ric}^{(2)}_{k\bar{\ell}} + \frac{(1-2t+t^2)}{4}T^\diamondsuit_{k\bar{\ell}} + \left(\frac{1-t}{2}\right)\left({}^c\mathrm{Ric}^{(3)}_{k\bar{\ell}} + {}^c\mathrm{Ric}^{(4)}_{k\bar{\ell}}\right) + \frac{(t-1)}{2}T^\circ_{k\bar{\ell}} \\
&= t^c\mathrm{Ric}^{(2)}_{k\bar{\ell}} + \frac{(1-t)^2}{4}T^\diamondsuit_{k\bar{\ell}} + \left(\frac{1-t}{2}\right)\left({}^c\mathrm{Ric}^{(1)}_{k\bar{\ell}} - \partial\partial^*\omega + {}^c\mathrm{Ric}^{(1)}_{k\bar{\ell}} - \bar{\partial}\bar{\partial}^*\omega\right) + \frac{(t-1)}{2}T^\circ_{k\bar{\ell}} \\
&= t^c\mathrm{Ric}^{(2)}_{k\bar{\ell}} + (1-t)^c\mathrm{Ric}^{(1)}_{k\bar{\ell}} + \frac{(t-1)}{2}(\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega) + \frac{(1-t)^2}{4}T^\diamondsuit_{k\bar{\ell}} + \frac{(t-1)}{2}T^\circ_{k\bar{\ell}}.
\end{aligned}$$

Proceed as before, and compute

$$\begin{aligned}
{}^t\mathrm{Ric}^{(3)}_{i\bar{\ell}} &= \frac{(1-t)}{2}\left[\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_k} - \frac{\partial^2 g_{k\bar{k}}}{\partial z_i \partial \bar{z}_\ell} - \frac{\partial^2 g_{i\bar{\ell}}}{\partial z_k \partial \bar{z}_k}\right] - \frac{(1+t)}{2}\frac{\partial^2 g_{k\bar{\ell}}}{\partial z_i \partial \bar{z}_k} \\
&\qquad + \frac{(1-t)^2}{4}\sum_{k,q} T^\ell_{iq}\overline{T^k_{qk}} + \frac{t^2}{4}\sum_{k,q} T^q_{ik}\overline{T^q_{k\ell}} \\
&= \frac{(1-t)}{2}\left[-\left({}^c\mathrm{Ric}^{(3)}_{i\bar{\ell}} - \frac{1}{4}T^\diamondsuit_{i\bar{\ell}}\right) - \left({}^c\mathrm{Ric}^{(1)}_{i\bar{\ell}} + \frac{1}{4}T^\diamondsuit_{i\bar{\ell}}\right) - \left({}^c\mathrm{Ric}^{(2)}_{i\bar{\ell}} + \frac{1}{4}T^\diamondsuit_{i\bar{\ell}}\right)\right] \\
&\qquad + \frac{(1+t)}{2}\left({}^c\mathrm{Ric}^{(3)}_{i\bar{\ell}} - \frac{1}{4}T^\diamondsuit_{i\bar{\ell}}\right) + \frac{(1-t)^2}{4}T^\heartsuit_{i\bar{\ell}} - \frac{t^2}{4}T^\diamondsuit_{i\bar{\ell}} \\
&= t^c\mathrm{Ric}^{(3)}_{i\bar{\ell}} + \frac{(1-t)}{2}\left({}^c\mathrm{Ric}^{(1)}_{i\bar{\ell}} + {}^c\mathrm{Ric}^{(2)}_{i\bar{\ell}}\right) - \frac{(1-t)^2}{4}T^\diamondsuit_{i\bar{\ell}} + \frac{(1-t)^2}{4}T^\heartsuit_{i\bar{\ell}}.
\end{aligned}$$

The formula for ${}^t\mathrm{Ric}^{(4)}_{\omega_g}$ is obtained in the same manner. □

Corollary 2.3.25. Let (X, ω) be a compact Hermitian manifold. If ${}^t\mathrm{Ric}^{(1)}_\omega > 0$ or ${}^t\mathrm{Ric}^{(1)}_\omega < 0$ for some $t \in \mathbb{R}$, then X is in the Fujiki class $\mathcal{C}$ if and only if X is Kähler.

PROOF. 2.3.24 shows that for all $t \in \mathbb{R}$, the first Gauduchon–Ricci curvature is $\partial\bar{\partial}$–closed. Hence, if ${}^t\mathrm{Ric}^{(1)}_\omega$ is definite, we can produce a pluriclosed metric by setting $\omega_t := {}^t\mathrm{Ric}^{(1)}_\omega$ or $\omega_t := -{}^t\mathrm{Ric}^{(1)}_\omega$, depending on the sign of ${}^t\mathrm{Ric}^{(1)}_\omega$. By a theorem of Chiose [96], a compact complex manifold in the Fujiki class $\mathcal{C}$ admits a pluriclosed metric if and only if it is Kähler. □

Corollary 2.3.26. Let (X,ω) be a Hermitian manifold. The Bismut–Ricci curvatures are related by

$$\begin{aligned} {}^b\mathrm{Ric}^{(1)}_\omega &= {}^c\mathrm{Ric}^{(1)}_\omega - (\partial\partial^*\omega + \bar\partial\bar\partial^*\omega), \\ {}^b\mathrm{Ric}^{(2)}_\omega &= {}^b\mathrm{Ric}^{(1)}_\omega + (\partial\partial^*\omega + \bar\partial\bar\partial^*\omega) + \sqrt{-1}\Lambda(\partial\bar\partial\omega) - {}^cT^\circ, \\ {}^b\mathrm{Ric}^{(3)}_\omega &= {}^b\mathrm{Ric}^{(1)}_\omega + \partial\partial^*\omega - \sqrt{-1}\Lambda(\partial\bar\partial\omega) + {}^cT^\heartsuit \\ {}^b\mathrm{Ric}^{(4)}_\omega &= {}^b\mathrm{Ric}^{(1)}_\omega + \bar\partial\bar\partial^*\omega - \sqrt{-1}\Lambda(\partial\bar\partial\omega) + \overline{{}^cT^\heartsuit}. \end{aligned}$$

Corollary 2.3.27. Let (X,ω) be a compact balanced manifold. Then the Gauduchon–Ricci curvatures are given by

$$\begin{aligned} {}^t\mathrm{Ric}^{(1)}_\omega &= {}^c\mathrm{Ric}^{(1)}_\omega, \\ {}^t\mathrm{Ric}^{(2)}_\omega &= t\,{}^c\mathrm{Ric}^{(2)}_\omega + (1-t){}^c\mathrm{Ric}^{(1)}_\omega + \frac{(1-t)^2}{4}T^\diamondsuit + \frac{(t-1)}{2}T^\circ, \\ {}^t\mathrm{Ric}^{(3)}_\omega &= t\,{}^c\mathrm{Ric}^{(3)}_\omega + \frac{(1-t)}{2}\left({}^c\mathrm{Ric}^{(1)}_\omega + {}^c\mathrm{Ric}^{(2)}_\omega\right) - \frac{(1-t)^2}{4}T^\diamondsuit, \\ {}^t\mathrm{Ric}^{(4)}_\omega &= t\,{}^c\mathrm{Ric}^{(4)}_\omega + \frac{(1-t)}{2}\left({}^c\mathrm{Ric}^{(1)}_\omega + {}^c\mathrm{Ric}^{(2)}_\omega\right) - \frac{(1-t)^2}{4}T^\diamondsuit. \end{aligned}$$

PROOF. It suffices to show that if ω is balanced, then $\partial\partial^*\omega + \bar\partial\bar\partial^*\omega = 0$ and $T^\heartsuit = 0$. The first assertion is well-known (see, e.g., [85, Lemma 2.3]). For the latter claim, we see that if the metric is balanced, then

$$T^\heartsuit_{k\bar k}\lambda_k^2 \;=\; \sum_{p,q,k} T^k_{kq}\overline{T^p_{qp}}\lambda_k^2 \;=\; \sum_{q,k}\lambda_k^2 T^k_{kq}\left(\sum_p \overline{T^p_{qp}}\right) \;=\; 0.$$

□

Corollary 2.3.28. Let (X,ω) be a balanced manifold. The Bismut–Ricci curvatures afford the relations

$$\begin{aligned} {}^b\mathrm{Ric}^{(1)}_\omega &= {}^c\mathrm{Ric}^{(1)}_\omega \\ {}^b\mathrm{Ric}^{(2)}_\omega &= {}^b\mathrm{Ric}^{(1)}_\omega + \sqrt{-1}\Lambda(\partial\bar\partial\omega) - {}^cT^\circ \\ {}^b\mathrm{Ric}^{(3)}_\omega &= {}^b\mathrm{Ric}^{(4)}_\omega \;=\; {}^b\mathrm{Ric}^{(1)}_\omega - \sqrt{-1}\Lambda(\partial\bar\partial\omega). \end{aligned}$$

Corollary 2.3.29. Let (X,ω) be a balanced manifold.

(i) ${}^b\mathrm{Ric}^{(2)}_\omega \le {}^b\mathrm{Ric}^{(1)}_\omega + \sqrt{-1}\Lambda(\partial\bar\partial\omega)$.
(ii) ${}^b\mathrm{Ric}^{(3)}_\omega = {}^b\mathrm{Ric}^{(1)}_\omega$ or ${}^b\mathrm{Ric}^{(4)}_\omega = {}^b\mathrm{Ric}^{(1)}_\omega$ if and only if ω is Kähler.

Recall that for a Hermitian line bundle $(\mathcal{L},h) \to X$ over a complex manifold X, the first Bott–Chern class $c_1^{\mathrm{BC}}(\mathcal{L}) \in H^{1,1}_{\mathrm{BC}}(X)$ and first Aeppli–Chern class $c_1^{\mathrm{AC}}(\mathcal{L}) \in H^{1,1}_{\mathrm{A}}(X)$ are represented by the curvature form $\Theta^{(\mathcal{L},h)} = \sqrt{-1}\partial\bar\partial\log(h)$. This definition is well-defined, independent of the specific choice of Hermitian metric. Indeed, if $\tilde{h}$ is another Hermitian metric on $\mathcal{L} \to X$, then the difference of the curvature forms $\Theta^{(\mathcal{L},\tilde{h})} - \Theta^{(\mathcal{L},h)} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log\left(\frac{\tilde{h}}{h}\right)$ is globally $\partial\bar{\partial}$–exact.

Remark 2.3.30. Liu–Yang [200] defined the first Aeppli Chern class as the cohomology class represented by the first Lichnerowicz Ricci curvature. The above result indicates that the choice of $t = 0$ is unnecessary; the first Gauduchon–Ricci curvature ${}^t\mathrm{Ric}_\omega^{(1)}$ represents the same cohomology class.

From 2.3.24, we observe that the results in [200] can be generalized substantially. For instance, we have the following extension of [200, Theorem 3.14]:

Proposition 2.3.31. Let (X, ω) be a Hermitian manifold. The first Gauduchon–Ricci form ${}^t\mathrm{Ric}_\omega^{(1)}$ represents $c_1^{\mathrm{AC}}(K_X^{-1}) \in H_A^{1,1}(X)$ for all $t \in \mathbb{R}$. Moreover,

(i) ${}^t\mathrm{Ric}_\omega^{(1)}$ is d–closed if and only if $\partial\bar{\partial}\bar{\partial}^*\omega = 0$.
(ii) If $\bar{\partial}\partial^*\omega$, then ${}^t\mathrm{Ric}_\omega^{(1)}$ represents the $c_1(K_X^{-1}) \in H^2_{\mathrm{DR}}(X,\mathbb{R})$, i.e., $c_1(K_X^{-1}) = c_1^{\mathrm{AC}}(K_X^{-1})$.
(iii) If ω is conformally balanced, then ${}^t\mathrm{Ric}_\omega^{(1)}$ represents $c_1(K_X^{-1}) \in H_{\bar{\partial}}^{1,1}(X)$ and also the first Bott–Chern class $c_1^{\mathrm{BC}}(K_X^{-1}) \in H_{\mathrm{BC}}^{1,1}(X)$.
(iv) ${}^t\mathrm{Ric}_\omega^{(1)} = {}^s\mathrm{Ric}_\omega^{(1)}$ for $t \neq s$ if and only if ω is balanced.

PROOF. Since ${}^c\mathrm{Ric}_\omega^{(1)} = -\sqrt{-1}\partial\bar{\partial}\log(\omega^n)$, we see that

$$d({}^t\mathrm{Ric}_\omega^{(1)}) = \frac{(t-1)}{2}\left(\bar{\partial}\partial\partial^*\omega + \partial\bar{\partial}\bar{\partial}^*\omega\right).$$

Decomposing the equation $d\left({}^t\mathrm{Ric}_\omega^{(1)}\right)$ into parts, proves (i). For statement (ii), if $\bar{\partial}\partial^*\omega = 0$, then

$${}^t\mathrm{Ric}_\omega^{(1)} = {}^c\mathrm{Ric}_\omega^{(1)} + \frac{(t-1)}{2}dd^*\omega.$$

Therefore, $\left[{}^t\mathrm{Ric}_\omega^{(1)}\right] = \left[{}^c\mathrm{Ric}_\omega^{(1)}\right]$ in $H^2_{\mathrm{DR}}(X,\mathbb{R})$.
Assume that ω is conformally balanced, i.e., there is a smooth function u such that $\omega_u := e^u\omega$ is balanced. The Christoffel symbols Γ_u of ω_u are given in any local frame by

$$(\Gamma_u)_{\bar{i}j}^k = \frac{1}{2}e^{-u}g^{k\bar{\ell}}\left(\frac{\partial}{\partial\bar{z}_i}(e^u g_{j\bar{\ell}}) - \frac{\partial}{\partial\bar{z}_\ell}(e^u g_{j\bar{i}})\right) = \Gamma_{\bar{i}j}^k + \frac{1}{2}\left(\delta_{jk}u_{\bar{i}} - g^{k\bar{\ell}}g_{j\bar{i}}u_{\bar{\ell}}\right).$$

Therefore,

$$(\Gamma_u)_{\bar{i}k}^k = \Gamma_{\bar{i}k}^k + \frac{n-1}{2}u_{\bar{i}},$$

and hence, $\bar{\partial}_u^*\omega_u = \bar{\partial}^*\omega + \sqrt{-1}(n-1)\partial u$. Differentiating this expression gives

$$\bar{\partial}\bar{\partial}_u^*\omega_u = \bar{\partial}\bar{\partial}^*\omega - (n-1)\sqrt{-1}\partial\bar{\partial}u, \qquad \partial\bar{\partial}_u^*\omega_u = \partial\bar{\partial}^*\omega.$$

Since ω_u is balanced, $\bar\partial_u^*\omega_u = 0$, and we have

$$\partial\partial^*\omega + \bar\partial\bar\partial^*\omega \quad = \quad 2(n-1)\sqrt{-1}\partial\bar\partial u.$$

In particular, ${}^t\mathrm{Ric}^{(1)}_\omega = {}^c\mathrm{Ric}^{(1)}_\omega - (n-1)\sqrt{-1}\partial\bar\partial u$, which proves (iii).
If ${}^t\mathrm{Ric}^{(1)}_\omega = {}^s\mathrm{Ric}^{(1)}_\omega$, then

$$\frac{(t-1)}{2}(\partial\partial^*\omega + \bar\partial\bar\partial^*\omega) \quad = \quad \frac{(s-1)}{2}(\partial\partial^*\omega + \bar\partial\bar\partial^*\omega).$$

Since $s \neq t$, we have $\partial\partial^*\omega = 0$ and $\bar\partial\bar\partial^*\omega = 0$. Hence, ω is balanced. □

An immediate consequence of the definition is that

$$c_1^{\mathrm{BC}}(\mathcal{L}) = 0 \implies c_1(\mathcal{L}) = 0 \implies c_1^{\mathrm{AC}}(\mathcal{L}) = 0.$$

If X supports the $\partial\bar\partial$–lemma, then the converse implications hold:

Proposition 2.3.32. Let $\mathcal{L} \to X$ be a holomorphic line bundle over a complex manifold X. If the $\partial\bar\partial$–lemma holds on X, then

$$c_1^{\mathrm{BC}}(\mathcal{L}) = 0 \iff c_1(\mathcal{L}) = 0 \iff c_1^{\mathrm{AC}}(\mathcal{L}) = 0.$$

PROOF. Suppose $c_1^{\mathrm{AC}}(\mathcal{L}) = 0$. It suffices to show that $c_1^{\mathrm{BC}}(\mathcal{L}) = 0$. Then there is a Hermitian metric h on $\mathcal{L}$ such that $\Theta^{(\mathcal{L},h)} = \partial\alpha + \overline{\partial\beta}$, where $\alpha, \beta \in \Omega_X^{0,1}$. Differentiating this equation, $\partial\alpha$ is $\bar\partial$–closed and $\overline{\partial\beta}$ is ∂–closed. Since the $\partial\bar\partial$–lemma holds, we can find smooth functions u and v such that $\partial\alpha = \partial\bar\partial u$ and $\partial\beta = \partial\bar\partial v$. Then $\Theta^{(\mathcal{L},h)} = \partial\bar\partial(u - \overline{v})$, proving the claim. □

Example 2.3.33. Let $X = \mathbb{S}^3 \times \mathbb{S}^1$ be the Hopf surface. From the Kunneth formula, the second Betti number $b_2(X) = 0$, and therefore, $c_1(\mathcal{L}) = c_1^{\mathrm{AC}}(\mathcal{L}) = 0$ for any holomorphic line bundle $\mathcal{L} \to X$. We claim that $c_1^{\mathrm{BC}}(K_X^{-1}) \neq 0$. Let ω_0 be the Boothby metric

$$\omega_0 := \sqrt{-1}\frac{\delta_{ij}}{|z|^2} dz_i \wedge d\overline{z}_j$$

on the Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$. The first Chern–Ricci curvature is easily computed to be

$${}^c\mathrm{Ric}^{(1)}_{\omega_0} \quad = \quad \frac{1}{|z|^2}\left(\delta_{ij} - \frac{\overline{z}_i z_j}{|z|^2}\right)\sqrt{-1} dz_i \wedge d\overline{z}_j.$$

By the Cauchy–Schwarz inequality, ${}^c\mathrm{Ric}^{(1)}_{\omega_0} \geq 0$. This implies $c_1^{\mathrm{BC}}(K_X^{-1}) \neq 0$. Indeed, if $c_1^{\mathrm{BC}}(K_X^{-1}) = 0$, then there exists a function u such that ${}^c\mathrm{Ric}^{(1)}_{\omega_0} = \sqrt{-1}\partial\bar\partial u \geq 0$. However, this would force u to be constant, which is an obvious contradiction. These observations were made by Tosatti in [283, Example 3.3].

Example 2.3.34. ([283, Example 3.5]). Another example of this type is given by the hypothetical complex structure on the six-sphere $\mathbb{S}^6$. Since $b_2(\mathbb{S}^6) = 0$, it is clear that $c_1(\mathcal{L}) = c_1^{\mathrm{AC}}(\mathcal{L}) = 0$ for any holomorphic line bundle on $\mathbb{S}^6$. On the other hand, if $c_1^{\mathrm{BC}}(K_X) = 0$, then K_X is holomorphically torsion. Since $H^1(\mathbb{S}^6, \mathbb{Z}) = H^3(\mathbb{S}^6, \mathbb{Z}) = 0$, the exponential sequence implies that $\mathrm{Pic}(\mathbb{S}^6) \simeq H^1(\mathbb{S}^6, \mathcal{O}_{\mathbb{S}^6})$. Therefore, K_X is holomorphically trivial, and there is a non-vanishing holomorphic 3–form α on $\mathbb{S}^6$. It is clear that α is d–closed and therefore must be d–exact. Writing $\alpha = d\beta$, we see that

$$0 \ < \ (\sqrt{-1})^9 \int_{\mathbb{S}^6} \alpha \wedge \overline{\alpha} \ = \ (\sqrt{-1})^9 \int_{\mathbb{S}^6} d(\beta \wedge d\overline{\beta}) \ = \ 0,$$

an obvious contradiction.

Remark 2.3.35. From 2.3.32, we deduce that the Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$ and any hypothetical complex structure on $\mathbb{S}^6$ does not support the $\partial\bar{\partial}$–lemma.

Liu–Yang [200] constructed first Lichnerowicz–Ricci-flat metrics on the Hopf manifolds $\mathbb{S}^{2n-1} \times \mathbb{S}^1$. We extend their construction, showing that the Hopf manifolds support first Gauduchon–Ricci-flat metrics for all $t \in \mathbb{R}\backslash\{1\}$:

Theorem 2.3.36. Let (X, ω_0) be the Hopf manifold $X = \mathbb{S}^{2n-1} \times \mathbb{S}^1$ endowed with the standard Boothby metric ω_0. The $(1,1)$–form

$$\omega \ := \ \omega_0 + \frac{4t(1-n)-4)}{n} {}^l\mathrm{Ric}_{\omega}^{(1)}$$

is a solution of the equation

$${}^t\mathrm{Ric}_{\omega}^{(1)} \ = \ 0, \qquad \text{for all } t \in \mathbb{R}\backslash\{1\}.$$

Moreover, for all $t < 1$, the $(1,1)$–form ω is positive-definite and thus defines a first Gauduchon–Ricci-flat metric.

Proof. Let $\alpha := \sqrt{-1}\partial\bar{\partial}\log|z|^2$. Then ${}^c\mathrm{Ric}_{\omega_0}^{(1)} = n\alpha$. Following [200], let

$$\omega_\lambda \ := \ \omega_0 + 4\lambda {}^l\mathrm{Ric}_{\omega_0}^{(1)}.$$

Then for $\lambda \in \mathbb{R}\backslash\{-1\}$, we have

$$\begin{aligned} {}^t\mathrm{Ric}_{\omega_\lambda}^{(1)} \ &= \ {}^c\mathrm{Ric}_{\omega_\lambda}^{(1)} + \frac{(t-1)}{2}(\partial\partial_\lambda^*\omega_\lambda + \bar{\partial}\bar{\partial}_\lambda^*\omega_\lambda) \ &&= \ n\alpha + \frac{(t-1)(n-1)}{1+\lambda}\alpha \\ & &&= \ \left(\frac{n(1+\lambda)+(t-1)(n-1)}{(\lambda+1)}\right)\alpha. \end{aligned}$$

The coefficient of α vanishes for all $t \in \mathbb{R}\backslash\{1\}$ if $\lambda = \frac{t(1-n)-1}{n}$, proving the theorem. $\square$

Remark 2.3.37. Observe that since $c_1^{\mathrm{BC}}(K_X^{-1}) \neq 0$ for $X = \mathbb{S}^{2n-1} \times \mathbb{S}^1$, there are no first Chern–Ricci-flat metrics on X.

Question 2.3.38. Do there exist first t–Gauduchon–Ricci-flat metrics on $\mathbb{S}^{2n-1} \times \mathbb{S}^1$ for $t > 1$?

Theorem 2.3.39. Let (X, ω) be a compact Hermitian manifold. If the first Gauduchon–Ricci curvature is quasi-positive, then $c_1^n(X) > 0$. In particular, $H^2_{\mathrm{DR}}(X, \mathbb{R})$, $H^{1,1}_{\bar\partial}(X)$, $H^{1,1}_{\mathrm{BC}}(X)$, and $H^{1,1}_A(X)$ are non-zero.

Proof. Let $\Theta = {}^tR^{K_X}$ denote the curvature of the connection on K_X induced by the t–Gauduchon connection ${}^t\nabla$ on $T^{1,0}X$. We decompose $\Theta = \Theta^{2,0} + \Theta^{1,1} + \Theta^{0,2}$, where $\Theta^{1,1} = {}^t\mathrm{Ric}^{(1)}_\omega$, and $\Theta^{0,2} = \overline{\Theta^{2,0}}$. Hence,

$$\int_X \Theta^n \quad = \quad \sum_{\ell=0}^{[n/2]} \binom{n}{2\ell}\binom{2\ell}{\ell} \int_X \left(\Theta^{2,0} \wedge \overline{\Theta^{2,0}}\right)^\ell \wedge \left(\Theta^{1,1}\right)^{n-2\ell}.$$

Since $\Theta^{1,1}$ is quasi-positive, $\Theta^{2,0} \wedge \overline{\Theta^{2,0}} \geq 0$, and $\int_X \left(\Theta^{1,1}\right)^n > 0$, it is clear that $\int_X \Theta^n > 0$. Hence,

$$\int_X c_1^n(K_X^{-1}) \quad = \quad \int_X \left(\Theta^{K_X}\right)^n \quad > \quad 0.$$

On the other hand, if $H^{1,1}_A(X) = 0$, then the first Chern–Ricci form ${}^c\mathrm{Ric}^{(1)}_\omega = \partial\alpha + \bar\partial\beta$ for 1–forms α, β. This implies that

$$\int_X \left({}^c\mathrm{Ric}^{(1)}_\omega\right)^n \quad = \quad \int_X \left(\partial\alpha + \bar\partial\beta\right) \wedge \left({}^c\mathrm{Ric}^{(1)}_\omega\right)^{n-1} \quad = \quad 0,$$

contradicting $c_1^n(K_X^{-1}) > 0$. $\square$

Gauduchon Scalar Curvatures.

Definition 2.3.40. Let (X, ω_g) be a Hermitian manifold. We define the *kth t–Gauduchon scalar curvature*

$${}^t\mathrm{Scal}^{(k)}_\omega \quad := \quad \mathrm{tr}_{\omega_g}\left({}^t\mathrm{Ric}^{(k)}_{\omega_g}\right).$$

Remark 2.3.41. For the Chern connection, the trace of the first and second Chern–Ricci curvatures yield the same function, which we simply call the *Chern scalar curvature* ${}^c\mathrm{Scal}_\omega$. Similarly, the trace of the third and fourth Chern–Ricci curvatures yield the same function, which we call the *altered Chern scalar curvature* ${}^c\widetilde{\mathrm{Scal}}_\omega$.

The following result is well-known, going back to Gauduchon [132]:

Theorem 2.3.42. Let (X, ω) be a Hermitian manifold. Let τ denote the (Chern) torsion $(1,0)$–form. The Chern scalar curvatures are related by

$${}^c\mathrm{Scal}_\omega - {}^c\widetilde{\mathrm{Scal}}_\omega \quad = \quad d^*\tau + |\tau|^2.$$

In particular, ${}^c\mathrm{Scal}_\omega = {}^c\widetilde{\mathrm{Scal}}_\omega$ if and only if ω is balanced.

Question 2.3.43. Does $d^*\tau = 0$ imply $\tau = 0$ if the manifold is compact?

Question 2.3.44. Are there examples of Hermitian metrics (on compact complex manifolds) where ${}^c\mathrm{Scal}_\omega > {}^c\widetilde{\mathrm{Scal}}_\omega$ or ${}^c\mathrm{Scal}_\omega < {}^c\widetilde{\mathrm{Scal}}_\omega$ at every point?

Theorem 2.3.45. Let (X, ω) be a compact Hermitian manifold with ${}^c\mathrm{Scal}_\omega > 0$. Then the Kodaira dimension $\kappa = -\infty$.

Proof. Let $\sigma \in H^0(X, K_X)$ be a holomorphic section of the canonical bundle K_X. The Bochner formula tells us that

$$\Delta_\omega |\sigma|^2 \quad = \quad |\nabla \sigma|^2 + {}^c\mathrm{Scal}_\omega |\sigma|^2.$$

Hence, if ${}^c\mathrm{Scal}_\omega > 0$, there are no holomorphic sections of K_X. □

Immediate from 2.3.24 is the following:

Proposition 2.3.46. Let (X, ω) be a Hermitian manifold. The Gauduchon scalar curvature ${}^t\mathrm{Scal}_\omega^{(k)} := \mathrm{tr}_\omega({}^t\mathrm{Ric}_\omega^{(k)})$ are given by

$$\begin{aligned}
{}^t\mathrm{Scal}_\omega^{(1)} &= {}^c\mathrm{Scal}_\omega + \frac{(t-1)}{2}\mathrm{tr}_\omega(\partial\partial^*\omega + \bar{\partial}\bar{\partial}^*\omega) \\
{}^t\mathrm{Scal}_\omega^{(2)} &= {}^c\mathrm{Scal}_\omega + \frac{(t^2-1)}{4}|{}^cT|^2 \\
{}^t\mathrm{Scal}_\omega^{(3)} &= {}^t\mathrm{Scal}_\omega^{(4)} \;=\; t\,{}^c\widetilde{\mathrm{Scal}}_\omega + (1-t)\,{}^c\mathrm{Scal}_\omega.
\end{aligned}$$

Corollary 2.3.47. Let (X, ω) be a Hermitian manifold.

(i) If ${}^t\mathrm{Scal}_\omega^{(1)} = {}^c\mathrm{Scal}_\omega$ for $t \in \mathbb{R}\backslash\{1\}$, then ω is balanced.
(ii) If ${}^t\mathrm{Scal}_\omega^{(2)} = {}^c\mathrm{Scal}_\omega$, then $t \in \{-1, 1\}$, or the metric is Kähler.

Proposition 2.3.48. Let (X, ω) be a Hermitian manifold. If ω is both balanced and pluriclosed, then ω is Kähler.

Proof. If ω is balanced, then 2.3.23 tells us that if ω is, in addition, pluriclosed, then

$${}^c\mathrm{Ric}_\omega^{(2)} = {}^c\mathrm{Ric}_\omega^{(1)} + {}^cT^\diamondsuit.$$

The trace of this expression gives

$${}^c\mathrm{Scal}_\omega \;=\; {}^c\mathrm{Scal}_\omega + |{}^cT|^2.$$

Hence, ${}^cT = 0$ and the metric is Kähler. □

2.4. The Holomorphic Bisectional Curvature and its Variants

In contrast with the Riemannian setting, the sectional curvature of a Hermitian metric is not the most natural curvature constraint. Goldberg–Kobayashi [135] introduced the replacement:

Definition 2.4.1. Let (X, ω) be a Hermitian manifold. The *t–Gauduchon holomorphic bisectional curvature* is defined by

$$ {}^t\mathrm{HBC}_\omega(u,v) \ := \ \frac{{}^tR(u,\overline{u},v,\overline{v})}{|u|^2_\omega |v|^2_\omega}, $$

where ${}^tR$ is the t–Gauduchon curvature tensor.

Example 2.4.2. The Euclidean metric on $\mathbb{C}^n$ has constant vanishing (Chern) holomorphic bisectional curvature.

Example 2.4.3. Let ω_{FS} denote the Fubini–Study metric on $\mathbb{P}^n$. Then, at $z = 0$, we have

$$ \begin{aligned} -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \bar{z}_k} \ &= \ -\frac{\partial^4 \log(1+|z|^2)}{\partial z_k \partial \bar{z}_\ell \partial z_i \partial \bar{z}_j} \\ &= \ \frac{1}{2}\frac{\partial^4 |z|^4}{\partial z_k \partial \bar{z}_\ell \partial z_i \partial \bar{z}_j} \\ &= \ \frac{\partial^3 (|z|^2 z_j)}{\partial z_k \partial \bar{z}_\ell \partial z_i} \\ &= \ \frac{\partial^2 (z_j z_\ell)}{\partial z_k \partial z_i} \ = \ (\delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{kj}). \end{aligned} $$

Since the isometry group of ω_{FS} acts transitively on $\mathbb{P}^n$, the above calculation holds at all points. In particular, the (Chern) holomorphic bisectional curvature of the Fubini–Study metric is pinched

$$ 1 \ \leq \ {}^c\mathrm{HBC}_{\omega_{\mathrm{FS}}} \ \leq \ 2. $$

Example 2.4.4. A similar computation shows that the Bergman metric ω_{B} on $\mathbb{B}^n$ has pinched (Chern) holomorphic bisectional curvature

$$ -2 \ \leq \ {}^c\mathrm{HBC}_{\omega_{\mathrm{B}}} \ \leq \ -1. $$

Example 2.4.5. Let ω_0 be the Boothby metric on the Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$. Then the components of the Chern curvature tensor are given by

$$ R_{i\bar{j}k\bar{\ell}} \ = \ \frac{4\delta_{k\ell}(\delta_{ij}|z|^2 - z_j \overline{z}_i)}{|z|^6}. $$

In particular, the Chern holomorphic bisectional curvature of the Boothby metric is non-negative

$$R_{i\bar{i}k\bar{k}} \;=\; \frac{4(|z|^2-|z_i|^2)}{|z|^6} \;=\; \frac{4}{|z|^6}\sum_{k\neq i}|z_k|^2 \;\geq\; 0.$$

Remark 2.4.6. The terminology is justified by the fact that if ω is a Kähler metric, then, via the (real) isomorphism of vector bundles

$$T^{\mathbb{R}}X \ni u_0 \;\longmapsto\; \frac{1}{\sqrt{2}}(u_0-\sqrt{-1}Ju_0) \in T^{1,0}X,$$

we have

$$\begin{aligned} R(u,\overline{u},v,\overline{v}) &= -R(u_0,Ju_0,v_0,Jv_0) \\ &= R(v_0,u_0,Ju_0,Jv_0)+R(Ju_0,v_0,u_0,Jv_0) \\ &= R(v_0,u_0,u_0,v_0)+R(Ju_0,v_0,v_0,Ju_0). \end{aligned}$$

In particular, the holomorphic bisectional curvature of a Kähler metric is a sum of two sectional curvatures.

We now observe that the Ricci curvature is dominated by the bisectional curvature, which follows from the following classical result of Berger [24, 110]:

Proposition 2.4.7. Let (X,ω) be a compact Hermitian manifold of (complex) dimension n. Then

$$\begin{aligned} \frac{2\pi}{n|v|^2}{}^t\mathrm{Ric}^{(1)}_\omega(v,\overline{v}) &= \fint_{\mathbb{S}^{2n-1}} {}^t\mathrm{HBC}_\omega([v],[w])d\sigma(w), \\ \frac{2\pi}{n|v|^2}{}^t\mathrm{Ric}^{(2)}_\omega(v,\overline{v}) &= \fint_{\mathbb{S}^{2n-1}} {}^t\mathrm{HBC}_\omega([v],[w])d\sigma(v), \end{aligned}$$

where $d\sigma$ is the Lebesgue measure on the unit sphere $\mathbb{S}^{2n-1}$ in $T^{1,0}_xX$ for each $x \in X$. Further, $\fint := \frac{(n-1)!}{2\pi^n}\int_{\mathbb{S}^{2n-1}}$.

Proof. Fix a point $x \in X$ and write ${}^tR_{i\bar{j}k\bar{\ell}}$ for the components of the $(1,1)$–part of the t–Gauduchon curvature tensor in a local frame near $x \in X$. Then

$$\begin{aligned} \fint_{\mathbb{S}^{2n-1}} {}^t\mathrm{HBC}_\omega([v],[w])d\sigma(w) &= \frac{1}{|v|^2}\sum_{i,j,k,\ell=1}^n {}^tR_{i\bar{j}k\bar{\ell}}v_i\overline{v}_j\fint_{\mathbb{S}^{2n-1}} w_k\overline{w}_\ell d\sigma(w) \\ &= \frac{1}{n|v|^2}\sum_{i,j,k,\ell=1}^n {}^tR_{i\bar{j}k\bar{\ell}}v_i\overline{v}_j w_k\overline{w}_\ell \delta_k^\ell \\ &= \frac{1}{n|v|^2}\sum_{i,j,k=1}^n {}^tR_{i\bar{j}k\bar{k}}v_i\overline{v}_j \;=\; \frac{2\pi}{n|v|^2}{}^t\mathrm{Ric}^{(1)}_\omega(v,\overline{v}). \end{aligned}$$

$\square$

Corollary 2.4.8. Let (X, ω) be a compact Hermitian manifold. The t–Gauduchon bisectional curvature dominates the first and second t–Gauduchon–Ricci curvatures.

Question 2.4.9. Let (X, ω) be a compact Hermitian manifold. Does the t–Gauduchon bisectional curvature dominate the third and fourth t–Gauduchon–Ricci curvatures?

One of the key properties of the holomorphic bisectional curvature is the following subbundle decreasing property, which follows from the positive semi-definiteness of the second fundamental form (2.2.7):

Proposition 2.4.10. Let $f : Y \hookrightarrow (X, \omega)$ be a complex submanifold of a Hermitian manifold (X, ω). Then

$$^c\mathrm{HBC}_{f^*\omega} \ \leq \ ^c\mathrm{HBC}_\omega.$$

Question 2.4.11. Does the subbundle decreasing property hold for the t–Gauduchon holomorphic bisectional curvature?

Example 2.4.12. The subbundle decreasing property implies that every Stein manifold admits a Kähler metric with non-positive holomorphic bisectional curvature $\mathrm{HBC}_\omega \leq 0$. The metric, of course, is the restriction of the Euclidean metric on the ambient $\mathbb{C}^n$.

Remark 2.4.13. The holomorphic bisectional curvature supports an algebraic interpretation: Let (X, ω) be a Hermitian manifold. Denote by $p : \mathbb{P}(T^{1,0}X) \to X$ the projectivization of its tangent bundle. Let $\mathcal{O}_{\mathbb{P}(T^{1,0}X)}(-1)$ denote the tautological bundle on $\mathbb{P}(T^{1,0}X)$. Then ω induces a Hermitian metric $e^{-\varphi}$ on $\mathcal{O}_{\mathbb{P}(T^{1,0}X)}(-1)$ whose curvature $\sqrt{-1}\partial\bar{\partial}\varphi$ satisfies

$$\sqrt{-1}\partial\bar{\partial}\varphi = -p^*\mathrm{HBC}_\omega + \omega_{\mathrm{FS},\mathbb{P}(T^{1,0}X)}.$$

In particular, if $\mathrm{HBC}_\omega > 0$, then the hyperplane bundle $\mathcal{O}_{\mathbb{P}(T^{1,0}X)}(1)$ over $\mathbb{P}(T^{1,0}X)$ is ample. Hence, the holomorphic bisectional curvature controls the positivity of the tangent bundle[5].

Remark 2.4.14. Wong [**302**] constructed examples of compact complex surfaces with ample tangent bundle.

Question 2.4.15. Is the ampleness of the tangent bundle equivalent to the existence of a Hermitian metric of positive (Chern) holomorphic bisectional curvature?

Example 2.4.16. Mumford's fake projective plane [**216**] is a compact Kähler surface with $b_1 = 0$ and a Kähler metric of negative holomorphic bisectional curvature. It was an old question as to whether a compact simply connected Kähler manifold (of complex dimension $\geq$ 2) could support a Kähler (or Hermitian) metric of negative (Chern) holomorphic bisectional

[5]More precisely, a vector bundle $\mathcal{E}$ is said to be *ample* if the hyperplane bundle $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ over its projectivization $\mathbb{P}(\mathcal{E})$ is an ample line bundle (see, e.g., [**153**]). If the holomorphic bisectional curvature is positive, then $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is a positive line bundle (see 3.4), and is, therefore, ample.

curvature. Recently, Mohsen [**209**, Corollary 2] constructed examples of simply connected complete intersections with Kähler metrics of negative holomorphic bisectional curvature. An important corollary of this is the following (c.f., [**321**, Question 35]):

Corollary 2.4.17. A simply connected Kähler manifold (X, ω) with $\mathrm{HBC}_{\omega} < 0$ need not be a Stein manifold.

The following example due to Wong [**302**] illustrates the difference between the sectional curvature and the holomorphic bisectional curvature:

Example 2.4.18. Let X be a compact quotient of $\mathbb{B}^3$. Since X admits a Kähler metric of negative holomorphic bisectional curvature, Kodaira's embedding theorem 3.4 gives an embedding $\Phi : X \longrightarrow \mathbb{P}^N$. Let H be a hyperplane in $\mathbb{P}^n$ such that $Y := X \cap H$ is smooth. By the subbundle decreasing property, Y supports a Kähler metric of negative holomorphic bisectional curvature.

However, we will show that Y does not support a Riemannian metric of non-positive sectional curvature. The key point is 2.1.27. Indeed, by the Lefschetz hyperplane section theorem 1.6.39, $\pi_1(X) \simeq \pi_1(Y)$. Therefore,

$$\mathbb{Z} \simeq H_6(X, \mathbb{Z}) \simeq H_6(\pi_1(X), \mathbb{Z}) \simeq H_6(\pi_1(Y), \mathbb{Z}) \not\simeq H_6(Y, \mathbb{Z}) = 0,$$

which implies that X cannot be an Eilenberg–Maclane space, and hence, does not support a Riemannian metric of non-positive sectional curvature.

For complex surfaces, we can see the distinction in the existence of a metric of negative sectional curvature and negative holomorphic bisectional curvature at the level of Chern classes:

Theorem 2.4.19. Let (X, ω) be a compact Hermitian surface. If the (Chern) sectional curvature of ω is

(i) negative or quasi-negative, then $c_1^2 > 2c_2$.
(ii) non-positive, then $c_1^2 \geq 2c_2$.

PROOF. Let $\Theta \in \Omega_X^{1,1} \otimes \mathrm{End}(T^{1,0}X)$ be the (Chern) curvature form of ω. From 2.2.47, the cohomology class $c_1^2 - 2c_2$ is represented by

$$\left(\sqrt{-1} \frac{\mathrm{tr}(\Theta)}{2\pi} \right)^2 - 2 \left(\frac{\mathrm{tr}(\Theta^2) - \mathrm{tr}(\Theta)^2}{8\pi^2} \right) \quad = \quad - \frac{\mathrm{tr}(\Theta^2)}{4\pi^2}.$$

Let $\Theta_i^j := \frac{\sqrt{-1}}{2\pi} R_{k\bar{\ell}i}{}^j dz_k \wedge d\bar{z}_\ell$ denote the curvature form in local holomorphic coordinates. Then

$$\begin{aligned} \operatorname{tr}(\Theta^2) \;=\; \operatorname{tr}(\Theta \wedge \overline{\Theta}) &= \sum_{a,b=1}^n \Theta_a^b \wedge \Theta_b^a \\ &= -\frac{1}{4\pi^2} \sum_{a,b=1}^n R_{k\bar{\ell}a}{}^b R_{p\bar{q}b}{}^a dz_k \wedge d\bar{z}_\ell \wedge dz_p \wedge d\bar{z}_q. \end{aligned}$$

If the metric has negative or quasi-negative (Chern) sectional curvature, then $R_{ak\bar{\ell}}{}^b < 0$, implying $R_{ak\bar{\ell}}^b R_{bp\bar{q}}^a > 0$, and thus

$$\begin{aligned} c_1^2 - 2c_2 &= -\int_X \frac{\operatorname{tr}(\Theta^2)}{4\pi^2} \omega^{n-2} \\ &= \frac{1}{16\pi^4} \sum_{a,b=1}^n \int_X R_{k\bar{\ell}a}{}^b R_{p\bar{q}b}{}^a dz_k \wedge d\bar{z}_\ell \wedge dz_p \wedge d\bar{z}_q \wedge \omega^{n-2} > 0. \end{aligned}$$

□

Example 2.4.20. Let $\mathbb{D}^2$ denote the bidisk, which satisfies $c_1^2 = 2c_2$. The product of the Poincaré metrics on each disk yields a Kähler metric ω on $\mathbb{D}^2$ with ${}^c\mathrm{Sec}_\omega \leq 0$ and ${}^c\mathrm{HBC}_\omega \leq 0$. The above theorem shows that $\mathbb{D}^2$ does not admit a Hermitian metric with quasi-negative (Chern) sectional curvature. Indeed, the curvature properties of the metric on $\mathbb{D}^2$ will descend to compact quotients.

Question 2.4.21. Can the above curvature obstructions to Chern class inequalities be extended to the curvature of more general connections (e.g., the Gauduchon connections)?

Question 2.4.22. Can the same technique be used to generate Chern class inequalities for the Bott–Chern c_1^{BC} and Aeppli Chern classes c_1^{AC}?

Example 2.4.23. Let $X := \Sigma_2 \times \Sigma_7$ be the complex surface given by the product of a curve of genus 2 and a curve of genus 7. The Chern numbers are $c_2 = 24$ and $c_1^2 = 48$, and hence, $c_1^2 = 2c_2$. It follows that there is no metric of (quasi-)negative sectional curvature on X.

Example 2.4.24. Other examples of $c_1^2 = 48$ and $c_2 = 24$ are given by the product of a curve of genus 3 with a curve of genus 4; Beauville's double covers of $\Sigma_2 \times \mathbb{P}^1$ (see [22]) and Beauville's double covers of $\mathcal{C} \times \mathbb{P}^1$, where $\mathcal{C}$ is a non-hyperelliptic curve of genus 3 (see [22]).

A similar argument to the proof of 2.4.19 gives the following result of Kleiman [181]:

Theorem 2.4.25. (Kleiman). Let (X, ω) be a compact Hermitian surface with quasi-negative (Chern) holomorphic bisectional curvature. Then $c_1^2 > c_2$. Further, if ${}^c\mathrm{HBC}_\omega \leq 0$, then $c_1^2 \geq c_2$.

The above theorem does not rule out the existence of a complete Hermitian of negative (Chern) holomorphic bisectional curvature on $\mathbb{D}^2$. We saw in 2.4.20, however, that the Chern classes of $\mathbb{D}^2$ obstruct the existence of a complete Hermitian metric of negative (Chern) sectional curvature. In fact, this is a well-known question of Mok:

Question 2.4.26. (Mok). Does there exist a complete Hermitian metric of negative (Chern) holomorphic bisectional curvature on the bidisk $\mathbb{D}^2$?

Example 2.4.27. The following complex surfaces do not have metrics of strictly negative holomorphic bisectional curvature (but potentially support metrics of non-positive holomorphic bisectional curvature): Products of curves $\Sigma_2 \times \Sigma_5$, products of curves $\Sigma_3 \times \Sigma_3$, Beauville's double covers of $\Sigma_2 \times \mathbb{P}^1$ or Beauville's double covers of $\Sigma_3 \times \mathbb{P}^1$, where Σ_3 is non-hyperelliptic [22], Bin surfaces [26]. More examples can be found in [23].

Example 2.4.28. Pignatelli–Polizzi surfaces [234] do not have metrics of non-positive holomorphic bisectional curvature. More examples can be found in [23].

There is an upper bound on the Chern slope of surfaces of general type [208, 317]:

Theorem 2.4.29. (Bogomolov–Miyaoka–Yau). Let X be a compact complex surface of general type. Then $c_1^2 \leq 3c_2$, and equality holds if and only if X is a ball quotient.

Example 2.4.30. Mostow and Siu [215] constructed a compact Kähler surface with negative sectional curvature whose universal cover is not $\mathbb{B}^2$. Their example has Chern numbers satisfying

$$\frac{c_1^2}{c_2} \ = \ \frac{852}{298} \ < \ 3.$$

They also constructed examples with $c_1^2/c_2 \sim 2.9$. In particular, the link between curvature and Chern class inequalities is not as clear as one might initially suspect from 2.4.19 and 2.4.25.

Remark 2.4.31. For some time there was a lot of interest in understanding the converse problem for these Chern class inequalities (see, e.g., [95, 258]). That is, let X be a compact complex surface of general type with $c_1 < 0$. Suppose

$$\frac{3c_2 - c_1^2}{c_1^2}$$

is suitably small. Does X support a Hermitian metric with negative sectional curvature or holomorphic bisectional curvature? Some positive answers to the question were hinted at by the results of Cheung [95]. A definitive negative answer was given by Hirzebruch [158] who produced a family of surfaces of general type which do not even support Hermitian metrics of negative holomorphic sectional curvature. For more developments in this direction, we invite the reader to see [95, §5].

The Holomorphic Bisectional Curvature and Moduli Theory. Mok's question 2.4.26 is a very special case[6] of the curious relationship between the holomorphic bisectional curvature and moduli theory. Shadows of this relationship are reflected in the Priessmann theorem 2.1.28. To describe the relationship, we recall the following theorem of Yang [310]:

Theorem 2.4.32. Let $F \hookrightarrow X \to B$ be a holomorphic fiber bundle with compact fiber F. Then X does not admit a Kähler metric of negative holomorphic bisectional curvature $\mathrm{HBC}_\omega < 0$.

Proof. Let $f : X \to B$ be a holomorphic fiber bundle with compact fiber F. Let $F \times \mathbb{D} \subset X$ be a local trivialization of X over a disk in B. Let $\mathcal{U}$ be a coordinate neighborhood of F, with $z_1, ..., z_n, z_{n+1} = w$ the coordinates on $\mathcal{U} \times \mathbb{D}$. Write $\omega = \sqrt{-1} \sum_{i,j=1}^{n+1} g_{i\bar{j}} dz^i \wedge d\bar{z}^j$ for the Kähler metric on $\mathcal{U} \times \mathbb{D}$. If we fix $w = b$, then the hypersurface $X_b := \{(z,w) \in \mathcal{U} \times \mathbb{D} : w = b\}$ supports the volume form ω_b^n given by the nth exterior power of $\omega_b := \sqrt{-1} \sum_{i,j=1}^n g_{i\bar{j}}(z,b) dz^i \wedge d\bar{z}^j$. The complex Hessian of ω_w^n in the base direction is given by

$$\frac{\partial^2 \omega_w^n}{\partial w \partial \overline{w}} = n\omega_w^{n-1} \wedge \sqrt{-1} \left(\sum_{i,j=1}^n \frac{\partial g_{i\bar{j}}}{\partial w \partial \overline{w}} dz^i \wedge d\bar{z}^j \right) + n(n-1)\omega_w^{n-2} \wedge \frac{\partial \omega_w}{\partial w} \wedge \frac{\partial \omega_w}{\partial \overline{w}}.$$

Setting $v := \frac{\partial}{\partial z_{n+1}}$, we may write

$$\begin{aligned} \frac{\partial^2 \omega_w^n}{\partial w \partial \overline{w}} &= n\omega_w^{n-1} \wedge \sqrt{-1} \sum_{i,j=1}^n R_{i\bar{j}v\bar{v}} dz^i \wedge d\bar{z}^j \\ &\quad + n\omega_w^{n-1} \wedge \sqrt{-1} \sum_{i,j=1}^n \sum_{k,\ell=1}^{n+1} g^{k\bar{\ell}} \frac{\partial g_{i\bar{k}}}{\partial w} \frac{\partial g_{\ell\bar{j}}}{\partial \overline{w}} dz^i \wedge d\bar{z}^j \\ &\quad + n(n-1)\omega_w^{n-2} \wedge \frac{\partial \omega_w}{\partial w} \wedge \frac{\partial \omega_w}{\partial \overline{w}}. \end{aligned}$$

Define a function $\mu : \mathbb{D} \to \mathbb{R}$ by sending a point $w \in \mathbb{D}$ to be the volume of the corresponding fiber:

$$\mu(w) := \int_{X_w} \omega_w^n.$$

Since Kähler submanifolds minimize the volume within their homology class 1.5.33, $\frac{\partial^2 \mu}{\partial w \partial \overline{w}} = 0$ at all points of $\mathbb{D}$, which is in violent contradiction with negative holomorphic bisectional curvature. □

Remark 2.4.33. Seshadri–Zheng [252] showed that if (X, ω) is a complete Hermitian manifold with $-c_0 \leq {}^c\mathrm{HBC}_\omega \leq -c_1 < 0$ and bounded (Chern) torsion, then X cannot be homeomorphic to a product of positive-dimensional complex manifolds. For related works, we invite the reader to see [27, 269, 118, 248, 268, 282].

[6] illustrating how little we understand.

Example 2.4.34. Klembeck [**183**] produced complete Kähler metrics of positive sectional curvature on $\mathbb{C}^n$. His example was modified by Seshadri [**251**] to give a complete Kähler metric of negative sectional curvature on $\mathbb{C}^n$. The sectional curvature of Seshadri's metric is pinched $-2 \leq \text{Sec} \leq 0$. Moreover, the sectional curvatures decay exponentially fast at infinity. Further, Greene–Wu [**140**] showed that if (X, ω) is a complete Hermitian manifold with

$$^c\text{HSC}_\omega \ \leq \ -\frac{C}{1+r^2},$$

where r is the distance to a fixed point in X, then X is Kobayashi hyperbolic. Since $\mathbb{C}^n$ is not Kobayashi hyperbolic the crux of the problem is concentrated in the *rate* of decay of the sectional curvatures. In particular, torsion and curvature lower bounds are not responsible for the relation to moduli theory.

Question 2.4.35. Does there exist a complete Hermitian metric on $\mathbb{C}^n$ with negative (Chern) sectional curvature have polynomial decay at infinity?

The above results are particularly interesting when compared with the following result of To–Yeung [**281**] (c.f., [**95**, p. 29]):

Theorem 2.4.36. Let $f : X \to B$ be a Kodaira fibration surface, i.e., a surjective holomorphic map with connected fibers such that the base and fiber have genus $g > 1$ and the variation in the complex structure of fibers is non-trivial. Then X admits a Kähler metric of negative holomorphic bisectional curvature.

Remark 2.4.37. We will not give the details of the proof of the above theorem, but it is worth mentioning the structure of the argument. Indeed, it is a well-known, difficult problem to produce metrics with a *strict sign* on the curvature since warped product constructions typically only generate degenerate signs on the curvature (i.e., ≥ 0 or ≤ 0). The critical point of the proof of 2.4.36 is to embed X into the moduli space $\mathcal{M}_{g,n}$ of genus $g > 1$ curves (with n marked points). This space supports a Weil–Petersson metric ω_{WP} whose holomorphic bisectional curvature is strictly negative. Hence, the induced metric given by restricting the Weil–Petersson metric to X will also have strictly negative holomorphic bisectional curvature by the subbundle decreasing property.

Before completing our discussion on negative holomorphic bisectional curvature, we mention the following structure result due to Liu [**198**]:

Theorem 2.4.38. Let (X^n, ω) be a compact Kähler manifold with $\text{HBC}_\omega \leq 0$. Then there exists a finite cover $\widetilde{X}$ of X such that $\widetilde{X}$ is the total space of a holomorphic and metric fiber bundle over a compact Kähler manifold Y^k with non-positive holomorphic bisectional curvature and strictly negative Ricci curvature on an open subset of Y. In particular, $c_1(Y) <$

0. The fiber is a flat complex torus $\mathbb{T}$, and $\widetilde{X}$ is diffeomorphic to $\mathbb{T} \times Y$. Finally, if r is the maximal rank of the Ricci curvature of ω, then $r = k = \mathrm{kod}(X)$, where $\mathrm{kod}(X)$ denotes the Kodaira dimension of X.

Corollary 2.4.39. For any compact Kähler manifold with $\mathrm{HBC}_\omega \leq 0$, the Kodaira dimension is equal to the maximal rank of the Ricci curvature.

Question 2.4.40. Is there a Riemannian (or Hermitian) analog of 2.4.38?

Let us now consider the case of positive holomorphic bisectional curvature. Observe that in the Riemannian category, Priessmann's theorem shows that a compact Riemannian manifold (M, g) with negative sectional curvature $\mathrm{Sec}_g < 0$ cannot be homeomorphic to a product of positive-dimensional Riemannian manifolds.

On the other hand, in the positive case, it is a long-standing question as to whether a product Riemannian manifold can admit a metric of positive sectional curvature:

Question 2.4.41. Let (M, g) be a complete Riemannian manifold with $\mathrm{Sec}_g > 0$. Can M be homeomorphic to a product of positive-dimensional Riemannian manifolds?

A famous important instance of the above question is the following conjecture of Hopf:

Conjecture 2.4.42. (Hopf). There is no metric of positive sectional curvature on $\mathbb{S}^2 \times \mathbb{S}^2$.

Returning to the complex analytic category, we know that a compact Kähler manifold with a Kähler metric of positive holomorphic bisectional curvature $\mathrm{HBC}_\omega > 0$ cannot be homeomorphic to a product. This follows from the celebrated solution of the Frankel conjecture due to Mori [**212**] and Siu–Yau [**259**]:

Theorem 2.4.43. Let (X, ω) be a compact Kähler manifold with $\mathrm{HBC}_\omega > 0$. Then X is biholomorphic to $\mathbb{P}^n$.

We will omit the general proof, but some important insight can be gained from looking at the surface case (i.e., when $n = 2$). The key point is the following extension of Frankel's theorem [**124**] due to Goldberg–Kobayashi [**135**]:

Theorem 2.4.44. Let (X, ω) be a complete Kähler manifold of dimension n with $\mathrm{HBC}_\omega > 0$. Let V^r and W^s be compact complex submanifolds of dimension r and s, respectively. If $r + s \geq n$ then V and W intersect non-trivially.

Proof. Proceed by contradiction and assume $V \cap W = \emptyset$. Let $\tau(t)$, for $0 \leq t \leq \ell$, be a minimal geodesic from V to W. Denote the endpoints by $p = \tau(0) \in V$ and $q = \tau(\ell) \in W$. Let v be a parallel vector field defined along τ, tangent to V at p, and tangent to W at q. The assumption that $r + s \geq n$ ensures such a vector field exists. Because V and W are complex submanifolds, $J\xi$ is also such a vector field. Let T be the vector field tangent to τ defined along τ. The second variation of the arc length with respect to infinitesimal variations ξ and $J\xi$ reads:

$$\begin{aligned} L''_{\xi}(0) &= g(\nabla_{\xi}\xi, T)_q - g(\nabla_{\xi}\xi, T)_p - \int_0^{\ell} R(T, \xi, T, \xi)dt, \\ L''_{J\xi}(0) &= g(\nabla_{J\xi}J\xi, T)_q - g(\nabla_{J\xi}J\xi, T)_p - \int_0^{\ell} R(T, J\xi, T, J\xi)dt. \end{aligned}$$

The second fundamental form of a complex submanifold of a Kähler manifold is skew-Hermitian (see, e.g., [135]). Hence,

$$g(\nabla_{\xi}\xi, T)_p + g(\nabla_{J\xi}J\xi, T)_p \;=\; 0 \;=\; g(\nabla_{\xi}\xi, T)_q + g(\nabla_{J\xi}J\xi, T)_q,$$

and therefore,

$$\begin{aligned} L''_{\xi}(0) + L''_{J\xi}(0) &= -\int_0^{\ell} (R(T, \xi, T, \xi) + R(T, J\xi, T, J\xi))dt \\ &= -\int_0^{\ell} R(T, JT, \xi, J\xi)dt \;\leq\; 0. \end{aligned}$$

It follows that $L''_{\xi}(0) < 0$ or $L''_{J\xi}(0) < 0$, violating the assumption that τ is a minimal geodesic from V to W. □

Corollary 2.4.45. A compact Kähler surface (X, ω) with $\text{HBC}_{\omega} > 0$ is biholomorphic to $\mathbb{P}^2$.

Proof. If $\text{HBC}_{\omega} > 0$, then $\text{Scal}_{\omega} > 0$, and hence, from 2.3.45, the plurigenera $p_m(X) = 0$ for all $m > 0$. Since $\text{Ric}_{\omega} > 0$, the first Betti number $b_1(X) = 0$, and in particular, $b_1 = 2h^{0,1} = 0$, i.e., X is regular. By Castelnuovo's theorem 1.7.37, X is rational. By the Enriques–Kodaira classification, X is birational to $\mathbb{P}^2$, or a Hirzebruch surface $\mathcal{F}_n$, where $n \in \mathbb{N}_0 - \{1\}$. The Hirzebruch surfaces $\mathcal{F}_n$ are ruled surfaces – they are $\mathbb{P}^1$–bundles over $\mathbb{P}^1$ – so they cannot support metrics of positive holomorphic bisectional curvature. To show that X is, in fact, biholomorphic to $\mathbb{P}^2$, it suffices to show that X is minimal, i.e., X does not contain any (-1)–curves. However, this is clear from the fact that there exist curves that do not intersect the exceptional curve (if one exists), which would violate 2.4.44. □

Remark 2.4.46. Let us mention that a proof of the Frankel conjecture via the Kähler–Ricci flow was given by Chen–Tian [90]. Ustinovskiy [290], using the Hermitian curvature flow [263] showed that a compact Kähler manifold with quasi-positive[7] holomorphic bisectional curvature is biholomorphic to $\mathbb{P}^n$. Further, it was shown by Goldberg–Kobayashi [135] that if

[7] That is, non-negative everywhere and positive at one point.

the metric with HBC > 0 is, in addition, Einstein, then (X, ω) is biholomorphically isometric to $(\mathbb{P}^n, \omega_{\text{FS}})$.

Question 2.4.47. Let (X, ω) be a compact Hermitian manfiold with positive t–Gauduchon holomorphic bisectional curvature ${}^t\text{HBC}_\omega > 0$. Is X biholomorphic to $\mathbb{P}^n$?

Remark 2.4.48. Even for the Chern connection, an answer to the above question appears to be unknown (at least to the author). If ${}^c\text{HBC}_\omega > 0$ then the underlying complex manifold supports a Kähler structure, given by the Kähler metric $\widetilde{\omega} := {}^c\text{Ric}_\omega^{(1)}$. It is almost certainly false, however, that the holomorphic bisectional curvature of $\widetilde{\omega}$ is positive, however.

Example 2.4.49. In the non-compact case, Klembeck [**183**] produced complete Kähler metrics on $\mathbb{C}^n$ with positive sectional curvature. Moreover, Greene–Wu [**141**] showed that a complete (non-compact) Kähler manifold with positive sectional curvature is Stein.

The case of nonnegative holomorphic bisectional curvature – the so-called *Generalized Frankel Conjecture* – was treated by Mok [**211**]:

Theorem 2.4.50. (Mok). Let (X, ω) be a compact Kähler manifold with $\text{HBC}_\omega \geq 0$. Suppose that the Ricci curvature is quasi-positive. Then X is biholomorphic to $\mathbb{P}^n$ or an irreducible compact Hermitian symmetric space of rank $k > 2$.

Remark 2.4.51. Recall that a Hermitian symmetric space is a Hermitian manifold whose underlying Riemannian structure is symmetric. In general, this does not imply that the manifold is complex homogeneous, but in the compact case, this is true. Indeed, the identity component of the isometry group of a compact Hermitian symmetric space acts transitively and is contained in the holomorphic automorphism group ([**324**, p. 212]). By the well-known theorem of Bochner–Montgomery [**34**], the automorphism group of a compact complex manifold is a complex Lie group, and thus, compact Hermitian symmetric spaces are complex homogeneous. This observation of the author (which is elementary and likely very well-known) came from a question posed to him by Franc Forstnerič and appears in [**123**, p. 36].

The search for the appropriate relaxation of the holomorphic bisectional curvature has led to several developments. The first candidate was introduced by Cao–Hamilton [**74**]:

Definition 2.4.52. Let (X, ω) be a Kähler manifold. The *orthogonal bisectional curvature* $\text{HBC}_\omega^\perp$ is defined to be the restriction of the holomorphic bisectional curvature HBC_ω to unitary pairs of $(1,0)$–tangent vectors.

Remark 2.4.53. Cao–Hamilton showed that the non-negativity of the orthogonal bisectional curvature was preserved by the Kähler–Ricci flow. In [**84**], Chen showed that any compact irreducible Kähler manifold with $\text{HBC}_\omega^\perp > 0$ and $c_1(K_X^{-1}) > 0$ must be biholomorphic to $\mathbb{P}^n$.

It was shown by Gu–Zhang [149], however, that the Kähler–Ricci flow, starting from a Kähler metric with $\text{HBC}^{\perp} \geq 0$ converges to a Kähler metric with $\text{HBC} \geq 0$. Hence, compact Kähler manifolds with $\text{HBC}^{\perp} \geq 0$ are fully classified by Mok's solution of the generalized Frankel conjecture [211].

The following question was brought to my attention by Jianchun Chu and Man-Chun Lee:

Question 2.4.54. Let (X, ω) be a compact Kähler manifold with $\text{HBC}_{\omega}^{\perp} < 0$. Is the canonical bundle K_X ample?

Remark 2.4.55. It is interesting to compare the orthogonal bisectional curvature with the isotropic curvature[8] in Riemannian geometry: Seshadri [250] has given a classification of manifolds with nonnegative isotropic curvature. In particular, he has shown that any compact irreducible Kähler manifold with nonnegative isotropic curvature must be either a Hermitian symmetric space or biholomorphic to $\mathbb{P}^n$.

On the other hand, a compact Kähler manifold with nonnegative isotropic curvature has $\text{HBC}_{\omega}^{\perp} \geq 0$ (see, e.g., [149]). Gu–Zhang [149] showed that this implication is strict.
For complex surfaces, Ivey [171] showed that $\text{HBC}_{\omega} \geq 0$ implies that the isotropic curvature is non-negative. Hence, Seshadri's theorem [250] can be viewed as a generalization of Mok's theorem [211] in the surface case. To the author's knowledge, it remains unknown whether this holds in higher dimensions.

The leading candidate for an appropriate relaxation of the holomorphic bisectional curvature is the following curvature constraint introduced by Wu–Yau–Zheng[9] [306]:

Definition 2.4.56. Let (X, ω) be a Hermitian manifold. The *t–Gauduchon Quadratic Orthogonal Bisectional Curvature* ${}^t\text{QOBC}_{\omega}$ is the function

$$^t\text{QOBC}_{\omega} : \mathcal{F}_X \times \mathbb{R}^n \to \mathbb{R}, \qquad {}^t\text{QOBC}_{\omega}(v) := \frac{1}{|v|^2} \sum_{\alpha,\gamma} {}^t R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} (v_{\alpha} - v_{\gamma})^2.$$

Remark 2.4.57. Although the QOBC was introduced by Wu–Yau–Zheng [306], it first appears implicitly in the paper of Bishop–Goldberg [28] as the Weitzenböck curvature operator

[8] Recall: Let (M, g) be a Riemannian manifold of (real) dimension ≥ 4. We say that g has non-negative *isotropic curvature* if, for all $x \in M$, and all orthonormal vectors $\{e_1, e_2, e_3, e_4\} \subset T_x M$, we have

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} \quad \geq \quad 2R_{1234}.$$

[9] Wu–Yau–Zheng only consider the Quadratic Orthogonal Bisectional Curvature for Kähler metrics.

(c.f., [80, 81, 228, 229, 230, 231]) acting on real $(1,1)$–forms. In contrast with the orthogonal bisectional curvature, the QOBC is significantly weaker than the holomorphic bisectional curvature, with an explicit example constructed in [191].

Given the awkward nature of the quadratic orthogonal bisectional curvature (from now on, QOBC), in [52, 51, 53], the author addressed the algebraic character of this curvature. During this investigation, it was discovered that there was a curious link between the QOBC, distance geometry, and combinatorics. To state these results, we introduce the following terminology:

Definition 2.4.58. A symmetric matrix $A = (A_{\alpha\gamma}) \in \mathbb{R}^{n \times n}$ is said to be a *Euclidean distance matrix* (EDM) (of embedding dimension 1) if there is a vector $v = (v_1, ..., v_n) \in \mathbb{R}^n$ such that $A_{\alpha\gamma} = (v_\alpha - v_\gamma)^2$.

Euclidean distance matrices play an important role in combinatorics. An important result in the subject is the following theorem due to Schoenberg [244], which characterizes EDMs among hollow matrices (i.e., matrices with no non-zero elements along the diagonal):

Proposition 2.4.59. (Schoenberg criterion). A real symmetric hollow matrix is an EDM if and only if it is negative semi-definite on the hyperplane $H := \{x \in \mathbb{R}^n : x^t \mathbf{e} = 0\}$, where $\mathbf{e} = (1, ..., 1)^t$.

Remark 2.4.60. By definition, an EDM A is a nonnegative matrix in the sense that each entry of the matrix is a nonnegative real number. Therefore, the Perron–Frobenius theorem informs us that the largest eigenvalue of A is nonnegative and occurs with the eigenvector in the nonnegative orthant. This eigenvalue is often referred to as the *Perron root* of A. Let A be a Euclidean distance matrix with eigenvalues $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$. From the above discussion, we know that $\delta_1 > 0$ and $\delta_k \leq 0$ for all $k \geq 2$. We make the following definition:

Definition 2.4.61. The *kth Perron weight* of the Euclidean distance matrix A is defined to be the ratio $r_k := -\delta_k/\delta_1 \in [0,1]$.

We can now characterize the conditions on the curvature tensor of a Kähler metric which ensure nonnegative QOBC:

Theorem 2.4.62. Let (X^n, ω) be a compact Kähler manifold. Let $\mathcal{R}$ be the matrix with entries $\mathcal{R}_{\alpha\gamma} := R_{\alpha\overline{\alpha}\gamma\overline{\gamma}}$. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $\mathcal{R}$ with respect to the frame which minimizes the QOBC. Then $\text{QOBC}_\omega \geq 0$ if and only if

$$\lambda_1 \quad \geq \quad \sum_{k=2}^n r_k \lambda_k$$

holds for all Perron weights $0 \leq r_k \leq 1$.

Proof. Fix a frame that minimizes the QOBC of ω. Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ denote the eigenvalues of $\mathcal{R} \in \mathbb{R}^{n \times n}$ and denote by $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$ the eigenvalues of an EDM Δ. Write $\mathcal{R} = U^t \text{diag}(\lambda) U$ and $\Delta = V^t \text{diag}(\delta) V$ for the eigenvalue decompositions of $\mathcal{R}$ and Δ. Then

$$\begin{aligned} \text{tr}(\mathcal{R}\Delta) \ = \ \text{tr}(U^t \text{diag}(\lambda) U V^t \text{diag}(\delta) V) &= \text{tr}(V U^t \text{diag}(\lambda) U V^t \text{diag}(\delta)) \\ &= \text{tr}(Q^t \text{diag}(\lambda) Q \text{diag}(\delta)) \\ &= \sum_{i,j} \lambda_i \delta_j Q_{ij}^2, \end{aligned}$$

where $Q = UV^t$ is orthogonal. The Hadamard square (by which we mean the matrix $Q \circ Q$ with entries Q_{ij}^2) of an orthogonal matrix is doubly stochastic (see, e.g., [156]). The class of $n \times n$ doubly stochastic matrices forms a convex polytope – the Birkhoff polytope $\mathcal{B}^n$. The minimum of $\text{tr}(\mathcal{R}\Delta)$ is given by

$$\min_{S \in \mathcal{B}^n} \sum_{i,j=1}^n \lambda_i \delta_j S_{ij}.$$

This function is linear in S, achieving its minimum on the boundary of $\mathcal{B}^n$. The well-known Birkhoff–von Neumann theorem tells us that $\mathcal{B}^n$ is the convex hull of the set of permutation matrices, and the vertices of $\mathcal{B}^n$ are precisely the permutation matrices. Hence,

$$\min_{S \in \mathcal{B}^n} \sum_{i,j=1}^n \lambda_i \delta_j S_{ij} \ = \ \min_{\sigma \in S_n} \sum_{i=1}^n \lambda_i \delta_{\sigma(i)},$$

where S_n denotes the symmetric group on n letters. An elementary argument (by induction, for instance) shows that

$$\min_{\sigma \in S_n} \sum_{i=1}^n \lambda_i \delta_{\sigma(i)} \ = \ \sum_{i=1}^n \lambda_i \delta_i.$$

From the above discussion, this completes the proof. □

Remark 2.4.63. The above theorem is of interest for several reasons: The first is an eigenvalue characterization in terms of the matrix $\mathcal{R}$. Of course, this matrix requires a frame to be fixed, but given the number of frame-dependent curvatures which have appeared in complex geometry in recent years (most notably the real bisectional curvature, the Schwarz bisectional curvatures, and the QOBC itself), this still offers insight into their relationship.

Remark 2.4.64. The existence of an eigenvalue characterization is surprising in itself [52], since EDMs are defined in a non-invariant manner: the class of positive matrices (i.e., matrices for which every entry is positive) are certainly not invariant under a change of basis.

The class of matrices A satisfying $\sum_{\alpha,\gamma=1}^n A_{\alpha\gamma}(v_\alpha - v_\gamma)^2 \geq 0$ form the so-called *dual EDM cone*. This is a completely elementary observation: An EDM (of embedding dimension 1) is a matrix of the form $B_{\alpha\gamma} := (v_\alpha - v_\gamma)^2$. Hence, we can write $\sum_{\alpha,\gamma=1}^n A_{\alpha\gamma}(v_\alpha - v_\gamma)^2 \geq 0$ as $\text{tr}(AB) \geq 0$, from which we immediately see the following:

Proposition 2.4.65. Let (X, ω) be a compact Kähler manifold. Then $\text{QOBC}_\omega \geq 0$ if and only if, with respect to the frame which minimizes the QOBC, the matrix $\mathcal{R}$ lies in the dual EDM cone.

The above result, albeit elementary, is important in that it gives us the appropriate language to speak when considering the QOBC. It also allows us to exploit the results of distance geometry and combinatorics to say something about the QOBC. For instance, using Dattorro's dual EDM cone criterion [**100**, **101**], we have:

Theorem 2.4.66. Let $\delta : \mathbb{R}^n \to \mathcal{S}^n_{\text{diag}}$ be the operator mapping a vector $v \in \mathbb{R}^n$ to the diagonal matrix $\text{diag}(v)$. Then a real symmetric matrix A lies in the dual EDM cone if and only if $\delta(A\mathbf{e}) - A$ is positive-semi-definite. In particular,

$$\text{QOBC}_\omega \geq 0 \iff \delta(\mathcal{R}\mathbf{e}) - \mathcal{R} \in \mathcal{PSD}.$$

We suspect these results (motivated exclusively by complex-geometric considerations) will have further generalizations and applications to combinatorics and distance geometry. We describe one potential direction in more detail:

Let G be a finite weighted graph (with possibly negative weights) with vertex set $V(G) = \{v_1, ..., v_n\}$. Let $A \in \mathbb{R}^{n \times n}$ be the adjacency matrix specifying the weights. The *Dirichlet energy* of a weighted graph is defined by

$$\mathcal{E}(f) \ := \ \sum_{\alpha,\gamma=1}^n A_{\alpha\gamma}(f(v_\alpha) - f(v_\gamma))^2,$$

where $f : V(G) \to \mathbb{R}$ is a function defined on the vertices of G. The above results show that for a compact Kähler manifold (X, ω), non-negative QOBC is equivalent to the non-negativity of the Dirichlet energy of every weighted graph (G, A) with G a finite graph with n vertices and A given by the matrix $\mathcal{R} = (\mathcal{R}_{\alpha\gamma})$.

Recall that the QOBC first appears in the paper of Bishop–Goldberg [**28**] as the Weitzenböck curvature operator (c.f., [**228**, **229**, **230**, **231**]) acting on real $(1,1)$–forms. In other words, if Δ_g denotes the Bochner Laplace operator, and Δ_d denotes the Laplace–Beltrami operator, acting on real $(1,1)$–forms, their difference realizes the QOBC. What is curious is that the above theorem indicates that the difference of these Laplace operators is the (discrete analog of the) Dirichlet energy associated with the curvature. We hope those more experienced in the discrete theory can give further insight into this direction.

It was shown by Niu [222] that if (X, ω) is compact Kähler with $\text{QOBC}_{\omega} \geq 0$, then $\text{Scal}_{\omega} \geq 0$. In [60], the author, joint with Kai Tang, gave a more direct argument and extended the computation to the Hermitian category:

Theorem 2.4.67. ([60, Theorem 4.8]). Let (X^n, ω) be a Hermitian manifold of (complex) dimension n with $\text{QOBC}_{\omega} \geq 0$. Then for any point $p \in X$ and any unitary pair $v, w \in T_p^{1,0}X$ we have

$$ {}^c\text{Ric}_{\omega}^{(1)}(v, \overline{v}) + {}^c\text{Ric}_{\omega}^{(1)}(w, \overline{w}) + {}^c\text{Ric}_{\omega}^{(2)}(v, \overline{v}) + {}^c\text{Ric}_{\omega}^{(2)}(w, \overline{w}) \quad \geq \quad 2(R_{v\overline{w}w\overline{v}} + R_{w\overline{v}v\overline{w}}). $$

Moreover, in any local frame, the scalar curvature satisfies

$$ \text{Scal}_{\omega} \quad \geq \quad \frac{1}{n-1} \sum_{1 \leq k < \ell \leq n} (R_{k\bar{\ell}\ell\bar{k}} + R_{\ell\bar{k}k\bar{\ell}}). $$

Proof. Suppose $\text{QOBC}_{\omega} \geq 0$. Then for any $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$, and any unitary frame, we have

$$ \sum_{i,j=1}^{n} R_{i\bar{i}j\bar{j}}(\xi_i - \xi_j)^2 \quad \geq \quad 0. $$

For distinct indices j, k, ℓ, set $\xi_k = 0$, $\xi_\ell = 2$, and $\xi_j = 1$. This gives

$$ \begin{aligned} &4R_{k\bar{k}\ell\bar{\ell}} + 4R_{\ell\bar{\ell}k\bar{k}} + \sum_{j \neq k, j \neq \ell} (R_{k\bar{k}j\bar{j}} + R_{j\bar{j}k\bar{k}} + R_{\ell\bar{\ell}j\bar{j}} + R_{j\bar{j}\ell\bar{\ell}}) \\ &\qquad = 4(R_{k\bar{k}\ell\bar{\ell}} + R_{\ell\bar{\ell}k\bar{k}}) + \sum_{j \neq k, j \neq \ell} (R_{k\bar{k}j\bar{j}} + R_{j\bar{j}k\bar{k}} + R_{\ell\bar{\ell}j\bar{j}} + R_{j\bar{j}\ell\bar{\ell}}) \ \geq \ 0. \quad (2.4.1) \end{aligned} $$

Let $f_k = \frac{1}{\sqrt{2}}(e_k - e_\ell)$, $f_\ell = \frac{1}{\sqrt{2}}(e_k + e_\ell)$ and $f_j = e_j$. Then (2.4.1) in this frame gives

$$
\begin{aligned}
& R(e_k - e_\ell, \overline{e_k - e_\ell}, e_k + e_\ell, \overline{e_k + e_\ell}) + R(e_k + e_\ell, \overline{e_k + e_\ell}, e_k - e_\ell, \overline{e_k - e_\ell}) \\
& \qquad + \frac{1}{2} \sum_{j\neq k, j\neq \ell} ((R(e_k - e_\ell, \overline{e_k - e_\ell}, e_j, \overline{e_j}) + R(e_j, \overline{e_j}, e_k - e_\ell, \overline{e_k - e_\ell})) \\
& \qquad\qquad + \frac{1}{2} \sum_{j\neq k, j\neq \ell} R(e_k + e_\ell, \overline{e_k + e_\ell}, e_j, \overline{e_j}) + R(e_j, \overline{e_j}, e_k + e_\ell, \overline{e_k + e_\ell}) \\
= \; & R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} + R_{k\bar k\ell\bar\ell} - R_{k\bar\ell\ell\bar k} - R_{k\bar\ell k\bar\ell} + R_{\ell\bar\ell k\bar k} - R_{\ell\bar k\ell\bar k} - R_{\ell\bar k k\bar\ell} \\
& + R_{k\bar k k\bar\ell} + R_{k\bar k\ell\bar k} - R_{k\bar\ell k\bar k} - R_{\ell\bar k k\bar k} - R_{k\bar\ell\ell\bar\ell} + R_{\ell\bar\ell\ell\bar k} + R_{\ell\bar\ell k\bar\ell} - R_{\ell\bar k\ell\bar\ell} \\
& + R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} + R_{k\bar k\ell\bar\ell} - R_{k\bar\ell\ell\bar k} - R_{k\bar\ell k\bar\ell} + R_{\ell\bar\ell k\bar k} - R_{\ell\bar k\ell\bar k} - R_{\ell\bar k k\bar\ell} \\
& - R_{k\bar k k\bar\ell} - R_{k\bar k\ell\bar k} + R_{k\bar\ell k\bar k} + R_{\ell\bar k k\bar k} + R_{k\bar\ell\ell\bar\ell} - R_{\ell\bar\ell\ell\bar k} - R_{\ell\bar\ell k\bar\ell} + R_{\ell\bar k\ell\bar\ell} \\
& + \frac{1}{2} \sum_{j\neq k, j\neq \ell} \left(R_{k\bar k j\bar j} + R_{\ell\bar\ell j\bar j} - R_{k\bar\ell j\bar j} - R_{\ell\bar k j\bar j} + R_{j\bar j k\bar k} + R_{j\bar j\ell\bar\ell} - R_{j\bar j k\bar\ell} - R_{j\bar j\ell\bar k} \right) \\
& + \frac{1}{2} \sum_{j\neq k, j\neq \ell} \left(R_{k\bar j j\bar j} + R_{\ell\bar\ell j\bar j} + R_{k\bar\ell j\bar j} + R_{\ell\bar k j\bar j} + R_{j\bar j k\bar k} + R_{j\bar j\ell\bar\ell} + R_{j\bar j k\bar\ell} + R_{j\bar j\ell\bar k} \right) \\
= \; & R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} + R_{k\bar k\ell\bar\ell} + R_{\ell\bar\ell k\bar k} - R_{k\bar\ell\ell\bar k} - R_{k\bar\ell k\bar\ell} - R_{\ell\bar k\ell\bar k} - R_{\ell\bar k k\bar\ell} \\
& \qquad\qquad + \frac{1}{2} \sum_{j\neq k, j\neq \ell} \left(R_{k\bar k j\bar j} + R_{\ell\bar\ell j\bar j} + R_{j\bar j k\bar k} + R_{j\bar j\ell\bar\ell} \right) \;\geq\; 0.
\end{aligned}
$$

Similarly, setting $f_k = \frac{1}{\sqrt{2}}(e_k - \sqrt{-1}e_\ell)$, $f_\ell = \frac{1}{\sqrt{2}}(e_k + \sqrt{-1}e_\ell)$ and $f_j = e_j$ gives

$$
\begin{aligned}
& R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} + R_{k\bar k\ell\bar\ell} + R_{\ell\bar\ell k\bar k} - R_{k\bar\ell\ell\bar k} + R_{k\bar\ell k\bar\ell} + R_{\ell\bar k\ell\bar k} - R_{\ell\bar k k\bar\ell} \\
& \qquad\qquad + \frac{1}{2} \sum_{j\neq k, j\neq \ell} \left(R_{k\bar k j\bar j} + R_{\ell\bar\ell j\bar j} + R_{j\bar j k\bar k} + R_{j\bar j\ell\bar\ell} \right) \;\geq\; 0.
\end{aligned}
$$

Adding these equations together, we get

$$
R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} + R_{k\bar k\ell\bar\ell} + R_{\ell\bar\ell k\bar k} - R_{k\bar\ell\ell\bar k} - R_{\ell\bar k k\bar\ell} + \sum_{j\neq k, j\neq \ell} \left(R_{k\bar k j\bar j} + R_{\ell\bar\ell j\bar j} + R_{j\bar j k\bar k} + R_{j\bar j\ell\bar\ell} \right) \;\;\geq\;\; 0.
$$

Observe that

$$
\begin{aligned}
& \mathrm{Ric}^{(1)}_{k\bar k} + \mathrm{Ric}^{(1)}_{\ell\bar\ell} + \mathrm{Ric}^{(2)}_{k\bar k} + \mathrm{Ric}^{(2)}_{\ell\bar\ell} \\
= \; & R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} + R_{k\bar k\ell\bar\ell} + R_{\ell\bar\ell k\bar k} + \sum_{j\neq k, j\neq \ell} (R_{k\bar k j\bar j} + R_{\ell\bar\ell j\bar j}) \\
& \qquad + R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} + R_{\ell\bar\ell k\bar k} + R_{k\bar k\ell\bar\ell} + \sum_{j\neq k, j\neq \ell} (R_{j\bar j k\bar k} + R_{j\bar j\ell\bar\ell}) \\
= \; & 2\left(R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} + R_{k\bar k\ell\bar\ell} + R_{\ell\bar\ell k\bar k} \right) + \sum_{j\neq k, j\neq \ell} (R_{k\bar k j\bar j} + R_{\ell\bar\ell j\bar j} + R_{j\bar j k\bar k} + R_{j\bar j\ell\bar\ell}).
\end{aligned}
$$

Hence, for $k \neq \ell$, we have

$$\mathrm{Ric}^{(1)}_{k\overline{k}} + \mathrm{Ric}^{(1)}_{\ell\overline{\ell}} + \mathrm{Ric}^{(2)}_{k\overline{k}} + \mathrm{Ric}^{(2)}_{\ell\overline{\ell}} \quad \geq \quad 2(R_{k\overline{\ell}\ell\overline{k}} + R_{\ell\overline{k}k\overline{\ell}}).$$

For the statement concerning the scalar curvature, we observe that

$$\begin{aligned} 2\mathrm{Scal}_\omega &= \frac{1}{n-1} \sum_{1 \leq k < \ell \leq n} \left(\mathrm{Ric}^{(1)}_{k\overline{k}} + \mathrm{Ric}^{(1)}_{\ell\overline{\ell}} \right) + \frac{1}{n-1} \sum_{1 \leq k < \ell \leq n} \left(\mathrm{Ric}^{(2)}_{k\overline{k}} + \mathrm{Ric}^{(2)}_{\ell\overline{\ell}} \right) \\ &= \frac{1}{n-1} \sum_{1 \leq k < \ell \leq n} \left(\mathrm{Ric}^{(1)}_{k\overline{k}} + \mathrm{Ric}^{(1)}_{\ell\overline{\ell}} + \mathrm{Ric}^{(2)}_{k\overline{k}} + \mathrm{Ric}^{(2)}_{\ell\overline{\ell}} \right) \\ &\geq \frac{2}{n-1} \sum_{1 \leq k < \ell \leq n} \left(R_{k\overline{\ell}\ell\overline{k}} + R_{\ell\overline{k}k\overline{\ell}} \right). \end{aligned}$$

□

Corollary 2.4.68. Let (X, ω) be a Kähler-like manifold with $\mathrm{QOBC}_\omega \geq 0$. Then $\mathrm{Scal}_\omega \geq 0$.

It is natural to consider the following altered variant of the QOBC [60]:

Definition 2.4.69. Let (X, ω) be a Hermitian manifold. The *altered quadratic orthogonal bisectional curvature* is the function

$$\widetilde{\mathrm{QOBC}}_\omega : \mathcal{F}_X \times \mathbb{R}^n \backslash \{0\} \to \mathbb{R}, \qquad \widetilde{\mathrm{QOBC}}_\omega(v) \ := \ \frac{1}{|v|^2_\omega} \sum_{\alpha, \gamma} R_{\alpha\overline{\gamma}\gamma\overline{\alpha}} (v_\alpha - v_\gamma)^2.$$

We have the following analog of 2.4.67:

Proposition 2.4.70. Let (X^n, ω) be a Hermitian manifold of (complex) dimension n and $\widetilde{\mathrm{QOBC}}_\omega \geq 0$. Then for any point $p \in X$ and any unitary pair $v, w \in T^{1,0}_p X$, we have

$$\mathrm{Ric}^{(3)}_\omega(v, \overline{v}) + \mathrm{Ric}^{(3)}_\omega(w, \overline{w}) + \mathrm{Ric}^{(4)}_\omega(v, \overline{v}) + \mathrm{Ric}^{(4)}_\omega(w, \overline{w}) \quad \geq \quad 2(R_{v\overline{w}w\overline{v}} + R_{w\overline{v}v\overline{w}}).$$

Moreover, in any local frame, the altered scalar curvature satisfies

$$\widetilde{\mathrm{Scal}}_\omega \quad \geq \quad \frac{1}{n-1} \sum_{1 \leq k < \ell \leq n} (R_{k\overline{k}\ell\overline{\ell}} + R_{\ell\overline{\ell}k\overline{k}}).$$

Proof. Suppose $\widetilde{\mathrm{QOBC}}_\omega \geq 0$. Then, in each unitary frame,

$$\sum_{i,j=1}^n R_{i\overline{j}j\overline{i}} (\xi_i - \xi_j)^2 \ \geq \ 0,$$

for all $\xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n$. Let $\xi_k = 0$, $\xi_\ell = 2$, and $\xi_j = 1$ for $k \neq \ell$, $j \neq k$. Then

$$4 \left(R_{k\overline{\ell}\ell\overline{k}} + R_{\ell\overline{k}k\overline{\ell}} \right) + \sum_{j \neq k, j \neq \ell} \left(R_{k\overline{j}j\overline{k}} + R_{j\overline{k}k\overline{j}} + R_{\ell\overline{j}j\overline{\ell}} + R_{j\overline{\ell}\ell\overline{j}} \right) \quad \geq \quad 0.$$

Let $f_k = \frac{1}{\sqrt{2}}(e_k - e_\ell)$, $f_\ell = \frac{1}{\sqrt{2}}(e_k + e_\ell)$, and $f_j = e_j$. Then

$$
\begin{aligned}
& R(e_k - e_\ell, \overline{e_k + e_\ell}, e_k + e_\ell, \overline{e_k - e_\ell}) + R(e_k + e_\ell, \overline{e_k - e_\ell}, e_k - e_\ell, \overline{e_k + e_\ell}) \\
& + \frac{1}{2} \sum_{j \neq k, j \neq \ell} \left(R(e_k - e_\ell, \overline{e_j}, e_j, \overline{e_k - e_\ell}) + R(e_j, \overline{e_k - e_\ell}, e_k - e_\ell, \overline{e_j}) \right) \\
& + \frac{1}{2} \sum_{j \neq k, j \neq \ell} \left(R(e_k + e_\ell, \overline{e_j}, e_j, \overline{e_k + e_\ell}) + R(e_j, \overline{e_k + e_\ell}, e_k + e_\ell, \overline{e_j}) \right) \\
= \; & R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} - R_{k\bar k\ell\bar\ell} - R_{k\bar\ell k\bar\ell} + R_{k\bar\ell\ell\bar k} - R_{\ell\bar k\ell\bar k} + R_{\ell\bar k k\bar\ell} - R_{\ell\bar\ell k\bar k} \\
& - R_{k\bar k k\bar\ell} + R_{k\bar k\ell\bar k} + R_{k\bar\ell k\bar k} - R_{\ell\bar k k\bar k} - R_{k\bar\ell\ell\bar\ell} + R_{\ell\bar k\ell\bar\ell} + R_{\ell\bar\ell k\bar\ell} - R_{\ell\bar\ell\ell\bar k} \\
& + R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} - R_{k\bar k\ell\bar\ell} - R_{k\bar\ell k\bar\ell} + R_{k\bar\ell\ell\bar k} - R_{\ell\bar k\ell\bar k} + R_{\ell\bar k k\bar\ell} - R_{\ell\bar\ell k\bar k} \\
& + R_{k\bar k k\bar\ell} - R_{k\bar k\ell\bar k} - R_{k\bar\ell k\bar k} + R_{\ell\bar k k\bar k} + R_{k\bar\ell\ell\bar\ell} - R_{\ell\bar k\ell\bar\ell} - R_{\ell\bar\ell k\bar\ell} + R_{\ell\bar\ell\ell\bar k} \\
& + \frac{1}{2} \sum_{j \neq k, j \neq \ell} \Big(R_{k\bar j j\bar k} + R_{\ell\bar j j\bar\ell} - R_{k\bar j j\bar\ell} - R_{\ell\bar j j\bar k} + R_{j\bar k k\bar j} + R_{j\bar\ell\ell\bar j} - R_{j\bar k\ell\bar j} - R_{j\bar\ell k\bar j} \Big) \\
& + \frac{1}{2} \sum_{j \neq k, j \neq \ell} \Big(R_{k\bar j j\bar k} + R_{\ell\bar j j\bar\ell} + R_{k\bar j j\bar\ell} + R_{\ell\bar j j\bar k} + R_{j\bar k k\bar j} + R_{j\bar\ell\ell\bar j} + R_{j\bar k\ell\bar j} + R_{j\bar\ell k\bar j} \Big) \\
= \; & 2 \left(R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} - R_{k\bar k\ell\bar\ell} - R_{k\bar\ell k\bar\ell} + R_{k\bar\ell\ell\bar k} - R_{\ell\bar k\ell\bar k} + R_{\ell\bar k k\bar\ell} - R_{\ell\bar\ell k\bar k} \right) \\
& + \sum_{j \neq k, j \neq \ell} \Big(R_{k\bar j j\bar k} + R_{\ell\bar j j\bar\ell} + R_{j\bar k k\bar j} + R_{j\bar\ell\ell\bar j} \Big) \;\geq\; 0.
\end{aligned}
$$

Similarly, setting $f_k = \frac{1}{\sqrt{2}}(e_k - \sqrt{-1}e_\ell)$, $f_\ell = \frac{1}{\sqrt{2}}(e_k + \sqrt{-1}e_\ell)$, $f_j = e_j$, we have

$$
\begin{aligned}
& R(e_k - \sqrt{-1}e_\ell, \overline{e_k + \sqrt{-1}e_\ell}, e_k + \sqrt{-1}e_\ell, \overline{e_k - \sqrt{-1}e_\ell}) \\
& + R(e_k + \sqrt{-1}e_\ell, \overline{e_k - \sqrt{-1}e_\ell}, e_k - \sqrt{-1}e_\ell, \overline{e_k + \sqrt{-1}e_\ell}) \\
& + \frac{1}{2} \sum_{j \neq k, j \neq \ell} \Big(R(e_k - \sqrt{-1}e_\ell, \overline{e_j}, e_j, \overline{e_k - \sqrt{-1}e_\ell}) + R(e_j, \overline{e_k - \sqrt{-1}e_\ell}, e_k - \sqrt{-1}e_\ell, \overline{e_j}) \Big) \\
& \frac{1}{2} \sum_{j \neq k, j \neq \ell} \Big(R(e_k + \sqrt{-1}e_\ell, \overline{e_j}, e_j, \overline{e_k + \sqrt{-1}e_\ell}) + R(e_j, \overline{e_k + \sqrt{-1}e_\ell}, e_k + \sqrt{-1}e_\ell, \overline{e_j}) \Big) \\
= \; & 2 \left(R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} - R_{k\bar k\ell\bar\ell} + R_{k\bar\ell k\bar\ell} + R_{k\bar\ell\ell\bar k} + R_{\ell\bar k\ell\bar k} + R_{\ell\bar k k\bar\ell} - R_{\ell\bar\ell k\bar k} \right) \\
& + \sum_{j \neq k, j \neq \ell} \Big(R_{k\bar j j\bar k} + R_{\ell\bar j j\bar\ell} + R_{j\bar k k\bar j} + R_{j\bar\ell\ell\bar j} \Big) \;\geq\; 0.
\end{aligned}
$$

Hence, we see that

$$
\begin{aligned}
2 \left(R_{k\bar k k\bar k} + R_{\ell\bar\ell\ell\bar\ell} - R_{k\bar k\ell\bar\ell} - R_{\ell\bar\ell k\bar k} + R_{k\bar\ell\ell\bar k} + R_{\ell\bar k k\bar\ell} \right) & \\
+ \sum_{j \neq k, j \neq \ell} \Big(R_{k\bar j j\bar k} + R_{\ell\bar j j\bar\ell} + R_{j\bar k k\bar j} + R_{j\bar\ell\ell\bar j} \Big) \;\geq\; 0. &
\end{aligned}
$$

Since

$$\begin{aligned}
& \mathrm{Ric}^{(3)}_{k\bar{k}} + \mathrm{Ric}^{(3)}_{\ell\bar{\ell}} + \mathrm{Ric}^{(4)}_{k\bar{k}} + \mathrm{Ric}^{(4)}_{\ell\bar{\ell}} \\
= \ & R_{k\bar{k}k\bar{k}} + R_{\ell\bar{\ell}\ell\bar{\ell}} + R_{\ell\bar{k}k\bar{\ell}} + R_{k\bar{\ell}\ell\bar{k}} + \sum_{j\neq k, j\neq \ell} (R_{j\bar{k}k\bar{j}} + R_{j\bar{\ell}\ell\bar{j}}) \\
& + R_{k\bar{k}k\bar{k}} + R_{\ell\bar{\ell}\ell\bar{\ell}} + R_{k\bar{\ell}\ell\bar{k}} + R_{\ell\bar{k}k\bar{\ell}} + \sum_{j\neq k, j\neq \ell} (R_{k\bar{j}j\bar{k}} + R_{\ell\bar{j}j\bar{\ell}}) \\
= \ & 2\left(R_{k\bar{k}k\bar{k}} + R_{\ell\bar{\ell}\ell\bar{\ell}} + R_{\ell\bar{k}k\bar{\ell}} + R_{k\bar{\ell}\ell\bar{k}}\right) + \sum_{j\neq k, j\neq \ell} (R_{j\bar{k}k\bar{j}} + R_{j\bar{\ell}\ell\bar{j}} + R_{k\bar{j}j\bar{k}} + R_{\ell\bar{j}j\bar{\ell}}),
\end{aligned}$$

it follows that

$$\mathrm{Ric}^{(3)}_{k\bar{k}} + \mathrm{Ric}^{(3)}_{\ell\bar{\ell}} + \mathrm{Ric}^{(4)}_{k\bar{k}} + \mathrm{Ric}^{(4)}_{\ell\bar{\ell}} \ \geq \ 2\left(R_{k\bar{k}\ell\bar{\ell}} + R_{\ell\bar{\ell}k\bar{k}}\right).$$

For the statement concerning the altered scalar curvature, observe that

$$\begin{aligned}
2\widetilde{\mathrm{Scal}}_{\omega} &= \frac{1}{n-1}\sum_{k<\ell}(\mathrm{Ric}^{(3)}_{k\bar{k}} + \mathrm{Ric}^{(3)}_{\ell\bar{\ell}}) + \frac{1}{n-1}\sum_{k<\ell}(\mathrm{Ric}^{(4)}_{k\bar{k}} + \mathrm{Ric}^{(4)}_{\ell\bar{\ell}}) \\
&= \frac{1}{n-1}\sum_{1\leq k<\ell n}\left(\mathrm{Ric}^{(3)}_{k\bar{k}} + \mathrm{Ric}^{(3)}_{\ell\bar{\ell}} + \mathrm{Ric}^{(4)}_{k\bar{k}} + \mathrm{Ric}^{(4)}_{\ell\bar{\ell}}\right) \\
&\geq \frac{2}{n-1}\sum_{1\leq k<\ell\leq n}(R_{k\bar{k}\ell\bar{\ell}} + R_{\ell\bar{\ell}k\bar{k}}).
\end{aligned}$$

□

Corollary 2.4.71. Let (X,ω) be a Hermitian manifold with $\mathrm{QOBC}_{\omega}\geq 0$ and $\widetilde{\mathrm{QOBC}}_{\omega}\geq 0$. Then $\mathrm{Scal}_{\omega}\geq 0$ and $\widetilde{\mathrm{Scal}}_{\omega}\geq 0$.

PROOF. The above theorem implies that if $\mathrm{QOBC}_{\omega}\geq 0$, then

$$\mathrm{Scal}_{\omega} \ \geq \ \frac{1}{n-1}\sum_{1\leq k<\ell\leq n}(R_{k\bar{\ell}\ell\bar{k}} + R_{\ell\bar{k}k\bar{\ell}}). \tag{2.4.2}$$

If $\widetilde{\mathrm{QOBC}}_{\omega}\geq 0$, then $\sum_{i,j=1}^{n} R_{i\bar{j}j\bar{i}}(\xi_i-\xi_j)^2\geq 0$ for all $\xi=(\xi_1,...,\xi_n)\in\mathbb{R}^n$. Setting $\xi_k=1$, $\xi_j=0$ for any $j\neq k$, we see that

$$\sum_{j\neq k}(R_{j\bar{k}k\bar{j}} + R_{k\bar{j}j\bar{k}}) \ \geq \ 0.$$

In particular, the right-hand side of (2.4.2) is non-negative. □

Question 2.4.72. Can one give a combinatorial proof (i.e., using Euclidean distance matrices and graph-theoretic results) of Niu's theorem $\mathrm{QOBC}_{\omega}>0 \implies \mathrm{Scal}_{\omega}>0$? Does this have any ramifications ramifications on the combinatorial side?

Example 2.4.73. The Boothby metric ω_0 on the Hopf surface $\mathbb{S}^3 \times \mathbb{S}^1$ has constant $\widetilde{\mathrm{QOBC}}_{\omega_0} \equiv 0$ and $\mathrm{QOBC}_{\omega_0} \geq 0$.

PROOF. The (Chern) QOBC of the standard metric on the Hopf surface is

$$ {}^c\mathrm{QOBC}_\omega(v) \;=\; \frac{1}{|v|^2}({}^cR_{1\bar{1}2\bar{2}} + {}^cR_{2\bar{2}1\bar{1}})(v_1 - v_2)^2 \;=\; \frac{4}{|v|^2|z|^4}(v_1 - v_2)^2. $$

From the scale invariance, assume $v = (v_1, v_2)$ is a unit vector, then

$$ {}^c\mathrm{QOBC}_\omega(v) \;=\; \frac{8}{|z|^4}(1 - v_1\sqrt{1 - v_1^2}), $$

this is maximized when $v_1 = -\frac{1}{\sqrt{2}}$ with value ${}^c\mathrm{QOBC}_\omega(v) = \frac{8}{|z|^4}$, and is minimized at $v_1 = \frac{1}{\sqrt{2}}$ with value ${}^c\mathrm{QOBC}_\omega(v)$. The (Chern) altered QOBC of the standard metric on the Hopf surface is

$$ {}^c\widetilde{\mathrm{QOBC}}_\omega \;=\; ({}^cR_{1\bar{2}2\bar{1}} + {}^cR_{2\bar{1}1\bar{2}})(1 - 2v_1\sqrt{1 - v_1^2}) \;\equiv\; 0. $$

□

Question 2.4.74. Let (X, ω) be a compact Hermitian manifold with ${}^c\mathrm{QOBC}_\omega > 0$ or ${}^c\widetilde{\mathrm{QOBC}}_\omega > 0$. Does X have negative Kodaira dimension $\kappa_X = -\infty$?

Let us end this section by mentioning the following important conjecture:

Conjecture 2.4.75. A compact simply connected homogeneous Kähler manifold (which is called a *Kähler C-space*) admits a Kähler metric with $\mathrm{QOBC} \geq 0$.

2.5. The Holomorphic Sectional Curvature and its Variants

Although the holomorphic bisectional curvature is natural from the perspective of the sectional curvature, the most natural object from the point of view of complex geometry is the holomorphic sectional curvature:

Definition 2.5.1. Let (X, ω) be a Hermitian manifold. The *t–Gauduchon holomorphic sectional curvature* is defined by

$$ {}^t\mathrm{HSC}_\omega(v) \ := \ \frac{1}{|v|_\omega^4} {}^tR(v, \overline{v}, v, \overline{v}), \qquad v \in T^{1,0}X, $$

where ${}^tR$ is the t–Gauduchon curvature tensor.

It is immediate that the (t–Gauduchon) holomorphic bisectional curvature ${}^t\mathrm{HBC}_\omega$ dominates the (t–Gauduchon) holomorphic sectional curvature ${}^t\mathrm{HSC}_\omega$.

Example 2.5.2. Let $\mathbb{B}^2 \subset \mathbb{C}^2$ be equipped with the Poincaré metric ω_{P}. Then

(i) $-4 \leq \mathrm{Sec}_{\omega_{\mathrm{P}}} \leq -1$.
(ii) $-2 \leq \mathrm{HBC}_{\omega_{\mathrm{P}}} \leq -1$.
(iii) $\mathrm{HSC}_{\omega_{\mathrm{P}}} \equiv -2$.

Example 2.5.3. Let $\mathbb{D} \times \mathbb{D}$ be the bidisk endowed with the product of the Poincaré metrics on each factor. If ω denotes the product metric, then

(i) $\mathrm{Sec}_\omega \leq 0$.
(ii) $\mathrm{HBC}_\omega \leq 0$.
(iii) $-2 \leq \mathrm{HSC}_\omega \leq -1$.

In particular, the product of two metrics with negative bisectional curvature will not have negative bisectional curvature in general. For the holomorphic sectional curvature, however, we have the following:

Proposition 2.5.4. (Grauert–Reckziegel [**136**]). Let Σ be a Riemann surface endowed with Hermitian metrics ω and η. Let K_ω and K_η denote the Gauss curvatures of these metrics. For any point $p \in \Sigma$,

(i) if $K_\omega(p) \leq 0$ and $K_\eta(p) \leq 0$, then $K_{\omega+\eta}(p) \leq 0$.
(ii) if $K_\omega(p) \leq -K_1 < 0$ and $K_\eta(p) \leq -K_2 < 0$, then

$$ K_{\omega+\eta}(p) \leq -\frac{K_1 K_2}{K_1 + K_2}. $$

Proof. Write $\omega = \sqrt{-1} g(z) dz \wedge d\overline{z}$ and $\eta = \sqrt{-1} h(z) dz \wedge d\overline{z}$. The Gauss curvature of ω (and similarly for η) is given by the formula

$$ K_g \ = \ -\frac{2}{g^3} \left(g \frac{\partial^2 g}{\partial z \partial \overline{z}} - \left| \frac{\partial g}{\partial z} \right|^2 \right). $$

It is easy to verify the following identity:

$$gh(g+h)(g^2K_g + h^2K_h - (g+h)^2K_{g+h}) \quad = \quad 2\left|g\frac{\partial g}{\partial z} - h\frac{\partial g}{\partial z}\right|^2 \ \geq \ 0.$$

It is then clear that

$$K_{g+h}(p) \quad \leq \quad \frac{g^2K_g(p) + h^2K_h(p)}{(g+h)^2},$$

from which (i) is immediate. For (ii), apply the elementary inequality

$$-\frac{(g^2K_1 + h^2K_2}{(g+h)^2} \quad \leq \quad -\frac{K_1K_2}{K_1+K_2}.$$

□

We can deduce a similar statement for the holomorphic sectional curvature by giving the following enlightening interpretation: Let (X,ω) be a Hermitian manifold. Fix a point $p \in X$ and a vector $v \in T_p^{1,0}X$. Let $f : \mathbb{D} \to X$ be a holomorphic disk, i.e., a holomorphic map from the unit disk $\mathbb{D} \subset \mathbb{C}$ into X such that $f(0) = p$ and $f'(0) = v$. We may assume f is a holomorphic immersion. Hence, the pullback metric $f^*\omega$ defines a non-degenerate smooth Hermitian metric on $\mathbb{D}$. We may therefore compute its Gauss curvature. It goes back to Wu [301] that the holomorphic sectional curvature can be defined by

$$\mathrm{HSC}_\omega \quad = \quad \sup_f K_{f^*\omega},$$

where the supremum is over all holomorphic disks satisfying $f(0) = p$ and $f'(0) = v$. As a consequence, we have:

Corollary 2.5.5. Let X be a Hermitian manifold with Hermitian metrics ω and η.

(i) If $\mathrm{HSC}_\omega \leq 0$ and $\mathrm{HSC}_\eta \leq 0$, then $\mathrm{HSC}_{\omega+\eta} \leq 0$.
(ii) If $\mathrm{HSC}_\omega \leq -K_1 < 0$ and $\mathrm{HSC}_\eta \leq -K_2 < 0$, then

$$\mathrm{HSC}_{\omega+\eta} \leq -\frac{K_1K_2}{K_1+K_2}.$$

Remark 2.5.6. Note the special property of the holomorphic sectional curvature here compared to the sectional curvature. In general, the sectional curvature of the sum of Riemannian metrics is forsaken by the sectional curvature of the summands.

Remark 2.5.7. Since the holomorphic sectional curvature is the restriction of the bisectional curvature to the diagonal in $T^{1,0}X \times T^{1,0}X$, we see that the holomorphic sectional curvature inherits the subbundle decreasing property: If $f : Y \hookrightarrow (X,\omega)$ is a complex submanifold, then

$$\mathrm{HSC}_{f^*\omega} \ \leq \ \mathrm{HSC}_\omega.$$

Remark 2.5.8. The value of the sectional curvature depends only on the two-plane spanned by the vectors u, v, and not on the specific choice of vectors; in other words, the sectional curvature descends to a function on the Grassmannian bundle of two-planes in the tangent bundle

$$\mathrm{Sec}_\omega : \mathcal{G}\mathrm{r}_2(T^{1,0}X) \to \mathbb{R}.$$

This Grassmannian contains a holomorphic vector bundle $\mathcal{G}\mathrm{r}_2^J(T^{1,0}X)$, given by the two-planes in $T^{1,0}X$ which are invariant under the complex structure. In fact, $\mathcal{G}\mathrm{r}_2^J(T^{1,0}X)$ is a projective bundle with fiber $\mathbb{P}^{n-1}$. The restriction of the sectional curvature Sec_ω to $\mathcal{G}\mathrm{r}_2^J(T^{1,0}X)$ defines the *holomorphic sectional curvature*. In particular, since the sectional curvature determines the curvature tensor complete, the same can be said for the holomorphic sectional curvature of a Kähler metric:

Theorem 2.5.9. Let (X, ω) be a Kähler manifold. The holomorphic sectional curvature determines the curvature tensor.

The above theorem is an immediate consequence of the following (multi-)linear algebra theorem:

Lemma 2.5.10. Let S_1 and S_2 be two symmetric bi-Hermitian forms in the sense that

$$S_k(\xi, \overline{\eta}, \zeta, \overline{\omega}) \;=\; S_k(\zeta, \overline{\eta}, \xi, \overline{\omega}), \qquad \text{and} \qquad S_k(\eta, \overline{\xi}, \omega, \overline{\zeta}) \;=\; \overline{S}_k(\xi, \overline{\eta}, \zeta, \overline{\omega}),$$

for $k \in \{1, 2\}$. If $S_1(\xi, \overline{\xi}, \xi, \overline{\xi}) = S_2(\xi, \overline{\xi}, \xi, \overline{\xi})$ for all ξ, then $S_1 = S_2$.

Proof. Let $S := S_1 - S_2$. For $\alpha \in \mathbb{C}$, we have

$$\begin{aligned}
0 &= S(\xi + \alpha\zeta, \overline{\xi + \alpha\zeta}, \xi + \alpha\zeta, \overline{\xi + \alpha\zeta}) \\
&= S(\xi, \overline{\xi}, \xi, \overline{\xi}) + \alpha\left[S(\xi, \overline{\xi}, \zeta, \overline{\xi}) + S(\zeta, \overline{\xi}, \xi, \overline{\xi})\right] + \overline{\alpha}\left[S(\xi, \overline{\xi}, \xi, \overline{\zeta}) + S(\xi, \overline{\zeta}, \xi, \overline{\xi})\right] \\
&\quad + |\alpha|^2\left[S(\xi, \overline{\xi}, \zeta, \overline{\zeta}) + S(\xi, \overline{\zeta}, \zeta, \overline{\xi}) + S(\zeta, \overline{\zeta}, \xi, \overline{\xi}) + S(\zeta, \overline{\xi}, \xi, \overline{\zeta})\right] \\
&\quad + \overline{\alpha}^2 S(\xi, \overline{\zeta}, \xi, \overline{\zeta}) + \alpha^2 S(\zeta, \overline{\xi}, \zeta, \overline{\xi}) + |\alpha|^4 S(\zeta, \overline{\zeta}, \zeta, \overline{\zeta}) \\
&\quad + \alpha|\alpha|^2\left[S(\zeta, \overline{\xi}, \zeta, \overline{\zeta}) + S(\zeta, \overline{\zeta}, \zeta, \overline{\xi})\right] + \overline{\alpha}|\alpha|^2\left[S(\zeta, \overline{\zeta}, \xi, \overline{\zeta}) + S(\xi, \overline{\zeta}, \zeta, \overline{\zeta})\right].
\end{aligned}$$

The $|\alpha|^2$ coefficient is

$$\begin{aligned}
S(\xi, \overline{\xi}, \zeta, \overline{\zeta}) + S(\xi, \overline{\zeta}, \zeta, \overline{\xi}) + S(\zeta, \overline{\zeta}, \xi, \overline{\xi}) + S(\zeta, \overline{\xi}, \xi, \overline{\zeta}) &= 2S(\xi, \overline{\xi}, \zeta, \overline{\zeta}) + 2S(\zeta, \overline{\zeta}, \xi, \overline{\xi}) \\
&= 4S(\xi, \overline{\xi}, \zeta, \overline{\zeta}) \;=\; 0.
\end{aligned}$$

Similarly, by expanding

$$S(\xi + \alpha\zeta, \overline{\xi + \alpha\zeta}, \eta + \mu\omega, \overline{\eta + \mu\omega}) \;=\; 0,$$

the coefficient of $\overline{\alpha\mu}$ yields $S(\xi, \overline{\zeta}, \eta, \overline{\omega}) = 0$. □

Example 2.5.11. For a constant $\kappa \in \mathbb{R}$, let us denote by $(\mathcal{S}^n_\kappa, \omega_\kappa)$ the *complex space form* of constant curvature κ. If $\kappa = 0$, $\kappa > 0$, or $\kappa < 0$, then $\mathcal{S}^n_\kappa \simeq \mathbb{C}^n$, $\mathcal{S}^n_\kappa \simeq \mathbb{P}^n$, and $\mathcal{S}^n_\kappa \simeq \mathbb{B}^n$, respectively. For any $\kappa \in \mathbb{R}$, the components of the metric ω_κ are given in a local coordinate frame by

$$g_{i\bar{j}} \ := \ \frac{\delta_{ij}(1+\kappa|z|^2) - \kappa z_j \overline{z}_i}{(1+\kappa|z|^2)^2}.$$

It is easy to see that

$$R_{i\bar{j}k\bar{\ell}} \ = \ \kappa(g_{i\bar{j}}g_{k\bar{\ell}} + g_{k\bar{j}}g_{i\bar{\ell}}).$$

In particular, the holomorphic sectional curvature of ω_κ is constant, equal to $\mathrm{HSC}_\omega \equiv 2\kappa$.

From 2.5.9, together with the Cartan–Ambrose–Hicks theorem 1.1.80, we have the following:

Corollary 2.5.12. Any two simply connected complete Kähler manifolds with constant holomorphic sectional curvature are biholomorphically isometric.

Given our complete understanding of Kähler manifolds with constant holomorphic sectional curvature, it is natural to ask about non-Kähler Hermitian manifolds with constant holomorphic sectional curvature. The long-standing conjecture in this direction is the following:

Conjecture 2.5.13. Let (X, ω) be a compact Hermitian manifold with pointwise constant Chern holomorphic sectional curvature ${}^c\mathrm{HSC}_\omega \equiv \kappa$ for some $\kappa \in \mathbb{R}$. If $\kappa \neq 0$, then ω is Kähler, otherwise, ω is Chern-flat.

Remark 2.5.14. Note that by the Schur lemma, if the holomorphic sectional curvature of a Kähler metric is pointwise constant, then it is globally constant. This is not true for a general non-Kähler Hermitian metric. The first example was discovered by Gray–Vanhecke [**139**]: Let $\omega_{\mathbb{C}^n}$ denote the Euclidean metric on $\mathbb{C}^n$. Let $f : \mathbb{C}^n \to \mathbb{C}$ be any non-linear holomorphic function. The metric $\omega := (1 + \mathrm{Re}(f))^{-2}\omega_{\mathbb{C}^n}$ has pointwise constant (Chern) holomorphic sectional curvature, but not constant (Chern) holomorphic sectional curvature.

For complex surfaces, Balas–Gauduchon [**18**] verified the conjecture in the cases that $\kappa = 0$ and $\kappa < 0$. The positive $\kappa > 0$ case (still for compact complex surfaces) was treated by Apostolov–Davidov–Muskarov [**13**]. Kai Tang [**271**] proved the conjecture for compact Kähler-like Hermitian manifolds. For compact locally conformally Kähler manifolds, the conjecture was verified in the $\kappa < 0$ and $\kappa = 0$ case by Chen–Chen–Nie [**85**]. Rao–Zheng [**239**] verified the conjecture for compact Bismut Kähler-like manifolds. They also obtain some information on the Chern curvature tensor and Chern torsion of pluriclosed metrics with pointwise constant (Chern) holomorphic sectional curvature [**239**, Theorem 2]. Under the stronger assumption of (pointwise constant) vanishing (Chern) real bisectional curvature, Zhou–Zheng [**326**] verified the conjecture for compact Hermitian threefolds. For complex nilmanifolds, Li–Zheng [193] showed that if the (Chern) holomorphic sectional curvature is pointwise constant, the constant must be zero. Chen–Chen–Nie also illustrated the necessity of the compactness assumption by producing the following example [85, Example 3.9]:

Example 2.5.15. Let $\omega := \sqrt{-1}\partial\overline{\partial}\log(1+|z|^2)$ be the restriction to $\mathbb{C}^n$ of the Fubini–Study metric on $\mathbb{P}^n$. The components of this metric are locally given by

$$g_{i\bar{j}} \;=\; \frac{(1+|z|^2)\delta_{ij} - \overline{z}_i z_j}{(1+|z|^2)^2},$$

and the holomorphic sectional curvature is constant $\mathrm{HSC}_\omega \equiv 2$. Let $f := 2\log(1+|z|^2)$ and conformally rescale the metric, defining $\omega_f := e^f\omega$. Then $\mathrm{HSC}_{\omega_f} \equiv 0$, but the curvature of ω_f:

$$R_{i\bar{j}k\bar{\ell}} \;=\; e^f(g_{i\bar{\ell}}g_{k\bar{j}} - g_{i\bar{j}}g_{k\bar{\ell}})$$

is nowhere vanishing.

Note that by an old theorem of Boothby [40], we have a complete understanding of compact complex manifolds admitting a Chern-flat Hermitian metric:

Theorem 2.5.16. (Boothby). A compact Hermitian manifold X with vanishing Chern curvature has a complex Lie group as its universal cover $\widetilde{X}$. In particular, X cannot be simply connected.

Proof. The Bianchi identities 2.2.67 for the curvature and torsion are

$$\begin{aligned} R_{i\bar{j}k\bar{\ell}} - R_{k\bar{j}i\bar{\ell}} &= 2T^j_{ik,\bar{\ell}}, \\ T^i_{hk}T^h_{\ell j} + T^i_{h\ell}T^j_{jk} + T^i_{hj}T^h_{k\ell} &= \frac{1}{2}(T^i_{j\ell,k} + T^i_{\ell k,j} + T^i_{kj,\ell}). \end{aligned}$$

Since the (Chern) curvature vanishes, we see that $T^j_{ik,\bar{\ell}} = 0$. In particular, the components of the (Chern) torsion are complex-analytic. Let

$$\Phi \;:=\; \sum_{i,j,k} |T^i_{jk}|^2.$$

Since $T^i_{jk,\bar{\ell}} = 0$, we see that

$$L(\Phi) \;:=\; \sum_h \Phi_{h\bar{h}} \;=\; \sum_{i,j,k,h} \overline{T^i_{jk}}T^i_{jk,h\bar{h}} + \overline{T^i_{jk,h}}T^i_{jk,h}.$$

From the interchange formulae, we have

$$T^i_{jk,h\bar{h}} \;=\; T^i_{jk,\bar{h}h} \;=\; 0,$$

and therefore

$$L(\Phi) \;=\; \sum_{i,j,k,h} |T^i_{jk,h}|^2 \;\geq\; 0.$$

By the maximum principle, it follows that $T^i_{jk,h} \equiv 0$. From the second Bianchi identity 2.2.67, it follows that the torsion components T^i_{jk} satisfy the Jacobi identity. Restricting to parallel orthonormal frames, the structure equations become

$$d\omega_i \;=\; T^i_{jk}\omega_j \wedge \omega_k, \qquad T^i_{jk} + T^i_{kj} \;=\; 0,$$

and we have $\omega^i_j = 0$. Since relative to these frames, the T^i_{jk} are constants and satisfy the Jacobi identity; the ω_i are the left-invariant forms of a local Lie group. The universal cover $\widetilde{X}$ is complete and a complex Lie group. The assertion in the statement of the theorem follows from the fact that the only compact complex Lie groups are tori, which are certainly not simply connected. $\square$

A useful tool for the study of Hermitian manifolds with (pointwise constant) Chern holomorphic sectional curvature is the following result due to Balas [17]:

Lemma 2.5.17. (Balas lemma). Let (X, ω_g) be a Hermitian manifold. Then the (Chern) holomorphic sectional curvature is (pointwise) constant $\mathrm{HSC}_\omega \equiv \kappa$ if and only if

$$R_{i\bar{j}k\bar{\ell}} + R_{k\bar{j}i\bar{\ell}} + R_{i\bar{\ell}k\bar{j}} + R_{k\bar{\ell}i\bar{j}} \;=\; 2\kappa(g_{i\bar{j}}g_{k\bar{\ell}} + g_{i\bar{\ell}}g_{k\bar{j}})$$

in any local frame.

Question 2.5.18. Does the Balas lemma hold for the t–Gauduchon connections?

The author, together with Kai Tang, proved the following [60, Theorem 2.1]:

Theorem 2.5.19. Let (X, ω) be a compact Hermitian manifold with pointwise constant Chern holomorphic sectional curvature ${}^c\mathrm{HSC}_\omega \equiv \kappa$. Then

$$\int_X {}^c\mathrm{Scal}_\omega \omega^n \;=\; \frac{\kappa n(n+1)}{2}\mathrm{vol}(X, \omega) + 2\int_X |\tau|^2\omega^n,$$

where τ denotes the (Chern) torsion $(1,0)$–form.

Proof. If (X^n, ω) is a compact Hermitian manifold with pointwise constant Chern holomorphic sectional curvature ${}^c\mathrm{HSC}_\omega \equiv c$, the Balas lemma [17] gives

$$ {}^cR_{i\bar{i}k\bar{k}} + {}^cR_{k\bar{i}i\bar{k}} + {}^cR_{i\bar{k}k\bar{i}} + {}^cR_{k\bar{k}i\bar{i}} \;=\; 2\kappa. \tag{2.5.1}$$

Hence, from (2.5.5), we have

$$\begin{aligned} 2\sum_i \tau_{i,\bar{i}} \;&=\; \sum_{i,k}({}^cR_{i\bar{i}k\bar{k}} - {}^cR_{k\bar{i}i\bar{k}}) \;=\; \sum_{i\neq k}({}^cR_{i\bar{i}k\bar{k}} - {}^cR_{k\bar{i}i\bar{k}}) \\ &=\; \sum_{i<k}({}^cR_{i\bar{i}k\bar{k}} + {}^cR_{k\bar{k}i\bar{i}}) - \sum_{i<k}({}^cR_{k\bar{i}i\bar{k}} + {}^cR_{i\bar{k}k\bar{i}}). \end{aligned}$$

From (2.5.1), we see that

$$\sum_{i<k}({}^cR_{k\bar{i}i\bar{k}} + {}^cR_{i\bar{k}k\bar{i}}) = \sum_{i<k}\left[2\kappa - ({}^cR_{i\bar{i}k\bar{k}} + {}^cR_{k\bar{k}i\bar{i}})\right].$$

Therefore,

$$\begin{aligned} 2\sum_i \tau_{i,\bar{i}} &= 2\sum_{i\neq k}({}^cR_{i\bar{i}k\bar{k}} + {}^cR_{k\bar{k}i\bar{i}}) - 2\kappa\frac{n(n-1)}{2} \\ &= 2\left({}^c\mathrm{Scal}_\omega - n\kappa\right) - \kappa n(n-1) = 2{}^c\mathrm{Scal}_\omega - \kappa n(n+1). \end{aligned} \tag{2.5.2}$$

From (2.5.6), it follows that

$$\int_X \left({}^c\mathrm{Scal}_\omega - \frac{1}{2}\kappa n(n+1)\right)\omega^n = 2\int_X |\tau|^2\omega^n,$$

or equivalently,

$$\int_X {}^c\mathrm{Scal}_\omega\, \omega^n = \frac{\kappa n(n+1)}{2}\int_X \omega^n + 2\int_X |\tau|^2\omega^n.$$

□

Remark 2.5.20. We will exhibit further refinements of the above theorem in [**57**].

Corollary 2.5.21. Let (X,ω) be a compact Hermitian manifold with pointwise constant Chern holomorphic sectional curvature ${}^c\mathrm{HSC}_\omega \equiv 0$. If the total Chern scalar curvature of ω vanishes, then ω is balanced. Further, there are three distinct cases:

(i) $\kappa_X = -\infty$ and K_X is unitary flat.
(ii) $\kappa_X = -\infty$ and neither K_X nor K_X^{-1} are pseudoeffective.
(iii) $\kappa_X = 0$ and K_X is holomorphically torsion.

Proof. If the holomorphic sectional curvature of ω vanishes identically, then 2.5.19 implies that the total Chern scalar curvature is non-negative. Assuming the total Chern scalar curvature vanishes, we see that ω is balanced and, in particular, Gauduchon. From [**313**, Theorem 1.4], there are the three cases (i)–(iii) in the statement. □

Corollary 2.5.22. Let (X,ω) be a compact Hermitian manifold with pointwise constant (Chern) holomorphic sectional curvature ${}^c\mathrm{HSC}_\omega \equiv 0$. If the total Chern scalar curvature of ω vanishes and ω is k–Gauduchon for some $1 \leq k \leq n-1$, then the metric is Chern-flat.

The Altered Holomorphic Sectional Curvature. It will be fruitful to introduce a variant of the holomorphic sectional curvature which is more readily comparable to the other curvatures:

Definition 2.5.23. Let (X, ω) be a Hermitian manifold. The t–*Gauduchon altered holomorphic sectional curvature* is the function in any local unitary frame by

$$ {}^t\widetilde{\mathrm{HSC}}_\omega(v) \quad := \quad \frac{1}{|v|^2_\omega} \sum_{\alpha,\gamma} \left({}^tR_{\alpha\bar{\alpha}\gamma\bar{\gamma}} + {}^tR_{\alpha\bar{\gamma}\gamma\bar{\alpha}} \right) v_\alpha v_\gamma, $$

where $v = (v_1, ..., v_n) \in \mathbb{R}^n \backslash \{0\}$.

Remark 2.5.24. The (Chern) altered holomorphic sectional curvature was formally introduced by the author and Kai Tang in [60], although it appeared implicitly in earlier works (see, e.g., [315]).

Proposition 2.5.25. Let (X, ω) be a Hermitian manifold. The t–Gauduchon altered holomorphic sectional curvature ${}^t\widetilde{\mathrm{HSC}}_\omega$ and t–Gauduchon holomorphic sectional curvature ${}^t\mathrm{HSC}_\omega$ are comparable in the sense that they have the same sign.

Proof. Let $\{e_\alpha\}$ be a local unitary frame for $T^{1,0}X$. The t–Gauduchon holomorphic sectional curvature in the direction of $u \in T^{1,0}X$ is given by

$$ {}^t\mathrm{HSC}_\omega(u) \quad = \quad \frac{1}{|u|^4_\omega} \sum_{\alpha,\beta,\gamma,\delta} {}^tR_{\alpha\bar{\beta}\gamma\bar{\delta}} u_\alpha \overline{u}_\beta u_\gamma \overline{u}_\delta. $$

Let $A = (\varepsilon^A_\alpha) \in \mathbb{Z}^n_4$, where $\mathbb{Z}_4 := \mathbb{Z}/4\mathbb{Z}$, and set $u_A := \sum_\alpha \varepsilon^A_\alpha e_\alpha$. Then

$$ \sum_{A \in \mathbb{Z}^n_4} {}^tR(u_A, \overline{u}_A, u_A, \overline{u}_A) \quad = \quad \sum_{A \in \mathbb{Z}^n_4} \sum_{\alpha,\beta,\gamma,\delta} {}^tR_{\alpha\bar{\beta}\gamma\bar{\delta}} \varepsilon^A_\alpha \overline{\varepsilon}^A_\beta \varepsilon^A_\gamma \overline{\varepsilon}^A_\delta \;=\; 4^n \sum_{\alpha,\gamma} \left({}^tR_{\alpha\bar{\alpha}\gamma\bar{\gamma}} + {}^tR_{\alpha\bar{\gamma}\gamma\bar{\alpha}} \right), $$

proving the claim. □

Although the (t–Gauduchon) holomorphic sectional curvature and (t–Gauduchon) altered holomorphic sectional curvature are comparable, there is the following subtlety concerning when they are (pointwise) constant:

Proposition 2.5.26. Let (X, ω) be a compact Hermitian manifold. If ${}^c\mathrm{HSC}_\omega \equiv \kappa$, then for any unit vector $v \in \mathbb{R}^n \backslash \{0\}$,

$$ {}^c\widetilde{\mathrm{HSC}}_\omega(v) \quad = \quad \kappa \left(1 + \sum_{1 \leq i,k \leq n} v_i v_k \right). $$

In particular, ${}^c\mathrm{HSC}_\omega \equiv \kappa$ and ${}^c\widetilde{\mathrm{HSC}}_\omega \equiv \kappa$ if and only if $\kappa = 0$.

Proof. Write R for the Chern curvature tensor. Suppose the holomorphic sectional curvature of ω is constant, equal to c. The Balas lemma [17] implies that in any unitary frame, we have

$$ R_{i\bar{i}k\bar{k}} + R_{k\bar{i}i\bar{k}} + R_{i\bar{k}k\bar{i}} + R_{k\bar{k}i\bar{i}} \quad = \quad 2\kappa. $$

Therefore, for $i=j$, $k=\ell$, and $i \neq k$, we have

$$R_{i\bar{i}k\bar{k}} + R_{k\bar{i}i\bar{k}} + R_{i\bar{k}k\bar{i}} + R_{k\bar{k}i\bar{i}} \quad = \quad 2\kappa.$$

For a unit vector $v=(v_1,...,v_n) \in \mathbb{R}^n \backslash \{0\}$, we have

$$\sum_{i \neq k} (R_{i\bar{i}k\bar{k}} + R_{k\bar{i}i\bar{k}} + R_{i\bar{k}k\bar{i}} + R_{k\bar{k}i\bar{i}}) v_i v_k \quad = \quad 2\kappa \sum_{i \neq k} v_i v_k.$$

The expressions $\sum_{i\neq k}(R_{i\bar{i}k\bar{k}} + R_{k\bar{k}i\bar{i}})v_i v_k$ and $\sum_{i \neq k}(R_{k\bar{i}i\bar{k}} + R_{i\bar{k}k\bar{i}})v_i v_k$ are symmetric, hence,

$$\sum_{i \neq k} (R_{i\bar{i}k\bar{k}} + R_{i\bar{k}k\bar{i}}) v_i v_k \quad = \quad \kappa \sum_{i \neq k} v_i v_k.$$

Since, by definition, $R_{i\bar{i}i\bar{i}} = \kappa$, we have

$$\begin{aligned}
\sum_{1 \leq i,k \leq n} (R_{i\bar{i}k\bar{k}} + R_{i\bar{k}k\bar{i}}) v_i v_k \quad &= \quad \kappa \sum_{i \neq k} v_i v_k + 2\kappa \sum_{i=1}^n v_i^2 \\
&= \quad \kappa \left(\sum_{1 \leq i,k \leq n} v_i v_k + \sum_{i=1}^n v_i^2 \right) \\
&= \quad \kappa \left(1 + \sum_{1 \leq i,k \leq n} v_i v_k \right),
\end{aligned}$$

where the last equality uses the fact that v has unit length. □

Using the altered holomorphic sectional curvature, we establish the following monotonicity theorem for the t–Gauduchon holomorphic sectional curvature:

Theorem 2.5.27. Let (X,ω) be a Hermitian manifold. The t–Gauduchon altered holomorphic sectional curvature is given by

$$^t\widetilde{\text{HSC}}_\omega \quad = \quad {}^c\widetilde{\text{HSC}}_\omega - \frac{(t-1)^2}{4|v|_\omega^2} \sum_q \left({}^cT_{iq}^i \overline{{}^cT_{kq}^k} + {}^cT_{iq}^k \overline{{}^cT_{kq}^i} \right) v_i v_k. \tag{2.5.3}$$

In particular, ${}^t\widetilde{\text{HSC}}_\omega \leq {}^c\widetilde{\text{HSC}}_\omega$ for all $t \in \mathbb{R}$ and equality holds if and only if $t=1$.

Proof. The altered holomorphic sectional curvature is given by the quadratic form-valued function on the unitary frame bundle:

$$^t\widetilde{\text{HSC}}_\omega(v) \quad = \quad \frac{1}{|v|^4} \sum_{i,k} ({}^tR_{i\bar{i}k\bar{k}} + {}^tR_{i\bar{k}k\bar{i}}) v_i^2 v_k^2,$$

where $v = (v_1, ..., v_n) \in \mathbb{R}^n \backslash \{0\}$. We compute

$$\begin{aligned} {}^tR_{i\bar{i}k\bar{k}} &= \frac{(1-t)}{2}\left[\frac{\partial^2 g_{k\bar{k}}}{\partial z_i \partial \bar{z}_i} - \frac{\partial^2 g_{k\bar{i}}}{\partial z_i \partial \bar{z}_k} - \frac{\partial^2 g_{i\bar{k}}}{\partial z_k \partial \bar{z}_i}\right] - \frac{(1+t)}{2}\frac{\partial^2 g_{k\bar{k}}}{\partial z_i \partial \bar{z}_i} \\ &\quad + \frac{(1-t)^2}{4}\sum_q {}^cT_{iq}^k \overline{{}^cT_{qi}^k} + \frac{t^2}{4}\sum_q {}^cT_{ik}^q \overline{{}^cT_{ik}^q} \\ {}^tR_{i\bar{k}k\bar{i}} &= \frac{(1-t)}{2}\left[\frac{\partial^2 g_{k\bar{i}}}{\partial z_i \partial \bar{z}_k} - \frac{\partial^2 g_{k\bar{k}}}{\partial z_i \partial \bar{z}_i} - \frac{\partial^2 g_{i\bar{i}}}{\partial z_k \partial \bar{z}_k}\right] - \frac{(1+t)}{2}\left[\frac{\partial^2 g_{k\bar{i}}}{\partial z_i \partial \bar{z}_k}\right] \\ &\quad + \frac{(1-t)^2}{4}\sum_q {}^cT_{iq}^i \overline{{}^cT_{qk}^k} + \frac{t^2}{4}\sum_q {}^cT_{ik}^q \overline{{}^cT_{ki}^q}. \end{aligned}$$

Therefore,

$$\begin{aligned} {}^tR_{i\bar{i}k\bar{k}} + {}^tR_{i\bar{k}k\bar{i}} &= -\frac{\partial^2 g_{i\bar{i}}}{\partial z_k \partial \bar{z}_k} - \frac{\partial^2 g_{i\bar{k}}}{\partial z_k \partial \bar{z}_i} - \frac{(1-t)^2}{4}\sum_q {}^cT_{ik}^q \overline{{}^cT_{ik}^q} - \frac{(1-t)^2}{4}\sum_q {}^cT_{qi}^i \overline{{}^cT_{qk}^k} \\ &= {}^cR_{i\bar{i}k\bar{k}} + {}^cR_{i\bar{k}k\bar{i}} - \frac{(1-s)^2}{4}\sum_q {}^cT_{ik}^q \overline{{}^cT_{ik}^q} - \frac{(1-s)^2}{4}\sum_q {}^cT_{qi}^i \overline{{}^cT_{qk}^k}. \end{aligned}$$

In particular, we see that (taking v to be of unit length for simplicity):

$$ {}^t\widetilde{\mathrm{HSC}}_\omega(v) = {}^c\widetilde{\mathrm{HSC}}_\omega - \frac{(1-s)^2}{4}\sum_{i,k,q} |{}^cT_{ik}^q v_i v_k|^2 - \frac{(1-t)^2}{4}\sum_{i,k,q} {}^cT_{qi}^i \overline{{}^cT_{qk}^k} v_i^2 v_k^2. $$

Since the two terms involving torsion are negative, the result follows. Finally, since the altered holomorphic sectional curvature has the same sign as the holomorphic sectional curvature, this proves the last claim. □

Corollary 2.5.28. Let (X, ω) be a Hermitian manifold. If ${}^t\widetilde{\mathrm{HSC}}_\omega = {}^s\widetilde{\mathrm{HSC}}_\omega$, then $t = s$, $t = 2 - s$, or ω is Kähler.

Since the altered holomorphic sectional curvature is comparable to the holomorphic sectional curvature, the following useful consequences of the above monotonicity result are easily obtained:

Corollary 2.5.29. Let (X, ω) be a Hermitian manifold.

(i) If ${}^c\mathrm{HSC}_\omega \leq 0$, then ${}^t\mathrm{HSC}_\omega \leq 0$ for all $t \in \mathbb{R}$.
(ii) If ${}^t\mathrm{HSC}_\omega > 0$ for some $t \in \mathbb{R}$, then ${}^c\mathrm{HSC}_\omega > 0$.

If one considers relaxations of the pinching of the holomorphic sectional curvature, we have the following result due to Cao–Yang [75]:

Theorem 2.5.30. For any $n \in \mathbb{N}_{\geq 2}$, there is a positive constant $\varepsilon = \varepsilon(n) > 0$ such that any compact Kähler manifold with $\frac{1}{2} - \varepsilon \leq \mathrm{HSC}_\omega \leq 1$ of (complex) dimension n is biholomorphic to one of the following:

(i) $\mathbb{P}^n$.
(ii) $\mathbb{P}^k \times \mathbb{P}^{n-k}$.
(iii) An irreducible rank 2 compact Hermitian symmetric space of dimension n.

One of the most enlightening interpretations of the holomorphic sectional curvature is in terms of holomorphic curves. The two main results concerning the holomorphic sectional curvature in this respect are the following:

Theorem 2.5.31. Let (X, ω) be a Hermitian manifold with (Chern) holomorphic sectional curvature bounded above by a negative constant ${}^c\text{HSC}_\omega \leq -\kappa < 0$. Then X is Brody hyperbolic.

Proof. Let $f : \mathbb{C} \to X$ be a non-constant entire curve. Endow $\mathbb{C}$ with the Euclidean metric $\omega_{\mathbb{C}}$, and write $\Delta_{\omega_{\mathbb{C}}}$ for the $\omega_{\mathbb{C}}$–trace of $\sqrt{-1}\partial\bar{\partial}$. Then

$$\Delta_{\omega_{\mathbb{C}}} |\partial f|^2 \;=\; |\nabla \partial f|^2 - R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \delta^{ij} f_i^\alpha \overline{f_j^\beta} \delta^{pq} f_p^\gamma \overline{f_q^\delta}.$$

Choose a frame such that $f_i^\alpha = \lambda_i \delta_i^\alpha$. Then

$$R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} f_i^\alpha \overline{f_i^\alpha} f_j^\gamma \overline{f_j^\gamma} \;=\; R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \lambda_i^2 \lambda_\gamma^2 \;=\; R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \lambda_\alpha^4,$$

since ∂f has rank one. By the maximum principle, f is constant. □

Theorem 2.5.32. (Greene–Wu [140]). Let (X, ω) be a Hermitian manifold. Let r be the distance (with respect to ω from a fixed point). If

$${}^c\text{HSC}_\omega \;\leq\; -\frac{C}{1+r^2},$$

then X is Kobayashi hyperbolic.

Remark 2.5.33. The assumption in the above theorem cannot be significantly relaxed. Indeed, Seshadri [251] showed that $\mathbb{C}^n$ supports a Kähler metric with holomorphic bisectional curvature bounded above by $-A[(1+r^2)\log(2+r)]^{-1}$. Of course, $\mathbb{C}^n$ is not Kobayashi hyperbolic. In particular, the optimal exponent must lie in the interval $\left[\frac{1}{2}, 1\right]$.

Question 2.5.34. Let (X, ω) be a Hermitian manifold with

$${}^t\text{HSC}_\omega \;\leq\; -\frac{C}{(1+r^k)^\ell}$$

for some $k, \ell \in \mathbb{R}$. Determine the sharpest values of $k, \ell \in \mathbb{R}$ such that X is Kobayashi hyperbolic. Are these the same values that ensure that X is Brody hyperbolic?

The second significant result is due to Yang [312]:

Theorem 2.5.35. Let (X, ω) be a compact Kähler manifold with $\text{HSC}_\omega > 0$. Then X is rationally connected in the sense that any two points lie in this image of a rational curve.

The proof of the above theorem makes use of Yang's notion of RC-positivity:

Definition 2.5.36. Let $(\mathcal{E}, h) \to X$ be a Hermitian holomorphic vector bundle over a complex manifold X. Let $\Theta_h \in \Omega_X^{1,1} \otimes \mathrm{End}(\mathcal{E})$ be its Chern curvature tensor. We say that $(\mathcal{E}, h)$ is *RC–positive* if for any non-zero local section $\sigma \in H^0(\mathcal{E})$, there exists a local section $v \in H^0(T^{1,0}X)$ such that

$$\Theta^{(\mathcal{E},h)}(v, \overline{v}, \sigma, \overline{\sigma}) \quad > \quad 0.$$

Remark 2.5.37. The following results are straightforward (see, [312] for details):

(i) Quotient bundles of RC-positive bundles are RC-positive.
(ii) Subbundles of RC-negative bundles are RC-negative.
(iii) If $(\mathcal{L}, h)$ is RC-positive, then the dual bundle $(\mathcal{L}^*, h^*)$ (with the induced metric) is RC-negative.

By [311, Corollary 1.9]:

Theorem 2.5.38. Let X be a projective manifold. Then the following are equivalent:

(i) X is uniruled[10];
(ii) there exists a smooth Hermitian metric ω with ${}^c\mathrm{Ric}_\omega$ having at least one positive eigenvalue at each point;
(iii) K_X is not pseudo-effective.

Lemma 2.5.39. (Yang [312]). Let (X, ω_g) be a compact Kähler manifold. Let $e_1 \in T_p^{1,0}X$ be a unit vector which minimizes HSC_{ω_g} at $p \in X$. Then for all $v \in T_p^{1,0}X$,

$$2R_{1\overline{1}v\overline{v}} \quad \geq \quad (1 + |g_{v\overline{1}}|^2) R_{1\overline{1}1\overline{1}}.$$

Proof. Let $e_2 \in T_p^{1,0}X$ be a unit vector orthogonal to e_1. For $\vartheta \in \mathbb{R}$, set

$$f_1(\vartheta) \;:=\; \mathrm{HSC}_{\omega_g}(\cos(\vartheta)e_1 + \sin(\vartheta)e_2).$$

This expands to

$$\begin{aligned} f_1(\vartheta) \quad = \quad & R_{1\overline{1}1\overline{1}} \cos^4(\vartheta) + R_{2\overline{2}2\overline{2}} \sin^4(\vartheta) + \sin^2(\vartheta)\cos^2(\vartheta) \left[4R_{1\overline{1}2\overline{2}} + R_{1\overline{2}1\overline{2}} + R_{2\overline{1}2\overline{1}}\right] \\ & + 2\sin(\vartheta)\cos^3(\vartheta) \left[R_{1\overline{1}1\overline{2}} + R_{2\overline{1}1\overline{1}}\right] + 2\cos(\vartheta)\sin^3(\vartheta) \left[R_{1\overline{2}2\overline{2}} + R_{2\overline{1}2\overline{2}}\right]. \end{aligned}$$

Since $f_1(\vartheta) \geq R_{1\overline{1}1\overline{1}}$ for all $\vartheta \in \mathbb{R}$ and $f_1(0) = R_{1\overline{1}1\overline{1}}$, we see that $f_1'(0) = 0$ and $f_1''(0) \geq 0$. Computing the derivatives of $f_1(\vartheta)$, we see that

$$\begin{aligned} f_1'(0) \quad &= \quad 2(R_{1\overline{1}1\overline{2}} + R_{2\overline{1}1\overline{1}}) \;=\; 0, \\ f_1''(0) \quad &= \quad 2(4R_{1\overline{1}2\overline{2}} + R_{1\overline{2}1\overline{2}} + R_{2\overline{1}2\overline{1}}) - 4R_{1\overline{1}1\overline{1}} \;\geq\; 0. \end{aligned}$$

[10] A compact complex manifold is said to be *uniruled* if there is a rational curve passing through every point.

Similarly, define

$$f_2(\vartheta) \ := \ \mathrm{HSC}_{\omega_g}(\cos(\vartheta)e_1 + \sqrt{-1}\sin(\vartheta)e_2).$$

This expands to

$$\begin{aligned} f_2(\vartheta) \ &= \ R_{1\bar{1}1\bar{1}}\cos^4(\vartheta) + R_{2\bar{2}2\bar{2}}\sin^4(\vartheta) + \sin^2(\vartheta)\cos^2(\vartheta)\,[4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}] \\ &\quad -2\sqrt{-1}\sin(\vartheta)\cos^3(\vartheta)\,[R_{1\bar{1}1\bar{2}} - R_{2\bar{1}1\bar{1}}] - 2\sqrt{-1}\cos(\vartheta)\sin^3(\vartheta)\,[R_{1\bar{2}2\bar{2}} - R_{2\bar{1}2\bar{2}}]\,. \end{aligned}$$

We have $f_2'(0) = 0$ and $f_2''(0) \geq 0$, and therefore,

$$\begin{aligned} f_2'(0) \ &= \ -2(R_{1\bar{1}1\bar{2}} - R_{2\bar{1}1\bar{1}}) \ = \ 0, \\ f_2''(0) \ &= \ 2(4R_{1\bar{1}2\bar{2}} - R_{1\bar{2}1\bar{2}} - R_{2\bar{1}2\bar{1}}) - 4R_{1\bar{1}1\bar{1}} \ \geq \ 0. \end{aligned}$$

Combining these relations with the ones inherited from $f_1'(0) = 0$ and $f_1''(0) \geq 0$, we see that

$$R_{1\bar{1}1\bar{2}} \ = \ R_{1\bar{1}2\bar{1}} \ = \ 0, \qquad \text{and} \qquad 2R_{1\bar{1}2\bar{2}} \ \geq \ R_{1\bar{1}1\bar{1}}.$$

Let $v \in T_p^{1,0}X$ be a unit vector. If v is parallel to e_1, then

$$2R_{1\bar{1}v\bar{v}} \ = \ 2R_{1\bar{1}1\bar{1}} \ = \ (1 + |g_{v\bar{1}}|^2)R_{1\bar{1}1\bar{1}}.$$

Consider, therefore, the case when v is not parallel to e_1. Let e_2 be the unit vector defined by

$$e_2 \ := \ \frac{v - g_{v\bar{1}}e_1}{|v - g_{v\bar{1}}e_1|}.$$

Then e_2 is a unit vector orthogonal to e_1 and $v = \alpha e_1 + \beta e_2$, with $\alpha = g_{v\bar{1}}$, $\beta = |v - g_{v\bar{1}}e_1|$, and $|\alpha|^2 + |\beta|^2 = 1$. In particular, since $R_{1\bar{1}1\bar{2}} = R_{1\bar{1}2\bar{1}} = 0$, we have

$$2R_{1\bar{1}v\bar{v}} \ = \ 2|\alpha|^2R_{1\bar{1}1\bar{1}} + 2|\beta|^2R_{1\bar{1}2\bar{2}}.$$

Since $2R_{1\bar{1}2\bar{2}} \geq R_{1\bar{1}1\bar{1}}$, we deduce that

$$2R_{1\bar{1}v\bar{v}} \ \geq \ (2|\alpha|^2 + |\beta|^2)R_{1\bar{1}1\bar{1}} \ = \ (1 + |\alpha|^2)R_{1\bar{1}1\bar{1}},$$

which yields the desired result. □

Remark 2.5.40. Let us remark that a similar argument, due to Brunebarbe–Klingler–Totaro [63, Lemma 1.4] can be used to show that if (X, ω) is Kähler manifold with $\mathrm{HSC}_\omega \leq -\kappa_0 < 0$, then there is a non-zero $u \in T_pX$ such that $\mathrm{HBC}_\omega(u, v) \leq -\frac{\kappa_0}{2}$ for all $v \in T_p^{1,0}X \backslash \{0\}$.

From 2.5.39, we have the following:

Proposition 2.5.41. Let (X^n, ω) be a compact Kähler manifold with $\mathrm{HSC}_\omega > 0$. Then the induced metric on $\Lambda_X^{p,0}$ if RC–positive and $H_{\bar{\partial}}^{p,0}(X) = 0$ for all $1 \leq p \leq n$.

PROOF. Proceed by contradiction and suppose that $\Lambda_X^{p,0}$, equipped with the metric induced by ω, is not RC–positive. Then there is a point $x_0 \in X$ and a non-zero section $\sigma \in \Omega_X^{p,0}$ such that $\Theta^{\Omega_X^{p,0}}(\cdot,\cdot,\sigma,\bar{\sigma}) \in \Omega_X^{1,1}$ is non-positive at x_0 (in the sense of forms). Let $e_1 \in \mathscr{X}^{1,0}(X)$ be the vector field which minimizes the holomorphic sectional curvature at x_0. By 2.5.39, we have

$$2R_{1\bar{1}v\bar{v}} \ \geq \ (1+|g_{v\bar{1}}|^2)R_{1\bar{1}1\bar{1}} \ > \ 0,$$

for all unit vectors $v \in T_{x_0}^{1,0}X$. Therefore $R(e_1,\bar{e}_1,\cdot,\cdot) > 0$ at x_0, which implies that $\Theta^{\Omega_X^{p,0}}(e_1,\bar{e}_1,\cdot,\cdot) > 0$ at x_0, and gives the desired contradiction. The Bochner formula easily implies that if $\Lambda_X^{p,0}$ is RC–positive, then $H_{\bar{\partial}}^{p,0}(X) = 0$. □

We require the following lemma from [**312**, Theorem 1.3]:

Theorem 2.5.42. Let X be a compact complex manifold. Let $\mathcal{E} \to X$ be an RC–positive vector bundle. Then for any vector bundle $\mathcal{A}$, there is a positive integer $c = c(\mathcal{A},\mathcal{E})$ such that

$$H^0(X, \mathrm{Sym}^{\otimes \ell}\mathcal{E}^* \otimes \mathcal{A}^{\otimes k}) \ = \ 0$$

for $\ell \geq c(k+1)$ and $k \geq 0$. Moreover, if X is a projective manifold, then any invertible subsheaf $\mathcal{F}$ of $\mathcal{O}_X(\mathcal{E}^*)$ is not pseudo-effective.

PROOF OF 2.5.35. The key point is that positive holomorphic sectional curvature implies that $(\Lambda^p TX, \Lambda^p\omega)$ is RC-positive for all $1 \leq p \leq \dim(X)$. For $p=2$, this implies that X is projective by Kodaira's projectivity criterion. By 2.5.42, if $(\Lambda^p TX, \Lambda^p\omega)$ is RC-positive for all $1 \leq p \leq \dim(X)$, then any invertible sheaf $\mathcal{L} \subset \Omega_X^p$ cannot be pseudo-effective[11]. In particular, taking $\mathcal{L} = K_X$, the canonical bundle is not pseudo-effective, and therefore, X is not uniruled. Let $\pi : X \longrightarrow Z$ be the associated MRC fibration 1.7.50, which, after resolving the singularities of π and Z, we may assume that π is a proper morphism with Z smooth. There are two cases:

(i) Z is a point, or
(ii) Z is a positive-dimensional variety which is uniruled.

In case (i) X is rationally connected. In case (ii) K_Z is pseudo-effective, and thus exhibits a pseudo-effective invertible sheaf $\pi^* K_Z \subset \Omega_X^{\dim(Z)}$. Hence, from the above discussion, case (ii) cannot occur, and X must be rationally connected. □

Remark 2.5.43. The above result is false for compact Hermitian manifolds with positive Chern holomorphic sectional curvature: The Boothby metric ω_0 on the Hopf manifold $\mathbb{S}^3 \times \mathbb{S}^1$ has ${}^c\mathrm{HSC}_{\omega_0} > 0$, but $\mathbb{S}^3 \times \mathbb{S}^1$ has no rational curves $\mathbb{P}^1 \to \mathbb{S}^3 \times \mathbb{S}^1$. Indeed, since $\mathbb{P}^1$ is simply

[11] Recall that a line bundle $\mathcal{L} \to X$ is said to be *pseudo-effective* if $\mathcal{L}$ admits a singular Hermitian metric h such that $\Theta^{(\mathcal{L},h)}$ is semi-positive in the sense of currents.

connected, any rational curve $\mathbb{P}^1 \to \mathbb{S}^3 \times \mathbb{S}^1$ lifts to the universal cover $\mathbb{P}^1 \to \mathbb{C}^2 \backslash \{0\}$. However, any such holomorphic map is constant.

The monotonicity result for the Gauduchon–Holomorphic sectional curvature indicates that ${}^c\mathrm{HSC}_\omega > 0$ is the weakest condition on the Gauduchon holomorphic sectional curvature. It is, therefore, natural to ask the following:

Question 2.5.44. Let (X, ω) be a compact Hermitian manifold. If there is a range of $t \in \mathbb{R}$ such that ${}^t\mathrm{HSC}_\omega > 0$ implies that X is projective and rationally connected?

Not much is known about manifolds that admit metrics of positive holomorphic sectional curvature. For instance, we have the following question raised by Yau [**321**]:

Question 2.5.45. (Yau). If a projective manifold is obtained from blowing up a compact manifold with positive holomorphic sectional curvature along a subvariety, does it support a metric with positive holomorphic sectional curvature? In general, can we find a geometric criterion to distinguish the concepts of unirationality and rationality?

Even the following special case of the above question remains unknown:

Question 2.5.46. Does the blow-up of $\mathbb{P}^2$ at two points admit a Kähler metric with positive holomorphic sectional curvature?

Remark 2.5.47. We do know, however, by an old result of Tsukamoto [**288**] that a compact Kähler manifold with $\mathrm{HSC}_\omega > 0$ is simply connected. Ni–Zheng [**221**] extended Tsukamoto's argument to show that any holomorphic isometry of a compact Kähler manifold with $\mathrm{HSC}_\omega > 0$ necessarily has a fixed point.

The following result for non-negative (Chern) holomorphic sectional curvature was obtained by Wang–Yang [**296**]:

Theorem 2.5.48. Let (X, ω) be a compact Hermitian manifold with ${}^c\mathrm{HSC}_\omega \geq 0$. If the (Chern) holomorphic sectional curvature is not identically zero, then

(i) there is a Gauduchon metric ω_G on X such that
$$\int_X \mathrm{Ric}^{(1)}_{\omega_G} \wedge \omega_G^{n-1} > 0.$$
(ii) K_X is not pseudo-effective.
(iii) There exists a Hermitian metric on K_X^{-1} that is RC-positive.

If, in addition, X is projective, then X is uniruled.

In [**161**], Hitchin investigated a curvature characterization of rational surfaces. He showed that every Hirzebruch surface admits a Kähler metric with positive holomorphic sectional curvature:

Example 2.5.49. (Hitchin). Let $\mathcal{F}_n := \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1})$ denote the nth Hirzebruch surface. Let z_1 denote an inhomogeneous coordinate on an open subset of the base $\mathbb{P}^1$. A point $w \in \mathcal{O}_{\mathbb{P}^1}(n) \oplus \mathcal{O}_{\mathbb{P}^1}$ can be represented by coordinates w_1, w_2 in the fiber direction as

$$w = \left(z_1, w_1(dz_1)^{-\frac{n}{2}}, w_2\right),$$

where $(dz_1)^{-1}$ is understood as a section of $T\mathbb{P}^1 = \mathcal{O}_{\mathbb{P}^1}(2)$. After projectivization, each fiber carries the inhomogeneous coordinate $z_2 = w_2/w_1$. For a positive real number $\alpha \in \mathbb{R}_+$, the metric

$$\omega_\alpha := \frac{\sqrt{-1}}{2}\partial\bar{\partial}\left[\log(1+|z_1|^2) + \alpha\log((1+|z_1|^2)^n + |z_2|^2)\right]$$

is globally well-defined on $\mathcal{F}_n$.

Theorem 2.5.50. ([**161**, **7**]). The holomorphic sectional curvature of $(\mathcal{F}_n, \omega_\alpha)$ is positive for each $n \in \mathbb{N}$.

Remark 2.5.51. The above result of Hitchin's, together with 1.7.35, shows that the existence of a Kähler metric with positive holomorphic sectional curvature does not imply that there exists a Kähler metric (or Hermitian metric) with positive (first Chern) Ricci curvature. We will discuss this further in relation to the positive analog of the Wu–Yau theorem.

The above result was extended by Yang–Zheng in [**316**]:

Theorem 2.5.52. Let $\mathcal{F}_{n,k} := \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n-1}}(k) \oplus \mathcal{O}_{\mathbb{P}^{n-1}})$ denote the kth Hirzebruch manifold (of dimension n). In each Kähler class, there is a Kähler metric with positive holomorphic sectional curvature. Moreover, the space of all $\mathrm{U}(n)$–invariant Kähler metrics with positive holomorphic sectional curvature on $\mathcal{F}_{n,k}$ is path connected.

Question 2.5.53. Are there Hermitian metrics of positive second (or third) Chern–Ricci curvature on $\mathcal{F}_n$ for all $n \in \mathbb{N}$? Are there metrics of positive weighted orthogonal Ricci curvature on $\mathcal{F}_n$? If so, are the parameters constrained?

The Real Bisectional Curvature and Altered Real Bisectional Curvature. From considerations of the Schwarz lemma in the Hermitian category, Yang–Zheng [**315**] (see also [**190**]) introduced the following curvature (c.f., 2.6.13):

Definition 2.5.54. Let (X, ω) be a Hermitian manifold. The *t–Gauduchon real bisectional curvature* ${}^t\mathrm{RBC}_\omega$ is the function

$${}^t\mathrm{RBC}_\omega : \mathcal{F}_X \times \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}, \qquad {}^t\mathrm{RBC}_\omega(v) := \frac{1}{|v|^2}\sum_{\alpha,\gamma} {}^tR_{\alpha\bar{\alpha}\gamma\bar{\gamma}} v_\alpha v_\gamma.$$

Here, $\mathcal{F}_X$ denotes the unitary frame bundle, ${}^tR_{\alpha\bar{\beta}\gamma\bar{\delta}}$ denote the components of the t–Gauduchon curvature tensor with respect to the local unitary frame, and $v = (v_1, ..., v_n) \in \mathbb{R}^n \backslash \{0\}$. We say that ${}^t\mathrm{RBC}_\omega \leq \kappa$ if $\max_{(e,\lambda)\in \mathcal{F}_X \times \mathbb{R}^n \backslash \{0\}} {}^t\mathrm{RBC}_\omega(e,\lambda) \leq \kappa$. Similar definitions apply for ${}^t\mathrm{RBC}_\omega \geq \kappa$ and ${}^t\mathrm{RBC}_\omega \equiv \kappa$.

Remark 2.5.55. If we let $v \in T^{1,0}X$ and choose a local unitary frame $\{e_\alpha\}$ for $T^{1,0}X$ such that v is parallel to e_1, then with respect to this frame, we have ${}^t\mathrm{RBC}_\omega(v) = {}^tR_{1\bar{1}1\bar{1}} = {}^t\mathrm{HSC}_\omega(v)$. Hence, the ($t$–Gauduchon) real bisectional curvature dominates the (t–Gauduchon) holomorphic sectional curvature.

Remark 2.5.56. From the definition of the t–Gauduchon altered holomorphic sectional curvature, we see that if the metric is t–Gauduchon Kähler-like, then ${}^t\mathrm{HSC}_\omega$ and ${}^t\mathrm{RBC}_\omega$ are comparable.

Remark 2.5.57. The (Chern) real bisectional curvature is not strong enough to dominate the (Chern) Ricci curvatures. A local example was constructed in [**315**].

Proposition 2.5.58. Let (X,ω) be a compact Hermitian manifold with pointwise constant Chern real bisectional curvature ${}^c\mathrm{RBC}_\omega \equiv \kappa$. Then $\kappa \leq 0$.

Motivated by the above definition, and the altered holomorphic sectional curvature, the author, joint with Kai Tang [**60**], introduced the following definition:

Definition 2.5.59. Let (X,ω) be a Hermitian manifold. The *t–Gauduchon altered real bisectional curvature* ${}^t\widetilde{\mathrm{RBC}}_\omega$ is the function

$$ {}^t\widetilde{\mathrm{RBC}}_\omega : \mathcal{F}_X \times \mathbb{R}^n \backslash \{0\} \longrightarrow \mathbb{R}, \qquad {}^t\widetilde{\mathrm{RBC}}_\omega(v) \ := \ \frac{1}{|v|^2} \sum_{\alpha,\gamma} {}^tR_{\alpha\bar{\gamma}\gamma\bar{\alpha}} v_\alpha v_\gamma. $$

Remark 2.5.60. Like the real bisectional curvature, the altered real bisectional curvature dominates the holomorphic sectional curvature: Indeed, for any unit $(1,0)$–tangent vector $u \in T^{1,0}M$, we can choose a unitary frame $e = \{e_1, ..., e_n\}$ such that u is a scalar multiple of e_1. Taking $v = (v_1, ..., v_n) = (1, 0, ..., 0)$ then gives

$$ \widetilde{\mathrm{RBC}}_\omega(v) \ = \ \sum_{\alpha,\gamma=1}^n R_{\alpha\bar{\gamma}\gamma\bar{\alpha}} v_\alpha v_\gamma \ = \ R_{1\bar{1}1\bar{1}} \ = \ \mathrm{HSC}_\omega(u). $$

As we will see in the next section, the altered real bisectional curvature will play a valuable computational role in the Schwarz lemma.

Proposition 2.5.61. Let (X,ω) be a Hermitian manifold. Then $\widetilde{\mathrm{RBC}}_\omega \equiv \kappa$ is equivalent to

$$ R_{k\bar{\ell}s\bar{t}} \xi_{kt} \xi_{s\ell} = \kappa \mathrm{tr}(\xi^2), $$

for any Hermitian matrix $\xi = (\xi_{ij})$.

Proof. For a fixed local unitary frame $e = \{e_1, ..., e_n\}$, any other unitary frame is given by $f_k = A_{kj}e_j$, where $A = (A_{kj})$ is a unitary matrix. Write

$$\begin{aligned} R(f_i, \overline{f_i}, f_j, \overline{f_j})v_i v_j &= R(A_{ik}e_k, \overline{A_{j\ell}e_\ell}, A_{js}e_s, \overline{A_{it}e_t})v_i v_j \\ &= \sum_{i,j,k,\ell,s,t} (v_i A_{ik}\overline{A_{it}})(v_j A_{js}\overline{A_{j\ell}})R_{k\bar{\ell}s\bar{t}} = \sum \xi_{kt}\xi_{s\ell}R_{k\bar{\ell}s\bar{t}}, \end{aligned}$$

where we set $\xi_{kt} := \sum_{i=1}^n v_i A_{ik}\overline{A_{it}}$. Note that $\xi = (\xi_{kt})$ defines a Hermitian matrix. The condition $\widetilde{\mathrm{RBC}}_\omega \equiv \kappa$ is therefore equivalent to

$$\begin{aligned} \sum \xi_{kt}\xi_{s\ell}R_{k\bar{\ell}s\bar{t}} &= \kappa \sum_s v_s^2 \\ &= \kappa \sum_{s,m} v_s v_m \delta_{sm}\delta_{ms} \\ &= \kappa \sum_{s,m} v_s v_m A_{s\bar{i}}\overline{A_{mi}}A_{mt}\overline{A_{st}} \\ &= \kappa \sum_{s,m,i,t} (v_s A_{si}\overline{A_{st}})(v_m A_{mt}\overline{A_{mi}}) = \kappa \sum_{i,t} \xi_{it}\xi_{ti} = \kappa \mathrm{tr}(\xi^2). \end{aligned}$$

□

In [**315**], it was shown that the if the Chern real bisectional curvature is pointwise constant ${}^c\mathrm{RBC}_\omega \equiv \kappa$, then $\kappa \leq 0$. For the altered real bisectional curvature, we have the following:

Theorem 2.5.62. Let (X, ω) be a compact Hermitian manifold with pointwise constant Chern altered real bisectional curvature ${}^c\widetilde{\mathrm{RBC}}_\omega \equiv \kappa$ for some $\kappa \in \mathbb{R}$. Then $\kappa \geq 0$. Further, if $\kappa = 0$, then ω is balanced with vanishing first, second, and third Ricci curvatures. In particular, if $n = 3$ and ${}^c\widetilde{\mathrm{RBC}}_\omega \equiv 0$, then ω is Chern-flat.

Proof. Let $\tau = \sum_j \tau_j \varphi_j = \sum_{i,j} {}^cT_{ij}^i \varphi_j$ denote the (Chern) torsion $(1,0)$–form (with respect to a unitary coframe $\{\varphi_1, ..., \varphi_n\}$. Let $\alpha_k = \sum_{i,j} {}^kT_{ij}^k \varphi_i \wedge \varphi_j$ denote the (Chern) torsion $(2,0)$–forms. From the Bianchi identity 2.2.67, we have

$$2{}^cT_{ij,\bar{\ell}}^k = {}^cR_{j\bar{l}i\bar{k}} - {}^cR_{i\bar{l}j\bar{k}}. \tag{2.5.4}$$

Setting $k = i$ and summing over k gives

$$2\tau_{j,\bar{i}} = \sum_k ({}^cR_{j\bar{i}k\bar{k}} - {}^cR_{k\bar{i}j\bar{k}}), \tag{2.5.5}$$

where the comma denotes covariant differentiation with respect to ${}^c\nabla$. Since $\partial(\omega^{n-1}) = -2\tau \wedge \omega^{n-1}$ and X is compact, integrating gives

$$\int_X \left(\sum_i \tau_{i,\bar{i}} \right) \omega^n = 2\int_X |\tau|^2 \omega^n. \tag{2.5.6}$$

In a similar manner to [315], if ${}^c\widetilde{\mathrm{RBC}}_\omega \equiv \kappa$ then

$$ {}^cR_{i\bar{i}k\bar{k}} + {}^cR_{k\bar{k}i\bar{i}} \ = \ 2\kappa, \qquad {}^cR_{i\bar{j}k\bar{\ell}} + {}^cR_{k\bar{\ell}i\bar{j}} \ = \ 0. $$

Hence,

$$ \begin{aligned} 2\sum_i \tau_{i,\bar{i}} \ = \ \sum_{i,k}({}^cR_{i\bar{i}k\bar{k}} - {}^cR_{k\bar{i}i\bar{k}}) \ &= \ \sum_{i\neq k}({}^cR_{i\bar{i}k\bar{k}} - {}^cR_{k\bar{i}i\bar{k}}) \\ &= \ \sum_{i\neq k}(2\kappa - {}^cR_{k\bar{k}i\bar{i}} + {}^cR_{i\bar{k}k\bar{i}}) \\ &= \ 2\kappa n(n-1) - \sum_{i\neq k}({}^cR_{k\bar{k}i\bar{i}} - {}^cR_{i\bar{k}k\bar{i}}) \\ &= \ 2\kappa n(n-1) - 2\kappa\sum_i \tau_{i,\bar{i}}, \end{aligned} $$

which implies that

$$ \sum_i \tau_{i,\bar{i}} \ = \ \frac{1}{2}\kappa n(n-1). $$

The remaining claims follow from [315, §3]. $\square$

More generally, if the real bisectional curvature coincides with the altered real bisectional curvature, the metric is balanced:

Proposition 2.5.63. Let (X,ω) be a Hermitian manifold. If

$$ {}^c\mathrm{RBC}_\omega \ \equiv \ {}^c\widetilde{\mathrm{RBC}}_\omega, $$

then ω is balanced.

Proof. Suppose ${}^c\mathrm{RBC}_\omega \equiv {}^c\widetilde{\mathrm{RBC}}_\omega$ at every point on X. Then for any local unitary frame, and any vectors $u = (u_1, ..., u_n) \in \mathbb{R}^n\backslash\{0\}$ and $v = (v_1, ..., v_n) \in \mathbb{R}^n\backslash\{0\}$, we have

$$ \frac{1}{|v|^2}\sum_{\alpha,\gamma} {}^cR_{\alpha\bar{\alpha}\gamma\bar{\gamma}} v_\alpha v_\gamma \ = \ \frac{1}{|u|^2}\sum_{\alpha,\gamma} {}^cR_{\alpha\bar{\gamma}\gamma\bar{\alpha}} u_\alpha u_\gamma. $$

Taking $u = v = \frac{1}{\sqrt{n}}(1, ..., 1)$ gives

$$ {}^c\mathrm{Scal}_\omega \ = \ \sum_{\alpha,\gamma} {}^cR_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \ = \ \sum_{\alpha,\gamma} {}^cR_{\alpha\bar{\gamma}\gamma\bar{\alpha}} \ = \ {}^c\widetilde{\mathrm{Scal}}_\omega. $$

By the well-known balanced criterion 2.3.42 of equality of the scalar curvatures, ω is balanced. $\square$

Theorem 2.5.64. Let (X,ω) be a Hermitian manifold.

(i) If ${}^c\widetilde{\mathrm{RBC}}_\omega \equiv \kappa$ for some $\kappa \in \mathbb{R}$, then ${}^c\mathrm{RBC}_\omega \geq 0$ if $\kappa > 0$, or ${}^c\mathrm{RBC}_\omega \leq 0$ if $\kappa < 0$.

(ii) If ${}^c\mathrm{RBC}_\omega \equiv \kappa$ for some $\kappa \in \mathbb{R}$, then ${}^c\widetilde{\mathrm{RBC}}_\omega \geq 0$ if $\kappa > 0$, or ${}^c\widetilde{\mathrm{RBC}}_\omega \leq 0$ if $\kappa < 0$.

In particular, if X is compact, then ${}^c\mathrm{RBC}_\omega \equiv \kappa_1$ and ${}^c\widetilde{\mathrm{RBC}}_\omega \equiv \kappa_2$ if and only if $\kappa_1 = \kappa_2 = 0$.

Proof. For the case (i), fix a local unitary frame $e = \{e_1, ..., e_n\}$. If the Chern altered real bisectional curvature is pointwise constant ${}^c\widetilde{\mathrm{RBC}}_\omega \equiv \kappa$, then

$$ {}^cR_{i\bar{i}k\bar{k}} + {}^cR_{k\bar{k}i\bar{i}} \ = \ 2\kappa, \qquad {}^cR_{i\bar{j}k\bar{\ell}} + {}^cR_{k\bar{\ell}i\bar{j}} \ = \ 0. $$

For the (Chern) real bisectional curvature, we have

$$ \begin{aligned} {}^c\mathrm{RBC}_\omega(v) &= \frac{1}{|v|^2} \sum_{\alpha,\gamma} {}^cR_{\alpha\bar{\alpha}\gamma\bar{\gamma}} v_\alpha v_\gamma \\ &= \frac{1}{|v|^2} \left(\sum_{\alpha<\gamma} ({}^cR_{\alpha\bar{\alpha}\gamma\bar{\gamma}} + {}^cR_{\gamma\bar{\gamma}\alpha\bar{\alpha}}) v_\alpha v_\gamma + \sum_\alpha {}^cR_{\alpha\bar{\alpha}\alpha\bar{\alpha}} v_\alpha^2 \right) \\ &= \frac{1}{|v|^2} \left(2\kappa \sum_{\alpha<\gamma} v_\alpha v_\gamma + \kappa \sum_\alpha v_\alpha^2 \right) \\ &= \frac{\kappa}{|v|^2} \left(\sum_{\alpha\neq\gamma} v_\alpha v_\gamma + \sum_\alpha v_\alpha^2 \right) \\ &= \frac{\kappa}{|v|^2} (v_1 + \cdots + v_n)^2. \end{aligned} $$

This proves case (i); the proof of case (ii) is similar. □

We will see in the next section that the real bisectional curvature plays an important role in the Schwarz lemma. The following result due to Kai Tang [270] shows that Yang's result [312] can be extended to the Hermitian category under the stronger assumption of positive real bisectional curvature:

Theorem 2.5.65. Let (X, ω) be a compact Hermitian manifold of (complex) dimension $n > 2$. Suppose ${}^c\mathrm{RBC}_\omega > 0$. Then

$$ h^{1,0} \ = \ h^{2,0} \ = \ h^{n-1,0} \ = \ h^{n,0} \ = \ 0. $$

In particular, if X is Kähler, then X is projective. Moreover, if $n = 3$, then X is rationally connected.

In light of these results, let us pose the following questions:

Question 2.5.66. Let (X, ω) be a compact Hermitian manifold with positive Chern real bisectional curvature ${}^c\mathrm{RBC}_\omega > 0$. Is X projective or rationally connected?

Similarly, we ask the corresponding question for the (Chern) altered real bisectional curvature:

Question 2.5.67. Let (X, ω) be a compact Hermitian manifold with positive Chern altered real bisectional curvature ${}^c\widetilde{\mathrm{RBC}}_\omega > 0$. Is X projective or rationally connected?

Of course, there is no particular reason for isolating the Chern real bisectional curvature and altered real bisectional curvature. In fact, by considering the montonicity theorem for the t–Gauduchon altered holomorphic sectional curvature 2.5.27 it is more natural to pose these questions for the t–Gauduchon real bisectional curvature and altered real bisectional:

Question 2.5.68. Let (X, ω) be a compact Hermitian manifold. Is there a range of $t \in \mathbb{R}$ such that ${}^t\mathrm{RBC}_\omega > 0$ (or ${}^t\widetilde{\mathrm{RBC}}_\omega > 0$) implies that X is projective or rationally connected?

2.6. The Schwarz Lemma in Kähler and Non-Kähler Geometry

Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic between Hermitian manifolds. A *Schwarz lemma* is an estimate on the pointwise norm squared[12] of the derivative of f, i.e., an estimate on $|\partial f|^2$, in terms of the curvature of the source manifold (X, ω_g) and the target manifold (Y, ω_h). One of the pieces of propaganda in [**49, 50**] is that a Schwarz lemma arises from the coalescence of a Bochner formula and a maximum principle. As a consequence, we start by proving the following novel (and very general) Bochner formula:

Theorem 2.6.1. *Let $(\mathcal{E}, h) \longrightarrow X$ be a holomorphic vector bundle over a complex manifold X. Let ∇ be a Hermitian connection on $\mathcal{E}$ with curvature $\Theta = \Theta^{(\mathcal{E},\nabla)}$. Then for any holomorphic section $\sigma \in H^0(\mathcal{E})$, we have*

$$\Delta^{(\mathcal{E},\nabla)} |\sigma|_h^2 \quad = \quad 2\mathrm{Re}\{\nabla^{1,0}\nabla^{0,1}\sigma, \sigma\} - \{\sigma, \mathrm{Ric}_{\nabla}^{(2)}\sigma\} + |\nabla^{1,0}\sigma|^2 + |\nabla^{0,1}\sigma|^2,$$

where $\mathrm{Ric}_{\nabla}^{(2)}$ is the endomorphism given by tracing over the differential form part of $\Theta^{1,1}$ and $\{\cdot, \cdot\}$ is an alternative notation for the Hermitian form h.

Proof. Let $(z_1, ..., z_n)$ denote local holomorphic coordinates on X, and let $\{e_1, ..., e_r\}$ be a local holomorphic frame for $\mathcal{E}$. The components of a Hermitian metric h on $\mathcal{E}$ are denoted $h_{\alpha\overline{\beta}} = h(e_\alpha, e_\beta)$. If $\sigma = \sigma^\alpha e_\alpha$ is a holomorphic section of $\mathcal{E}$, then

$$|\sigma|_h^2 := h_{\alpha\overline{\beta}}\sigma^\alpha \overline{\sigma}^\beta.$$

Since the connection ∇ is compatible with h, we have

$$\begin{aligned} \nabla_i \nabla_{\overline{j}} |\sigma|_h^2 \quad &= \quad h_{\alpha\overline{\beta}}(\nabla_i \nabla_{\overline{j}} \sigma^\alpha)\sigma^{\overline{\beta}} + h_{\alpha\overline{\beta}}(\nabla_i \nabla_{\overline{j}} \sigma^{\overline{\beta}})\sigma^\alpha \\ &\qquad + h_{\alpha\overline{\beta}}(\nabla_i \sigma^\alpha)(\nabla_{\overline{j}}\sigma^{\overline{\beta}}) + h_{\alpha\overline{\beta}}(\nabla_i \sigma^{\overline{\beta}})(\nabla_{\overline{j}}\sigma^\alpha). \end{aligned}$$

We have the following commutation formula

$$\nabla_i \nabla_{\overline{j}} \sigma^{\overline{\alpha}} - \nabla_{\overline{j}} \nabla_i \sigma^{\overline{\alpha}} \quad = \quad -\overline{\Theta_{j\bar{i}\gamma}{}^{\alpha}}\sigma^{\overline{\gamma}}. \tag{2.6.1}$$

Hence,

$$\begin{aligned} \nabla_i \nabla_{\overline{j}} |\sigma|_h^2 \quad &= \quad h_{\alpha\overline{\beta}}(\nabla_i \nabla_{\overline{j}} \sigma^\alpha)\sigma^{\overline{\beta}} + h_{\alpha\overline{\beta}}(\nabla_{\overline{j}} \nabla_i \sigma^{\overline{\beta}})\sigma^\alpha - h_{\alpha\overline{\beta}}\overline{\Theta_{j\bar{i}\gamma}{}^{\beta}}\sigma^{\overline{\gamma}}\sigma^\alpha \\ &\qquad + h_{\alpha\overline{\beta}}(\nabla_i \sigma^\alpha)(\nabla_{\overline{j}}\sigma^{\overline{\beta}}) + h_{\alpha\overline{\beta}}(\nabla_i \sigma^{\overline{\beta}})(\nabla_{\overline{j}}\sigma^\alpha). \end{aligned}$$

[12] In the harmonic map literature, this would be referred to as the *energy density*.

Summing over $i = j$, and choosing coordinates such that $h_{\alpha\overline{\beta}}(x_0) = \delta_{\alpha\beta}(x_0)$ at a point $x_0 \in X$, we get

$$\begin{aligned} \nabla_i \nabla_{\bar{i}} |\sigma|_h^2 &= (\nabla_i \nabla_{\bar{i}} \sigma^\alpha)\sigma^{\overline{\alpha}} + (\nabla_{\bar{i}} \nabla_i \sigma^{\overline{\alpha}})\sigma^\alpha - \overline{\Theta_{i\bar{i}\gamma\overline{\alpha}}} \sigma^{\overline{\gamma}} \sigma^\alpha \\ &\qquad + (\nabla_i \sigma^\alpha)(\nabla_{\bar{i}} \sigma^{\overline{\alpha}}) + (\nabla_i \sigma^{\overline{\alpha}})(\nabla_{\bar{i}} \sigma^\alpha) \\ &= 2\mathrm{Re}\left\{(\nabla^{1,0}\nabla^{0,1}\sigma)\overline{\sigma}\right\} - \{\sigma, \mathrm{Ric}_{\nabla}^{(2)}\sigma\} + |\nabla^{1,0}\sigma|^2 + |\nabla^{0,1}\sigma|^2. \end{aligned}$$

□

Since $\nabla^{0,1}\sigma = \bar{\partial} f$ in the case that ∇ is the Chern connection, we have the following immediate corollary:

Corollary 2.6.2. Let $(\mathcal{E}, h) \longrightarrow (X, \omega_g)$ be a holomorphic vector bundle over a Hermitian manifold (X, ω_g). Let ∇ denote the Chern connection on $\mathcal{E}$. Then for any holomorphic section $\sigma \in H^0(\mathcal{E})$, we have

$$\Delta_{\omega_g} |\sigma|_h^2 = |\nabla^{1,0}\sigma|^2 - \sqrt{-1}\{\Theta^{(\mathcal{E},h)}\sigma, \sigma\}.$$

When $\sigma = \partial f$, and $\mathcal{E} = \Omega_X^{1,0} \otimes f^* T^{1,0} Y$, we have

$$\sqrt{-1}\partial\overline{\partial}|\partial f|^2 = \langle \nabla^{1,0}\partial f, \nabla^{1,0}\partial f\rangle - \sqrt{-1}\langle \Theta^{\Omega_X^{0,1} \otimes f^* T^{1,0}Y} \partial f, \partial f\rangle. \tag{2.6.2}$$

Since the curvature of the tensor product of bundles splits additively, we get opposing contributions to the curvature from the source, and target metrics:

$$\Theta^{\Omega_X^{1,0} \otimes f^* T^{1,0} Y} = -\Theta^{T^{1,0}X} \otimes \mathrm{id} + \mathrm{id} \otimes f^* \Theta^{T^{1,0}Y}. \tag{2.6.3}$$

Taking the trace of (2.6.2) with respect to ω_g, we recover the *Chern–Lu formula* [**92**, **203**]:

Theorem 2.6.3. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Then

$$\begin{aligned} \Delta_{\omega_g} |\partial f|^2 &= |\nabla \partial f|^2 + {}^c\mathrm{Ric}^{(2)}_{k\bar{\ell}} g^{k\overline{q}} g^{p\bar{\ell}} h_{\alpha\overline{\beta}} f_p^\alpha \overline{f_q^\beta} \\ &\qquad - {}^c\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} \left(g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta}\right)\left(g^{p\overline{q}} f_p^\gamma \overline{f_q^\delta}\right), \end{aligned} \tag{2.6.4}$$

where ${}^c\mathrm{Ric}^{(2)}$ denotes the second Chern–Ricci curvature of ω_g and ${}^c\widetilde{R}$ denotes the Chern curvature tensor of ω_h.

It is clear that to apply the Bochner technique to (2.6.4), we require a lower bound on the second Chern–Ricci curvature of the source manifold and an upper bound on the curvature of the target manifold. To refine our understanding of the target curvature term in (2.6.4), we recall the following polarization result due to Royden [**241**]:

Theorem 2.6.4. (Royden). Let $\xi_1, ..., \xi_\nu$ be othogonal tangent vectors. Let $\mathcal{S}(\xi, \overline{\eta}, \zeta, \overline{\omega})$ be a symmetric bi-Hermitian form in the sense that

$$\mathcal{S}(\xi, \overline{\eta}, \zeta, \overline{\omega}) \;=\; \mathcal{S}(\zeta, \overline{\eta}, \xi, \overline{\omega}), \qquad \text{and} \qquad \mathcal{S}(\eta, \overline{\xi}, \omega, \overline{\zeta}) \;=\; \overline{\mathcal{S}}(\xi, \overline{\eta}, \zeta, \overline{\omega}).$$

If $\mathcal{S}(\xi, \overline{\xi}, \xi, \overline{\xi}) \le \kappa \|\xi\|^4$, then

$$\sum_{\alpha,\beta} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\beta, \overline{\xi}_\beta) \;\le\; \frac{\kappa}{2}\left[\left(\sum_\alpha \|\xi_\alpha\|^2\right)^2 + \sum_\alpha \|\xi_\alpha\|^4\right].$$

Further, if $\kappa \le 0$, then

$$\sum_{\alpha,\beta} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\beta, \overline{\xi}_\beta) \;\;\le\;\; \kappa \frac{n+1}{2n}\left(\sum_\alpha \|\xi_\alpha\|^2\right)^2.$$

Proof. For any $A \in \mathbb{Z}_4^n$, write $A = \{\varepsilon_1, ..., \varepsilon_n\}$ with $\varepsilon_\alpha^4 = 1$ for each α. Set $\xi_A := \sum_\alpha \varepsilon_\alpha \xi_\alpha$. Then $\|\xi_A\|^2 = \sum_\alpha \|\xi_\alpha\|^2$, and therefore,

$$\mathcal{S}(\xi_A, \overline{\xi}_A, \xi_A, \overline{\xi}_A) \;\;\le\;\; \kappa\left(\sum_\alpha \|\xi_\alpha\|^2\right)^2.$$

Write

$$\begin{aligned}
\kappa\left(\sum_\alpha \|\xi_\alpha\|^2\right)^2 \;\;&\ge\;\; \frac{1}{4^n}\sum_A \mathcal{S}(\xi_A, \overline{\xi}_A, \xi_A, \overline{\xi}_A) \\
&=\;\; \frac{1}{4^n}\sum_A \varepsilon_\alpha \overline{\varepsilon}_\beta \varepsilon_\gamma \overline{\varepsilon}_\delta \mathcal{S}(\xi_\alpha, \overline{\xi}_\beta, \xi_\gamma, \overline{\xi}_\delta) \\
&=\;\; \sum_\alpha \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\alpha, \overline{\xi}_\alpha) + \sum_{\alpha\neq\gamma} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\gamma, \overline{\xi}_\gamma) + \mathcal{S}(\xi_\alpha, \overline{\xi}_\gamma, \xi_\gamma, \overline{\xi}_\alpha).
\end{aligned}$$

Symmetry of $\mathcal{S}$ implies

$$\sum_\alpha \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\alpha, \overline{\xi}_\alpha) + 2\sum_{\alpha\neq\gamma} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\gamma, \overline{\xi}_\gamma) \;\le\; \kappa\left(\sum_\alpha \|\xi_\alpha\|^2\right)^2,$$

and

$$2\sum_{\alpha,\gamma} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\gamma, \overline{\xi}_\gamma) \;\;\le\;\; \kappa\left[\left(\sum_\alpha \|\xi_\alpha\|^2\right)^2 + \sum_\alpha \|\xi_\alpha\|^4\right].$$

Since $n\sum_\alpha \|\xi_\alpha\|^4 \ge \left(\sum_\alpha \|\xi_\alpha\|^2\right)^2$, we get

$$\sum_{\alpha,\gamma} \mathcal{S}(\xi_\alpha, \overline{\xi}_\alpha, \xi_\gamma, \overline{\xi}_\gamma) \;\;\le\;\; \kappa \frac{n+1}{2n}\left(\sum_\alpha \|\xi_\alpha\|^2\right)^2.$$

□

Applying this polarization argument to the target curvature term, assuming the metric is Kähler, shows that the target curvature term can indeed be controlled by the holomorphic sectional curvature [**241**, **320**, **315**, **49**, **50**]:

Theorem 2.6.5. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Assume there are constants $C_1, C_2 \in \mathbb{R}$ such that

$$ {}^c\mathrm{Ric}^{(2)}_{\omega_g} \ \geq \ -C_1\omega_g - C_2 f^*\omega_h. $$

Let ω_h be a (Chern) Kähler-like metric with ${}^c\mathrm{HSC}_{\omega_h} \leq \kappa_0$, for some $\kappa_0 \in \mathbb{R}$. Then for $\kappa_0 \geq 0$,

$$ \Delta_{\omega_g}|\partial f|^2 \ \ \geq \ \ -C_1|\partial f|^2 - (C_2 + \kappa_0)|\partial f|^4, \tag{2.6.5} $$

while for $\kappa_0 \leq 0$,

$$ \Delta_{\omega_g}|\partial f|^2 \ \ \geq \ \ -C_1|\partial f|^2 - \left(C_2 + \frac{(n+1)\kappa_0}{2n}\right)|\partial f|^4, \tag{2.6.6} $$

Proof. Suppose ${}^c\mathrm{Ric}^{(2)}_{\omega_g} \ \geq \ -C_1\omega_g - C_2 f^*\omega_h$ for some constants $C_1, C_2 \in \mathbb{R}$. Then

$$ {}^c\mathrm{Ric}^{(2)}_{k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} \ \ \geq \ \ -C_1 g_{k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - C_2 h_{\gamma\bar{\delta}} f_k^\gamma \overline{f_\ell^\delta} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta}. $$

Choose the local frame such that at both metrics are Euclidean at a point and $f_i^\alpha = \lambda_i \delta_i^\alpha$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = \cdots = \lambda_n = 0$. Then

$$ \begin{aligned} -C_1 g_{k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - C_2 h_{\gamma\bar{\delta}} f_k^\gamma \overline{f_\ell^\delta} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} \ \ &= \ \ -C_1\lambda_\alpha^2 - C_2\lambda_\alpha^4 \\ &\geq \ \ -C_1|\partial f|^2 - C_2|\partial f|^4. \end{aligned} $$

For the target curvature term, we apply Royden's polarization argument to deduce that if ${}^c\mathrm{HSC}_{\omega_h} \leq \kappa_0$, then in the frame we considered before, we have

$$ {}^c\widetilde{R}^h_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta} \ \ = \ \ \sum_{\alpha,\gamma} {}^c\widetilde{R}_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \lambda_\alpha^2 \lambda_\gamma^2 \ \leq \ \frac{\kappa_0}{2}\left(\left(\sum_\alpha \lambda_\alpha^2\right)^2 + \sum_\alpha \lambda_\alpha^4\right). $$

If $\kappa_0 \geq 0$, then

$$ \frac{\kappa_0}{2}\left(\left(\sum_\alpha \lambda_\alpha^2\right)^2 + \sum_\alpha \lambda_\alpha^4\right) \ \ \leq \ \ \kappa_0\left(\sum_\alpha \lambda_\alpha^2\right)^2 \ \ = \ \ \kappa_0|\partial f|^2. $$

If $\kappa_0 \leq 0$, then

$$ \sum_{\alpha,\gamma} {}^c R^h_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \lambda_\alpha^2 \lambda_\gamma^2 \ \leq \ \frac{(n+1)\kappa_0}{2n}|\partial f|^4. $$

$\square$

Corollary 2.6.6. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds such that X is compact. Assume there are constants $C_1, C_2 \in \mathbb{R}$ such that

$$^c\text{Ric}^{(2)}_{\omega_g} \ \geq \ -C_1 \omega_g - C_2 f^* \omega_h.$$

Let ω_h be a (Chern) Kähler-like metric with $^c\text{HSC}_{\omega_h} \leq \kappa_0 < -C_2$. Then

$$|\partial f|^2 \ \ \leq \ \ -\frac{C_1}{C_2 + \kappa_0}.$$

In particular, if $^c\text{Ric}^{(2)}_{\omega_g} \geq \varepsilon f^* \omega_h$ for some $\varepsilon > 0$ and $^c\text{HSC}_{\omega_h} \leq \varepsilon$, then f is constant.

Remark 2.6.7. The above result is a refinement on the existing Schwarz lemmas which are known in the literature. It is well-known that it suffices to assume the target metric is (Chern) Kähler-like to ensure that Royden's polarization argument 2.6.4 holds. In the existing forms of the Schwarz lemma, however, the lower bound on the second Chern–Ricci curvature required the coefficient of $f^* \omega_h$ to be non-negative (see, e.g., [**315**, **304**]). This assumption, as we just showed, however, is superfluous. Let us emphasize that $^c\text{Ric}^{(2)}_{\omega_g} \geq \varepsilon f^* \omega_h$ is weaker than $^c\text{Ric}^{(2)}_{\omega_g} \geq \varepsilon \omega_g$.

In [**57**, **58**], the author, joint with James Stanfield, introduced the following class of manifolds:

Definition 2.6.8. Let (X, ω) be a Hermitian manifold. We say that ω is *partially t–Gauduchon Kähler-like* if

$$^t\text{Ric}^{(1)}_{\omega} \ = \ ^t\text{Ric}^{(2)}_{\omega}.$$

Example 2.6.9. Kähler metrics are certainly partially Kähler-like. Moreover, t–Gauduchon Kähler-like metrics are partially t–Gauduchon Kähler-like. More examples and properties of this class of manifolds are given in [**57**, **58**].

With this definition in place, we can rephrase the above Schwarz lemma:

Proposition 2.6.10. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map from a compact partially Chern Kähler-like manifold into a Chern Kähler-like Hermitian manifold. Assume

$$^c\text{Ric}^{(1)}_{\omega_g} \ \geq \ -C_1 \omega_g - C_2 f^* \omega_h$$

and $^c\text{HSC}_{\omega_h} \leq -\kappa_0 < -C_2$. Then

$$|\partial f|^2 \ \leq \ -\frac{C_1}{C_2 + \kappa_0}.$$

Remark 2.6.11. Phrasing the assumption on the source metric ω_g in terms of the first Chern–Ricci curvature in place of the second Chern–Ricci curvature is of growing importance in the non-Kähler Hermitian setting. Indeed, the first Chern–Ricci curvature, in contrast with the second Chern–Ricci curvature, is governed by a complex Monge–Ampère equation.

We want to further understand the target curvature term arising in the Schwarz lemma when the Chern curvature tensor of the target metric does not have the symmetries of the Kähler curvature tensor. Recall that the (Chern) holomorphic sectional curvature is sufficient to control the target curvature term because of Royden's polarization argument 2.6.4. In this direction, let us ask the following:

Question 2.6.12. Royden's polarization argument makes use of a polarization identity coming from multi-linear algebra. There are many such polarization identities. Can Royden's idea be extended by employing different polarization identities?

To understand precisely what the target curvature term is in the Schwarz lemma, we consider the following:Choose coordinates $(z_1, ..., z_n)$ centered at a point $p \in X$ and $(w_1, ..., w_n)$ at $f(p) \in Y$ such that $g_{i\overline{j}} = \delta_{ij}$ and $h_{\alpha\overline{\beta}} = \delta_{\alpha\beta}$ at p and $f(p)$, respectively. If $f = (f^1, ..., f^n)$, then with $f_i^\alpha = \frac{\partial f^\alpha}{\partial z_i}$, the coordinates can be chosen such that $f_i^\alpha = \lambda_i \delta_i^\alpha$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r \geq \lambda_{r+1} = \cdots = 0$, and r is the rank of $\partial f = (f_i^\alpha)$. Hence, the target curvature term reads

$$g^{i\overline{j}} g^{p\overline{q}} \left({}^c\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} \right) f_i^\alpha \overline{f_j^\beta} f_p^\gamma \overline{f_q^\delta} \quad = \quad \sum_{\alpha,\gamma} \left({}^c\widetilde{R}_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \right) \lambda_\alpha^2 \lambda_\gamma^2. \tag{2.6.7}$$

In particular, we recover the (Chern) real bisectional curvature [315]. We therefore have the following Schwarz lemma due to Yang–Zheng [315]:

Theorem 2.6.13. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds such that X is compact. Assume there are constants $C_1, C_2 \in \mathbb{R}$ such that

$${}^c\mathrm{Ric}_{\omega_g}^{(2)} \ \geq \ -C_1 \omega_g - C_2 f^* \omega_h.$$

Let ω_h be a Hermitian metric with ${}^c\mathrm{RBC}_{\omega_h} \leq \kappa_0 < -C_2$. Then

$$|\partial f|^2 \quad \leq \quad -\frac{C_1}{C_2 + \kappa_0}.$$

In particular, if ${}^c\mathrm{Ric}_{\omega_g}^{(2)} \geq \varepsilon f^* \omega_h$ for some $\varepsilon > 0$ and ${}^c\mathrm{RBC}_{\omega_h} \leq \varepsilon$, then f is constant.

Remark 2.6.14. The above statement is a small refinement of the statement that appears in [315]. Indeed, Yang–Zheng assume that $C_2 \geq 0$.

Remark 2.6.15. The nature of the (Chern) real bisectional curvature is troubling. As we saw in 2.5.56, for a Kähler-like metric, it is comparable to the holomorphic sectional curvature. However, for a general Hermitian metric, the real bisectional curvature strictly dominates the holomorphic sectional curvature. The frame dependence of the real bisectional curvature is an undesirable feature of the present state of affairs, similar to the Quadratic Orthogonal Bisectional Curvature. Even within a fixed frame, the expression for the real bisectional curvature is difficult to draw meaning from.

The Aubin–Yau Inequality. To obtain insight into the real bisectional curvature, we consider not just the Chern–Lu incarnation of the Schwarz lemma, but the Aubin–Yau inequality. All forms of the Schwarz lemma so far have arisen (more or less) from applying the maximum principle to (2.6.4) or (2.6.5). If we additionally assume that f is biholomorphic, however, we have another Laplacian at our disposal; namely, the target metric Laplacian: $\Delta_{\omega_h} = \text{tr}_{\omega_h}\sqrt{-1}\partial\bar{\partial}$. The Schwarz lemma with the target metric Laplacian was first considered in [**15, 318**], and hence is referred to as the *Aubin–Yau second-order estimate* (c.f., [**242**]):

Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map, biholomorphic onto its image. From the Bochner formula 2.6.2, we see that

$$\sqrt{-1}\partial\bar{\partial}|\partial f|^2 \quad = \quad \langle \nabla \partial f, \nabla \partial f \rangle - \sqrt{-1}\langle \Theta^{\Omega_X^{1,0} \otimes f^* T^{1,0} Y} \partial f, \partial f \rangle. \tag{2.6.8}$$

Taking the trace of (2.6.8) with respect to the target metric ω_h, we see that, in coordinates,

$$\begin{aligned} h^{\gamma\bar{\delta}} \partial_\gamma \partial_{\bar{\delta}} \left(g^{i\bar{j}} h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta} \right) \quad = \quad & h^{\gamma\bar{\delta}} g^{i\bar{j}} h_{\alpha\bar{\beta}} f_{ik}^\alpha \overline{f_{j\ell}^\beta} (f^{-1})_\gamma^k \overline{(f^{-1})_\delta^\ell} - g^{i\bar{j}}\, {}^c\widetilde{\text{Ric}}^{(2)}_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta} \\ & + h^{\gamma\bar{\delta}} g^{i\bar{q}} g^{p\bar{j}} \left({}^c R_{k\bar{\ell}p\bar{q}} \right) h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta} (f^{-1})_\gamma^k \overline{(f^{-1})_\delta^\ell}. \end{aligned} \tag{2.6.9}$$

As before, the first term on the right-hand side is the second fundamental form of f, and the second term is (minus the) second Chern–Ricci curvature of ω_h. This time, we want to understand

$$h^{\gamma\bar{\delta}} g^{i\bar{q}} g^{p\bar{j}} \left({}^c R_{k\bar{\ell}p\bar{q}} \right) h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta} (f^{-1})_\gamma^k \overline{(f^{-1})_\delta^\ell}.$$

Again, choose coordinates at p and $f(p)$ such that $g_{i\bar{j}} = \delta_{ij}$, $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$, and $f_i^\alpha = \lambda_i \delta_i^\alpha$. Then

$$h^{\gamma\bar{\delta}} g^{i\bar{q}} g^{p\bar{j}} \left({}^c R_{k\bar{\ell}p\bar{q}} \right) h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta} (f^{-1})_\gamma^k \overline{(f^{-1})_\delta^\ell} \quad = \quad \sum_{i,k} {}^c R_{i\bar{i}k\bar{k}} \frac{\lambda_i^2}{\lambda_k^2}. \tag{2.6.10}$$

This is not controlled by the real bisectional curvature (what is the vector here?).

Algebraic Framework. To better understand both (2.6.7) and (2.6.10), let us introduce the matrix $\mathcal{R} \in \mathbb{R}^{n \times n}$ with entries $\mathcal{R}_{\alpha\gamma} := R_{\alpha\bar{\alpha}\gamma\bar{\gamma}}$. If $v = (v_1, ..., v_n) \in \mathbb{R}^n \backslash \{0\}$, then the real bisectional curvature can be written as the Rayleigh quotient

$$f(v) \quad := \quad \frac{v^t \mathcal{R} v}{v^t v}.$$

In particular, an upper bound on the real bisectional curvature translates to an upper bound on

$$\max_{v \in \mathbb{R}^n \backslash \{0\}} f(v) \quad = \quad \max_{v \in \mathbb{R}^n \backslash \{0\}} \frac{v^t \mathcal{R} v}{v^t v},$$

for each fixed frame. From the scale invariance of the Rayleigh quotient, we may equivalently assume that $v \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$. This observation makes it immediate from the compactness of $\mathbb{S}^{n-1}$ that the maximum of $f(v)$ is always achieved. Further, the maximum occurs when v is the eigenvector of $\mathcal{R}$ corresponding to the largest eigenvalue.

The Schwarz Bisectional Curvatures. Recall that the quantity arising in the Chern–Lu inequality is

$$g^{i\bar{j}} g^{p\bar{q}} \left({}^c\widetilde{R}_{\alpha\overline{\beta}\gamma\overline{\delta}} \right) f_i^{\alpha} \overline{f_j^{\beta}} f_p^{\gamma} \overline{f_q^{\delta}} \quad = \quad \sum_{\alpha,\gamma} {}^c\widetilde{R}_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \lambda_{\alpha}^2 \lambda_{\gamma}^2, \tag{2.6.11}$$

where $\lambda_1^2 \geq \lambda_2^2 \geq \cdots \lambda_r^2 \geq \lambda_{r+1}^2 = \cdots = 0$, with $r = \text{rank}(\partial f)$. The real bisectional assumes control of the Rayleigh quotient $\dfrac{v^t \mathcal{R} v}{v^t v}$ for all $v \in \mathbb{R}^n \backslash \{0\}$. In particular, the positivity of the real bisectional curvature is equivalent to $\mathcal{R}$ being positive-definite (in each frame). But it is clear from (2.6.11) that we need only control the Rayleigh quotient over the cone

$$\Gamma \quad := \quad \{(x_1, ..., x_n) \in \mathbb{R}_+^n : x_1 \geq x_2 \geq \cdots \geq x_n \geq 0\}. \tag{2.6.12}$$

Here, $\mathbb{R}_+^n := \{(x_1, ..., x_n) \in \mathbb{R}^n : x_k \geq 0 \ \forall k, \ \text{and} \ x_1^2 + \cdots + x_n^2 > 0\}$ denotes the positive orthant. The precise (say, upper) bound on (2.6.11) is then given by controlling

$$\max_{v \in \Gamma} \frac{v^t \mathcal{R} v}{v^t v}.$$

Remark 2.6.16. In fact, if $f : X \longrightarrow Y$ is a holomorphic map with ∂f of rank r, then we may consider following smaller cone:

$$\Gamma_r \quad := \quad \{(x_1, ..., x_n) \in \mathbb{R}_+^n : x_1 \geq x_2 \geq \cdots \geq x_r \geq x_{r+1} = \cdots = 0\}.$$

For the Aubin–Yau inequality, we want to estimate (from below) the quantity:

$$h^{\gamma\bar{\delta}} g^{i\bar{q}} g^{p\bar{j}} {}^c R_{k\bar{\ell}p\bar{q}} h_{\alpha\overline{\beta}} f_i^{\alpha} \overline{f_j^{\beta}} (f^{-1})_{\gamma}^k \overline{(f^{-1})_{\delta}^{\ell}} \quad = \quad \sum_{i,k} {}^c R_{i\bar{i}k\bar{k}} \frac{\lambda_i^2}{\lambda_k^2}. \tag{2.6.13}$$

Let $\Gamma_+ := \{(x_1, ..., x_n) \in \mathbb{R}_+^n : x_1 \geq x_2 \geq \cdots \geq x_n > 0\}$ denote the cone of ordered positive n-tuples. For a vector $v \in \Gamma_+$, we denote by $u_v := v_{\circ}^{-1}$ the vector which inverts v with respect to the Hadamard product. That is, if $v = (v_1, ..., v_n) \in \Gamma_+$, then $u_v = (v_1^{-1}, ..., v_n^{-1})$. Then a bound on (2.6.13) translates to a bound on the generalized Rayleigh quotient

$$\frac{u_v^t \mathcal{R} v}{|u_v||v|}.$$

To distinguish this from the real bisectional curvature, we define the following:

Definition 2.6.17. Let (X, ω) be a Hermitian manifold. Define the (t–Gauduchon) *first Schwarz bisectional curvature*

$$ {}^t\mathrm{SBC}_{\omega}^{(1)} : \mathcal{F}_X \times \Gamma_+ \longrightarrow \mathbb{R}, \qquad \mathrm{SBC}_{\omega}^{(1)}(v) \ := \ \frac{u_v^t \mathcal{R} v}{|u_v||v|}, $$

and the (t–Gauduchon) *second Schwarz bisectional curvature*

$$ {}^t\mathrm{SBC}_{\omega}^{(2)} : \mathcal{F}_X \times \Gamma \longrightarrow \mathbb{R}, \qquad \mathrm{SBC}_{\omega}^{(2)}(v) \ := \ \frac{v^t \mathcal{R} v}{|v|^2}, $$

where $\mathcal{R}$ is the matrix with entries $\mathcal{R}_{\alpha\gamma} := {}^tR_{\alpha\overline{\alpha}\gamma\overline{\gamma}}$ with respect to the local unitary frame.

Remark 2.6.18. We say that these curvatures are bounded below if they are bounded below for all vectors in their domain and all frames. Similar definitions are made for bounded above, constant, etc., with the apparent modifications.

We can now state the sharpest forms of the Chern–Lu and Aubin–Yau inequalities in the Hermitian category (known at present) [**49**, **50**]:

Theorem 2.6.19. (Hermitian Chern–Lu). Let $f : (X^n, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Assume

$$ {}^c\mathrm{Ric}_{\omega_g}^{(2)} \geq -C_1 \omega_g - C_2 f^* \omega_h $$

for constants $C_1, C_2 \in \mathbb{R}$. Assume that ${}^c\mathrm{SBC}_{\omega_h}^{(2)} \leq -\kappa_0$ for some constant $\kappa_0 \leq -C_2$. If X is compact, then

$$ |\partial f|^2 \ \leq \ -\frac{C_1 r}{\kappa_0 + C_2}, $$

where r is the rank of ∂f.

Theorem 2.6.20. (Hermitian Aubin–Yau). Let $f : (X^n, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map between compact Hermitian manifolds, which is biholomorphic onto its image. Assume ${}^c\mathrm{SBC}_{\omega_g}^{(1)} \geq -\kappa_0$ and ${}^c\mathrm{Ric}_{\omega_h}^{(2)} \leq -C_1\omega_h + C_2 (f^{-1})^* \omega_g$ for κ_0, C_1, C_2 constants such that $C_1 > 0$,

$$ |\partial f|^2 \ \leq \ \frac{n(C_2 + \kappa_0)}{C_1}. $$

Proof. From 2.6.9, we have

$$ \begin{aligned} h^{\gamma\overline{\delta}} \partial_\gamma \partial_{\overline{\delta}} \left(g^{i\overline{j}} h_{\alpha\overline{\beta}} f_i^\alpha \overline{f_j^\beta} \right) \ &= \ h^{\gamma\overline{\delta}} g^{i\overline{j}} h_{\alpha\overline{\beta}} f_{ik}^\alpha \overline{f_{j\ell}^\beta} (f^{-1})_\gamma^k \overline{(f^{-1})_\delta^\ell} - g^{i\overline{j}}\, {}^c\widetilde{\mathrm{Ric}}_{\alpha\overline{\beta}}^{(2)} f_i^\alpha \overline{f_j^\beta} \\ &\quad + h^{\gamma\overline{\delta}} g^{i\overline{q}} g^{p\overline{j}} \left({}^cR_{k\overline{\ell}p\overline{q}} \right) h_{\alpha\overline{\beta}} f_i^\alpha \overline{f_j^\beta} (f^{-1})_\gamma^k \overline{(f^{-1})_\delta^\ell}. \end{aligned} $$

The upper bound on the second Chern–Ricci curvature then gives

$$ \begin{aligned} -{}^c\widetilde{\mathrm{Ric}}_{\alpha\overline{\beta}}^{(2)} g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta} \ &\geq \ C_1 h_{\alpha\overline{\beta}} g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta} - C_2 g_{i\overline{j}} g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta} (f^{-1})_\alpha^i \overline{(f^{-1})_\beta^j} \\ &= \ C_1 |\partial f|^2 - C_2 n \end{aligned} $$

Choose coordinates such that $g_{i\bar{j}} = \delta_{ij}$, $h_{\alpha\bar{\beta}} = \delta_{\alpha\beta}$, and $f_i^\alpha = \lambda_i \delta_i^\alpha$. Then

$$h^{\gamma\bar{\delta}} g^{i\bar{q}} g^{p\bar{j}} \left({}^c R_{k\bar{\ell}p\bar{q}} \right) h_{\alpha\bar{\beta}} f_i^\alpha \overline{f_j^\beta} (f^{-1})_\gamma^k \overline{(f^{-1})_\delta^\ell} \quad = \quad {}^c R_{k\bar{k}i\bar{i}} \frac{\lambda_i^2}{\lambda_k^2}.$$

Assuming a lower bound on the (Chern) first Schwarz bisectional curvature, we have

$$\sum_{i,k=1}^n {}^c R_{i\bar{i}k\bar{k}} \frac{\lambda_i^2}{\lambda_k^2} \quad \geq \quad -\kappa_0.$$

Combining these estimates, we have

$$\Delta_{\omega_h} |\partial f|^2 \quad \geq \quad C_1 |\partial f|^2 - nC_2 - \kappa_0.$$

The maximum principle completes the proof. □

One can combine these Schwarz lemmas to obtain the following 8–dimensional family of Schwarz lemmas:

Theorem 2.6.21. Let $f : (X^n, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map of rank r between Hermitian manifolds with $\text{SBC}_{\omega_g}^{(1)} \geq -\kappa_1$ and $\text{SBC}_{\omega_h}^{(2)} \leq -\kappa_2$, for some constants $\kappa_1, \kappa_2 \geq 0$. Assume there is a Hermitian metric ω_0 on X such that, for constants $C_1, C_2, C_3, C_4 \in \mathbb{R}$, with $C_3 > 0$ and $C_2 - \kappa_2 > 0$, we have

$$-C_1 \omega_0 - C_2 f^* \omega_h \ \leq \ {}^c\text{Ric}_{\omega_0}^{(2)} \ \leq \ -C_3 \omega_0 + C_4 \omega_g.$$

Then, if X is compact,

$$|\partial f|^2 \quad \leq \quad \frac{C_1 nr(\kappa_1 + C_4)}{C_3(\kappa_2 - C_2)}.$$

One particular corollary of the above theorem is the following Hermitian analog of the Chen–Cheng–Lu Schwarz lemma (c.f., [86]):

Corollary 2.6.22. Let $f : (X^n, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map of rank r between Hermitian manifolds with $\text{SBC}_{\omega_g}^{(1)} \geq -\kappa_1$ and $\text{SBC}_{\omega_h}^{(2)} \leq -\kappa_2$, for some constants $\kappa_1, \kappa_2 \geq 0$. Assume there is a Hermitian metric ω_0 on X such that

$$-C_1 \omega_0 + C_2 f^* \omega_h \ \leq \ {}^c\text{Ric}_{\omega_0}^{(2)} \ \leq \ -C_3 \omega_0,$$

where $C_1 = \frac{\kappa_2 + C_2}{\kappa_2 nr} C_3$, and $C_2 \geq \kappa_2(nr - 1)$. Then, if X is compact,

$$|\partial f|^2 \quad \leq \quad \frac{\kappa_1}{\kappa_2}.$$

The theorem also yields a Schwarz lemma expressed exclusively in terms of second Chern–Ricci curvatures:

Corollary 2.6.23. Let $f : (X^n, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map of rank r between Hermitian manifolds with $\mathrm{SBC}^{(1)}_{\omega_g} \geq -\kappa_1$ and $\mathrm{SBC}^{(2)}_{\omega_h} \leq -\kappa_2$, for some constants $\kappa_1, \kappa_2 \geq 0$. Assume there is a Hermitian metric ω_0 on X such that

$$-C_1\omega_0 + C_2 f^*\omega_h \ \leq \ {}^c\mathrm{Ric}^{(2)}_{\omega_0} \ \leq \ -C_3\omega_0 + C_4\omega_g,$$

where $C_3 > 0$ and $nr(\kappa_1 + C_4) \leq \kappa_2 + C_2$. Then, if X is compact,

$$|\partial f|^2 \ \leq \ \frac{C_1}{C_3}.$$

Example 2.6.24. Let us exhibit an example of a metric with negative second Schwarz bisectional curvature but whose real bisectional curvature does not have a sign: In a small disk centered at the origin in $\mathbb{C}^2$, with coordinates (z, w), define the metric

$$g \ = \ \begin{pmatrix} 1 + |z|^2 + 2|w|^2 & 0 \\ 0 & 1 + |w|^2 + 2|z|^2 \end{pmatrix}.$$

At the origin, the metric is Euclidean, and the 1–jets vanish. Hence, the (Chern) curvature tensor has non-zero components

$$R_{1\bar{1}1\bar{1}} \ = \ -1 \quad R_{1\bar{1}2\bar{2}} \ = \ -2 \quad R_{2\bar{2}1\bar{1}} \ = \ -2 \quad R_{2\bar{2}2\bar{2}} \ = \ -1.$$

The eigenvalues of $\mathfrak{Q}$

$$\lambda_1 \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \ = \ -3, \qquad \lambda_2 \begin{pmatrix} -1 & -2 \\ -2 & -1 \end{pmatrix} \ = \ 1,$$

with corresponding eigenvectors $v_1 = (1,1)$, $v_2 = (-1,1)$. In particular, the real bisectional curvature does not have a sign. To compute the (Chern) second Schwarz bisectional curvature ${}^c\mathrm{SBC}^{(2)}$, we need to find the maximum of

$$f(x,y) := -x^2 - 4xy - y^2,$$

over $\Gamma = \{(x,y) \in \mathbb{S}^1 : x \geq y \geq 0\}$. This is readily observed to be -1 (occurring at $(1,0)$).

Remark 2.6.25. This yields an interesting comparison between the Chern–Lu and Aubin–Yau inequalities. Indeed, the Chern–Lu inequality requires an upper bound on the second Schwarz bisectional curvature. The second Schwarz bisectional curvature is a Rayleigh quotient, which is well-known to give a variational characterization of the eigenvalues. The Aubin–Yau inequality requires a lower bound on the first Schwarz bisectional curvature. The first Schwarz bisectional curvature is a generalized Rayleigh quotient, known to give a variational characterization of the singular values. Therefore, at least philosophically, it appears that the Chern–Lu inequality is to the Aubin–Yau inequality what the eigenvalue decomposition is to the singular value decomposition.

Remark 2.6.26. The Schwarz bisectional curvatures may appear, upon first glance, as contrived, technical objects. The rich study of such structures in convex optimization and control theory gives credence to the Schwarz bisectional curvatures (besides the Schwarz lemma). Indeed, borrowing from this literature, we have the following definition: Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix. We say that A is *copositive* (respectively, *conegative*) if $v^t A v \geq 0$ (respectively, $v^t A v \leq 0$) for all $v \in \mathbb{R}^n_+$. We say that A is *strictly copositive* (respectively, *strictly conegative*) if $v^t A v > 0$ (respectively, $v^t A v < 0$) for all $v \in \mathbb{R}^n_+$. Copositive matrices generalize positive semi-definite matrices. For us, the following class of copositive matrices will be of particular importance: Let $\Gamma \subset \mathbb{R}^n_+$ be a cone. A symmetric matrix $A \in \mathbb{R}^{n \times n}$ is said to be Γ–copositive if $v^t A v \geq 0$ for all $v \in \Gamma$. Let (X, ω) be a Hermitian manifold. The second Schwarz bisectional curvature $\mathrm{SBC}_{\omega}^{(2)}$ is semi-negative (respectively, negative) if, for all frames, the matrix $\mathcal{Q} = \frac{1}{2}(\mathcal{R} + \mathcal{R}^t)$ is Γ–conegative (respectively, strictly Γ–conegative). Of course, one can formulate analogous definitions and make similar statements for the first Schwarz bisectional curvature, extending the notions of copositivity from quadratic forms to bilinear forms.

Our understanding of the first and second Schwarz bisectional curvatures is still in its infancy.

Question 2.6.27. Suppose (X, ω) be a compact Hermitian manifold with pointwise constant ${}^t\mathrm{SBC}_{\omega}^{(1)} \equiv \kappa$ (or ${}^t\mathrm{SBC}_{\omega}^{(2)} \equiv \kappa$) for some $\kappa \in \mathbb{R}$. For $\kappa \in \mathbb{R} \backslash \{0\}$, is the metric Kähler and t–Gauduchon-flat otherwise?

Question 2.6.28. Let (X, ω) be a compact Hermitian manifold. Suppose ${}^c\mathrm{SBC}_{\omega}^{(1)} > 0$ or ${}^c\mathrm{SBC}_{\omega}^{(2)} > 0$. Is X projective or rationally connected?

Since we wish to understand the relationship between the Schwarz bisectional curvatures and the holomorphic sectional curvature, it is natural to ask:

Question 2.6.29. Are there examples of compact Hermitian manifolds with ${}^t\mathrm{HSC}_{\omega} > 0$ but do not admit metrics with ${}^t\mathrm{SBC}_{\omega}^{(1)} > 0$ or ${}^t\mathrm{SBC}_{\omega}^{(2)} > 0$?

Gauduchon Schwarz Lemmas. All the results in the previous section were for the Chern connection. Given the growing interest in more general Hermitian connections on non-Kähler complex manifolds, it is desirable to obtain Schwarz lemmas for more general connections. The results of this section are joint with James Stanfield and appear in [58]:
Let us emphasize that the main point of the above formula is that it simplifies our computation to a computation of $\{\nabla^{1,0}\nabla^{0,1}\sigma, \sigma\}$. Maintaining the insight that we want to avoid coordinates for as long as we can, we introduce the following useful computational gadget:

Definition 2.6.30. Let $(\mathcal{E}, h) \longrightarrow X$ be a Hermitian vector bundle endowed with a metric connection ∇. We define the *CR–torsion* of ∇ to be the $\mathrm{End}(\mathcal{E})$–valued $(0,1)$–form $A \in$

$\Omega_X^{0,1}(\mathcal{E})$ defined by

$$A \ := \ \nabla^{0,1} - \bar{\partial}.$$

Remark 2.6.31. Observe that with respect to any local frame for $\mathcal{E}$, the CR–torsion A is a matrix of $(0,1)$–forms. Moreover, if P is the matrix describing a change of frame, then A transforms according to the adjoint action $A \mapsto PAP^{-1}$. Hence, the type of A is invariant under a change of frame.

The CR–torsion yields a fruitful language for elucidating more general Bochner formulae:

Theorem 2.6.32. Let $(\mathcal{E}, h) \longrightarrow X$ be a Hermitian vector bundle, endowed with a Hermitian connection ∇. Let $A \in \Omega_X^{0,1}(\mathcal{E})$ denote the CR-torsion of ∇ and let $\sigma \in H^0(\mathcal{E})$ be a holomorphic section of $\mathcal{E}$. Then, in a local frame $\{e_i\}$ for $T^{1,0}X$, we have

$$\begin{aligned} \partial\overline{\partial}|\sigma|^2(e_i, \overline{e_j}) \ &= \ \{A_{\bar{j},i}\sigma, \sigma\} + \{\sigma, A_{\bar{i},j}\sigma\} + \{A_{\bar{j}}(\sigma_{,i}), \sigma\} + \{\sigma, A_{\bar{i}}(\sigma_{,j})\} + \{\sigma_{,i}, \sigma_{,j}\} \\ &\quad + \{A_{\bar{j}}\sigma, A_{\bar{i}}\sigma\} - \{A_{[e_i, \overline{e_j}]^{0,1}}\sigma, \sigma\} + \{\sigma, A_{[\overline{e_i}, e_j]^{0,1}}\sigma\} + \{\sigma, \Theta_{\bar{i}j}\sigma\}. \end{aligned}$$

Theorem 2.6.33. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Let $\{e_i\}$ be a local unitary frame on X and $\{w_\alpha\}$ a local unitary frame on $f(X) \subseteq Y$. Choose Hermitian connections on $T^{1,0}X$ and $T^{1,0}Y$. Then $\partial f \in \Omega_X^{1,0}(f^* T^{1,0}Y)$ satisfies

$$\begin{aligned} \Delta_\omega |\partial f|^2 \ &= \ |\nabla \partial f|^2 + 2\,\mathrm{Re}\left(-f_k^\alpha \overline{f_\ell^\alpha} T_{\bar{i}\ell,i}^k + f_\ell^\alpha \overline{f_\ell^\beta} f_i^\gamma \overline{f_i^\delta} \tilde{T}_{\bar{\delta}\alpha,\gamma}^\beta - \overline{f_j^\beta} f_{k,i}^\beta T_{\bar{i}j}^k + \overline{f_j^\delta}\,\overline{f_i^\gamma} f_{j,i}^\alpha \tilde{T}_{\bar{\gamma}\alpha}^\delta \right) \\ &\quad + 2\,\mathrm{Re}\left(f_k^\alpha \overline{f_\ell^\alpha T_{\bar{i}i}^r} T_{\bar{r}\ell}^k - f_\ell^\alpha \overline{f_\ell^\beta} f_i^\gamma \overline{f_i^\delta} \overline{\tilde{T}_{\bar{\gamma}\delta}^\mu} \tilde{T}_{\bar{\mu}\alpha}^\beta \right) + \overline{f_k^\alpha} f_\ell^\alpha \overline{R_{i\bar{i}}{}^k{}_\ell} - \overline{f_\ell^\alpha} f_\ell^\beta \overline{f_i^\gamma} f_i^\delta \overline{\tilde{R}_{\gamma\bar{\delta}}{}^\beta{}_\alpha}. \end{aligned}$$

The letters T and R are respectively the torsion and curvature of the source connection, and $\tilde{T}$, $\tilde{R}$ are the torsion and curvature of the target connection.

We consider now the case when $T^{1,0}X$ is endowed with the s–Gauduchon connection ${}^s\nabla$ and $T^{1,0}Y$ is endowed with the t–Gauduchon connection. In this case, we have the following:

Theorem 2.6.34. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. If $T^{1,0}X$ is endowed with ${}^s\nabla$ and $T^{1,0}Y$ is endowed with ${}^t\nabla$, then

$$
\begin{aligned}
\Delta_{\omega_g} |\partial f|^2 \quad = \quad & |\nabla \partial f|^2 + \frac{s^2+2s-1}{2s(2s-1)} {}^s\mathrm{Ric}^{(2)}_{k\overline{k}} \lambda_k^2 + \frac{1-s}{4s(2s-1)} \left(2(1-s) {}^s\mathrm{Ric}^{(1)}_{k\overline{k}} - 2s({}^s\mathrm{Ric}^{(4)}_{k\overline{k}} + {}^s\mathrm{Ric}^{(3)}_{k\overline{k}}) \right) \lambda_k^2 \\
& + \frac{(s-1)^3}{8s^2(2s-1)} \left({}^sT^i_{ir} \overline{{}^sT^k_{kr}} + {}^sT^k_{kr} \overline{{}^sT^i_{ir}} \right) \lambda_k^2 + \frac{(s-1)(s^3+7s^2-5s+1)}{8s^3(2s-1)} {}^sT^k_{ir} \overline{{}^sT^k_{ir}} \lambda_k^2 \\
& + \frac{(1-s)(3s^3+7s^2-7s+1)}{8s^3(2s-1)} {}^sT^i_{kr} \overline{{}^sT^i_{kr}} \lambda_k^2 \\
& + \frac{t}{1-2t} \left({}^t\widetilde{R}_{\alpha\overline{\alpha}\beta\overline{\beta}} + {}^t\widetilde{R}_{\alpha\overline{\beta}\beta\overline{\alpha}} \right) \lambda_\alpha^2 \lambda_\beta^2 + \frac{1}{2t-1} {}^t\widetilde{R}_{\alpha\overline{\beta}\beta\overline{\alpha}} \lambda_\alpha^2 \lambda_\beta^2 \\
& + \frac{(1-t)^3}{8t^2(2t-1)} \left({}^t\widetilde{T}^\alpha_{\alpha\gamma} \overline{{}^t\widetilde{T}^\beta_{\beta\gamma}} + {}^t\widetilde{T}^\beta_{\beta\gamma} \overline{{}^t\widetilde{T}^\alpha_{\alpha\gamma}} \right) \lambda_\alpha^2 \lambda_\beta^2 \\
& + \frac{(t-1)^2(t+1)}{4t^2(2t-1)} {}^t\widetilde{T}^\beta_{\alpha\gamma} \overline{{}^t\widetilde{T}^\beta_{\alpha\gamma}} \lambda_\alpha^2 \lambda_\beta^2 + \frac{t-1}{t} {}^t\widetilde{T}^\gamma_{\alpha\beta} \overline{\widetilde{T}^\gamma_{\alpha\beta}} \lambda_\alpha^2 \lambda_\beta^2 \\
& + \left(\frac{1-t}{2t} - \frac{1-s}{2s} \right) \mathrm{Re} \left({}^sT^k_{ij} \overline{\widetilde{{}^tT^k_{ij}}} \right) \lambda_i \lambda_j \lambda_k .
\end{aligned}
$$

The proof of the above theorem requires dealing with the terms involving derivatives of torsion and the Hessian terms.

Theorem 2.6.35. For a holomorphic section $\sigma \in H^0(\mathcal{E})$,

$$
\begin{aligned}
\partial\overline{\partial}|\sigma|^2(u, \overline{v}) = & \left\langle \nabla_{\overline{v}}\sigma, \overline{\nabla_{\overline{u}}\sigma} \right\rangle + \left\langle \nabla_u \sigma, \overline{\nabla_v \sigma} \right\rangle - \left\langle \sigma, \overline{\Theta(u, \overline{v})\sigma} \right\rangle \\
& + 2\,\mathrm{Re} \left\langle \left((\nabla_u A)_{\overline{v}} + A_{\overline{A_{\overline{u}} v}} \right) \sigma, \overline{\sigma} \right\rangle + 2\,\mathrm{Re} \left\langle A_{\overline{v}} \nabla_u \sigma, \overline{\sigma} \right\rangle \\
& - \left\langle \sigma, \overline{\nabla_{[u,\overline{v}]^{1,0}} \sigma} \right\rangle - \left\langle \sigma, \overline{\nabla_{[\overline{u},v]^{1,0}} \sigma} \right\rangle .
\end{aligned}
$$

In particular,

$$
\begin{aligned}
\Delta_\omega |\sigma|^2 = & |\nabla \sigma|^2 - \left\langle \Theta(e_i, \overline{e_i})\sigma, \overline{\sigma} \right\rangle \\
& + 2\,\mathrm{Re} \left\langle \left((\nabla_{e_i} A)_{\overline{e_i}} + A_{\overline{A_{\overline{e_i}} e_i}} \right) \sigma, \overline{\sigma} \right\rangle + 2\,\mathrm{Re} \left\langle A_{\overline{e_i}} \nabla_{e_i} \sigma, \overline{\sigma} \right\rangle ,
\end{aligned}
$$

where $\{e_i\}_{i=1}^n$ is any unitary frame.

PROOF. Using the standard formulae for exterior derivatives, we have

$$\begin{aligned}
\partial\overline{\partial}|\sigma|^2(u,\overline{v}) &= \mathrm{d}\overline{\partial}|\sigma|^2(u,\overline{v}) \\
&= u(\overline{\partial}|\sigma|^2(\overline{v})) - \overline{v}(\overline{\partial}|\sigma|^2(u)) - \overline{\partial}|\sigma|^2([u,\overline{v}]) \\
&= u(\overline{v}|\sigma|^2) - [u,\overline{v}]^{0,1}|\sigma|^2 \\
&= u(\langle\nabla_{\overline{v}}\sigma,\overline{\sigma}\rangle + \langle\sigma,\overline{\nabla_v\sigma}\rangle) - [u,\overline{v}]^{0,1}|\sigma|^2 \\
&= \left\langle\nabla_{\overline{v}}\sigma,\overline{\nabla_{\overline{u}}\sigma}\right\rangle + \left\langle\nabla_u\sigma,\overline{\nabla_v\sigma}\right\rangle + \langle\nabla_u(A_{\overline{v}}\sigma),\overline{\sigma}\rangle + \left\langle\sigma,\overline{\nabla_{\overline{u}}\nabla_v\sigma}\right\rangle \\
&\quad - \left\langle\nabla_{[u,\overline{v}]^{0,1}}\sigma,\overline{\sigma}\right\rangle - \left\langle\sigma,\overline{\nabla_{\overline{[u,\overline{v}]^{0,1}}}\sigma}\right\rangle
\end{aligned}$$

By definition of curvature and the fact that σ is holomorphic,

$$\nabla_{\overline{v}}\nabla_u\sigma = \nabla_u(A_{\overline{v}}\sigma) - \Theta(u,\overline{v})\sigma - \nabla_{[u,\overline{v}]^{1,0}}\sigma - A_{[u,\overline{v}]^{0,1}}\sigma.$$

Hence,

$$\begin{aligned}
\partial\overline{\partial}|\sigma|^2(u,\overline{v}) &= \left\langle\nabla_{\overline{v}}\sigma,\overline{\nabla_{\overline{u}}\sigma}\right\rangle + \left\langle\nabla_u\sigma,\overline{\nabla_v\sigma}\right\rangle \\
&\quad + 2\,\mathrm{Re}\,\langle\nabla_u(A_{\overline{v}}\sigma),\overline{\sigma}\rangle - \left\langle\sigma,\overline{\Theta(u,\overline{v})\sigma}\right\rangle \\
&\quad - \left\langle\sigma,\overline{\nabla_{[u,\overline{v}]^{1,0}}\sigma}\right\rangle - \left\langle\sigma,\overline{\nabla_{[\overline{u},v]^{1,0}}\sigma}\right\rangle - 2\,\mathrm{Re}\left\langle A_{[u,\overline{v}]^{0,1}}\sigma,\sigma\right\rangle \\
&= \left\langle\nabla_{\overline{v}}\sigma,\overline{\nabla_{\overline{u}}\sigma}\right\rangle + \left\langle\nabla_u\sigma,\overline{\nabla_v\sigma}\right\rangle - \left\langle\sigma,\overline{\Theta(u,\overline{v})\sigma}\right\rangle \\
&\quad + 2\,\mathrm{Re}\,\langle(\nabla_uA)_{\overline{v}}\sigma,\overline{\sigma}\rangle + 2\,\mathrm{Re}\,\langle A_{\nabla_u\overline{v}}\sigma,\overline{\sigma}\rangle + 2\,\mathrm{Re}\,\langle A_{\overline{v}}\nabla_u\sigma,\overline{\sigma}\rangle - 2\,\mathrm{Re}\left\langle A_{[u,\overline{v}]^{0,1}}\sigma,\sigma\right\rangle \\
&\quad - \left\langle\sigma,\overline{\nabla_{[u,\overline{v}]^{1,0}}\sigma}\right\rangle - \left\langle\sigma,\overline{\nabla_{[\overline{u},v]^{1,0}}\sigma}\right\rangle \\
&= \left\langle\nabla_{\overline{v}}\sigma,\overline{\nabla_{\overline{u}}\sigma}\right\rangle + \left\langle\nabla_u\sigma,\overline{\nabla_v\sigma}\right\rangle - \left\langle\sigma,\overline{\Theta(u,\overline{v})\sigma}\right\rangle \\
&\quad + 2\,\mathrm{Re}\left\langle\left((\nabla_uA)_{\overline{v}} + A_{\overline{A_{\overline{u}}v}}\right)\sigma,\overline{\sigma}\right\rangle + 2\,\mathrm{Re}\,\langle A_{\overline{v}}\nabla_u\sigma,\overline{\sigma}\rangle \\
&\quad - \left\langle\sigma,\overline{\nabla_{[u,\overline{v}]^{1,0}}\sigma}\right\rangle - \left\langle\sigma,\overline{\nabla_{[\overline{u},v]^{1,0}}\sigma}\right\rangle
\end{aligned}$$

□

For a Hermitian vector bundle $\mathcal{E} \to X$ with Hermitian connection ∇. Define $B^{\nabla} \in H^0(T^*X \otimes T^*X \otimes \mathrm{End}(\mathcal{E}))$ by

$$B^{\nabla}(u,\overline{v}) := (\nabla_uA)_{\overline{v}} + A_{\overline{A_{\overline{u}}v}}.$$

Then, with $(\cdot)_{\mathrm{H}}$ denoting the Hermitian component, we see that

$$\Delta_\omega|\sigma|^2 \quad = \quad |\nabla\sigma|^2 - \langle\Theta(e_i,\overline{e_i})\sigma,\overline{\sigma}\rangle + 2\,\langle B(e_i,\overline{e_i})_{\mathrm{H}}\sigma,\overline{\sigma}\rangle + 2\,\mathrm{Re}\,\langle A_{\overline{e_i}}\nabla_{e_i}\sigma,\overline{\sigma}\rangle\,.$$

Further, it is easy to see that if f is holomorphic, then for all $u, v, w \in T^{1,0}Y$,

$$\left(B^{f^*TX\otimes T^*Y}(u,\overline{v})\partial f\right)w \quad = \quad B(\partial f(u),\overline{\partial f(v)})(\partial f(w)) - \partial f(B(u,\overline{v})w). \qquad (2.6.14)$$

Recall that the CR–torsion of the t–Gauduchon connection is given by $A_{\overline{u}}v = T(\overline{u}, v)^{1,0}$, where T is the torsion of the t–Gauduchon connection. Then

$$A_{\bar{i}j}^k \;=\; T_{\bar{i}j}^k \;=\; -T_{\bar{k}j}^i.$$

Further, since $2tT_{\bar{i}j}^k = (1-t)\overline{T_{ik}^j}$. In any unitary frame, let us write $B_{i\ell}^{jk} \;:=\; \langle B(e_i, \overline{e_j})e_\ell, \overline{e_k}\rangle$ for the components of B with respect to the unitary frame $\{e_\alpha\}$.

Proposition 2.6.36. For $t \in \mathbb{R}\backslash\{0, \frac{1}{2}\}$, we have

$$\begin{aligned}
-R_{i\bar{j}\ell\bar{k}} + B_{i\ell}^{jk} + \overline{B_{jk}^{i\ell}} \;\;=\;\; & \frac{1-2t-t^2}{2t(2t-1)}R_{i\bar{j}\ell\bar{k}} + \frac{t-1}{2t(2t-1)}\Big((1-t)R_{\ell\bar{k}i\bar{j}} - t(R_{i\bar{k}\ell\bar{j}} + R_{\ell\bar{j}i\bar{k}})\Big) \\
& + \frac{(1-t)^3}{8t^2(2t-1)}\left(T_{ir}^j\overline{T_{kr}^\ell} + T_{\ell r}^k\overline{T_{jr}^i}\right) + \frac{(t-1)^2(7t^2-4t+1)}{8t^3(2t-1)}T_{ir}^k\overline{T_{jr}^\ell} \\
& + \frac{(t-1)^3(1-5t)}{8t^3(2t-1)}T_{\ell r}^j\overline{T_{kr}^i} + \frac{t-1}{2t}T_{i\ell}^r\overline{T_{jk}^r}.
\end{aligned}$$

Proof. By definition,

$$B_{i\ell}^{jk} = A_{\bar{j}\ell,i}^k + \overline{A_{\bar{i}j}^r}A_{\bar{r}\ell}^k = T_{\bar{j}\ell,i}^k - \frac{(1-t)^2}{4t^2}\overline{T_{kr}^\ell}T_{ir}^j,$$

where, here, the comma denotes covariant differentiation with respect to ${}^t\nabla$. We first consider the derivative terms: Let

$$C_{i\ell}^{jk} \;\;:=\;\; T_{\bar{j}\ell,i}^k + \overline{T_{\bar{i}k,j}^\ell}, \qquad D_{i\ell}^{jk} \;\;:=\;\; T_{i\bar{j},\ell}^k + \overline{T_{j\bar{i},k}^\ell}$$

From the Gauduchon symmetries,

$$\begin{aligned}
2tC_{i\ell}^{jk} \;\;&=\;\; 2tT_{\bar{j}\ell,i}^k + (1-t)T_{i\ell,\bar{j}}^k \\
&=\;\; 2tT_{\bar{j}\ell,i}^k + (t-1)T_{\ell i,\bar{j}}^k \\
&=\;\; (t-1)T_{\{\bar{j}\ell,i\}}^k - (t-1)T_{i\bar{j},\ell}^k + (1+t)T_{\bar{j}\ell,i}^k.
\end{aligned}$$

It follows that

$$4tC_{i\ell}^{jk} = (t-1)\left(T_{\{\bar{j}\ell,i\}}^k + \overline{T_{\{\bar{i}k,j\}}^\ell}\right) - (t-1)D_{i\ell}^{jk} + (1+t)C_{i\ell}^{jk},$$

and hence,

$$(3t-1)C_{i\ell}^{jk} + (t-1)D_{i\ell}^{jk} = (t-1)\left(T_{\{\bar{j}\ell,i\}}^k + \overline{T_{\{\bar{i}k,j\}}^\ell}\right). \tag{2.6.15}$$

Similarly, we have

$$\begin{aligned} 2tD^{jk}_{i\ell} &= 2tT^k_{i\bar{j},\ell} + 2t\overline{T^\ell_{j\bar{i},k}} \\ &= -2tT^j_{i\bar{k},\ell} + (1-t)T^j_{\ell i,\bar{k}} \\ &= 2tT^j_{\bar{k}i,\ell} + (t-1)T^j_{i\ell,\bar{k}} \\ &= (t-1)T^j_{\{\bar{k}i,\ell\}} + (1+t)T^j_{\bar{k}i,\ell} + (t-1)T^j_{\bar{k}\ell,i} \\ &= (t-1)T^j_{\{\bar{k}i,\ell\}} + (1+t)T^k_{i\bar{j},\ell} - (t-1)T^k_{\bar{j}\ell,i}, \end{aligned}$$

and so

$$4tD^{jk}_{i\ell} = (t-1)\left(T^j_{\{\bar{k}i,\ell\}} + \overline{T^i_{\{\bar{\ell}j,k\}}}\right) + (t+1)D^{jk}_{i\ell} - (t-1)C^{jk}_{i\ell}.$$

Thus,

$$(t-1)C^{jk}_{i\ell} + (3t-1)D^{jk}_{i\ell} = (t-1)\left(T^j_{\{\bar{k}i,\ell\}} + \overline{T^i_{\{\bar{\ell}j,k\}}}\right). \tag{2.6.16}$$

Solving the linear system gives

$$C^{jk}_{i\ell} = \frac{t-1}{4t(2t-1)}\left((3t-1)\left(T^k_{\{\bar{j}\ell,i\}} + \overline{T^\ell_{\{\bar{i}k,j\}}}\right) + (1-t)\left(T^j_{\{\bar{k}i,\ell\}} + \overline{T^i_{\{\bar{\ell}j,k\}}}\right)\right).$$

We now focus on eliminating the terms involving derivatives of the torsion. Recall from Kobayashi–Nomizu [**185**, p. 135] that

$$T^k_{\{\bar{j}\ell,i\}} = R_{\{i\bar{j}\ell\}\bar{k}} + T^r_{\{i\bar{j}}T^k_{\ell\}r} + T^{\bar{r}}_{\{i\bar{j}}T^k_{\ell\}\bar{r}},$$

where R denotes the curvature of ${}^t\nabla$. The Gauduchon symmetries give

$$\begin{aligned} T^k_{\{\bar{j}\ell,i\}} &= R_{i\bar{j}\ell\bar{k}} - R_{\ell\bar{j}i\bar{k}} + T^r_{i\bar{j}}T^k_{\ell r} + T^r_{\bar{j}\ell}T^k_{ir} + T^r_{\ell i}T^k_{\bar{j}r} + T^{\bar{r}}_{i\bar{j}}T^k_{\ell\bar{r}} + T^{\bar{r}}_{\bar{j}\ell}T^k_{i\bar{r}} \\ &= R_{i\bar{j}\ell\bar{k}} - R_{\ell\bar{j}i\bar{k}} + \frac{t-1}{2t}\left(T^k_{\ell r}\overline{T^i_{jr}} + T^k_{ir}\overline{T^\ell_{jr}} + T^r_{i\ell}\overline{T^r_{jk}}\right) + \frac{(t-1)^2}{4t^2}\left(T^j_{ir}\overline{T^\ell_{kr}} - T^j_{\ell r}\overline{T^i_{kr}}\right). \end{aligned}$$

Hence,

$$\begin{aligned} T^k_{\{\bar{j}\ell,i\}} + \overline{T^\ell_{\{\bar{i}k,j\}}} &= 2R_{i\bar{j}\ell\bar{k}} - R_{\ell\bar{j}i\bar{k}} - R_{i\bar{k}\ell\bar{j}} + \frac{(t-1)(3t-1)}{4t^2}\left(T^j_{ir}\overline{T^\ell_{kr}} + T^k_{\ell r}\overline{T^i_{jr}}\right) \\ &\quad + \frac{t-1}{t}\left(T^k_{ir}\overline{T^\ell_{jr}} + T^r_{i\ell}\overline{T^r_{jk}}\right) - \frac{(t-1)^2}{2t^2}T^j_{\ell r}\overline{T^i_{kr}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} T^j_{\{\bar{k}i,\ell\}} + \overline{T^i_{\{\bar{\ell}j,k\}}} &= 2R_{\ell\bar{k}i\bar{j}} - R_{i\bar{k}\ell\bar{j}} - R_{\ell\bar{j}i\bar{k}} + \frac{(t-1)(3t-1)}{4t^2}\left(T^k_{\ell r}\overline{T^i_{jr}} + T^j_{ir}\overline{T^\ell_{kr}}\right) \\ &\quad + \frac{t-1}{t}\left(T^j_{\ell r}\overline{T^i_{kr}} + T^r_{\ell i}\overline{T^r_{kj}}\right) - \frac{(t-1)^2}{2t^2}T^k_{ir}\overline{T^\ell_{jr}}. \end{aligned}$$

It follows that

$$\begin{aligned}\frac{4t(2t-1)}{t-1}C_{i\ell}^{jk} =& \,2(3t-1)R_{i\bar{j}\ell\bar{k}} + 2(1-t)R_{\ell\bar{k}i\bar{j}} - 2t(R_{i\bar{k}\ell\bar{j}} + R_{\ell\bar{j}i\bar{k}}) \\ &+ \frac{(t-1)(3t-1)}{2t}\left(T_{\ell r}^{k}\overline{T_{jr}^{i}} + T_{ir}^{j}\overline{T_{kr}^{\ell}}\right) + \frac{(t-1)(7t^2-4t+1)}{2t^2}T_{ir}^{k}\overline{T_{jr}^{\ell}} \\ &+ \frac{(t-1)^2(1-5t)}{2t^2}T_{\ell r}^{j}\overline{T_{kr}^{i}} + \frac{2t(2t-1)}{t}T_{i\ell}^{r}\overline{T_{jk}^{r}}.\end{aligned}$$

Thus,

$$\begin{aligned}B_{i\ell}^{jk} + \overline{B_{jk}^{i\ell}} =& \,C_{i\ell}^{jk} - \frac{(t-1)^2}{4t^2}\left(T_{ir}^{j}\overline{T_{kr}^{\ell}} + T_{\ell r}^{k}\overline{T_{jr}^{i}}\right) \\ =& \,\frac{t-1}{4t(2t-1)}\Big(2(3t-1)R_{i\bar{j}\ell\bar{k}} + 2(1-t)R_{\ell\bar{k}i\bar{j}} - 2t(R_{i\bar{k}\ell\bar{j}} + R_{\ell\bar{j}i\bar{k}})\Big) \\ &+ \frac{(1-t)^3}{8t^2(2t-1)}\left(T_{ir}^{j}\overline{T_{kr}^{\ell}} + T_{\ell r}^{k}\overline{T_{jr}^{i}}\right) + \frac{(t-1)^2(7t^2-4t+1)}{8t^3(2t-1)}T_{ir}^{k}\overline{T_{jr}^{\ell}} \\ &+ \frac{(t-1)^3(1-5t)}{8t^3(2t-1)}T_{\ell r}^{j}\overline{T_{kr}^{i}} + \frac{t-1}{2t}T_{i\ell}^{r}\overline{T_{jk}^{r}},\end{aligned}$$

as required. □

For any Hermitian connection ∇ on the tangent bundle of a Hermitian manifold, define, with respect to any local unitary frame,

$$K^{\nabla}_{i\bar{j}l}{}^{k} \;:=\; -R_{i\bar{j}l\bar{k}} + B_{il}^{jk} + \overline{B_{jk}^{il}}.$$

Let $f:(X,\omega_g)\to(Y,\omega_h)$ be a holomorphic map. Let ∇^X and ∇^Y be Hermitian connections on $T^{1,0}X$ and $T^{1,0}Y$, respectively. Let $K = K^{\nabla^X}$ and $\widetilde{K} = K^{\nabla^Y}$. Let $\{e_i\}_{i=1}^n$ and $\{e_\alpha\}_{\alpha=1}^n$ be local unitary frames of X and Y respectively. Writing $f_i^\alpha := \left\langle \partial f(e_i), \overline{f^*e_\alpha}\right\rangle$, from (2.6.14), we have

$$\left\langle K_{i\bar{i}}\partial f, \overline{\partial f}\right\rangle = f_i^\alpha\overline{f_i^\beta}f_j^\gamma\overline{f_j^\delta}\widetilde{K}_{\alpha\bar{\beta}\gamma}{}^{\delta} - f_k^\alpha\overline{f_j^\alpha}K_{i\bar{i}j}{}^{k}.$$

Choosing the frames such that $f_i^\alpha = \delta_i^\alpha\lambda_\alpha$ gives

$$\left\langle K_{i\bar{i}}\partial f, \overline{\partial f}\right\rangle = \sum_{\alpha,\beta}\lambda_\alpha^2\lambda_\beta^2\widetilde{K}_{\alpha\bar{\alpha}\beta\bar{\beta}} - \sum_{i,k,\ell}K_{i\bar{i}\ell\bar{k}}\lambda_\ell\lambda_k.$$

If $K = K^{{}^s\nabla}$, and $\widetilde{K} = K^{{}^t\widetilde{\nabla}}$, then

$$
\begin{aligned}
\left\langle K_{i\bar{i}}\partial f, \overline{\partial f}\right\rangle = & \frac{s^2+2s-1}{2s(2s-1)} R_{i\bar{i}k\bar{k}}\lambda_k^2 + \frac{1-s}{4s(2s-1)}\left(2(1-s)R_{k\bar{k}i\bar{i}} - 2s(R_{i\bar{k}k\bar{i}} + R_{k\bar{i}i\bar{k}})\right)\lambda_k^2 \\
& + \frac{(s-1)^3}{8s^2(2s-1)}\left(T^i_{ir}\overline{T^k_{kr}} + T^k_{kr}\overline{T^i_{ir}}\right)\lambda_k^2 - \frac{(s-1)^2(7s^2-4s+1)}{8s^3(2s-1)} T^k_{ir}\overline{T^k_{ir}}\lambda_k^2 \\
& + \frac{(1-s)(3s^3+7s^2-7s+1)}{8s^3(2s-1)} T^i_{kr}\overline{T^i_{kr}}\lambda_k^2 \\
& + \frac{1-2t-t^2}{2t(2t-1)}\widetilde{R}_{\alpha\bar{\alpha}\beta\bar{\beta}}\lambda_\alpha^2\lambda_\beta^2 + \frac{t-1}{4t(2t-1)}\left(2(1-t)\widetilde{R}_{\alpha\bar{\alpha}\beta\bar{\beta}} - 2t(\widetilde{R}_{\alpha\bar{\beta}\beta\bar{\alpha}} + \widetilde{R}_{\alpha\bar{\beta}\beta\bar{\alpha}})\right)\lambda_\alpha^2\lambda_\beta^2 \\
& + \frac{(1-t)^3}{8t^2(2t-1)}\left(\widetilde{T}^\alpha_{\alpha\gamma}\overline{\widetilde{T}^\beta_{\beta\gamma}} + \widetilde{T}^\beta_{\beta\gamma}\overline{\widetilde{T}^\alpha_{\alpha\gamma}}\right)\lambda_\alpha^2\lambda_\beta^2 + \frac{(t-1)^2(7t^2-4t+1)}{8t^3(2t-1)}\widetilde{T}^\beta_{\alpha\gamma}\overline{\widetilde{T}^\beta_{\alpha\gamma}}\lambda_\alpha^2\lambda_\beta^2 \\
& + \frac{(t-1)^3(1-5t)}{8t^3(2t-1)}\widetilde{T}^\alpha_{\beta\gamma}\overline{\widetilde{T}^\alpha_{\beta\gamma}}\lambda_\alpha^2\lambda_\beta^2 + \frac{t-1}{2t}\widetilde{T}^\gamma_{\alpha\beta}\overline{\widetilde{T}^\gamma_{\alpha\beta}}\lambda_\alpha^2\lambda_\beta^2 \\
= & \frac{s^2+2s-1}{2s(2s-1)} R_{i\bar{i}k\bar{k}}\lambda_k^2 + \frac{1-s}{4s(2s-1)}\left(2(1-s)R_{k\bar{k}i\bar{i}} - 2s(R_{i\bar{k}k\bar{i}} + R_{k\bar{i}i\bar{k}})\right)\lambda_k^2 \\
& + \frac{(s-1)^3}{8s^2(2s-1)}\left(T^i_{ir}\overline{T^k_{kr}} + T^k_{kr}\overline{T^i_{ir}}\right)\lambda_k^2 - \frac{(s-1)^2(7s^2-4s+1)}{8s^3(2s-1)} T^k_{ir}\overline{T^k_{ir}}\lambda_k^2 \\
& + \frac{(1-s)(3s^3+7s^2-7s+1)}{8s^3(2s-1)} T^i_{kr}\overline{T^i_{kr}}\lambda_k^2 \\
& + \frac{t}{1-2t}\left(\widetilde{R}_{\alpha\bar{\alpha}\beta\bar{\beta}} + \widetilde{R}_{\alpha\bar{\beta}\beta\bar{\alpha}}\right)\lambda_\alpha^2\lambda_\beta^2 + \frac{1}{2t-1}\widetilde{R}_{\alpha\bar{\beta}\beta\bar{\alpha}}\lambda_\alpha^2\lambda_\beta^2 \\
& + \frac{(1-t)^3}{8t^2(2t-1)}\left(\widetilde{T}^\alpha_{\alpha\gamma}\overline{\widetilde{T}^\beta_{\beta\gamma}} + \widetilde{T}^\beta_{\beta\gamma}\overline{\widetilde{T}^\alpha_{\alpha\gamma}}\right)\lambda_\alpha^2\lambda_\beta^2 \\
& + \frac{(t-1)^2(t+1)}{4t^2(2t-1)}\widetilde{T}^\beta_{\alpha\gamma}\overline{\widetilde{T}^\beta_{\alpha\gamma}}\lambda_\alpha^2\lambda_\beta^2 + \frac{t-1}{2t}\widetilde{T}^\gamma_{\alpha\beta}\overline{\widetilde{T}^\gamma_{\alpha\beta}}\lambda_\alpha^2\lambda_\beta^2.
\end{aligned}
$$

Let us now focus on the final term appearing in the Schwarz lemma:

Lemma 2.6.37.

$$
\begin{aligned}
2\operatorname{Re}\left\langle A_{\bar{i}}\nabla_i\partial f, \overline{\partial f}\right\rangle = & \operatorname{Re}\left(\frac{t-1}{2t}\widetilde{T}^\gamma_{\mu\nu}\overline{\widetilde{T}^\gamma_{\alpha\beta}} f^\mu_j f^\nu_i \overline{f^\beta_i}\,\overline{f^\alpha_j} + \frac{s-1}{2s}T^k_{ir}\overline{T^j_{ir}} f^\alpha_k\overline{f^\alpha_j}\right) \\
& + \operatorname{Re}\left(\frac{1-t}{2t}\overline{\widetilde{T}^\gamma_{\beta\alpha}} T^k_{ij} f^\gamma_k \overline{f^\beta_i}\,\overline{f^\alpha_j} - \frac{1-s}{2s}\overline{T^j_{ir}}\widetilde{T}^\alpha_{\beta\gamma} f^\beta_r f^\gamma_i \overline{f^\alpha_j}\right)
\end{aligned}
$$

Proof. First, we claim that

$$
(\nabla_i\partial f)^\alpha_j - (\nabla_j\partial f)^\alpha_i \;=\; -T^k_{ij}f^\alpha_k + \widetilde{T}^\alpha_{\beta\gamma}f^\beta_i f^\gamma_j.
$$

Indeed, if we assume $\{e_i\}_{i=1}^n$ is a coordinate frame satisfying $\langle e_i, \overline{e_j}\rangle = \delta_{ij}$ at the point where we compute, we have

$$(\nabla_i f)_j^\alpha \;=\; \partial_i\partial_j f^\alpha - f_k^\alpha \Gamma_{ij}^k + f_j^\gamma f_i^\beta \widetilde{\Gamma}_{\beta\gamma}^\alpha,$$

and thus the statement follows from the definition of torsion. Then,

$$(A_{\bar{i}}\nabla_i \partial f)_j^\alpha = \overline{f_i^\beta}\tilde{A}_{\overline{\beta}\gamma}^\alpha f_{j,i}^\gamma - f_{r,i}^\alpha A_{\bar{i}j}^r.$$

Hence, using the symmetry $A_{\bar{i}j}^k = -A_{\bar{k}j}^i$, we have

$$\begin{aligned}
2\,\mathrm{Re}\left\langle A_{\bar{i}}\nabla_i\partial f, \overline{\partial f}\right\rangle &= 2\,\mathrm{Re}\left(\tilde{A}_{\overline{\beta}\gamma}^\alpha f_{j,i}^\gamma \overline{f_i^\beta}\,\overline{f_j^\alpha} - f_{r,i}^\alpha A_{\bar{i}j}^r \overline{f_j^\alpha}\right)\\
&= \mathrm{Re}\left(\widetilde{A}_{\overline{\beta}\gamma}^\alpha (f_{j,i}^\gamma - f_{i,j}^\gamma)\overline{f_i^\beta}\,\overline{f_j^\alpha} - A_{\bar{i}j}^r (f_{r,i}^\alpha - f_{i,r}^\alpha)\overline{f_j^\alpha}\right)\\
&= \mathrm{Re}\Bigg(\widetilde{A}_{\overline{\beta}\gamma}^\alpha T_{ij}^k f_k^\gamma \overline{f_i^\beta}\,\overline{f_j^\alpha} + \widetilde{A}_{\overline{\beta}\gamma}^\alpha \widetilde{T}_{\mu\nu}^\gamma f_j^\mu f_i^\nu \overline{f_i^\beta}\,\overline{f_j^\alpha}\\
&\quad + A_{\bar{i}j}^r T_{ri}^k f_k^\alpha \overline{f_j^\alpha} - A_{\bar{i}j}^r \widetilde{T}_{\beta\gamma}^\alpha f_r^\beta f_i^\gamma \overline{f_j^\alpha}\Bigg)\\
&= \mathrm{Re}\Bigg(\widetilde{T}_{\overline{\beta}\gamma}^\alpha T_{ij}^k f_k^\gamma \overline{f_i^\beta}\,\overline{f_j^\alpha} + \widetilde{T}_{\overline{\beta}\gamma}^\alpha \widetilde{T}_{\mu\nu}^\gamma f_j^\mu f_i^\nu \overline{f_i^\beta}\,\overline{f_j^\alpha}\\
&\quad + T_{\bar{i}j}^r T_{ri}^k f_k^\alpha \overline{f_j^\alpha} - T_{\bar{i}j}^r \widetilde{T}_{\beta\gamma}^\alpha f_r^\beta f_i^\gamma \overline{f_j^\alpha}\Bigg)\\
&= \mathrm{Re}\left(\frac{t-1}{2t}\widetilde{T}_{\mu\nu}^\gamma \overline{\widetilde{T}_{\alpha\beta}^\gamma} f_j^\mu f_i^\nu \overline{f_i^\beta}\,\overline{f_j^\alpha} + \frac{s-1}{2s} T_{ir}^k \overline{T_{ir}^j} f_k^\alpha \overline{f_j^\alpha}\right)\\
&\quad + \mathrm{Re}\left(\frac{1-t}{2t}\overline{\widetilde{T}_{\beta\alpha}^\gamma} T_{ij}^k f_k^\gamma \overline{f_i^\beta}\,\overline{f_j^\alpha} - \frac{1-s}{2s}\overline{T_{ir}^j}\widetilde{T}_{\beta\gamma}^\alpha f_r^\beta f_i^\gamma \overline{f_j^\alpha}\right).
\end{aligned}$$

□

We consider the following specific cases:

Theorem 2.6.38. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Then

$$\begin{aligned}
\Delta_{\omega_g}|\partial f|^2 \;=\;& |\nabla\partial f|^2 - \frac{1}{3}{}^b\mathrm{Ric}_{k\bar{k}}^{(2)}\lambda_k^2 + \frac{2}{3}{}^b\mathrm{Ric}_{k\bar{k}}^{(1)}\lambda_k^2 + \frac{1}{3}{}^b\mathrm{Ric}_{k\bar{k}}^{(4)}\lambda_k^2 + \frac{1}{3}{}^b\mathrm{Ric}_{k\bar{k}}^{(3)}\lambda_k^2\\
&+\frac{1}{3}\left({}^bT_{ir}^i\overline{{}^bT_{kr}^r} + {}^bT_{kr}^r\overline{{}^bT_{ir}^i}\right)\lambda_k^2 - {}^bT_{ir}^k\overline{{}^bT_{ir}^k}\lambda_k^2 + {}^bT_{kr}^i\overline{{}^bT_{kr}^i}\lambda_k^2\\
&-\frac{1}{3}{}^b\mathrm{RBC}_{\omega_h}\left(\sum_\alpha \lambda_\alpha^4\right)^2 - \frac{2}{3}{}^b\widetilde{\mathrm{RBC}}_{\omega_h}\left(\sum_\alpha\lambda_\alpha^4\right)^2\\
&-\frac{1}{3}\left({}^b\widetilde{T}_{\alpha\gamma}^\alpha\overline{{}^b\widetilde{T}_{\beta\gamma}^\beta} + {}^b\widetilde{T}_{\beta\gamma}^\beta\overline{{}^b\widetilde{T}_{\alpha\gamma}^\alpha}\right)\lambda_\alpha^2\lambda_\beta^2 + 2{}^b\widetilde{T}_{\alpha\beta}^\gamma\overline{{}^b\widetilde{T}_{\alpha\beta}^\gamma}\lambda_\alpha^2\lambda_\beta^2.
\end{aligned}$$

Theorem 2.6.39. (Bismut Schwarz Lemma). Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Suppose

$$2{}^b\mathrm{Ric}^{(1)}_{\omega_g} - {}^b\mathrm{Ric}^{(2)}_{\omega_g} + {}^b\mathrm{Ric}^{(3)}_{\omega_g} + {}^b\mathrm{Ric}^{(4)}_{\omega_g} \quad \geq \quad -C_1\omega_g - C_2 f^*\omega_h$$

for some constants $C_1, C_2 \in \mathbb{R}$. Assume $|{}^b\widetilde{T}| \leq B$ and ${}^b\mathrm{RBC}_{\omega_h} + 2{}^b\widetilde{\mathrm{RBC}}_{\omega_h} \leq \kappa_0$ for some constants $B, \kappa_0 \in \mathbb{R}$. Then

$$\Delta_{\omega_g}|\partial f|^2 \quad \geq \quad -\frac{C_1}{3}|\partial f|^2 - \left(\frac{C_2}{3} + \frac{\kappa_0}{3} + \frac{2B}{3}\right)|\partial f|^4.$$

If X is compact and $C_2 + \kappa_0 + 2B < 0$, then

$$|\partial f|^2 \quad \leq \quad -\frac{C_1}{C_2 + \kappa_0 + 2B}.$$

From the Ricci curvature relations, we see that if (X, ω_g) is a compact balanced manifold, then

$$\begin{aligned}
2{}^b\mathrm{Ric}^{(1)}_{\omega_g} - {}^b\mathrm{Ric}^{(2)}_{\omega_g} + {}^b\mathrm{Ric}^{(3)}_{\omega_g} + {}^b\mathrm{Ric}^{(4)}_{\omega_g} \quad &= \quad 2{}^c\mathrm{Ric}^{(1)}_{\omega_g} - \left({}^c\mathrm{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\Lambda(\partial\bar{\partial}\omega_g) - {}^cT^\circ\right) \\
&\qquad +2\left({}^c\mathrm{Ric}^{(1)}_{\omega_g} - \sqrt{-1}\Lambda(\partial\bar{\partial}\omega_g)\right) \\
&= \quad 3{}^c\mathrm{Ric}^{(1)}_{\omega_g} - 3\sqrt{-1}\Lambda(\partial\bar{\partial}\omega_g) + {}^cT^\circ.
\end{aligned}$$

Hence, we can reformulate the previous Schwarz lemma as follows:

Theorem 2.6.40. (Bismut Schwarz Lemma). Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Assume (X, ω_g) is a compact balanced manifold with

$$3{}^c\mathrm{Ric}^{(1)}_{\omega_g} - 3\sqrt{-1}\Lambda(\partial\bar{\partial}\omega_g) + {}^cT^\circ \quad \geq \quad -C_1\omega_g - C_2 f^*\omega_h,$$

for some constants $C_1, C_2 \in \mathbb{R}$. Assume $\left|{}^b\tilde{T}\right| \leq B$ and ${}^b\mathrm{RBC}_{\omega_h} + 2{}^b\widetilde{\mathrm{RBC}}_{\omega_h} \leq \kappa_0$ for some constant $B, \kappa_0 \in \mathbb{R}$. Then

$$3\Delta_{\omega_g}|\partial f|^2 \quad \geq \quad -C_1|\partial f|^2 - (C_2 + \kappa_0 + 2B)\,|\partial f|^4.$$

Hence, if $C_2 + \kappa_0 + 2B < 0$, then

$$|\partial f|^2 \quad \leq \quad -\frac{C_1}{C_2 + \kappa_0 + 2B}.$$

Remark 2.6.41. Since, for a compact balanced manifold (X, ω_g), the first Chern–Ricci curvature coincides with (the $(1,1)$–part of) the first Bismut–Ricci curvature, the assumption in the above theorem can be replaced with

$$3{}^b\mathrm{Ric}^{(1)}_{\omega_g} - 3\sqrt{-1}\Lambda(\partial\bar{\partial}\omega_g) + {}^cT^\circ \quad \geq \quad -C_1\omega_g - C_2 f^*\omega_h.$$

Further, we note that the above assumption is strictly weaker than

$$3{}^b\mathrm{Ric}^{(1)}_{\omega_g} - 3\sqrt{-1}\Lambda(\partial\bar{\partial}\omega_g) \ \geq \ -C_1\omega_g - C_2 f^*\omega_h,$$

since ${}^cT^{\circ}$ is positive-definite (unless the metric is Kähler).

Theorem 2.6.42. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Then

$$\begin{aligned} \Delta_{\omega_g}|\partial f|^2 \ &= \ |\nabla\partial f|^2 + \frac{7}{12}{}^2\mathrm{Ric}^{(2)}_{k\bar{k}}\lambda_k^2 + \frac{1}{12}{}^2\mathrm{Ric}^{(1)}_{k\bar{k}}\lambda_k^2 + \frac{1}{6}{}^2\mathrm{Ric}^{(4)}_{k\bar{k}}\lambda_k^2 + \frac{1}{6}{}^2\mathrm{Ric}^{(3)}_{k\bar{k}}\lambda_k^2 \\ &+ \frac{1}{96}\left({}^2T^i_{ir}\overline{{}^2T^r_{kr}} + {}^2T^r_{kr}\overline{{}^2T^i_{ir}}\right)\lambda_k^2 + \frac{9}{64}{}^2T^k_{ir}\overline{{}^2T^k_{ir}}\lambda_k^2 - \frac{13}{64}{}^2T^i_{kr}\overline{{}^sT^i_{kr}}\lambda_k^2 \\ &- \frac{2}{3}{}^2\mathrm{RBC}_{\omega_h}\left(\sum_{\alpha}\lambda_\alpha^4\right)^2 - \frac{1}{3}{}^2\widetilde{\mathrm{RBC}}_{\omega_h}\left(\sum_{\alpha}\lambda_\alpha^4\right)^2 \\ &- \frac{1}{96}\left({}^t\widetilde{T}^\alpha_{\alpha\gamma}\overline{{}^t\widetilde{T}^\beta_{\beta\gamma}} + {}^t\widetilde{T}^\beta_{\beta\gamma}\overline{{}^t\widetilde{T}^\alpha_{\alpha\gamma}}\right)\lambda_\alpha^2\lambda_\beta^2 + \frac{1}{16}{}^2\widetilde{T}^\beta_{\alpha\gamma}\overline{{}^2\widetilde{T}^\beta_{\alpha\gamma}}\lambda_\alpha^2\lambda_\beta^2 + \frac{1}{2}{}^2\widetilde{T}^\gamma_{\alpha\beta}\overline{\widetilde{{}^2T}^\gamma_{\alpha\beta}}\lambda_\alpha^2\lambda_\beta^2 \end{aligned}$$

Theorem 2.6.43. (2+2 Schwarz Lemma). Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Suppose

$$7{}^2\mathrm{Ric}^{(2)}_{\omega_g} + {}^2\mathrm{Ric}^{(1)}_{\omega_g} + 2{}^2\mathrm{Ric}^{(4)}_{\omega_g} + 2{}^2\mathrm{Ric}^{(3)}_{\omega_g} \ \ \geq \ \ -C_1\omega_g - C_2 f^*\omega_h,$$

for some constants $C_1, C_2 \in \mathbb{R}$. Let $C_3 > 0$ be a constant such that $|{}^2T| \leq C_3$. If

$$2{}^2\mathrm{SBC}^{(2)}_{\omega_h} + {}^2\widetilde{\mathrm{SBC}}^{(2)}_{\omega_h} \ \leq \ -\kappa_0,$$

then

$$\Delta_{\omega_g}|\partial f|^2 \ \ \geq \ \ -\left(\frac{C_1}{12} + \frac{C_3}{24}\right)|\partial f|^2 + \left(\frac{\kappa_0}{3} - \frac{C_2}{12}\right)|\partial f|^4.$$

Hence, if X is compact, and $4\kappa_0 > C_2$, then

$$|\partial f|^2 \ \ \leq \ \ \frac{C_1 + 2C_3}{4\kappa_0 - C_2}.$$

Theorem 2.6.44. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between Hermitian manifolds. Then

$$\begin{aligned} \Delta_{\omega_g}|\partial f|^2 \ &= \ |\nabla\partial f|^2 + {}^{\frac{1}{3}}\mathrm{Ric}^{(2)}_{k\bar{k}}\lambda_k^2 - 2{}^{\frac{1}{3}}\mathrm{Ric}^{(1)}_{k\bar{k}}\lambda_k^2 + {}^{\frac{1}{3}}\mathrm{Ric}^{(4)}_{k\bar{k}}\lambda_k^2 + {}^{\frac{1}{3}}\mathrm{Ric}^{(3)}_{k\bar{k}}\lambda_k^2 \\ &+ \left({}^{\frac{1}{3}}T^i_{ir}\overline{{}^{\frac{1}{3}}T^r_{kr}} + {}^{\frac{1}{3}}T^r_{kr}\overline{{}^{\frac{1}{3}}T^i_{ir}}\right)\lambda_k^2 + {}^2T^k_{ir}\overline{{}^2T^k_{ir}}\lambda_k^2 + 3{}^{\frac{1}{3}}T^i_{kr}\overline{{}^{\frac{1}{3}}T^i_{kr}}\lambda_k^2 \\ &+ {}^{\frac{1}{3}}\mathrm{SBC}^{(2)}_{\omega_h}\left(\sum_{\alpha}\lambda_\alpha^4\right)^2 - 2{}^{\frac{1}{3}}\widetilde{\mathrm{SBC}}^{(2)}_{\omega_h}\left(\sum_{\alpha}\lambda_\alpha^4\right)^2 \\ &- \left({}^{\frac{1}{3}}\widetilde{T}^\alpha_{\alpha\gamma}\overline{{}^{\frac{1}{3}}\widetilde{T}^\beta_{\beta\gamma}} + {}^{\frac{1}{3}}\widetilde{T}^\beta_{\beta\gamma}\overline{{}^{\frac{1}{3}}\widetilde{T}^\alpha_{\alpha\gamma}}\right)\lambda_\alpha^2\lambda_\beta^2 - 4{}^{\frac{1}{3}}\widetilde{T}^\beta_{\alpha\gamma}\overline{{}^{\frac{1}{3}}\widetilde{T}^\beta_{\alpha\gamma}}\lambda_\alpha^2\lambda_\beta^2 + -2{}^{\frac{1}{3}}\widetilde{T}^\gamma_{\alpha\beta}\overline{{}^{\frac{1}{3}}\widetilde{T}^\gamma_{\alpha\beta}}\lambda_\alpha^2\lambda_\beta^2. \end{aligned}$$

Remarks on the Maximum Principle. Although the pointwise equality (2.6.4) does not require the source manifold to be compact, in the absence of compactness, there is no guarantee that a maximum exists, and the maximum principle cannot be applied directly. One way to circumvent this in this in the non-compact case is to consider manifolds with a certain exhaustion property:

Definition 2.6.45. ([203, §5]). A manifold M^n has the K*–exhaustion property* if M is exhausted by a sequence of open submanifolds $M_1 \subset M_2 \subset \cdots \subset M$, with compact closures and such that:

(i) for each $k \in \mathbb{N}$, there is a smooth function $v_k \geq 0$ on M_k with $\frac{1}{2}\Delta v_k \leq \frac{R}{n} + K\exp(v_k)$, for some fixed constant $K > 0$;
(ii) if p_i is a divergent sequence[13] of points in M_k, then $v_k(p_i) \to \infty$.

Example 2.6.46. The unit ball $\mathbb{B}^n$ has the K–exhaustion property with $K = 2n(n+1)$. As a consequence, we can apply (2.6.5) if the source manifold is $\mathbb{B}^n$. It is worth emphasizing that this is why Chern considers this somewhat restrictive case in [92]. The K–exhaustion property is, of course, very restrictive. The breakthrough that was required for the Schwarz lemma in the non-compact case was made by Omori [225] and Yau [319]:

Theorem 2.6.47. Let (M, g) be a complete Riemannian manifold. Assume $\mathrm{Ric}_g \geq -C$ for some $C \in \mathbb{R}$. Let $f : M \to \mathbb{R}$ be a smooth function that is bounded above. Then for any $\varepsilon > 0$, there is a point $p \in M$ such that $|f(p) - \sup_M f| < \varepsilon$, $\|\mathrm{grad}|_p f\| < \varepsilon$, and $\Delta|_p f < \varepsilon$.

Remark 2.6.48. The subject of maximum principles for non-compact Riemannian manifolds is rich: Omori [225] was the first to show that the naive extensions of the maximum principle to complete Riemannian manifolds did not hold. Omori established the above theorem under the more restrictive assumption of a lower bound on the sectional curvature. The lower bound on the Ricci curvature given in Yau's formulation [319] is, of course, a substantial improvement.

Pigola–Rigoli–Setti further observed that the validity of the Omori–Yau maximum principle on M does not depend on curvature bounds as much as one would expect. For instance, the Omori–Yau maximum principle holds on every Riemannian manifold (M^n, g) which admits a non-negative proper $\mathcal{C}^2$ function φ satisfying:

(i) There exists $A > 0$ such that $|\nabla\varphi| \leq A\sqrt{\varphi}$ away from a compact set.
(ii) There exists $B > 0$ such that $\Delta\varphi \leq B\sqrt{\varphi}\sqrt{G(\sqrt{\varphi})}$ away from a compact set, where $G : [0, +\infty) \to [0, +\infty)$ is a smooth function such that

[13] An infinite sequence p_i in M_k is said to be *divergent* if every compact open set in M_k contains only a number of points in this sequence.

(a) $G(0) > 0$.
(b) $G'(t) \geq 0$.
(c) $\int_0^{+\infty} 1/\sqrt{G(t)}dt = \infty$.
(d) $\limsup_{t\to\infty} \frac{tG(\sqrt{t})}{G(t)} < \infty$.

A weaker form of the Omori–Yau maximum principle is given by dropping the requirement on the gradient in the Omori–Yau maximum:

Definition 2.6.49. Let (M^n, g) be a Riemannian manifold. We say that the *weak Omori–Yau maximum principle* holds on M if for any function $u \in \mathcal{C}^2(M)$ with $\sup_M u < \infty$, there is a sequence of points $\{p_k\}_{k\in\mathbb{N}}$ in M with the following properties:

$$u(p_k) > \sup_M u - \frac{1}{k}, \qquad \Delta u(p_k) < \frac{1}{k}.$$

The validity of the weak Omori–Yau maximum is equivalent to *stochastic completeness*. We remind the reader that:

Definition 2.6.50. A (not necessarily complete) Riemannian manifold M is said to be *stochastically complete* if for some (and therefore any) $(x,t) \in M \times (0,\infty)$, it holds that

$$\int_M p(x,y,t)dy = 1,$$

where $p(x,y,t)$ is the heat kernel of the Laplacian.

Remark 2.6.51. One can give the following alternative descriptions of stochastic completeness:

(i) Any bounded solution $u(x,t)$ in $M\times[0,\infty)$ of the associated heat equation $\partial_t u = \Delta u$ is uniquely determined by the initial value $u|_{t=0}$.
(ii) The lifetime of the corresponding Brownian motion associated with the Laplace–Beltrami operator is infinite.

Example 2.6.52. Recall that a Riemannian manifold (M,g) is said to be *parabolic* if the Laplacian does not admit a positive fundamental solution. Any parabolic manifold (e.g., $\mathbb{R}$ and $\mathbb{R}^2$) is stochastically complete. The converse is not true: $\mathbb{R}^n$ (with the Euclidean measure) is stochastically complete for all $n \in \mathbb{N}$, but $\mathbb{R}^n$ is parabolic only for $n \in \{1,2\}$.

Remark 2.6.53. If (M^n, g) is geodesically complete, then one can state sufficient conditions for parabolicity and stochastic completeness in terms of the volume function $V(r) := \mu(\mathbb{B}(x_0,r))$, where $\mathbb{B}(x_0,r)$ is the geodesic ball of radius r centered at a fixed point $x_0 \in M$. For instance, we have:

(i) If $\int_0^\infty \frac{r}{V(r)}dr = \infty$, then M is parabolic.

(ii) If $\int_0^\infty \frac{r}{\log V(r)} dr = \infty$, then M is stochastically complete.

We note that $V(r) \leq Cr^2$ and $V(r) \leq e^{Cr^2}$ will imply (i) and (ii), respectively (see, e.g., [143]).

Example 2.6.54. ([67]). The Weil–Petersson metric ω_{WP} is stochastically complete.

Historical Developments. The Schwarz lemma, as we have considered it, is built on a rich history. In its classical form, it states:

Theorem 2.6.55. A holomorphic map $f : \mathbb{D}(R_1) \to \mathbb{D}(R_2)$ fixing the origin, satisfies, for all $z \in \mathbb{D}(R_1)$,

$$|f(z)| \leq \frac{R_2}{R_1}|z|. \tag{2.6.17}$$

Proof. The standard proof that we teach is the following: If $f(0) = 0$, the function $g(z) := f(z)/z$ admits a holomorphic extension to all of $\mathbb{D}(R_1)$. Applying the maximum principle to $g(z)$ on each disk $|z| \leq R_1 - \varepsilon$, and letting $\varepsilon \to 0$ proves the statement. □

Remark 2.6.56. Let us remark that this was not the proof originally given by Schwarz in [247] (who proved the Schwarz lemma for one-to-one holomorphic maps). The proof here was first presented by Carathèodory [76, p. 114, Note 13], where it is attributed to Erhard Schmidt.

In (2.6.17), if we keep R_2 fixed, and let R_1 get arbitrarily large, we recover the following well-known corollary:

Corollary 2.6.57. (Liouville's theorem). A bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ assumes at most one value.

Remark 2.6.58. The Schwarz lemma is a local, finite statement about holomorphic maps. The Liouville theorem, in contrast, is a global statement obtained from letting $R_1 \to \infty$. This is the prototypical example of the so-called *Bloch principle*; namely, the principle that any global statement concerning holomorphic maps arises from a stronger, finite version:

Nihil est in infinito quod non prius fuerit in finito.[14]

To further illustrate Bloch's principle, let us recall the following generalization of the Liouville theorem: The *Picard theorem* states

Corollary 2.6.59. A non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ assumes all but possibly one value.

The corresponding finite version prophesized by the Bloch principle is the *Schottky theorem*:

[14] There is nothing in the infinite that has not previously been in the finite. See [33, p. 84]

Theorem 2.6.60. Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map which omits the values 0 and 1. Then $|f(z)|$ affords a bound in terms of $|f(0)|$ and $|z|$.[15]

The first instance of the Bloch principle was the following *Valiron theorem*:

Corollary 2.6.61. A non-constant entire function has holomorphic branches of the inverse in arbitrarily large Euclidean disks.

In [32], Bloch improved Valiron's arguments and proved the underlying finite agent, which we now call the *Bloch theorem*:

Theorem 2.6.62. Every holomorphic function $f : \mathbb{D} \to \mathbb{C}$ has an inverse branch in some Euclidean disk of radius $\mathcal{B}|f'(0)|$, where $\mathcal{B} > 0$ is an absolute constant.

The constant $\mathcal{B}$ is now called *Bloch's constant*, and its precise value remains unknown.

Remark 2.6.63. For the reader's convenience, let us give a more transparent definition of the Bloch constant: Let $\mathbb{D}$ denote the unit disk in $\mathbb{C}$. Let $\mathcal{O}(\mathbb{D})$ denote the space of holomorphic functions on $\mathbb{D}$. Suppose that for some $z_0 \in \mathbb{D}$ and $f \in \mathcal{O}(\mathbb{D})$, $f'(z_0) \neq 0$. Then there is a neighborhood $\mathcal{U}$ of z_0 such that f is univalent in $\mathcal{U}$ and $f(\mathcal{U}$ is a disk of center $f(z_0)$ and radius r_0. Denote this disk by $\mathbb{D}(f(z_0), r_0)$ and refer to it as the *univalent disk of f with radius r_0*. Let $r(z_0, f)$ denote the radius of the largest univalent disk of f with center $f(z_0)$, where $r(z_0, f) = 0$ if $f'(z_0) = 0$. Define $r(f) := \sup\{r(z, f) : z \in \mathbb{D}\}$. The *Bloch constant* is then defined

$$\mathcal{B} \ := \ \inf\{r(f) : f \in \mathcal{O}(\mathbb{D}) : f'(0) = 1\}.$$

Remark 2.6.64. It was observed by Pick [233] that in the Schwarz lemma, we do not require $f(0) = 0$. Indeed, suppose $f : \mathbb{D} \to \mathbb{D}$ is a holomorphic self-map of the unit disk. For $\alpha \in \mathbb{D}$, the Möbius transformation

$$\varphi_\alpha : \mathbb{D} \to \mathbb{D}, \qquad \varphi_\alpha(z) := \frac{z - \alpha}{1 - \overline{\alpha} z}$$

defines an automorphism of $\mathbb{D}$ which sends α the origin. The inverse of φ_α is, moreover, $\varphi_\alpha^{-1} = \varphi_{-\alpha}$. If $f(0) \neq 0$, we can produce a holomorphic self-map of $\mathbb{D}$ which fixes the origin by considering the composite map

$$\varphi_{f(z)} \circ f \circ \varphi_{-z}.$$

Setting $w = \varphi_{-z}(\zeta)$, the familiar Schwarz lemma gives

$$|\varphi_{f(z)} \circ f(w)| \quad \leq \quad |\varphi_z(w)|.$$

[15]The original proof [245] gave no explicit bound for $|f(z)|$.

Explicitly, this reads

$$\left| \frac{f(w) - f(z)}{1 - \overline{f(z)} f(w)} \right| \quad \leq \quad \left| \frac{w - z}{1 - \overline{z} w} \right|.$$

The function

$$d_{\mathrm{H}} : \mathbb{D} \times \mathbb{D} \to \mathbb{R}, \qquad d_{\mathrm{H}}(z, w) := \left| \frac{z - w}{1 - \overline{w} z} \right|$$

defines a distance function on $\mathbb{D}$, the *pseudo-hyperbolic distance*. The function d_{H} defines an honest distance function[16]. It does not, however, come from integrating a Riemannian metric. This can be circumvented by replacing the pseudo-hyperbolic distance with the distance function coming from the Poincaré metric:

$$\rho \;:=\; \frac{|dz|}{(1 - |z|^2)}.$$

Remark 2.6.65. It is an elementary exercise to show that the associated *Poincaré distance function* is given by

$$\mathrm{dist}_{\rho}(z, w) \;=\; \tan^{-1} \left(\frac{z - w}{1 - z\overline{w}} \right).$$

Summarizing this discussion and replacing the pseudo-hyperbolic distance with the Poincaré distance, we have recovered the theorem of Pick [**233**]:

Theorem 2.6.66. (Schwarz–Pick lemma). Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of the unit disk. Then for all $z, w \in \mathbb{D}$,

$$\mathrm{dist}_{\rho}(f(z), f(w)) \quad \leq \quad \mathrm{dist}_{\rho}(z, w).$$

That is, with respect to the Poincaré distance, all holomorphic maps are distance-decreasing.

Observe that since the Poincaré distance function comes from a Hermitian metric, we can look at its curvature. The *Gauss curvature* of a Hermitian metric $g = \lambda |dz|$ is the function

$$K_g \;=\; -\frac{1}{\lambda^2} \Delta \log \lambda \;=\; -\frac{1}{\lambda^2} \frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda.$$

In particular, if $K_g \leq -C$, then setting $u = \log(\lambda)$, we have $\Delta u \geq C e^{2u}$.

Applying the above formula, the curvature of Poincaré metric

$$\rho = \frac{|dz|}{(1 - |z|^2)}$$

is seen to be $K_{\rho} \equiv -4$. That is, the Poincaré metric has constant (negative) Gauss curvature.

[16] i.e., a symmetric non-degenerate function satisfying the triangle inequality.

If we suppose that $g = e^u|dz|$ is some metric of negative curvature, in the sense that $K_g \leq -4$, it is natural to ask how g compares with the Poincaré metric. If we let $R = 1 - \varepsilon$ for some $\varepsilon \in (0,1)$, the Poincaré metric on $\mathbb{D}(R)$ is given by $\rho_R = e^v|dz| := \frac{R}{R^2-|z|^2}|dz|$. Since $K_{\rho_R} \equiv -4$, we have $\Delta v = 4e^{2v}$. Hence, on the open set $\Omega := \{z \in \mathbb{D}(R) : u(z) > v(z)\}$, the function $u-v$ is subharmonic: $\Delta(u-v) \geq e^{2u} - e^{2v}$. By the maximum principle, $u-v$ cannot achieve an interior maximum; hence, the supremum must be approached on the boundary. But Ω cannot have boundary points on $|z| = R$, since $v \to \infty$ as $|z| \to R$. By continuity, at a boundary point z of Ω with $|z| < R$, we have $u - v = 0$, yielding a contradiction if Ω is non-empty. We therefore deduce that $u(z) \leq v(z)$ for all $|z| < R$. Letting $\varepsilon \to 0$ recovers the following theorem of Ahlfors:

Theorem 2.6.67. (Ahlfors–Schwarz lemma). Let (Σ, g) be a Riemann surface with Gauss curvature $K_g \leq -4$. Then for all holomorphic maps $f : \mathbb{D} \to \Sigma$,

$$\mathrm{dist}_g(f(z), f(w)) \;\leq\; \mathrm{dist}_\rho(z, w),$$

for all $z, w \in \mathbb{D}$.

Remark 2.6.68. In the collected works of Ahlfors [3, p. 341], one finds the following reflection concerning his version of the Schwarz lemma: Ahlfors confesses that his generalization of the Schwarz lemma had "*more substance that I was aware of*", but "*without applications, my lemma would have been too lightweight for publication*". The applications that Ahlfors alludes to here were the following: First, a proof of Schottky's theorem with definite numerical bounds: If $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function such that $f(\mathbb{D}) \cap \{0,1\} = \emptyset$, then

$$\log|f(z)| \;<\; \frac{1+\vartheta}{1-\vartheta}\left(7 + \max\{0, \log|f(0)|\}\right),$$

for all $|z| \leq \vartheta < 1$. The second was an improved lower bound on the Bloch constant:

$$\mathcal{B} \;\geq\; \frac{\sqrt{3}}{4}. \tag{2.6.18}$$

Remark 2.6.69. Ahlfors–Grunsky [4] established an upper bound on the Bloch constant:

$$\mathcal{B} \;\leq\; \frac{1}{\sqrt{1+\sqrt{3}}}\frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})} \;\sim\; 0.4719.$$

In 1990, Bonk [39] improved the lower bound to $\mathcal{B} > \frac{\sqrt{3}}{4} + 10^{-14}$. In 1996, Chen–Gauthier [88] obtained $\mathcal{B} > \frac{\sqrt{3}}{4} + 2 \cdot 10^{-4}$. At present, the best lower bound has been achieved by Xiong [307]: $\mathcal{B} \geq \frac{\sqrt{3}}{4} + 3 \cdot 10^{-4}$. This remains far from the conjectured value:

Conjecture 2.6.70. The Bloch constant is conjectured to be equal to the Ahlfors–Grunsky upper bound:

$$\mathcal{B} = \frac{1}{\sqrt{1+\sqrt{3}}} \frac{\Gamma(\frac{1}{3})\Gamma(\frac{11}{12})}{\Gamma(\frac{1}{4})} \sim 0.4719.$$

Further Directions. The results of this section remain very preliminary and continue to be worked out and developed in [**57, 58**]. We suspect that the Schwarz lemmas for the Bismut connection established here (and [**57, 58**]) will play a role in the pluriclosed flow [**264, 265**]. In the spirit of [**49, 50**] we intend to extend these results to prove Gauduchon versions of the Aubin–Yau and Chen–Cheng–Lu inequalities.

Exploring the Bochner technique in the Riemannian category for connections more general than the Levi-Civita connection would also be of tremendous interest.

CHAPTER 3

The Wu–Yau Theorem

We saw in Chapter 2 that the holomorphic sectional curvature of a Kähler metric controls the value distribution of holomorphic curves. More precisely, we saw that a Hermitian manifold (X, ω) with Chern holomorphic sectional curvature bounded above by a negative constant ${}^c\text{HSC}_\omega \leq -\kappa_0 < 0$ is Brody hyperbolic. In particular, if X is compact, then X is Kobayashi hyperbolic. On the other hand, if (X, ω) is a compact Kähler manifold with positive (Chern) holomorphic sectional curvature ${}^c\text{HSC}_\omega > 0$, then X is rationally connected.

The holomorphic sectional curvature is the most natural curvature constraint from the perspective of complex geometry. On the other hand, in algebraic geometry, the (first Chern) Ricci curvature ${}^c\text{Ric}_\omega^{(1)}$ is most natural. Indeed, the ampleness of the canonical bundle of a compact Kähler manifold is equivalent to the Ricci curvature being everywhere negative.

To understand the landscape of complex manifolds, we want to not only understand the complex-analytic and algebro-geometric classifications but how they are related. As we discussed in the introduction, one of the primary motivators is the following (Hermitian extension of the) Kobayashi conjecture:

Conjecture 3.0.1. Let (X, ω) be a compact Kobayashi hyperbolic manifold. Then the canonical bundle K_X is ample. In particular, X is projective and canonically polarized.

We observe that the Kobayashi conjecture predicts that if a compact complex manifold X supports no entire curves $\mathbb{C} \to X$, then K_X is ample. In particular, X is projective with a Kähler–Einstein metric ω_φ such that $\text{Ric}_{\omega_\varphi} = -\omega_\varphi$.

Remark 3.0.2. To the author's knowledge, Kobayashi conjectured that a compact Kähler manifold, which is Kobayashi hyperbolic, must be projective and canonically polarized.

By the classification of surfaces, a proof of the Kobayashi conjecture for Kähler surfaces was given by Campana [70] and Wong [302]:

Theorem 3.0.3. ([70, 302]). Let X be a compact Kähler surface. If X is Kobayashi hyperbolic, then K_X is ample.

PROOF. Classify X by its Kodaira dimension κ: If $\kappa = -\infty$, then X is uniruled and hence, cannot be hyperbolic. If $\kappa = 0$, then X is covered either by a torus or a K3 surface; both are non-hyperbolic. For tori, this is clear; for K3 surfaces, this follows from the fact that if X is a K3 surface, then X has arbitrarily small close deformations, which are Kummer surfaces, which are not hyperbolic. Since small deformations of hyperbolic manifolds are hyperbolic, the result follows. If $\kappa = 1$, then X is an elliptic surface and thus, is not hyperbolic. If $\kappa = 2$ and K_X is not ample, then by Kodaira's theorem, X contains a (-2)–curve. Hence, X cannot be hyperbolic. □

Remark 3.0.4. If, in addition, one assumes that X admits a Kähler metric of negative holomorphic sectional curvature, an alternative argument was given by Heier–Lu–Wong [**155**].

There are a number of cases for which the (Hermitian extension of the) Kobayashi conjecture can be verified. First, we state the following definition:

Definition 3.0.5. Let X be a compact complex manifold. We say that X is *Moishezon* if there is a bimeromorphic map $\varphi : X \longrightarrow Y$ onto a smooth projective variety Y.

Remark 3.0.6. We remind the reader that the *algebraic dimension* a_X of a (connected) compact complex manifold X is defined to be the transcendence degree of the field of meromorphic functions on X. This is a non-negative integer $0 \leq a_X \leq \dim_{\mathbb{C}}(X)$. From [**?**, Chapter 3], Moishezon manifolds can be equivalently described by having maximal algebraic dimension, i.e., $a_X = \dim_{\mathbb{C}}(X)$.

Remark 3.0.7. Since the Kodaira dimension κ_X bounds the algebraic dimension from below:

$$\kappa_X \ \leq \ a_X \ \leq \ \dim_{\mathbb{C}} X,$$

it is clear that if X is a compact complex manifold of general type, then X is Moishezon. In fact, it is easy to see that X is Moishezon if and only if X supports a big line bundle $\mathcal{L} \to X$.

Moishezon manifolds play an important role in the Kobayashi conjecture. For instance, we have:

Theorem 3.0.8. Let X be a compact Kobayashi hyperbolic manifold with K_X big. Then X is projective and canonically polarized.

PROOF. If K_X is big, then X is Moishezon. Moishezon's theorem [**210**] asserts that if X is Moishezon without rational curves, then X is projective. Hence, we deduce that X is projective and of general type. From the base-point-free theorem and the relative cone theorem [**?**, **?**, **187**], a projective manifold of general type with no rational curves has ample canonical bundle. □

We can replace the big assumption on the canonical bundle with the quasi-negativity of the first Chern–Ricci curvature of a Hermitian metric using the following Siu–Demailly criterion:

Theorem 3.0.9. Let (X, ω) be a compact Hermitian manifold with quasi-negative (or quasi-positive) first Chern–Ricci curvature. Then X is Moishezon.

In fact, this can be vastly improved by making use of holomorphic Morse inequalities. To describe the improvement, we introduce the following definition:

Definition 3.0.10. Let $(\mathcal{L}, h) \to X$ be a Hermitian line bundle over a compact complex manifold X. We define the *k–index locus* $\mathcal{Z}_k(\mathcal{L}, h)$ of $(\mathcal{L}, h)$ to be the set of points in X such that the curvature form $\Theta^{(\mathcal{L},h)}$ has k negative eigenvalues and $n-k$ nonnegative eigenvalues.

With the above definition, we have the following [**107**, p. 167]:

Theorem 3.0.11. Let $(\mathcal{L}, h) \to X$ be a Hermitian line bundle over a complex manifold X. Suppose

$$\int_{\mathcal{Z}_1(\mathcal{L},h)} \left(\Theta^{(\mathcal{L},h)} \right)^n \quad > \quad 0,$$

then $\kappa(\mathcal{L}) = n$, the line bundle $\mathcal{L}$ is big, and X is Moishezon.

In particular, we have

Corollary 3.0.12. Let X be a compact Kobayashi hyperbolic manifold.

(i) If there is a Hermitian line bundle $(\mathcal{L}, h) \to X$ such that $\int_{\mathcal{Z}_1(\mathcal{L},h)} \left(\Theta^{(\mathcal{L},h)} \right)^n > 0$, then X is projective with nef canonical bundle.

(ii) If there is a Hermitian metric ω on X such that $\int_{\mathcal{Z}_1(K_X,\omega^n)} \left(-{}^c\mathrm{Ric}_{\omega}^{(1)} \right)^n > 0$, then X is projective and canonically polarized.

One of the critical cases to rule out is the Kobayashi hyperbolicity of complex manifolds with vanishing Kodaira dimension:

Lemma 3.0.13. Let X be a projective Kobayashi hyperbolic manifold. Assume the abundance conjecture holds. If Kobayashi hyperbolic manifolds must have positive Kodaira dimension $\kappa > 0$, then the Kodaira dimension must be maximal $\kappa = \dim_{\mathbb{C}} X$.

Proof. If X is projective Kobayashi hyperbolic, then by Mori [**213**], the canonical bundle K_X is nef. In particular, the abundance conjecture (which is known in dimensions ≤ 3) implies that K_X is semi-ample. That is, the linear system $|K_X^{\otimes \ell}|$ yields a surjective holomorphic map $\Phi_{|K_X^{\otimes \ell}|} : X \longrightarrow Y \subset \mathbb{P}^{N_\ell}$ onto a normal irreducible and reduced projective variety Y of dimension $\dim_{\mathbb{C}} Y = \kappa(X)$, where the fibers of this map have vanishing Kodaira dimension. Since complex submanifolds of Kobayashi hyperbolic manifolds are Kobayashi hyperbolic, the fibers must be zero-dimensional and $\kappa(X) = \dim_{\mathbb{C}} Y = \dim_{\mathbb{C}} X$, i.e., X must be of general type. $\square$

3.1. Kobayashi Hyperbolicity and the Holomorphic Sectional Curvature

Kobayashi hyperbolicity is *strictly weaker* than the existence of a Hermitian metric with negative (Chern) holomorphic sectional curvature. For a long time, it was conjectured that a compact Kobayashi hyperbolic manifold admits a Hermitian metric with negative (Chern) holomorphic sectional curvature. Evidence for this conjecture was given by Grauert–Reckziegel [**136**] who constructed a Hermitian metric in a neighborhood of a fiber of an analytic family of compact Riemann surfaces of genus $g \geq 2$ over a Riemann surface, such that the Hermitian metric has negative holomorphic sectional curvature in this neighborhood. This local construction was extended to higher dimensions by Cowen [**99**]. Deschamps–Martin [**109**], and Schneider [**243**] showed that Kodaira surfaces of general type have negative tangent bundle in the sense of Grauert.
The first general construction of Hermitian metrics with negative holomorphic sectional curvature was given by Cheung [**94, 95**]:

Theorem 3.1.1. Let $f : X \longrightarrow Y$ be a holomorphic submersion from a compact complex manifold X into a Hermitian manifold (Y, ω_h) with ${}^c\text{HSC}_{\omega_h} < 0$. If each fiber $X_y := f^{-1}(y)$ supports a Hermitian metric with negative (Chern) holomorphic sectional curvature, and these metrics vary smoothly as a function of Y, then X supports a Hermitian metric with everywhere negative (Chern) holomorphic sectional curvature.

In other words, if f is a holomorphic submersion with

$$(\text{HSC} < 0 \text{ base}) + (\text{HSC} < 0 \text{ fiber}) \implies (\text{HSC} < 0 \text{ total space}).$$

Example 3.1.2. Let $f : X \to Y$ be a Kodaira surface. The base Y and fibers X_y are compact Riemann surfaces of genus $g \geq 2$. Hence, admit Hermitian metrics of negative (Chern) holomorphic sectional curvature. Therefore, by 3.1.1, X admits a Hermitian metric with negative (Chern) holomorphic sectional curvature.

Remark 3.1.3. From the subbundle decreasing property of the holomorphic sectional curvature, we also have

$$(\text{HSC} < 0 \text{ total space}) \implies (\text{HSC} < 0 \text{ fiber}).$$

It is natural to ask, therefore, the following:

Question 3.1.4. Let $f : (X, \omega_g) \to Y$ be a holomorphic submersion from a compact Hermitian manifold (X, ω_g) with ${}^c\text{HSC}_{\omega_g} < 0$. Does there exist a Hermitian metric ω_h on Y such that ${}^c\text{HSC}_{\omega_h} < 0$?

Let us remark that the positive analog of 3.1.1 was recently established by Chaturvedi–Heier [**79**]:

Theorem 3.1.5. Let $f : X \longrightarrow Y$ be a holomorphic submersion from a compact complex manifold to a Hermitian manifold (Y, ω_h) with ${}^c\text{HSC}_{\omega_h} > 0$. Assume the fibers admit Hermitian metrics with positive (Chern) holomorphic sectional curvature. Then X admits a Hermitian metric with positive (Chern) holomorphic sectional curvature.

Example 3.1.6. The semi-positive (or non-negative) analog of 3.1.5 does not hold, as the following example illustrates: Let $p : \mathbb{D} \times \mathbb{D} \to \mathbb{D}$ be the holomorphic map given by projecting to the second factor, i.e., $p(z, w) = w$. The Hermitian metric $\omega_h := \sqrt{-1}\frac{dw \wedge d\overline{w}}{(1+|w|^2)}$ on $\mathbb{D}$ has positive (Chern) holomorphic sectional curvature. On the other hand,

$$\Phi := \sqrt{-1}\frac{e^{2|w|^2} dz \wedge d\overline{z}}{1 + |z|^4 e^{4|w|^2}}$$

restricts to a Hermitian metric of semi-positive (Chern) holomorphic sectional curvature on each fiber. For any $\lambda > 0$,

$$\omega_\lambda \ := \ \Phi + \lambda p^* \omega_h$$

is a Hermitian metric, but does not have semi-positive (Chern) holomorphic sectional curvature for any $\lambda > 0$.

Despite the growing evidence for the conjecture that every compact Kobayashi hyperbolic manifold admits a Hermitian metric with negative (Chern) holomorphic sectional curvature, it was later shown by Demailly [105] that the conjecture was false. The starting point is the following algebraic hyperbolicity criterion:

Proposition 3.1.7. (Demailly). Let (X, ω) be a compact Hermitian manifold and let $f : \mathcal{C} \to X$ be a non-constant holomorphic map from a Riemann surface $\mathcal{C}$ of genus g. Suppose that $\text{HSC}_\omega \leq A$ for some $A \in \mathbb{R}$. Then

$$2 - 2g + \sum_{p \in \mathcal{C}} (m_p - 1) \quad \leq \quad \frac{A}{2\pi} \deg_\omega(\mathcal{C}),$$

where $\deg_\omega(\mathcal{C}) \ = \ \int_{\mathcal{C}} f^* \omega$.

Proof. We follow the proof given in [110]: The differential df of $f : \mathcal{C} \to X$ gives an injection $df : T^{1,0}\mathcal{C} \to T^{1,0}X$ at the level of sheaves. Because of the vanishing of df, this is not an injection of bundles. Let $\mathcal{D} = \sum_{p \in \mathcal{C}} (m_p - 1) \cdot p$, where m_p denotes the multiplicity of df at p. Twisting $T^{1,0}\mathcal{C}$ with the line bundle associated to $\mathcal{D}$ gives an injection of bundles $df : T^{1,0}\mathcal{C} \otimes \mathcal{O}_{\mathcal{C}}(\mathcal{D}) \to T^{1,0}X$.

If t is a local holomorphic coordinate centered at a point $p \in \mathcal{C}$, then ∂_t gives a coordinate frame for $T^{1,0}\mathcal{C}$. A local holomorphic frame for $T^{1,0}\mathcal{C} \otimes \mathcal{O}_{\mathcal{C}}(\mathcal{D})$, centered at $p \in \mathcal{C}$, is given by $\eta(t) = t^{1-m_p}\partial_t$. Set $\xi(t) := df(\eta(t))$, which gives a local holomorphic coordinate frame for $f^*\mathcal{T}_X$. The Griffiths curvature of $(T^{1,0}\mathcal{C} \otimes \mathcal{O}_{\mathcal{C}}(\mathcal{D}), h)$ is

$$\langle \Theta(T^{1,0}\mathcal{C} \otimes \mathcal{O}_{\mathcal{C}}(\mathcal{D}), h)(\partial_t, \partial_{\overline{t}})\xi, \xi \rangle_h \quad = \quad \Theta(T^{1,0}\mathcal{C} \otimes \mathcal{O}_{\mathcal{C}}(\mathcal{D}), h)(\partial_t, \partial_{\overline{t}}) \|\xi\|_\omega^2.$$

By the subbundle decreasing property of the holomorphic sectional curvature, we have

$$\begin{aligned}\langle \Theta(T^{1,0}\mathcal{C}\otimes\mathcal{O}_{\mathcal{C}}(\mathcal{D}), h)(\partial_t, \partial_{\bar{t}})\xi, \xi\rangle_h &\leq \langle \Theta(f^*\mathcal{T}_X, f^*\omega)(\partial_t, \partial_{\bar{t}})\xi, \xi\rangle_{f^*\omega} \\ &= \langle f^*\Theta(T^{1,0}X, \omega)(\partial_t, \partial_{\bar{t}})\xi, \xi\rangle_{f^*\omega} \\ &= |t^{m_p-1}|^2\langle \Theta(T^{1,0}X, \omega)(\xi, \bar{\xi})\xi, \xi\rangle_\omega \\ &\leq A|t^{m_p-1}|^2\|\xi\|_\omega^4 \\ &= -\sqrt{-1}A(f^*\omega)(\partial_t, \partial_{\bar{t}}),\end{aligned}$$

where $f^*\omega$ is understood to be the pullback at the level of forms. This implies that

$$\sqrt{-1}\Theta(T^{1,0}\mathcal{C}\otimes\mathcal{O}_{\mathcal{C}}(\mathcal{D}), h) \leq Af^*\omega.$$

Hence,

$$\int_{\mathcal{C}} \frac{\sqrt{-1}}{2\pi}\Theta(T^{1,0}\mathcal{C}\otimes\mathcal{O}_{\mathcal{C}}(\mathcal{D}), h) \leq \frac{A}{2\pi}\int_{\mathcal{C}} f^*\omega = \frac{A}{2\pi}\deg_\omega(\mathcal{C}).$$

Further,

$$\int_{\mathcal{C}} \frac{\sqrt{-1}}{2\pi}\Theta(T^{1,0}\mathcal{C}\otimes\mathcal{O}_{\mathcal{C}}(\mathcal{D}), h) = \deg(T^{1,0}\mathcal{C}\otimes\mathcal{O}_{\mathcal{C}}(\mathcal{D})) = 2-2g+\sum_{p\in\mathcal{C}}(m_p-1),$$

proving the theorem. □

Remark 3.1.8. It is perhaps worth noting that Demailly's algebraic criterion does not distinguish between non-positive and quasi-negative holomorphic sectional curvature.

Remark 3.1.9. Demailly's example of a Kobayashi hyperbolic manifold that does not support a Hermitian metric of negative holomorphic sectional curvature effectively reduces to the following: Construct a projective surface X which is fibered by hyperbolic curves over a hyperbolic base. This ensures that X is Kobayashi hyperbolic. Then construct this fibration so that there is a sufficiently singular fiber that violates Demailly's algebraic obstruction. The reader may wish to consult [**105, 110**] for further details.

The fact that Kobayashi hyperbolicity is strictly weaker than the existence of a Hermitian metric with negative (Chern) holomorphic sectional curvature makes the Kobayashi conjecture particularly difficult – there is even less structure to work with. On the other hand, even if we have a Hermitian metric of negative (Chern) holomorphic sectional curvature, its relationship to the canonical bundle is not so clear.

If X is a compact complex manifold, then K_X is ample if and only if there exists a Hermitian metric ω with negative (first Chern) Ricci curvature ${}^c\text{Ric}_\omega^{(1)} < 0$. In the presence of a Hermitian metric of negative holomorphic sectional curvature, the Kobayashi conjecture can be expressed as follows:

Conjecture 3.1.10. Let (X, ω) be a compact Hermitian manifold with ${}^c\text{HSC}_\omega < 0$. Then there exists a Kähler metric η such that $\text{Ric}_\eta < 0$.

The first evidence for this version of the conjecture comes from the fact that the holomorphic sectional curvature and Ricci curvature reside on similar strata of the curvature hierarchy. For instance, if ω is a Kähler metric, then both the Ricci curvature and holomorphic sectional curvature are dominated by the holomorphic bisectional curvature and dominate the scalar curvature. On the other hand, they do not dominate each other:

Example 3.1.11. Let

$$X_d \ := \ \{z_0^d + \cdots + z_n^d = 0\} \ \subseteq \ \mathbb{P}^n$$

be the Fermat hypersurface of degree $d \geq n+2$ in $\mathbb{P}^n$. By adjunction, K_{X_d} is ample, and therefore, by the Aubin–Yau theorem, X_d admits a Kähler–Einstein metric of negative Ricci curvature. There is no Hermitian metric on X_d with negative (Chern) holomorphic sectional curvature, however, since X_d contains complex lines.

In light of these examples, the following theorem illustrates the curious relationship between the holomorphic sectional curvature and the Ricci curvature:

Theorem 3.1.12. (Hermitian Wu–Yau theorem). Let (X, ω_g) be a compact partially Kähler-like Hermitian manifold with a Hermitian metric ω_h such that ${}^c\text{SBC}_{\omega_h}^{(2)} < 0$. Then K_X is ample. In particular, X is projective and canonically polarized.

We do not have full access to cohomology to prove the Hermitian Wu–Yau theorem. The technique relies on establishing the existence of a Kähler–Einstein of negative scalar curvature. The construction of this Kähler–Einstein metric will be given by a solution of a complex Monge–Ampère equation, following the framework set up in [**304**].

We give a proof of the Wu–Yau theorem assuming the solvability of the complex Monge–Ampère equation. Namely, we will assume the following theorem [**93, 285**]:

Theorem 3.1.13. (Cherrier, Tosatti–Weinkove). Let (X, ω) be a compact Hermitian manifold. Let $F \in \mathcal{C}^\infty(X, \mathbb{R})$ be a smooth function. Then there exists a unique $\varphi \in \mathcal{C}^\infty(X, \mathbb{R})$ and a unique $b \in \mathbb{R}$ such that $\omega_\varphi := \omega + \sqrt{-1}\partial\bar{\partial}\varphi > 0$ and

$$\omega_\varphi^n \ = \ e^{F+b}\omega^n, \qquad \sup_X \varphi = 0.$$

We will return to a discussion of the complex Monge–Ampère equation after completing the proof of the Wu–Yau theorem.

The proof of the Hermitian extension of the Wu–Yau theorem hinges upon the Schwarz lemma in the Hermitian category. As a consequence, we require a lower bound on the second

Chern–Ricci curvature of the source metric appearing in the Schwarz lemma and a negative upper bound on the Chern second Schwarz bisectional curvature of the target metric.

Let us outline the structure of the argument: We want to first show that if (X, ω_g) is a compact Hermitian manifold with a Hermitian metric ω_h such that ${}^c\text{SBC}^{(2)}_{\omega_h} \leq -\kappa_0 < 0$, then the canonical bundle K_X is nef. Following [**304, 305, 286, 315, 49, 50**], we proceed by contradiction and suppose that K_X is not nef. Then there exists an $\varepsilon_0 > 0$ such that the real $(1,1)$–form

$$\varepsilon_0 \omega_g - {}^c\text{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar{\partial}\varphi$$

is not positive-definite for any smooth function $\varphi \in \mathcal{C}^\infty(X, \mathbb{R})$. However, for any $\varepsilon > 0$, the real $(1,1)$–form

$$(\varepsilon + \varepsilon_0)\omega_g - {}^c\text{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar{\partial}\varphi$$

is a positive-definite Hermitian metric. By the Aubin–Yau theorem [**15, 318**], there is a smooth function $\psi_\varepsilon \in \mathcal{C}^\infty(X, \mathbb{R})$ such that

$$\omega_\varepsilon \ := \ (\varepsilon + \varepsilon_0)\omega_g - {}^c\text{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar{\partial}u_\varepsilon \ > \ 0, \qquad u_\varepsilon \ := \ \varphi_\varepsilon + \psi_\varepsilon, \tag{3.1.1}$$

is a solution of the complex Monge–Ampère equation

$$\omega_\varepsilon^n \ = \ e^{u_\varepsilon}\omega_g^n. \tag{3.1.2}$$

Differentiating (3.1.2) yields

$${}^c\text{Ric}^{(1)}_{\omega_\varepsilon} \ = \ -\sqrt{-1}\partial\bar{\partial}u_\varepsilon + {}^c\text{Ric}^{(1)}_{\omega_g} \ = \ -\omega_\varepsilon + (\varepsilon + \varepsilon_0)\omega_g. \tag{3.1.3}$$

Hence, ω_ε is a twisted first Chern–Ricci–Einstein metric.

Let ω_h be the Hermitian metric on X with ${}^c\text{SBC}^{(2)}_{\omega_h} \leq -\kappa_0 < 0$. To apply 2.6.19 to the identity map $\text{id} : (X, \omega_\varepsilon) \longrightarrow (X, \omega_h)$, we need control of ${}^c\text{Ric}^{(2)}_{\omega_\varepsilon}$. Recall from the relations on the Chern–Ricci curvatures that

$${}^c\text{Ric}^{(2)}_{\omega_\varepsilon} \ = \ {}^c\text{Ric}^{(1)}_{\omega_g} - \sqrt{-1}\Lambda_\varepsilon(\partial\bar{\partial}\omega_\varepsilon) - (\partial\partial^*_\varepsilon\omega_\varepsilon + \bar{\partial}\bar{\partial}^*_\varepsilon\omega_\varepsilon) + {}^cT^{\diamondsuit}_\varepsilon,$$

where Λ_ε is the formal adjoint of the Lefschetz operator $L_{\omega_\varepsilon}(\cdot) := \omega_\varepsilon \wedge \cdot$, and ∂^*_ε, $\bar{\partial}^*_\varepsilon$ denote the formal adjoints (with respect to ω_ε) of ∂, $\bar{\partial}$, respectively.

We can generate several variants on the Hermitian extension of the Wu–Yau theorem by considering various constraints on the metric ω_ε. The simplest case is when ω_ε is (Chern) partially Kähler-like (e.g., when ω_ε is Chern Kähler-like, or Kähler). We proceed with the argument in this case, and will return to consider alternative assumptions on ω_ε:

Suppose that for all $\varepsilon > 0$, the metric ω_ε is (Chern) partially Kähler-like. Then (3.1.3) implies that

$$ {}^c\mathrm{Ric}^{(2)}_{\omega_\varepsilon} \;=\; -\omega_\varepsilon + (\varepsilon+\varepsilon_0)\omega_g. $$

Since X is compact, the smooth metrics ω_g and ω_h are uniformly equivalent. In particular, we have a uniform[1] constant $C > 0$ such that $\omega_g \geq C^{-1}\omega_h$, and therefore,

$$ {}^c\mathrm{Ric}^{(2)}_{\omega_g} \;\geq\; -\omega_\varepsilon + C^{-1}(\varepsilon+\varepsilon_0)\omega_h. $$

Hence, 2.6.19 yields

$$ \mathrm{tr}_{\omega_\varepsilon}(\omega_h) \;\leq\; \frac{1}{C^{-1}(\varepsilon+\varepsilon_0)+\kappa_0} \;\leq\; C, $$

where the last (generic) constant $C > 0$ is uniform, independent of $\varepsilon \searrow 0$. Let $x_0 \in X$ be the point at which u_ε attains its maximum. At this point, we have $\sqrt{-1}\partial\bar{\partial}u_\varepsilon(x_0) \leq 0$, and hence $\left((\varepsilon+\varepsilon_0)\omega_g - {}^c\mathrm{Ric}^{(1)}_{\omega_g}\right)(x_0) > 0$. Further,

$$ e^{\sup_X u_\varepsilon} \;=\; e^{u_\varepsilon(x)} \;\leq\; \frac{\left((\varepsilon+\varepsilon_0)\omega_g - {}^c\mathrm{Ric}^{(1)}_{\omega_g}\right)^n}{\omega_g^n}(x_0) \;\leq\; C, \tag{3.1.4} $$

where $C > 0$ is a uniform constant. From (3.1.4) we have a uniform constant $C > 0$ such that

$$ \sup_X \frac{\omega_\varepsilon^n}{\omega_g^n} \;\leq\; C. $$

Therefore, from the uniform equivalence of ω_g and ω_h, we have

$$ \mathrm{tr}_{\omega_g}(\omega_\varepsilon) \;\leq\; \mathrm{tr}_{\omega_\varepsilon}(\omega_g)^{n-1}\frac{\omega_\varepsilon^n}{\omega_g^n} \;\leq\; C\mathrm{tr}_{\omega_\varepsilon}(\omega_g)^{n-1} \;\leq\; C\mathrm{tr}_{\omega_\varepsilon}(\omega_h)^{n-1} \;\leq\; C, $$

where $C > 0$ denotes a generic uniform constant. It follows that there is a uniform constant $C > 0$ such that

$$ C^{-1}\omega_g \;\leq\; \omega_\varepsilon \;\leq\; C\omega_g. $$

This, together with the complex Monge–Ampère equation $\omega_\varepsilon^n = e^{u_\varepsilon}\omega_g^n$ implies that $\inf_X u_\varepsilon \geq -C$. The higher-order estimates

$$ \|\omega_\varepsilon\|_{\mathcal{C}^k(X,\omega_g)} \;\leq\; C_k, \tag{3.1.5} $$

where C_k is independent of ε for all $k \in \mathbb{N}_0$, follow from [**93**], generalizing Calabi's third-order estimate [**318**] (c.f., [**232**, **285**, **286**]). Given (3.1.5), we can now obtain the desired

[1]Throughout this section, uniform will be understood to mean independent of $\varepsilon > 0$ for $\varepsilon > 0$ small.

contradiction. Indeed, from (3.1.5), via the Arzelà–Ascoli theorem and a diagonal argument, there is a subsequence $\varepsilon_i \searrow 0$ such that ω_{ε_i} converges smoothly to a Hermitian metric

$$\omega_0 \;=\; \varepsilon_0\omega_g - {}^c\mathrm{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar\partial u > 0,$$

violating the fact that $\varepsilon_0\omega - {}^c\mathrm{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar\partial u$ is not positive-definite.

We formulate the statement we just proved more precisely:

Theorem 3.1.14. Let (X, ω_g) be a compact Hermitian manifold with a Hermitian metric ω_h such that ${}^c\mathrm{SBC}^{(2)}_{\omega_h} \leq -\kappa_0 < 0$. If, for any $\varepsilon > 0$, the Hermitian metric

$$\omega_\varepsilon \;:=\; (\varepsilon + \varepsilon_0)\omega_g - {}^c\mathrm{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar\partial u_\varepsilon$$

defined in (3.1.1) is Chern partially Kähler-like, then K_X is nef.

We may now give a proof of the following Hermitian extension of the Wu–Yau theorem following [304]:

Theorem 3.1.15. Let (X, ω_g) be a compact Hermitian manifold with a Hermitian metric ω_h such that ${}^c\mathrm{SBC}^{(2)}_{\omega_h} \leq -\kappa_0 < 0$. If, for any $\varepsilon > 0$, the Hermitian metric

$$\omega_\varepsilon \;:=\; (\varepsilon + \varepsilon_0)\omega_g - {}^c\mathrm{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar\partial u_\varepsilon$$

defined in (3.1.1) is Chern partially Kähler-like, then K_X is ample.

Proof. From 3.1.14, the canonical bundle K_X is nef. Hence, for any $\varepsilon > 0$, there is a smooth function $f_\varepsilon \in \mathcal{C}^\infty(X, \mathbb{R})$ such that

$$\omega_{f_\varepsilon} \;:=\; \varepsilon\omega_g - {}^c\mathrm{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar\partial f_\varepsilon \;>\; 0$$

on X. By [15, 318, 285], there is a smooth function $v_\varepsilon \in \mathcal{C}^\infty(X)$ such that, for $u_\varepsilon := f_\varepsilon + v_\varepsilon$, we have

$$\omega^n_{u_\varepsilon} = e^{u_\varepsilon}\omega^n_g, \qquad \omega_{u_\varepsilon} := \varepsilon\omega_g - {}^c\mathrm{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar\partial u_\varepsilon > 0.$$

Repeating the argument from 3.1.14, we extract a subsequence $\varepsilon_i \searrow 0$ such that $\omega_{u_{\varepsilon_i}}$ converges smoothly to

$$\omega_u = -{}^c\mathrm{Ric}^{(1)}_{\omega_g} + \sqrt{-1}\partial\bar\partial u > 0,$$

where $\omega_u^n = e^u\omega_g^n$. Differentiating the complex Monge–Ampère equation, we see that

$$ {}^c\mathrm{Ric}^{(1)}_{\omega_u} \;=\; {}^c\mathrm{Ric}^{(1)}_{\omega_g} - \sqrt{-1}\partial\bar\partial u \;=\; -\omega_u. \tag{3.1.6}$$

In particular, ω_u is first Chern–Ricci–Einstein. Taking the exterior derivative of (3.1.6) shows that ω_u is Kähler, and hence, Kähler–Einstein with negative scalar curvature. Therefore, K_X is ample, and X is projective and canonically polarized. $\square$

Remark 3.1.16. The structure of the above argument is both elementary and marvelous. The argument requires no cohomology theory, and hence, no Kähler assumption. What is, moreover, quite striking is that the Kähler–Einstein is constructed directly on a non-Kähler Hermitian manifold. This should be compared with the Kähler–Ricci flow proof of the Wu–Yau theorem (in the Kähler setting) given by Nomura [223] which argues in the reverse direction: One obtains the Kähler–Einstein metric by proving that the canonical bundle K_X is ample. This latter approach requires more algebro-geometric theory, and hence, is less amenable on a general Hermitian manifold.

Remark 3.1.17. Let us further remark that although the argument for the proof of both 3.1.14 and 3.1.15 is contained in [304, 305, 286], the statements are only given for Kähler metrics. The reason for this is likely because of the awkward nature of additional assumptions on a Hermitian metric, outside of the Kähler condition $d\omega = 0$. Indeed, both 3.1.14 and 3.1.15 are stated in terms of the perturbed metric ω_ε. The metric ω_ε is perturbed (from ω_g) in two ways:

(i) It is scaled in the ω_g direction by $\varepsilon + \varepsilon_0$;
(ii) and it is translated within its $\partial\bar{\partial}$–cohomology class by $-{}^c\text{Ric}_{\omega_g}^{(1)} + \sqrt{-1}\partial\bar{\partial}u_\varepsilon$.

Outside of the Kähler condition, not many assumptions on ω_g are preserved by the map

$$\omega_g \ \longmapsto \ \omega_\varepsilon \ := \ (\varepsilon+\varepsilon_0)\omega_g - {}^c\text{Ric}_{\omega_g}^{(1)} + \sqrt{-1}\partial\bar{\partial}u_\varepsilon. \tag{3.1.7}$$

The balanced or partially Kähler-like conditions are certainly not preserved with the $\partial\bar{\partial}$–cohomology class, for instance. The pluriclosed condition is preserved by the map (3.1.7). Unfortunately, the pluriclosed is not as fruitful as the balanced condition in comparing the first and second Chern–Ricci curvatures. Indeed, if ω_g is pluriclosed, then

$${}^c\text{Ric}_{\omega_\varepsilon}^{(2)} \ = \ {}^c\text{Ric}_{\omega_g}^{(1)} - (\partial\partial_\varepsilon^*\omega_\varepsilon + \bar{\partial}\bar{\partial}_\varepsilon^*\omega_\varepsilon) + {}^cT_\varepsilon^{\diamondsuit},$$

and one still has to deal with the $\partial\partial_\varepsilon^*\omega_\varepsilon + \bar{\partial}\bar{\partial}_\varepsilon^*\omega_\varepsilon$ term. Note that ${}^cT_\varepsilon^{\diamondsuit}$ is positive, so can be done away with in the estimate. The $\partial\partial_\varepsilon^*\omega_\varepsilon + \bar{\partial}\bar{\partial}_\varepsilon^*\omega_\varepsilon$ term does not have a sign in general and vanishes only if ω_ε is balanced. Of course, by 2.3.48, a Hermitian metric that is both balanced and pluriclosed is Kähler. For further results in this direction, we invite the reader to consult [57, 58].

Let us give an example application of the Bismut Schwarz lemma 2.6.40 to the Hermitian extension of the Wu–Yau theorem:

Theorem 3.1.18. Let (X, ω_g) be a compact Hermitian manifold with a Hermitian metric ω_h such that

$${}^b\text{SBC}_{\omega_h}^{(2)} + {}^b\widetilde{\text{SBC}}_{\omega_h}^{(2)} \ \leq \ -\kappa_0 \ < \ 0.$$

Suppose that for any $\varepsilon > 0$, the Hermitian metric ω_ε defined in (3.1.1) is balanced with

$$\sqrt{-1}\Lambda_\varepsilon(\partial\bar{\partial}\omega_g) \quad \leq \quad \frac{1}{3}(C_1 - 3)\omega_\varepsilon + \frac{1}{3}(C_2 + 3(\varepsilon + \varepsilon_0))\omega_g,$$

for some constants $C_1, C_2 \in \mathbb{R}$, where Λ_ε is the formal adjoint of the Lefschetz operator associated to ω_ε. If there is a uniform bound B_0 on the norm of the Bismut torsion of ω_ε and $\kappa_0 - C_2 - 2B_0 > 0$, then K_X is ample.

The proof is identical to the proof of 3.1.14 and 3.1.15, making the appropriate modifications in the Schwarz lemma. We could include a substantially larger number of variants on the Hermitian extension of the Wu–Yau theorem, but due to time constraints, we leave these to [**58**].

Further directions. From the monotonicity theorem for the t–Gauduchon holomorphic sectional curvature, we see that negative Chern holomorphic sectional curvature is the strongest condition on the Gauduchon holomorphic sectional curvatures. On the other hand, positive Chern holomorphic sectional curvature is the weakest condition on the Gauduchon holomorphic sectional curvatures. Recall that if (X, ω) is a Hermitian manifold (not necessarily complete) with

$${}^c\mathrm{HSC}_\omega \leq -\kappa_0 < 0,$$

then the Schwarz lemma implies that X is Brody hyperbolic (i.e., every holomorphic map $\mathbb{C} \to X$ is constant). If X is compact, then Brody's theorem implies that X is Kobayashi hyperbolic. For a long time, it was conjectured that every compact Kobayashi hyperbolic manifold supports a Hermitian metric with negative Chern holomorphic sectional curvature. A counterexample to this conjecture was given by Demailly [**105**] (see also [**110**]). The counterexample is given by a projective Kobayashi hyperbolic surface, fibered by genus $g > 1$ curves over a genus $g > 1$ curve, with a fiber sufficiently singular to violate Demailly's algebraic hyperbolicity criterion.

In light of the above monotonicity result for the Gauduchon–Holomorphic sectional curvature, it is natural to ask the following question:

Question 3.1.19. Let X be a compact Kobayashi hyperbolic manifold. Does X admit a Hermitian metric with ${}^t\mathrm{HSC}_\omega < 0$ for some (range of) $t \in \mathbb{R}$?

In [**57, 58**], significantly more general results will appear concerning the Wu–Yau theorem in the Hermitian category. These results, at the time of writing this, however, are still in preparation.

3.2. A Brief Discussion of Complex Monge–Ampère Equations

We start by recalling the general framework which has been the foundation of complex differential geometry since [318].

Yau's solution of the Calabi conjecture. We saw in Chapter 2 that if ω is a Kähler metric, then the Ricci form $\mathrm{Ric}_\omega = -\sqrt{-1}\partial\bar{\partial}\log\omega^n$ represents the first Chern class of the anti-canonical bundle $c_1(K_X^{-1})$. The Calabi conjecture asserts that the converse is true, namely:

Conjecture 3.2.1. (Calabi [68]). Let (X,ω) be a compact Kähler manifold. Let α be a representative of $c_1(K_X^{-1})$. Then there exists a Kähler metric $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ such that $\alpha = \mathrm{Ric}_{\omega_\varphi}$.

Assuming the Calabi conjecture, we see that by taking $\alpha = 0$, it follows that for any Kähler metric ω, there is a cohomologous metric ω_φ such that $\mathrm{Ric}_{\omega_\varphi} = 0$.

It was shown by Calabi that the Calabi conjecture is equivalent to the solvability of a complex Monge–Ampère equation. Indeed, for any Kähler metric ω, and any representative α of $c_1(K_X^{-1})$, we have $\mathrm{Ric}_\omega = \alpha + \sqrt{-1}\partial\bar{\partial}F$ for some smooth function $F \in \mathcal{C}^\infty(X,\mathbb{R})$. Since $\alpha = \mathrm{Ric}_{\omega_\varphi}$, we have

$$\begin{aligned}\mathrm{Ric}_\omega = \mathrm{Ric}_{\omega_\varphi} + \sqrt{-1}\partial\bar{\partial}F \quad &\Longleftrightarrow \quad -\sqrt{-1}\partial\bar{\partial}\log\omega^n = -\sqrt{-1}\partial\bar{\partial}\log\omega_\varphi^n + \sqrt{-1}\partial\bar{\partial}F\\ &\Longleftrightarrow \quad \sqrt{-1}\partial\bar{\partial}\log\frac{\omega_\varphi^n}{\omega^n} = \sqrt{-1}\partial\bar{\partial}F.\end{aligned}$$

By the maximum principle, there is a constant $b \in \mathbb{R}$ such that

$$\log\frac{\omega_\varphi^n}{\omega^n} = F + b,$$

and thus, the solvability of the Calabi conjecture is equivalent to the solvability of the complex Monge–Ampère equation

$$\omega_\varphi^n = e^{F+b}\omega^n. \tag{3.2.1}$$

Uniqueness. The uniqueness of (3.2.1) was established by Calabi [68] and follows from a straightforward maximum principle argument.

Existence. The method of existence to (3.2.1) was also formulated by Calabi. The approach is the well-known continuity method. We consider a family of complex Monge–Ampère equations, starting from one whose solution is known and ending at (3.2.1):

$$(\omega + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n = e^{tF+c_t}\omega^n,$$

where $\omega_t := \omega + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0$ and c_t is the constant such that $\int_X \omega_{\varphi_t}^n = \int_X e^{tF+c_t}\omega^n$.

Openness. The openness will follow from the implicit function theorem for Banach spaces [267]:

Theorem 3.2.2. (Implicit Function Theorem). Let $\Psi : U \times V \to W$ be a $\mathcal{C}^k$–operator between Banach spaces. Write $D_y\Psi(x_0, y) \in \mathrm{Hom}(V, W)$ for the differential of $y \mapsto \Psi(x_0, y)$, for some fixed $x_0 \in U$. If $D_y\Psi(x_0, y)$ is invertible at $(x_0, y_0) \in U \times V$. Then there are open neighborhoods $U_0 \subseteq U \times V$ and $V_0 \subseteq V \times W$ of (x_0, y_0) and $(x_0, \Psi(x_0, y_0))$, respectively, such that $U_0 \ni (x, y) \mapsto (x, \Psi(x, y)) \in V_0$ is invertible with inverse of $\mathcal{C}^k$ regularity.

Lemma 3.2.3. Suppose that $(\mathrm{MA})_t$ has a solution for some $t < 1$. Then for any sufficiently small $\varepsilon > 0$, there is a solution to $(\mathrm{MA})_{t+\varepsilon}$.

Proof. Following [267, Lemma 3.3], we define an operator $\Phi : \mathcal{C}^{3,\alpha}(X) \times [0, 1] \longrightarrow \mathcal{C}^{1,\alpha}(X)$,

$$\Phi(\varphi, t) \ := \ \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t)^n}{\omega_0^n} - \varphi - tF.$$

By assumption, there exists a smooth function $\varphi_t \in \mathcal{C}^\infty(X, \mathbb{R})$ such that $\Phi(\varphi_t, t) = 0$ and $\omega_t = \omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_t > 0$ is Kähler. The metric ω_t is used to define the Hölder norms on X. To apply the implicit function theorem, we compute

$$D\Phi_{(\varphi_t, t)}(\psi, 0) \ = \ \frac{n\sqrt{-1}\partial\bar{\partial}\psi \wedge \omega_t^{n-1}}{\omega_t^n} - \psi \ = \ \Delta_{\omega_t}\psi - \psi.$$

The linear operator $L(\psi) := \Delta_{\omega_t}\psi - \psi$ has trivial kernel. Indeed, if $L(\psi) = 0$, then

$$\int_X |\psi|^2_{\omega_t}\omega_t^n \ = \ \int_X \psi\Delta_{\omega_t}\psi\omega_t^n \ = \ -\int_X |\nabla\psi|^2_{\omega_t}\omega_t^n \ \leq \ 0,$$

so $\psi = 0$. The operator L is self-adjoint, so L^* has a trivial kernel. From [267, Theorem 2.13], it follows that L is an isomorphism of Hölder spaces

$$L : \mathcal{C}^{3,\alpha}(X) \longrightarrow \mathcal{C}^{1,\alpha}(X).$$

By the implicit function theorem, for s sufficiently close to t, there exist functions $\varphi_s \in \mathcal{C}^{3,\alpha}(X)$ such that $\Phi(\varphi_s, s) = 0$. For s sufficiently close to t, this φ_s will be close enough to φ_t in $\mathcal{C}^{3,\alpha}$ to ensure that $\omega_0 + \sqrt{-1}\partial\bar{\partial}\varphi_s$ is a positive form.
To see that the solution φ_s is indeed smooth, write $\omega_0 = \sqrt{-1}g_{i\bar{j}}dz_i \wedge d\bar{z}_j$ in local coordinates. In coordinates, the equation $\Phi(\varphi_s, s) = 0$ reads

$$\log\det(g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi_s) - \log\det(g_{i\bar{j}}) - \varphi_s - sF \ = \ 0.$$

Let $\tilde{g}_{i\bar{j}} := g_{i\bar{j}} + \partial_i\partial_{\bar{j}}\varphi_s$. Since $\varphi_s \in \mathcal{C}^{3,\alpha}$, we can differentiate the equation to obtain

$$\tilde{g}^{i\bar{j}}(\partial_k g_{i\bar{j}} + \partial_k\partial_i\partial_{\bar{j}}\varphi_s) - \partial_k \log\det(g_{i\bar{j}}) - \partial_k\varphi_s - s\partial_k F \ = \ 0.$$

Write the above equation as

$$\tilde{g}^{i\bar{j}}\partial_i\partial_{\bar{j}}(\partial_k\varphi_s) - \partial_k\varphi_s \quad = \quad s\partial_k F + \partial_k \log\det(g_{i\bar{j}}) - \tilde{g}^{i\bar{j}}\partial_k g_{i\bar{j}}.$$

View this as a linear elliptic equation $P(\partial_k\varphi_s) = H$ for the function $\varphi_k\varphi_s$, where $H = s\partial_k F + \partial_k \log\det(g_{i\bar{j}}) - \tilde{g}^{i\bar{j}}\partial_k g_{i\bar{j}}$. The coefficients of P are in $\mathcal{C}^{1,\alpha}$ and $H \in \mathcal{C}^{1,\alpha}$. Hence, $\partial_k\varphi_s \in \mathcal{C}^{3,\alpha}$, and similarly, $\varphi_{\bar{k}}\varphi_s \in \mathcal{C}^{3,\alpha}$. Therefore, $\varphi_s \in \mathcal{C}^{4,\alpha}$, and repeating the argument shows that $\varphi_s \in \mathcal{C}^{5,\alpha}$, and iterating shows that φ_s is smooth. □

Closedness. To show that $S \subseteq [0,1]$ is closed, we require a priori estimates on (3.2.1). This is the most difficult part of the problem, and is carried out as follows: We first establish the uniform L^∞–estimate $\|\varphi_t\|_{L^\infty(X,\omega)} \leq C$, where the constant $C > 0$ does not depend on the parameter t. This is the main contribution of Yau's in [**318**] and is well-known to be the most difficult. The second step is to establish the $\mathcal{C}^2$–estimate $\Delta_\omega\varphi_t \leq C$ (again, the constant $C > 0$ is uniform, independent of t). We can then establish all higher-order estimates from these two estimates via standard elliptic bootstrapping techniques or the complex Evans–Krylov theory.

In more detail, by the results of [**93**], it suffices to obtain a uniform L^∞–bound on φ. Cherrier [**93**] showed, by extending the second-order estimate of Yau [**318**] and Aubin [**15**], that $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ is then uniformly equivalent to ω. Cherrier [**93**] further extends the Calabi third-order estimate [**318**] to show that a uniform $\mathcal{C}^1$ bound on ω_φ can be obtained. All higher-order estimates via standard elliptic bootstrapping techniques or the complex Evans–Krylov theory [**150, 284**].

Due to time constraints, we only discuss the L^∞–estimate due to Tosatti–Weinkove [**285**]. Some of the exposition is borrowed from [**300**]. We first recall the following analytic preliminaries:

Theorem 3.2.4. (Poincaré inequality). Let (X,ω) be a compact Hermitian manifold. There exists a constant C such that for any real-valued function $f : X \to \mathbb{R}$ with $\int_X f\omega^n = 0$ we have

$$\int_X |f|^2\omega^n \quad \leq \quad C\int_X |\partial f|^2\omega^n,$$

where $|\partial f|^2 = g^{i\bar{j}}\partial_i f\partial_{\bar{j}} f$.

Theorem 3.2.5. (Sobolev inequality). Let (X,ω) be a compact Hermitian manifold of (complex) dimension $n > 1$. There exists a uniform constant C such that for any real-valued function $f : X \to \mathbb{R}$ we have

$$\left(\int_X |f|^{2p}\omega^n\right)^{\frac{1}{p}} \leq C\left(\int_X |\partial f|^2\omega^n + \int_X |f|^2\omega^n\right),$$

where $p = \frac{n}{(n-1)} > 1$.

The L^∞–estimate. Let $\varphi := \varphi_t$ and let $\widetilde{F} := tF + c_t$.

Claim: There is a constant $C = C(X, g, \sup \tilde{F})$, independent of t, such that if $\int_X \varphi\omega^n = 0$, then

$$\|\varphi\|_{L^\infty(X,\omega)} \leq C.$$

The main idea is just an exercise in book-keeping relative to the case $n = 2$. Hence, considering the case $n = 2$, we have

$$\begin{aligned}
\int_X \varphi(\omega^2 - \omega_\varphi^2) &= \int_X \varphi \wedge (\omega - \omega_\varphi) \wedge (\omega + \omega_\varphi) \\
&= \int_X \varphi \wedge (-\sqrt{-1}\partial\bar{\partial}\varphi) \wedge (\omega + \omega_\varphi) \\
&= \int_X \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge (\omega + \omega_\varphi),
\end{aligned}$$

where the last equality follows from integrating by parts (and the Kähler condition). Hence,

$$\int_X \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge (\omega + \omega_\varphi) \geq \int_X \sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega = \int_X |\nabla\varphi|_g^2\omega^2.$$

On the other hand, from the equation, we know that

$$\int_X \varphi(\omega^2 - \omega_\varphi^2) = \int_X \varphi(\omega^2 - e^{\widetilde{F}}\omega^2) \leq C\int_X |\varphi|^2\omega^2,$$

where $C = C(X, g, \sup \tilde{F})$. Hence,

$$C\int_X |\varphi|^2\omega^2 \geq \int_X |\nabla\varphi|_g^2\omega^2.$$

The Poincaré inequality tells us that, whenever the average is zero, the L^2–norm of the gradient is controlled from below by the L^2–norm, multiplied by a small positive constant. Hence,

$$C\int_X |\varphi|\omega^2 \geq \int_X |\nabla\varphi|_g^2\omega^2 \geq C\int_X |\varphi|^2\omega^2,$$

where the constant coming from the Poincaré inequality depends only on ω. By Cauchy–Schwarz, we have

$$C\int_X |\varphi|\omega^2 \leq C\left(\int_X \omega^2\right)^{\frac{1}{2}}\left(\int_X |\varphi|^2\omega^2\right)^{\frac{1}{2}}.$$

Combining these inequalities, we have

$$\|\varphi\|_{L^2(X,\omega)} \leq C.$$

To complete the L^∞–estimate, we want to

(a) replace φ by a power of φ, namely, $\varphi|\varphi|^q$, for some $q \in \mathbb{N}_0$.
(b) Use the Sobolev inequality in place of the Poincaré inequality, which in complex dimension 2 reads:

$$\left(\int_X |f|^4\omega^2\right)^{\frac{1}{2}} \leq C\left(\int_X |\nabla f|^2\omega^2 + \int_X |f|^2\omega^2\right).$$

To this end, we see that

$$\begin{aligned} C\int_X |\varphi|^{q+1}\omega^2 &\geq \int_X \varphi|\varphi|^q(\omega^2 - \omega_\varphi^2) \\ &= -\int_X \varphi|\varphi|^q(\sqrt{-1}\partial\bar{\partial}\varphi) \wedge (\omega + \omega_\varphi) \\ &= (q+1)\int_X |\varphi|^q\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge (\omega + \omega_\varphi) \\ &\geq (q+1)\int_X |\varphi|^q\sqrt{-1}\partial\varphi \wedge \bar{\partial}\varphi \wedge \omega \\ &= \frac{(q+1)}{(\frac{q}{2}+1)^2}\int_X \sqrt{-1}\partial\left(\varphi|\varphi|^{\frac{q}{2}}\right) \wedge \bar{\partial}\left(\varphi|\varphi|^{\frac{q}{2}}\right) \wedge \omega. \end{aligned}$$

Apply the Sobolev inequality to $|f| = |\varphi|^{\frac{q+2}{2}}$ to give

$$\begin{aligned} \left(\int_X |f|^4\omega^2\right)^{\frac{1}{2}} &= \left(\int_X |\varphi|^{2(q+2)}\omega^2\right)^{\frac{1}{2}} \\ &\leq C(q+2)\left(\int_X |\varphi|^{q+1}\omega^2 + \int_X |\varphi|^{q+2}\omega^2\right). \end{aligned}$$

Set $p = q+2$. Then

$$\|\varphi\|_{L^{2p}} \leq C^{\frac{1}{p}}p^{\frac{1}{p}}\max\left(1, \|\varphi\|_{L^p}\right). \tag{3.2.2}$$

Replace p by $2p$:

$$\|\varphi\|_{L^{4p}} \leq C^{\frac{1}{2p}}(2p)^{\frac{1}{2p}}\max(1, \|\varphi\|_{L^{2p}}) \leq C^{\frac{1}{p}}C^{\frac{1}{2p}}p^{\frac{1}{p}}(2p)^{\frac{1}{2p}}\max(1, \|\varphi\|_{L^p}),$$

where the second inequality follows from (3.2.2).
Replacing p by $2^{k-1}p$, we see that

$$\|\varphi\|_{L^{2^kp}} \leq C^{\frac{1}{p}}C^{\frac{1}{2p}}\cdots C^{\frac{1}{2^{k-1}p}}p^{\frac{1}{p}}\cdots(2^{k-1}p)^{\frac{1}{2^{k-1}p}}\max(1, \|\varphi\|_{L^p}).$$

Setting $p = 2$ yields

$$\begin{aligned}\|\varphi\|_{L^{2^{k+1}}} &\leq C^{\frac{1}{2}} C^{\frac{1}{4}} \cdots C^{\frac{1}{2^k}} 2^{\frac{1}{2}} \cdots (2^k)^{\frac{1}{2^k}} \max(1, \|\varphi\|_{L^2}) \\ &= C^{\sum_{r=1}^k 2^{-r}} 2^{\sum_{r=1}^k r 2^{-r}} \max(1, \|\varphi\|_{L^2}).\end{aligned}$$

The series $\sum_{r=1}^{\infty} 2^{-r}$ and $\sum_{r=1}^{\infty} r2^{-r}$ are both convergent, and therefore, letting $k \to \infty$ yields

$$\|\varphi\|_{L^\infty} \leq C \max(1, \|\varphi\|_{L^2}).$$

3.3. The Quasi-Negative Case dans l'esprit de Diverio–Trapani

In this section, we will consider a refinement of the Wu–Yau theorem, first established by Diverio–Trapani [111] (see also [305]), relaxing the negativity of the holomorphic sectional curvature to quasi-negativity (non-positive everywhere and negative at one point). One of the key results on which the theorem of Diverio–Trapani [111] hinges is the following:

Theorem 3.3.1. (Diverio–Trapani). Let (X^n, ω) be a compact Kähler manifold of (complex) dimension n. If the holomorphic sectional curvature of ω is quasi-negative, then

$$\int_X c_1(K_X)^n > 0.$$

We extended this result to the Hermitian category in a joint work with Kai Tang and Yashan Zhang [61]. The main theorem curiously does not require the curvature to have a sign. To state the main theorem, let us introduce the following terminology:

Definition 3.3.2. Let (X, ω) be a compact Kähler manifold. For positive constants $\delta_1, \delta_2 > 0$, we say that a Hermitian metric α on X is

(i) δ_1–*bounded* (relative to ω) if there is a smooth function $\psi : X \to \mathbb{R}$ such that

$$\alpha \ \leq \ \delta_1 \omega + \sqrt{-1}\partial\bar{\partial}\psi.$$

(ii) δ_2–*volume non-collapsed* on an open set $\mathcal{U} \subset X$ if $\alpha^n \geq \delta_2 \omega_0^n$.

A Hermitian metric α satisfying both (i) and (ii) is said to have (δ_1, δ_2)–*bounded geometry* (relative to ω and $\mathcal{U}$). The space of Hermitian metrics with (δ_1, δ_2)–bounded geometry (relative to ω_0 and $\mathcal{U}$) is denoted by $\mathcal{H}_{\delta_1, \delta_2}(\omega_0, \mathcal{U})$.

Remark 3.3.3. We observe that δ_1–boundedness is cohomological in nature, while the assumption of δ_2–volume non-collapsed is pointwise on the open set $\mathcal{U}$. It would be interesting to know whether the assumption of δ_2–volume non-collapsed can be removed in the subsequent results we present here.

Definition 3.3.4. Let (X, ω) be a compact Hermitian manifold. Let $\mathcal{F}_\omega$ be curvature function of the Hermitian metric[2] ω. For a positive constant $\delta > 0$, and a non-empty open set $\mathcal{U} \subset X$, we say that $\mathcal{F}_\omega$ is (ε, δ)–quasi-negative (relative to $\mathcal{U}$) if, there is a sufficiently small $\varepsilon > 0$ such that $\mathcal{F}_\omega \leq \varepsilon$ on X, and if $\mathcal{F}_\omega \leq -\delta$ on $\mathcal{U}$.

Remark 3.3.5. In the main theorems of the present section, the $\varepsilon > 0$ will also depend on a metric ω_0 and constants δ_1, δ_2. In this case, we say that a curvature is (ε, δ)–quasi-negative

[2] For instance, $\mathcal{F}_\omega$ can be the scalar curvature, Ricci curvature, holomorphic sectional curvature, etc.

(relative to $\omega_0, \mathcal{U}, \delta, \delta_1, \delta_2$). Let us, moreover, note that, for a curvature function $\mathcal{F}_\omega$, we understand $\mathcal{F}_\omega \leq C$ on a set X to mean

$$\sup_{x \in X} \sup_{v_x \in T_x X} \mathcal{F}_\omega(v_x) \leq C.$$

The above terminology naturally extends the notion of quasi-negative, which can be viewed as a limit (as $\varepsilon \to 0$) of (ε, δ)–quasi-negativity.

Given the above definitions, we now state the main theorem of [61]:

Theorem 3.3.6. Let (X^n, ω_0) be a compact Kähler manifold. Assume that there is a Hermitian metric $\eta \in \mathcal{H}_{\delta_1, \delta_2}(\omega_0, \mathcal{U})$ with (δ_1, δ_2)–bounded geometry. If the (Chern) second Schwarz bisectional curvature of η is (ε, δ)–quasi-negative (relative to $\omega_0, \mathcal{U}, \delta_1, \delta_2, \delta$), then

$$\int_X c_1(K_X)^n > 0.$$

Proof. To show that $\int_X c_1(K_X)^n > 0$, it suffices to obtain the estimate

$$\int_X (-\mathrm{Ric}_{\omega_0})^n = (2\pi)^n \int_X c_1(K_X)^n > 0. \tag{3.3.1}$$

Let $\rho : \mathbb{R} \to \mathbb{R}$ be the function

$$\rho(t) \quad := \quad \begin{cases} \frac{1}{n}, & t \leq 0, \\ 1, & t > 0. \end{cases} \tag{3.3.2}$$

For the Hermitian metric η in the statement of 3.3.6, let $\kappa_\eta : X \to \mathbb{R}$ be the function

$$\kappa_\eta(x) \quad := \quad \rho \left(\max_{(\vartheta, v_x) \in \mathcal{F}_X \times \mathbb{R}^n} \mathrm{RBC}_\eta(\vartheta, v_x) \right) \cdot \max_{(\vartheta, v_x) \in \mathcal{F}_X \times \mathbb{R}^n} \mathrm{RBC}_\eta(\vartheta, v_x),$$

where ϑ is a unitary frame (i.e., a section of the unitary frame bundle, and $v_x \in \mathbb{R}^n$). Let us also introduce the notation

$$\mu_\eta \quad := \quad \max_{x \in X} \max_{(\vartheta, v_x) \in \mathcal{F}_X \times \mathbb{R}^n} \mathrm{RBC}_\eta(\vartheta, v_x). \tag{3.3.3}$$

To establish (3.3.1), we will show that there are constants $\varepsilon, c_3, c_4 > 0$ such that

$$\begin{aligned} \int_X (-\mathrm{Ric}_{\omega_0})^n \quad &\geq \quad \int_X (n\delta_1 \kappa_\eta \omega_0 - \mathrm{Ric}_{\omega_0})^n - c_4 \varepsilon && (3.3.4) \\ &\geq \quad c_3 - c_4 \varepsilon, && (3.3.5) \end{aligned}$$

where $\varepsilon > 0$ can be chosen such that $c_3 - c_4 \varepsilon > 0$.

To this end, consider the twisted Wu–Yau continuity method, given by the complex Monge–Ampère equation

$$(t(\omega_0 + \delta_1^{-1} \sqrt{-1} \partial \overline{\partial} \psi) - \mathrm{Ric}(\omega_0) + \sqrt{-1} \partial \overline{\partial} \varphi_t)^n \quad = \quad e^{\varphi_t} \omega_0^n. \tag{3.3.6}$$

From the assumption of δ_1–boundedness, we see that

$$\omega_0 + \delta_1^{-1}\sqrt{-1}\partial\overline{\partial}\psi \quad \geq \quad \delta_1^{-1}\eta. \tag{3.3.7}$$

Set $\omega_t := t(\omega_0 + \delta_1^{-1}\sqrt{-1}\partial\overline{\partial}\psi) - \text{Ric}(\omega_0) + \sqrt{-1}\partial\overline{\partial}\varphi_t$, allowing us to write (3.3.6) as

$$\omega_t^n \quad = \quad e^{\varphi_t}\omega_0^n.$$

From (3.3.7), the metrics ω_t afford the lower bound

$$\text{Ric}_{\omega_t} \quad \geq \quad -\omega_t + t\delta_1^{-1}\eta.$$

Taking f to be the identity map, $\lambda = 1$, and $\mu = t\delta_1^{-1}$ in the Chern–Lu Schwarz lemma, we have

$$\Delta_{\omega_t} \log \text{tr}_{\omega_t}(\eta) \quad \geq \quad (-\kappa_\eta + t(n\delta_1)^{-1})\text{tr}_{\omega_t}(\eta) - 1. \tag{3.3.8}$$

Here, and throughout, we are abusing notation, and identify κ_η with $\max_{x \in X} \kappa_\eta(x)$. By the maximum principle,

$$\sup_X \text{tr}_{\omega_t}(\eta) \quad \leq \quad \frac{n\delta_1}{t - \kappa_\eta n\delta_1}. \tag{3.3.9}$$

For $t > n\delta_1\mu_\eta$, the estimate (3.3.9) is independent of t. As a consequence, the continuity method admits a smooth solution for $t > n\delta_1\mu_\eta$ (c.f., [**286, 323**]). Introduce the potential $u_t := \varphi_t + t\delta_1^{-1}\psi$. The crux of the argument is to estimate $\sup_{\mathcal{U}} u_t$ from below, and $\sup_X u_t$ from above. Indeed, the constant c_3 in (3.3.5) is given by

$$c_3 \quad := \quad \liminf_{t \to n\delta_1\mu_\eta} \int_X e^{u_t}\omega_0^n,$$

where μ_η is defined in (3.3.3). The positivity of c_3 demands u_t to not be identically $-\infty$, while the finiteness of u_t requires an upper bound on $\sup_X u_t$. The upper bound on $\sup_X u_t$ is straightforward:

Lemma 3.3.7. For $t \in (n\delta_1\mu_\eta, 2n\delta_1\mu_\eta]$, we have

$$\sup_{x \in X} u_t(x) \quad \leq \quad \log(2n\delta_1\varepsilon + b_0)^n \;\leq\; \log(c_0 + b_0)^n, \tag{3.3.10}$$

where $c_0 > 0$ is such that Tian's α–invariant [**278**] satisfies $\alpha(X, c_0\omega_0) \geq 2$, and $\varepsilon \leq c_0/(2n\delta_1)$.

Proof. Since $u_t = \varphi_t + t\delta_1^{-1}\psi$, (3.3.8) reads

$$e^{u_t - t\delta_1^{-1}\psi}\omega_0^n \quad = \quad e^{\varphi_t}\omega_0^n \quad = \quad (t\omega_0 - \text{Ric}_{\omega_0} + \sqrt{-1}\partial\overline{\partial}u_t)^n. \tag{3.3.11}$$

Since the metric ω_0 is smooth, and X is compact, there is some $b_0 \geq 0$ such that

$$\text{Ric}_{\omega_0} \quad \geq \quad -b_0\omega_0. \tag{3.3.12}$$

From (3.3.11), (3.3.12), and the fact that $\psi \leq 0$, at the point $x \in X$ where u_t achieves its maximum, we have

$$\sup_{x\in X} u_t(x) \;\leq\; \sup_X \left(t\delta_1^{-1}\psi + \log \frac{(t\omega_0 - \mathrm{Ric}_{\omega_0} + \sqrt{-1}\partial\overline{\partial}u_t)^n}{\omega_0^n} \right) \;\leq\; \log(t_0 + b_0)^n.$$

Since $t \leq 2n\delta_1\mu_\eta \leq 2n\delta_1\varepsilon$, this proves (3.3.10).

□

We now want to estimate $\sup_{\mathfrak{U}} u_t$ from below.

Lemma 3.3.8.

$$\sup_{\mathfrak{U}} u_t \;\geq\; n\log(n) + \log\left(\frac{-\int_{\mathfrak{U}} \kappa_\eta e^{t(n\delta_1)^{-1}\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_t^n}{\int_X \omega_t^n} \right). \tag{3.3.13}$$

Proof. Start by integrating (3.3.8) over X with respect to the volume form ω_t^n. By the divergence theorem,

$$\begin{aligned}
\int_X \omega_t^n \;&\geq\; \int_X (-\kappa_\eta + t(n\delta_1)^{-1})\mathrm{tr}_{\omega_t}(\eta)\omega_t^n \\
&\geq\; \int_{\mathfrak{U}} (-\kappa_\eta + t(n\delta_1)^{-1})\mathrm{tr}_{\omega_t}(\eta)\omega_t^n \;\geq\; -\int_{\mathfrak{U}} \kappa_\eta \mathrm{tr}_{\omega_t}(\eta)\omega_t^n,
\end{aligned}$$

using the fact that $(-\kappa_\eta + t(n\delta_1)^{-1})\mathrm{tr}_{\omega_t}(\eta) > 0$. By the arithmetic-geometric mean inequality,

$$\begin{aligned}
-\int_{\mathfrak{U}} \kappa_\eta \mathrm{tr}_{\omega_t}(\eta)\omega_t^n \;&\geq\; -n\int_{\mathfrak{U}} \kappa_\eta \left(\frac{\eta^n}{\omega_t^n}\right)^{\frac{1}{n}} \omega_t^n \\
&=\; -n\int_{\mathfrak{U}} \kappa_\eta \left(\frac{\omega_0^n}{\omega_t^n}\right)^{\frac{1}{n}} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_t^n.
\end{aligned} \tag{3.3.14}$$

Using (3.3.11) in (3.3.14), we achieve the estimate

$$\begin{aligned}
\int_X \omega_t^n \;&\geq\; -n\int_{\mathfrak{U}} \kappa_\eta e^{-\frac{1}{n}(u_t - t\delta_1^{-1}\psi)} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_t^n \\
&\geq\; -ne^{-\frac{1}{n}\sup_{\mathfrak{U}} u_t} \int_{\mathfrak{U}} \kappa_\eta e^{t(n\delta_1)^{-1}\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_t^n.
\end{aligned} \tag{3.3.15}$$

Then (3.3.13) follows from (3.3.15). □

From the above lemma, it suffices to estimate

$$\frac{-\int_{\mathfrak{U}} \kappa_\eta e^{t(n\delta_1)^{-1}\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_t^n}{\int_X \omega_t^n} \tag{3.3.16}$$

from below. The following lemma gives an estimate for the numerator:

Lemma 3.3.9. There are positive constants $c_1, \delta_2 > 0$ such that

$$-\int_{\mathfrak{U}} \kappa_\eta e^{t(n\delta_1)^{-1}\psi} \left(\frac{\eta^n}{\omega_0^n} \right)^{\frac{1}{n}} \omega_t^n \ \ \geq \ \ \delta_2\delta_3 \int_{\mathfrak{U}} e^{u_t^*}\omega_0^n \ \geq \ \frac{c_1 \delta_2^{\frac{1}{n}}\delta_3}{n}. \tag{3.3.17}$$

PROOF. Let us write $u_t^* := u_t - \sup_X u_t$, so that $\sup_X u_t^* \leq 0$. In this notation, (3.3.16) reads

$$\begin{aligned} \frac{-\int_{\mathfrak{U}} \kappa_\eta e^{t(n\delta_1)^{-1}\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_t^n}{\int_X \omega_t^n} &= \frac{-\int_{\mathfrak{U}} \kappa_\eta e^{u_t} e^{-\left(1-\frac{1}{n}\right)t\delta_1^{-1}\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_0^n}{\int_X e^{u_t} e^{-t\delta_1^{-1}\psi}\omega_0^n} \\ &= \frac{-\int_{\mathfrak{U}} \kappa_\eta e^{u_t^*} e^{-\left(1-\frac{1}{n}\right)t\delta_1^{-1}\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_0^n}{\int_X e^{u_t^*} e^{-t\delta_1^{-1}\psi}\omega_0^n} \end{aligned} \tag{3.3.18}$$

Since $\psi \leq 0$, and $t \in (n\delta_1\mu_\eta, 2n\delta_1\mu_\eta]$, we have

$$\begin{aligned} -\int_{\mathfrak{U}} \kappa_\eta e^{u_t^*} e^{-\left(1-\frac{1}{n}\right)t\delta_1^{-1}\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_0^n &\geq -\int_{\mathfrak{U}} \kappa_\eta e^{u_t^*} e^{(1-n)\kappa_\eta\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_0^n \\ &\geq \delta_3 \int_{\mathfrak{U}} e^{u_t^*} e^{(1-n)\kappa_\eta\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_0^n, \end{aligned} \tag{3.3.19}$$

where the last inequality follows from the negative curvature estimate $\mathrm{RBC}_\eta \leq -\delta_3$ on $\mathfrak{U}$. Let

$$c_1 \ \ := \ \ \inf\left\{ \int_{\mathfrak{U}} e^v \omega_0^n \ : \ v \in \mathrm{PSH}_{(c_0+b_0)\omega_0}(X), \ \sup_X v = 0 \right\}. \tag{3.3.20}$$

Then c_1 is a positive constant depending only on $\mathfrak{U}$ and ω_0. Since η has (δ_1, δ_2)–bounded geometry, we have

$$-\int_{\mathfrak{U}} \kappa_\eta e^{u_t^*} e^{-(n-1)\mu_\eta\psi} \left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}} \omega_0^n \ \ \geq \ \ \frac{\delta_2^{\frac{1}{n}}\delta_3}{n} \int_{\mathfrak{U}} e^{u_t^*}\omega_0^n \ \geq \ \frac{c_1\delta_2^{\frac{1}{n}}\delta_3}{n}.$$

From (3.3.19) and (3.3.20), this gives the desired lower bound for the numerator in (3.3.18). □

We now complete the estimate for lower bound on $\sup_{\mathfrak{U}} u_t$:

Lemma 3.3.10. There are constants $c_1, c_2, \delta_1, \delta_2 > 0$ such that

$$\sup_{\mathfrak{U}} u_t \ \ \geq \ \ n\log(n) + \log\left(\frac{c_1\delta_2\delta_3}{c_2}\right).$$

PROOF. For the denominator in (3.3.18), we will again use Tian's α–invariant [278]. Indeed, first observe that, since $0 < t \leq 2n\delta_1\mu_\eta \leq c_0$, (3.3.7) implies

$$c_0\omega_0 + t\delta_1^{-1}\sqrt{-1}\partial\bar{\partial}\psi \ \ \geq \ \ t\omega_0 + t\delta_1^{-1}\sqrt{-1}\partial\bar{\partial}\psi \ \geq \ t\delta_1^{-1}\eta \ > \ 0.$$

In other words, $t\delta_1^{-1}\psi$ is $c_0\omega_0$–plurisubharmonic. Hence, since $c_0 > 0$ is chosen such that Tian's α–invariant satisfies $\alpha(X, c_0\omega_0) \geq 2$, we have

$$\int_X e^{u_t^*} e^{-t\delta_1^{-1}\psi}\omega_0^n \ \leq \ \int_X e^{-t\delta_1^{-1}\psi}\omega_0^n \ = \ c_0^{-n}\int_X e^{-t\delta_1^{-1}\psi}(c_0\omega_0)^n \ \leq \ c_2, \tag{3.3.21}$$

for some constant $c_2 > 0$ depending only on c_0 and ω_0. Combining (3.3.16), (3.3.18), (3.3.17), and (3.3.21), we see that

$$\frac{-\int_{\mathcal{U}} \kappa_\eta e^{t(n\delta_1)^{-1}\psi}\left(\frac{\eta^n}{\omega_0^n}\right)^{\frac{1}{n}}\omega_t^n}{\int_X \omega_t^n} \ \geq \ \frac{c_1\delta_2^{\frac{1}{n}}\delta_3}{nc_2}. \tag{3.3.22}$$

Inserting (3.3.22) into (3.3.13),

$$\sup_{\mathcal{U}} u_t \ \geq \ n\log(n) + \log\left(\frac{c_1\delta_2^{\frac{1}{n}}\delta_3}{nc_2}\right).$$

□

We now complete the proof: Let $c_4 > 0$ be the constant such that

$$\int_X \sum_{k=1}^n \binom{n}{k}(2n\delta_1\varepsilon\omega_0)^k \wedge (-\mathrm{Ric}_{\omega_0})^{n-k} \ \leq \ c_4\varepsilon.$$

Then

$$\begin{aligned}
\int_X (2\pi c_1(K_X))^n \ = \ \int_X (-\mathrm{Ric}_{\omega_0})^n \ &\geq \ \int_X (2n\delta_1\varepsilon\omega_0 - \mathrm{Ric}_{\omega_0})^n - c_4\varepsilon \\
&\geq \ \lim_{t\searrow n\delta_1\mu_\eta} (t\omega_0 - \mathrm{Ric}_{\omega_0} + \sqrt{-1}\partial\bar{\partial}u_t)^n - c_4\varepsilon \\
&= \ \lim_{t\searrow n\delta_1\mu_\eta} \int_X e^{u_t}e^{-t\delta_1^{-1}\psi}\omega_0^n - c_4\varepsilon \\
&\geq \ \liminf_{t\searrow n\delta_1\mu_\eta} \int_X e^{u_t}\omega_0^n - c_4\varepsilon \\
&= \ c_3 - c_4\varepsilon.
\end{aligned}$$

Taking $\varepsilon \leq \min\left\{\frac{c_3}{2c_4}, \frac{c_0}{2n\delta_1}\right\}$ completes the proof. □

Corollary 3.3.11. Let (X, ω) be a compact Kähler surface with quasi-negative holomorphic sectional curvature. Then $b_1(X) = 0$.

PROOF. Since quasi-negative holomorphic sectional curvature implies K_X is ample, there is a Kähler metric of negative Ricci curvature. By the Nakano vanishing theorem,

$$H^q(X, \Omega_X^p \otimes K_X) \ = \ 0, \qquad p + q = 1.$$

Hence, by the Dolbeault theorem, $H^1(X, \mathcal{O}_X) = H^{0,1}(X)$. On the other hand, by Serre duality,

$$H^{0,1}(X, K_X) = H^{0,1}(X, K_X \otimes K_X^{-1}) = H^{0,1}(X, \mathcal{O}_X).$$

Hence, $h^{0,1} = 0$ which implies that $b_1 = 2h^{0,1} = 0$. □

Corollary 3.3.12. Let (X, ω) be a compact Kähler surface with quasi-negative holomorphic sectional curvature. The Euler characteristic of X is positive.

PROOF. By Poincaré duality and the above result, we see that

$$\chi(X) = b_0 - b_1 + b_2 - b_3 + b_4 = 2 + b_2 \geq 2.$$

□

Further directions. The results contained in this section will appear in a joint work with Kai Tang, and Yashan Zhang [61]. The presence of Tian's α–invariant [278] (known to be related to the log canonical threshold [83]) in the above argument is curious. It would be interesting if a hidden algebraic phenomenon occurs (see [61] for further discussion).
Let us mention the following question which Diverio raised:

Question 3.3.13. Let (X, ω) be a compact Kähler manifold with quasi-negative holomorphic sectional curvature. Does there always exist a Kähler metric on X with negative holomorphic sectional curvature?

There are no known examples of Kähler metrics with quasi-negative, but not negative, holomorphic sectional curvature. For some time, the author has proposed to address this problem by attempting to produce a Kazdan–Warner-type theorem for the holomorphic sectional curvature:

Question 3.3.14. Let (X, ω) be a compact Kähler manifold. The holomorphic sectional curvature defines a function $\text{HSC}_\omega : \mathbb{P}(T^{1,0}X) \longrightarrow \mathbb{R}$. Can any smooth function $f : \mathbb{P}(T^{1,0}X) \to \mathbb{R}$ be the holomorphic sectional curvature of a (Kähler or Hermitian) metric?

3.4. Remarks on the Wu–Yau Theorem

Several questions remain wide open concerning the Wu–Yau theorem. The first question is whether the positive-analog of the Wu–Yau theorem holds. The naive positive-analog would be the following:

Naive positive analog of the Wu–Yau Theorem. Let (X, ω) be a compact Hermitian manifold with ${}^c\mathrm{HSC}_\omega > 0$. Does there exist a Hermitian metric η with ${}^c\mathrm{Ric}_\eta^{(1)} > 0$?

Hitchin's example [**161**] on Hirzebruch surfaces that we discussed previously shows that this naive positive analog of the Wu–Yau theorem is false. Analyzing Hitchin's argument, however, indicates what the positive analog of the Wu–Yau theorem should be. To motivate the conjecture, let X be a compact complex surface. If $\varphi : \widetilde{X} \to X$ is the blow-up of X at a point $p \in X$, the exceptional divisor $\mathcal{E} = \varphi^{-1}(p)$ is a rational curve of self-intersection -1. In particular, there is no Hermitian metric on $\widetilde{X}$ with negative (Chern) holomorphic sectional curvature. In this sense, blow-ups increase the holomorphic sectional curvature. This does not mean that if (X, ω) is a compact complex surface with ${}^c\mathrm{HSC}_\omega > 0$ then ${}^c\mathrm{HSC}_{\pi^*\omega} > 0$. In this direction, however, we ask the following:

Question 3.4.1. Let (X, ω) be a compact Hermitian manifold with ${}^c\mathrm{HSC}_\omega > 0$. Let $\varphi : \widetilde{X} \longrightarrow X$ be a modification of X. Does $\widetilde{X}$ support a Hermitian metric η with ${}^c\mathrm{HSC}_\eta > 0$?

On the other hand, the positivity of the first Chern class in Hitchin's examples was violated by blowing up. In this sense, blow-ups decrease the first Chern–Ricci curvature. Given this asymmetry coming from blow-ups, the natural conjectural picture for the positive analog of the Wu–Yau theorem appears to be the following:

Conjecture 3.4.2. Let (X, ω) be a compact Hermitian manifold with ${}^c\mathrm{Ric}_\omega^{(1)} > 0$. Then X admits a Hermitian metric η with ${}^c\mathrm{HSC}_\eta > 0$.

Remark 3.4.3. From Yang's theorem 2.5.35, a compact Kähler manifold with HSC > 0 is rationally connected. On the other hand, we know from 1.7.42 that Fano manifolds are rationally connected. We suspect that the above conjecture, at least for Kähler manifolds, should not be too difficult to prove. Indeed, since $\mathrm{Ric}_\omega > 0$ implies that X is projective. Hence, one can look at the curvature of the metric induced by the Fubini–Study metric on the ambient $\mathbb{P}^N$.

Recall that the Wu–Yau theorem states that the existence of a (say, Kähler) metric ω of negative holomorphic sectional curvature on a compact Kähler manifold implies that the canonical bundle K_X is ample. In particular, there is a Kähler metric $\omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ such that $\mathrm{Ric}_{\omega_\varphi} < 0$. Diverio raised the following question:

Question 3.4.4. (Diverio). Does the metric ω have to be perturbed (non-trivially) in general? Can φ be taken to be $\varphi \equiv 0$?

This question translates to a statement about the algebraic properties of the curvature operator, and in particular, it suffices to work at a single point. The above question then translates to whether

$$\max_{v_p \in T_p^{1,0}X} \mathrm{HSC}_\omega(v_p) < 0 \implies \max_{v_p \in T_p^{1,0}X} \mathrm{Ric}_\omega(v_p, \overline{v}_p) < 0. \tag{3.4.1}$$

Because the question is purely algebraic, however, if (3.4.1) holds, then

$$\min_{v_p \in T_p^{1,0}X} \mathrm{HSC}_\omega(v_p) > 0 \implies \min_{v_p \in T_p^{1,0}X} \mathrm{Ric}_\omega(v_p, \overline{v}_p) > 0 \tag{3.4.2}$$

would hold also. However, Hitchin's examples [**161**] demonstrate that this is false in general. Consequently, the perturbation φ within the cohomology class must be non-trivial. In a joint work with Simone Diverio [**55**], the author raised the following variant of 3.4.4:

Question 3.4.5. Can one classify or produce necessary or sufficient conditions for the perturbation in the Wu–Yau theorem to be trivial?

The Aubin–Yau theorem asserts that on a compact Kähler manifold with a Kähler metric ω of negative Ricci curvature, one can perturb the metric within its cohomology class to the unique Kähler–Einstein ω_{KE}. Hence, one starting point where one can begin to make non-trivial progress on the above question is by considering the following much more restrictive case:

Question 3.4.6. Let (X, ω) be a compact Kähler–Einstein manifold with $\mathrm{Ric}_\omega = -\omega$. Can one classify or produce necessary or sufficient conditions for ω to have negative holomorphic sectional curvature?

To facilitate the discussion of this question, the author and Diverio introduced the following terminology [**55**]:

Definition 3.4.7. A compact Kähler manifold (X, ω) is said to be *Wu–Yau–Kähler–Einstein* if it admits a Kähler–Einstein metric with holomorphic sectional curvature having a sign.

Since the holomorphic sectional curvature and Ricci curvature both dominate the scalar curvature, it is clear that their signs must be the same. Obvious examples of Wu–Yau–Kähler–Einstein manifolds are space-forms. An interesting class of examples are given by Kähler C–spaces [**8**]:

Proposition 3.4.8. (Alvarez–Heier–Zheng). All Kähler C-spaces admit Kähler–Einstein metrics with positive holomorphic sectional curvature. In particular, Kähler C-spaces are Wu–Yau–Kähler–Einstein.

Let us now restrict to the case of surfaces, where we can obtain quite refined conditions based on Chern class inequalities.

Corollary 3.4.9. ([95]). Let (X, ω) be a compact Kähler–Einstein surface with $\text{Ric}_\omega = -\omega$. Assume the Bochner curvature tensor is of constant length. If $c_2 < c_1^2$, then X has negative holomorphic sectional curvature.

Remark 3.4.10. The above result is not very satisfactory since we are unaware of a single example of a compact Kähler manifold with Bochner curvature tensor of constant length. One might hope to find non-trivial examples by looking for Kähler metrics with constant vanishing Bochner curvature tensor. In this case, Bryant's results [65] show that they must be products of space forms. One can look for examples of compact Kähler manifolds with parallel Bochner curvature tensor.

Example 3.4.11. Let X be a compact quotient of a bounded homogeneous domain. Since the automorphism group acts transitively, the Schwarz lemma implies that any complete Kähler–Einstein metric must be invariant under automorphisms, and therefore, α_2 is constant.

Theorem 3.4.12. ([95, Theorem 4.13]). Let (X, ω) be a compact Kähler–Einstein surface with $\text{Ric}_\omega = -\omega$. If (X, ω) is Wu–Yau–Kähler–Einstein, then

$$c_2(X) \ \leq \ 3c_1^2(X).$$

Example 3.4.13. The above theorem implies that a compact complex surface of general type with $c_2 - 3c_1^2 > 0$ is not Wu–Yau–Kähler–Einstein. In particular, the Horikawa surfaces [164, 165, 166, 167] which realize the equality case in Noether's inequality, are not Wu–Yau–Kähler–Einstein.

Further directions. Our understanding of the Wu–Yau theorem is very small at present. To the author's knowledge, we do not have a classification of the compact complex surfaces, which are Kobayashi hyperbolic or admit Hermitian metrics of holomorphic sectional curvature. Hence, we pose the following problem:

Question 3.4.14. Let (X, ω) be a compact complex surface. Are there conditions on the Chern numbers which characterize the existence of a (Kähler or Hermitian) metric of negative (Chern) holomorphic sectional curvature?

Question 3.4.15. Classify all compact Wu–Yau–Kähler complex surfaces and all compact Wu–Yau–Kähler–Einstein surfaces.

The role of the Kähler condition would be interesting to explore, especially in light of the Hermitian Schwarz lemma. Hence, we also ask:

Question 3.4.16. Is there a compact Hermitian manifold that admits a Hermitian metric of negative (Chern) holomorphic sectional curvature but does not admit a Kähler metric of negative holomorphic sectional curvature?

The Hermitian/Kähler condition can be further explored by replacing negative Chern holomorphic sectional curvature with negative t–Gauduchon holomorphic sectional curvature in the above question.

Appendix – Kodaira's Embedding Theorem

One of the central underlying themes and motivating problems in complex geometry is to understand how various notions of positivity for vector bundles are related. We will see that the concepts of abundance (in the ordinary sense) and positivity (in terms of curvature, or numerical invariants) are intimately related. We start with the case of line bundles:

Definition A.1. A line bundle $\mathcal{L} \to X$ is said to be *very ample* if the sections of $\mathcal{L}$ furnish a holomorphic embedding of X into some projective space. A line bundle $\mathcal{L}$ is said to be *ample* if there is an integer $m > 0$ such that $\mathcal{L}^{\otimes m}$ is very ample.

Let us give some details on how sections of line bundles give rise to embeddings in projective space. Let $s_0, ..., s_k$ be a basis for the vector space $H^0(X, \mathcal{L})$. Each section s_j is a map from X to $\mathcal{L}$ given by sending a point $x \in X$ into the fiber $\mathcal{L}_x \ni s_j(x)$. The fibers are one-dimensional vector spaces, and locally $\mathcal{L}|_U \simeq U \times \mathbb{C}$. So we can make sense of a map $f : X \longrightarrow \mathbb{P}^k$ by defining

$$f(x) \ := \ (s_0(x) : \cdots : s_k(x)) \in \mathbb{C}^{k+1},$$

at least locally. If we pick another trivializing of $\mathcal{L}$, the transition functions are elements of $\mathbb{C}^*$, so the values of f differ only by a scalar multiple. Hence, so long as there are no $x \in X$ such that $s_j(x) = 0$ for all $0 \leq j \leq k$, we have an embedding of X into $\mathbb{P}^k$.

Example A.2. (Curves). Let $\mathcal{L} \to \Sigma$ be a holomorphic line bundle over a compact Riemann surface Σ_g of genus g. We define the *degree of* $\mathcal{L}$ to the image of $c_1(\mathcal{L})$ under the isomorphism $H^2(\Sigma_g, \mathbb{Z}) \simeq \mathbb{Z}$ given by the orientation determined by the complex structure. From the Chern–Weil theorem, the degree is equivalently defined as

$$\deg(\mathcal{L}) \ = \ \frac{\sqrt{-1}}{2\pi} \int_{\Sigma_g} \Theta^{(\mathcal{L},h)},$$

where $\Theta^{(\mathcal{L},h)}$ is the curvature form of a Hermitian metric h on $\mathcal{L}$. If $\mathcal{L} = K_{\Sigma_g}$, the canonical bundle of Σ_g, then the Gauss–Bonnet formula tells us that

$$\deg(K_{\Sigma_g}) \ = \ -\chi(\Sigma_g) \ = \ 2g - 2.$$

Example A.3. The tautological bundle $\mathcal{O}_{\mathbb{P}^n}(-1) \to \mathbb{P}^n$ is a non-trivial bundle with no global sections. Indeed, every section $\sigma : \mathbb{P}^n \to \mathcal{O}_{\mathbb{P}^n}(-1)$ determines a a section $\sigma : \mathbb{P}^n \to \mathbb{P}^n \times \mathbb{C}^{n+1}$. It is straightforward to show (the reader may find details in [**255**, Corollary 1.2, §5.2, Chapter 1]) σ must be of the form $\sigma(x) = (x, [v])$ for some fixed $v \in \mathbb{C}^{n+1}$. Since the fiber has the identification $\mathcal{O}_{\mathbb{P}^n}(-1)_x \simeq \ell_x$, we see that $v \in \ell_x$ for all $x \in \mathbb{P}^n$, which implies $v = 0$.

Definition A.4. Let $\mathcal{E} \to X$ be a holomorphic vector bundle over a complex manifold X. We say that $\mathcal{E}$ is *ample* if $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ is an ample line bundle.

Remark A.5. Let us note that not every compact Kähler manifold supports an ample line bundle. One obstruction to a compact complex manifold being projective is that there are not enough hypersurfaces. That is, if X is a compact smooth algebraic variety, then X supports a complex subvariety of (complex) codimension one whose homology class is non-trivial.

Example A.6. Let X be the quotient of $\mathbb{C}^2 - \{0\}$ by powers of $\operatorname{diag}(\alpha, \alpha)$ for $\alpha > 0$, i.e., X is a *diagonal Hopf surface*. We see that X is homeomorphic to $\mathbb{S}^3 \times \mathbb{S}^1$. Hence, by Künneth's formula, $b_2(X) = 0$. In particular, X is not Kähler (and therefore, certainly not projective), and moreover, X does not contain any hypersurface with non-trivial homology class.

Example A.7. Let us now give an example of a non-projective Kähler manifold. Let Λ be a lattice of rank 4 in $\mathbb{C}^2$, and take $X = \mathbb{C}^2/\Lambda$. Here, X is a torus, so $H^2(X) \simeq \mathbb{Z}^6$. Let (z, w) denote the holomorphic coordinates on $\mathbb{C}^2$. The 1–forms dz and dw are translation-invariant, and therefore, descend to 1–forms on X. Let $\omega = dz \wedge dw$. Suppose X admits a complex curve Y. Then $\omega|_Y$ vanishes identically and thus $\int_Y \omega = 0$. Let now (α_1, α_2), (β_1, β_2), (γ_1, γ_2), (δ_1, δ_2) be a basis for the lattice Λ. For any basis of $H_2(X, \mathbb{Z})$, pairing with ω yields six maximal minors of the matrix $\begin{pmatrix} \alpha_1 & \beta_1 & \gamma_1 & \delta_1 \\ \alpha_2 & \beta_2 & \gamma_2 & \delta_2 \end{pmatrix}$. Since $\int_Y \omega = 0$, this pairing vanishes for complex curves Y. For Λ chosen generically, however, there is no reason for any integral combination of these minors to be zero.

Example A.8. A further obstruction is given by the *algebraic dimension* $a(X)$ – the transcendence degree (over $\mathbb{C}$) of the field of meromorphic functions. A projective manifold has maximal algebraic dimension $a(X) = \dim_{\mathbb{C}} X$, i.e., X is Moishezon. In particular, complex manifolds with $a(X) < \dim_{\mathbb{C}} X$ will not be algebraic. A notable example of a complex manifold with vanishing algebraic dimension is any complex manifold diffeomorphic to $\mathbb{S}^6$.

Definition A.9. Let X be a complex manifold and $\mathcal{L} \to X$ a line bundle. We say that $\mathcal{L}$ is *positive* if there is a Hermitian metric h on $\mathcal{L}$ whose curvature form

$$\Theta^{(\mathcal{L},h)} := \sqrt{-1}\partial\overline{\partial}\log(h)$$

is positive (in the sense of $(1,1)$–forms). A line bundle $\mathcal{L} \to X$ is said to be *negative* if the dual bundle $\mathcal{L}^* \to X$ is positive.

From [**324**, p. 201]:

Theorem A.10. (Kodaira–Nakano vanishing). Let X be a compact Kähler manifold with $\dim_{\mathbb{C}} X = n$. Let $\mathcal{L}$ be a positive line bundle over X. Then

$$H^{p,q}(X, \mathcal{L}) = 0$$

for all $p + q > n$.

Theorem A.11. (Kodaira Vanishing). Let $\mathcal{L} \to X$ be a positive line bundle over a compact complex manifold X. Then for any holomorphic vector bundle $\mathcal{E}$ on X, there is a positive integer $k_0 \in \mathbb{N}$ such that

$$H^q(X, \mathcal{L}^{\otimes k} \otimes \mathcal{E}) = 0$$

for all $q > 0$ and all $k \geq k_0$.

Proposition A.12. Let $\mathcal{L} \to X$ be an ample line bundle over a complex manifold X. Then $\mathcal{L}$ is positive.

Proof. Since $\mathcal{L}$ is ample, the sections of a sufficiently high multiple $\mathcal{L}^{\otimes k}$ of $\mathcal{L}$ embed X into $\mathbb{P}^N$ via a holomorphic map $\Phi : X \to \mathbb{P}^N$. Let ω_{FS} denote the Fubini–Study metric on $\mathbb{P}^N$. We obtain a smooth Hermitian metric h on $\mathcal{L}^{\otimes k}$ whose curvature form is $\frac{\sqrt{-1}}{2\pi}\Theta^{(\mathcal{L},h)} = \Phi^*\omega_{\mathrm{FS}}$. The metric on $\mathcal{L}$ is given by taking the kth root of h, i.e., $h^{\frac{1}{k}}$, which has curvature form

$$\frac{\sqrt{-1}}{2\pi}\Theta^{(\mathcal{L}^{\otimes k}, h^{1/k})} \;=\; \frac{1}{k}\Phi^*\omega_{\mathrm{FS}} \;>\; 0.$$

$\square$

The converse is true when X is compact Kähler or X is Stein. This is achieved by the Kodaira embedding theorem:

Theorem A.13. (Kodaira embedding theorem). Let X be a compact complex manifold. The following are equivalent:

(i) X is projective, i.e., there exists a holomorphic embedding $f : X \to \mathbb{P}^N$.
(ii) There exists a Kähler form ω on X which represents an integral class, i.e., is in the image of the morphism $H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{R})$.
(iii) There exists a holomorphic line bundle $\mathcal{L} \to X$ that is positive.

Note that the only difficult part of the theorem is that (iii) $\implies$ (i). Indeed, since the restriction of the Fubini–Study metric ω_{FS} represents an integral class, (i) $\implies$ (ii). The implication (ii) $\implies$ (iii) is an immediate consequence of the Lefschetz theorem on $(1,1)$–classes. The standard proof of (iii) $\implies$ (i) is by first proving the Kodaira vanishing theorem (see, e.g., [**142**]). Let us exhibit the less well-known[3] proof due to Donaldson [**115**]:

Proof. Let $\mathcal{L} \to X$ be a positive line bundle, i.e., $\mathcal{L}$ is a holomorphic line bundle with a connection ∇ whose curvature is $\Theta^{(\mathcal{L},h)} = \sqrt{-1}\omega > 0$, where ω is a Kähler form on X. Note that by taking higher tensor powers $\mathcal{L}^{\otimes k}$ corresponds to a scaling of the curvature by a factor of k. This can be thought of as scaling the lengths in X by a factor of $\sqrt{k}$. Fix a point $p \in X$, and consider a small ball in X centered at p. By increasing the value of k, we increase radius $R = R(k)$ of this ball, which can be thought of as an embedding of $B(R) \subset \mathbb{C}^n$ into X. Moreover, if (in this ball) we write $\omega = \sqrt{-1}\sum_{i=1}^n dz_i \wedge d\overline{z}_j$ for the Kähler form, with $J = J_{\mathbb{C}^n} + O(|z|)$ the underlying complex structure, we can assume that the embedding $B(R) \hookrightarrow X$ is an isometry. Now, over $\mathbb{C}^n$, we consider the trivial line bundle endowed with a connection whose curvature is $\sqrt{-1}\omega$. Let σ_0 denote the *standard holomorphic section* of this (twisted) trivial line bundle, i.e., the holomorphic section satisfying $\sqrt{-1}\partial\overline{\partial}\log|\sigma_0|^2 = \sqrt{-1}\omega$. We observe that σ_0 is a Gaussian holomorphic section, in the sense that

$$|\sigma_0|^2 \quad = \quad \exp\left(-\frac{1}{4}|z|^2\right).$$

We can think of this as a "compactly supported holomorphic section", and we use our embedding to transport this holomorphic section over to X. Of course, this only defines something in a neighborhood of the point in p (in the rescaled metric), so we introduce a cut-off function β_R and simply extend σ_0 by 0, i.e., if we supress the transporting maps, we define $\sigma_1 := \beta_R\sigma_0$. We have to pay for the crime we've committed here, introducing the cut-off bumps our section out of the kernle of $\overline{\partial}$. We observe, however, the extent of the damage:

$$|\overline{\partial}\sigma_1|^2 \quad = \quad |(\overline{\partial}\beta_R)\sigma_0|^2 \ \leq \ |\overline{\partial}\beta_R|^2 e^{-\frac{1}{4}|z|^2}$$

is mitigated by the fact that σ_0 is Gaussian. We will construct $\sigma := \sigma_1 - \eta$, where η is the correction which ensures that we recover a genuine holomorphic section.

Hodge theory tells us the formula to write down, provided we know that the Hodge Laplacian $\Delta_{\overline{\partial}} = \overline{\partial}^*\overline{\partial} + \overline{\partial}\overline{\partial}^*$ is invertible on $\Omega^{0,1}(L^{\otimes k})$. Indeed, assuming this is the case, the error term is given by $\eta = \overline{\partial}^*\Delta_{\overline{\partial}}^{-1}\overline{\partial}\sigma_1$. Now, if $\eta = \sigma_1$, we have gotten very far, since $\sigma \equiv 0$. The point of this construction is to produce a non-trivial holomorphic section, so we want to show that η is much smaller than σ_1, so that σ defines a genuine non-trivial holomorphic section.

[3] The only reference I'm aware of is the lecture given by Donaldson here concerning the Hörmander technique [**115**].

As in the standard proof of the Kodaira embedding theorem, we will now use a Bochner formula for $\Delta_{\overline{\partial}}$. That is, if we let ∇ denote the connection on $\Omega^{0,1}(L^{\otimes k})$, we can write (with respect to the rescaled metric):

$$\Delta_{\overline{\partial}} \;=\; (\nabla^{0,1})^*\nabla^{0,1} + \frac{1}{k}\mathrm{Ric} + 1.$$

Since particular, by choosing k sufficiently large, we can bound the operator norm

$$\|\Delta_{\overline{\partial}}^{-1}\| \;\leq\; 2.$$

Hence,

$$\|\eta\|_{L^2}^2 \;=\; \langle \overline{\partial}^*\Delta_{\overline{\partial}}^{-1}\overline{\partial}\sigma, \overline{\partial}^*\Delta_{\overline{\partial}}^{-1}\overline{\partial}\sigma\rangle_{L^2} \;=\; \langle \Delta_{\overline{\partial}}^{-1}\overline{\partial}\sigma, \overline{\partial}\sigma\rangle_{L^2} \;\leq\; 2\|\overline{\partial}\sigma\|_{L^2}^2 \;=\; O\left(e^{-\frac{k}{4}|z|^2}\right),$$

and by elliptic estimates, this is enough to achieve the L^∞–estimate. □

Remark A.14. Recall that Nash's embedding theorem states that every Riemannian manifold (M, g) supports an *isometric* embedding into some Euclidean space $\mathbb{R}^n$. This is a substantial strengthening of the Whitney embedding theorem, which gives no control on the metric. We have seen that the Whitney embedding theorem certainly fails in the complex category: no compact complex manifold, for instance, holomorphic embeds into $\mathbb{C}^n$. In fact, those which do support such embeddings form the important class of Stein manifolds. The Kodaira embedding theorem states that any Stein manifold or compact Kähler manifold with a positive line bundle supports an embedding into some $\mathbb{P}^N$. In a sense, this an algebro-geometric and complex-analytic analog of the Whitney embedding theorem. One can therefore ask the daring question of whether such an embedding is also an isometry. Taken literally, the answer is certainly no. Indeed, there is no complex curve in $\mathbb{P}^n$ for which the induced metric has constant negative curvature. If one affords some flexibility, then we have the following beautiful theorem obtained by Tian [**277**, Theorem A]:

Theorem A.15. (Tian). Let (X, ω) be a polarized algebraic manifold with polarization $\mathcal{L} \to X$. Define the Bergman metric $\omega_k := \frac{1}{k}\Phi_k^*\omega_{\mathrm{FS}}$. Then

$$\max_X \left\{ \|\omega_k - \omega\|_\omega, \|\nabla\omega_k - \nabla\omega\|_\omega, \|\nabla^2\omega_k - \nabla^2\omega\|_\omega, \|\mathrm{Rm}(\omega_k) - \mathrm{Rm}(\omega)\|_\omega \right\} \;=\; O\left(\frac{1}{\sqrt{k}}\right),$$

where ∇ is the covariant derivative with respect to ω, Rm denotes the curvature tensor, and $O(1/\sqrt{k})$ means a constant bounded by $C/\sqrt{k}$ with C depending only on the metric ω.

Remark A.16. This theorem implies that ω_k converges to ω in the $\mathcal{C}^2$–topology on the space $\mathrm{Sym}^2(X)$ of all symmetric covariant 2–tensors. This was later improved to $\mathcal{C}^\infty$–convergence by Zelditch [**322**, Corollary 3], which was conjectured by Tian in [**277**, p. 100].

Remark A.17. Kodaira's theorem tells you that for sufficiently large k, the map $\Phi_k = \Phi_{|\mathcal{L}^{\otimes k}|}$ gives you an embedding for any choice of basis of holomorphic sections of $\mathcal{L}^{\otimes k}$. If you pull back the cohomology class of the Fubini–Study metric and multiply by $\frac{1}{k}$, you get $c_1(\mathcal{L})$. Kodaira's theorem says nothing about the pullback of the metric itself. To get a well-defined metric via an embedding, one must specify the basis (up to an action of the unitary group). To recover the original metric you started with, you use an L^2–orthonormal basis of sections for each k, pullback the metric, rescale by $\frac{1}{k}$ and then take a limit as $k \to \infty$.

Corollary A.18. Let X be a compact Kähler manifold such that $H^2(X, \mathcal{O}_X) = 0$. Then X is projective.

Remark A.19. It is common to refer to the above $h^{2,0} = 0$ as the *Kodaira projectivity criterion*. Observe that the vanishing of $h^{2,0} = 0$ not enough if X is not compact Kähler. Indeed, consider a compact complex surface with universal cover $\mathbb{C}^2 \backslash \{0\}$. For instance, quotient $\mathbb{C}^2 \backslash \{0\}$ by the infinite cyclic group generated by the homothety

$$(z_1, z_2) \;\mapsto\; \left(\frac{1}{2} z_1, \frac{1}{2} z_2\right).$$

The resulting compact complex surface $H = (\mathbb{C}^2 \backslash \{0\}) / \sim$ is diffeomorphic to $\mathbb{S}^1 \times \mathbb{S}^3$. It therefore violates the Betti number criterion for supporting a Kähler metric. Since $b_2(H) = 0$, however, $h^{2,0} = 0$.

Bibliography

[1] Abresch, U., Gromoll, D., On complete manifolds with nonnegative Ricci curvature. J. Amer. Math. Soc. 3 (1990), no. 2, 355–374. [Cited on page 128.]

[2] Ahlfors, L. V. , An extension of Schwarz's lemma. Trans. Amer. Math. Soc. 43 (1938), no. 3, 359–364. MR1501949 [Cited on page xi.]

[3] Ahlfors, L. V., Collected papers. Vol. 1. 1929–1955. Edited with the assistance of Rae Michael Shortt. Contemporary Mathematicians. Birkhäuser, Boston, Mass., 1982. xix+520 pp. ISBN: 3-7643-3075-9 MR0688648 [Cited on page 243.]

[4] Ahlfors, L., Grunsky, H., Über die Blochsche Konstante. (German) Math. Z. **42** (1937), no. 1, pp. 671–673. [Cited on page 243.]

[5] Alessandrini, L., Bassanelli, G., Modifications of compact balanced manifolds. C. R. Acad. Sci. Paris Sér. I Math. 320 (1995), no. 12, 1517–1522 [Cited on pages 81 and 113.]

[6] Alexander, J. W., An Example of a Simply Connected Surface Bounding a Region which is not Simply Connected, Proceedings of the National Academy of Sciences of the United States of America, National Academy of Sciences, 10 (1): 8–10, (1924). [Cited on pages 22 and 53.]

[7] Alvarez, A., Chaturvedi, A., Heier, G., Optimal pinching for the holomorphic sectional curvature of Hitchin's metrics on Hirzebruch surfaces, Contemp. Math., **654**, pp. 133–142, (2015). [Cited on page 209.]

[8] Alvarez, A., Heier, G., Zheng, F., On projectivized vector bundles and positive holomorphic sectional curvature, Proc. Amer. Math. Soc., 146(7): pp. 2877–2882, (2018) [Cited on page 274.]

[9] Anderson, M. T., Metrics of negative curvature on Vector Bundles, Proceedings of the American Mathematical Society, Vol. 99, No. 2 (Feb., 1987), pp. 357–363 [Cited on page 125.]

[10] Andrews, B., Lectures on Differential Geometry, available at the author's webpage https://maths-people.anu.edu.au/~andrews/DG/ [Cited on pages viii, 13, 27, and 28.]

[11] Angella, D., Tomsassini, A., On the $\partial\bar{\partial}$–lemma and Bott–Chern cohomology. Invent. Math. **192** (2013), no. **1**, pp. 71–81. [Cited on page 99.]

[12] Apostolov, V., Dloussky, G., Twisted differentials and Lee classes of locally conformally symplectic complex surfaces, arXiv preprint, https://arxiv.org/pdf/2204.02122.pdf [Cited on page 117.]

[13] Apostolov, V., Davidov, J., Muskarov, O., Compact self-dual Hermitian surfaces. Trans Amer Math Soc, 1996, 348: 3051–3063 [Cited on page 197.]

[14] Artin, M., Mumford, D., Some elementary examples of unirational varieties which are not rational. Proc. London Math. Soc. (3) 25 (1972), 75–95. [Cited on pages 108 and 109.]

[15] Aubin, T., Équations du type Monge–Ampère sur les varietés kählériennes compactes, C. Rendus Acad. Sci. Paris **283** (1976), pp. 119–121. [Cited on pages 113, 222, 253, 255, and 261.]

[16] Aviles, P., McOwen, R. C., Conformal deformation to constant negative scalar curvature on noncompact Riemannian manifolds, J. Diff. Geom. **27** (1988), pp. 225–239. [Cited on page 131.]

[17] Balas, A., Compact Hermitian manifolds of constant holomorphic sectional curvature, Math. Zeit., 189 (1985), 193–210. [Cited on pages 199 and 201.]

[18] Balas, A., Gauduchon, P., Any Hermitian metric of constant nonpositive (Hermitian) holomorphic sectional curvature on a compact complex surface is Kähler, Math. Zeit., 190 (1985), pp. 39–43. [Cited on page 197.]

[19] Barbaro, G., On the curvature of the Bismut connection: Bismut Yamabe problem and Calabi-Yau with torsion metrics, arXiv:2109.06159 [Cited on page 153.]

[20] Barth, W. P., Convex domains and Kobayashi hyperbolicity. Proc. Amer. Math. Soc. 79(4), 556–558 (1980) [Cited on page 38.]

[21] Barth, W. P., Hulek, K., Peters, C. A. M., Van de Ven, A., Compact Complex Surfaces, Second Enlarged Edition, Vol 4, A Series of Modern Surveys in Mathematics, Springer-Verlag Berlin Heidelberg 2004 [Cited on pages viii, 101, 106, 107, 114, and 115.]

[22] Beauville, A., L'application canonique pour les surfaces de type général. (French) Invent. Math. 55 (1979), no. 2, 121–140. [Cited on pages 176 and 177.]

[23] Belmans, P., Commelin, J., *le superficie algebriche*, https://superficie.info [Cited on page 177.]

[24] Berger, M., Sur les variétés d'Einstein compactes. In Comptes Rendus de la IIIe Réunion du Groupement des Mathématiciens d'Expression Latine (Namur, 1965), pages 35–55. Librairie Universitaire, Louvain, 1966. [Cited on page 173.]

[25] Bettiol, R. G., On different notions of positivity of curvature, Ph.D. thesis at the University of Notre Dame, available online at https://lehman.edu/faculty/rbettiol/files/Bettiol-Thesis.pdf [Cited on page 127.]

[26] Bin, N., A new example of an algebraic surface with canonical map of degree 16. Arch. Math. (Basel) 113 (2019), no. 4, 385–390. [Cited on page 177.]

[27] Bingyuan, L., Two applications of the Schwarz lemma. Pacific J. Math. 296 (2018), no. 1, 141–153. [Cited on page 178.]

[28] Bishop, R. L., Goldberg, S. I., On the second cohomology group of a Kaehler manifold of positive curvature, Proc. Amer. Math. Soc. 16 (1965), 119–122. [Cited on pages 183 and 186.]

[29] Biswas, I., McKay, B., Holomorphic Cartan geometries and rational curves. Complex Manifolds 3 (2016), no. 1, 145–168 [Cited on page 82.]

[30] Biquard, O., Höring, A., Kähler geometry and Hodge theory, expository paper [Cited on page viii.]

[31] Bismut, J.-M., A local index theorem for non-Kähler manifolds, Math. Ann., **284**, 4 (1989), pp. 681–699. [Cited on page 149.]

[32] Bloch, A., Les théorèmes de M. Valiron sur les fonctions entières et la théorie de l'uniformisation. (French) Ann. Fac. Sci. Toulouse Sci. Math. Sci. Phys. (3) 17 (1925), 1–22. MR1508386 [Cited on page 241.]

[33] Bloch, A., La conception actuelle de la théorie des fonctions entières et méromorphes, L'Enseignement Mathématique, **25** (1926) https://zbmath.org/?q=an:52.0315.03 [Cited on page 240.]

[34] Bochner, S., Montgomery, D., Groups on analytic manifolds. Ann. of Math. (2) 48 (1947), 659–669 [Cited on page 182.]

[35] Bogomolov, F., Classification of surfaces of class VII_0 with $b_2 = 0$. Math. USSR Izv 10, p. 255–269 (1976). [Cited on page 116.]

[36] Bogomolov, F., Surfaces of class VII_0 and affine geometry, Math. USSR Izv., 21, p. 31–73 (1983). [Cited on page 116.]

[37] Böhm, C., Inhomogeneous Einstein metrics on low-dimensional spheres and other low-dimensional spaces, Invent. Math. 134 (1998), 145–176. [Cited on page 129.]

[38] Böhm, C., Wilking, B., Manifolds with positive curvature operators are space forms. Ann. of Math. (2) 167 (2008), no. 3, 1079–1097 [Cited on page 123.]

[39] Bonk, M., On Bloch's constant. Proc. Amer. Math. Soc. 110 (1990), no. 4, pp. 889–894 [Cited on page 243.]

[40] Boothby, W., Hermitian manifolds with zero curvature, Michigan Math. J. 5 (1958), no. 2, pp. 229–233. [Cited on page 198.]

[41] Boothby, W., An Introduction to Differentiable Manifolds and Riemannian Geometry, Revised Second Edition, Pure and Applied Mathematics, 120. Academic Press, Inc., Orlando, FL, 1986. [Cited on page viii.]

[42] Boucksom, S., Demailly, J.-P., Paun, M., Peternell, T., The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension, J. Alg. Geom., 22(2), pp. 201–248, (2013). [Cited on page 160.]

[43] Bourguignon, J.-P., Karcher, H., Curvature operators: pinching estimates and geometric examples, Ann. Sci. Ećole Norm. Sup. 11 (1978), 71–92. [Cited on page 129.]

[44] Boy, W., Über die Curvatura integra und die Topologie geschlossener Flächen, Math. Ann. **57**, pp. 151–184, (1903). [Cited on page 6.]

[45] Boyer, C. P., Galicki, K., Kollár, J., Einstein metrics on spheres, Annals of Mathematics, **162** (2005), pp. 557–580. [Cited on page 129.]

[46] Bredon, G. E., Sheaf theory, Second Edition, Springer, Graduate Texts in Mathematics, 170, 1997. [Cited on pages viii, 51, and 54.]

[47] Broder, K., Twisted Kähler–Einstein metrics and collapsing, arXiv:2003.14009 (submitted). [Cited on pages v and vii.]

[48] Broder, K., Second-order estimates for collapsed limits of Ricci-flat Kähler metrics, arXiv:2106.13343 (submitted). [Cited on pages v and vii.]

[49] Broder, K., The Schwarz lemma in Kähler and non-Kähler geometry, arXiv:2109.06331 (submitted). [Cited on pages vii, xv, 216, 219, 224, 244, and 253.]

[50] Broder, K., The Schwarz lemma: An Odyssey, arXiv:2110.04989, (to appear in the Rocky Mountain Journal of Mathematics). [Cited on pages vii, xv, 216, 219, 224, 244, and 253.]

[51] Broder, K., On the nonnegativity of the Dirichlet energy of a weighted graph, Bull. of the Aust. Math. Soc., 1-5. doi:10.1017/S0004972721001015 [Cited on pages vii and 184.]

[52] Broder, K., An eigenvalue characterization of the dual EDM cone, Bull. of the Aust. Math. Soc., 1-3. doi:10.1017/S0004972721000915 [Cited on pages vii, 184, and 185.]

[53] Broder, K., Remarks on the Quadratic Orthogonal Bisectional Curvature, (to appear in the Journal of Geometry). [Cited on pages vii and 184.]

[54] Broder, K., Lectures on Vector Calculus, (Book). [Cited on pages vi, vii, 21, 22, and 53.]

[55] Broder, K., Diverio, S., Remarks on the Wu–Yau theorem, (in preparation). [Cited on pages vii, xvii, and 274.]

[56] Broder, K., Rubinstein, Y., Curvature aspects of the hypothetical complex structure on $\mathbb{S}^6$ (in preparation). [Cited on page vii.]

[57] Broder, K., Stanfield, J., On the Gauduchon curvature of Hermitian manifolds, (in preparation). [Cited on pages vii, viii, xiii, xvi, 200, 220, 244, 256, and 257.]

[58] Broder, K., Stanfield, J., On the Bochner technique in the Hermitian category with applications to the Wu–Yau theorem (in preparation). [Cited on pages vii, viii, xiv, xvi, 220, 227, 244, 256, and 257.]

[59] Broder, K., Tang, K., On the weighted orthogonal Ricci curvature, arXiv:2111.00346 (submitted). [Cited on page vii.]

[60] Broder, K., Tang, K., On the altered holomorphic curvatures of Hermitian manifolds, arXiv:2201.03666 (submitted). [Cited on pages vii, 187, 189, 199, 201, and 210.]

[61] Broder, K., Tang, K., Zhang, Y., (ε, δ)–quasi-negativity and positivity of the canonical bundle, (in preparation). [Cited on pages vii, xvii, 265, 266, and 271.]

[62] Brody, R., Compact Manifolds and Hyperbolicity, Transactions of the American Mathematical Society Vol. 235 (Jan., 1978), pp. 213–219 [Cited on pages xi and 37.]

[63] Brunebarbe, Y., Klinger, B., Totaro, B., Symmetric differentials and the fundamental group, [Cited on page 206.]

[64] Brunella, M., A characterization of hyperbolic Kato surfaces, Publ. Mat. **58** (2014), pp. 251–261. [Cited on page 117.]

[65] Bryant, R., Bochner-Kähler metrics, J. Amer. Math. Soc. 14 (2001), pp. 623–715 [Cited on page 275.]

[66] Buchdahl, N., On compact Kähler surfaces, Ann. Inst. Fourier **49** (1999), pp. 287–302. [Cited on page 114.]

[67] Burns, K., Masur, H., Matheus, C., Wilkinson, A., Rates of mixing for the Weil–Petersson geodesic flow I: No rapid mixing in non-exceptional moduli spaces, Advances in Mathematics, vol 306, pp. 589–602, (2017) [Cited on page 240.]

[68] Calabi, E., On Kähler manifolds with vanishing canonical class, Algebraic geometry and topology. A symposium in honor of S. Lefschetz, Princeton University Press, Princeton, N. J., 1957, pp. 78–89 [Cited on pages xi, 158, and 259.]

[69] Calabi, E., Eckmann, B., A class of compact complex manifolds which are not algebraic, Ann. of Math., **58** (1953), pp. 494–500. [Cited on page 34.]

[70] Campana, F., Twistor spaces and nonhyperbolicity of certain symplectic Kähler manifolds, In Complex analysis (Wuppertal 1991), Aspects Math., E17, pp. 64–69. Friedr. Veiweg, Braunschweig, 1991. [Cited on pages xii and 246.]

[71] Campana, F., Demailly, J.-P., Peternell, T., Rationally connected manifolds and semipositivity of the Ricci curvature. In Recent advances in algebraic geometry, volume 417 of London Math. Soc. Lecture Note Ser., p. 71–91. Cambridge Univ. Press, Cambridge, 2015. [Cited on page 160.]

[72] Campbell, L. A., Howard, A., Ochiai, T., Moving holomorphic disks off analytic subsets. Proc. Amer. Math. Soc. 60 (1976), 106–108 (1977). [Cited on page 38.]

[73] Candel, A., Uniformization of surface laminations, Ann. Sci. École Norm. Sup. (4) **26** (1993), no. 4, pp. 489–516. [Cited on page 116.]

[74] Cao, H.-D., Hamilton, R., Gradient Kähler-Ricci solitons and periodic orbits. Comm. Anal. Geom. 8 (2000), no. 3, 517–529 [Cited on page 182.]

[75] Cao, X., Yang, B., A note on the almost-one-half holomorphic pinching. C. R. Math. Acad. Sci. Paris 355 (2017), no. 10, 1094–1098 [Cited on page 203.]

[76] Caratheòdory, C., Sur quelques applications du théorème de Landau-Picard, C. R. Acad. Sci. Paris **144** (1907), 1203–1206, [Cited on page 240.]

[77] Cascini, P., Rational curves on complex manifolds, Milan J. Math. 81 (2013), no. 2, 291–315. [Cited on page 82.]

[78] Ceballos, M., Otal, A., Ugarte, L., Villacampa, R., Invariant complex structures on 6-nilmanifolds: classification, Frölicher spectral sequence and special Hermitian metrics. J. Geom. Anal. 26 (2016), no. 1, 252–286. [Cited on page 99.]

[79] Chaturvedi, A., Heier, G., Hermitian metrics of positive holomorphic sectional curvature on fibrations, Math. Z. 295 (2020), no. 1-2, 349–364 [Cited on pages xi and 249.]

[80] Chau, A., Tam, L.-F., Kähler C–spaces and quadratic bisectional curvature. J. Differential Geom. 94 (2013), no. 3, 409–468 [Cited on page 184.]

[81] Chau, A., Tam, L.-F., On quadratic orthogonal bisectional curvature. J. Differential Geom. 92 (2012), no. 2, 187–200 [Cited on page 184.]

[82] Cheeger, J., Gromoll, D., On the structure of complete manifolds of nonnegative curvature, Annals of Mathematics, second series, **96** (3), pp. 413–443, 1972. [Cited on page 126.]

[83] Cheltsov, I., Shramov, C. With an appendix by J.-P. Demailly. Log canonical thresholds for nonsingular Fano threefolds. Uspekhi Mat. Nauk, **63** (5), pp. 73–180, 2009. [Cited on page 271.]

[84] Chen, X. X., On Kähler manifolds with positive orthogonal bisectional curvature, Adv. Math. 215 (2007), no. 2, 427–445 [Cited on page 182.]

[85] Chen, H., Chen, L., Nie, X., Chern-Ricci curvatures, holomorphic sectional curvature and Hermitian metrics, Sci China Math, **63**, (2020). [Cited on pages 165, 197, and 198.]

[86] Chen, Z. H., Cheng, S. Y., Lu, Q. K., On the Schwarz lemma for complete Kähler manifolds. Sci. Sinica 22 (1979), no. 11, 1238–1247. [Cited on page 225.]

[87] Chen, X., Donaldson, S., Sun, S., Kähler–Einstein metrics on Fano manifolds. I: Approximation of metrics with cone singularities. J. Amer. Math. Soc. 28 (2015), no. 1, 183–197 [Cited on page 159.]

[88] Chen, H., Gauthier, P. M., On Bloch's constant, J. Anal. Math. **69** (1996), pp. 275–291. [Cited on page 243.]

[89] Chen, H., Nie, X., Compact Hermitian surfaces with pointwise constant Gauduchon holomorphic sectional curvature, arXiv:2201.13083v2. [Cited on page 152.]

[90] Chen, X., Sun, S., Tian, G., A note on Kähler-Ricci soliton. Int. Math. Res. Not. IMRN 2009, no. 17, 3328–3336 [Cited on page 181.]

[91] Cheng, S. Y. Eigenvalue comparison theorems and its geometric applications, Mathematische Zeitschrift, 143 (3): 289–297, 1975. [Cited on page 129.]

[92] Chern, S.-S., On holomorphic mappings of hermi- tian manifolds of the same dimension. In Entire Functions and Related Parts of Analysis (Proc. Sympos. Pure Math., La Jolla, Calif., 1966). Amer. Math. Soc., Providence, R.I. 1968. pp. 157–170. [Cited on pages 217 and 238.]

[93] Cherrier, P., Équations de Monge-Ampère sur les variétés hermitiennes compactes, Bull. Sci. Math. (2) 111 (1987), no. 4, 343–385 [Cited on pages 252, 254, and 261.]

[94] Cheung, C.-K., Hermitian metrics of negative holomorphic sectional curvature on some hyperbolic manifolds, Math. Z., 201(1): pp. 105–119, 1989 [Cited on pages xi and 249.]

[95] Cheung, C.-K., Negative holomorphic sectional curvature and hyperbolic manifold, Ph.D. thesis, Berkeley, 1988 [Cited on pages xi, 177, 179, 249, and 275.]

[96] Chiose, I., Obstructions to the existence of Kähler structures on compact complex manifolds. Proc. Amer. Math. Soc. 142 (2014), no. 10, 3561–3568. [Cited on pages 82, 113, and 164.]

[97] Clemens, C., Griffiths, P., The intermediate Jacobian of the cubic threefold. Ann. of Math. (2) 95 (1972), 281–356. [Cited on page 109.]

[98] Correa, E. M., Levi-Civita Ricci-flat metrics on non-Kähler Calabi-Yau manifolds, arXiv:2204.04824v3 [Cited on pages xiii and 152.]

[99] Cowen, M. J., Families of negatively curved Hermitian manifolds, Proceedings of the AM.S. **39**, no. 2, pp. 362–366. [Cited on pages xi and 249.]

[100] Dattorro, J., Equality relating Euclidean distance cone to positive semidefinite cone, Linear Algebra Appl. 428 (11–12) (2008), 2597–2600. [Cited on page 186.]

[101] Dattorro, J., Convex Optimization & Euclidean Distance Geometry 2ε (MeBoo, Palo Alto, CA, 2019), available at the author's webpage at https://convexoptimization.com/TOOLS/0976401304.pdf. [Cited on page 186.]

[102] Debarre, O., Higher-dimensional Algebraic Geometry, Universitext, pp. xiv+233. Springer, New York (2001) [Cited on pages 108 and 109.]

[103] Deligne, P., Griffiths, P., Morgan, J., Sullivan, D., Real homotopy theory of Kähler manifolds. Invent. Math. **29** (1975), no. 3, pp. 245–274. [Cited on page 99.]

[104] Demailly, J.-P., Complex analytic and Differential Geometry, availabe online at https://www-fourier.ujf-grenoble.fr/~demailly/manuscripts/agbook.pdf [Cited on page 90.]

[105] Demailly, J.-P., Algebraic criteria for Kobayashi hyperbolic projective varieties and jet differentials, Algebraic geometry–Santa Cruz 1995, Proc. Sympos. Pure Math., vol. **62**, Amer. Math. Soc., Providence, RI, 1997, pp. 285–360. [Cited on pages xii, 250, 251, and 257.]

[106] Demailly, J. -P., Analytic Methods in Algebraic Geometry, SMM1, Surveys of Modern Mathematics, International Press (2010). [Cited on page 102.]

[107] Demailly, J.-P., Applications of Pluripotential Theory to Algebraic Geometry, in Pluripotential Theory, Lecture Notes in Mathematics 2075, DOI 10.1007/978-3-642-36421-1,3, Springer-Verlag Berlin Heidelberg 2013 [Cited on page 248.]

[108] Demailly, J.-P., Peternell, T., Schneider, M., Compact complex manifolds with numerically effective tangent bundles, J. Alg. Geom. **3** (1994) 295–345. [Cited on page 102.]

[109] Deschamps-Martin, M., Propriétés de descente des variétés à fibré cotangent ample. Ann. Inst. Fourier (Grenoble) **33**, pp. 39–64 (1985). [Cited on pages xi and 249.]

[110] Diverio, S., Kobayashi hyperbolicity, negativity of the curvature and positivity of the canonical bundle, https://arxiv.org/abs/2011.11379, to appear in a forthcoming monograph of the SMF in the series "Panoramas et Synthèses". [Cited on pages 173, 250, 251, and 257.]

[111] Diverio, S., Trapani, S., Quasi-negative holomorphic sectional curvature and positivity of the canonical bundle, J. Differential Geom. **111** (2019), no. 2, 303–314. [Cited on pages xvi and 265.]

[112] Dloussky, G., Structure des surfaces de Kato, Mémoire de la S.M.F. $2^{\text{ème}}$ série, **14** (1984). [Cited on pages 116 and 117.]

[113] Dloussky, G., Oeljeklaus, Toma, Class VII_0 surfaces with b_2 curves, Tohoku Math. J. **55** (2003), pp. 283–309. [Cited on page 117.]

[114] Do Carmo, M., Riemannian geometry. Translated from the second Portuguese edition by Francis Flaherty. Mathematics: Theory and Applications. Birkhäuser Boston, Inc., Boston, MA, 1992. xiv+300 pp. ISBN: 0-8176-3490-8 [Cited on pages viii, 12, 124, 125, and 129.]

[115] Donaldson, S., [Cited on page 280.]

[116] Eilenberg, S., Steenrod, N., Foundations of algebraic topology. Princeton University Press, Princeton, N.J., 1952. xv+328 pp [Cited on page 63.]

[117] Enoki, I., On surfaces of class VII_0 with curves, Japan Academy. Proceedings. Series A. Math. Sci., **56** (6), pp. 275–279, (1980). [Cited on page 117.]

[118] Fan, X.-Q., Tam, L.-F., Yu, C., Product of almost-Hermitian manifolds. J. Geom. Anal. 24 (2014), no. 3, 1425–1446. [Cited on page 178.]

[119] Fang, S., Tosatti, V., Weinkove, B., Zheng, T., Inoue surfaces and the Chern–Ricci flow, J. Funct. Anal. 271 (2016), no. 11, 3162–3185. [Cited on page 116.]

[120] Ferlengez, B., Granja, G., Milivojevic, A., On the topology of the space of almost complex structures on the six sphere, [Cited on page 8.]

[121] Fino, A., Vezzoni, L., Special Hermitian metrics on compact solvmanifolds, J. Geom. Phys. **91** (2015), pp. 40–53. [Cited on pages 79 and 117.]

[122] Fischer, W., Grauert, H., Lokal-triviale Familien kompakter komplexer Mannigfaltigkeiten. In: Nachr. Akad. Wiss. Göttingen Math.-Phys. Kl. II 1965 [Cited on page 112.]

[123] Forstnerič, F., Developments in Oka theory since 2017, arXiv:2006.07888v1 [Cited on page 182.]

[124] Frankel, T., Manifolds with positive curvature. Pacific J. Math. 11 (1961), 165–174 [Cited on pages 125 and 180.]

[125] Freedman, M. H., The topology of four-dimensional manifolds, Journal of Differential Geometry, **17** (3): pp. 357–453. [Cited on page 3.]

[126] Fu, J., Li, J., Yau, S.-T., Balanced metrics on non-Kähler Calabi–Yau threefolds, J. Diff. Geom. **90** (2012), pp. 81–129. [Cited on page 80.]

[127] Fu, J.-X., Zhou, X., Scalar curvatures in almost Hermitian geometry and some applications, arXiv:1901.10130. [Cited on page 152.]

[128] Futaki, A., An obstruction to the existence of Einstein–Kähler metrics, Inv. Math., **73**, pp. 437–443 (1983). [Cited on page 159.]

[129] Gao, L. Z., The construction of negatively Ricci curved manifolds, Math. Ann. **271** (1985), pp. 185–208 [Cited on page 128.]

[130] Gao, L. Z., Yau, S.-T., The existence of negatively Ricci curved manifolds on three manifolds, Invent. Math. **85** (1986), pp. 637–652. [Cited on page 128.]

[131] Gauduchon, P., Le théorème de l'excentricité nulle. C. R. Acad. Sci. Paris Sér. A-B 285 (1977), no. 5, A387–A390 [Cited on page 79.]

[132] Gauduchon, P., La 1-forme de torsion d'une variété hermitienne compacte, Math. Ann. **267** (1984), no. 4, pp. 495–518. [Cited on pages 97 and 169.]

[133] Gauduchon, P., Hermitian connections and Dirac operators, Boll. Un. Mat. Ital. B (7) 11 (1997), no. 2, suppl., 257–288 [Cited on pages 148 and 149.]

[134] Gauduchon, P., Calabi's extremal Kähler metrics: An elementary introduction, available online https://cims.nyu.edu/~rodion/lib/P. [Cited on page viii.]

[135] Goldberg, S., Kobayashi, S., Holomorphic bisectional curvature. J. Differential Geometry 1 (1967), 225–233. MR0227901 [Cited on pages 172, 180, and 181.]

[136] Grauert, H., Reckziegel, H., Hermitesche metriken und nortnale familien holomorpher abbildungen, Math. Z. **89** (1965) pp. 108–125. [Cited on pages xi, 194, and 249.]

[137] Grauert, H., Remmert, R., Theory of Stein spaces, volume 236 of Grundlehren Math. Wiss. Springer-Verlag, Berlin-New York, 1979. Translated from German by Alan Huckleberry. [Cited on pages viii, ix, 35, and 59.]

[138] Grauert, H., Remmert, R., Coherent Analytic Sheaves, Grundlehren der mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 265. Springer-Verlag, Berlin, 1984. xviii+249 pp. ISBN: 3-540-13178-7 [Cited on pages viii, 56, and 64.]

[139] Gray, A., Vanhecke, L., Almost Hermitian manifolds with constant holomorphic sectional curvature. Casopis Pest. Mat. 104 (1979), no. 2, 170–179 [Cited on page 197.]

[140] Greene, R. E., Wu, H., Function Theory on Manifolds Which Possess a Pole, Lecture Notes in Mathematics, 699, Springer-Verlag Berlin Heidelberg New York, 1979. [Cited on pages 179 and 204.]

[141] Greene, R. E., Wu, H., Some function-theoretic properties of noncompact Kähler manifolds, Proc. of Symposia in Pure. Math., vol. 27, 1975. [Cited on page 182.]

[142] Griffiths, P., Harris, J., Principles of Algebraic Geometry, Wiley Classics Library, (1978). [Cited on pages viii, 64, 69, 70, 145, and 280.]

[143] Grigorýan, A., Masamune, J., Parabolicity and stochastic completeness of manifolds in terms of the Green formula, Journal de Mathématiques Pures et Appliquées, vol. 100, no. 5, (2013), pp. 607–632. [Cited on page 240.]

[144] Gromoll, D., Meyer, W. T., Examples of complete manifolds with positive Ricci curvature. J. Differential Geom. 21 (1985), no. 2, 195–211 [Cited on page 128.]

[145] Gromov, M., Lawson, H. B., The classification of simply connected manifolds of positive scalar curvature, Ann. of Math. (2) **111** (1980), no. 3, pp. 423–434. [Cited on page 131.]

[146] Gromov, M., Partial Differential Relations, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], 9. Springer-Verlag, Berlin, 1986. x+363 pp. ISBN: 3-540-12177-3 [Cited on pages 125 and 128.]

[147] Grove, K., Verdiani, L., Ziller, W., An exotic $T_1\mathbb{S}^4$ with positive curvature, Geom. Funct. Anal., **21** (2011), pp. 499–524. [Cited on page 127.]

[148] Gruel, G.M., Lossen, C., Shustin, E., Introduction to Singularities and Deformations, Springer, (2007). [Cited on page 64.]

[149] Gu, H., Zhang, Z., An extension of Mok's theorem on the generalized Frankel conjecture. Sci. China Math. **53**, 1–12 (2010). [Cited on page 183.]

[150] Guan, B., Li, Q., Complex Monge–Ampère equations and totally real submanifolds, Adv. Math. (2010). [Cited on page 261.]

[151] Gunning, R. C. , Rossi, H., Analytic functions of several complex variables, Americal Mathematical Society, (1965). [Cited on pages viii, 47, 49, 60, and 61.]

[152] Hartshorne, R., Algebraic geometry. Graduate Texts in Mathematics, No. 52. Springer–Verlag, New York-Heidelberg, 1977. xvi+496 pp. ISBN: 0-387–90244–9 [Cited on pages viii, 51, 53, 54, 56, and 100.]

[153] Hartshorne, R., Ample vector bundles. Inst. Hautes Études Sci. Publ. Math. No. 29 (1966), 63–94. [Cited on page 174.]

[154] Hehl, F. W., Obukhov, Y. N., Élie Cartan's torsion in geometry and in field theory, as essay, Ann. Fond. Louis de Broglie, 32(2-3), 157–194 (2007) [Cited on page 121.]

[155] Heier, G., Lu, S., Wong, B., On the canonical line bundle and negative holomorphic sectional curvature, Math. Res. Lett. **17** (2010), no. 06, pp. 1101–1110 [Cited on pages xii and 247.]

[156] Heinz, T. F., Topological Properties of Orthostochastic Matrices. Linear algebra and its applications 20, 265–269 (1978) [Cited on page 185.]

[157] Hironaka, H., ON THE THEORY OF BIRATIONAL BLOWING-UP. Thesis (Ph.D.)–Harvard University. 1960. [Cited on page 81.]

[158] Hirzebruch, F., Chern numbers of algebraic surfaces: an example, Math. Ann., **266**, no. 3 (1984), pp. 351–356. [Cited on page 177.]

[159] Hirzebruch, F., Topological methods in algebraic geometry, Translated from the German and Appendix One by R. L. E. Schwarzenberger, With a preface to the third English edition by the author and Schwarzenberger, Appendix Two by A. Borel, Reprint of the 1978 edition, Classics in Mathematics. Springer-Verlag, Berlin, 1995. [Cited on page 34.]

[160] Hitchin, N., Harmonic spinors, Advances in Math. **14** (1974), pp. 1–55. [Cited on page 130.]

[161] Hitchin, N., *On the curvature of rational surfaces*, Differential geometry (Proc. Sympos. Pure Math., Vol. XXVII, Part 2, Stanford Univ., Stanford, Calif., 1973), pp. 65–80. Amer. Math. Soc., Providence, R. I., 1975 [Cited on pages xvii, 106, 107, 208, 209, 273, and 274.]

[162] Hitchin, N., Kählerian twistor spaces, Proc. London Math. Soc. (3) **43** (1981), pp. 133–150. [Cited on page 79.]

[163] Hopf, H., Zur Topologie der komplexen Mannigfaltigkeiten, Studies and Essays presented to R. Courant, Interscience, New York, 1948, pp. 167–185. [Cited on pages 75 and 115.]

[164] Horikawa, E., Algebraic surfaces of general type with small C_1^2. I. Ann. of Math. (2) 104 (1976), no. 2, 357–387. [Cited on page 275.]

[165] Horikawa, E., Algebraic surfaces of general type with small c_1^2. II. Invent. Math. 37 (1976), no. 2, 121–155. [Cited on page 275.]

[166] Horikawa, E., Algebraic surfaces of general type with small c_1^2. III. Invent. Math. 47 (1978), no. 3, 209–248. [Cited on page 275.]

[167] Horikawa, E., Algebraic surfaces of general type with small c_1^2. IV. Invent. Math. 50 (1978/79), no. 2, 103–128. [Cited on page 275.]

[168] Huybrechts, D., Complex Geometry: an introduction, Universitext, Springer-Verlag, Berlin (2005). [Cited on pages viii and 40.]

[169] Inoue, M., New surfaces with no meromorphic functions. II, Complex analysis and algebraic geometry, Tokyo, Iwanami Shoten, pp. 91–106, (1977). [Cited on page 117.]

[170] Iskovskikh, V. A., Manin, Ju. I., Three-dimensional quartics and counterexamples to the Lüroth problem. (Russian) Mat. Sb. (N.S.) 86(128) (1971), 140–166. [Cited on pages 108 and 109.]

[171] Ivey, [Cited on page 183.]

[172] Jensen, G. R., Einstein metrics on principal fibre bundles, J. Differential Geom. 8 (1973), 599–614. [Cited on page 129.]

[173] Kashiwara, M., Schapira, P., Sheaves on Manifolds, Springer, (1992). [Cited on pages 51, 53, and 54.]

[174] Kato, M., Compact complex manifolds containing global spherical shells, Proc. Japan Acad. 53(1), 1516 (1977) [Cited on page 116.]

[175] Kato, M., Compact complex manifolds containing global spherical shells. I, Proceedings of the International Symposium on Algebraic Geometry (Kyoto Univ., Kyoto, 1977), Kinokuniya Book Store, Tokyo, 1978, pp. 4584 [Cited on page 116.]

[176] Kato, M., On a certain class of nonalgebraic non-Kähler compact complex manifolds, Recent progress of algebraic geometry in Japan, North-Holland. Math. Stud. 73, North-Holland Amsterdam, 1983, pp. 2850 [Cited on page 116.]

[177] Kähler, E., Über eine bemerkenswerte Hermitische Metrik. Abh. Math. Sem. Hamburg, **9** (1933), pp. 173–186. [Cited on page xi.]

[178] Kazdan, J., Prescribing the curvature of a Riemannian manifold, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, American Mathematical Society, Providence, Rhode Island., (1984). [Cited on page 131.]

[179] Kazdan, J., Warner, F., 'Prescribing curvature', Proc. of Symposia in Pure Math. 27 (1975) 309–319 [Cited on page 131.]

[180] Kirchoff, A., Sur l'existence des certains champs tensoriels sur les sphères à n dimensions. (French) C. R. Acad. Sci. Paris 225 (1947), 1258–1260 [Cited on page 43.]

[181] Kleiman, S., Ample vector bundles in surface, Proc. of the AMS (1969), pp. 673–676. [Cited on page 176.]

[182] Kleiman, S., Misconceptions about K_X, Enseign. Math. (2) 25 (1979), no. 3-4, 203–206 (1980) [Cited on page 56.]

[183] Klembeck, P. F., A complete Kähler metric of positive curvature on $\mathbb{C}^n$. Proc. Amer. Math. Soc. 64 (1977), no. 2, 313–316 [Cited on pages 179 and 182.]

[184] Kobayashi, S., Intrinsic metrics on complex manifolds. Bull. Amer. Math. Soc., **73** (1967), pp. 347–349 [Cited on pages xii, 36, and 38.]

[185] Kobayashi, S., Nomizu, K., Foundations of Differential Geoemtry, Volume II, Wiley Classics Library Edition, (1996). [Cited on pages 154 and 232.]

[186] Kodaira, K., A certain type of irregular algebraic surfaces, Journal d'Analyse Math. **19** (1967), pp. 207–215. [Cited on pages xi and 114.]

[187] Kóllar, J., Miyaoka, Y., Mori, S., Rational connectedness and boundedness of Fano manifolds, J. Diff. Geom. **36** (3), pp. 765–779, (1992). [Cited on pages 108, 109, and 247.]

[188] Lárusson, F., What is ... an Oka manifold? Notices Amer. Math. Soc. 57 (2010), no. 1, 50–52 [Cited on pages viii, ix, and 35.]

[189] Lazersfeld, R., Positivity in Algebraic Geometry, I. Classical Setting: Line Bundles and Linear Series, Springer, (2003). [Cited on page 114.]

[190] Lee, M. C., Streets, J., Complex manifolds with negative curvature operator, Int. Math. Res. Not, arXiv: 1903.12645. [Cited on page 209.]

[191] Li, Q., Wu, D., Zheng, F., An example of compact Kähler manifold with non-negative quadratic bisectional curvature, Proc. Amer. Math. Soc. 141 (2013), no. 6, 2117–2126. MR3034437 [Cited on page 184.]

[192] Li, J., Yau, S.-T., Zheng, F., On projectively flat Hermitian manifolds, Comm. in Analysis and Geometry, 2, p. 103–109 (1994). [Cited on page 116.]

[193] Li, Y., Zheng, F., Complex nilmanifolds with constant holomorphic sectional curvature, arXiv:2103.09571. [Cited on page 198.]

[194] Libermann, P., Sur les structures presque-complexes et autres structures infinitésimales régulières, Bull. Soc. Math. France, **83** (1955), pp. 195–224. [Cited on page 149.]

[195] Libermann, P., Classification and conformal properties of almost hermitian structures, Colloquia Mathematica Societatis János Bolyai, Differential Geometry **31**, Budapest (1979). [Cited on page 149.]

[196] Lichnerowicz, A., Théorie globale des connexions et des groupes d'holonomie, Edizioni Cremonese, Roma (1962). [Cited on page 148.]

[197] Lichnerowicz, A., Spineurs harmoniques, C. R. Acad. Sci. Paris **257** (1963), pp. 7–9. [Cited on page 130.]

[198] Liu, G., Compact Kähler manifolds with nonpositive bisectional curvature. Geom. Funct. Anal. 24 (2014), no. 5, 1591–1607 [Cited on page 179.]

[199] Liu, K., Yang, X., Geometry of Hermitian manifolds. Internat. J. Math. 23 (2012), no. 6, 1250055, 40 pp [Cited on page 152.]

[200] Liu, K., Yang, X., Ricci curvatures on Hermitian manifolds. Trans. Amer. Math. Soc. 369 (2017), no. 7, 5157–5196. MR3632564 [Cited on pages xiii, 144, 152, 161, 162, 166, and 168.]

[201] Liu, K., Yang, X., Minimal Complex Surfaces with Levi–Civita Ricci-flat Metrics, Acta Mathematica Sinica, English Series, Aug., 2018, Vol. 34, No. 8, pp. 1195–1207 [Cited on page 152.]

[202] Lohkamp, J., Metrics of negative Ricci curvature, Annals of Mathematics, Second Series, Vol. 140, No. 3 (Nov., 1994), pp. 655–683. [Cited on page 128.]

[203] Lu, Y.-C., Holomorphic mappings of complex manifolds. J. Differential Geometry 2 (1968), 299–312 [Cited on pages 217 and 238.]

[204] Matsushima, Y., Sur la structure du group d'homeomorphismes analytiques d'une certaine varietie Kahleriennes. Nagoya Math. J., **11**, pp. 145–150 (1957). [Cited on page 159.]

[205] Michelsohn, M. L., On the existence of special metrics in complex geometry, Acta Math. **149** (1982), pp. 261–295. [Cited on pages 79, 80, and 154.]

[206] Milnor, J., On manifolds homeomorphic to the 7-sphere, Ann. of Math. **64** (1956), 399–405. [Cited on page 4.]

[207] Miranda, R., Algebraic curves and Riemann surfaces. Graduate Studies in Mathematics, 5. American Mathematical Society, Providence, RI, 1995. xxii+390 pp. ISBN: 0-8218-0268-2 [Cited on page 65.]

[208] Miyaoka, Y., On the Chern numbers of surfaces of general type, Invent. Math. **42**, (1977), pp. 225–237. [Cited on page 177.]

[209] Mohsen, J.-P., Construction of negatively curved complete intersections, arXiv:1805.10411v3 [Cited on page 175.]

[210] Moishezon, B. G., On n–dimensional compact varieties with n algebraically independent meromorphic functions, I, II, and III (English translated version). Seven Papers on Algebra, Algebraic Geometry and Algebraic Topology. American Mathematical Society Translations: Series 2. Vol. 63, (1967). [Cited on pages 82, 113, and 247.]

[211] Mok, N., The uniformization theorem for compact Kähler manifolds of nonnegative holomorphic bisectional curvature. J. Differential Geom. 27 (1988), no. 2, 179–214 [Cited on pages 182 and 183.]

[212] Mori, S., Projective manifolds with ample tangent bundles. Ann. of Math. (2), 110(3):593–606, 1979. [Cited on page 180.]

[213] Mori, S., Threefolds whose canonical bundles are not numerically effective. Ann. of Math. (2), 116(1):133–176, 1982. [Cited on pages 82 and 248.]

[214] Moroianu, A., Lectures on Kähler geometry, Lectures on Kähler geometry. London Mathematical Society Student Texts, 69. Cambridge University Press, Cambridge, 2007. x+171 pp. ISBN: 978-0-521-68897-0 [Cited on pages viii, 133, and 136.]

[215] Mostow, D., Siu, Y.-T., A compact Kähler surface of negative curvature not covered by the ball. Ann. of Math. (2) 112 (1980), no. 2, 321–360. [Cited on page 177.]

[216] Mumford, D., An algebraic surface with K ample, $(K^2) = 9, p_g = q = 0$, Amer. J. Math. 101 (1979), 233–244. [Cited on page 174.]

[217] Nakamura, I., Towards classification of non-Kählerian surfaces, Sugaku Expositions vol. 2, No 2 , 209–229 (1989) [Cited on page 117.]

[218] Neeman, A., Affines and Hilbert's Fourteenth Problem, Annals of Mathematics, 1988, Second Series, **127**, no. 2, pp. 229–244. [Cited on page 56.]

[219] Neeman, A., Algebraic and Analytic Geometry, London Mathematical Society Lecture Note Series, 345. Cambridge University Press, Cambridge, 2007. xii+420 pp. [Cited on page 56.]

[220] Newlander, A., Nirenberg, L., Complex analytic coordinates in almost complex manifolds. Ann. of Math., **65** (1957), pp. 391–404.

[221] Ni, L., Zheng, F., Comparison and vanishing theorems for Kähler manifolds, Calc. Var. **57** (151), (2018) [Cited on page 208.]

[222] Niu, Y., A note on nonnegative quadratic orthogonal bisectional curvature, Proc. of the Amer. Math. Soc., vol. 142, no. 11, (2014), pp. 3975–3979. [Cited on page 187.]

[223] Nomura, R., Kähler manifolds with negative holomorphic sectional curvature, Kähler–Ricci flow approach. Int Math. Res. Not. IMRN 00 (2017): 16. MR3873539 [Cited on page 256.]

[224] Ogawa, Y., Operators on almost Hermitian manifolds. J. Differential Geometry 4 (1970), 105–119. [Cited on page 97.]

[225] Omori, H., Isometric immersions of Riemannian manifolds, J. Math. Soc. Japan **19** (1967), pp. 205–214. [Cited on page 238.]

[226] Özaydin, M., Walschap, G., Vector bundles with no soul, Proc. Amer. Math. Soc. **120** (1994), pp. 565–567. [Cited on page 127.]

[227] Petersen, P., Riemannian Geometry, 3rd edn, Graduate Texts in Mathematics, 171 (Springer, Cham, 2016). [Cited on pages viii, 121, and 125.]

[228] Petersen, P., Wink, M., New curvature conditions for the Bochner technique, Invent. Math. 224 (2021), 33–54. [Cited on pages 184 and 186.]

[229] Petersen, P., Wink, M., Vanishing and estimation results for Hodge numbers, J. reine angew. Math. 2021 (2021), 197–219. [Cited on pages 184 and 186.]

[230] Petersen, P., Wink, M., The Bochner technique and weighted curvatures. SIGMA Symmetry Integrability Geom. Methods Appl. 16 (2020), Paper No. 064, 10 pp [Cited on pages 184 and 186.]

[231] Petersen, P., Wink, M., Tachibana-type theorems and special holonomy. Ann. Global Anal. Geom. 61 (2022), no. 4, 847–868 [Cited on pages 184 and 186.]

[232] Phong, D. H., Sesum, N., Sturm, J., Multiplier ideal sheaves and the Kähler–Ricci flow. Comm. Anal. Geom. 15 (2007), no. 3, 613–632. [Cited on page 254.]

[233] Pick, G., Über eine Eigenschaft der konformen Abbildung kreisförmiger Bereiche. (German) Math. Ann. 77 (1915), no. 1, 1–6. [Cited on pages 241 and 242.]

[234] Pignatelli, R., Polizzi, F., A family of surfaces with $p_g = q = 2$, $K^2 = 7$ and Albanese map of degree 3. Math. Nachr. 290 (2017), no. 16, 2684–2695. [Cited on page 177.]

[235] Polizzi, F., https://mathoverflow.net/users/7460/francesco-polizzi, Fibrations in complex geometry, URL (version: 2020-08-14): https://mathoverflow.net/q/369148 [Cited on page 111.]

[236] Popa, M., Chapter 1. Topology of Algebraic Varieties, Hodge Decomposition, and Applications, available here: https://people.math.harvard.edu/~mpopa/571/chapter1.pdf [Cited on page 94.]

[237] Range, R. M., Holomorphic functions and integral representations in several complex variables, Springer, (1986). [Cited on page 48.]

[238] Ranicki, A., Commentary on "On the parallelizability of the spheres" by R. Bott and J. Milnor and "On the nonexistence of elements of Hopf invariant one" by J. F. Adams, Bulletin of the American Mathematical Society, vol. 48, no. 4, 2011, pp. 509–511. [Cited on page 8.]

[239] Rao, P., Zheng, F., Pluriclosed manifolds with constant holomorphic sectional curvature, arXiv:2104.01319. [Cited on page 197.]

[240] Rosenberg, J., Manifolds of positive scalar curvature: A progress report, Surveys in differential geometry. Vol. XI, 259–294, Surv. Differ. Geom., 11, Int. Press, Somerville, MA, 2007. [Cited on page 130.]

[241] Royden, H. L., Intrinsic metrics on Teichmüller space. Proceedings of the International Congress of Mathematicians (Vancouver, B. C., 1974), vol. 2, pp. 217–221., Canad. Math. Congress, Montreal, Que., 1975. [Cited on pages 217 and 219.]

[242] Rubinstein, Y., Smooth and singular Kähler-Einstein metrics. Geometric and spectral analysis, 45–138, Contemp. Math., 630, Centre Rech. Math. Proc., Amer. Math. Soc., Providence, RI, 2014. [Cited on page 222.]

[243] Schneider, M., Complex surfaces with negative tangent bundle. Complex Analysis and Algebraic Geometry (Göttingen 1985). Lecture Notes Math. 1194, 150–157. Berlin Heidelberg New York. Springer 1986.

[244] Schoenberg, I. J., Remarks to Maurice Fréchet's article "Sur la définition axiomatique dúne classe déspace distanciés vectoriellement applicable sur l'espace de Hilbert", Ann. of Math. (2) 36(3) (1935), 724–732. [Cited on page 184.]

[245] Schottky, F., Über den Picard'schen Satz und die Borel'schen Ungleichungen, Sitz.-Ber. Königl. Preuss. Akad. Wiss. Berlin, Jhrg. 1904, 1244–1263 [Cited on page 241.]

[246] Schouten, Über unitäre Geometrie. Math. Ann., **103** (1930), pp. 319–346. [Cited on page xi.]

[247] Schwarz, H., Gesammelte Mathematische Abhandlungen. II, Springer-Verlag, Berlin, 1890, [Cited on page 240.]

[248] Seo, A., On a theorem of Paul Yang on negatively pinched bisectional curvature. Pacific J. Math. 256 (2012), no. 1, 201–209 [Cited on page 178.]

[249] Serre, J.-P., Faisceaux algébriques cohérents. (French) Ann. of Math. (2) 61 (1955), 197–278 [Cited on page 63.]

[250] Seshadri, H., Manifolds with nonnegative isotropic curvature, Comm. Ananl. Geom., **17** (2009), pp. 621–635. [Cited on page 183.]

[251] Seshadri, H., Negative sectional curvature and the product complex structure. Math. Res. Lett. 13 (2006), no. 2-3, 495–500 [Cited on pages 179 and 204.]

[252] Seshadri, H., Zheng, F., Complex product manifolds cannot be negatively curved. Asian J. Math. 12 (2008), no. 1, 145–149 [Cited on page 178.]

[253] Sha, J.-P., Yang, D., Sha, J.-P., Yang, D.-G., Examples of manifolds of positive Ricci curvature, J. Diff. Geom., **29**, (1989), pp. 95–103. [Cited on page 128.]

[254] Shabat, B. V., Introduction to Complex Analysis – Part II – Functions of Several Variables, American Mathematical Society, (1992). [Cited on pages 32, 35, 48, and 76.]

[255] Shafarevich, I. R., Basic Algebraic Geometry 1, Varieties in Projective Space, Springer-Verlag Berlin Heidelberg 2013 [Cited on page 278.]

[256] Singer, I. M., Thorpe, J. A., Lecture notes on elementary topology and geometry, Scott, Foresman and Company. [Cited on page viii.]

[257] Siu, Y.-T., Every K3 surface is Kähler. Invent. Math. 73 (1983), no. 1, 139–150. [Cited on pages 111 and 114.]

[258] Siu, Y.-T., Yang, P., Compact Kähler–Einstein surface of nonpositive bisectional curvature, Invent. Math. **64** (1981), pp. 471–487. [Cited on page 177.]

[259] Siu, Y.-T., Yau, S.-T., Compact Kähler manifolds of positive bisectional curvature. Invent. Math. 59 (1980), no. 2, 189–204. [Cited on page 180.]

[260] Song, J., Weinkove, B., An introduction to the Kähler-Ricci flow. An introduction to the Kähler-Ricci flow, 89–188, Lecture Notes in Math., 2086, Springer, Cham, 2013 [Cited on page viii.]

[261] Stallings, J. R., The piecewise-linear structure of Euclidean space, Proceedings of the Cambridge Philosophical Society 58: 481–488 (1962) [Cited on page 4.]

[262] Streets, J., Pluriclosed flow and the geometrization of complex surfaces. Geometric analysis – in honor of Gang Tian's 60th birthday, 471–510, Progr. Math., 333, Birkhäuser, Springer, Cham, 2020. [Cited on page 117.]

[263] Streets, J., Tian, G., Hermitian curvature flow. J. Eur. Math. Soc. (JEMS) 13 (2011), no. 3, 601–634. [Cited on pages 117 and 181.]

[264] Streets, J., Tian, G., A parabolic flow of pluriclosed metrics. Int. Math. Res. Not. IMRN 2010, no. 16, 3101–3133 [Cited on pages 117, 149, and 244.]

[265] Streets, J., Tian, G., Regularity theory for pluriclosed flow, C. R. Acad. Sci. Paris, Ser. I 349 (2011), pp. 1–4. [Cited on pages 117, 149, and 244.]

[266] Strominger, A., Superstrings with torsion. Nuclear Phys. B 274 (1986), no. 2, 253–284. [Cited on page 149.]

[267] Székelyhidi, G., An Introduction to Extremal Kähler Metrics, Graduate Studies in Mathematics, volume 152, American Mathematical Society. [Cited on page 260.]

[268] Tam, L.-F., Yu, C., Complex product manifolds and bounds of curvature. Asian J. Math. 14 (2010), no. 2, 235–242. [Cited on page 178.]

[269] Tang, K., An extension of a theorem of Paul Yang on negatively pinched curvature. Geom. Dedicata 202 (2019), 193–202. [Cited on page 178.]

[270] Tang, K., Positive curvature operator, projective manifold and rational connectedness, arXiv:1905.04894v2 [Cited on page 213.]

[271] Tang, K., Holomorphic sectional curvature and Kähler-like metric, 2020, Sci. China-Math. (Chinese series). [Cited on page 197.]

[272] Tao, T., Hilbert's fifth problem and related topics. Graduate Studies in Mathematics, 153. American Mathematical Society, Providence, RI, 2014 [Cited on pages 2 and 5.]

[273] Teleman, A., Projectively flat surfaces and Bogomolov's theorem on class VII_0 surfaces, Int. J. Math., vol. 5, no. 2, p. 253–264 (1994). [Cited on page 116.]

[274] Teleman, A., Gauge theoretical methods in the classification of non-Kählerian surfaces, Algebraic topology – old and new, p. 109–120, Banach Center Publ., 85, Polish Acad. Sci. Inst. Math., Warsaw, 2009. [Cited on page viii.]

[275] Thorpe, J. A., The zeros of nonnegative curvature operators, J. Diff. Geom., **5** (1971), pp. 113–125. [Cited on page 127.]

[276] Thorpe, J. A., On the curvature tensor of a positively curved 4–manifold, in Proceedings of the Thirteenth Biennial Seminar of the Canadian Mathematical Congress (Dalhousie Univ., Halifax, N.S., 1971), Vol. 2, Canad. Math. Congr., Montreal, Que., 1972, 156–159. [Cited on page 127.]

[277] Tian, G., On a set of polarized Kähler metrics on algebraic manifolds, J. Diff. Geom. **32** (1990), pp. 99–130. [Cited on page 281.]

[278] Tian, G., On Kähler–Einstein metrics on certain Kähler manifolds with $C_1(M) > 0$. Invent. Math., **89** (2), pp. 225–246, (1987). [Cited on pages 267, 269, and 271.]

[279] Tian, G., Kähler–Einstein metrics with positive scalar curvature. Invent. Math. 130 (1997), no. 1, 1–37 [Cited on page 159.]

[280] Tian, G., K-stability and Kähler-Einstein metrics. Comm. Pure Appl. Math. 68 (2015), no. 7, 1085–1156 [Cited on page 159.]

[281] To, W.-K., Yeung, S.-K., Kähler metrics of negative holomorphic bisectional curvature on Kodaira surfaces, Bulletin of the London Mathematical Society, vol 43, Issue 3, June 2011, pp. 507–512. [Cited on page 179.]

[282] Tosatti, V., A general Schwarz lemma for almost-Hermitian manifolds. Comm. Anal. Geom. 15 (2007), no. 5, 1063–1086. MR2403195 [Cited on page 178.]

[283] Tosatti, V. , Non-Kähler Calabi–Yau Manifolds, [Cited on pages 102, 167, and 168.]

[284] Tosatti, V., Weinkove, B., Estimates for the complex Monge–Ampère equation on Hermitian and balanced manifolds, Asian J. Math., (2010) [Cited on page 261.]

[285] Tosatti, V., Weinkove, B., The complex Monge-Amp̀re equation on compact Hermitian manifolds. J. Amer. Math. Soc. 23 (2010), no. 4, 1187–1195 [Cited on pages 252, 254, 255, and 261.]

[286] Tosatti, V., Yang, X.-K., An extension of a theorem of Wu–Yau, J. Differential Geom. 107(3): 573–579 [Cited on pages xii, 253, 254, 256, and 267.]

[287] Trudinger, N., Remarks concerning the conformal deformation of Riemannian structures on compact manifolds. Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 265–274. [Cited on page 131.]

[288] Tsukamoto, Y., On Kählerian manifolds with positive holomorphic sectional curvature. Proc. Japan Acad. **33** (1957), pp. 333–335. [Cited on page 208.]

[289] Ueno, K., Classification theory of algebraic varieties and compact complex spaces, Springer Lect. Notes in Math. **439** (1975). [Cited on pages 104 and 105.]

[290] Ustinovskiy, Y., The Hermitian curvature flow on manifolds with non-negative Griffiths curvature. Amer. J. Math. 141 (2019), no. 6, 1751–1775 [Cited on page 181.]

[291] Verbitsky, M., Rational curves and special metrics on twistor spaces, Geom. Topol. 18 (2014), no. 2, pp. 897–909. [Cited on page 79.]

[292] Viaclovsky, J. A., 218C Introduction to Manifolds and Geometry, Spring 2020, lecture notes available online at https://www.math.uci.edu/~jviaclov/courses/218C_Spring_2020.pdf. [Cited on page 27.]

[293] Viehweg, E., Weak positivity and the additivity of the Kodaira dimension for certain fibre spaces, Advanced Studies in Pure Mathematics 1, 1983, Algebraic Varieties and Analytic Varieties, pp. 329–353. [Cited on page 114.]

[294] Vladimirov, V. S., Methods of the theory of functions of several complex variables, M.I.T. (1966) (Translated from Russian) [Cited on page 48.]

[295] Voisin, C., Hodge theory and complex algebraic geometry I, Cambridge studies in Advanced Mathematics **76**, (2002). [Cited on pages viii, 49, and 53.]

[296] Wang, J., Yang, X., RC-positivity and scalar-flat metrics on ruled surfaces, Math. Z. 301 (2022), no. 1, 917–934. 53C55 [Cited on page 208.]

[297] Warner, F. W., Foundations of differentiable manifolds and Lie groups, Graduate Texts in Mathematics, vol. 94, Springer-Verlag, New York-Berlin, 1983. Corrected reprint of the 1971 edition. MR722297 [Cited on page 90.]

[298] Watson, B., Almost Hermitian submersions, J. Diff. Geom., **11** (1976), pp. 147–165. [Cited on page 161.]

[299] Weil, A., Final report on contract AF 18(603)-57, Scientific works. Collected papers, vol. II, Berlin, New York: Springer-Verlag, pp. 545–547 [Cited on page 111.]

[300] Weinkove, B., Monge-Ampere equations on complex manifolds, lecture available online at http://scgp.stonybrook.edu/video_portal/video.php?id=748 [Cited on pages viii and 261.]

[301] Wu, H. A remark on holomorphic sectional curvature. Indiana Univ. Math. J. 22 (1972/73), 1103–1108. [Cited on page 195.]

[302] Wong, B., The uniformization of compact Kähler surfaces of negative curvature, J. Differential Geom. 16 (1981), no. 3, 407–420, MR0654634 [Cited on pages xii, 174, 175, and 246.]

[303] Wong, P. M., Wu, D., Yau, S.-T., Picard number, holomorphic sectional curvature, and ampleness, Proc. Amer. Math. Soc. 140 (2012), no. 2, 621–626, MR2846331 [Cited on page xii.]

[304] Wu, D., Yau, S.-T., Negative holomorphic sectional curvature and positive canonical bundle, Invent. Math. **204** (2016), no. 2, 595–604 [Cited on pages xii, 220, 252, 253, 255, and 256.]

[305] Wu, D., Yau, S.-T., A remark on our paper "Negative holomorphic sectional curvature and positive canonical bundle", Comm. Anal. Geom. **24** (2016), no. 4, 901–912. [Cited on pages 253, 256, and 265.]

[306] Wu, D., Yau, S.-T., Zheng, F., A degenerate Monge-Ampère equation and the boundary classes of Kähler cones. Math. Res. Lett. 16 (2009), no. 2, 365–374 [Cited on page 183.]

[307] Xiong, C., Lower bound of Bloch's constant, Nanjing Daxue Xuebao Shuxue Bannian Kan 15 (1998), no. 2, pp. 174–179 [Cited on page 243.]

[308] Yamabe, H., On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960), 21–37. [Cited on page 131.]

[309] Yang, B., Zheng, F., On curvature tensors of Hermitian manifolds, Communications in Analysis and Geometry, vol. **26** (2018), no. 5, pp. 1195–1222 [Cited on pages 151 and 154.]

[310] Yang, P., Kähler metrics on fibered manifolds, Proc. of the A. M. S. **63** (1977), pp. 131–133. [Cited on page 178.]

[311] Yang, X., A partial converse to the Andreotti-Grauert theorem, Compos. Math. 155 (2019), no. 1, 89–99. [Cited on page 205.]

[312] Yang, X., RC-positivity, rational connectedness and Yau's conjecture. Camb. J. Math. 6 (2018), no. 2, 183–212. [Cited on pages 204, 205, 207, and 213.]

[313] Yang, X., Scalar curvature on compact complex manifolds, Trans. Amer. Math. Soc. 371 (2019), no. 3, 2073–2087 [Cited on page 200.]

[314] Yang, X., Compact Kähler manifolds with quasi-positive second Chern–Ricci curvature, arXiv:2006.13884v1 [Cited on page 160.]

[315] Yang, X., Zheng, F., On the real bisectional curvature for Hermitian manifolds, Trans. Amer. Math. Soc. **371** (2019), no. 4, 2703–2718 [Cited on pages xv, 201, 209, 210, 211, 212, 219, 220, 221, and 253.]

[316] Yang, B., Zheng, F., Hirzebruch manifolds and positive holomorphic sectional curvature. Ann. Inst. Fourier (Grenoble) 69 (2019), no. 6, 2589–2634 [Cited on page 209.]

[317] Yau, S.-T., The Calabi conjecture and some new results in algebraic geometry, Proc. Nat. Acad. Sci. USA, **74** (1977), pp. 1798–1799. [Cited on page 177.]

[318] Yau, S.-T., On the Ricci curvature of a compact Kähler manifold and the complex Monge–Ampère equation, I, Comm. Pure Appl. Math. 31 (1978), pp. 339–411. [Cited on pages xi, 113, 161, 222, 253, 254, 255, 259, and 261.]

[319] Yau, S.-T., Harmonic functions on complete Riemannian manifolds, Communications on pure and applied mathematics **28** (1975), 201–228 [Cited on page 238.]

[320] Yau, S. -T., A general Schwarz lemma for Kähler manifolds. Amer. J. Math. 100 (1978), no. 1, 197–203. MR0486659 [Cited on page 219.]

[321] Yau, S.-T., Problem section, pp. 669-706 in Seminar on differential geom- etry, S.T. Yau, editor, Annals of Mathematics Studies, Princeton Univ. Press 102 (1982). [Cited on pages 175 and 208.]

[322] Zelditch, S., Szego kernels and a theorem of Tian, Internat. Math. Res. Notices 1998, no. 6, 317–331. 32L10 (47B35) [Cited on page 281.]

[323] Zhang, Y., Holomorphic sectional curvature, nefness and Miyaoka–Yau type inequality, Math. Z., **298**, 2021, pp. 953–974. [Cited on page 267.]

[324] Zheng, F., Complex differential geometry. AMS/IP Studies in Advanced Mathematics, 18. American Mathematical Society, Providence, RI; International Press, Boston, MA, 2000. xii+264 pp. ISBN: 0-8218-2163-6 [Cited on pages vii, viii, 37, 70, 105, 182, and 279.]

[325] Zhiren, J., A counterexample to the Yamabe problem for complete noncompact manifolds, Partial differential equations (Tianjin, 1986), Lecture Notes in Mathematics, vol. 1306, Berlin: Springer, pp. 93–101. [Cited on page 131.]

[326] Zhou, W. , Zheng, F., Hermitian threefolds with vanishing real bisectional curvature, arXiv:2103.04296. [Cited on page 197.]

[327] Ziller, W., Homogeneous Einstein metrics on spheres and projective spaces, Math. Ann. 259 (1982), 351–358. [Cited on page 129.]

[328] Ziller, W., Riemannian manifolds with positive sectional curvature, Geometry of manifolds with nonnegative sectional curvature, 1–19, Lecture Notes in Math., 2110, Springer, Cham, 2014. 53C20 [Cited on page viii.]