The Schwarz Lemma in Kähler geometry and non-Kähler Hermitian Geometry

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The Schwarz lemma tells us that

Lemma (Schwarz Lemma): A holomorphic map $f : \mathbb{D}(R_1) \to \mathbb{D}(R_2)$ from the disk of radius R_1 to the disk of radius R_2 (both centered at the origin in $\mathbb{C}$), with $f(0) = 0$, satisfies

$$|f(z)| \leq \frac{R_2}{R_1}|z|,$$

for all $|z| < R_1$.

If we keep R_2 fixed and let R_1 be arbitrarily large, we get the following corollary:

Corollary 1 (Liouville's theorem): A bounded entire function assumes at most one value.

The passage

$$\text{Schwarz} \implies \text{Liouville}$$

is encapsulated in a more general heuristic — **Bloch's Principle** — which states that a global result (such as Liouville's theorem) always has its origins in a stronger, finite version (such as the Schwarz lemma).

Example: The **Picard theorem** — a generalization of Liouville's theorem — states that a non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ satisfies either $f(\mathbb{C}) = \mathbb{C}$ or $f(\mathbb{C}) = \mathbb{C} \backslash \{\text{point}\}$.

The finite version of Picard's theorem is **Schottky's theorem**: If $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function such that

$$f(\mathbb{D}) \cap \{0, 1\} = \emptyset,$$

then $|f(z)|$ can be bounded in terms of z and $f(0)$.

In 1916, Pick made a crucial observation relating the Schwarz lemma to hyperbolicity — an observation one might have thought would have been made by Klein or Poincaré:

Lemma 3 (Schwarz—Pick lemma): Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic map. Let d_{H} denote the distance function associated to the hyperbolic metric (of constant curvature -1) on the disk. Then

$$d_{\mathrm{H}}(f(z), f(w)) \leq d_{\mathrm{H}}(z, w), \qquad \forall z, w \in \mathbb{D}.$$

One may formulate the Schwarz—Pick lemma equivalently as asserting that every holomorphic self-map of the unit disk is either a linear fractional transformation (a non-Euclidean isometry), or it shrinks the hyperbolic length of a curve.

Example: Valiron's theorem states that for a non-constant entire function f, there exist disks D of arbitrarily large radius, and analytic functions g on D such that $f(g(z)) = z$ for all $z \in D$.

The finite version is **Bloch's theorem** – Let f be a holomorphic functions on the unit disk $|z| \leq 1$. Suppose that $|f'(0)| = 1$. Then there exists a disk of radius b and an analytic function g on this disk, such that $f(g(z)) = z$ for all z in this disk.

In 1938, Ahlfors proved the following generalization of the Schwarz—Pick lemma:

Lemma 4 (Schwarz—Pick—Ahlfors lemma): Let $f : \mathbb{D} \to X$ be a holomorphic map from the unit disk to a Riemann surface (X, g), with the Gauss curvature of g bounded above $K_g \leq -1$. Then

$$|df|^2 \leq 1 \qquad \text{everywhere.}$$

Equivalently,

$$d_g(f(z), f(w)) \leq d_{\mathrm{H}}(z, w), \qquad \forall z, w \in \mathbb{D}.$$

Remark: In reading the published collected work of Ahlfors (p. 341, published in 1982), one finds a reflection of Ahlfors concerning his version of the Schwarz lemma.

He confesses that his generalization of the Schwarz lemma "*has more substance than I was aware of*", but still says that "*without applications my lemma would have been too lightweight for publication*".

One of the applications he gave, was an elementary and novel proof of Bloch's theorem with an explicit lower bound for Bloch's constant B, namely

$$B \geq \frac{\sqrt{3}}{4}.$$

In 1973, Yau proved the following generalization of Ahlfors—Schwarz Lemma:

Theorem (Yau): Let (M, g) be a complete surface with $K_g \geq -A$ and (N, h) a complete surface with $K_h \leq -B$. Then for any holomorphic map $f : M \to N$,

$$|df|^2 \leq \frac{A}{B}, \qquad \text{or equivalently,} \qquad f^*h \leq \frac{A}{B} g.$$

Remark: The underlying philosophy is that the more negative the curvature of the target manifold, the more a holomorphic map will shrink distances and lengths of curves.

We will give another way of looking at this shortly.

This type of result seems oddly reminiscent, but in apparent reverse, of the standard comparison theorems (specifically, the Rauch comparison theorem) from Riemannian geometry, which roughly assert that:

$$\text{The more negative the curvature} \implies \text{the more certain curves are stretched.}$$

For more in this direction, see Robert Osserman's Notices of the AMS article – *From Schwarz to Pick to Ahlfors and Beyond* (vol. 46, no. 8).

Higher-dimensional generalizations of the Ahlfors—Schwarz Lemma:

Let (M, ω_g) and (N, ω_h) be two Hermitian manifolds of respective dimension m and n. The simplest metric invariant of a part of Hermitian manifolds is the ratio of the distance and volume elements.

The former is encoded in the **trace**:

$$\mathrm{tr}_{\omega_g}(f^*\omega_h) \qquad \text{or} \qquad \mathrm{tr}_{f^*\omega_h}(\omega_g),$$

while the latter is encoded in the **determinant**:

$$\det_{\omega_g}(f^*\omega_h) = \frac{f^*\omega_h^k}{\omega_g^k}, \qquad \text{or} \qquad \det_{f^*\omega_h}(\omega_g) = \frac{\omega_g^k}{f^*\omega_h^k},$$

where k is the rank of f.

In Kähler geometry, one typically works with the same manifold, i.e, $M = N$ and ω_g, ω_h are cohomologous:

$$\omega_g = \omega_h + \sqrt{-1}\partial\overline{\partial}\varphi,$$

where $\varphi \in \mathrm{PSH}(\omega_h)$.

Moreover, these metrics are usually determined by **some complex Monge–Ampére equation**

$$\omega_g^n = (\omega_h + \sqrt{-1}\partial\overline{\partial}\varphi)^n = e^f \omega_h^n.$$

In particular, the **determinant is controlled** and so the effort is primarily placed on **controlling the trace.**

It is worth pointing out that the first steps towards higher-dimensional versions of the Schwarz lemma were carried out by **Chern** (1968) and **Lu** (1968).

In fact, as we will see later, these computations are special cases of even earlier computations of **Eells** and **Sampson** (1958).

From the Ahlfors and Yau versions of the Schwarz lemma that we saw above, curvature plays a central role.

Curvature of a Riemannian manifold:

Let (M, g) be a Riemannian manifold. The tangent bundle to M carries the Levi-Civita connection — the unique torsion-free connection which is compatible with the metric.

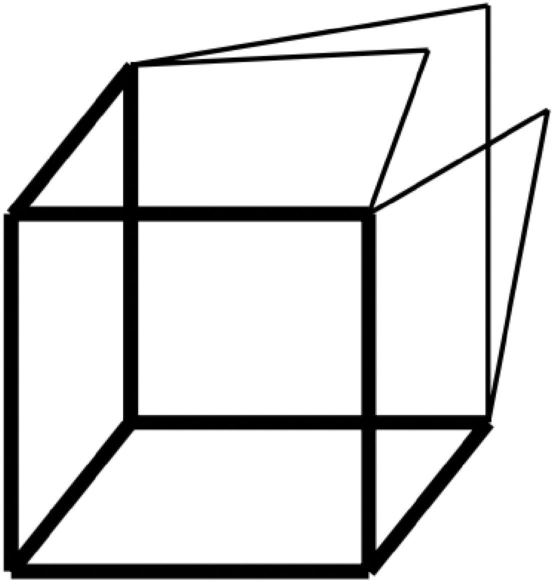
The curvature R of the Levi-Civita connection is referred to as the Riemannian curvature tensor and is a $(4, 0)$—tensor satisfying the following conditions:

(i) **Symmetry** of the first two and last two entries:
$R(X, Y, Z, W) = R(Z, W, X, Y)$,

(ii) **Anti-symmetry** of the interchange of the first two entries:
$R(X, Y, Z, W) = -R(Y, X, Z, W)$,

(iii) The **Bianchi identity**:
$R(X, Y, Z, W) + R(X, Z, W, Y) + R(X, W, Y, Z) = 0$.

Bianchi identity:



$$R(u,v)w + R(v,w)u + R(w,u)v = 0$$

Remark: The Riemannian curvature measures the distortion of distances under the exponential map.

Definition: The contraction of the **first** and **third** components of the Riemannian curvature tensor yields the (Riemannian) Ricci curvature tensor

$$\text{Ric}(X, Y) = \sum_{i=1}^{n} R(e_i, X, e_i, Y),$$

where e_i is a local orthonormal frame for $\mathcal{T}_M$.

Remark: The Ricci curvature measures the distortion of the volume form under the exponential map.

If Ric > 0, then **Myer's theorem** states that π_1 is finite.

The model example is $\mathbb{S}^n$.

If Ric < 0, there are non-trivial Killing vector fields.

Lockhamp (extending Gao's result in dimension 3) showed that every Riemannian manifold admits a complete metric with Ricci curvature bounded between two negative constants.

In particular, there are no topological obstructions to the existence of negative Ricci curvature metrics.

Negative Ricci curvature controls the isometry group:

(**Bochner, Lockhamp**): If M^n is a compact manifold of dimension $n \geq 3$, with $G = \text{Isom}(M, g) \subset \text{Diff}(M)$ the isometry group of (M, g), then

$$\text{Ric}_g < 0 \iff G \text{ is finite.}$$

Reminder of Kähler manifolds:

Recall that a Hermitian manifold (M, g) is said to be **Kähler** if the associated $(1,1)$—form (the Kähler form)

$$\omega(u, v) := g(u, Jv) > 0$$

is closed in the sense that $d\omega = 0$.

Examples:

(†) $\mathbb{P}^n$ with the Fubini–Study metric

$$\omega_{\mathrm{FS}} = \sqrt{-1}\partial\overline{\partial}\log\left(\sum_{k=0}^{n}|Z_k|^2\right),$$

where

$$\mathrm{Ric}(\omega_{\mathrm{FS}}) = (n+1)\omega_{\mathrm{FS}} > 0.$$

The Kähler property is preserved under restrictions, so smooth hypersurfaces $X = \{F = 0\} \subset \mathbb{P}^{n+1}$ yield further examples of Kähler manifolds:

(i) $\deg(X) < n + 2 \implies \mathrm{Ric} > 0,$

(ii) $\deg(X) = n + 2 \implies \mathrm{Ric} = 0,$

(iii) $\deg(X) > n + 2 \implies \mathrm{Ric} < 0.$

In local coordinates, the Kähler form can be written

$$\omega = \sqrt{-1} \sum_{i,j} g_{i\bar{j}} dz_i \wedge d\overline{z}_j.$$

The Kähler condition $d\omega = 0$ implies that

$$\frac{\partial g_{i\bar{j}}}{\partial z_k} = \frac{\partial g_{k\bar{j}}}{\partial z_i}, \qquad \text{and} \qquad \frac{\partial g_{i\bar{j}}}{\partial \overline{z}_k} = \frac{\partial g_{i\bar{k}}}{\partial \overline{z}_i}.$$

Recall that the tangent bundle $\mathcal{T}_M$ of a Hermitian manifold carries two distinguished connections:

- (†) The ($\mathbb{C}$—linear extension of the) Levi-Civita connection ∇ — The unique torsion-free connection compatible with the metric.
- (†) The Chern connection ∇^h — The unique connection compatible with the metric and the complex structure.

In general, the connections do not coincide, but on Kähler manifolds they do (this can be taken as a definition of Kähler).

It is well-known that if $\nabla = \nabla^h$ on $\mathcal{T}_M$, then M is Kähler.

It was recently shown by Bo Yang and Fangyang Zheng (2018) that if $R = R^h$, then M is Kähler.

The **curvature tensor** R of a Kähler manifold (M, g) is a $(4, 0)$—tensor which in local coordinates is described by

$$R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2 g_{i\bar{j}}}{\partial z_k \partial \overline{z}_{\bar{\ell}}} + g^{p\bar{q}} \frac{\partial g_{p\bar{j}}}{\partial z_k} \frac{\partial g_{i\bar{q}}}{\partial \overline{z}_{\ell}}.$$

This Kähler condition imposes the following **symmetries** on the curvature tensor (of the Chern or Levi-Civita connection):

Specifically, the only non-zero components of the curvature are given by $R_{i\bar{j}k\bar{\ell}}$, and

$$R_{i\bar{j}k\bar{\ell}} = R_{k\bar{\ell}i\bar{j}}, \qquad R_{i\bar{j}k\bar{\ell}} = R_{k\bar{j}i\bar{\ell}}, \qquad R_{i\bar{j}k\bar{\ell}} = -R_{\bar{j}ik\bar{\ell}}.$$

In the general Hermitian setting, these symmetries are violently false.

From the curvature tensor, we can arrive at various other notions of curvature:

(†) Using the metric to contract the **third** and **fourth** indices, we get the Ricci curvature:

$$\mathrm{Ric}_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}.$$

(†) Contracting the Ricci curvature yields the **scalar curvature**:

$$s = g^{i\bar{j}} \mathrm{Ric}_{i\bar{j}}.$$

The contraction of the third and fourth indices of the Ricci curvature yields the explicit expression:

$$\mathrm{Ric}_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} = -\frac{\partial^2}{\partial z_i \partial \overline{z}_j} \log \det(g_{k\bar{\ell}}),$$

which is the curvature form of a Hermtitian metric on the anti-canonical bundle.

In particular, by Chern–Weil theory, the Ricci curvature represents the first Chern class of K_M^{-1}.

(†) The **holomorphic bisectional curvature** is defined:

$$\mathrm{HBC}(u,v) = \frac{1}{|u|^2|v|^2} R_{i\bar{j}k\bar{\ell}} u_i \overline{u}_j v_k \overline{v}_\ell.$$

(†) The **holomorphic sectional curvature** is defined:

$$\mathrm{HSC}(v) := \frac{1}{|v|^4} R_{i\bar{j}k\bar{\ell}} v_i \overline{v}_j v_k \overline{v}_\ell.$$

The holomorphic bisectional curvature affords the following algebro-geometric interpretation:

Let $\pi : \mathbb{P}(\mathcal{T}_M) \to M$ denote the projectivized tangent bundle. The metric ω on M induces a Hermitian metric on the tautological bundle

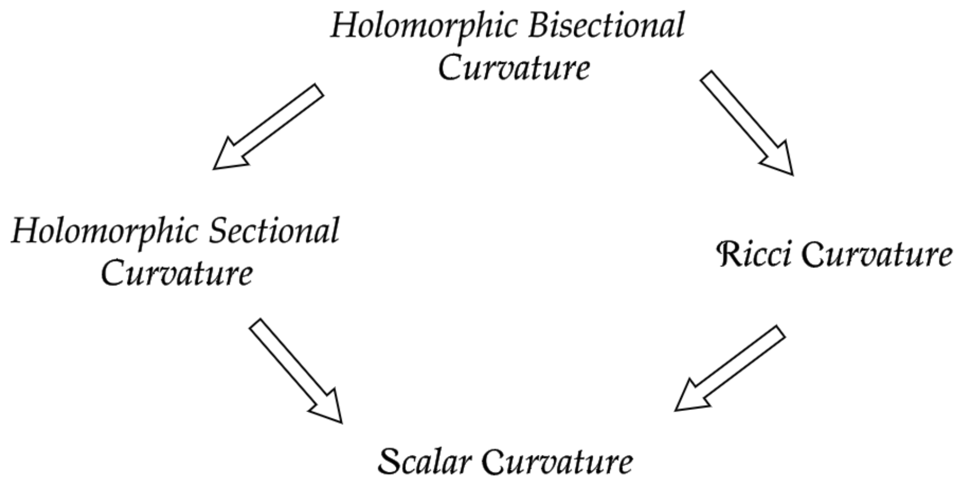
$$\mathcal{O}(-1) \hookrightarrow \pi^* \mathcal{T}_M$$

of $\mathbb{P}(\mathcal{T}_M)$ with curvature form Θ given by

$$\sqrt{-1}\Theta = -\mathrm{HBC} + \omega_{\mathrm{FS}}.$$

So the bisectional curvature controls the positivity of $\mathcal{T}_X$.

We have the following relationship between the various notions of curvature of Kähler metrics:



In general, the (sign of the) Ricci curvature does not control the (sign of the) holomorphic sectional curvature, nor does the (sign of the) holomorphic sectional curvature control the (sign of the) Ricci curvature. (See the examples of Hirzebruch surfaces given by Hitchin).

Example: Let $\mathbb{B}^2$ be the unit ball in $\mathbb{C}^2$ equipped with the Poincaré metric. Then

- † $-4 \leq \text{Rm}(\mathbb{B}^2) \leq -1$,
- † $-2 \leq \text{HBC}(\mathbb{B}^2) \leq -1$,
- † $\text{HSC}(\mathbb{B}^2) = -2$.

Example: Let $\mathbb{B} \times \mathbb{B}$ be the bidisk with the product of Poincaré metrics on each factor. Then

- † $\text{Rm}(\mathbb{B} \times \mathbb{B}) \leq 0$,
- † $\text{HBC}(\mathbb{B} \times \mathbb{B}) \leq 0$,
- † $-2 \leq \text{HSC}(\mathbb{B} \times \mathbb{B}) \leq -1$.

The curvature of Kähler surfaces is constrained by the following Chern class inequalities:

(i) HSC $< 0 \implies c_1^2 > 2c_2$,

(ii) HBC $< 0 \implies c_1^2 > c_2$.

A compact complex surface is a ball quotient if and only if $c_1^2 = 3c_2$.

In 1980, Mostow–Siu constructed an example of a Kähler surface with negative sectional curvature which is not a smooth compact quotient of a ball, their example satisfies

$$c_1^2 = \frac{852}{298} c_2, \qquad \frac{852}{298} \sim 2.86.$$

The holomorphic sectional curvature is intimately related to the notions of hyperbolicity in complex geometry.

Recall that a complex manifold M is said to be **Kobayashi hyperbolic** if the Kobayshi pseudo-metric d_K, defined to be the largest pseudometric on X which satisfies

$$d_K(f(x), f(y)) \leq d_{\mathrm{P}}(x, y),$$

where d_{P} is the Poincaré metric on the disk, is a metric.

The idea behind Kobayashi hyperbolicity is that there is a genuine bound on the size of disks which map holomorphically into X.

For example, the Kobayashi pseudometric is identically zero on $\mathbb{C}$.

Moreover, the Schwarz lemma says that the unit disk is Kobayashi hyperbolic (with its Kobayashi metric being the Poincaré metric).

It is known that if M admits a Hermitian metric of negative holomorphic sectional curvature, then M is Kobayashi hyperbolic, i.e.,

$$\text{HSC} < 0 \implies \text{Kobayashi hyperbolic.}$$

The converse is **not true.** A counterexample was given by Demailly in the 90s.

The idea is to use Demailly's algebraic criterion for Kobayashi hyperbolicity:

If (M, ω) is a Hermitian manifold with $\mathrm{HSC}_{\omega} < k < 0$, then M is Kobayashi hyperbolic if for any holomorphic map $F : C \to M$ from a Riemann surface of genus g into M, we have

$$2g - 2 \geq \frac{k}{2\pi} \deg_{\omega}(C) + \sum_p (m_p - 1).$$

One can violate the inequality

$$2g - 2 \geq \frac{k}{2\pi} \deg_{\omega}(C) + \sum_{p} (m_p - 1)$$

by producing a smooth projective surface, fibered by hyperbolic curves, over a hyperbolic curve with at least one very singular fiber.

It would be interesting to know if there is a curvature characterization of Kobayashi hyperbolicity.

The Schwarz lemma relates dilation of holomorphic maps to the curvature of the source and target manifold. This has its origins in the theory of harmonic maps:

Definition: A smooth map $f : (M, g) \to (N, h)$ is said to be harmonic if it minimises the energy functional:

$$E(f) := \frac{1}{2} \int_M |\nabla f|^2 d\mu_g,$$

where $d\mu_g$ is the Riemannian volume form associated to the metric g.

Examples:

- † Harmonic maps $f : \mathbb{R} \to M$ are geodesics.
- † Holomorphic maps between Kähler manifolds are harmonic.

The Euler—Lagrange operator associated to the energy functional is called the tension field:

$$\tau(f) := \mathrm{div}(df).$$

Hence,

$$f \text{ is harmonic } \iff \tau(f) \equiv 0.$$

Remark: This offers the following physical interpretation — A smooth map $f : M \to N$ places the source manifold M onto N. The harmonic maps are therefore exactly those maps f for which $f(M)$ rests on N in a state of elastic equillibrium.

The energy density of $f : M \to N$ measures the sum of the squares of stretches of length along a complete set of mutually orthogonal directions (e_i):

$$e(f)(x) := \sum_{i=1}^{m} \frac{|df(x)e_i|^2}{|e_i|^2}.$$

If M is complete with $\mathrm{Rm}^M \geq 0$, any harmonic function with $E(f) < \infty$ is constant.

If $f : M \to N$ is harmonic, we have

$$\Delta e(f) = |\nabla df|^2 + \langle \mathrm{Ric}^M(\nabla_v f), \nabla_v f \rangle - \langle \mathrm{Rm}^N(f)(\nabla_v f, \nabla_w f)\nabla_v f, \nabla_w f \rangle.$$

The first term $|\nabla df|^2$ is the second fundamental form of $f(M)$ inside N.

A map with vanishing second fundamental form is said to be totally geodesic — such maps are characterised by the property of taking geodesics to geodesics.

If M is complete with $\mathrm{Ric}^M \geq 0$, then every non-negative harmonic function on M is constant.

Uniform Equivalence of Kähler metrics:

Let ω_g, ω_h be two Kähler metrics on a Kähler manifold M. We say that the metrics are uniformly equivalent if

$$C_1 \omega_g \leq \omega_h \leq C_2 \omega_g,$$

for some continuous functions $C_2 \geq C_1 > 0$.

If ω_g and ω_h are cohomologous, with $\omega_g = \omega_h + \sqrt{-1}\partial\overline{\partial}\varphi$, then the uniform equivalence is implied by either:

$$m + \Delta_{\omega_g}\varphi = \mathrm{tr}_{\omega_g}(\omega_h) \leq C_2 \qquad \text{and} \qquad \det_{\omega_g}(\omega_h) \geq C_1 C_2^{m-1}/(m-1)^{m-1}$$

or

$$m - \Delta_{\omega_h}\varphi = \mathrm{tr}_{\omega_h}(\omega_g) \leq 1/C_1 \qquad \text{and} \qquad \det_{\omega_g}(\omega_h) \leq C_1 C_2^{m-1}(m-1)^{m-1}$$

The Monge—Ampére equation controls the determinant, so the focus is primarily on estimating the trace. This is carried out by various versions of the Chern—Lu inequality:

Yau Schwarz lemma:

Let $f : (M, \omega_g) \to (N, \omega_h)$ be a holomorphic map from a complete Kähler manifold M with $\text{Ric}(\omega_g) \geq -A$ into a Kähler manifold (N, ω_h) with $\text{HBC}(\omega_h) \leq -B < 0$. Then

$$|df|^2 \leq \frac{A}{B}.$$

Chen—Cheng—Lu version:

Chen—Cheng—Lu obtained a different variant, assuming that M is complete Kähler with $\mathrm{HSC}_{\omega_g} \leq -B < 0$ and $\mathrm{Rm}_{\omega_g} \geq -C$, and N is Hermitian with $\mathrm{HSC}_{\omega_h} < -K < 0$. Then

$$|df|^2 \leq \frac{B}{K}.$$

Lei Ni version:

Let (M,g) be a complete Kähler manifold such that $\text{HSC}_g \geq -K$ for some $K \geq 0$. Let (N,h) be a Kähler manifold such that $\text{HSC}_h \leq -B$ for some $B > 0$. Then for any holomorphic map $f : M \to N$,

$$\|\partial f\|_m^2 \leq \frac{K}{B}, \qquad \|\partial f\|_m^2 := \sup_{v \neq 0} \frac{|\partial f(v)|^2}{|v|^2}.$$

provided that the bisectional curvature of M is bounded from below.

Royden's version of the Schwarz lemma:

Theorem: Let $f : (M, \omega_g) \to (N, \omega_h)$ be a non-constant holomorphic map between Kähler manifolds. Suppose that $\mathrm{Ric}_{\omega_g} \geq A$ and $\mathrm{HSC}_{\omega_h} \leq B < 0$. Then

$$\Delta_{\omega_g} \log \mathrm{tr}_{\omega_g}(\omega_h) \geq 2A - \frac{k+1}{k} B \mathrm{tr}_{\omega_g}(\omega_h),$$

where k is the rank of f at the point at which the Laplacian is computed.

Summary of Kähler versions of the Schwarz lemma:

Version	Source Manifold (M, g).	Target Manifold (N, h)	Estimate:
Yau	Complete Kähler with $Ric \geq -A$	Kähler with $HBC \leq -B < 0$	$\| df \|^2 \leq \frac{A}{B}$
Chen--Cheng--Lu	Complete Kähler with $HBC \leq -B < 0$ and $Rm \geq -C$	Hermitian with $HSC < -K < 0$	$\|df\|^2 \leq \frac{B}{K}$
Royden	Kähler with $Ric \geq -A$	Kähler with $HSC < -B < 0$	$\Delta_g \log \|\partial f\|^2 \geq -2A + \frac{k+1}{k} B \|\partial f\|^2$
Lei Ni	Complete Kähler with $HSC \geq -K$ and $HBC \geq -A$	Kähler with $HSC < -B < 0$	$\sup_{v \neq 0} \frac{\|\partial f(v)\|^2}{\|v\|^2} \leq \frac{K}{B}$

A unified perspective of the various versions of the Schwarz lemma is facilitated by the Bochner formula.

Let $f : (M, g) \to (N, h)$ be a holomorphic map between Hermitian manifolds. The differential of f is a holomorphic section ∂f of the bundle

$$\mathcal{E} := \mathcal{T}_M^* \otimes f^* \mathcal{T}_N.$$

Let dz_i be a local frame for $\mathcal{T}_M^*$ and $e_\alpha = f^* \partial_{w_\alpha}$ be a local frame for $f^* \mathcal{T}_N$. Then

$$\sigma := \partial f = f_i^\alpha dz_i \otimes e_\alpha,$$

where $f_i^\alpha = \frac{\partial f^\alpha}{\partial z_i}$.

Bochner formula:

The Bochner formula states that:

$$\partial\overline{\partial}|\sigma|^2 = \langle \nabla'\sigma, \nabla'\sigma \rangle - \langle \Theta^{\mathcal{E}}\sigma, \sigma \rangle,$$

where $\Theta^{\mathcal{E}}$ is the curvature of $\mathcal{E}$, namely:

$$\Theta^{\mathcal{E}} = -\Theta^{\mathcal{T}_M} \otimes \mathrm{id}_{f^*\mathcal{T}_N} + \mathrm{id}_{\mathcal{T}_M^*} \otimes f^*\Theta^{\mathcal{T}_N}.$$

We will apply this to the holomorphic section $\sigma = \partial f$ of $\mathcal{E}$.

Chern—Lu inequality:

Applying the Bochner formula with $\sigma = \partial f$, we see that

$$\partial\overline{\partial}|\partial f|^2 = |\nabla df|^2 + \langle \Theta^{\mathcal{T}_M} \partial f, \partial f \rangle - \langle f^* \Theta^{\mathcal{T}_N} \partial f, \partial f \rangle.$$

Taking the trace with respect to the metric ω_g, we see that

$$\Delta_{\omega_g}(|\partial f|^2) = \mathrm{tr}_{\omega_g}(\sqrt{-1}\partial\overline{\partial}|\partial f|^2) = |\nabla df|^2 + \mathrm{Ric}_g^{(2)}|\partial f|^2 - \langle \mathrm{Rm}_h \partial f, \partial f \rangle.$$

In coordinates, this is given by

$$\Delta_{\omega_g}(|\partial f|^2) = |\nabla df|^2 + g^{i\overline{j}} R_{i\overline{j}k\overline{\ell}} g^{k\overline{q}} g^{p\overline{\ell}} h_{\alpha\overline{\beta}} f_p^\alpha \overline{f_q^\beta} - R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} (g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta})(g^{p\overline{q}} f_p^\gamma \overline{f_q^\delta}).$$

Ricci term arising in the Schwarz lemma:

Observe that the above calculation yields the term

$$g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}.$$

If g is Kähler, then this is just the Ricci curvature of g.

But the Ricci curvature of Hermitian manifolds is more complicated.

Ricci curvature of a Hermitian metric:

Recall that the tangent bundle of a Hermitian non—Kähler manifold carries two distinguished connections — the Chern connection and the Levi-Civita connection — which do not coincide.

As a consequence, there are multiple ways in which we can take the trace of the Chern curvature.

Chern—Ricci curvatures:

There are four ways to contract the curvature of the Chern connection using the metric:

(i) Contracting the third and fourth indices:

$$\mathrm{Ric}^{(1)}_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}.$$

(ii) Contracting the first and second indices:

$$\mathrm{Ric}^{(2)}_{k\bar{\ell}} = g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}.$$

(iii) Contracting the second and third indices:

$$\mathrm{Ric}^{(3)}_{i\bar{\ell}} = g^{k\bar{j}} R_{i\bar{j}k\bar{\ell}}.$$

(iv) Contracting the first and fourth indices:

$$\mathrm{Ric}^{(4)}_{k\bar{j}} = g^{i\bar{\ell}} R_{i\bar{j}k\bar{\ell}}.$$

The **first Chern—Ricci** curvature

$$\mathrm{Ric}^{(1)}_{i\bar{j}} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}.$$

is given by

$$\mathrm{Ric}^{(1)}_{i\bar{j}} = -\frac{\partial^2 \log(\det(g_{k\bar{\ell}}))}{\partial z_i \partial \overline{z}_j}$$

and represents the first Chern class of the anti-canonical bundle.

But it is the **second Chern—Ricci** curvature tensor

$$\mathrm{Ric}^{(2)}_{k\bar{\ell}} = g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}.$$

that arises in the Schwarz Lemma calculation.

Let $\mathcal{E} \to X$ denote a holomorphic vector bundle, and denote by ∇^h its Chern connection – the unique connection compatible with its complex structure and metric. The curvature $\Theta^{\mathcal{E}}$ of the Chern connection is a section of the bundle

$$\Lambda^{1,1}\mathcal{T}_M^* \otimes \mathcal{E}^* \otimes \mathcal{E}.$$

If we take the trace over the endomorphism part $(\mathcal{E}^* \otimes \mathcal{E})$, we get a $(1,1)$–form – the first Chern–Ricci curvature.

If we take the trace of $\Theta^{\mathcal{E}}$ over the $\Lambda^{1,1}\mathcal{T}_M^*$ part, get an endomorphism $\text{tr}_\omega(\Theta^{\mathcal{E}})$ – the second Chern–Ricci curvature.

The Hopf manifold $\mathbb{S}^{2k+1} \times \mathbb{S}^1$ has strictly positive second Ricci curvature, but only has nonnegative first Ricci curvature.

From which, one arrives at the following second-order estimate:

$$\Delta_{\omega_g}(\log \mathrm{tr}_{\omega_g}(f^*\omega_h)) \geq \frac{1}{\mathrm{tr}_{\omega_g}(f^*\omega_h)} \Bigg[(\mathrm{Ric}_g^{(2)})_{k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\overline{\beta}} f_p^{\alpha} \overline{f_q^{\beta}}$$

$$- R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} (g^{i\bar{j}} f_i^{\alpha} \overline{f_j^{\beta}})(g^{p\bar{q}} f_p^{\gamma} \overline{f_q^{\delta}}) \Bigg]$$

If we fix a point $p \in M$, we can choose coordinates such that $f_i^\alpha = \lambda_i \delta_i^\alpha$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r > \lambda_{r+1} = \cdots = 0$, where $r = \text{rank}(df)$.
We see that

$$R^h_{\alpha\overline{\beta}\gamma\overline{\delta}}(g^{i\overline{j}} f_i^\alpha \overline{f_j^\beta})(g^{p\overline{q}} f_p^\gamma \overline{f_q^\delta}) \ = \ \sum R^h_{\alpha\overline{\beta}\gamma\overline{\delta}} \lambda_i^2 \lambda_k^2 \delta_i^\alpha \delta_k^\gamma \ = \ \sum_{\alpha,\gamma} R_{\alpha\overline{\alpha}\gamma\overline{\gamma}} \lambda_\alpha^2 \lambda_\gamma^2.$$

This observation motivates the following:

Real bisectional curvature:

Let (M^n, g) be a Hermitian manifold, and let R be the curvature of the Chern connection. For $p \in M$, let $\{e_1, ..., e_n\}$ be a unitary tangent frame for $\mathcal{T}^{(1,0)}_{M,p}$ and let $a = (a_1, ..., a_n) \in \mathbb{R}^n$ be a non-zero vector.

The real bisectional curvature of g, relative to the frame e and vector a is defined

$$B_g(e, a) := \frac{1}{|a|^2} \sum_{i,j=1}^{n} R_{i\bar{i}j\bar{j}} a_i a_j.$$

Let (M^n, g) be a Hermitian manifold and $v \in \mathcal{T}_M^{(1,0)}$ a $(1,0)$—tangent vector.

Choosing the unitary frame e such that e_1 is parallel to v and choosing the vector $a = (1, 0, ..., 0) \in \mathbb{R}^n$, we recover the holomorphic sectional curvature:

$$B_g(e, a) = \frac{1}{|v|^2} \sum_{i,j=1}^{n} R_{i\bar{j}k\bar{\ell}} v_i \overline{v}_j v_k \overline{v}_\ell.$$

In particular,

$$\text{RBC} \implies \text{HSC}.$$

In general, the holomorphic sectional curvature does not dominate the real bisectional curvature.

Indeed, the metric on the ball centered at the origin in $\mathbb{C}^{n \geq 2}$ given by

$$g_{i\bar{j}} = (1 + |z|^2)\delta_{ij} + (\varepsilon - 2)\overline{z}_i z_j, \qquad \varepsilon \in (0, 1),$$

has HSC > 0, but RBC does not have a sign.

Real bisectional curvature of Kähler metrics:

If the metric is Kähler, the symmetries of the curvature tensor ensure that the sign of the real bisectional curvature coincides with that of the holomorphic sectional curvature.

In the Hermitian non—Kähler setting, the following is true:

$$\text{HBC} \implies \text{RBC} \implies \text{Scal},$$

but

$$\text{RBC} \not\Longrightarrow \text{Ric}.$$

Returning to the main Schwarz Lemma Estimate:

Recall that we had

$$\Delta_{\omega_g}(\log \mathrm{tr}_{\omega_g}(f^*\omega_h)) \geq \frac{1}{\mathrm{tr}_{\omega_g}(f^*\omega_h)}\Bigg[(\mathrm{Ric}_g^{(2)})_{k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^{\alpha}\overline{f_q^{\beta}}$$

$$-R^h_{\alpha\bar{\beta}\gamma\bar{\delta}}(g^{i\bar{j}} f_i^{\alpha}\overline{f_j^{\beta}})(g^{p\bar{q}} f_p^{\gamma}\overline{f_q^{\delta}})\Bigg]$$

together with

$$R^h_{\alpha\bar{\beta}\gamma\bar{\delta}}(g^{i\bar{j}} f_i^{\alpha}\overline{f_j^{\beta}})(g^{p\bar{q}} f_p^{\gamma}\overline{f_q^{\delta}}) \ = \ \sum R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} \lambda_i^2 \lambda_k^2 \delta_i^{\alpha} \delta_k^{\gamma} \ = \ \sum_{\alpha,\gamma} R^h_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \lambda_{\alpha}^2 \lambda_{\gamma}^2.$$

If we assume that the real bisectional curvature of h is bounded above by $-B < 0$, then we see that

$$\sum R^h_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \lambda_{\alpha}^2 \lambda_{\gamma}^2 \ \leq \ -B\left(\sum_{\alpha} \lambda_{\alpha}^4\right) \ \leq \ -\frac{B}{r}\left(\sum_{\alpha} \lambda_{\alpha}^2\right)^2 \ = \ -\frac{B}{r}(\mathrm{tr}_{\omega_g}(f^*\omega_h))^2.$$

Final statement of Schwarz lemma:

Let $f : (M, g) \to (N, h)$ be a holomorphic map between Hermitian manifolds. Assume

$$\mathrm{Ric}_g^{(2)} \geq -\lambda \omega_g + \mu f^* \omega_h,$$

for some continuous functions λ and μ, with $\mu \geq 0$. Assume that $\mathrm{RBC}_h \leq -B < 0$, then

$$\Delta_{\omega_g}(\mathrm{tr}_{\omega_g}(f^*\omega_h)) \geq -\lambda \mathrm{tr}_{\omega_g}(f^*\omega_h) + \left(\frac{f^* B}{r} + \frac{\mu}{n} \right) \mathrm{tr}_{\omega_g}(f^*\omega_h)^2.$$

Thanks for listening!

In short:

(†) The Schwarz lemma controls the norm of the derivative of a holomorphic map in terms of:

(a) The curvature of the source – Ricci curvature typically.

(b) The curvature of the target – Bisectional, Holomorphic sectional, real bisectional.

A good mental model is provided by elastic equillibrium analogy (harmonic maps).

(†) The Schwarz lemma provides a certain 'duality' between the (second) Ricci curvature and the holomorphic sectional curvature (or real bisectional curvature).

The 'duality' is 'paired' via the Bochner formula.

(†) The holomorphic sectional curvature is related to Kobayashi hyperbolicity, but not exactly (Demailly's example).

(†) In the general Hermitian setting, the natural curvatures which arise are the Ricci and the real bisectional curvature.