TWISTED KÄHLER–EINSTEIN METRICS AND COLLAPSING

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Abstract. Let X be a compact Kähler manifold with semi-ample canonical bundle K_X. Write $f : X \to X_{\text{can}}$ for the Iitaka fibration induced by the pluricanonical system $|K_X^\ell|$, for $\ell \gg 0$. Let $\omega(t)$ be a family of Kähler metrics on X evolving under the Kähler–Ricci flow or under the La Nave–Tian continuity method. Denote by $(\mathcal{Z}, d_{\mathcal{Z}})$ the Gromov–Hausdorff limit of the sequence $(X, \omega(t))$. These (incomplete) notes serve to detail the progress the author has made concerning the structure of $(\mathcal{Z}, d_{\mathcal{Z}})$.

Contents

1. Introduction

The purpose of this note is to detail the problem that the author has been working on for the past year. The problem concerns a particular metric geometry conjecture pertaining to the minimal model program (MMP) – a program aimed at the classification of varieties up to birational isomorphism. To motivate the problem we consider here, we recall the *abundance conjecture*: Let X be a normal projective variety over $\mathbb{C}$.[1] If the canonical bundle K_X is nef, then K_X is semi-ample. This has been verified by Kawamata [42, 43] for minimal models of general type. Moreover, Song [58] gave an analytic proof of Kawamata's result for smooth minimal models of general type. We state [58, Theorem 1.1] explicitly:

Theorem 1.1. Let X be a projective manifold with K_X nef and big. Then K_X is semi-ample, i.e., for sufficiently large $\ell \in \mathbb{N}$, the bundle K_X^ℓ is globally generated.

The Riemannian analogue of the above theorem may be formulated in the following manner:

[1] Unless otherwise stated, all varieties will be over the complex numbers.

Theorem 1.2. Let X be a smooth minimal model of general type. Let n be the complex dimension of X. There exists a unique smooth Kähler–Einstein metric ω_{KE} on the regular part of the canonical model X_{can}° of X such that

(i) ω_{KE} extends uniquely to a Kähler current on X_{can} with locally bounded potentials.
(ii) The metric completion of $(X_{\text{can}}^{\circ}, \omega_{\text{KE}})$ is a compact length metric space $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ homeomorphic to the projective variety X_{can}.
(iii) The singular set $\mathcal{S}$ of $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ has (real) Hausdorff dimension no greater than $2n-4$. In particular, $(\mathcal{Z} \backslash \mathcal{S}, \text{dist}_{\mathcal{Z}})$ is convex in $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ and is homeomorphic to $(X_{\text{can}}^{\circ}, \text{dist}_{\omega_{\text{KE}}})$.

In this note, we are almost exclusively interested in statement (ii) of the above theorem. The main conjecture of this note is the following:

Conjecture 1.3. If $(\mathcal{Z}, d_{\mathcal{Z}})$ denotes the metric completion of $(X_{\text{can}}^{\circ}, \omega_{\text{can}})$ and $S_{\text{can}} := \mathcal{Z} \backslash X_{\text{can}}^{\circ}$, then

(i) $(\mathcal{Z}, d_{\mathcal{Z}})$ is a compact length metric space and S_{can} has real Hausdorff codimension at least 2.
(ii) $(X, \omega(t))$ converges to $(\mathcal{Z}, d_{\mathcal{Z}})$ in the Gromov–Hausdorff topology.
(iii) $\mathcal{Z}$ is homeomorphic to X_{can}.

Again, statement (iii) is the primary problem of interest, i.e., determining the homeomorphism type of the Gromov–Hausdorff limit. Under strong control of the singularities of the canonical model, part (iii) of the conjecture has been verified by Song–Tian–Zhang [61] and Gross–Tosatti–Zhang [35]. The problem is intimately related to many fields of geometry and PDE, including Kähler geometry, the minimal model program, metric geometry, Ricci flow, moduli theory, and several complex variables. In section 2 we detail the relevant background from each of these fields. In section 3 we include the recent aforementioned results of Song–Tian–Zhang and Gross–Tosatti–Zhang.

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2. Background

2.1. **Reminder: Kähler Geometry.**

A *Kähler manifold* is a smooth manifold M endowed with a triple of tensors (g, J, ω) – a Riemannian metric g, an almost complex structure J, and a non-degenerate 2–form ω, satisfying the following conditions:

(A) *Coherence:*
  (i) $g(Ju, Jv) = g(u, v)$ for all $u, v \in TM$, i.e., g is a Hermitian metric.
  (ii) $\omega(u, Ju) > 0$ for all nonzero $u \in TM$, i.e., J is ω–tame.

(iii) $\omega(Ju, Jv) = \omega(u, v)$ for all $u, v \in TM$, i.e., ω is compatible with J.[2]

(B) *Closedness.* The non-degenerate 2–form is closed: $d\omega = 0$. In particular, ω is a symplectic form which represents a non-trivial de Rham cohomology class $[\omega] \in H^2_{\mathrm{DR}}(M, \mathbb{R})$.

(C) *Integrability.* The almost complex structure $J : TM \to TM$ is integrable, i.e., the holomorphic tangent bundle $T^{1,0}M$ is involutive and hence, M is a complex manifold (see, e.g., [55]).

The metric g above is called a *Kähler metric* and the symplectic form ω above is called a *Kähler form.*

Remark. The existence of an almost complex structure J implies that M is even-dimensional: Indeed, if n is the real dimension of M, we see from $\det(J^2) = \det(J)^2 = (-1)^n$ that n is necessarily even. Moreover, the existence of the Kähler form ω requires that $H^2(M, \mathbb{R})$ is non-zero.

Convention 2.1.1. It is common practice in Kähler geometry to interchange the Kähler metric and the Kähler form without distinction.

Example 2.1.2.

(i) Complex projective space $\mathbb{P}^n$ equipped with the Fubini–Study metric ω_{FS} is Kähler. In the affine chart $U_0 := \{[Z_0 : \cdots : Z_n] : Z_0 = 1\}$, the Fubini–Study metric can be written as

$$\omega_{\mathrm{FS}} \quad := \quad \sqrt{-1}\partial\overline{\partial} \log \left(1 + \sum_{j=1}^{n} |Z_j|^2 \right).$$

(ii) Complex Euclidean space $\mathbb{C}^n$ equipped with the Euclidean metric

$$\omega_{\mathrm{Eucl.}} := \sqrt{-1} \sum_{i,j=1}^{n} dz^i \wedge d\overline{z}^i,$$

is Kähler.

(iii) Complex submanifolds of Kähler manifolds are Kähler. In particular, projective manifolds and Stein manifolds[3] are Kähler.

[2]Note that the compatibility of ω and J is expressed by (ii) and (iii), not just (iii).

[3]Stein manifolds are those complex manifolds which admit holomorphic embeddings into $\mathbb{C}^n$. Domains of holomorphy and open Riemann surfaces are examples of Stein manifolds.

Reminder. Let (M,g) be a Riemannian manifold. A *geodesic* on M is a smooth curve $\gamma : [0,1] \to M$ which locally minimizes the length between any points.[4] Using geodesics, we can define a canonical set of local coordinates on M. Indeed, fix $p \in M$, and let $\exp_p : T_pM \to M$ be the map sending a tangent vector $v \in T_pM$ to the endpoint of the unique geodesic $\gamma : [0,1] \to M$ with $\gamma(0) = p$, $\gamma'(0) = v$. This map is always a diffeomorphism in some neighbourhood of p and the coordinates obtained from such a diffeomorphism are called *geodesic normal coordinates*. By Hopf–Rinow, if M is compact (completeness suffices), the exponential map is defined for all $v \in T_pM$.

In these coordinates, we may compute the Taylor expansion of the components of the metric g, upon which we find that

$$g_{ij} \quad = \quad \delta_{ij} - \frac{1}{3} R_{ikj\ell} x^k x^\ell + O(|x|^3),$$

where δ_{ij} is the Euclidean metric on $\mathbb{R}^n$. The correction term $R_{ikj\ell}$ at the quadratic level is called the *Riemannian curvature tensor* and is a tensor of type $(4,0)$. The Riemannian curvature tensor measures the extent to which distances between geodesics are distorted under the exponential map.

While this definition of curvature gives very nice geometric intuition, it is not the most useful when it comes to computing the curvature. For the purposes of computation, we will recall the (more standard) definition of curvature in terms of connections.

Reminder. Let $\pi : E \to M$ be a smooth vector bundle of rank k on a smooth manifold M. A *connection* (more precisely, a covariant derivative) on E is a map $\nabla : \Gamma(E) \to \Gamma(E \otimes T^*M)$ such that, for any smooth function $f \in \mathcal{C}^\infty(M)$ and any smooth section $\sigma \in \Gamma(E)$, the Leibniz rule is satisfied

$$\nabla(f\sigma) = df \otimes \sigma + f \cdot \nabla\sigma,$$

where d is the exterior derivative.

Let $\{e_1, ..., e_k\}$ be a smooth (local) frame for E. The connection can then be defined in terms of such local frames by

$$\nabla e_i = \sum_{j=1}^k \vartheta_{ij} e_j,$$

where ϑ_{ij} is a matrix of 1–forms called the *connection matrix* (relative to this frame). Let $\{e_1', ..., e_k'\}$ be another local frame for E, and let φ_{ij} be the bundle charts for E. Then, by

[4]Geodesics in Euclidean space are simply straight lines, geodesics on $\mathbb{S}^2$ are the great circles.

definition $e_i' = \sum_{j=1}^k \varphi_{ij} e_j$. The action of the covariant derivative in terms of this new frame is given by $\nabla e_i' = \sum_{j=1}^k \vartheta_{ij}' e_j'$. Moreover,

$$\nabla e_i' \;=\; \sum_{\ell=1}^k \left(\sum_{j=1}^k d\varphi_{ij} + \varphi_{ij}\vartheta_{j\ell} \right) e_\ell,$$

and combining these two expressions for $\nabla e_i'$, we have

$$\sum_{\ell=1}^k \sum_{j=1}^k \varphi_{ij}\vartheta_{j\ell}' e_\ell \;=\; \sum_{\ell=1}^k \left(\sum_{j=1}^k d\varphi_{ij} + \varphi_{ij}\vartheta_{j\ell} \right) e_\ell,$$

which (in matrix notation) implies that

$$\vartheta' = d\varphi \cdot \varphi^{-1} + \varphi \cdot \vartheta \cdot \varphi^{-1}.$$

Reminder. Let $\pi : E \to M$ be a smooth vector bundle with a connection ∇. We say that ∇ is a *metric connection* if

Reminder. Let $\pi : E \to M$ be a smooth vector bundle of rank k on a smooth manifold M. Let ∇ be a connection on E. The *curvature* of ∇ is defined by $\nabla^2 := \nabla \circ \nabla$.

Example.

(i) The exterior derivative d is a flat connection on the trivial bundle.
(ii)

Important special case. Let $L \to M$ be a holomorphic line bundle on a complex manifold M. Let (U_α) be a trivialising chart for L. A frame for L (over U_α, say) consists of a single holomorphic section $e_\alpha : U_\alpha \to L$. Let ∇ be a connection on L specified in U_α by

$$\nabla e_\alpha = \vartheta_\alpha e_\alpha,$$

for some 1–form ϑ_α. Let $g_{\alpha\beta}$ be the transition functions for L, i.e., $e_\alpha = g_{\alpha\beta} e_\beta$. Applying the connection to this relation, we see that

$$\vartheta_\alpha g_{\alpha\beta} e_\beta \;=\; \vartheta_\alpha e_\alpha \;=\; \nabla e_\alpha \;=\; dg_{\alpha\beta} \otimes e_\beta + g_{\alpha\beta} \nabla e_\beta \;=\; dg_{\alpha\beta} \otimes e_\beta + g_{\alpha\beta}\vartheta_\beta e_\beta.$$

That is, on $U_\alpha \cap U_\beta$,

$$\vartheta_\alpha g_{\alpha\beta} \;=\; dg_{\alpha\beta} + g_{\alpha\beta}\vartheta_\beta \quad \Longrightarrow \quad \vartheta_\alpha - \vartheta_\beta \;=\; g_{\alpha\beta}^{-1} dg_{\alpha\beta} \;=\; d\log(g_{\alpha\beta}).$$

Example.

(i) Let $\mathbb{S}^n$ be the unit n–sphere equipped with the round metric g.
(ii) Let $\mathbb{H}^n$ be hyperbolic space.
(iii) Products?

Remark. It is a standard fact of Riemannian geometry that the Riemannian curvature tensor is determined by the sectional curvatures.

Reminder. The Riemannian metric g on a smooth manifold gives rise to a volume form $d\mu_g$, which in the geodesic normal coordinates $(x_1, ..., x_n)$, can be written as

$$d\mu_g = \sqrt{\det(g_{ij})} dx_1 \wedge \cdots \wedge dx_n.$$

Computing the Taylor expansion of $d\mu_g$, we see that

$$d\mu_g \quad = \quad \left[1 - \frac{1}{6} R_{jk} x^j x^k + O(|x|^3)\right] dx_1 \wedge \cdots \wedge dx_n.$$

The correction term R_{jk} at the quadratic level is called the *Ricci curvature tensor* and is a tensor of type $(2, 0)$. The Ricci curvature measures the extent to which volumes along geodesics are distorted under the exponential map. If $\text{Ric}(g) > 0$, volumes decrease, while if $\text{Ric}(g) < 0$, volumes increase.

Reminder. A Riemannian metric g on M is said to be *Einstein* if there exists a constant $\lambda \in \mathbb{R}$ such that $\text{Ric}(g) = \lambda g$. The number λ is called the *Einstein constant* for g. If $\lambda = 0$, we say that g is *Ricci-flat.* Since the Ricci curvature is invariant under scaling (while the metric is not), we can always scale the Einstein constant to be one of the following $\lambda \in \{-1, 0, 1\}$.

Examples.

(i) Metrics of constant Riemannian curvature, such as the round metric on $\mathbb{S}^n$, are Einstein metrics.
(ii) Any Ricci-flat metric on a torus is in fact flat, i.e., the Riemannian curvature of the metric vanishes identically.
(iii) An old result of Lockhamp [**?**] tells us that any Riemannian manifold of dimension $n \geq 3$ admits a metric whose Ricci curvature is bounded between two negative constants.

Reminder. If we look at the distortion of the volume of a small ball $B_\varepsilon^M(p)$ in the manifold (M, g) compared with the volume of the ball $B_\varepsilon^{\mathbb{R}^n}(0)$ in the corresponding Euclidean space, we find that

$$\text{Vol}(B_\varepsilon^M(p)) \quad = \quad \left[1 - \frac{\text{Scal}(g)}{6(n+2)} \varepsilon^2 + O(\varepsilon^4)\right] \text{Vol}(B_\varepsilon^{\mathbb{R}^n}(0)).$$

The correction term $\text{Scal}(g)$ at the quadratic level is the *scalar curvature*, and is a $(0, 0)$–tensor, i.e., a real-valued function on the manifold.

Examples.

(i) Any Riemannian manifold admits a metric of negative scalar curvature. Of course, this can be deduced from the aforementioned result of Lockhamp, but was in fact known prior to this.

(ii) There are topological obstructions to the existence of a metric with positive scalar curvature.

(iii) The search for metrics of constant scalar curvature is the famous Yamabe problem. The solution of the Yamabe problem (for compact Riemannian manifolds) tells us that there is a unique metric of constant scalar curvature in any conformal class.

Reminder 2.1.3. Let (X, g, ω) be a Kähler manifold. In local (holomorphic) coordinates, the Ricci curvature of the metric g can be written as

$$\mathrm{Ric}(g) = \mathrm{Ric}_{i\bar{j}} dz^i \otimes d\bar{z}^j.$$

The *Ricci form* is the $(1,1)$–form associated to this $(2,0)$–tensor:

$$\mathrm{Ric}(\omega) = \frac{\sqrt{-1}}{2\pi} \mathrm{Ric}_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

The Kähler condition $d\omega = 0$ is equivalent to the following symmetries of the metric:

$$\partial_k g_{i\bar{j}} \quad = \quad \partial_i g_{k\bar{j}}, \qquad \forall i, j, k, \tag{1}$$

where $\partial_k := \dfrac{\partial}{\partial z^k}$. Moreover, Reminder 2.1.3 is independent of the choice of local holomorphic coordinates. The Christoffel symbols of the Levi-Civita connection of a Kähler metric are given by

$$\Gamma^i_{jk} \quad = \quad g^{i\bar{\ell}} \partial_j g_{k\bar{\ell}},$$

where $(g^{i\bar{\ell}})$ denotes the inverse of the matrix $(g_{i\bar{\ell}})$. The Riemannian curvature tensor is given

$$\begin{aligned} R_{i\bar{j}k\bar{\ell}} \quad &= \quad -g_{m\bar{j}} \partial_{\bar{\ell}} \Gamma^m_{ik} && (2) \\ &= \quad -\partial_i \partial_{\bar{j}} g_{k\bar{\ell}} + g^{p\bar{q}} (\partial_i g_{k\bar{q}})(\partial_{\bar{j}} g_{p\bar{\ell}}). && (3) \end{aligned}$$

The Ricci curvature of g is

$$R_{i\bar{j}} \quad = \quad g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}} \quad = \quad g^{k\bar{\ell}} R_{k\bar{\ell}i\bar{j}} \quad = \quad R_k{}^k{}_{i\bar{j}}.$$

Reminder 2.1.4. Let X be a Kähler manifold. The *canonical bundle* K_X *of* X is the determinant line bundle of holomorphic n–forms on X. The *anti-canonical bundle* K_X^{-1} *of* X is the dual of the canonical bundle.

Theorem 2.1.5. The Ricci form $\text{Ric}(\omega)$ of a Kähler manifold (X, ω) satisfies the following:

(i) $\text{Ric}(\omega)$ is the curvature form of a Hermitian metric on the anti-canonical bundle K_X^{-1}.
(ii) $\text{Ric}(\omega)$ is a d–closed form.

Proof. First, recall that for an invertible Hermitian matrix $(A_{i\bar{j}}(t))$, whose entries are dependent on a variable t, we have

$$\frac{\partial}{\partial t} \det(A) = A^{i\bar{j}} \left(\frac{\partial}{\partial t} A_{i\bar{j}} \right) \det(A).$$

From this,

$$R_{i\bar{j}} = -\partial_{\bar{j}} \Gamma_{ki}^k = -\partial_{\bar{j}} \left(g^{k\bar{q}} \partial_i g_{k\bar{q}} \right) = -\partial_i \partial_{\bar{j}} \log \det(g).$$

The Ricci form is then expressed as $\text{Ric}(\omega) = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log(\omega^n)$, which is obviously closed. Moreover, (i) follows from the fact that the curvature form of a Hermitian metric h on a Hermitian line bundle E is given by $\Theta_{E,h} = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \det(h)$, together with the fact that a volume form on X is equivalent to a Hermitian metric on K_X^{-1} (see, e.g., [33]). □

Reminder 2.1.6. The cohomology class represented by $\text{Ric}(\omega)$ is the *first Chern class of* K_X^{-1}, commonly written $c_1(K_X^{-1})$. We often simply refer to this cohomology class as the *first Chern class of* X and denote it by c_1.

(i) The first Chern class c_1 is said to be *positive* if it represented by a closed $(1,1)$–form (written locally as) $\eta = \sqrt{-1} \sum_{i,j=1}^n \eta_{ij} dz^i \wedge d\bar{z}^j$ with $(\sqrt{-1}\eta_{ij})$ positive-definite. In this case, we write $c_1 > 0$; compact Kähler manifolds with $c_1 > 0$ are called *Fano manifolds.*
(ii) Similarly, the first Chern class is said to be *negative* if it can be represented by a $(1,1)$–form, as above, with negative-definite coefficient matrix. In this case, we write $c_1 < 0$; compact Kähler manifolds with $c_1 < 0$ are said to be *canonically polarized.*
(iii) Finally, the first Chern class is said to be *trivial* if it can be represented by 0. In this case, we write $c_1 = 0$; compact Kähler manifolds with $c_1 = 0$ are called *Calabi–Yau manifolds.*

If the first Chern class of X satisfies one of the above three conditions, we say that the first Chern class is *definite.*

Aside: Čech Cohomology. Let X be a topological space covered by open sets $\mathcal{U} = (U_\alpha)_{\alpha \in A}$ for some indexing set A. Fix a well-ordering on A. For $\alpha_0, ..., \alpha_p \in A$, we will denote the intersection $U_{\alpha_0} \cap \cdots \cap U_{\alpha_p}$ by $U_{\alpha_0,...,\alpha_p}$. We will associate to a sheaf $\mathcal{F}$ of abelian groups on X, a complex $(C^\bullet(\mathcal{U}, \mathcal{F}), d)$ of abelian groups as follows: For each $p \geq 0$, let

$$C^p(\mathcal{U}, \mathcal{F}) := \prod_{\alpha_0 < \cdots < \alpha_p} \mathcal{F}(U_{\alpha_0,...,\alpha_p}).$$

The boundary map $d : C^p(\mathcal{U}, \mathcal{F}) \to C^{p+1}(\mathcal{U}, \mathcal{F})$ is defined by

$$(d\eta)_{\alpha_0,\ldots,\alpha_{p+1}} \;:=\; \sum_{k=0}^{p+1} (-1)^k \eta_{\alpha_0,\ldots,\hat{\alpha}_k,\ldots,\alpha_{p+1}} |U_{\alpha_0,\ldots,\hat{\alpha}_k,\ldots,\alpha_{p+1}},$$

where $\hat{\alpha}_k$ means that we omit α_k. It can be readily verified that $d^2 = 0$.

Example. Let $\sigma = \{\sigma_{U_\alpha}\} \in C^0(\mathcal{U}, \mathcal{F})$, then

$$(d\sigma)_{\alpha\beta} = -\sigma_\alpha + \sigma_\beta.$$

If $\tau = \{\tau_{\alpha\beta}\} \in C^1(\mathcal{U}, \mathcal{F})$, then

$$(d\tau)_{\alpha\beta\gamma} = \tau_{\alpha\beta} + \tau_{\beta\gamma} - \tau_{\alpha\gamma}.$$

Reminder. Let $\mathcal{F}$ be a sheaf of abelian groups on a topological space X. Let $\mathcal{U}$ be an open covering of X. The cohomology groups $H^k(\mathcal{U}, \mathcal{F})$ associated to the complex $(C^\bullet(\mathcal{U}, \mathcal{F}), d)$ are called the *Čech cohmology groups.*

Remark. Let $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ be an exact sequence of sheaves of abelian groups on X. In general, this will fail to induce a long exact sequence of Čech cohomology groups

$$\cdots \to H^p(\mathcal{U}, \mathcal{F}') \to H^p(\mathcal{U}, \mathcal{F}) \to H^p(\mathcal{U}, \mathcal{F}'') \to H^{p+1}(\mathcal{U}, \mathcal{F}') \to \cdots.$$

Indeed, if this were the case, we could take $\mathcal{U} = X$ and this would imply the global sections functor $X \mapsto \mathcal{F}(X)$ is exact, which is, of course, false in general.

Question. To what extent do the Cech cohomology groups depend on the open covering? Are the Cech cohomology groups isomorphic to the sheaf cohomology groups in general?

Reminder. There is a group structure on the set of (isomorphism classes of) holomorphic line bundles on a compact complex manifold X. This group is called the *Picard group* $\mathrm{Pic}(X)$. The group operation is given by the tensor product, while the inverse operation is given by taking duals.

Remark 2.2.2. Associated to a holomorphic line bundle $\pi : L \to X$, there is a collection of open sets U_α which cover X, and a corresponding collection of holomorphic maps

$$\varphi_\alpha : \pi^{-1}(U_\alpha) \longrightarrow U_\alpha \times \mathbb{C}.$$

On any overlap, $U_\alpha \cap U_\beta \neq \emptyset$, there are induced maps

$$g_{\alpha\beta} : U_\alpha \cap U_\beta \longrightarrow \mathbb{C}^*, \qquad g_{\alpha\beta}(z) = \varphi_\alpha^{-1} \circ \varphi_\beta(z)|_{L_z},$$

where $L_z = \pi^{-1}(z)$ is the fiber of L over z.

These non-vanishing holomorphic maps satisfy the *cocycle condition*:

$$\begin{cases} g_{\alpha\beta}g_{\beta\alpha} & = 1 \\ g_{\alpha\beta}g_{\beta\gamma}g_{\gamma\alpha} & = 1. \end{cases}$$

The converse is also true; namely, given a collection of such non-vanishing holomorphic maps $g_{\alpha\beta} \in \mathcal{O}^*(U_\alpha \cap U_\beta)$, the line bundle can be recovered.

Theorem. Let X be a complex manifold. There is an isomorphism between $\text{Pic}(X)$ and $H^1(X, \mathcal{O}_X^*)$.

Remark. Let X be a compact complex manifold. The exponential sheaf sequence is the following short exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^* \longrightarrow 0,$$

of sheaves of abelian groups. This induces a long exact sequence of sheaf cohomology groups

$$\cdots \to H^1(X, \mathcal{O}_X) \to H^1(X, \mathcal{O}_X^*) \xrightarrow{\delta} H^2(X, \mathbb{Z}) \longrightarrow \cdots,$$

where δ is the connecting homomorphism. Theorem implies that we have a group homomorphism $\delta : \text{Pic}(X) \to H^2(X, \mathbb{Z})$, allowing us to identify with each isomorphism class of holomorphic line bundles on X an integral cohomology class $c_1(L) := \delta(L)$ in $H^2(X, \mathbb{Z})$ called the *first Chern class of* X.

Remark. The first Chern class is a particular example of a *characteristic class* – cohomology classes which provide a measure for determining when two vector bundles are not isomorphic. It is elementary to show the following properties of the first Chern class:

(i)

Theorem. Let $L \to X$ be a holomorphic line bundle over a complex manifold X. Let Θ_h be the curvature form of a Hermitian metric h on L. Then the first Chern class $c_1(L)$ can be represented by $\frac{\sqrt{-1}}{2\pi}\Theta_h$.

Proof. □

Remark. In particular, since the Ricci curvature $\text{Ric}(\omega)$ of a Kähler metric ω satisfies $\text{Ric}(\omega) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\omega^n)$, and ω^n is a volume form (i.e., a metric on the canonical bundle, the Ricci form represents the first Chern class of the anti-canonical bundle K_X^{-1}.
In particular, if ω is Einstein, i.e., $\text{Ric}(\omega) = \lambda\omega$, for some constant $\lambda \in \mathbb{R}$, we have $c_1(K_X^{-1}) = \lambda[\omega]$. That is, the existence of a Kähler–Einstein metric on X imposes topological restrictions on the underlying manifold.

Reminder 2.1.7. Let X be a compact Kähler manifold with local holomorphic coordinates denoted $(z^1, ..., z^n)$. A smooth form on X decomposes into components of type (p,q) depending on the number of dz's and $d\overline{z}$'s. The set of smooth forms on X of type (p,q) is denoted $\Omega^{p,q}(X)$, and we have the decomposition

$$\Omega^k(X) \ = \ \bigoplus_{p+q=k} \Omega^{p,q}(X).$$

The cohomology $H^{p,q}(X)$ is defined to be the quotient of closed (p,q)–forms by exact (p,q)–forms.

Theorem 2.1.8. (Hodge decomposition [32, 33]). Let X be a compact Kähler manifold. The de Rham cohomology groups enjoy the following decomposition

$$H^k_{\mathrm{DR}}(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(X),$$

and $\overline{H^{p,q}(X)} \simeq H^{q,p}(X)$.

Lemma 2.1.9. ($\partial\overline{\partial}$–lemma). Let X be a compact Kähler manifold. A real $(1,1)$–form η is closed if and only if η is locally $\sqrt{-1}\partial\overline{\partial}$–exact, i.e.,

$$d\eta = 0 \quad \Longleftrightarrow \quad \eta = \sqrt{-1}\partial\overline{\partial}u,$$

for some locally-defined smooth function u.

Proof. It is clear that if η is locally $\sqrt{-1}\partial\overline{\partial}$–exact, then η is closed. For the reverse implication, suppose η is closed. By the Poincaré lemma, there exists (at least locally) a 1–form ξ such that $\eta = d\xi$. Since X is compact Kähler, the Hodge decomposition theorem (with $k=1$) allows us to write $\xi = \xi^{1,0} + \xi^{0,1}$. Then

$$\eta = d\xi = \overline{\partial}\xi^{1,0} + \partial\xi^{0,1},$$

since by comparing types, one necessarily has $\overline{\partial}\xi^{0,1} = 0 = \partial\xi^{1,0}$. By the Dolbeault lemma (see, e.g., [38, Chapter 1D]) $\xi^{0,1} = \overline{\partial}f$ for some smooth locally-defined $\mathbb{R}$–valued function f. Hence, $\xi^{1,0} = \partial\overline{f}$ and

$$\eta \ = \ \overline{\partial}\partial\overline{f} + \partial\overline{\partial}f \ = \ \sqrt{-1}\partial\overline{\partial}u,$$

where $u := 2\mathrm{im}(f)$. □

Remark 2.1.10. The $\partial\overline{\partial}$–lemma informs us that if ω and ω' are two Kähler metrics representing the same cohomology class in $H^2_{\mathrm{DR}}(X, \mathbb{R})$, then there exists a smooth $\mathbb{R}$–valued function φ such that $\omega' = \omega + \sqrt{-1}\partial\overline{\partial}\varphi$. We often write $\omega_\varphi := \omega + \sqrt{-1}\partial\overline{\partial}\varphi$ for a metric cohomology to ω.

Remark 2.1.11. It is common to express the $\partial\overline{\partial}$–lemma in terms of the operators $d := \partial + \overline{\partial}$ and $d^c := \sqrt{-1}(\overline{\partial} - \partial)$. In this case, $2\sqrt{-1}\partial\overline{\partial} = dd^c$, and we can write two cohomologous Kähler metrics as $\omega_\varphi = \omega + dd^c\varphi$. We will interchange the use of $\sqrt{-1}\partial\overline{\partial}$ and dd^c without mention.

Remark 2.1.12. The above quick and dirty way of defining the first Chern class does not illucidate the deep topological theory (of vector bundles) behind it. It is actually a remarkable fact[5] that the first Chern class of X can be represented by its Ricci form, at least if X is Kähler. In the 50s, Calabi [10] conjectured that the converse was also true: namely, let (X, ω) be a compact Kähler manifold with definite first Chern class. If η is any closed $(1,1)$–form representing c_1, Calabi conjectured the existence of a Kähler metric ω_φ cohomologous to ω such that $\text{Ric}(\omega_\varphi) = \eta$.

The Calabi conjecture is intimately related to the existence of Einstein metrics (i.e., metrics of constant Ricci curvature) on compact Kähler manifolds and complex Monge–Ampère equations. Indeed, if the first Chern class is definite (since the Ricci curvature is scale-invariant) we can find $\lambda \in \{-1, 0, 1\}$ such that $c_1 = \lambda[\omega]$. Let $\omega_\varphi = \omega + dd^c\varphi$ be the Kähler metric cohomologous to ω whose Ricci form is η (whose existence is the content of the Calabi conjecture). Then

$$\begin{aligned} \text{Ric}(\omega_\varphi) = \lambda\omega_\varphi \quad &\iff \quad \text{Ric}(\omega_\varphi) - \text{Ric}(\omega) = \lambda\omega_\varphi - \lambda\omega + \lambda\omega - \text{Ric}(\omega) \\ &\iff \quad \text{Ric}(\omega_\varphi) - \text{Ric}(\omega) = \sqrt{-1}\partial\overline{\partial}\lambda\varphi - \sqrt{-1}\partial\overline{\partial}f \\ &\iff \quad -\sqrt{-1}\partial\overline{\partial}\log(\omega_\varphi^n) + \sqrt{-1}\partial\overline{\partial}\log(\omega^n) = \sqrt{-1}\partial\overline{\partial}(\lambda\varphi - f) \\ &\iff \quad \omega_\varphi^n = e^{f - \lambda\varphi}\omega^n. \end{aligned}$$

The existence of a Kähler–Einstein metric[6] ω_φ in the cohomology class of ω is given by finding a smooth solution φ to the complex Monge–Ampère equation

$$\omega_\varphi^n = e^{f - \lambda\varphi}\omega^n,$$

where $f : X \to \mathbb{R}$ is a smooth, and

$$\int_X e^{f - \lambda\varphi}\omega^n \;=\; \int_X \omega_\varphi^n \;=\; \text{vol}(X).$$

For $\lambda < 0$ and $\lambda = 0$, Calabi [10] proved uniqueness of the solution to the above complex Monge–Ampère equation, using a straightforward maximum principle argument. For $\lambda > 0$, Bando–Mabuchi [2] proved that the solution is unique up to holomorphic automorphism. For $\lambda = 0$, i.e., when X is Calabi–Yau, the existence of the smooth solution was established by Yau [70]. Yau's existence theorem for $\lambda = 0$ is now commonly referred to as the *Calabi–Yau*

[5] Following from Chern–Weil theory (see, e.g., [33]) and Theorem 2.1.5.

[6] That is, a Kähler metric with constant Ricci curvature.

theorem. For $\lambda < 0$, existence was proved by Yau [70] and Aubin [1] independently.

For $\lambda > 0$, there are obstructions to the existence of Kähler–Einstein metrics. The first obstruction was observed by Matsushima [53], noticing that it was necessary for the Lie group of holomorphic automorphisms to be reductive, i.e., the complexification of a real compact Lie group. Later obstructions were found by Futaki [31] who introduced a Lie algebra character for the Lie algebra of holomorphic vector fields, which necessarily vanished for the existence of Kähler–Einstein metrics on Fano manifolds. This lead to a folklore conjecture that the only obstructions were to arise from the Lie algebra of holomorphic vectors. An example of Mukai (see, e.g., [64]), however, provides an example of a Fano manifold with no non-trivial holomorphic vector fields, but nevertheless, does not admit a Kähler–Einstein metric. In [64], Tian proposed the notion of K–stability which he showed to be a necessary condition for existence, and conjectured that it was also sufficient. This so-called Yau–Tian–Donaldson conjecture (YTD conjecture) was verified in 2012 by Tian [65] and Chen–Donaldson–Sun [15].

Example 2.1.13.

(i) The complex torus $\mathbb{T}^2 = \mathbb{C}/\Lambda$ equipped with the flat metric is a Calabi–Yau manifold of (complex) dimension 1. In complex dimension 2, the only simply-connected Calabi–Yau manifolds are K3 surfaces. The Calabi–Yau theorem states that any compact Calabi–Yau manifold admits a unique Ricci-flat Kähler metric in each Kähler class.

(ii) A compact Riemann surface of genus $g \geq 2$, equipped with a metric of constant negative curvature is an example of a canonically polarized manifold.

(iii) Projective space $\mathbb{P}^n$ equipped with its Fubini–Study metric is an example of a Fano manifold, with $\mathrm{Ric}(\omega_{\mathrm{FS}}) = (n+1)\omega_{\mathrm{FS}}$.

Theorem 2.1.14. ([68, 71]). Let (X^n, ω_X) be a Calabi–Yau manifold and $f : X \to Y$ a holomorphic map with connected fibres into a compact Kähler manifold (Y^κ, ω_Y), where $0 < \kappa < n$. Let $\omega_0 = f^*\omega_Y$. For $0 < t \leq 1$, we let $\widetilde{\omega}_t$ be the unique Ricci-flat Kähler metric cohomologous to $[\omega_0] + t[\omega_X]$. Then there is a uniform constant $C > 0$ such that for all $t \in (0, 1]$, we have

$$\mathrm{tr}_{\widetilde{\omega}_t}\omega_0 \leq C.$$

Proof. Set $\omega_t = \omega_t + t\omega_X$ for $0 < t \leq 1$. Since ω_t and $\widetilde{\omega}_t$ are cohomologous, there is a smooth ω_t–plurisubharmonic function φ_t such that $\widetilde{\omega}_t = \omega_t + \sqrt{-1}\partial\overline{\partial}\varphi_t$. A simple computation

$$\Delta_{\widetilde{\omega}_t}(\log \mathrm{tr}_{\widetilde{\omega}_t}\omega_0 - (A+1)\varphi_t) \ \geq \ \mathrm{tr}_{\widetilde{\omega}_t}\omega_0 - n(A+1),$$

allows us to deduce the desired estimate from the maximum principle. □

2.2. **Reminder: Divisors and Line Bundles.** We recall the equivalence between holomorphic line bundles and divisors on a compact complex manifold.

Reminder 2.2.1.

Reminder 2.2.3. A *Weil divisor* on a projective manifold X is a locally finite formal linear combination

$$D = \sum c_j D_j,$$

where $c_j \in \mathbb{Z}$ and D_j are irreducible codimension 1 submanifolds of X. If the coefficients c_j lie in $\mathbb{Q}$, or $\mathbb{R}$, then the divisor is referred to as a $\mathbb{Q}$–*divisor*, or $\mathbb{R}$–*divisor*.

Construction 2.2.4. Let D be a divisor on a projective manifold X. Cover X be open sets (U_α) and let $f_\alpha \in \mathcal{M}^*(U_\alpha)$ denote the local defining functions[7] for D. The functions $g_{\alpha\beta} := f_\alpha f_\beta^{-1}$ are non-vanishing holomorphic functions on $U_\alpha \cap U_\beta$ and satisfying $g_{\alpha\beta} g_{\beta\gamma} g_{\gamma\alpha} = 1$ in $U_\alpha \cap U_\beta \cap U_\gamma$. The line bundle with transition functions $g_{\alpha\beta}$ is called the *line bundle associated to the divisor* D, and is denoted $\mathcal{O}(D)$. The reader may easily verify that the map $D \mapsto \mathcal{O}(D)$ is well-defined, independent of the choice of local defining functions.

Reminder 2.2.5. We mention some important classes of divisors:

(i) A divisor $D = \sum c_j D_j$ is said to be *effective* if $c_j \geq 0$ for all j.
(ii) A divisor D is said to be *very ample* if the linear system, i.e., sections $H^0(X, \mathcal{O}(D))$ of $\mathcal{O}(D)$, provide an embedding of X into some projective space.
(iii) A divisor D is said to be *ample* if some sufficiently high (tensor) power of D is very ample.
(iv) A divisor D is said to be *big* if $\dim H^0(X, \mathcal{O}(D^k)) \geq ck^{\dim X}$ for k sufficiently large.
(v) A divisor D is said to be *nef* if the intersection product of D with every curve is non-negative.

2.3. **Reminder: The Neron–Severi Group and Cones.** Let X be a compact complex manifold. The exponential sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

induces the following exact sequence on cohomology

$$\begin{aligned} 0 &\longrightarrow H^0(X, \mathbb{Z}) \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X^*) \longrightarrow H^1(X, \mathbb{Z}) \longrightarrow \\ H^1(X, \mathcal{O}_X) &\longrightarrow H^1(X, \mathcal{O}_X^*) \longrightarrow H^2(X, \mathbb{Z}) \xrightarrow{\varphi} H^2(X, \mathcal{O}_X) \longrightarrow \cdots \end{aligned}$$

Reminder 2.3.1. The *Neron–Severi group* $\text{NS}(X)$ is the kernel of the map φ in the above exact sequence.

[7]Notation: Here, we denote by $\mathcal{M}^*(U)$ to be sections of the sheaf of non-vanishing meromorphic functions on U, see Reminder 2.10.2.

Proposition 2.3.2. An integral cohomology class in $H^2(X, \mathbb{Z})$ is the first Chern class of a holomorphic line bundle if and only if it lies in $\mathrm{NS}(X)$.

Reminder 2.3.3. Set $\mathrm{NS}_{\mathbb{R}}(X) := \mathrm{NS}(X) \otimes_{\mathbb{Z}} \mathbb{R}$. The rank of this vector space is called the *Picard number*

$$\rho(X) \quad := \quad \dim_{\mathbb{R}} \mathrm{NS}_{\mathbb{R}}(X).$$

Lemma 2.3.4. If X is a compact Kähler manifold, $\mathrm{NS}_{\mathbb{R}}(X)$ is a real subspace of $H^{1,1}(X, \mathbb{R})$, the space of real $(1,1)$–cohomology classes. In particular,

$$0 \leq \rho(X) \leq \dim H^{1,1}(X, \mathbb{R}).$$

Remark 2.3.5. The example of complex tori shows that all intermediate values for the Picard number can occur in the wild.

Reminder 2.3.6. Let (X, ω) be a compact Kähler manifold.

(i) The *Kähler cone* $\mathcal{K} \subset H^{1,1}(X, \mathbb{R})$ is the subset of real $(1,1)$–cohomology classes which can be represented by a Kähler form.
(ii) The *nef cone* $\overline{\mathcal{K}}$ consists of real $(1,1)$–cohomology classes $[\alpha]$ such that for all $\varepsilon > 0$, the cohomology class $[\alpha + \varepsilon\omega]$ is a Kähler class.
(iii) The *pseudo-effective cone* $\mathscr{E} \subset H^{1,1}(X, \mathbb{R})$ is the set of cohomology classes which can be represented by a closed positive current of type $(1,1)$.
(iv) The *big cone* $\mathcal{E}^\circ$ consists of the real $(1,1)$–cohomology classes which can be represented by a closed positive $(1,1)$–current T such that $T \geq \varepsilon\omega$ for $\varepsilon > 0$ small, i.e., a cohomology class α is big if and only if it can be represented by a strictly positive current.

Remark 2.3.7. The Kähler cone $\mathcal{K}$ is an open cone, a fact following easily from the definition. Bounded sets of positive measures are weakly compact, and this property extends to currents, i.e., bounded sets of currents are weakly compact. In particular, $\mathcal{E}$ is a closed cone, and $\overline{\mathcal{K}} \subset \mathcal{E}$. The inclusion is strict: Let $\alpha = [\omega]$ be a Kähler class. Then for all p–dimensional analytic sets Y,

$$\int_Y \omega^p > 0.$$

Let X be the blow up of $\mathbb{P}^2$ at one point. The exceptional curve $E \simeq \mathbb{P}^1$ has a cohomology class $\alpha = [\eta]$ such that $\int_E \eta = E^2 = -1$. Thus, $\alpha \notin \overline{\mathcal{K}}$, while $[\eta] \in \mathcal{E}$.

Remark 2.3.8. Nef classes may be represented by smooth forms with arbitrarily small negative parts. A compactness argument shows that such a class contains a positive current, i.e., a nef class is pseudo-effective. The negative part of a smooth representative cannot be taken to be zero in general, however. Demailly–Peternell–Schneider have shown [20] that a classical construction of Serre yields an example of a smooth curve X on a projective surface whose cohomology class is nef and contains exactly one positive current: the current of integration on X.

We conclude this section with a discussion of the algebraic analogues of these cones. Denote by $\mathcal{K}_{\text{NS}} := \mathcal{K} \cap \text{NS}_{\mathbb{R}}(X)$, $\mathcal{E}_{\text{NS}} := \mathcal{E} \cap \text{NS}_{\mathbb{R}}(X)$, etc.

Theorem 2.3.9. Let X be a projective manifold. Then

(i) $\mathcal{K}_{\text{NS}} = \text{Amp}(X)$, the open cone generated by ample divisors on X.
(ii) $\mathcal{E}^{\circ}_{\text{NS}} = \text{Big}(X)$, the cone generated by classes of big divisors.
(iii) $\mathcal{E}_{\text{NS}} = \overline{\text{Eff}(X)}$, the closure of the cone generated by effective divisors.
(iv) $\overline{\mathcal{K}}_{\text{NS}} = \overline{\text{Nef}(X)}$, the closure of the cone generated by nef divisors.

2.4. **Mori's Minimal Model Program: Generalities.** The aim of the minimal model program (MMP) is to classify algebraic varieties up to birational isomorphism.

Reminder 2.4.1. Let X, Y be two projective varieties. A map $\phi : X \longrightarrow Y$ is a *birational isomorphism* if there are Zariski open subset $U \subset X$, $V \subset Y$ such that $\phi : U \longrightarrow V$ is an isomorphism. We often write $\phi : X \dashrightarrow Y$ to emphasise that ϕ is a birational isomorphism.

Reminder 2.4.2. Let X be a variety and D a divisor on X. A projective morphism $f : Y \to X$ is a *log resolution* of (X, D) if

(i) f is birational.
(ii) Y is smooth.
(iii) the exceptional locus[8] $\text{exc}(f)$ is a divisor and $\text{exc}(f) \cup f^{-1}(\text{supp}(D))$ is a simply normal crossings divisor.

We will often simply refer to a log resolution as a *blow up*. One of the primary tools in the MMP is the following resolution of singularities theorem due to Hironaka:[9]

[8]A closed subvariety Z of Y is an exceptional variety for f if $\dim f(Z) < \dim Z$. If Z has codimension one in Y, then Y is called an exceptional divisor. Exceptional divisors for surfaces are called exceptional curves.

[9]It is interesting that at the time of writing this, the generalisation of Hironaka's result to varieties over a field of characteristic $p > 0$ remains a widely open problem, at least for varieties of dimension greater than 3.

Theorem 2.4.3. ([40, 46]). Let X be a variety and $D \subset X$ a divisor. Then a log resolution of (X, D) exists.

We have the following immediate corollary of Theorem 2.4.3:

Corollary 2.4.4. Every projective variety is birational to a smooth projective variety.

It therefore suffices to classify smooth projective varieties. We recall the classificiation of curves:

Theorem 2.4.5. Let X be a projective curve (i.e., a compact Riemann surface) of genus $g := \dim H^0(X, K_X)$. Then

(i) $g = 0$ if and only if X is birational to $\mathbb{P}^1$.
(ii) $g = 1$ if and only if X is birational to an elliptic curve.
(iii) $g \geq 2$ if and only if X is of *general type*.

In cases (ii), there is a one-parameter family of elliptic curves: In affine coordinates (z, w) on $\mathbb{C}^2$, we can write an elliptic curve in (Weierstrass form?):

$$y^2 = x(x-1)(x-s),$$

which can be viewed as a degree 2 cover of $\mathbb{P}^1$, branched over the four points 0, 1, ∞ and s.

For higher-dimensional projective varieties, the genus is no longer sufficient to classify varieties up to birational isomorphism. We must consider a collection of invariants which generalise the genus. Indeed, we will consider the canonical bundle K_X together with all its tensor powers $mK_X := K_X^{\otimes m}$. Let $\sigma_0, ..., \sigma_N$ be a basis for $H^0(X, mK_X)$. Such a basis defines a rational map $\phi_m : X \dashrightarrow \mathbb{P}^N$ by $\phi_m(x) := [\sigma_0(x) : \cdots : \sigma_N(x)]$. We call ϕ_m the *mth pluricanonical map*. This map is defined at all points $x \in X$ which do not lie in the common zero set $\text{Bs}(X) = \bigcap_{j=0}^N \sigma_j^{-1}(0)$, which is called the *base locus of* X. **Moreover, for this map to be well-defined, we note that any two bases for $H^0(X, mK_X)$ differ by an action of the unitary group.**

Reminder 2.4.6. Let X be a normal projective variety. The *Kodaira dimension* of X, denoted by $\kappa(X)$, is defined the maximal rank of the pluricanonical maps, i.e.,

$$\kappa(X) = \max_{m \in \mathbb{N}} \dim \phi_m(X).$$

The Kodaira dimension meaures the rate of growth of sections of K_X^m:

$$\dim H^0(X, K_X^m) \sim m^{\kappa(X)}.$$

Remark. If K_X^m admits global sections for at least one $m \in \mathbb{N}$, the Kodaira dimension if a nonnegative integer bounded between 0 and $\dim(X)$. If K_X^m admits no global sections for any $m \in \mathbb{N}$, we declare $\kappa(X) = -\infty$.

Theorem 2.4.7. (see, e.g., [5]). The plurigenera and the Kodaira dimension are birational invariants. Moreover, if $\kappa(X) \neq -\infty$, then the Kodaira dimension is a non-negative integer bounded between 0 and $\dim X$.

Let us now turn to surfaces, i.e., projective varieties of dimension 2. In contrast to the situation for curves, there are an infinite number of smooth surfaces in each birational class. Blow ups produce (-1)–curves, i.e., curves $E \simeq \mathbb{P}^1$ with self-intersection -1. A result of Castelnuovo [47, p. 8] informs us that (-1)–curves correspond exactly to the exceptional curves of a blow-up. In particular, if a surface X contains a (-1)–curve, we can blow it down. That is, we have a birational map (a *blow down*) $f : X \dashrightarrow X^{(1)}$ over a smooth base $X^{(1)}$, which contracts the (-1)–curve to a point in $X^{(1)}$. After each blow down, the Picard number decreases by 1, and since the Picard number is always finite, this process of blowing down (-1)–curves terminates after a finite number of steps. The resulting surface, call it $Y := X^{(n)}$, which contains no (-1)–curves, is called a *minimal model for* X. For surfaces, these minimal models are classified to a large extent:

Theorem 2.4.8. (see, e.g., [5]). Let Y be the minimal model of a projective surface X.

(i) $\kappa(Y) = -\infty \implies Y = \mathbb{P}^2$ or Y is a ruled surface over a curve.
(ii) $\kappa(Y) = 0 \implies Y$ is a $K3$ surface, an Enriques surface, or an étale quotient of an abelian surface.
(iii) $\kappa(Y) = 1 \implies Y$ is a minimal elliptic surface, i.e., the total space of a fibration whose generic fibre is an elliptic curve.
(iv) $\kappa(Y) = 2 \implies Y$ is of general type.

Remark 2.4.9. For varieties of dimension greater than 2, the classification problem becomes much more complicated. The first major problem is how to generalise the notion of (-1)–curves and their contractions in the higher-dimensional setting. This problem was solved by Mori who found that (-1)–curves could be replaced by so-called *extremal rays*. This also leads to the notion of a *Mori fibre space*, generalising the notion of a ruled surface to higher dimensions.

Conjecture 2.4.10. (MMP Conjecture). Let X be a smooth projective variety.

(i) If $\kappa(X) = -\infty$, then X is birational to a Mori fibre space $Y \to Z$.
(ii) If $\kappa(X) \geq 0$, then X is birational to a minimal variety Y.

Reminder 2.4.11. Let X be a smooth projective variety. The canonical bundle K_X of X is *semi-ample* if there exists $\ell \geq 1$ such that for every $x \in X$ there is a section $s \in H^0(X, K_X^\ell)$ such that $s(x) \neq 0$, i.e., K_X^ℓ is *base-point free.*

The primary difficulty in establishing the above conjecture is concentrated in the following famous conjecture:

Conjecture 2.4.12. (Abundance Conjecture). Let Y be a minimal variety. Then there is a fibration $f : Y \to Z$ (with connected fibres) and an ample divisor A on Z such that for some $m > 0$,

$$mK_Y = f^* A.$$

Remark 2.4.13. The abundance conjecture, therefore, predicts that any variety is birationally equivalent to the total space of a fibration, where the canonical bundle of the generic fibre is either (i) anti-ample $K_X < 0$, (ii) torsion $K_X = 0$, or (iii) ample $K_X > 0$ (c.f., Theorem 2.4.5).

In Conjecture 2.4.10, we will only be concerned with part (ii). In fact, motivated by the abundance conjecture, we will consider projective manifolds X with semi-ample canonical bundle.

Remark 2.4.14. The sections of K_X^ℓ for $\ell \geq 1$ sufficiently large, produce a holomorphic map $f : X \to \mathbb{P}^N$, where $N = \dim H^0(X, K_X^\ell) - 1$. Indeed, simply choose a basis $\{s_0, ..., s_N\}$ of $H^0(X, K_X^\ell)$ and define $f(x) := [s_0(x) : \cdots : s_N(x)]$, which is well-defined since K_X^ℓ is base-point free. A consequence of the definition of f is that $K_X^\ell \simeq f^* \mathcal{O}_{\mathbb{P}^n}(1)$, i.e., the pullback of the hyperplane bundle. Remmert's proper mapping theorem (see, e.g., [33, p. 34]) informs us that the image $f(X) \subset \mathbb{P}^N$ is an irreducible analytic subvariety of $\mathbb{P}^N$, denote it by Y. And for sufficiently high powers of K_X, we can ensure that $f : X \to Y$ has connected fibres, Y is normal (see, e.g., [50, Theorem 2.1.27]), and $\dim(Y) = \kappa(X)$ (see, e.g., [50, Theorem 2.1.33]).

2.5. **Reminder: Metric Geometry.**

Convention 2.5.1. An *interval* is understood to be a connected subset of $\mathbb{R}$; it is permitted to be open, closed, finite, infinite, or even a singleton. A *path* is a continuous map $\gamma : I \to X$, where X is a topological space and I is an interval. Further, all topological spaces are assumed Hausdorff, unless otherwise stated.

Reminder 2.5.2. A *length structure* on a topological space X is specified by the following data:

(i) a class $\mathcal{A}$ of *admissible paths.*
(ii) a map length : $A \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$.

We require the following from the class $\mathcal{A}$ of admissible curves:

(A1) $\mathcal{A}$ is closed under restrictions: if $\gamma : I \to X$ is an admissible path, and $J \subset I$ is a sub-interval of I, then $\gamma|_J \to X$ also lies in $\mathcal{A}$.
(A2) $\mathcal{A}$ is closed under concatenation of paths: if $\gamma_1 : [a, b] \to X$, $\gamma_2 : [b, c] \to X$ are two admissible paths with $\gamma_1(b) = \gamma_2(b)$, then $\gamma : [a, c] \to X$ defined by

$$\gamma(t) = \begin{cases} \gamma_1(t), & a \leq t \leq b, \\ \gamma_2(t), & b < t \leq c \end{cases}$$

is also an admissible curve.
(A3) $\mathcal{A}$ is closed under linear reparametrisations: Let $\gamma : [a, b] \to X$ lie in $\mathcal{A}$ and $h : [c, d] \to [a, b]$ be a homeomorphism which can be expressed as $h(t) = \lambda t + \delta$. Then $\gamma \circ h$ is also an admissible path.

We require the following from the length function:

(B1) length$(-)$ is an additive function: if $\gamma : [a, b] \to X$ is an admissible curve and $a \leq c \leq b$, then

$$\text{length}(\gamma|_{[a,b]}) = \text{length}(\gamma|_{[a,c]}) + \text{length}(\gamma|_{[c,b]}).$$

(B2) length$(-)$ varries continuously with respect to the restriction of an admissible path: let $\gamma : [a, b] \to X$ be an admissible curve of finite length. Set $\text{length}_t(\gamma) := \text{length}(\gamma|_{[a,t]})$ for each $t \in [a, b]$. The map length_t is required to be continuous with respect to the variable t.
(B3) length$(-)$ is invariant under reparametrisations by linear homeomorphisms: $\text{length}(\gamma \circ h) = \text{length}(\gamma)$ for all linear homeomorphisms h.
(B4) length$(-)$ agrees with the topology of X: Let U be a neighbourhood of a fixed point $p \in X$. The lengths of paths connecting p with points in $X \backslash U$ is separated from zero, i.e.,

$$\inf \{\text{length}(\gamma) : \gamma(a) = x, \ \gamma(b) \in X \backslash U\} > 0.$$

Reminder 2.5.3. Let X be a topological space endowed with a length structure $(\mathcal{A}, \text{length})$. The *distance function associated to the length structure* is defined

$$d_{\mathcal{A}}(x, y) \ := \ \inf \{\text{length}(\gamma) : \gamma : [0, 1] \to X, \ \gamma \in \mathcal{A}, \ \gamma(0) = x, \ \gamma(1) = y\}.$$

Remark 2.5.4. The reader may easily verify that for any topological space X with a length structure $(\mathcal{A}, \text{length})$, the pair $(X, d_{\mathcal{A}})$ defines a metric space. The metric $d_{\mathcal{A}}$ need not be finite, however. A dumb example is found by taking X to have two components: no continuous path can connect two points from the distinct components, so the resulting length is infinite.

Reminder 2.5.5. A metric that is obtained as the distance function associated to a length structure is called a *length metric*. A metric space whose metric is a length metric is called a *length space*.

Remark 2.5.6. Not every metric arises as the distance function associated to a length structure. Indeed, it is straightforward to prove that length spaces are locally path connected. Hence, we obtain a simple formula for producing metric spaces which are not length spaces: take any topological space with a metrizable topology which is not locally path connected. Using this formula, we see that the following are not length spaces:

(i) The rational numbers $\mathbb{Q}$.
(ii) The union of the topologist's sine curve $\{(x, y) : y = \sin(x^{-1}),\ x > 0\}$ with $\{x = 0\}$ (endowed with topology induced from $\mathbb{R}^2$).

Reminder 2.5.7. A length structure $(\mathcal{A}, \text{length})$ is said to be *complete* if for any two points x, y, there is a path $\gamma \in \mathcal{A}$ such that $\text{length}(\gamma) = d_{\mathcal{A}}(x, y)$.

Reminder 2.5.8. Let (X, d) be a metric space (not necessarily a length space). A point $z \in X$ is said to be a *midpoint* of $x, y \in X$ if

$$d(x, z) \ = \ d(z, y) \ = \ \frac{1}{2} d(x, y).$$

Lemma 2.5.9. Let $(X, d_{\mathcal{A}})$ be a complete length metric space. Then for any pair of points $x, y \in X$, a midpoint exists.

Proof. This is an obvious consequence of the intermediate value theorem, together with requirement (B2) of the length function. $\square$

Theorem 2.5.10. Let (X, d) be a complete metric space. If for every pair of points $x, y \in X$, a midpoint exists, then d is a length metric.

Proposition 2.5.11. The completion of a length metric space is a length metric space.

Reminder 2.5.12. Let (X, d) be a metric space which contains two subsets A and B. Denote by

$$\mathrm{dist}(A, B) := \inf\{\mathrm{dist}(a, b) : a \in A, b \in B\}.$$

Set $A(\varepsilon) := \{x \in X : \mathrm{dist}(x, A) < \varepsilon\}$ and similarly define $B(\varepsilon)$. The *Hausdorff distance* $\mathrm{dist}_H(A, B)$ of A and B is

$$\mathrm{dist}_H(A, B) \quad := \quad \inf_{\varepsilon \geq 0} \{A \subset B(\varepsilon) \text{ and } B \subset A(\varepsilon)\}.$$

The reader may readily verify that the Hausdorff distance defines a distance function on the closed subsets of X.

Reminder 2.5.13. Let (X, d_X) and (Y, d_Y) be two compact metric spaces. The *Gromov–Hausdorff distance* $\mathrm{dist}_{\mathrm{GH}}(X, Y)$ of (X, d_X) and (Y, d_Y) is

$$d_{\mathrm{GH}}(X, Y) \quad := \quad \inf_{f,g} d_H(f(X), g(Y)),$$

where the infimum is taken over all isometric embeddings $f : X \to Z$, $g : Y \to Z$, for all possible choices of metric space Z.

Proposition 2.5.14. Let X, Y be compact metric spaces. If $d_{\mathrm{GH}}(X, Y) = 0$, then X and Y are isometric.

Reminder 2.5.15. The *pointed Gromov–Hausdorff distance* of two pointed metric spaces (X, x), (Y, y) is defined

$$d_{\mathrm{GH}}((X, x), (Y, y)) \quad := \quad \inf\{d_{\mathrm{H}}(X, Y) + d(x, y)\},$$

where the infimum is taken over all Hausdorff distances. We say that

$$(M_i, p_i, d_i) \longrightarrow (M_\infty, p_\infty, d_\infty)$$

in the *pointed Gromov–Hausdorff topology* if for all R, the closed metric balls

$$(\overline{B}_R(p_i), p_i, d_i) \longrightarrow (\overline{B}_R(p_\infty), p_\infty, d_\infty)$$

converge with respect to the pointed Gromov–Hausdorff metric.

Theorem 2.5.16. (Gromov's precompactness theorem). Let $n \in \mathbb{Z}_{\geq 2}$, $k \in \mathbb{R}$ and $D \in \mathbb{R}_{>0}$ be fixed. The collection of compact Riemannian manifolds (M^n, g) with $\mathrm{Ric}(g) \geq (n-1)kg$ and $\mathrm{diam}(M) \leq D$ is compact in the Gromov–Hausdorff topology.

The following relative volume comparison of Gromov [34] is useful in proving geometric convexity.

Proposition 2.5.17. Let (M^n, g) be a Riemannian manifold with $\text{Ric}(g) \geq (n-1)g$ and $\text{diam}(M, g) \leq D$. Let $E \subset M$ be a compact set with smooth boundary. Assume that there are two points $p, q \in M$ with

$$B_g(p, r) \cap E \ = \ \emptyset \ = \ B_g(q, r),$$

for some $r > 0$ and that every minimal geodesic from p to some point in $B_g(q, r)$ intersects E. Then there is a constant $c = c(n, r, D) > 0$ such that

$$\text{Vol}_g(\partial E) \ \geq \ c\text{Vol}_g(B_g(q, r)).$$

Reminder 2.5.18. Let X, Y be two sets. A *correspondence* between X and Y is a set $\mathcal{R} \subset X \times Y$ such that

(i) for any $x \in X$ there is some $y \in Y$ such that $(x, y) \in \mathcal{R}$.
(ii) for any $y \in Y$ there is some $x \in X$ such that $(x, y) \in \mathcal{R}$.

Example 2.5.19. Let $f : X \to Y$ be a surjective map. A correspondence $\mathcal{R}$ between X and Y is given by

$$\mathcal{R} = \{(x, f(x)) : x \in X\}.$$

Remark 2.5.20. Not every correspondence can be given by the graph of a surjective map. We can say something more general, however: Let $f : Z \to X$ and $g : Z \to Y$ be two surjective maps emanating from a set Z. The set

$$\mathcal{R} = \{(f(z), g(z)) : z \in Z\}$$

defines a correspondence. Moreover, any correspondence can be obtained in this way.

Reminder 2.5.21. Let (X, d_X) and (Y, d_Y) be two metric spaces, with $\mathcal{R}$ a correspondence between them. The *distortion of* $\mathcal{R}$ is defined

$$\text{distort}(\mathcal{R}) \ \ := \ \ \sup \left\{ |d_X(x, \hat{x}) - d_Y(y, \hat{y})| : (x, y), (\hat{x}, \hat{y}) \in \mathcal{R} \right\}.$$

Theorem 2.5.22. Let $(X_n)_{n \in \mathbb{N}}$ be a sequence of length spaces, and X a complete metric space. Assume that X_n converges to X in the Gromov–Hausdorff topology. Then X is a length space.

2.6. **Reminder: The Kähler–Ricci Flow and Continuity Method.**

Reminder 2.6.1. Let (X, ω_0) be a compact Kähler manifold of complex dimension n. A *solution of the Kähler–Ricci flow on X, starting at an initial metric* ω_0, is a family of Kähler metrics $\omega(t)$ solving the equation

$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)), \qquad \omega(0) = \omega_0.$$

The above equation will be referred to as the (unnormalised) Kähler–Ricci flow. By rescaling, we have the normalised Kähler–Ricci flow:

$$\frac{\partial \omega(t)}{\partial t} = -\mathrm{Ric}(\omega(t)) - \omega(t), \qquad \omega(0) = \omega_0.$$

Theorem 2.6.2. There exists a unique solution $\omega(t)$ to the Kähler–Ricci flow on some maximal time interval $[0, T)$ for some $0 < T \leq +\infty$. An explicit description of the maximal time T is given by cohomological data:

$$T = \sup\{t > 0 : [\omega_0] + tK_X > 0\}.$$

In particular, the Kähler–Ricci flow exists for all time if and only if K_X is nef.

Using the estimates in Yau's proof of the Calabi conjecture, Cao [11] proved the following for the Kähler–Ricci flow:

Theorem 2.6.3. Let (X, ω_0) be a compact Calabi–Yau manifold. Then the (unnormalised) Kähler–Ricci flow converges in $\mathcal{C}^\infty$ to the unique Kähler–Einstein metric in $[\omega_0]$.

Remark 2.6.4. Similar results hold for canonically polarised manifolds.

A compact Kähler manifold, in general, however, does not have a definite first Chern class. Motivated by the use of the Kähler–Ricci to find canonical metrics on compact Kähler manifolds, Song–Tian [60] proposed the following:

Reminder 2.6.5. Let X^n be a compact Kähler manifold with semi-ample canonical bundle. Assume that $0 < \kappa < n$, which implies that X is the total space of a Calabi–Yau fibration $f : X \to X_{\mathrm{can}}$. A closed positive $(1,1)$–current ω_{can} on X_{can} is said to be a *canonical metric* if:

(i) $f^*\omega_{\mathrm{can}} \in -2\pi c_1(X)$.
(ii) ω_{can} is smooth and $(f^*\omega_{\mathrm{can}})^\kappa \wedge \omega_{\mathrm{SF}}^{n-\kappa}$ is continuous on X.
(iii) $\mathrm{Ric}(\omega_{\mathrm{can}}) = -\sqrt{-1}\partial\overline{\partial}\log(\omega_{\mathrm{can}}^\kappa)$ is a well-defined current on X_{can} and

$$\mathrm{Ric}(\omega_{\mathrm{can}}) = -\omega_{\mathrm{can}} + \omega_{\mathrm{WP}}$$

holds on X_{can}°.

In [60], Song–Tian establish the existence of canonical metrics by establishing the existence of a unique solution to a singular complex Monge–Ampère equation.

Theorem 2.6.6. ([60, Theorem 3.1]). Let X be as in Reminder 2.6.5. Then there exists a unique canonical metric on X_{can}.

Theorem 2.6.7. Let (X^n, ω_0) be a smooth projective manifold with semi-ample canonical bundle K_X. Then X is the total space of a Calabi–Yau fibration $f : X \to X_{\text{can}}$. Assume that $0 < \kappa < n$. Then for any initial metric ω_0, the (normalised) Kähler–Ricci flow exists for all time $t \in [0, +\infty)$ and

(i) $\omega(t) \longrightarrow f^*\omega_{\text{can}} \in -2\pi c_1(X)$ in the sense of currents.

(ii) ω_{can} is smooth on X_{can}° and satisfies the twisted Kähler–Einstein equation

$$\text{Ric}(\omega_{\text{can}}) \ = \ -\omega_{\text{can}} + \omega_{\text{WP}},$$

where ω_{WP} is the pullback of the Weil–Petersson metric from the moduli space of polarised Calabi–Yau manifolds.

(iii) For any compact subset $K \subset X^\circ$, there is a cosntant $C_K > 0$ such that

$$\|\text{Rm}(\omega(t))\|_{L^\infty(K)} + e^{(n-\kappa)t} \sup_{f^{-1}(s) \in K} \|\omega(t)^{n-\kappa}|_{f^{-1}(s)}\|_{L^\infty} \ \leq \ C_K.$$

An alternative approach to deforming Kähler metrics within in a fixed Kähler class was proposed by La Nave–Tian [49], referred to as the *continuity method*. Let (X, ω_A) be a compact Kähler manifold. A family of Kähler metrics $\omega(t)$ *evolves under the continuity method* if it solves the following one-parameter family of equations

$$\text{Ric}(\omega(t)) \ = \ -\omega(t) + t\omega_A,$$

for a fixed Kähler metric ω_A on X. It is clear that the cohomology classes of these metrics satisfy $[\omega(t)] = t[\omega_A] - c_1(X)$. The following sharp local existence result is analagous to that obtained for the Ricci flow (c.f., Theorem 2.6.2).

Theorem 2.6.8. For any initial metric ω_0, there is a smooth family of solutions $\omega(t)$ of the continuity method on $X \times [0, T)$, where

$$T := \sup\{t > 0 \mid t[\omega_A] - c_1(X) > 0\}.$$

Remark 2.6.9. Despite being a less-natural way of deforming a Kähler metric, one of the primary advantages of using the continuity method over the Kähler–Ricci flow is that the Ricci curvature is uniformly bounded below:

$$\text{Ric}(\omega(t)) \ = \ -\omega(t) + t\omega_A \ \geq \ -\omega(t).$$

This allows one to be able to readily apply the compactness theory of Cheeger–Colding [12, 13].

2.7. Reminder: Moduli Theory.

Reminder 2.7.1. Let (X, ω) be a Calabi–Yau manifold with $\mathcal{L} \to X$ an ample line bundle. The pair $(X, \mathcal{L})$ is called a *polarized Calabi–Yau manifold*. We recall the construction of the (coarse) moduli space of polarized Calabi–Yau manifolds, denoted $\mathcal{M}_{\text{CY}}$: Embed X into some $\mathbb{P}^N$ using the sections of $\mathcal{L}^k$ for sufficiently large $k \in \mathbb{Z}$. Denote by $\mathfrak{hilb}(X)$ the Hilbert scheme of X which is acted on by $\text{PSL}_{N+1}(\mathbb{C})$. Then we define $\mathcal{M}_{\text{CY}}$ to be the quotient of the stable points of $\mathfrak{hilb}(X)$ by $\text{PSL}_{N+1}(\mathbb{C})$.

Remark 2.7.2. The unobstructedness theorem of Tian [63] tells us that the deformation of complex structures of Calabi–Yau manifolds is unobstructed. That is, the universal deformation space (Kuranishi space) is smooth. The existence of finite automorphisms, however, illuminates the fact that $\mathcal{M}_{\text{CY}}$ may admit quotient singularities – $\mathcal{M}_{\text{CY}}$ is a complex orbifold, or smooth Deligne–Mumford stack, in general.

Reminder 2.7.3. A *family of compact complex manifolds* $\mathcal{X} = (X, p, S)$ is specified by a surjective holomorphic map $p : X \to S$ between connected complex spaces X, S, where p is flat with connected fibres.

Remark 2.7.4. If X and S are both smooth, flatness implies that p is everywhere of maximal rank, the family $\mathcal{X}$ is then called a *smooth family*. A classical result of Ehresmann tells us that a smooth family of smoothly trivial – in particular, all fibres are diffeomorphic.

Reminder 2.7.5. Fix a compact complex manifold V. A *deformation of* V, parametrised by a complex space S is a family $\mathcal{X} = (X, p, S)$ with a distinguished basepoint $s_0 \in S$ and a biholomorphism $V \simeq X_{s_0} = p^{-1}(s_0)$.

Remider 2.7.6. A deformation $\mathcal{X} = (X, p, S)$ of a compact complex manifold V with basepoint s_0 is said to be (locally) *complete* if every defromation $\mathcal{X}' = (X', p', S')$ of V with basepoint s_0' (locally) is obtained as the pullback from $\mathcal{X}$ via a suitable holomorphic map $f : S' \to S$, where $f(s_0') = s_0$. If f is in fact always uniquely determined by $\mathcal{X}'$, the deformation $\mathcal{X}$ is called (locally) *universal*.

Remark 2.7.7. The universal deformation of V, if it exists, is unique up to isomorphism.[10] Universal deformations, however, do not always exist (see, e.g., [3, p. 30]). Hence, a weaker notion is introduced:

[10]Morphisms of deformations are defined to be basepoint preserving morphisms between the families which are compatible with the isomorphisms from V onto the fibres over the basepoints.

Reminder 2.7.8. Let $\mathcal{X} = (X, p, S)$ be a deformation of V with basepoint s_0. We say that $\mathcal{X}$ is a *versal deformation* if for every $\mathcal{X}'$ (as above), there is an f (as above), which need not be unique, but whose derivative at s_0' is uniquely determined.

Theorem 2.7.9. (Kuranishi's theorem). Every compact complex manifold admits a versal deformation.

Let $\mathcal{X} = (X, p, S)$ be a smooth deformation of a compact complex manifold V with basepoint s_0. The exact sequence of sheaves

$$0 \longrightarrow \mathcal{T}_{X_{s_0}} \longrightarrow \mathcal{T}_X|_{X_{s_0}} \longrightarrow \mathcal{N}_{X_{s_0}/S} \longrightarrow 0$$

induces a map

$$\rho_{s_0} : \mathcal{T}_S(s_0) \longrightarrow H^1(V, \mathcal{T}_V),$$

called the *Kodaira–Spencer map.*

Remark 2.7.10. The following results are straightforward consequences of the definition of the Kodaira–Spencer map (for details, see, e.g., [3, p. 30–31]):

(i) A smooth family is complete if and only if the Kodaira–Spencer map is surjective.
(ii) If $\mathcal{X}$ is a complete deformation and the Kodaira–Spencer map is bijective, then $\mathcal{X}$ is a versal deformation.

Reminder 2.7.11. Let $\mathcal{X} \to \mathcal{M}_{\mathrm{CY}}$ be the universal family of (polarized) Calabi–Yau manifolds. Let $U \subset \mathcal{M}_{\mathrm{CY}}$ be a local chart, in which we have local coordinates $(t_1, ..., t_k)$, $k := \dim \mathcal{M}_{\mathrm{CY}}$. Via the Kodaira–Spencer map, each $\partial/\partial t_j$ corresponds to an element

$$\imath \left(\frac{\partial}{\partial t_j} \right) \in H^1(\mathcal{X}_t, T\mathcal{X}_t).$$

The *Weil–Petersson metric* is the L^2–inner product of harmonic forms representing classes in $H^1(\mathcal{X}_t, T\mathcal{X}_t)$. The following explicit description of the Weil–Petersson metric, for the moduli space of Calabi–Yau manifolds, is due to Tian [63]. Let Ψ be a nowhere vanishing $(n-k, 0)$–form on $\mathcal{X}_t$. Then

$$\omega_{\mathrm{WP}} \left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial \bar{t}_j} \right) \quad = \quad \left(\int_{\mathcal{X}_t} \Psi \wedge \overline{\Psi} \right)^{-1} \int_{\mathcal{X}_t} \Psi \lrcorner \imath \left(\frac{\partial}{\partial t_i} \right) \wedge \overline{\Psi \lrcorner \imath \left(\frac{\partial}{\partial t_j} \right)}.$$

2.8. **Regularisation of Plurisubharmonic Functions: Smooth Case.**

Reminder 2.8.1. Let $D \subseteq \mathbb{C}^n$ be a domain, i.e., a connected open set. A function $f : D \to [-\infty, +\infty)$ is *plurisubharmonic* if f is

(i) upper semi-continuous.
(ii) subharmonic on the intersection of each complex line with D, i.e., for all $p \in D$ and all $\zeta \in \mathbb{C}^n$ with $|\zeta| < \text{dist}(p, \mathbb{C}^n \backslash D)$, the function f satisfies the mean-value inequality

$$f(p) \leq \frac{1}{2\pi} \int_0^{2\pi} f(p + e^{\sqrt{-1}\vartheta}\zeta) d\vartheta.$$

Proposition 2.8.2. A $\mathcal{C}^2$ function $f : D \to [-\infty, +\infty)$ is plurisubharmonic at p if and only if $\sqrt{-1}\partial\overline{\partial} f$ is positive semi-definite in a neighbourhood of p.

Remark 2.8.3. Taking the trace of $\sqrt{-1}\partial\overline{\partial} f$ with respect to the Euclidean metric on $\mathbb{C}^n$ tells us that plurisubharmonic functions are subharmonic.

Example 2.8.4. If $g : \mathbb{C}^n \to \mathbb{C}$ is holomorphic, the function $f = \log |g|$ is plurisubharmonic.

Reminder 2.8.5. A plurisubharmonic function $f \in \text{PSH}(X)$ is said to have *analytic singularities* if f can be expressed locally in the form

$$f = \frac{\lambda}{2} \log \left(\sum_{j=1}^n |f_j|^2 \right) + g, \tag{4}$$

where λ is a positive real number, g is a locally bounded function, and each f_j is holomorphic.

Remark 2.8.6. If X happens to be algebraic, we say that f has algebraic singularities if f admits the above description (4) on a small Zariski open set with λ a positive rational number, and the f_j polynomials.

Reminder 2.8.7. A *current of degree* p on a complex manifold X is a p–form whose coefficients are distributions.[11] A degree p–current ω is said to be *closed* if $d\omega = 0$ is satisfied in the sense of distributions. Similarly, ω is *exact* if there is a $(p-1)$–current η such that $\omega = d\eta$ in the sense of distributions.

[11] An alternative definition for a degree p is a current is: an element of the dual space of $\Omega_c^p(X)$, the space of smooth q–forms on X with compact support. The duality pairing is given by $\langle \Theta, \omega \rangle = \int_X \Theta \wedge \omega$, where $\omega \in \Omega_c^q(X)$ and $p + q = \dim(X)$.

Example 2.8.8. Let Y be a complex submanifold of a complex manifold X. The *current of integration* $[Y]$ is

$$\langle [Y], \omega \rangle := \int_Y \omega,$$

where $\omega \in \Omega^p_c(X)$ and $p := \dim_{\mathbb{R}} Y$. It is clear that $[Y]$ is a current with measure coefficients. Stokes' theorem, which is easily extended to currents (see, e.g., [26]) tells us that

$$d[Y] = (-1)^{q-1}[\partial Y].$$

In particular, $[Y]$ is a closed current if Y has no boundary.

Remark 2.8.9. The above example leads one to introduce the following terminology: The *dimension* of a current of degree p on a complex manifold X is $\dim_{\mathbb{R}} X - p$. Further, the notions of bidegree for currents carry over from their definitions from smooth forms. The notion of bidimension is therefore defined analogously to how the dimension of current was defined.

Reminder 2.8.10. A current of bidimension (p, p), i.e., bidegree $(n-p, n-p)$, Θ on a complex manifold X is said to be *positive* if for all $(1,0)$–forms $\omega_1, ..., \omega_p$ on X, the measure

$$\Theta \wedge \sqrt{-1}\omega_1 \wedge \overline{\omega_1} \wedge \cdots \wedge \sqrt{-1}\omega_p \wedge \overline{\omega}_p$$

is positive (in the sense of measures).

Remark 2.8.11. A current $\Theta = \sqrt{-1}\sum_{1 \leq j,k \leq n} \Theta_{j\overline{k}} dz^j \wedge d\overline{z}^k$ of bidegree $(1,1)$ is positive if and only if the complex measure $\sum_{1 \leq j,k \leq n} \xi^j \overline{\xi}^k \Theta_{j\overline{k}}$ is a positive measure for all $(\xi^1, ..., \xi^n) \in \mathbb{C}^n$.

Example 2.8.12. Let X be a complex manifold. Let $f : X \to [-\infty, +\infty)$ be a plurisubharmonic function on X which is not identically $-\infty$ on X.[12] Associated to such an f is a closed posittive current of bidegree $(1,1)$, namely, $\Theta = \sqrt{-1}\partial\overline{\partial}u$. The converse is also true for currents defined on domains $D \subset X$ such that $H^2_{\text{DR}}(D, \mathbb{R}) = H^1(D, \mathcal{O}) = 0$.

Reminder 2.8.13. Let $f : X \to [-\infty, +\infty)$ be an upper semi-continuous function on a complex manifold X. We say that f is *quasi-plurisubharmonic* if f can locally be expressed as $f = u + v$, where u is plurisubharmonic and v is smooth. Let ϑ be a real closed $(1,1)$–form on X. A quasi-plurisubharmonic function f is said to be *ϑ–plurisubharmonic* if $\vartheta + dd^c f \geq 0$ in the sense of currents.[13] The set of ϑ–plurisubharmonic functions on X is denoted $\text{PSH}(X, \vartheta)$.

[12] If X is not assumed connected, then we assume that f is not identically $-\infty$ on any connected component of X.

[13] The real operator dd^c is defined: $dd^c := \dfrac{\sqrt{-1}}{\pi}\partial\overline{\partial}$.

Theorem 2.8.14. (Local regularisation, [6, Theorem 3.3]). Let (X, ω) be a Kähler manifold and let ϑ be a real closed $(1,1)$–form on X. Take f to be a ϑ–plurisubharmonic function on X. The convolutions $f_j := f \star \rho_j$ (defined with respect to a local coordinate system z) are smooth functions satisfying the following:

(i) $f_j + c_j \searrow f$ pointwise for some sequence $c_j \searrow 0$.
(ii) $\vartheta + dd^c f_j \geq -\varepsilon_j \omega$ for some sequence $\varepsilon_j \searrow 0$.
(iii) If f is locally bounded, and the convolutions f_j are defined with respect to a another local coordinate system w (in a neighbourhood of the same point as the convolutions above), then $f \star_z \rho_j - f \star_w \rho_j \to 0$ locally uniformly.

Theorem 2.8.15. (Global regularisation, [6, Theorem 3.4]). Let (X, ω) be a Kähler manifold. Let f be a locally bounded quasi-plurisubharmonic function on X. There exists a sequence of smooth functions $f_j \in \mathcal{C}^\infty(X)$

(i) $f_j \searrow f$ pointwise on X.
(ii) If f is ϑ–plurisubharmonic, for some closed real $(1,1)$–form ϑ, then for each relatively compact open subsets $U \subset X$, there is a sequence $\varepsilon_j \searrow 0$ such that $\vartheta + dd^c f_j \geq -\varepsilon_j \omega$ on U.

Remark 2.8.16. An old result of Richberg [56] tells us that a continuous ϑ–plurisubharmonic function f on a complex manifold X can be approximated (locally uniformly) from above by a sequence $f_j \in \mathcal{C}^\infty(X)$ such that the negative part of $\vartheta + dd^c f_j$ decays to zero on compacts. The above theorem extends this result to the case when f is merely locally bounded and was initially obtained using the ideas in [16].

Example 2.8.17. The local boundedness assumption in Theorem 2.8.15 cannot be omitted. Indeed, blow up a compact Kähler surface (X, ω) at a point to produce a cohomology class $[E] \in H^2(X)$ such that $[E]^2 = -1$. Endow $\mathcal{O}(E)$ with a Hermitian metric h whose associated curvature form is Θ and let $s \in H^0(X, \mathcal{O}(E))$ be the defining section of $\mathcal{O}(E)$. The function $f = \log h(s, \overline{s})$ is Θ–plurisubharmonic but fails to be locally bounded. Assume there is a sequence $f_j \in \mathcal{C}^\infty(X)$ satisfying (i) and (ii) of Theorem 2.8.15. Then

$$0 \leq \int_X (\vartheta + dd^c f_j + \varepsilon_j \omega)^2 \quad = \quad ([E] + \varepsilon_j [\omega])^2 \ \searrow \ [E]^2 \ = \ -1.$$

Theorem 2.8.15 also permits a good definition of the wedge product of currents, under some mild assumptions. That is, if f is a plurisubharmonic function and Θ is a closed positive current on a complex manifold X, we want to make sense of the expression $dd^c f \wedge \Theta$ even in the case when neither f or Θ are smooth. Of course, in great generality this is not possible, since one cannot multiply measures (see, e.g., [44] for entertaining examples).

Reminder 2.8.18. ([4]). Let f be a locally bounded quasi-plurisubharmonic function and Θ a current on a complex manifold X. Define

$$dd^c f \wedge \Theta := \lim_{j \to +\infty} dd^c f_j \wedge \Theta,$$

where lim is understood to be in the weak topology of currents, and $f_j \searrow f$ is the regularised sequence obtained from Theorem 2.8.15. Note that this definition depends only on the closed positive current $dd^c f$ and not on the specific choice of potential f (c.f., Example 2.8.12).

For quasi-plurisubharmonic functions functions which are not locally bounded, the notion of a non-pluripolar product has been introduced (see, e.g., [7]).

2.9. **Regularisation of Plurisubharmonic Functions: Singular Case.** One of the main difficulties concerning the main problem is that the background geometry is degenerate in a number of ways: We need to work with non-smooth Kähler spaces (X, ω) (which will be defined shortly) – these are, in particular, normal complex varieties. Naturally, one wants to resolve the singularities via a birational map $\pi : \widetilde{X} \longrightarrow X$ and work on the desingularised space $\widetilde{X}$. The cohomology class represented by $\pi^*\omega$ is no longer Kähler, however, and in fact, is not even nef. In general, $[\pi^*\omega]$ is a big cohomology class and much less is known about the complex geometry of these cohomology classes. There has been progress, however, in recent years, see, e.g., [23, 7], where the study of complex Monge–Ampère equations in big cohomology classes was initiated.

For the convenience of the reader, let us restate that a cohomology class $\alpha \in H^{1,1}(X, \mathbb{R})$ is *big* if it can be represented by a Kähler current, i.e., a closed positive current which dominates a Kähler form. In general, such Kähler currents can be rather unwieldy and too singular, so one prefers to work with another class of currents:

Definition 2.9.1. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class. A positive current $T = \vartheta + dd^c f \in \alpha$ is said to have *minimal singularities* if for every positive current $S = \vartheta + dd^c \psi \in \alpha$, there exists a constant $c \in \mathbb{R}$ such that $\psi \leq \varphi + c$ on X. A ϑ–plurisubharmonic function φ is said to have *minimal singularities* if the associated current has minimal singularities.

Example 2.9.2. The function

$$V_\vartheta := \sup \left\{ v \in \text{PSH}(X, \vartheta) : \sup_X v \leq 0 \right\}$$

is ϑ–plurisubharmonic with minimal singularities.

Definition 2.9.3. Let $\alpha \in H^{1,1}(X, \mathbb{R})$ be a big cohomology class. The *polar locus of* α is the set

$$P(\alpha) := \{x \in X : V_\vartheta(x) = -\infty\}.$$

The *non-bounded locus of* α is the set

$$\mathrm{NB}(\alpha) := \{x \in X : V_\vartheta \notin L^\infty_{\mathrm{loc}}(\{x\})\}.$$

It is clear that these definitions do not depend on the choice of ϑ.

Remark 2.9.4. It is immediate that $P(\alpha) \subseteq \mathrm{NB}(\alpha)$, and it is easy to provide examples of α for which $P(\alpha) = \mathrm{NB}(\alpha)$. For instance, take α to be a real $(1,1)$–cohomology class represented by a smooth semi-positive form. Then for such a choice of α, we have $P(\alpha) = \mathrm{NB}(\alpha) = \emptyset$. It remains unknown as to whether there exists a big cohomology class with $P(\alpha) \subsetneq \mathrm{NB}(\alpha)$ (c.f., [25]).

Remark 2.9.5. Let (X, ω) be a Kähler manifold. Theorem 2.8.15 tells us that for $f \in \mathrm{PSH}(X, \vartheta)$, there is a sequence f_j of $(\vartheta + \varepsilon_j \omega)$–plurisubharmonic functions with analytic singularities which approximate f from above pointwise. Applying this to the Kähler current representing a big cohomology class, it follows that any big cohomology class contains a strictly positive current with analytic singularities.

The main theorem of [25] is the following:

Theorem 2.9.6. ([25, p. 2]). Let X be a compact Kähler manifold with α a big cohomology class, as above. Let $\vartheta \in \alpha$ be a smooth representative and let $T = \vartheta + dd^c f$ be a positive current in α. If $P(\alpha) = \mathrm{NB}(\alpha)$, then there exists a sequence $f_j \in \mathrm{PSH}(X, \vartheta)$ of exponentially continuous[14] ϑ–plurisubharmonic functions which have minimal singularities and satisfy $f_j \searrow f$ pointwise on X.

Remark 2.9.7. The technical condition $P(\alpha) = \mathrm{NB}(\alpha)$ is necessary: the existence of an exponentially continuous ϑ–plurisubharmonic function with minimal singularities in α implies that $P(\alpha) = \mathrm{NB}(\alpha)$. See [25, p. 2] for further discussion.

For us, the importance of Theorem 2.9.6 comes from the coupling of Theorem 2.9.6 with the aforementioned approximation result for continuous quasi-plurisubharmonic functions due to Richberg.

[14] We say that f is *exponentially continuous* if and only if $e^f : X \to \mathbb{R}$ is continuous.

Corollary 2.9.8. Let (V, ω_V) be a normal compact Kähler space.[15] Let f be an ω_V–plurisubharmonic function on V. Then there exists a sequence of smooth ω_V–plurisubharmonic functions $f_j \searrow f$ pointwise on V.

Theorem 2.9.9. ([48]). Let (X^n, ω) be a compact Kähler manifold and μ a smooth volume form on X. Consider the complex Monge–Ampère equation

$$(\omega + \sqrt{-1}\partial\overline{\partial}\varphi)^n \quad = \quad F\Omega,$$

where F is a non-negative $L^p(X, \Omega)$–function, for some $p > 1$, with

$$\int_X F\Omega = \int_X \omega^n.$$

Then there exists $0 < \alpha < 1$ such that $\|\varphi\|_{\mathcal{C}^\alpha(X,\omega)} \leq C$.

Remark 2.9.10. The above theorem is easily generalised to compact Kähler orbifolds.

Theorem 2.9.11. ([51]). Let (X, ω) be a compact Kähler manifold. Let ω_φ be another Kähler metric cohomologous to ω with $\varphi \in \mathcal{C}^\alpha(X, \omega)$. Then

$$\text{dist}_{\omega_\varphi}(p, q) \leq C \text{dist}_\omega(p, q)^{\frac{\alpha}{2}},$$

for all $p, q \in X$.

Proof. The problem is local, so we fix a chart $B(x, R)$ to work, in which ω is uniformly equivalent to the Euclidean metric. Set

$$\rho_x(y) := \text{dist}_{\omega_\varphi}(x, y),$$

which satisfies

$$|\nabla^{\omega_\varphi} \rho_x| \leq 1 \leq \text{tr}_\omega \omega_\varphi.$$

[15]Complex Spaces are treated in §2.9.

For $r < R/3$, we fix a smooth cutoff function η supported in $B(x, 2r)$ with $\eta \equiv 1$ on $B(x, r)$, such that $dd^c\eta \leq Cr^{-2}\omega$. We compute:

$$\begin{aligned}
\int_{B(x,r)} (\mathrm{tr}_\omega \omega_\varphi)\omega^n &= n \int_{B(x,r)} \omega_\varphi \wedge \omega^{n-1} \\
&\leq n \int_{B(x,2r)} \eta(\omega + dd^c\varphi) \wedge \omega^{n-1} \\
&\leq Cr^{2n} + n \int_{B(x,2r)} \eta dd^c\varphi \wedge \omega^{n-1} \\
&= Cr^{2n} + n \int_{B(x,2r)} \left(\varphi - \inf_{B(x,2r)} \varphi \right) dd^c\eta \wedge \omega^{n-1} \\
&\leq Cr^{2n} + Cr^{2(n-1)} \mathrm{osc}_{B(x,2r)}\varphi \\
&\leq Cr^{2(n-1)+\alpha}.
\end{aligned}$$

Hence,

$$\int_{B(x,r)} |\nabla^{\omega_\varphi} \rho_x| \omega^n \leq Cr^{2(n-1)+\alpha},$$

and from the Morrey embedding theorem:

$$\|\rho_x\|_{\mathcal{C}^{\alpha/2}(B(x,R/3))} \leq C,$$

with C depending only on (X, ω) and the Hölder bound on φ. Therefore, on coordinate charts,

$$\mathrm{dist}_{\omega_\varphi}(x, y) \leq C\mathrm{dist}_\omega(x, y)^{\frac{\alpha}{2}},$$

and by the triangle inequality, this holds globally on X. □

Remark 2.9.12. The fact that Hölder regularity of the potential (as in Theorem 2.9.9) allows us to obtain control of the induced distance functions (as in Theorem 2.9.11) is very useful. In fact, this is one of the primary stumbling blocks in extending the main result of this note to more singular settings.

Indeed, when working with singular compact Kähler spaces (V, ω_V), one naturally works with a desingularisation $(\widetilde{V}, \pi^*\omega_V)$ of this space. While the space $\widetilde{V}$ is now a complex manifold, the cohomology class of $\pi^*\omega_V$ ruined. It is no longer represents a Kähler class, but merely a big cohomology class. Hence, in order to recover Theorem 2.9.11 in this more general setting, we require Hölder regularity for complex Monge–Ampère equations in big cohomology classes. This is in fact a problem of independent interest:

Question 2.9.13. ([21, Problem 19]). Assume ω is merely semi-positive and big, and

$$(\omega + dd^c\varphi)^n = f dV, \qquad f \in L^p, \qquad p > 1.$$

Is φ Hölder continuous on X?

Reminder 2.9.14. Let α be a big class in $H^{1,1}(X, \mathbb{R})$. The *ample locus* $\mathrm{Amp}(\alpha)$ is the set of all $x \in X$ for which there exists a strictly positive current $T \in \alpha$ with analytic singularities which is smooth in a neighbourhood of x.

Remark 2.9.15. When α is the first Chern class of a big line bundle, the ample locus coincides with the complement of the so-called augmented base locus $\mathbb{B}_+(\alpha)$ (see, e.g., [22]).

In [18], Demailly–Dinew–Guedj–Hiep–Kolodziej–Zeriahi establish a positive solution to the above question on the ample locus of $[\omega]$. The fundamental idea is that currents with minimal singularities are only locally bounded on the ample locus of $[\omega]$. The problem is thus to understand the regularity at the boundary of the ample locus.

2.10. **Kähler Spaces.**

Reminder 2.10.1. A *reduced complex space* is a $\mathbb{C}$–analytic ringed space $(X, \mathcal{O}_X)$, where X is a Hausdorff topological space and $\mathcal{O}_X$ is a subsheaf of $\mathscr{C}_X$ satisfying the following: for each $p \in X$, there is an open neighbourhood $U \subset X$ such that $(U, \mathcal{O}_X|_U)$ is isomorphic to $(A, \mathcal{O}_A)$ as a $\mathbb{C}$–analytic ringed space, where A is an analytic subvariety in some $\mathbb{C}^n$ and $\mathcal{O}_A$ is the sheaf of holomorphic functions on A.

Reminder 2.10.2. The sheaf $\mathcal{M}_X$ of meromorphic functions on a reduced complex space is defined (see, e.g., [45, p. 204]) as the sheaf associated to the presheaf of rings of fractions $U \mapsto \mathcal{O}_X(U)[S(U)^{-1}]$, where $S(U)$ is the set of all elements of $\mathcal{O}_X(U)$ which do not restrict to zero-divisors in the stalks $\mathcal{O}_{X,x}$ for all $x \in X$.

Reminder 2.10.3. A complex space $(X, \mathcal{O}_X)$ is said to be *normal* if every local ring $\mathcal{O}_{X,x}$ is integrally closed in $\mathcal{M}_{X,x}$.

Remark 2.10.4. It is known (see, e.g., [29, p. 125]) that normal complex spaces are locally irreducible, locally pure-dimensional, and the singular set has complex codimension at least 2. Moreover, Riemann's second extension theorem holds for normal complex spaces (see, e.g., [29, p. 143]), i.e., a function holomorphic on the regular set of a normal complex space necessarily extends to a function holomorphic holomorphic on all of X.

Reminder 2.10.5. Let $(X, \mathcal{O}_X)$ be a reduced complex space. We say that an upper semi-continuous function $\varphi : X \to \mathbb{R} \cup \{-\infty\}$ is *(strictly) plurisubharmonic* at a point $p \in X$ if there is an open neighbourhood $U \subset X$ containing p, a local embedding $\tau : U \to G \subset \mathbb{C}^n$, and a (strictly) plurisubharmonic function $\widetilde{\varphi}$ on G such that $\varphi|_U = \widetilde{\varphi} \circ \tau$. If φ is (strictly) plurisubharmonic at all points of X, we say that φ is *(strictly) plurisubharmonic*.

Remark 2.10.6. A theorem of Narasimhan [54] tells us that the definition is independent of the local embedding $\tau : U \to G \subset \mathbb{C}^n$. There are alternative characterisations of plurisubharmonic functions on complex spaces, but due to results in [27] they turn out to be equivalent for normal complex spaces.

Reminder 2.10.7. A *Kähler form* on a complex space X is a current ω who local potentials extend to smooth strictly pluri-subharmonic functions in local embeddings of X into Euclidean space. A complex space X is called a *Kähler space* if it admits a Kähler form.

The motivation for considering Kähler spaces stems out of necessity in the minimal model program. In what follows, $(V, \mathcal{O}_V)$ denotes a normal complex space whose regular locus is denoted V°. We denote by $j : V^\circ \to V$ the natural open immersion. The canonical sheaf of V is defined $K_V := j_* K_{V^\circ}$, which is a coherent analytic sheaf on V. Similarly, the *pluricanonical sheaf* of V is defined $K_V^{[q]} := j_* K_{V^\circ}^{[q]}$, for $q > 0$, which is a coherent analytic sheaf on V, also.

Reminder 2.10.8. The complex space $(V, \mathcal{O}_V)$ of pure dimension n is called 1–*Gorenstein* if the canonical sheaf K_V is a locally free sheaf of rank one. A local section of ω_V defining a non-vanishing holomorphic n–form on V° is called a *local generator of* K_V. We say that V is *Gorenstein* if V is 1–Gorenstein and Cohen–Macaulay.

Reminder 2.10.9. The complex space $(V, \mathcal{O}_V)$ is said to be $\mathbb{Q}$–*Gorenstein* if for all $x \in V$ there is a positive integer N_x and an open neighbourhood U of x such that $K_U^{[N_x]}$ is a locally free sheaf of rank one. The smallest N_x satisfying this condition is called the *local index of* V *near* x. The lowest common multiple of all local indices is finite and is called the *index of* V.

Reminder 2.10.10. Let V be $\mathbb{Q}$–Gorenstein with index N. We say that V has *canonical singularities* if for any resolution $\pi : \widetilde{V} \to V$ of V, and any $m \in \mathbb{N}$,

$$\pi_* K_X^{[Nm]} = K_V^{[Nm]}.$$

Theorem 2.10.11. Let V be a projective algebraic manifold of general type whose canonical ring

$$R := \bigoplus_{k \in \mathbb{N}} H^0(V, K_V^m)$$

is of finite type. Then the canonical model $V_{\text{can}} := \text{Proj}(R)$ has only canonical singularities.

Reminder 2.10.12. Let V be $\mathbb{Q}$–Gorenstein with index N. We say that V has *log-terminal singularities* if for any log resolution $\pi : \widetilde{V} \to V$ and any local generator α of $K_V^{[N]}$, the pole along any component E (of the exceptional divisor of π) of the metromorphic N–canonical form $\pi^*\alpha$ on X is of order $\leq N-1$.

Remark 2.10.13. If V happens to be algebraic, we may give alternative definitions for canonical and log-terminal singularities: Let $\pi : \widetilde{V} \to V$ be a resolution of V.

(i) V has canonical singularities if

$$K_{\widetilde{V}} \ \sim_{\mathbb{Q}} \ \pi^* K_V + \sum_E a_E E, \qquad a_E \geq 0,$$

where the sum runs over all exceptional divisors of π and $\sim_{\mathbb{Q}}$ means numerical equivalence of $\mathbb{Q}$–Cartier divisors.

(ii) V has log-terminal singularities if

$$K_{\widetilde{V}} \ \sim_{\mathbb{Q}} \ \pi^* K_V + \sum_E a_E E, \qquad a_E > -1.$$

Example 2.10.14. If S is a normal complex algebraic surface. Then S has canonical singularities if and only if S is locally biholomorphic to $\mathbb{C}^2/\Gamma$ for some finite subgroup $\Gamma \subset \mathrm{SL}(2,\mathbb{C})$. If Γ is a finite subgroup of $\mathrm{GL}(2,\mathbb{C})$, then S has log-terminal singularities.

3. Recent Progress of STZ and GTZ

Song–Tian–Zhang Proof. Let (X^n, ω) be a projective manifold of complex dimension n, Kodaira dimension $0 < \kappa < n$, and with K_X semi-ample. For ℓ sufficiently large, the sections of K_X^ℓ furnish an Iitaka fibration[16]

$$f : X \longrightarrow X_{\text{can}},$$

where $X_{\text{can}} = \text{Proj}\left(\bigoplus_{m \geq 0} H^0(X, mK_X)\right)$ is the canonical model of X. In general, X_{can} is an irreducible normal projective variety of dimension κ (in particular, a compact Kähler space). Fix a Kähler metric ω_A on X. We will consider a continuous family of Kähler metrics $\omega(t)$ evolving under

$$\text{Ric}(\omega(t)) = -\omega(t) + t\omega_A, \qquad t \in (0,1]. \tag{5}$$

Let χ denote the pullback of the Fubini–Study metric to X_{can}. Taking cohomology classes of (5):

$$[\omega(t)] = [\chi] + t[\omega_A].$$

[16]To be explicit: This is a proper surjective holomorphic map with connected Calabi–Yau fibres (over X_{can}°) which is a holomorphic submersion over X_{can}°. It is common to refer to such a map as a *Calabi–Yau fibration.*

As mentioned before, the Ricci curvature of $\omega(t)$ is uniformly bounded from below: $\text{Ric}(\omega(t)) \geq -\omega(t)$. From the diameter estimate of Fu–Guo–Song [28], the lower Ricci bound gives

$$\text{diam}(X, \omega(t)) \leq D,$$

for some uniform constant D. By Gromov's precompactness theorem Theorem 2.5.16, as $t \searrow 0$ (after possibly passing to a subsequence), there is a limit $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ of the sequence of metric spaces $\big(X, \text{dist}_{\omega(t)}\big)$.

We first show that $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ is isometric to the metric completion of $(X_{\text{can}}^{\circ}, \text{dist}_{\omega_{\text{can}}})$, where X_{can}° is the regular locus of X_{can} with the critical values of the Iitaka fibration removed. Set $X^{\circ} = f^{-1}(X_{\text{can}}^{\circ})$. We recall some preliminary lemmas:

Lemma 3.1.1. ([28, Theorem 1.3]). Let $\omega(t)$ evolve under (5). For any compact subset $K \subset X^{\circ}$, we have

$$\|\omega_t - f^* \omega_{\text{can}}\|_{\mathcal{C}^0(K, \omega_A)} \ \longrightarrow \ 0,$$

as $t \searrow 0$.

We want to prove geodesic convexity of X_{can}° using Proposition 2.5.17. This is proved in [13, §3] for Gromov–Hausdorff limits of Riemannian manifolds with a lower bound on the Ricci curvature. We are yet to determine the Gromov–Hausdorff limit of $(X, \omega(t))$, however.

Construction 3.1.2. Let us briefly mention how to construct an ε–neighbourhood of the discriminant locus of the Iitaka map: Since K_X is semi-ample, for sufficiently large ℓ, there is an ample line bundle $\mathcal{L} \to X_{\text{can}}$ such that $K_X^{\ell} = f^* \mathcal{L}$. We choose an effective $\mathbb{Q}$–divisor σ on X_{can} such that σ lies in the isomorphism class of $\mathcal{L}$ and the discriminant locus lies in the support of σ. Equip $\mathcal{L}$ with a (singular) Hermitian metric h. We set $D_{\varepsilon} := \{|\sigma|_h^2 < \varepsilon\}$ to be the desired tubular neighbourhood.

Lemma 3.1.3. ([61, Proposition 2.1]). Let $\varepsilon > 0$ be given and let D_{ε} be an ε–neighbourhood of the discriminant locus of the Iitaka map $f : X \to X_{\text{can}}$. For any pair of points $p, q \in D_{\varepsilon}$ there is a continuous map $\gamma_t \subset D_{\varepsilon}$ connecting p, q and $0 < \delta < \infty$ such that

$$\text{length}_{\omega(t)}(\gamma_t) \ \leq \ \text{dist}_{\omega(t)}(p, q) + \delta.$$

Theorem 3.1.4. The metric completion of $(X_{\text{can}}^{\circ}, \text{dist}_{\omega_{\text{can}}})$ is isometric to the Gromov–Hausdorff limit $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$.

Proof. Let $\overline{(X_{\text{can}}^{\circ}, \text{dist}_{\omega_{\text{can}}})}$ denote the metric completion of $(X_{\text{can}}^{\circ}, \text{dist}_{\omega_{\text{can}}})$. We need to construct a bijective map

$$h : (\mathcal{Z}, \text{dist}_{\mathcal{Z}}) \longrightarrow \overline{(X_{\text{can}}^{\circ}, \text{dist}_{\omega_{\text{can}}})}$$

preserving the distance between points in $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$. We do this by first showing that the distance functions $\text{dist}_{\mathcal{Z}}$ and $\text{dist}_{\omega_{\text{can}}}$ coincide on X_{can}°. The bijectivity of h is easy enough to prove: By [28], X_{can}° locally isometrically embeds into $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ as a dense open subset. Let p be some point in $\overline{(X_{\text{can}}^{\circ}, \text{dist}_{\omega_{\text{can}}})}$. Then there is a sequence $\{p_j\} \subset X_{\text{can}}^{\circ}$ which limits to p with respect to $\text{dist}_{\omega_{\text{can}}}$. Since the distance functions $\text{dist}_{\omega_{\text{can}}}$ and $\text{dist}_{\mathcal{Z}}$ coincide, the sequence $p_j = h^{-1}(p_j) \subset h^{-1}(X_{\text{can}}^{\circ}) = X_{\text{can}}^{\circ}$ limits to $q \in \mathcal{Z}$ with respect to $\text{dist}_{\mathcal{Z}}$ such that $h(q) = p$. The same argument shows that h is injective.

We show that the distance functions $\text{dist}_{\omega_{\text{can}}}$ and $\text{dist}_{\mathcal{Z}}$ coincide on X_{can}° in two steps:
Let us first show that

$$\text{dist}_{\omega_{\text{can}}} \ \leq \ \text{dist}_{\mathcal{Z}} \quad \text{on } X_{\text{can}}^{\circ}.$$

Fix two points $p, q \in X_{\text{can}}^{\circ}$ and lift these to arbitrary points in their fibre $p' \in f^{-1}(p)$, $q' \in f^{-1}(q)$. Let γ_t be a continuous path[17] connecting p' and q' in X. By Lemma 3.1.3, for any $\delta > 0$, we can find $\varepsilon > 0$ such that for $p', q' \in f^{-1}(D_{\varepsilon})$ (the preimage of an ε–neighbourhood of the discriminant locus)

$$\text{length}_{\omega(t_j)}(\gamma_{t_j}) \ \leq \ \text{dist}_{\omega(t_j)}(p', q') + \delta,$$

for all t_j, where $\omega(t_j)$ limits to ω_{can} in the $\mathcal{C}^0$–topology as $t_j \to \infty$. In particular, for j sufficiently large,

$$\begin{aligned}
\text{dist}_{\omega_{\text{can}}}(p, q) \ &\leq \ (1+\delta)\text{length}_{\omega(t_j)}(\gamma_{t_j}) \\
&\leq \ (1+\delta)(\text{dist}_{\omega(t_j)}(p', q') + \delta) \\
&\leq \ \text{dist}_{\omega(t_j)}(p', q') + \delta(\text{diam}(X, \omega(t_j)) + 2 + 2\delta) \\
&\to \ \text{dist}_{\mathcal{Z}}(p, q) + \delta(\text{diam}(\mathcal{Z}, \text{dist}_{\mathcal{Z}}) + 2 + 2\delta) \quad (\text{as } t_j \to +\infty) \\
&\to \ \text{dist}_{\mathcal{Z}}(p, q) \quad (\text{as } \delta \to 0),
\end{aligned}$$

recalling the diameter estimate in [28] to justify the last line.

We now complete the proof by establishing the reverse inequality:

$$\text{dist}_{\mathcal{Z}} \ \leq \ \text{dist}_{\omega_{\text{can}}} \quad \text{on } X_{\text{can}}^{\circ}.$$

As before, let $p, q \in X_{\text{can}}^{\circ}$ with lifts $p' \in f^{-1}(p)$ and $q' \in f^{-1}(q)$. Let γ_{δ} be the continuous path connection p and q such that

$$\left|\text{length}_{\omega_{\text{can}}}(\gamma_{\delta}) - \text{dist}_{\omega_{\text{can}}}(p, q)\right| \ < \ \delta,$$

[17] In fact, for each finite t, the path is smooth. The path will only fail to be smooth in the limit if it crosses the discriminant locus of the Iitaka fibration f.

the existence of such a path is by definition. Let $\widehat{\gamma}_\delta$ denote the lift of γ_δ to a path in X connecting p' and q'. For sufficiently large j,

$$\begin{aligned}\mathrm{dist}_{\mathcal{Z}}(p,q) &\leq \mathrm{dist}_{\omega(t_j)}(p',q') + \delta \\ &\leq \mathrm{length}_{\omega(t_j)}(\widehat{\gamma}_\delta) + \delta \\ &\leq \mathrm{length}_{\omega_{\mathrm{can}}}(\gamma_\delta) + 2\delta \\ &\to \mathrm{dist}_{\omega_{\mathrm{can}}}(p,q) \quad (\text{as } \delta \to 0).\end{aligned}$$

□

We now turn our attention to the main problem: determining the homeomorphism type of the Gromov–Hausdorff limit $(\mathcal{Z}, \mathrm{dist}_{\mathcal{Z}})$. Prior to the mentioning of any theorems, what is required of the problem.

Let

$$\Psi : (X_{\mathrm{can}}^\circ, \mathrm{dist}_{\omega_{\mathrm{can}}}) \longrightarrow (\mathcal{Z}, \mathrm{dist}_{\mathcal{Z}})$$

be the isometric embedding. For any Kähler metric ω on X_{can}, the Yau Schwarz lemma estimate $\omega(t) \geq C^{-1} f^* \omega$ provides a uniform (independent of t) Lipschitz constant bound for the map $h_t : (X, \omega(t)) \longrightarrow (X_{\mathrm{can}}, \omega)$. Allowing $t \searrow 0$, we have a surjective Lipschitz map

$$h = h_\infty : (\mathcal{Z}, \mathrm{dist}_{\mathcal{Z}}) \longrightarrow (X_{\mathrm{can}}, \mathrm{dist}_\omega),$$

satisfying $h \circ \Psi = \mathrm{id}$. The conjecture is that h is a homeomorphism; it is enough to show that h is injective.

We consider first the case when X_{can} has at worst orbifold singularities. In which case, by standard practices, we may simply assume that X_{can} is smooth with ω a smooth Kähler metric on X_{can}, cohomologous to ω_{can}. The main idea of this first approach is to establish the following estimate:

$$\mathrm{dist}_{\omega_{\mathrm{can}}}(p,q) \ \leq \ C_\alpha \mathrm{dist}_\omega(p,q)^\alpha, \tag{6}$$

for some $\alpha \in (0,1)$, and all $p, q \in X_{\mathrm{can}}^\circ$.

Granted this, let $h(p) = h(q)$ for some points $p, q \in \mathcal{Z}$. Choose sequences $\{p_j'\}$ and $\{q_j'\}$ in X_{can}° which converge to $h(p)$ and $h(q)$ with respect to dist_ω. From the above distance estimate, $\mathrm{dist}_{\omega_{\mathrm{can}}}(p_j', h(p)) \to 0$ and $\mathrm{dist}_{\omega_{\mathrm{can}}}(q_j', h(q)) \to 0$. Since $h \circ \Psi = \mathrm{id}$ on X_{can}°, the sequences $p_j = h^{-1}(p_j')$ and $q_j = h^{-1}(q_j')$ are Cauchy in $(\mathcal{Z}, \mathrm{dist}_{\mathcal{Z}})$, and thus converge to unique limits (which are necessarily p and q) in $\mathcal{Z}$. Since $h(p) = h(q)$, we observe that $p = q$ and h is injective, as claimed.

To prove (6), the idea is to approximate the metric ω_{can} by metrics $\omega_j = \omega + \sqrt{-1}\partial\overline{\partial}\varphi_j$ which solve the regularised complex Monge–Ampère equation

$$\omega_j^{\kappa} \;=\; c_j e^{-F_j}\omega^{\kappa},$$

where the density $c_j e^{-F_j}$ is bounded in $L^p(\omega^{\kappa})$ for some $p > 1$. The regularity theory of Kolodziej then implies that the potential φ_j is α–Hölder continuous, for $\alpha \in (0,1)$. Hence, by Li's distance estimate, we have

$$\text{dist}_{\omega_j}(p,q) \;\leq\; C_{\alpha}\text{dist}_{\omega}(p,q),$$

for all $p, q \in X_{\text{can}}^{\circ}$. Limiting this estimate as ω_j converges to ω_{can}, yields the desired estimate.

Proof of the Main Theorem in the Orbifold Case. Let ω be a smooth orbifold Kähler metric cohomologous to ω_{can}. Let

$$F = -\log\left(\frac{\omega_{\text{can}}^{\kappa}}{\omega^{\kappa}}\right),$$

which we claim is quasi-plurisubharmonic on X_{can}° and bounded from above.
To see that F is quasi-plurisubharmonic, we compute

$$\begin{aligned}\sqrt{-1}\partial\overline{\partial}F &= \text{Ric}(\omega_{\text{can}}) - \text{Ric}(\omega)\\ &= -\omega_{\text{can}} + \omega_{\text{WP}} - \text{Ric}(\omega).\end{aligned}$$

Since ω_{WP} is a semi-positive $(1,1)$–form, it follows that F is quasi-plurisubharmonic. To bound F from above, recall that $\omega(t)$ is cohomologous to $\chi + t\omega_A$, and thus we may find a smooth function $\varphi(t) \in \mathcal{C}^{\infty}(X)$ solving the complex Monge–Ampère equation

$$t^{-(n-\kappa)}\omega(t)^n = \exp(\varphi)\Omega, \tag{7}$$

subject to the contraint $\omega(t) = \chi + t\omega_A + \sqrt{-1}\partial\overline{\partial}\varphi(t) > 0$ for $t \in (0,1]$. Here, Ω is a smooth volume form such that

$$\text{Ric}(\Omega) = -\sqrt{-1}\partial\overline{\partial}\log(\Omega) = \text{Ric}(\omega(t)) + \sqrt{-1}\partial\overline{\partial}\varphi(t).$$

It suffices to show that $\text{tr}_{\omega_{\text{can}}}\omega \leq C$ for some constant C. We do this by finding a uniform constant $C > 0$ such that $\text{tr}_{\omega(t)}\chi \leq C$. Indeed, by Lemma 3.1.1, $\omega(t) \to f^*\omega_{\text{can}}$, and ω is uniformly equivalent to χ. To this end, let

$$H = \log\text{tr}_{\omega(t)}(\chi) - B\varphi(t),$$

for some sufficiently large $B > 0$ to be determined later, and $t \in (0,1]$. Here, the maximum principle applied to (7) yields an upper bound on φ, hence on the right hand side of the degenerate Monge–Ampère equation. The L^{∞} estimate of Demailly–Pali [19] then yields a lower bound on φ. A standard computation then shows that

$$\Delta^{\omega(t)}\log(H) \geq (B - C_1)\text{tr}_{\omega(t)}(\chi) - C_2,$$

and thus H is bounded from above.

With this F being a quasi-plurisubharmonic function that is bounded from above, we may regularise F by a decreasing sequence of smooth quasi-plurisubharmonic functions $F_j \searrow F$. Let $\omega_{\text{can}} = \chi + \sqrt{-1}\partial\overline{\partial}\varphi_{\text{can}}$ solve the complex Monge–Ampère equation

$$\omega_{\text{can}}^{\kappa} = e^{-F}\omega^{\kappa}.$$

By [60], there exists $p > 1$ such that e^{-F} is $L^p(\omega^{\kappa})$ bounded. In particular, e^{-F_j} is $L^p(\omega^{\kappa})$ bounded for each j, also. We consider the regularised complex Monge–Ampère equations

$$\omega_j^{\kappa} = (\omega + \sqrt{-1}\partial\overline{\partial}\varphi_j)^{\kappa} = c_j e^{-F_j}\omega^{\kappa},$$

where

$$c_j := \frac{1}{\text{vol}_{\omega_j}(X_{\text{can}})} \int_{X_{\text{can}}} e^{-F_j}\omega^{\kappa}.$$

It is easy to see that $c_j \in L^p(\omega^{\kappa})$ for each j. Therefore, the the density of the regularised complex Monge–Ampère equation lies in $L^p(\omega^{\kappa})$ for some $p > 1$. Hence, as we mentioned above,

$$\text{dist}_{\omega_j}(p,q) \ \leq \ C_{\alpha}\text{dist}_{\omega}(p,q)^{\alpha},$$

for all $p, q \in X_{\text{can}}^{\circ}$ and some $\alpha \in (0,1)$. To complete the proof, we simply need to show that $\text{Ric}(\omega_j) \geq -C\omega_j$ for some uniform constant C. Already, we have

$$\text{Ric}(\omega_j) \ = \ \sqrt{-1}\partial\overline{\partial}F_j + \text{Ric}(\omega) \ \geq \ -C_1\omega.$$

To get $\omega_j \geq C\omega$ for some $C > 0$, a simple Schwarz lemma argument shows that $\text{tr}_{\omega_j}(\omega) \leq C$, or equivalently

$$\omega_j \ \geq \ (nC)^{-1}\omega.$$

Hence, $\text{Ric}(\omega_j) \geq -C_2\omega_j$ for some uniform $C_2 > 0$ and the diameter estimate [28] applies:

$$\text{diam}(X, \omega_j) \leq D$$

for some uniform D.

Extension of Song–Tian–Zhang's approach. One of the key steps in the Song–Tian–Zhang approach is the result of Li [51] informing us that Hölder regularity of the Kähler potential yields distance comparison theorems. Therefore, to extend this geometric approach to the more singular and degenerate setting, we focus on Question 2.9.13, i.e., establishing Hölder regularity for complex Monge–Ampère equations in big cohomology classes over the non-ample locus. We propose the following approach:

Let $\pi^*\omega_{\text{can}}$ be the pullback of the canonical metric ω_{can} on X_{can} to the resolution $\widetilde{X}_{\text{can}}$. This represents a big cohomology class since X_{can} is singular. Let $\widetilde{\omega}$ be a reference Kähler metric on $\widetilde{X}_{\text{can}}$ and choose $\varepsilon > 0$ such that

$$\omega_\varepsilon \ := \ \pi^*\omega_{\text{can}} + \varepsilon\widetilde{\omega}$$

is a Kähler metric on $\widetilde{X}_{\text{can}}$. The complex Monge–Ampère equation

$$(\omega_\varepsilon + dd^c\varphi_\varepsilon)^\kappa = f_\varepsilon \omega_\varepsilon^\kappa$$

admits a Hölder continuous solution for each $\varepsilon > 0$, where $0 \leq f_\varepsilon \in L^{p>1}(\omega_\varepsilon^\kappa)$ satisfies

$$\int_{\widetilde{X}_{\text{can}}} f_\varepsilon \omega_\varepsilon^\kappa = \int_{\widetilde{X}_{\text{can}}} f_\varepsilon (\omega_\varepsilon + dd^c\varphi_\varepsilon)^\kappa.$$

We want to establish a stability theorem for the Hölder regularity of the potential as the cohomology class degenerates.

Update: This Hölder regularity issue is, in fact, not necessary to determine the homeomorphism type of the Gromov–Hausdorff limit. Indeed, the ample locus coincides with X_{can}°, and we only need the Hölder regularity on X_{can}°.

Gross–Tosatti–Zhang Proof. We consider an alternative, more recent proof, of the homeomorphism type of the Gromov–Hausdorff limit $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ due to Gross–Tosatti–Zhang [35]. This approach is a little more analytic in comparison with the geometric proof due to Song–Tian–Zhang detailed in the previous section. Recall the set-up: We let (X^n, ω) be a projective manifold of complex dimension n, Kodaira dimension $0 < \kappa < n$, and with K_X semi-ample. Let $f : X \longrightarrow X_{\text{can}}$ denote the Iitaka map.

In general, X_{can} is a normal compact Kähler space and the fibration f has a non-empty discriminant locus $\Delta := X_{\text{can}} \backslash X_{\text{can}}^\circ$. By Theorem 2.4.3, the singularities of the fibration can be resolved, producing a surjective holomorphic map with connected fibres: $\pi : \widetilde{X}_{\text{can}} \longrightarrow X_{\text{can}}$, where X_{can} is a smooth Kähler manifold of (complex) dimension κ. Moreover, $E = \pi^{-1}(\Delta)$ is a divisor with simple normal crossings in $\widetilde{X}_{\text{can}}$. Write $E = \bigcup_{j=1}^N E_j$ for the decomposition of E into irreducible components, where each E_j is smooth and intersects E_k $(j \neq k)$ in simple normal crossings. For some $0 \leq k \leq N$, the divisors $E_1, ..., E_k$ are exceptional divisors for π, while $E_{k+1}, ..., E_N$ are proper transforms of divisors in the discriminant locus. We point out that $k = 0$ corresponds to the situation where Δ already has simple normal crossings and $\pi = \text{id}$, while $k = N$ corresponds to the case when the complex codimension of Δ is at least 2.

Construction 3.2.1. We associate an orbifold Kähler structure to the pair $(\widetilde{X}_{\text{can}}, E)$ as follows: Fix positive integers m_j, where $1 \leq j \leq N$. We produce a smooth Kähler metric on $\widetilde{X}_{\text{can}} \backslash E$ with orbifold singularities along E_j of orbifold order m_j, denoted ω_{orb}. Let U be a chart for $\widetilde{X}_{\text{can}}$ with local holomorphic coordinates $(w^1, ..., w^\kappa)$ adapted to the normal crossings structure, i.e., U is (biholomorphic to) a unit polydisk in $\mathbb{C}^\kappa$ such that

$$E \cap U = \{w^1 \cdots w^\ell = 0\},$$

for some $1 \leq \ell \leq \kappa$, and $E_{j_i} \cap U = \{w^i = 0\}$ for some $1 \leq j_i \leq N$, and $1 \leq i \leq \ell$. Let $q : \widetilde{U} \to U$ be the local uniformising chart given by

$$q(w^1, ..., w^\kappa) \quad = \quad ((w^1)^{m_{j_1}}, ..., (w^\ell)^{m_{j_\ell}}, w^{\ell+1}, ..., w^\kappa).$$

The metric $q^* \omega_{\text{orb}}$ on $\widetilde{U} \backslash \{w^1 \cdots w^\ell = 0\}$ extends smoothly to a Kähler metric on $\widetilde{U}$. Hence, on $U \backslash \{w^1 \cdots w^\ell = 0\}$, the metric ω_{orb} is uniformly equivalent to the model orbifold metric:

$$\sqrt{-1} \sum_{j=1}^{\ell} \frac{1}{|w_j|^{2(1-1/m_{i_j})}} dw^j \wedge d\overline{w}^j + \sqrt{-1} \sum_{j=\ell+1}^{N} dw^j \wedge d\overline{w}^j.$$

Let s_j be defining sections for the divisors E_j, and let h_j be smooth Hermitian metrics on $\mathcal{O}(E_j)$. For any Kähler metric $\widetilde{\omega}$ on $\widetilde{X}_{\text{can}}$, and any sufficiently small $\varepsilon > 0$, an orbifold metric on $\widetilde{X}_{\text{can}}$ with orbifold order m_j along E_j is given by

$$\omega_{\text{orb}} \quad = \quad \widetilde{\omega} + \varepsilon \sqrt{-1} \sum_{j=1}^{N} \partial \overline{\partial} |s_j|_{h_j}^{2/m_j}.$$

This metric is said to be *compatible* with the orbifold structure. More generally, we may consider a cone structure associated to $(\widetilde{X}_{\text{can}}, E)$ by replacing $\frac{1}{m_j}$ with a general cone angle $2\pi\alpha_j$. The same construction yields a cone metric ω_{cone} with prescribed cone angles along the irreducible components E_j uniformly equivalent to the model cone metric with the same cone angles.

Theorem 3.2.2. ([35, Theorem 6.1]). Let $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ denote the Gromov–Hausdorff limit of $(X, \omega(t))$. Suppose there exists $C > 0$, $\alpha_j \in (0, 1] \cap \mathbb{Q}$, for $1 \leq j \leq N$, and $A > 0$ such that on $\pi^{-1}(X_{\text{can}}^\circ)$ we have

$$\pi^* \omega_{\text{can}} \quad \leq \quad C \left(1 - \sum_{j=1}^{N} \log |s_j|_{h_j} \right)^A \omega_{\text{cone}},$$

where ω_{cone} is a conical metric with cone angle $2\pi\alpha_j$ along each E_j. Then

(i) $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ is a compact metric space with singular locus of real Hausdorff codimension at least 2.

(ii) $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ is isometric to the metric completion of $(X_{\text{can}}^\circ, \text{dist}_{\omega_{\text{can}}})$.

Moreover, if X_{can} is smooth with the codimension one part of the discriminant locus Δ having only simple normal crossing support, i.e., $\pi = \text{id}$, then $\mathcal{Z}$ is homeomorphic to X_{can}.

We refer the reader to [35] and [69] for details on statements (i) and (ii) in the above theorem. We will, however, detail here a proof of the last statement. The advantage that this approach has over the geometric approach of Song–Tian–Zhang is that it is clear exactly what estimate (which we mention below) has to be improved. A useful observation made in [35] is the following:

In the previous section, we saw that X_{can}° locally isometrically embeds into $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ as a dense open subset (by a result of [28]). Let $\Psi : (X_{\text{can}}^{\circ}, \text{dist}_{\omega_{\text{can}}}) \longrightarrow (\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ denote the isometric embedding. We also let $h : (\mathcal{Z}, \text{dist}_{\mathcal{Z}}) \longrightarrow \overline{(X_{\text{can}}^{\circ}, \text{dist}_{\omega_{\text{can}}})}$ be a map into the metric completion of X_{can}° (c.f., Theorem 3.1.4). By the Yau Schwarz lemma, we have $\omega(t) \geq C^{-1} f^* \omega_{\text{can}}$, giving a uniform Lipschitz constant bound for $f : (X, \omega(t)) \longrightarrow (X_{\text{can}}, \omega_{\text{can}})$ independent of t. Hence, in the (subsequential) limit, the map h is surjective and Lipschitz with $h \circ \Psi = \text{id}$.

Proposition 3.2.3. There is a continuous surjection $p : \widetilde{X}_{\text{can}} \longrightarrow \mathcal{Z}$ such that $\pi = h \circ p$.

Observe that in the case that X_{can} is smooth and the codimension one part of the discriminant locus has simple normal crossings, the above proposition is enough to determine the homeomorphism type of $\mathcal{Z}$. Indeed, if $\pi = \text{id}$, $h \circ p = \text{id}$ forces p to be injective. Hence, with $\widetilde{X}_{\text{can}} = X_{\text{can}}$ and p a continuous bijection between compact Hausdorff spaces, p is the desired homeomorphism between X_{can} and $\mathcal{Z}$.

In general, one needs to prove the following:

Proposition 3.2.4. The Gromov–Hausdorff limit $(\mathcal{Z}, \text{dist}_{\mathcal{Z}})$ is homeomorphic to $(X_{\text{can}}, \text{dist}_{\omega_{\text{can}}})$ if given any two points $y, \hat{y} \in \widetilde{X}_{\text{can}}$ with $\pi(y) = \pi(\hat{y})$, we have $p(y) = p(\hat{y})$.

In this case, p necessarily factors through π, i.e., there is a continuous[18] surjective map $q : X_{\text{can}} \longrightarrow \mathcal{Z}$ such that $p = q \circ \pi$. Then

$$\pi = h \circ p \quad \Longrightarrow \quad \pi = h \circ q \circ \pi \quad \Longrightarrow \quad h \circ q = \text{id},$$

since π is surjective, and so q is a homeomorphism.

Proof. The fibres of π are connected subvarieties of $\widetilde{X}_{\text{can}}$, hence their singular locus can be stratified by locally closed smooth subvarieties of decreasing dimension. We may therefore

[18]The map is continuous since π is a topological quotient map.

find a piecewise smooth path γ in $\widetilde{X}_{\text{can}}$ connecting y and $\hat{y}$ such that $\pi(\gamma) = \pi(y)$. In particular, γ is contained in the simple normal crossings divisor $E = \widetilde{X}_{\text{can}} \backslash \pi^{-1}(X_{\text{can}}^{\circ})$. Denote by $E = E^1 \supset E^2 \supset \cdots \supset E^n \supset E^{n+1} = \emptyset$ the stratification, where each E^j is smooth and of codimension j in $\widetilde{X}_{\text{can}}$. Let y_j be points which lying successively on γ such that $y_0 = y$, $y_m = \hat{y}$, and between any two such points, γ is smooth and lies in a unique open stratum $E^{k(i)} \backslash E^{k(i)+1}$. It is then enough to show that $p(y_i) = p(y_{i+1})$ for each i.

By relabelling if necessary, we may assume that we have two points lie in a single open stratum $E^k \backslash E^{k+1}$. Since E^n is a finite set of points, the $k = n$ case is trivial, since by further subdivisions of the curve γ we can always assume that y and $\hat{y}$ are close together. We have three cases to consider:

(i) Both y and $\hat{y}$ lie on $E^k \backslash E^{k+1}$.

Proof. □

(ii) The point y lies on $E^k \backslash E^{k+1}$, and the point $\hat{y}$ lies on E^{k+1}.

Proof. □

(iii) The points y and $\hat{y}$ both lie on E^{k+1}.

Proof. □

□

Prior to proving these propositions, let us discuss the important estimates that allow us to determine the homeomorphism type when X_{can} is smooth with simple normal crossing discriminant locus. The main estimates are due to Guenancia–Paun [37]. We introduce some further notation:

Fix a conical structure $\{E_j, 2\pi\alpha_j\}$ on $\widetilde{X}_{\text{can}}$ and compatible conical Kähler metric ω_{cone}. For a fixed Kähler metric $\widetilde{\omega}$ on $\widetilde{X}_{\text{can}}$, write

$$\omega_{\text{cone}} = \widetilde{\omega} + \sqrt{-1}\partial\overline{\partial}\eta,$$

where $\eta = C^{-1}\sum_j |s_j|_{h_j}^{2\alpha_j}$, as before, for C sufficiently large. Define

$$H := \prod_j |s_j|_{h_j}^{2\beta_j},$$

where the product is only over those components F_j of E such that E_j is π–exceptional, and has non-trivial vanishing order $\beta_j > 0$. Let $\Theta_{h_j} = -\sqrt{-1}\partial\overline{\partial}\log(h_j)$ denote the curvature form of h_j. On the complement of the π–exceptional components of E in $\widetilde{X}_{\text{can}}$, we have

$$\sqrt{-1}\partial\overline{\partial}\log(H) \quad = \quad -\sum_j \beta_j \Theta_{h_j}.$$

Set

$$\psi \quad = \quad \frac{\pi^*\omega^n \prod_j |s_j|_{h_j}^{2(1-\alpha_j)}}{H\widetilde{\omega}^n}.$$

Using Hodge-theoretic techniques, we are able to establish the following:

Theorem 3.2.5. There is a constant $C > 0$ and natural numbers $A \in \mathbb{N}$, $0 \leq k \leq N$, and rational numbers $\beta_j > 0$, $1 \leq j \leq k$, and $0 < \alpha_j \leq 1$, $k+1 \leq j \leq N$, such that on $\pi^{-1}(X_{\text{can}}^{\circ})$, we have

$$C^{-1}\omega_{\text{cone}}^n \ \leq \ \frac{(\pi^*\omega)^n}{\prod_{j=1}^k |s_j|_{h_j}^{2\beta_j}} \ \leq \ C\left(1 - \sum_{j=1}^N \log|s_j|_{h_j}\right)^A \omega_{\text{cone}}^n.$$

In particular, we observe that

$$C^{-1} \ \leq \ \psi \ \leq \ C\left(1 - \sum_j \log|s_j|_{h_j}\right)^d.$$

We thus have the complex Monge–Ampère equation

$$(\pi^*\omega_{\text{can}} + \sqrt{-1}\partial\overline{\partial}(\pi^*\varphi))^n \quad = \quad \psi H \frac{\widetilde{\omega}^n}{\prod_j |s_j|_{h_j}^{2(1-\alpha_j)}}.$$

The function $-\log(\psi)$ is locally bounded above near E, so a result of Grauert–Remmert tells us that we may extend $-\log(\psi)$ to a quasi-plurisubharmonic function on $\widetilde{X}_{\text{can}}$. Using Demailly's regularisation theorem Theorem 2.8.14, we have a decreasing sequence of smooth functions u_j such that

$$\sqrt{-1}\partial\overline{\partial}u_j \quad \geq \quad -\text{Ric}(\widetilde{\omega}) + \sum_j (1-\alpha_j)\Theta_{h_j} - \sum \beta_j \Theta_{h_j} - \frac{1}{j}\widetilde{\omega}$$

on all of $\widetilde{X}_{\text{can}}$. Similarly, since $\sqrt{-1}\partial\overline{\partial}\log(H) \geq -C\widetilde{\omega}$, we regularise $\log(H)$ by smooth functions v_k such that $\sqrt{-1}\partial\overline{\partial}v_k \geq -C\widetilde{\omega}$ and $v_k \leq C$ on $\widetilde{X}_{\text{can}}$.
Set

$$\omega_{j,k} = \pi^*\omega_{\text{can}} + \frac{1}{j}\widetilde{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_{j,k}$$

be a solution of

$$\omega_{j,k}^n \quad = \quad c_{j,k}e^{v_k - u_j}\frac{\widetilde{\omega}^n}{\prod_j |s_j|_{h_j}^{2(1-\alpha_j)}},$$

where $\omega_{j,k}$ is a cone metric with the same cone angles as ω_{cone}. By [41], such a metric exists and is smooth on $\widetilde{X}_{\text{can}} \backslash E$. The ration $\omega_{j,k}^n / \widetilde{\omega}^n$ is L^p–bounded, independent of j, for some $p > 1$, and therefore

$$\sup_{\widetilde{X}_{\text{can}}} |\varphi_{j,k}| \ \leq \ C.$$

A straightforward extension of the Laplacian estimate in [37] yields the following:

Proposition 3.2.6. For each j there is a constant C_j such that on $\widetilde{X}_{\text{can}} \backslash E$,

$$\text{tr}_{\omega_{\text{cone}}}(\omega_{j,k}) \ \leq \ C_j,$$

for all k.

Proof. Let $\omega_{j,k} = \pi^* \omega_{\text{can}} + \frac{1}{j}\widetilde{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_{j,k}$ be a solution on $\widetilde{X}_{\text{can}}$ of

$$\omega_{j,k}^n = c_{j,k} e^{v_k - u_j} \frac{\widetilde{\omega}^n}{\prod_i |s_i|_{h_i}^{2(1-\alpha_i)}}.$$

We approximate the above equation by a smooth PDE:

$$\omega_{j,k,\varepsilon}^n \ = \ c_{j,k,\varepsilon} e^{v_k - u_j} \frac{\widetilde{\omega}^n}{\prod_i (|s_i|_{h_i}^2 + \varepsilon^2)^{(1-\alpha_i)}},$$

where $\omega_{j,k,\varepsilon} = \pi^* \omega_{\text{can}} + \frac{1}{j}\widetilde{\omega} + \sqrt{-1}\partial\overline{\partial}\varphi_{j,k,\varepsilon}$ is now a smooth Kähler metric on $\widetilde{X}_{\text{can}}$. The potentials $\varphi_{j,k,\varepsilon}$ converge uniformly to $\varepsilon_{j,k}$ as $\varepsilon \searrow 0$. We claim that

$$\text{tr}_{\omega_\varepsilon} \omega_{j,k,\varepsilon} \ \leq \ C_j, \tag{8}$$

where C_j is independent of k and ε, and

$$\omega_\varepsilon = \pi^* \omega_{\text{can}} + \frac{1}{j}\widetilde{\omega} + \sqrt{-1}\partial\overline{\partial}\chi_{j,\varepsilon}$$

is a family of Kähler metrics approximating a conical metric. Standard arguments then show that $\omega_{j,k,\varepsilon} \to \omega_{j,k}$ locally smoothly away from E as $\varepsilon \searrow 0$. We now prove (8): We have the following classical estimate:

$$\Delta^{\omega_\varphi}(\log \text{tr}_\omega \omega_\varphi) \ \geq \ \frac{1}{\text{tr}_\omega(\omega_\varphi)} \left[-\text{tr}_\omega \text{Ric}(\omega_\varphi) + \text{tr}_{\omega_\varphi} \left(\text{tr}_{g_\varphi^{-1}}(\sqrt{-1}\widetilde{\Theta}_\omega) \right) \right].$$

Note that in the last part of the above estimate we take the trace $\text{tr}_{\omega_\varphi}$ of the contravariant part of the curvature tensor, and then take the trace $\text{tr}_{g_\varphi^{-1}}$ of the covariant part. Moreover, Θ_ω is the Chern curvature of (T_X, ω), and $\sqrt{-1}\Theta_\omega$ is a real $(1,1)$–form with values in the bundle of Hermitian endomorphisms of T_X. Its contraction with ω, written $\sqrt{-1}\widetilde{\Theta}_\omega$ is naturally a $(1,1)$–form with values in the bundle of Hermitian endomorphisms of T_X^*. The Hermitian metric on T_X^* induced by g_φ is denoted g_φ^{-1}.

Choose holomorphic coordinates near $p \in X$ such that ω_φ is diagonal at p, writing:

$$\omega_\varphi = \sqrt{-1} \sum \lambda_j dw^j \wedge d\overline{w}^j.$$

In these coordinates, we also observe that

$$-\text{tr}_\omega \text{Ric}(\omega_\varphi) = \Delta^\omega f - \sum_{i,k} R_{i\bar{i}k\bar{k}},$$

and

$$\mathrm{tr}_{\omega_\varphi}\left(\mathrm{tr}_{g_\varphi^{-1}}(\sqrt{-1}\widetilde{\Theta}_\omega)\right) = \sum_{i,k}\frac{\lambda_i}{\lambda_k}R_{i\bar{i}k\bar{k}}.$$

It then follows that

$$\Delta^{\omega_\varphi}(\log \mathrm{tr}_\omega \omega_\varphi) \quad \geq \quad \frac{1}{\sum_p \lambda_p}\left(\sum_{i\leq k}\left(\frac{\lambda_i}{\lambda_k}+\frac{\lambda_k}{\lambda_i}-2\right)R_{i\bar{i}k\bar{k}} + \Delta^\omega f\right).$$

□

The above proposition allows us to apply the maximum principle to quantities involving $\mathrm{tr}_{\omega_{\mathrm{cone}}}(\omega_j)$, provided we add a term which goes to $-\infty$ along E (arbitrarily slowly). Using this, we obtain

Theorem 3.2.7. On $\widetilde{X}_{\mathrm{can}}\backslash E$, we have

$$\mathrm{tr}_{\omega_{\mathrm{cone}}}\pi^*\omega \;\leq\; \frac{C\psi}{|s_F|^{2A}},$$

where $|s_F|^2 := \prod_i |s_{F_i}|^2_{h_i}$.

Proof. □

The presence of the divisorial terms do not allow us to verify the conjecture in the general case. If the resolution is the identity, i.e., the base of the fibration is smooth and Δ has simple normal crossings, the divisorial terms in the above estimate are not present.

Theorem 3.2.8. Suppose there is a constant $C > 0$, rational numbers $0 < \alpha_i \leq 1$, $1 \leq i \leq \mu$, and $A > 0$ such that on $\pi^{-1}(X^\circ_{\mathrm{can}})$, we have

$$\pi^*\omega_{\mathrm{can}} \quad\leq\quad C\left(1-\sum_{j=1}^N \log|s_j|_{h_j}\right)^A \omega_{\mathrm{cone}}, \tag{9}$$

where ω_{cone} is a conical metric with cone angle $2\pi\alpha_j$ along each E_j. If X_{can} is smooth and the codimension one part of the discriminant locus has simple normal crossing support, then X_{can} is homeomorphic to $\mathcal{Z}$.

Extension of Gross–Tosatti–Zhang's approach. The extension of this analytic approach is to improve the maximum principle argument in [35] to remove the $|s_F|^{2A}$ term in Theorem 3.2.7. In more detail, we recall that in the proof of Theorem 3.2.7, the quantity used in the maximum principle computation is

$$Q \quad:=\quad \log \mathrm{tr}_{\omega_{\mathrm{cone}}}\omega_j + n\Psi + u_j - A^2\varphi_j + Ab\eta + \varepsilon\log|s_E|^2 + A\log|s_F|^2,$$

where A is large, $b > 0$ is small and $0 < \varepsilon \leq 1/j$. The terms $n\Psi + u_j - A^2\varphi_j + Ab\eta$ are bounded on $\widetilde{X}_{\mathrm{can}}$ with bounds independent of j (except for u_j). The $\varepsilon\log|s_E|^2$ is necessary:

it ensures that the maximum of Q occurs on $\widetilde{X}_{\text{can}}\backslash E$. Further, the bad divisorial term in the final estimate arises from $A\log|s_F|^2$, but one cannot simply omit this term and proceed with the argument in [35]. Indeed, setting $\Theta_F := \sum_i \Theta_{F_i}$, we observe that (see [35, p. 28]):

$$\Delta_{\omega_j} Q \ \ \geq \ \ -C\text{tr}_{\omega_j}\omega_{\text{cone}} + A\text{tr}_{\omega_j}\left(A_0\pi^*\omega_{\text{can}} - \Theta_F + b\sqrt{-1}\partial\overline{\partial}\eta\right) - A^2 n,$$

for $A_0 \leq A$. Then A_0 is chosen such that $A_0\pi^*\omega_{\text{can}} - \Theta_F$ is a Kähler metric.

Zhou Zhang has suggested to attempt to carry the computation through with $A > 0$ small. This may be enough to determine the homeomorphism type. To this end, I have been reviewing some results of this type that are present in [39]. In particular, they consider smooth minimal models of general type. The main arguments we are concerned with are contained in the proof of the following: (see [39, p. 5–6] for notation)

Lemma. Fix $j \in \{1, ..., M\}$ and $\eta > 0$. There exist constants C and $\delta > 0$ such that on $(\widetilde{B}\backslash\{0\}) \cap (X - U_\eta^j)$, depending only on the fixed initial data and $\eta > 0$, such that

(i) $\omega \leq C|\sigma_j|_{h_j}^{-1}\omega_{\text{Eucl}}$.

(ii) $\omega \leq C|\sigma_j|_{h_j}^{-(1-\delta)}(\omega_0 + \omega_{\text{Eucl}})$.

In particular, (ii) improves the divisorial term present in (i) in the way that we want. Hence, we may be able to adapt the arguments of [39] to strengthen the results of [35] in this way.

Let us also mention that the proof of (9) is contained in [69, §4] and uses techniques from *special Kähler geometry* and *algebraic completely integrable systems*. The proof appears complicated, hence, after understanding the proof, a goal will be to simplify the proof, attempting to only use techniques of Kähler geometry.

4. Approach to extend the work of STZ and GTZ

The Hölder regularity problem is, in fact, not required to determine the homeomorphism type of the Gromov–Hausdorff limit. Indeed, by [18], it is known that the potential is Hölder continuous on the ample locus. Since the ample locus coincides with X_{can}°, and the distance estimates are only required on X_{can}°, this regularity question is not an issue. What is necessary, however, is a strengthening of the estimate in [35]:

$$\pi^*\omega \ \leq \ \frac{C}{|s_F|^{2A}}\left(1 - \sum_{j=1}^{\mu}\log|s_j|_{h_j}\right)^A \omega_{\text{cone}}. \tag{10}$$

Indeed, we would like to improve this to be an estimate of the form

$$\pi^*\omega \ \leq \ C\left(1 - \sum_{j=1}^{\mu}\log|s_j|_{h_j}\right)^A \omega_{\text{cone}}.$$

Given an estimate of this form, one can then apply the arguments of [69] or for a more geometric approach, follow the arguments in [39] to obtain diameter estimates.

An approach to strengthen (10) would be to introduce another parameter $0 < \delta < 1$ and attempt to estimate $\pi^*\omega$ in the following way:

$$\pi^*\omega \ \leq \ \frac{C}{|s_F|^{2A(1-\delta)}} \left(1 - \sum_{j=1}^{\mu} \log |s_j|_{h_j} \right)^A (\omega_{\text{cone}} + \widetilde{\omega}),$$

where δ is permitted to be arbitrarily close to 1 and $\widetilde{\omega}$ is a reference form on $\widetilde{X}_{\text{can}}$. This is motivated by the estimates in [39, Lemma 2.2].

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