Recent developments concerning the Schwarz Lemma with applications to the Wu–Yau Theorem

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Classical Bochner Technique

Let (M, g) be a compact Riemannian manifold[1]. Let $\alpha \in \Omega^1_M$.

$$\Delta_d \alpha = (dd^* + d^* d)\alpha = \nabla^* \nabla \alpha + \mathrm{Ric}_g(\alpha^\sharp, \cdot).$$

If α is *harmonic*, i.e., $\Delta_d \alpha = 0$, then

$$\Delta_d |\alpha|^2 \quad = \quad |\nabla \alpha|^2 + \mathrm{Ric}_g(\alpha^\sharp, \alpha^\sharp).$$

Theorem. (Bochner). If $\mathrm{Ric}_g > 0$, then $b_1(M) = 0$.

[1] connected and orientable.

Complex-Analytic Bochner Formula

Let (X, ω) be a complex manifold. Let $\sigma \in H^0(\mathcal{E})$ be a *holomorphic section* of a *holomorphic vector bundle* $\mathcal{E} \to X$.

We want to compute $\Delta_\omega |\sigma|^2 = \mathbf{tr}_\omega(\sqrt{-1}\partial\bar{\partial}|\sigma|^2)$.

Let $\mathcal{E} \to X$ be a complex vector bundle.

Reminder. A *first-order* $\mathbb{C}$*–linear differential operator* $\bar{\partial}^{\mathcal{E}} : H^0(\mathcal{E}) \to \Omega_X^{0,1} \otimes \mathcal{E}$ is said to be *CR operator* if

$$\bar{\partial}^{\mathcal{E}}(f\sigma) = \bar{\partial}f \otimes \sigma + f\,\bar{\partial}^{\mathcal{E}}\sigma.$$

If, in addition,

$$\bar{\partial}^{\mathcal{E}} \circ \bar{\partial}^{\mathcal{E}} = 0,$$

then we call $\bar{\partial}^{\mathcal{E}}$ a *holomorphic structure*.

Theorem. (Koszul–Malgrange). Let $\mathcal{E}$ be a complex vector bundle. Then $\mathcal{E}$ is a *holomorphic vector bundle* if and only if *$\mathcal{E}$ admits a holomorphic structure $\bar{\partial}^{\mathcal{E}}$*.

Connections on Holomorphic Vector Bundles

If $\bar\partial^{\mathcal{E}}$ is a *holomorphic structure* on $\mathcal{E}$, we can complete it to a *Hermitian connection* ∇ in the sense that there is a Hermitian connection ∇ such that

$$\nabla^{0,1} = \bar\partial^{\mathcal{E}}.$$

If $\mathcal{E} = T^{1,0}X$, this connection is called the *Chern connection*.

The *Bochner formula* for this connection reads:

$$\Delta_\omega |\sigma|^2 = |\nabla\sigma|^2 - \sqrt{-1}\langle \Theta^{\mathcal{E}}\sigma, \sigma\rangle,$$

where $\Theta^{\mathcal{E}}$ is the curvature of the Hermitian metric on $\mathcal{E}$.

The Schwarz Lemma

Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a *holomorphic map between complex manifolds.*

We can identify ∂f with a section $\partial f \in H^0(\Omega_X^{1,0} \otimes f^* T^{1,0} Y)$.

Inserting this into the *Bochner formula* yields

$$\Delta_\omega |\partial f|^2 = |\nabla \partial f|^2 - \sqrt{-1} \langle \Theta^{\Omega_X^{1,0} \otimes f^* T^{1,0} Y} \partial f, \partial f \rangle.$$

The curvature *splits additively under tensor products*:

$$\Theta^{\Omega_X^{1,0}\otimes f^*T^{1,0}Y} = \Theta^{\Omega_X^{1,0}} \otimes \mathrm{id} + \mathrm{id} \otimes \Theta^{f^*T^{1,0}Y},$$

inverts additively under dualization:

$$\Theta^{\Omega_X^{1,0}\otimes f^*T^{1,0}Y} = -\Theta^{T^{1,0}X} \otimes \mathrm{id} + \mathrm{id} \otimes \Theta^{f^*T^{1,0}Y},$$

and *commutes with pullback*:

$$\Theta^{\Omega_X^{1,0}\otimes f^*T^{1,0}Y} = -\Theta^{T^{1,0}X} \otimes \mathrm{id} + \mathrm{id} \otimes f^*\Theta^{T^{1,0}Y}$$

Schwarz Lemma

The *Bochner formula* therefore yields

$$\begin{aligned}\Delta_{\omega_g} |\partial f|^2 &= |\nabla \partial f|^2 + \mathrm{Ric}_{\omega_g} \otimes \omega_g^\sharp \otimes \omega_g^\sharp \otimes \omega_h \otimes \partial f \otimes \overline{\partial f} \\ &\quad - \mathrm{Rm}_{\omega_h} \otimes \omega_g^\sharp \otimes \partial f \otimes \overline{\partial f} \otimes \omega_g^\sharp \otimes \partial f \otimes \overline{\partial f}.\end{aligned}$$

In local coordinates, we have

$$\Delta_{\omega_g} |\partial f|^2 = |\nabla \partial f|^2 + \underbrace{g^{i\bar{j}} R^g_{i\bar{j}k\bar{\ell}}}_{\text{Ricci}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^\alpha \overline{f_q^\beta} - R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{i\bar{j}} f_i^\alpha \overline{f_j^\beta} g^{p\bar{q}} f_p^\gamma \overline{f_q^\delta}.$$

Here $f_i^\alpha := \frac{\partial f^\alpha}{\partial z_i}$

Royden's Polarization Argument

Royden showed that if the *target metric is Kähler*[2], the target curvature term can be controlled by the *holomorphic sectional curvature*.

> **Recall:** Let ω is a Kähler metric with underlying complex structure J. The restriction of the *sectional curvature* to the J–invariant 2–planes (i.e., 2–planes of the form $\{u, Ju\}$) defines the *holomorphic sectional curvature*.

In terms of the curvature tensor,

$$\mathrm{HSC}_{\omega}(v) := R(v, \overline{v}, v, \overline{v}).$$

[2]Recall: A Hermitian metric is said to be Kähler if the torsion of the Chern connection vanishes.

The Holomorphic Sectional Curvature

The *holomorphic sectional curvature* is very natural to the study of complex geometry:

(†) (Ahlfors). $\mathrm{HSC}_\omega < 0 \implies X$ is *Brody hyperbolic*[3]

Every entire curve $\mathbb{C} \to X$ *is constant*.

(†) (Yang). $\mathrm{HSC}_\omega > 0 \implies X$ is *rationally connected*:

Any two points lie in the image of a rational curve $\mathbb{P}^1 \to X$.

[3] If X is compact, this is equivalent to Kobayashi hyperbolicity.

Royden's Polarization Argument

The argument hinges upon the following polarization argument – called *Royden's trick*:

Proposition. Let $\xi_1, ..., \xi_\nu$ be ν orthogonal tangent vectors. If $S(\xi, \overline{\eta}, \zeta, \overline{\omega})$ is a *symmetric bi-Hermitian form* in the sense that

(i) $S(\xi, \overline{\eta}, \zeta, \overline{\omega}) = S(\zeta, \overline{\eta}, \xi, \overline{\omega})$,

(ii) $S(\eta, \overline{\xi}, \omega, \overline{\zeta}) = \overline{S}(\xi, \overline{\eta}, \zeta, \overline{\omega})$,

such that for all ξ,

$$S(\xi, \overline{\xi}, \xi, \overline{\xi}) \ \leq \ -\kappa_0 \|\xi\|^4,$$

for $\kappa_0 \geq 0$, then

$$\sum_{\alpha,\beta} S(\xi_\alpha, \overline{\xi}_\alpha, \xi_\beta, \overline{\xi}_\beta) \ \leq \ -\frac{\nu+1}{2\nu}\kappa_0 \left(\sum_\alpha \|\xi_\alpha\|^2 \right)^2.$$

Royden's Schwarz Lemma

Theorem. (Royden 1980). Let $f : (X, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map between *Kähler manifolds*. Suppose $\mathrm{Ric}_{\omega_g} \geq -C_1 \omega_g$ and $\mathrm{HSC}_{\omega_h} \leq -\kappa_0$ for some constants $C_1, \kappa_0 > 0$. Then

$$\Delta_{\omega_g} \mathrm{tr}_{\omega_g}(f^* \omega_h) \; = \; \Delta_{\omega_g} |\partial f|^2 \; \geq \; -2C_1 + \frac{r+1}{r} \kappa_0 |\partial f|^2,$$

where $r = \mathrm{rank}(\partial f)$.

In particular, if X is compact, then

$$\mathrm{tr}_{\omega_g}(f^* \omega_h) \; = \; |\partial f|^2 \leq \frac{2C_1 r}{(r+1)\kappa_0}.$$

The Wu–Yau Theorem

The following result is due to Wong (surfaces), Heier–Lu–Wong (projective threefolds), Wu–Yau (projective), Tosatti–Yang (Kähler):[4]

Theorem. Let (X, ω) be a compact Kähler manifold with $\text{HSC}_\omega < 0$. Then the canonical bundle K_X is ample.

In particular, we see that

$$\text{HSC}_\omega < 0 \implies \exists\, \omega_\varphi = \omega + \sqrt{-1}\partial\bar{\partial}\varphi \text{ such that } \text{Ric}_{\omega_\varphi} < 0.$$

[4]Recall: A line bundle $\mathcal{L}$ is ample if the sections of $\mathcal{L}^{\otimes k}$ (k large) furnish a holomorphic embedding $\Phi : X \longrightarrow \mathbb{P}^{N_k}$.
In particular, K_X^{-1} is ample if and only if $\text{Ric}_\omega > 0$.

The Kobayashi Conjecture

The Wu–Yau theorem is an important step towards the more general *Kobayashi conjecture:*

> **Conjecture.** Let X be a *compact Kobayashi hyperbolic manifold*. Then *K_X is ample*.

Remarks:

- (Demailly 1997) Kobayashi hyperbolicity[5] is *strictly weaker* than the *existence of a metric with negative holomorphic sectional curvature*.

- Kobayashi (1970) conjectured that a *compact Kähler manifold which is Kobayashi hyperbolic has ample canonical bundle*.

[5] That is, every entire holomorphic curve $\mathbb{C} \to X$ is constant.

Curvature Heirarchy

The *holomorphic sectional curvature* and *Ricci curvature* occupy similar strata of the curvature heirarchy[6]:

$$\begin{tikzcd}
 & & \mathrm{HBC}_\omega \arrow[dl] \arrow[dr] & \text{(holomorphic bisectional curvature)} & \\
\text{(Ricci curvature)} & \mathrm{Ric}_\omega \arrow[dr] \arrow[rr, bend right, dotted] & & \mathrm{HSC}_\omega \arrow[dl] \arrow[ll, bend right, dotted] & \text{(holomorphic sectional curvature)} \\
 & & \mathrm{Scal}_\omega & \text{(Scalar curvature)} &
\end{tikzcd}$$

[6] Arrows indicate dominance: i.e., $A \to B$ means that $A > 0 \implies B > 0$, and similarly for < 0, ≤ 0, ≥ 0, etc.
Recall: $\mathrm{HBC}_\omega(u, v) = R(u, \overline{u}, v, \overline{v})$; $\quad \mathrm{HSC}_\omega(u) = R(u, \overline{u}, u, \overline{u})$;

Examples

Example 1. (Hitchin). Let $\mathcal{F}_n := \mathbb{P}(1 \oplus H^n)$ denote the *nth Hirzebruch surface* (a $\mathbb{P}^1$–bundle over $\mathbb{P}^1$).

Hitchin showed that $\mathcal{F}_n$ admits a Kähler metric ω with $\text{HSC}_\omega > 0$. For $n > 1$, however, $c_1(\mathcal{F}_n) \not> 0$, and thus, *does not support a Kähler metric of positive Ricci curvature*.

Example 2. Let

$$X_d := \{z_0^d + \cdots + z_n^d = 0\} \subseteq \mathbb{P}^n$$

denote the degree d *Fermat hypersurface*.

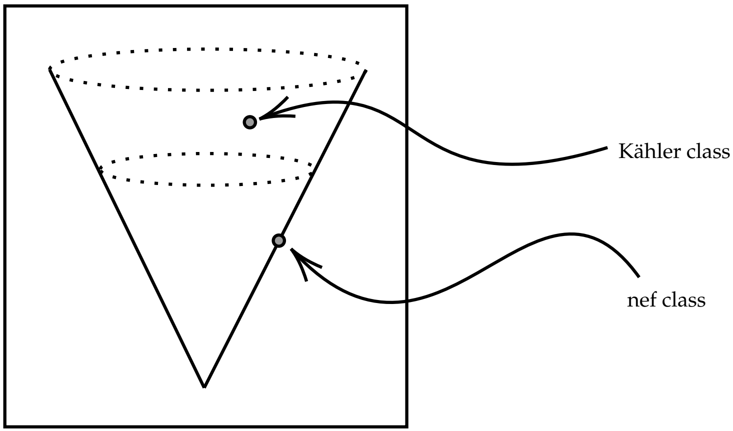
For $d \geq n+2$, adjunction implies that K_{X_d} is ample, and thus X_d *admits a Kähler(–Einstein) metric of negative Ricci curvature*. But X_d *admits complex lines*, and thus, cannot support a metric with $\text{HSC}_\omega < 0$.

Sketch of Wu–Yau Theorem Proof

Reminder: A *cohomology class in $H^2(X, \mathbb{R})$* is said to be a *Kähler class* if it is *represented by a Kähler form*.

The set of Kähler classes form an open convex cone – *the Kähler cone* – in the finite-dimensional vector space $H^{1,1}(X, \mathbb{R})$.

A cohomology class on the boundary of the Kähler cone is a nef class.



$H^{1,1}(X, \mathbb{R})$

We first show that K_X is nef[7]. Proceed by contradiction, and assume this is not the case. Then there is some $\varepsilon_0 > 0$ such that $\varepsilon_0[\omega_g] - c_1(K_X^{-1})$ *is nef but not Kähler*. Then for any $\varepsilon > 0$, the class $(\varepsilon + \varepsilon_0)[\omega_g] - c_1(K_X^{-1})$ is Kähler. Hence, we can find a smooth function φ_ε such that

$$(\varepsilon + \varepsilon_0)\omega_g - \mathrm{Ric}_{\omega_g} + \sqrt{-1}\partial\bar{\partial}\varphi_\varepsilon > 0.$$

By the *Aubin–Yau theorem*, we can find a smooth function ψ_ε such that

$$\omega_\varepsilon := (\varepsilon + \varepsilon_0)\omega_g - \mathrm{Ric}_{\omega_g} + \sqrt{-1}\partial\bar{\partial}(\psi_\varepsilon + \varphi_\varepsilon) > 0,$$

and $\omega_\varepsilon^n = e^{u_\varepsilon}\omega_g^n$, where $u_\varepsilon := \varphi_\varepsilon + \psi_\varepsilon$. This implies that

$$\mathrm{Ric}_{\omega_\varepsilon} = \mathrm{Ric}_{\omega_g} - \sqrt{-1}\partial\bar{\partial}u_\varepsilon = -\omega_\varepsilon + (\varepsilon + \varepsilon_0)\omega_g.$$

Hence, *Royden's Schwarz lemma* implies that

$$\sup_X \mathrm{tr}_{\omega_\varepsilon}(\omega_g) \ \leq \ \frac{n}{\varepsilon + \varepsilon_0},$$

which is *uniformly bounded independent of ε* (as $\varepsilon \searrow 0$).

[7] Recall: A line bundle $\mathcal{L}$ is nef if for any $\varepsilon > 0$, there is a Hermitian metric h_ε such that $\Theta_{h_\varepsilon} \geq -\varepsilon\omega$ (for some positive $(1,1)$–form ω).

Moreover, at any point $x \in X$ where u_ε attains its maximum, we have $((\varepsilon + \varepsilon_0)\omega_g - \mathrm{Ric}_{\omega_g})(x) > 0$ and

$$e^{\sup_X u_\varepsilon} = e^{u_\varepsilon(x)} \le \frac{((\varepsilon + \varepsilon_0)\omega_g - \mathrm{Ric}_{\omega_g})^n}{\omega_g^n} \le C,$$

independent of ε small. Hence,

$$\mathrm{tr}_{\omega_g}(\omega_\varepsilon) \le \frac{1}{(n-1)!}(\mathrm{tr}_{\omega_\varepsilon}\omega_g)^{n-1}\frac{\omega_\varepsilon^n}{\omega_g^n} \le C.$$

Hence, we get $C^{-1}\omega_g \le \omega_\varepsilon \le C\omega_g$, and *standard bootstrapping* yields

$$\|\omega_\varepsilon\|_{\mathcal{C}^k(X,\omega_g)} \le C_k,$$

for all $k \in \mathbb{N}_0$, with C_k independent of ε.

From the higher-order estimates, the *Arzelà–Ascoli theorem* allows us to extract a subsequence $\varepsilon_i \to 0$ such that *ω_{ε_i} converges smoothly to a Kähler metric ω_0* which satisfies

$$[\omega_0] = \varepsilon_0[\omega_g] - c_1(K_X^{-1}).$$

This *contradicts the assumption that this class was not Kähler*, however.

Proof of Ampleness

We now know that *K_X is nef*. Hence, for all $\varepsilon > 0$ we can find a smooth function u_ε such that

$$\omega_\varepsilon \ := \ \varepsilon\omega_g - \mathrm{Ric}_{\omega_g} + \sqrt{-1}\partial\bar{\partial}u_\varepsilon > 0, \qquad \omega_\varepsilon^n = e^{u_\varepsilon}\omega_g^n.$$

The same argument as above shows that we can extract a convergent subsequence, converging to

$$\omega_u \ := \ -\mathrm{Ric}_{\omega_g} + \sqrt{-1}\partial\bar{\partial}u > 0, \qquad \omega_u^n = e^u\omega_g^n.$$

This implies that $\mathrm{Ric}_{\omega_u} = \mathrm{Ric}_{\omega_g} - \sqrt{-1}\partial\bar{\partial}u = -\omega_u$. Hence, *$\omega_u$ is Kähler–Einstein* with negative scalar curvature, and *K_X is ample*.

The Schwarz Lemma Revisited

To extend *Royden's* argument beyond the Kähler setting, we need to understand

$$\Delta_{\omega} |\partial f|^2 = |\nabla \partial f|^2 + g^{i\bar{j}} R^{g}_{i\bar{j}k\bar{\ell}} g^{k\bar{q}} g^{p\bar{\ell}} h_{\alpha\bar{\beta}} f_p^{\alpha} \overline{f_q^{\beta}} - R^{h}_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{i\bar{j}} f_i^{\alpha} \overline{f_j^{\beta}} g^{p\bar{q}} f_p^{\gamma} \overline{f_q^{\delta}}.$$

Remarks:

- The *Monge–Ampère equation* controls the *first Chern–Ricci*

$$^{c}\mathrm{Ric}^{(1)}_{\omega} = g^{k\bar{\ell}} R_{i\bar{j}k\bar{\ell}}.$$

- But the *second Chern–Ricci curvature* appears in the Schwarz lemma

$$^{c}\mathrm{Ric}^{(2)}_{\omega} = g^{i\bar{j}} R_{i\bar{j}k\bar{\ell}}.$$

- *Royden's polarization argument* requires the curvature of the target metric to have the symmetry

$$R_{i\bar{j}k\bar{\ell}} = R_{k\bar{j}i\bar{\ell}}.$$

  In particular, a non-Kähler metric will *not support this symmetry* in general.

Target curvature term

To understand the target curvature term

$$R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{i\bar{j}} f_i^{\alpha} \overline{f_j^{\beta}} g^{p\bar{q}} f_p^{\gamma} \overline{f_q^{\delta}},$$

choose a frame such that at a point $p \in X$, $g_{i\bar{j}}(p) = \delta_{ij}$ and $f_i^{\alpha} = \lambda_i \delta_i^{\alpha}$, where $\lambda_i \in \mathbb{R}$. Then

$$R^h_{\alpha\bar{\beta}\gamma\bar{\delta}} g^{i\bar{j}} f_i^{\alpha} \overline{f_j^{\beta}} g^{p\bar{q}} f_p^{\gamma} \overline{f_q^{\delta}} = \sum_{\alpha,\gamma} R^h_{\alpha\bar{\alpha}\gamma\bar{\gamma}} \lambda_{\alpha}^2 \lambda_{\gamma}^2.$$

This motivated Yang–Zheng to introduce the following:

Definition. Let (X, ω) be a Hermitian manifold. The *real bisectional curvature* RBC_{ω} is the function

$$\mathrm{RBC}_{\omega}(v) := \frac{1}{|v|^2} \sum_{\alpha,\gamma} R_{\alpha\bar{\alpha}\gamma\bar{\gamma}} v_{\alpha} v_{\gamma},$$

where $v = (v_1, ..., v_n) \in \mathbb{R}^n \backslash \{0\}$.

The Real Bisectional Curvature

(i) If the metric is *Kähler*:
the *real bisectional curvature* is *comparable* to the *holomorphic sectional curvature*.

(ii) For a general Hermitian metric:
the *real bisectional curvature* *strictly dominates* the *holomorphic sectional curvature* and *scalar curvatures*.

(iii) The *real bisectional curvature* is *not strong enough*, however, to control the *Ricci curvatures*.

Hermitian Schwarz Lemma

Yang–Zheng (2017) proved the following:

Theorem. Let $f : (X, \omega_g) \to (Y, \omega_h)$ be a holomorphic map between *Hermitian manifolds*. Suppose $\mathrm{Ric}^{(2)}_{\omega_g} \geq -C_1\omega_g + C_2 f^*\omega_h$ for constants C_1, C_2, where $C_2 \geq 0$. If $\mathrm{RBC}_{\omega_h} \leq -\kappa_0 \leq 0$, then

$$\Delta_{\omega_g} \log |\partial f|^2 \ \geq \ -C_1 + \frac{1}{r}(C_2 + \kappa_0)|\partial f|^2.$$

Hence, if X is compact,

$$|\partial f|^2 \leq \frac{C_1 r}{(C_2 + \kappa_0)}.$$

The previous argument for the Wu–Yau theorem can be applied to show that

Corollary. (Yang–Zheng). Let X be a *compact Kähler manifold* with a *Hermitian metric* of *negative real bisectional curvature*. Then *K_X is ample*.

Gauduchon connections

Definition. The *t–Gauduchon connection* is defined

$$ {}^{t}\nabla = t\,{}^{c}\nabla + (1-t)\,{}^{l}\nabla, $$

where ${}^{c}\nabla$ is the *Chern connection* and ${}^{l}\nabla$ is the *Lichnerowicz connection* (*orthogonal projection to $T^{1,0}X$ of the complexified Levi-Civita connection*).

Constructing a Schwarz Lemma

Recall that we proved the *Schwarz lemma* by applying the *Bochner formula*

$$\Delta_\omega |\sigma|^2 = |\nabla \sigma|^2 - \sqrt{-1} \langle \Theta^{\mathcal{E}} \sigma, \sigma \rangle$$

to the section $\partial f \in H^0(\Omega_X^{1,0} \otimes f^* T^{1,0} Y)$.

Hence, the *first obstruction* to developing more *general Schwarz lemmas* is *generalizing the Bochner formula beyond the Chern connection*.

Novel Bochner Formula

Theorem. (B.–Stanfield). Let $(\mathcal{E}, h) \longrightarrow X$ be a holomorphic vector bundle over a *Hermitian manifold* (X, ω). Let ∇ be a *Hermitian connection* on $\mathcal{E}$. Then for any holomorphic section $\sigma \in H^0(\mathcal{E})$, we have

$$\Delta_\omega |\sigma|_h^2 = |\nabla^{1,0}\sigma|^2 + |\nabla^{0,1}\sigma|^2 + 2\mathrm{Re}\{\nabla^{1,0}\nabla^{0,1}\sigma, \sigma\} - \{\sigma, \Theta^{\mathcal{E}}\sigma\}.$$

Incorporating Torsion

The *end application* of the *Bochner formula* will be to sections of bundles *derived from tangent bundles* (e.g., $\Omega_X^{1,0} \otimes f^*T^{1,0}Y$).

We want to *extend this general Bochner formula as far as possible*, however, *before descending to calculations for specific choices of bundles*.

We therefore introduce the following:

Definition. Let $(\mathcal{E}, h) \longrightarrow X$ be a Hermitian vector bundle endowed with a Hermitian connection ∇. We define the *CR–torsion* of ∇ to be the $\mathrm{End}(\mathcal{E})$–valued $(0,1)$–form $A \in \Omega_X^{0,1}(\mathcal{E})$ defined by

$$A := \nabla^{0,1} - \bar{\partial}^{\mathcal{E}}.$$

Bochner formula with torsion

Theorem. (B.–Stanfield). Let $(\mathcal{E}, h) \longrightarrow X$ be a Hermitian vector bundle which is a *tensor bundle associated to* $T^{1,0}X$. If ∇ *is a Hermitian connection on* $\mathcal{E}$ *extended from a Hermitian connection on* $T^{1,0}X$, then

$$\begin{aligned}\Delta_\omega |\sigma|^2 &= |\nabla^{1,0}\sigma|^2 + |\nabla^{0,1}\sigma|^2 - \{\sigma, \Theta_{\bar{i}i}\sigma\} \\ &\quad +2\,\mathrm{Re}\left(\{A_{\bar{i},i}\sigma, \sigma\} + \{A_{\bar{i}}(\sigma_{,i}), \sigma\} + \{A_{\overline{A_{\bar{i}}(e_i)}}\sigma, \sigma\}\right)\end{aligned}$$

where $\{e_i\}$ is a unitary frame for $T^{1,0}X$.

The Schwarz Lemma with torsion

Theorem. (B.–Stanfield). Let $f : X \to Y$ be a holomorphic map between *Hermitian manifolds*. Let $\{e_i\}$ be a local unitary frame on X and $\{w_\alpha\}$ a local unitary frame on $f(X) \subseteq Y$. Choose *Hermitian connections* on $T^{1,0}X$ and $T^{1,0}Y$. Then $\partial f \in \Omega_X^{1,0} \otimes f^* T^{1,0} Y$ satisfies

$$\begin{aligned}
\Delta_\omega |\partial f|^2 \quad = \quad & |\nabla \partial f|^2 + \overline{f_k^\alpha} f_\ell^\alpha R_{i\bar{i}}{}^k{}_\ell - \overline{f_\ell^\alpha} f_\ell^\beta \overline{f_i^\gamma} f_i^\delta \tilde{R}_{\gamma\bar{\delta}}{}^\beta{}_\alpha \\
& + 2\,\mathrm{Re}\left(-f_k^\alpha \overline{f_\ell^\alpha} T^k_{\bar{i}\ell,i} + f_\ell^\alpha \overline{f_\ell^\beta} f_i^\gamma \overline{f_i^\delta} \tilde{T}^\beta_{\bar{\delta}\alpha,\gamma} - \overline{f_j^\beta} f^\beta_{k,i} T^k_{\bar{i}j} + \overline{f_j^\delta}\,\overline{f_i^\gamma} f^\alpha_{j,i} \tilde{T}^\delta_{\bar{\gamma}\alpha} \right) \\
& + 2\,\mathrm{Re}\left(f_k^\alpha \overline{f_\ell^\alpha}\, \overline{T^r_{\bar{i}i}} T^k_{\bar{r}\ell} - f_\ell^\alpha \overline{f_\ell^\beta} f_i^\gamma \overline{f_i^\delta}\, \overline{\tilde{T}^\mu_{\bar{\gamma}\delta}} \tilde{T}^\beta_{\bar{\mu}\alpha} \right)
\end{aligned}$$

The letters T and R are respectively the torsion and curvature of the source connection, and $\tilde{T}$, $\tilde{R}$ are the torsion and curvature of the target connection.

Gauduchon–Schwarz Lemma

If we take the connections to be *Gauduchon* with the *same parameter* $t \in \mathbb{R}$, we have the following:

$$\Delta_\omega |\partial f|^2 \;=\; |\nabla \partial f|^2 + \frac{\operatorname{Re}\lambda_i^2}{2t(2t-1)}\left((t^2+2t-1)\,{}^t\mathrm{Ric}^{(2)}_{i\bar{i}}\right)$$

$$+\frac{\operatorname{Re}\lambda_i^2}{2t(2t-1)}\left((t-1)(3t-1)\,{}^t\mathrm{Ric}^{(3)}_{i\bar{i}} + (t-1)^2\left({}^t\mathrm{Ric}^{(1)}_{i\bar{i}} - {}^t\mathrm{Ric}^{(4)}_{i\bar{i}}\right)\right)$$

$$+\operatorname{Re}\lambda_i^2\left(\frac{(1-3t)(t+1)}{2t(2t-1)}T^i_{\bar{r}k}\overline{T^j_{\bar{r}k}} + \frac{2t^2+t+1}{2t(1-t)}T^r_{\bar{k}j}\overline{T^r_{\bar{k}i}} + \frac{3-7t}{2t-1}T^r_{\bar{i}j}\overline{T^r_{\bar{k}k}}\right)$$

$$+\frac{t}{2t-1}\,{}^t\mathrm{RBC}_{\omega_h} + \frac{(1-t)}{t}\,{}^t\widetilde{\mathrm{RBC}}_{\omega_h} + \frac{(1+t)}{2t-1}|\widetilde{T}^r_{\bar{i}k}\lambda_i\lambda_k|^2 + \frac{7t-3}{2t-1}|\widetilde{T}^r_{\bar{i}i}\lambda_i^2|^2$$

Gauduchon–Schwarz Lemma with Kähler source

If we assume (X, ω_g) *is Kähler*, then in the frame where $f_i^\alpha = \lambda_i \delta_i^\alpha$ we have, for $t \in \mathbb{R} \setminus \left\{0, \frac{1}{2}, 1\right\}$,

$$\begin{aligned} \Delta_{\omega_g} |\partial f|^2 \quad &= \quad |\nabla \partial f|^2 + {}^c\mathrm{Ric}^{(2)}_{i\bar{i}} \lambda_i^2 + \frac{t}{2t-1} {}^t\mathrm{RBC}_{\omega_h} + \frac{(1-t)}{t} {}^t\widetilde{\mathrm{RBC}}_{\omega_h} \\ &\quad + \frac{(1+t)}{2t-1} |\widetilde{T}^r_{\bar{i}k} \lambda_i \lambda_k|^2 + \frac{7t-3}{2t-1} |\widetilde{T}^r_{\bar{i}i} \lambda_i^2|^2. \end{aligned}$$

Note that:

$${}^t\mathrm{RBC}_{\omega_h}(v) := \frac{1}{|v|^2} \sum_{i,k} {}^tR_{i\bar{i}k\bar{k}} v_i v_k,$$

and

$${}^t\widetilde{\mathrm{RBC}}_{\omega_h}(v) := \frac{1}{|v|^2} \sum_{i,k} {}^tR_{i\bar{k}k\bar{i}} v_i v_k.$$

Monotonicity Theorem for the Holomorphic Sectional Curvature

Theorem. (B.–Stanfield). Let (X, ω) be a Hermitian manifold. Let ${}^t\mathrm{HSC}_\omega$ denote the *t–Gauduchon holomorphic sectional curvature*. Then for all $t \in \mathbb{R}$, we have

$$ {}^t\mathrm{HSC}_\omega \ \leq \ {}^c\mathrm{HSC}_\omega, $$

with equality if and only if $t = 1$.

If we choose $t \in (-\infty, -1) \cup (1, \infty)$, then ${}^c\mathrm{HSC}_{\omega_h} < 0$ implies that

$$ \frac{t}{2t-1} {}^t\mathrm{RBC}_{\omega_h} + \frac{(1-t)}{t} {}^t\widetilde{\mathrm{RBC}}_{\omega_h} > 0. $$

Moreover, for this range of t, the *torsion coefficient functions* are *positive*.

Refined Schwarz Lemma

Theorem. (B.–Stanfield). Let $f : (X, \omega_g) \longrightarrow (Y, \omega_h)$ be a holomorphic map of rank r from an n–dimensional *Kähler manifold* (X, ω_g) into a *Hermitian manifold* (Y, ω_h). Suppose

$$\mathrm{Ric}_{\omega_g} \geq -C_1 \omega_g + C_2 f^* \omega_h,$$

where $C_1 \in \mathbb{R}$ and $C_2 \geq 0$. Assume that ${}^c\mathrm{HSC}_{\omega_h} \leq -\kappa_0$ for some $\kappa_0 \geq 0$. Then

$$\Delta_{\omega_g} |\partial f|^2 \quad \geq \quad -C_1 |\partial f|^2 + \left(\frac{C_2}{n} + \frac{\kappa_0}{r} \right) |\partial f|^4.$$

In particular, if X is compact, then

$$|\partial f|^2 \quad \leq \quad \frac{C_1 n r}{C_2 r + n \kappa_0}.$$

Refined Wu–Yau Theorem

Theorem. (B.–Stanfield 2022). Let X be a *compact Kähler manifold*. If X supports a *Hermitian metric* ω with ${}^c\mathrm{HSC}_{\omega} < 0$, then K_X *is ample*.

Further directions/applications/questions

Question. Can *compact Kobayashi hyperbolic manifolds* be characterized by ${}^t\mathrm{HSC}_\omega < 0$ for some range of $t \in \mathbb{R}$?

Demailly (97) showed that there are *compact Kobayashi hyperbolic manifolds*[8] which *do not admit Hermitian metrics* with ${}^c\mathrm{HSC}_\omega < 0$.

[8]Reminder: A compact complex manifold X is said to be Kobayashi hyperbolic if all holomorphic maps $\mathbb{C} \to X$ are constant.

Question. Can *compact complex manifolds* which are *rationally connected* be characterized by ${}^t\mathrm{HSC}_\omega > 0$ for some range of $t \in \mathbb{R}$?

Yang showed that if (X, ω) is compact Kähler with $\mathrm{HSC}_\omega > 0$, then X is *rationally connected*[9].

On the other hand, the standard metric on a *Hopf surface* $\mathbb{S}^3 \times \mathbb{S}^1$ has ${}^c\mathrm{HSC}_\omega > 0$, but since the universal cover is $\mathbb{C}^2 \backslash \{0\}$, the Hopf surface *does not have any rational curves*.

[9] Reminder: A complex manifold X is said to be rationally connected if any two points lie in the image of a rational curve $\mathbb{P}^1 \to X$.

Thank you for listening!