The Schwarz Lemma

An Odyssey

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Based on the recent work:

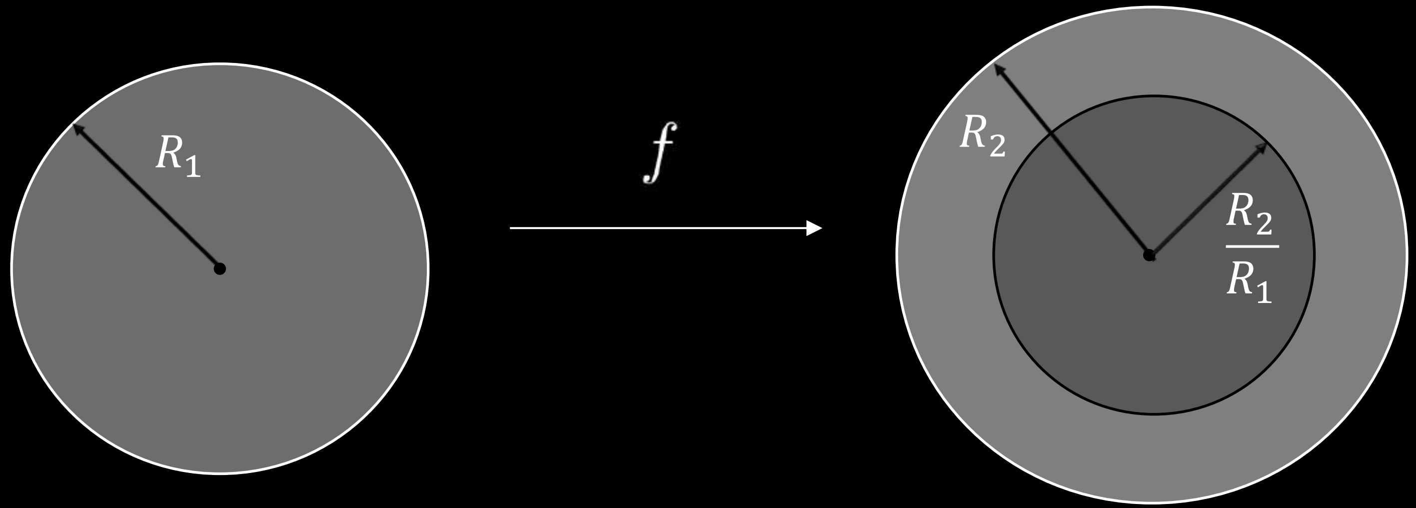
B., *The Schwarz Lemma in Kähler and Non-Kähler Geometry*, arXiv:2109.06331

B., *The Schwarz Lemma: An Odyssey* (to appear)

The Schwarz lemma is one of the pioneering results of one complex variable.

A holomorphic map $f : \mathbb{D}(R_1) \to \mathbb{D}(R_2)$ fixing the origin, satisfies

$$|f(z)| \le \frac{R_2}{R_1}|z|, \qquad \forall\ |z| < R_1.$$



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$$|f(z)| \; \leq \; \frac{R_2}{R_1}|z|, \qquad \forall \; |z| < R_1.$$

If we keep R_2 fixed, and let R_1 get arbitrarily large, we recover the following corollary:

Liouville's theorem:

A bounded holomorphic function $f : \mathbb{C} \to \mathbb{C}$ assumes at most one value.

Some motivation:

The fundamental theorem of algebra is precisely a theorem of this type:

A non-constant algebraic function $p : \mathbb{C} \to \mathbb{C}$ assumes all finite values.

The Schwarz lemma is a local, finite statement about holomorphic maps.

The Liouville theorem, in contrast, is a global statement obtained from letting $R_1 \to \infty$.

This is the prototypical example of the so-called

Bloch principle

any global statement concerning holomorphic maps arises from a stronger, finite version

Nihil est in infinito quod non prius fuerit in finito.

A second example of Bloch's principle

The Liouville theorem generalizes to the Picard theorem:

A non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ assumes all but possibly one value.

The finite version underpinning the Picard theorem is the Schottky theorem:

If $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic function such that $f(\mathbb{D}) \cap \{0, 1\} = \emptyset$,

then $|f(z)|$ affords a bound in terms of z and $f(0)$.

The first example of the Bloch principle:

The Valiron theorem:

The image of a non-constant entire function $f : \mathbb{C} \to \mathbb{C}$ contains Euclidean disks of arbitrarily large radii.

The corresponding finite version is Bloch's theorem:

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map with $f'(0) \neq 0$. Then the image $f(\mathbb{D})$ contains a disk of radius $\mathscr{B}|f'(0)|$ for some constant $\mathscr{B} > 0$.

André Bloch

Bloch collaborated with many great mathematicians, including Hadamard, Mittag-Leffler, Pólya, and Cartan.

Cartan

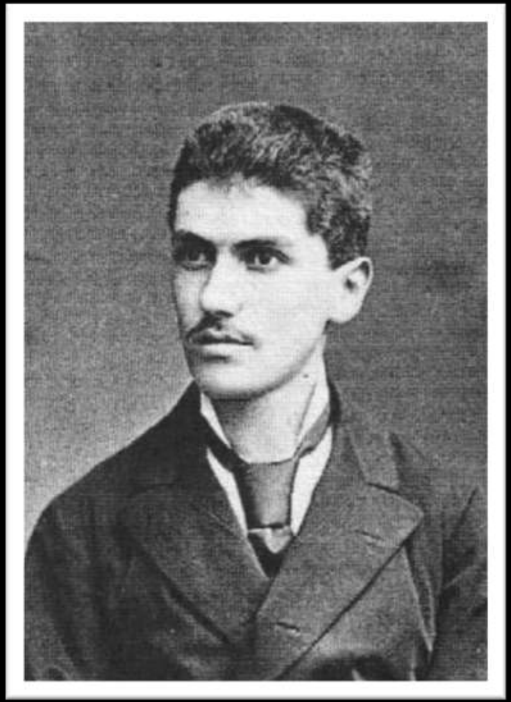
Hadamard

The First Divide

The Pick Schwarz Lemma

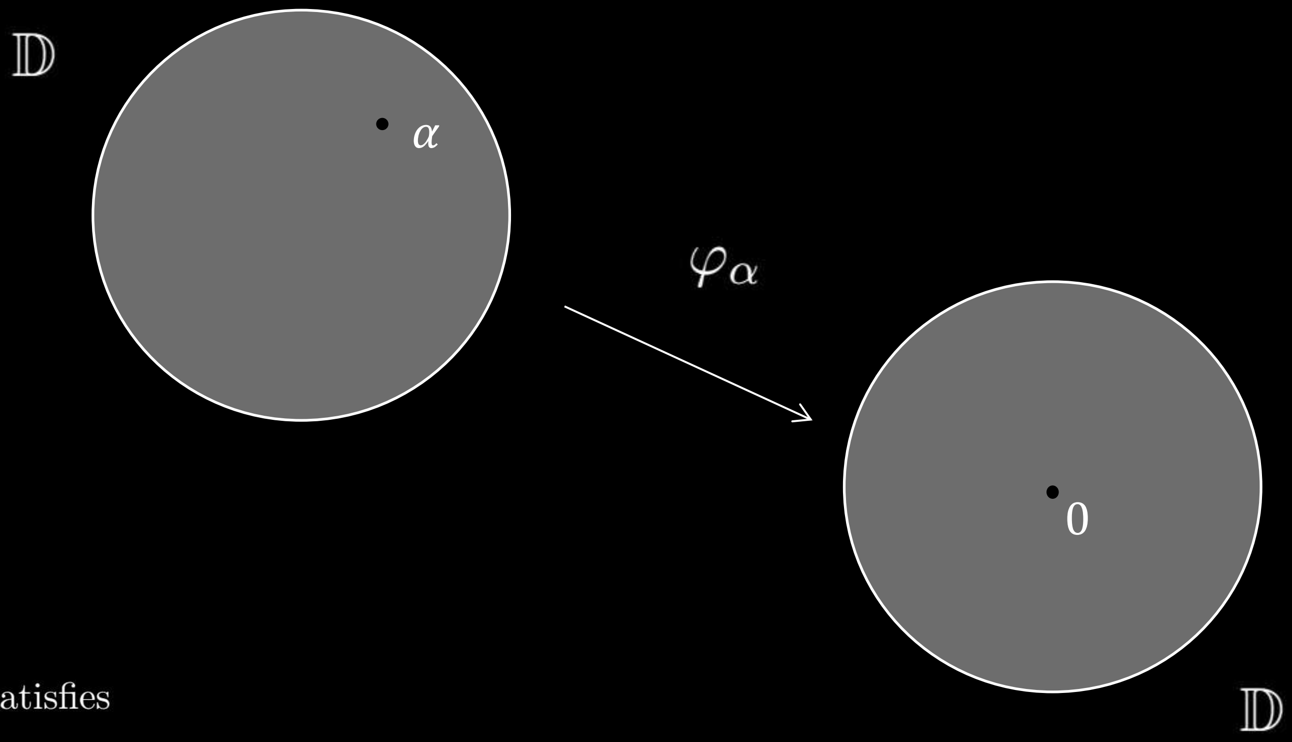
As it stands, the Schwarz lemma controls the growth of holomorphic functions.

It was observed by Pick in 1916, that the Schwarz lemma can be given a radically different interpretation.



We first observe that the assumption $f(0) = 0$ is superfluous:

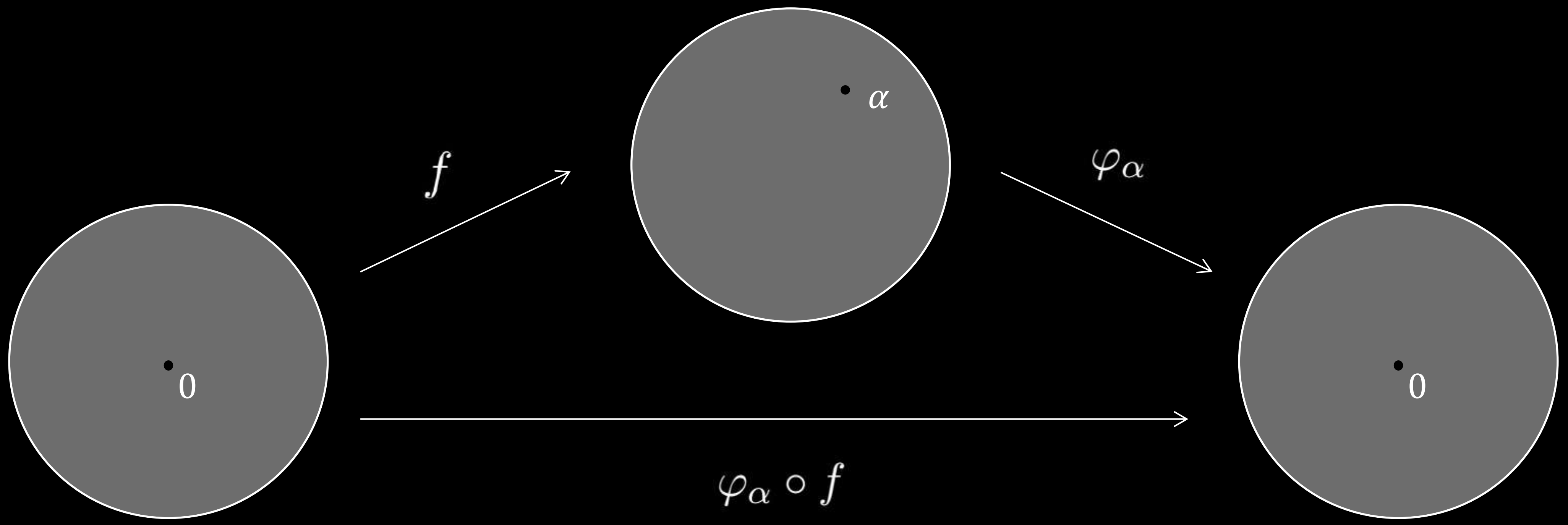
$$\varphi_\alpha(z) = \frac{z - \alpha}{1 - \overline{\alpha} z}$$



A holomorphic map $f : \mathbb{D}(R_1) \to \mathbb{D}(R_2)$ fixing the origin, satisfies

$$|f(z)| \ \le \ \frac{R_2}{R_1}|z|, \qquad \forall\ |z| < R_1.$$

We now have a holomorphic self-map of $\mathbb{D}$, fixing the origin:



Applying the familiar Schwarz lemma to this composite map $\varphi_{f(z)} \circ f \circ \varphi_{-z}$

$$\rightsquigarrow \quad |\varphi_{f(z)} \circ f(w)| \leq |\varphi_z(w)| \quad \text{where } w = \varphi_{-z}(\zeta)$$

$$\Big\Downarrow$$

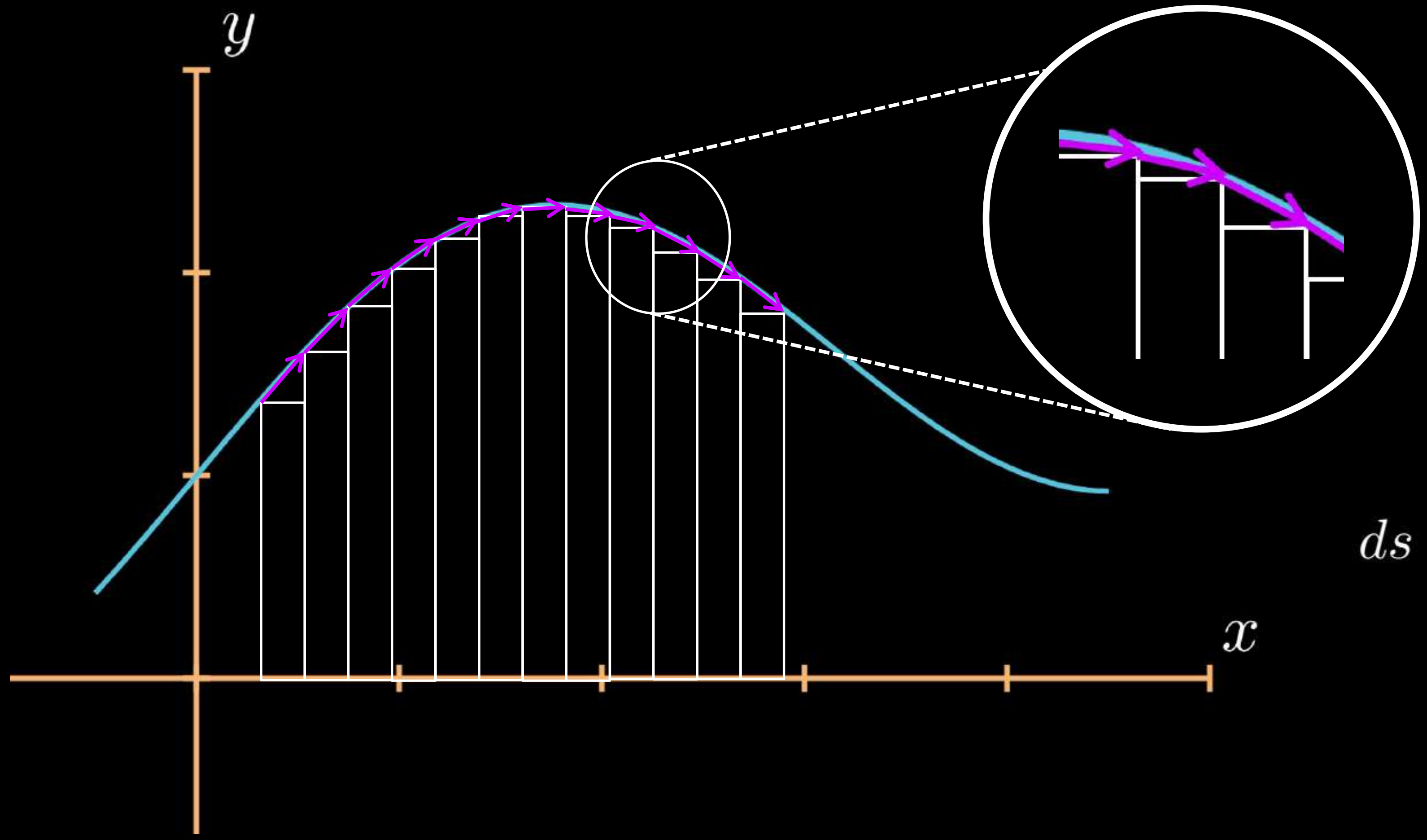
$$\left|\frac{f(w) - f(z)}{1 - \overline{f(z)} f(w)}\right| \leq \left|\frac{w - z}{1 - \bar{z} w}\right|$$

The function $d_{\mathrm{H}} : \mathbb{D} \times \mathbb{D} \to \mathbb{R}$,

$$d_{\mathrm{H}}(z, w) := \left| \frac{z - w}{1 - \overline{w} z} \right|$$

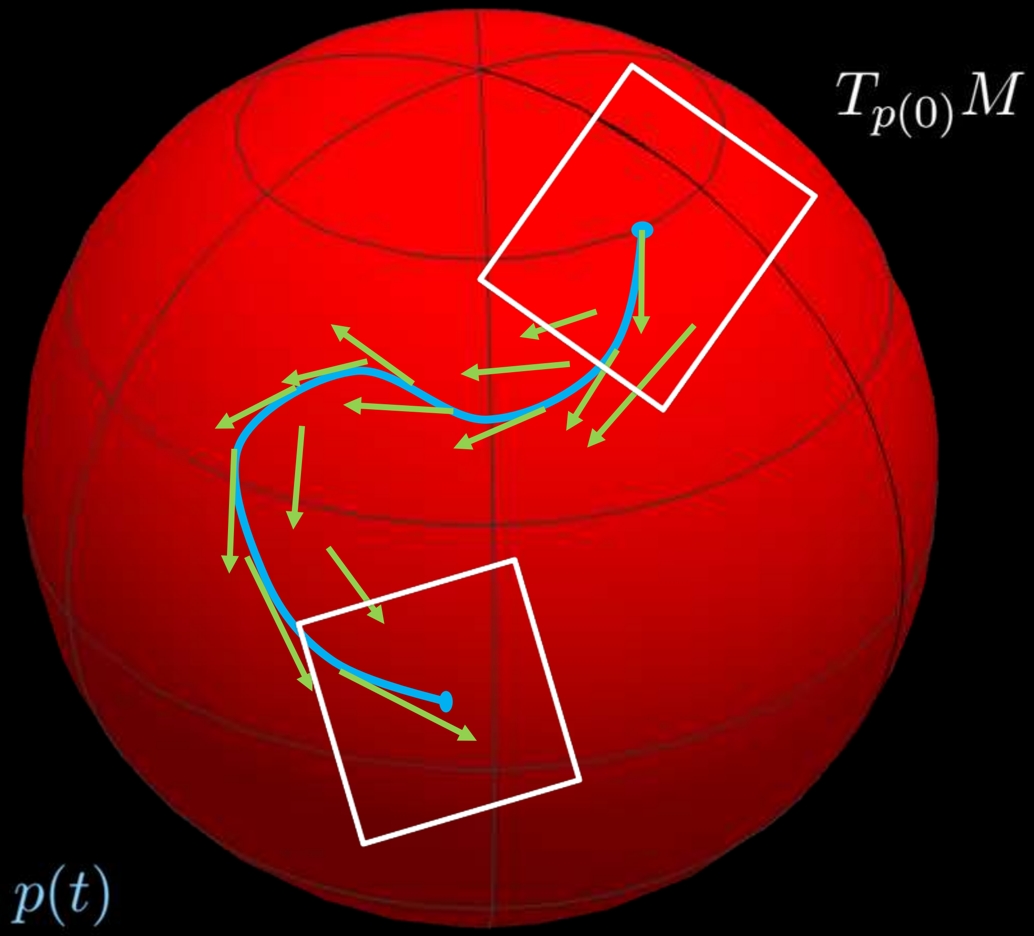
defines a distance function on $\mathbb{D}$, referred to as the pseudo-hyperbolic distance.

The pseudo-hyperbolic distance defines an honest distance function, i.e., a symmetric non-degenerate function satisfying the triangle inequality. It does not, however, come from integrating a Riemannian metric.



$$ds = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)} dt$$

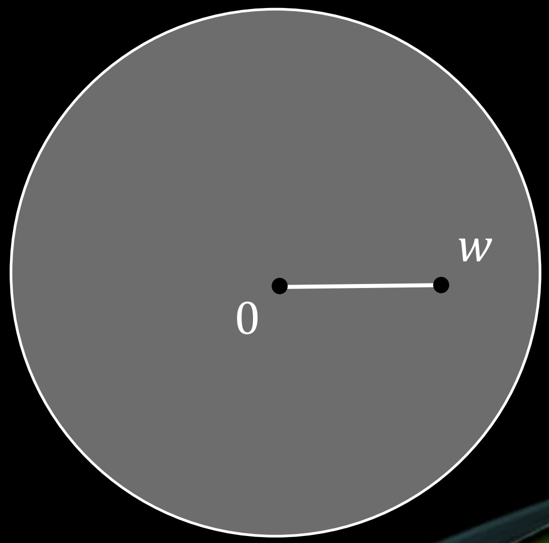
A Riemannian metric $g = \lambda|dz|$ is a smoothly varying family of inner products on each tangent space T_pM.



The Poincaré metric on $\mathbb{D}$ is defined

$$\rho = \frac{|dz|}{(1-|z|^2)^2}.$$

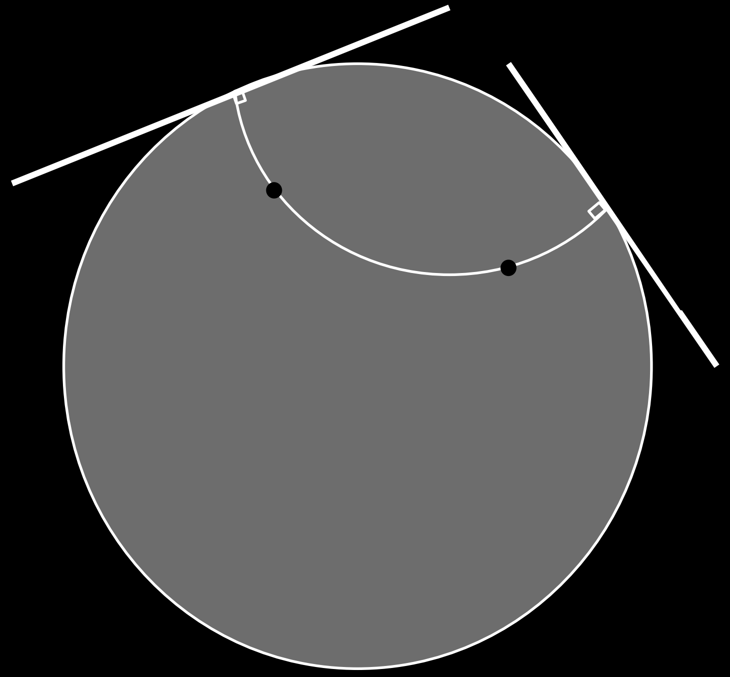
Let us compute the length of a line segment ℓ connecting 0 with $w \in (0,1)$.



$$\rho = \frac{|dz|}{(1-|z|^2)^2}.$$

Parametrize ℓ by $\gamma(t) = tw$. Then

$$\text{length}_\rho(\ell) = \int_\ell d\rho = \int_0^1 \frac{w}{1-w^2t^2}dt = \int_0^w \frac{dt}{1-t^2} = \frac{1}{2}\log\frac{1+w}{1-w}.$$



It is an elementary exercise to show that the associated distance function is given by

$$\text{dist}_\rho(z,w) = \frac{1}{2}\log\left|\frac{z-w}{1-z\overline{w}}\right|.$$

Summarizing this discussion, we have discovered the theorem of Pick:

Let $f : \mathbb{D} \to \mathbb{D}$ be a holomorphic self-map of the unit disk. Then for all $z, w \in \mathbb{D}$,

$$\mathrm{dist}_\rho(f(z), f(w)) \quad \leq \quad \mathrm{dist}_\rho(z, w).$$

That is, with respect to the Poincaré distance, all holomorphic maps are distance-decreasing.

Aside: Intrinsic Metrics

The Poincaré metric ρ has the property that

(i) it is invariant under the automorphism group $\mathrm{Aut}(\mathbb{D})$

(ii) all holomorphic maps are distance-decreasing

$$\mathrm{dist}_\rho(f(z), f(w)) \leq \mathrm{dist}_\rho(z, w).$$

The Caràtheodory pseudo-distance on a domain $\Omega \subset \mathbb{C}^n$

$$d_\Omega(z, w) := \sup_f \operatorname{dist}_\rho(f(z), f(w))$$

where the supremum ranges over all holomorphic maps $f : \Omega \to \mathbb{D}$

generalises these properties of the Poincaré metric to arbitrary domains

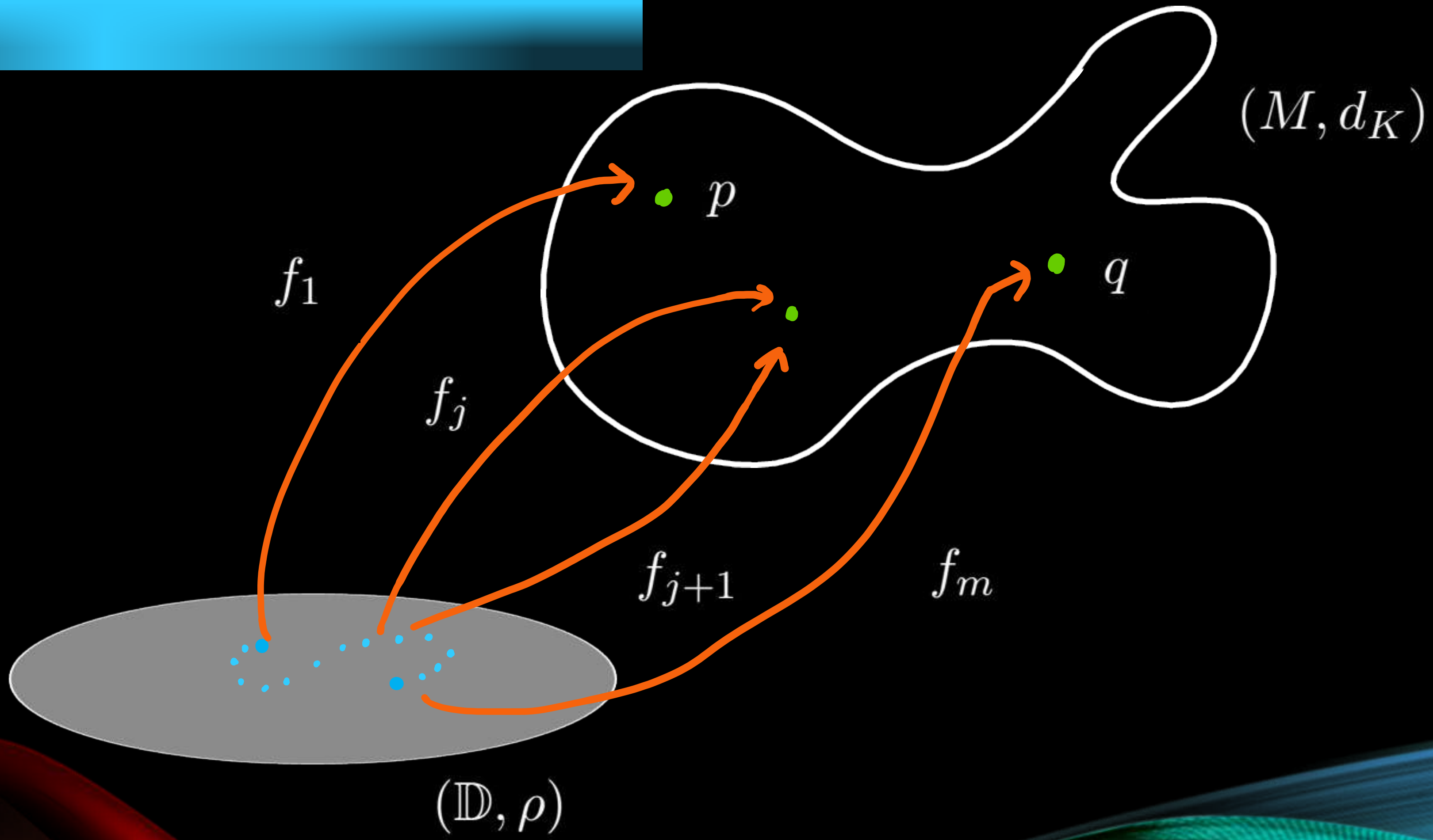
We call such a function an *intrinsic pseudo-distance.*

The existence of an intrinsic pseudo-distance on any complex manifold was introduced by Kobayashi

The Kobayashi pseudo-distance is defined

$$d_K(p,q) \;:=\; \inf \sum_{i=1}^{m} \mathrm{dist}_\rho(s_j, t_j),$$

where the infimum is taken over all $m \in \mathbb{N}$, all pairs of points $\{s_j, t_j\}$ in $\mathbb{D}$, and all collections of holomorphic maps $f_j : \mathbb{D} \to M$ such that $f_1(s_1) = p$, $f_m(t_m) = q$ and $f_j(t_j) = f_{j+1}(s_{j+1})$.


(M, d_K)
p
q
f_1
f_j
f_{j+1}
f_m
$(\mathbb{D}, \rho)$

A complex manifold is said to be Kobayashi hyperbolic if d_K is

an honest distance functions, i.e., d_K is non-degenerate.

If M is compact we can give a simpler definition:

M is Kobayashi hyperbolic

$\Longleftrightarrow$

Every entire curve $\mathbb{C} \to M$ is constant

(Brody hyperbolic)

The Picard theorem states that $\mathbb{C} \backslash \{0, 1\}$ is Brody hyperbolic.

Kobayashi hyperbolic manifolds form a rich class of complex manifolds.

Stein Manifolds

Complex submanifolds of $\mathbb{C}^n$

Analytic cousins (in the sense of GAGA) of affine varieties

Stein manifolds M have an abundance of holomorphic maps $M \to \mathbb{C}$.

Stein Manifolds

Abundance of holomorphic maps

$$M \to \mathbb{C}$$

Kobayashi hyperbolic manifolds

No holomorphic maps

$$\mathbb{C} \to M.$$

Stein Manifolds

Abundance of holomorphic maps

$$M \to \mathbb{C}$$

$$\cdots$$

Abundance of holomorphic maps

$$\mathbb{C} \to M.$$

Kobayashi hyperbolic manifolds

No holomorphic maps

$$\mathbb{C} \to M.$$

$$\cdots$$

No holomorphic maps

$$M \to \mathbb{C}$$

Stein Manifolds

Abundance of holomorphic maps

$$M \to \mathbb{C}$$

Kobayashi hyperbolic manifolds

No holomorphic maps

$$\mathbb{C} \to M.$$

Oka manifolds

Abundance of holomorphic maps

$$\mathbb{C} \to M.$$

Does not appear interesting

No holomorphic maps

$$M \to \mathbb{C}$$

Oka manifolds

Every continuous map $f : S \to M$, where S is Stein

is homotopy-equivalent to a holomorphic map $f : S \to X$

Examples:

Complex Lie groups and their homogeneous spaces.

Hopf manifolds, Hirzebruch surfaces.

Riemann surfaces of genus $g = 0$ and $g = 1$.

There is an abstract homotopy theory lurking in the background here:

Larusson showed that one can embed the category of complex manifolds into a model category (in the sense of Quillen) in such a way that a manifold is cofibrant if and only it is Stein and fibrant if and only if it is Oka.

a category you can do homotopy theory in

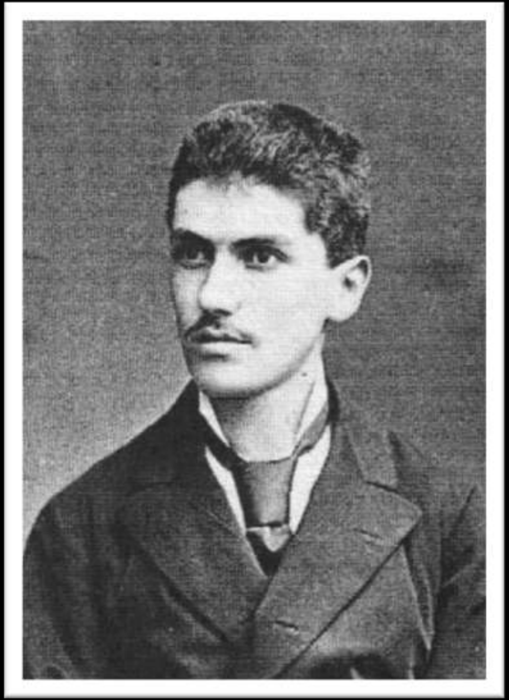
The Second Divide

The Ahlfors Schwarz Lemma

Schwarz

Dilations of holomorphic maps

$$|f(z)| \le \frac{R_2}{R_1}|z|$$

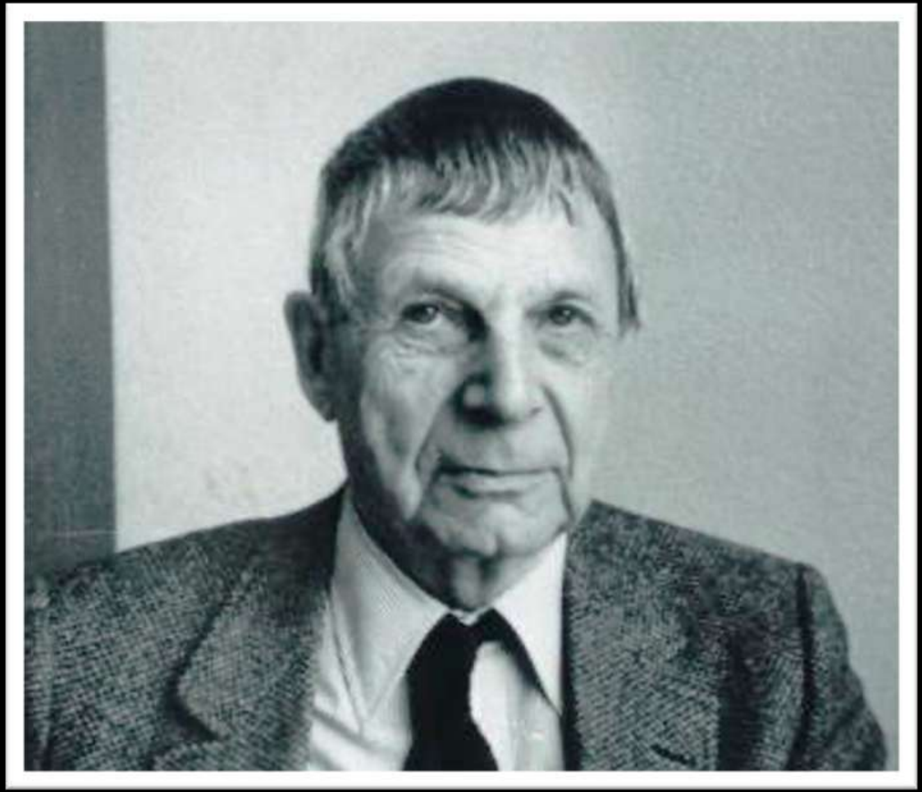


Pick

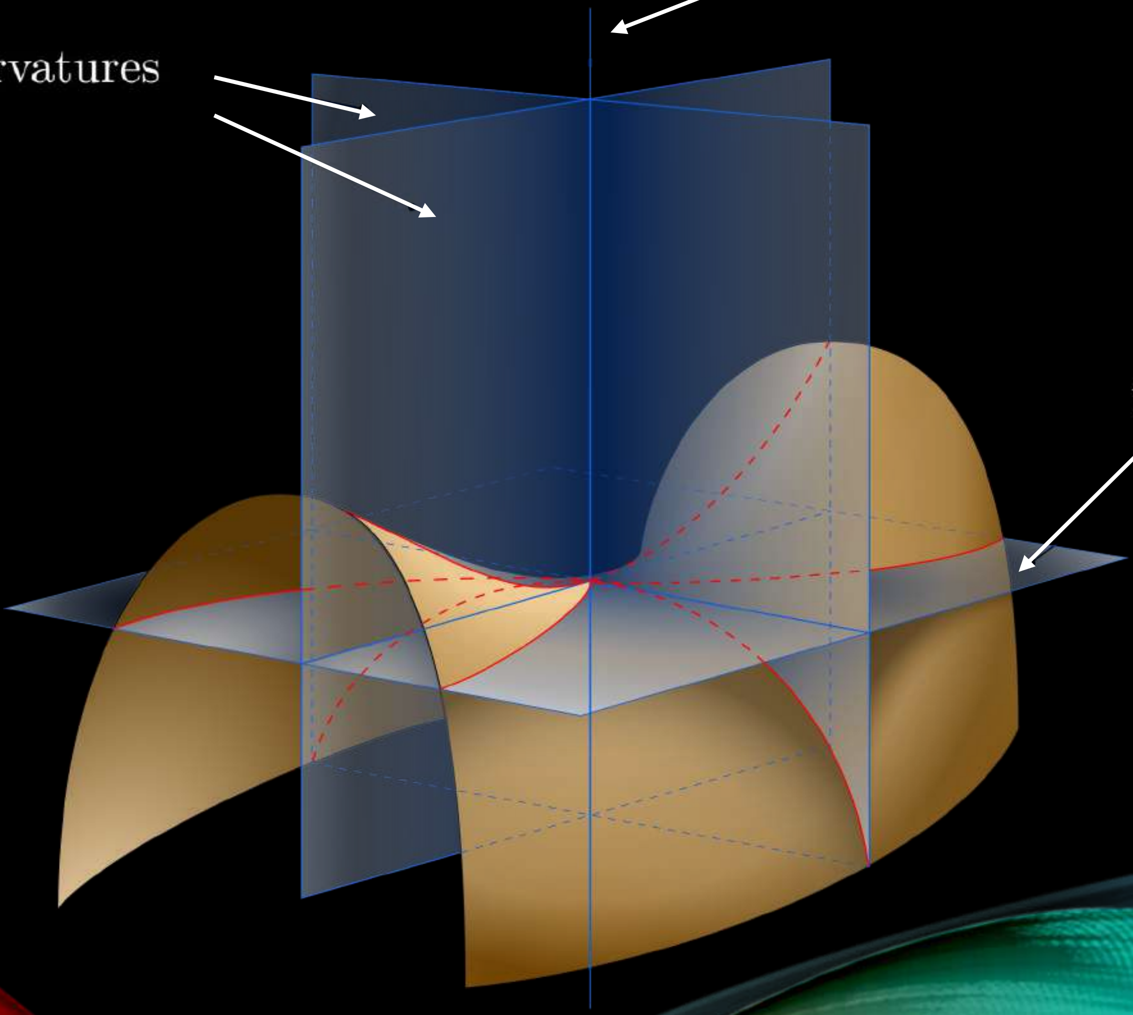
Poincaré distance

$$\mathrm{dist}_\rho(f(z), f(w)) \quad \le \quad \mathrm{dist}_\rho(z, w).$$

The real breakthrough came from Ahlfors version of the Schwarz lemma in 1937



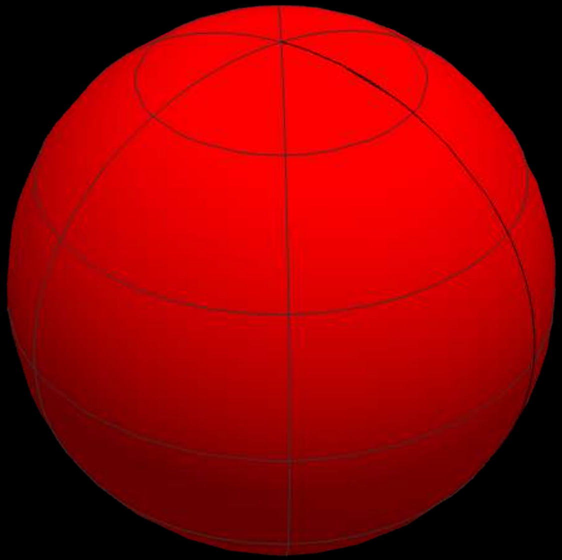
Reminder: Gauss curvature



$$g = \lambda |dz|$$

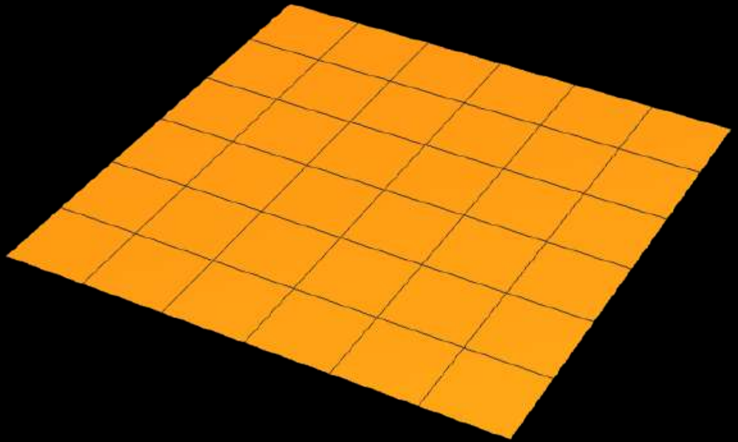
$$\Big\downarrow$$

$$K_g = -\frac{1}{\lambda}\frac{\partial^2}{\partial z \partial \overline{z}} \log(\lambda)$$



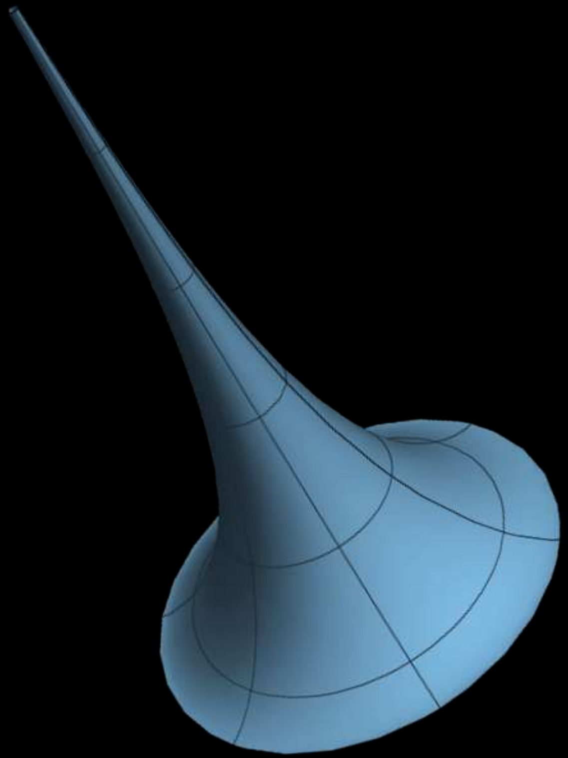
Positive curvature

$K_g > 0$



Vanishing curvature

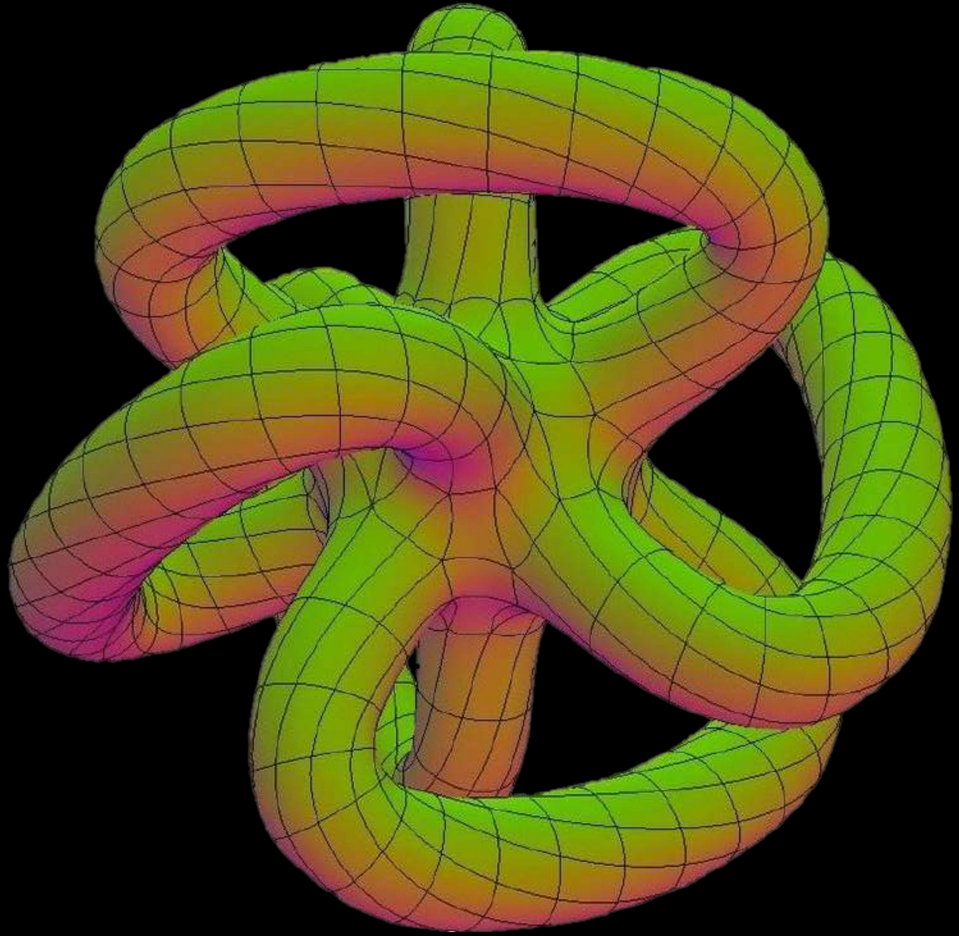
$K_g = 0$



Negative curvature

$K_g < 0$

genus $g = 3$



genus $g = 6$

Hyperbolic surfaces

Recall the Poincaré metric is given by the formula

$$\rho = \frac{|dz|}{(1 - |z|^2)^2} \qquad \rightsquigarrow \qquad \lambda(z) = \frac{1}{(1 - |z|^2)^2}$$

Let us compute its Gauss curvature.

$$K_g = -\frac{1}{\lambda(z)^2} \frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda(z).$$

$$\rho = \frac{|dz|}{(1-|z|^2)^2} \quad \rightsquigarrow \quad \lambda(z) = \frac{1}{(1-|z|^2)^2}$$

So the Gauss curvature of the Poincaré metric is

$$K_g = -\frac{1}{\lambda(z)^2}\frac{\partial^2}{\partial z \partial \overline{z}} \log \lambda(z).$$

$$= -\frac{1}{\lambda(z)}\frac{1}{(1-|z|^2)^2} \quad = -\frac{(1-|z|^2)^2}{(1-|z|^2)^2} \quad = -1.$$

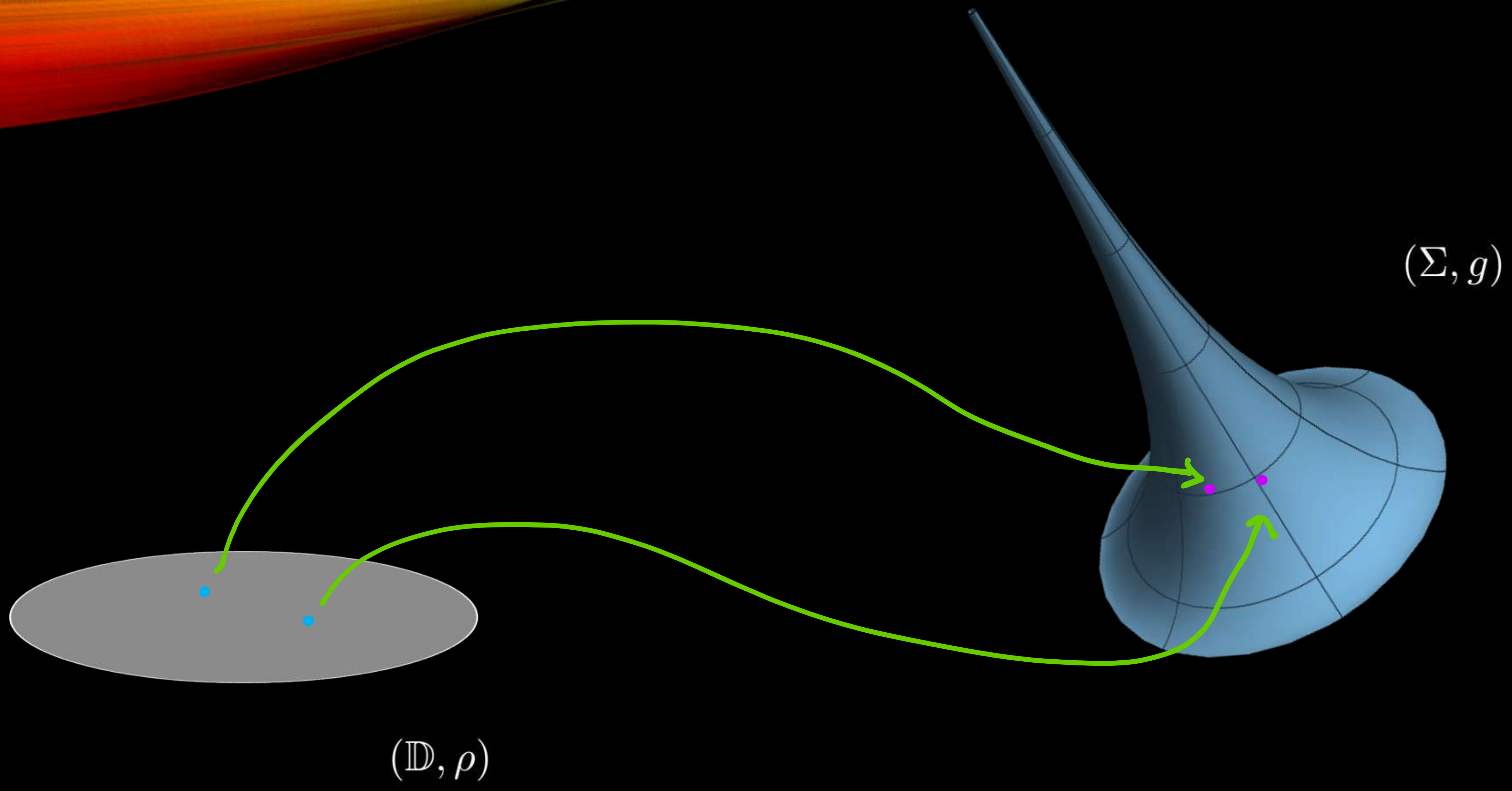
Constant (negative) Gauss curvature

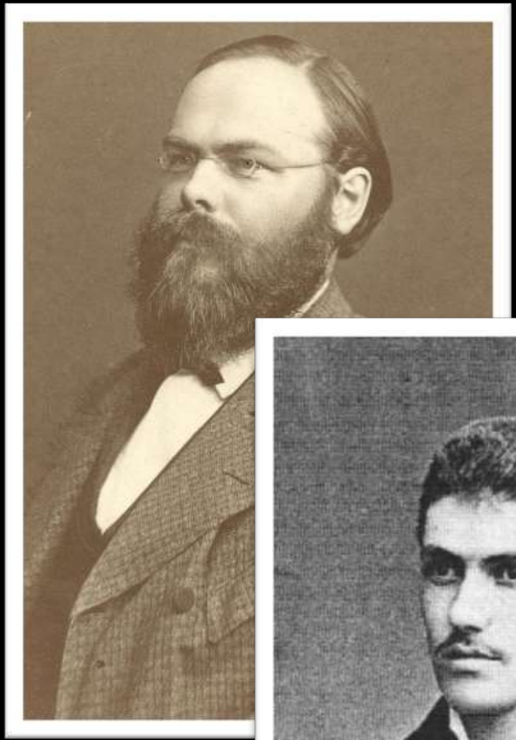
The Ahlfors Schwarz Lemma

Let (Σ, g) be a Riemann surface with Gauss curvature $K_g \leq -1$.

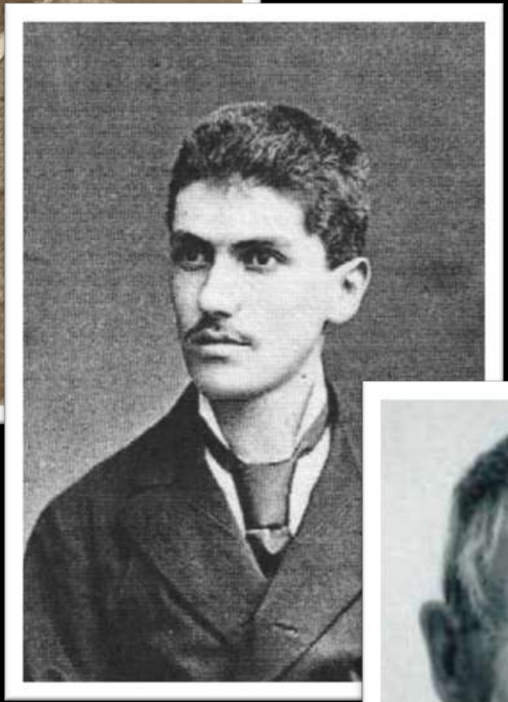
Then for all holomorphic maps $f : \mathbb{D} \to \Sigma$,

$$\mathrm{dist}_g(f(z), f(w)) \ \leq \ \mathrm{dist}_\rho(z, w)$$

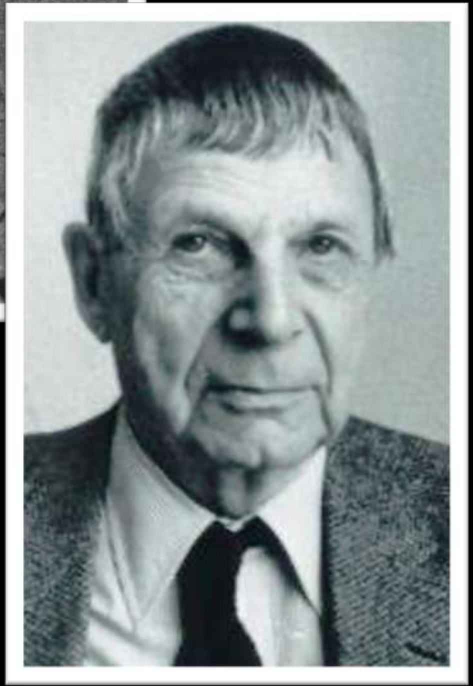

(Σ, g)
$(\mathbb{D}, \rho)$



Schwarz



Pick



Ahlfors

Dilations of holomorphic maps

$$|f(z)| \leq \frac{R_1}{R_2}|z|$$

Poincaré distance

$$\mathrm{dist}_\rho(f(z), f(w)) \quad \leq \quad \mathrm{dist}_\rho(z, w).$$

Negative curvature

$$\mathrm{dist}_g(f(z), f(w)) \ \leq \ \mathrm{dist}_\rho(z, w)$$

In Ahlfors collected works, one finds the following reflection

of Ahlfors concerning his version of the Schwarz lemma

He confesses that it had “more substance than I was aware of.”

But, “without aplications, my lemma would have been too lightweight for publication.”

The applications Ahlfors alludes to were the following:

A proof of Schottky's theorem, with definite numerical bounds:

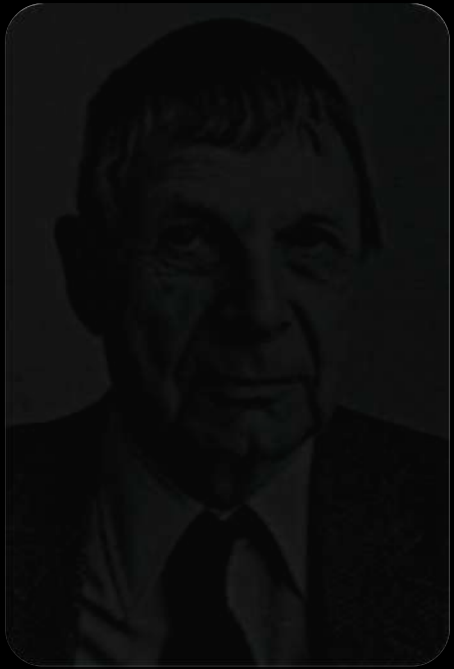
If $f : \mathbb{D} \to \mathbb{C}$ is a holomorphic map such that $f(\mathbb{D}) \cap \{0, 1\} = \emptyset$, then

$$\log|f(z)| < \frac{1+\vartheta}{1-\vartheta}(7 + \log|f(0)|)$$

for all $|z| \leq \vartheta < 1$.

The second application was an improved lower bound on the Bloch constant:

$$\mathscr{B} \geq \frac{\sqrt{3}}{4}.$$



Recall the statement of Bloch's theorem:

Let $f : \mathbb{D} \to \mathbb{C}$ be a holomorphic map with $f'(0) \neq 0$. Then the image

$$f(\mathbb{D}) \text{ contains a disk of radius } \mathscr{B}|f'(0)| \text{ for some constant } \mathscr{B} > 0.$$

Bloch showed that $\mathscr{B} \geq \frac{1}{72} \sim 0.01389$

Ahlfors showed that $\mathscr{B} \geq \frac{\sqrt{3}}{4} \sim 0.433$

Ahlfors and Grunsky showed that

$$\mathscr{B} \leq \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\Gamma\left(\frac{1}{4}\right)\sqrt{1+\sqrt{3}}} \sim 0.4719$$

It is conjectured that

$$\mathscr{B} = \frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{11}{12}\right)}{\sqrt{1+\sqrt{3}}\Gamma\left(\frac{1}{4}\right)} \sim 0.4719.$$

The current record is held by Xiong (2004), with

$$\mathscr{B} \geq \frac{\sqrt{3}}{4} + 3 \times 10^{-4} \sim 0.4333$$

Each of these formulations have lead to huge developments:

The Pick Schwarz lemma leads one to the notion of Kobayashi hyperbolicity

The Ahlfors Schwarz lemma leads to important second-order estimates in Kähler geometry

Generalizing the original formulation has often been done with the focus on the boundary

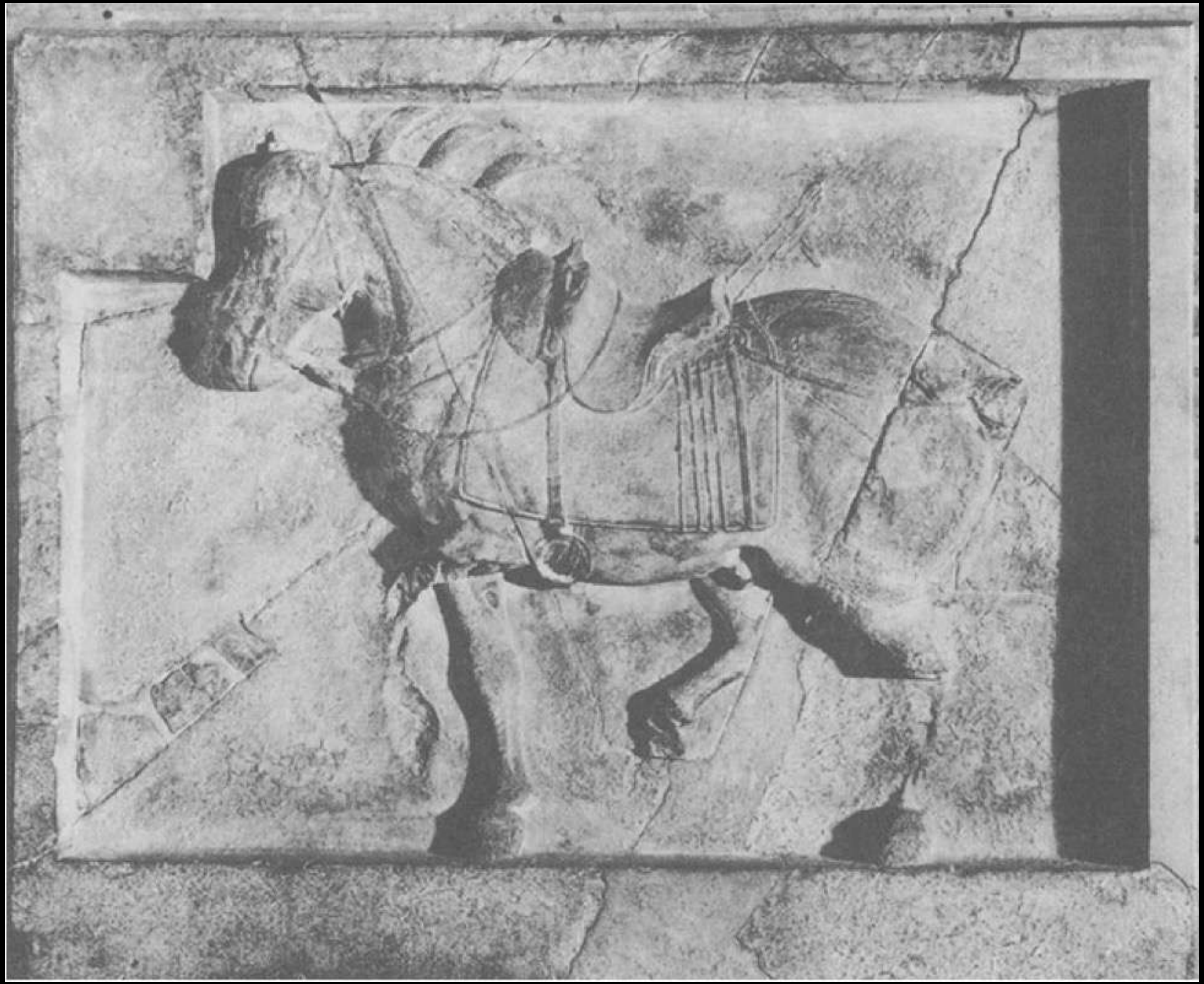
What has lead to the recent developments is by focusing on the map

Curvatures arise in the domain or range of a holomorphic map

Hence, one can produce desired curvature constraints

by composing various holomorphic maps.

老馬識途



Thanks for listening