CANONICAL METRICS IN KÄHLER GEOMETRY

KYLE BRODER – ANU MSI & PKU BICMR – NOVEMBER 2019

The purpose of this talk is to discuss the following (dangerously vague) problem: Given a Riemannian manifold (M, g) does there exist a *best metric*? Since this talk is intended to be expository, let us remind ourselves of the definition of a Riemannian manifold:

First, a **smooth manifold** is a Hausdorff topological space M with an open cover (U_α) and a collection of homeomorphisms $\varphi_\alpha : U_\alpha \to V_\alpha \subset \mathbb{R}^n$, where n is the dimension of the manifold, such that the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ are $\mathscr{C}^\infty$ smooth. That is, a smooth manifold locally looks likes $\mathbb{R}^n$ and in a sufficiently small neighbourhood of each point of the manifold, we can find smooth local coordinates $(x^1, ..., x^n)$.

We can impose further constraints on the manifold to introude a different geometry: If the homeomorphisms φ_α map into $\mathbb{C}^n$ (as opposed to $\mathbb{R}^n$) and we require that the transition maps are *holomorphic* (as opposed to smooth), the resulting smooth manifold is a **complex manifolds**. A particular example, one-dimensional complex manifolds[1] are called **Riemann surfaces**.

Fix a point $p \in M$ and define the **tangent space** T_pM to be the vector space spanned by the partial derivatives $\partial_{x^1}, ..., \partial_{x^n}$, viewed as tangent vectors to M at p, where $(x^1, ..., x^n)$ are the local coordinates in the neighbourhood of p. A **Riemannian metric** g on M is a smoothly varying family of inner products $g_p : T_pM \times T_pM \to \mathbb{R}$ on each tangent space. In local coordinates, we can write the metric as

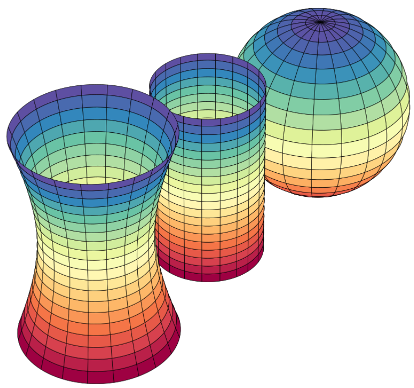
$$g = g_{ij} dx^i \otimes dx^j, \qquad g_{ij} = g(\partial_{x^i}, \partial_{x^j}).$$

We are, of course, using the Einstein summation convention here.

Every smooth manifold can be equipped with a Riemannian metric by patching together the local Euclidean metrics via a partition of unity. This of course requires the manifold to be paracompact.

[1]That is, manifolds with real dimension 2.

Using the metric, we can talk begin to talk about how the manifold is curved, and this leads to the notion of **curvature**. For the moment, let us restrict ourselves to surfaces, where the curvature is given by a single number called the **Gauss curvature**. The Gauss curvature $K = \kappa_1 \cdot \kappa_2$ is the product of the principal curvatures and is best understood by considering the following pictures:



Recall that we want to address the question of a manifold admitting a "best metric". The motivation for this question comes from the **Uniformisation theorem** of Riemann surfaces.

The uniformisation theorem tells us that, given any compact Riemann surface M, we can equip M with a metric of constant Gauss curvature. The proof requires some analysis, but the heuristic is rather straightforward: An extension of the Riemann mapping theorem tells us that any simply-connected Riemann surface is biholomorphic to one of the following candidates:

(i) $\mathbb{CP}^1 \cong \mathbb{S}^2$, equipped with the round metric of constant Gauss curvature $+1$.
(ii) $\mathbb{C}$, equipped with the Euclidean metric of constant Gauss curvature 0.
(iii) $\mathbb{H} \cong \mathbb{D}$, equipped with the Poincaré metric of constant Gauss curvature -1.

Given such a classification of simply connected Riemann surfaces, the universal cover of our given Riemann surface must be one of the above types. Thus, we can simply pull the metric on the universal cover back down to obtain a metric on our given surface.

It would be nice to know if such beautiful results hold for higher-dimensional manifolds. What we need to first observe is that the curvature of a higher-dimensional Riemannian manifold is much more complicated than the curvature of a surface. To introduce the **curvature tensor** of higher-dimensional manifolds, we remind ourselves that a manifold can always be equipped with a canonical set of coordiates called **geodesic normal coordinates**. Given a Riemannian manifold (M, g), a **geodesic** is a smooth curvature $\gamma : [0, 1] \to M$ which locally minimises the length between any two points. Geodesics in Euclidean space are given by straight lines, while geodesics on $\mathbb{S}^2$ are given by the **great circles**.[2]

Using geodesics, we can define the **exponential map** $\exp_p : T_pM \to M$ by sending a tangent $v \in T_pM$ to the endpoint of the geodesic $\gamma(1)$, where $\gamma(0) = p$, and $\gamma'(0) = v$. This map is a diffeomorphism in a neighbourhood of p and gives us the so-called **geodesic normal coordinates**.

We can compute the Taylor expansion of the components of the metric in these coordinates, upon which we find that

$$g_{ij} \quad = \quad \delta_{ij} - \frac{1}{3}R_{ikj\ell}x^k x^\ell + O(|x|^3),$$

and we define the **Riemannian curvature tensor** to be the correction term $R_{ikj\ell}$. The Riemannian curvature tensor measures the extent to which distances between geodesics is distorted under the exponential map.

Further, a Riemannian metric gives rise to a **volume form** which is defined (in local coordinates) to be $d\mu_g = \sqrt{\det(g_{ij})}dx^1 \wedge \cdots dx^n$. Computing the Taylor expansion of $d\mu_g$ as we did above, we find that

$$d\mu_g \quad = \quad \left[1 - \frac{1}{6}R_{jk}x^j x^k + O(|x|^3)\right] dx^1 \wedge \cdots \wedge dx^n,$$

and we define the **Ricci curvature tensor** to be the correction term R_{jk}. The Ricci curvature tensor measures the extent to which volumes along geodesics are distorted under the exponential map. If $\mathrm{Ric}(g) > 0$, then volumes decrease along geodesics, while if $\mathrm{Ric}(g) < 0$, volumes increase along geodesics.

Finally, we can look at the distortion of the volume of a small ball $B_\varepsilon^M(p)$ in the manifold compared with the volume of the ball $B_\varepsilon^{\mathbb{R}^n}(0)$ in Euclidean space. In which case, we find that

$$\mathrm{Vol}(B_\varepsilon^M(p)) \quad = \quad \left[1 - \frac{\mathrm{Scal}(g)}{6(n+2)}\varepsilon^2 + O(\varepsilon^4)\right] \mathrm{Vol}(B_\varepsilon^{\mathbb{R}^n}(0)),$$

and we define the **Scalar curvature** to be the correction term $\mathrm{Scal}(g)$, which unlike the Riemannian curvature and Ricci curvature, is a real number like the Gauss curvature.[3]

Given that the scalar curvature, like the Gauss curvature, is a real number, perhaps the most natural extension of the uniformisation theorem is to ask whether a Riemannian manifold

[2]One may think of geodesics as the path that a particle would follow if it was not acted on by an external force.

[3]For surfaces, the Scalar curvature is twice the Gauss curvature.

M^n, $n \geq 3$, admits **a metric of constant scalar curvature**? For compact manifolds, this is the solution of the famous **Yamabe problem**. The solution of the Yamabe problem tells us that there is a unique metric of constant scalar curvature in each conformal class. That is, given any Riemannian metric g on a compact Riemannian manifold M, we can conformally rescale the metric, i.e., multiply by a positive function to obtain a metric $\overline{g} := e^{2u} g$ which has constant scalar curvature. The problem with this, however, is that there are simply **too many** constant scalar curvature metrics. The space of all such metrics is **infinite-dimensional**.

Another candidate may be metrics of **constant Riemannian curvature**. From the point of view of differential equations, however, it is very unlikely that many such metrics exist. Indeed, the components of the Riemannian curvature tensor grow quartically in the dimension, while the components of the metric only grow quadratically in the dimension.

Einstein metrics, i.e., metrics of constant Ricci curvature, do not have the above problems. So these metrics seem to be the apropriate candidate to generalise the uniformisation theorem. The Einstein equation $\text{Ric}(g) = \lambda g$, where $\lambda \in \mathbb{R}$, is still very difficult to solve. If one imposes strong symmetry conditions on the manifold by, for example, considering homogeneous spaces, then the Einstein equation simplifies and one can obtain Einstein metrics. The first general setting where the problem of finding Einstein metrics becomes tractable is in **Kähler geometry**.

To define Kähler manifolds, we discuss three tensors: g, J and ω on a smooth manifold M:

- (†) The Riemannian metric g is a symmetric positive-definite quadratic form on each tangent space.
- (†) The symplectic form ω is an anti-symmetric non-degenerate quadratic form on each tangent space.
- (†) The almost complex structure is an endomorphism $J : TM \to TM$ satisfying $J^2 = -\text{id}$.

Given this triple of tensors, we can impose three additional criteria that enrich the geometry:

(i) **Coherence**. We require that g and ω are mutually compatbile with respect to the almost complex structure J:

$$g(Ju, Jv) = g(u, v), \quad \forall u, v \qquad \omega(u, Ju) > 0, \quad \forall u \neq 0, \qquad \omega(Ju, Jv) = \omega(u, v).$$

(ii) **Closedness**. We require that the symplectic form is closed, i.e., $d\omega = 0$.

(iii) **Integrability**. We require that the eigenbundle $T^{1,0}M$ is an integrable tangent distribution.

The coherent condition yields mutual compatibility between the three tensors – in particular, and two determine the third. The closedness condition on the symplectic form implies that ω represents a de-Rham class in $H^2_{\mathrm{DR}}(M, \mathbb{R})$. In particular, this puts constraints on the topology of the underlying manifold when the manifold is compact. The integrability condition ensures the existence of local holomorphic coordinates by the Newlander–Nirenberg theorem. We explain condititons (ii) and (iii) in more detail.

We remind ourselves of simple definitions of the de Rham cohomology groups:

If M is a smooth manifold, a p–form η on M is a smooth section of $\Lambda^p T^* M$. In local coordinates $(x^1, ..., x^n)$, we can write η as

$$\eta \ = \ \sum_I \eta_I dx^I \ = \ \sum_{1 \leq i_1 < i_2 < \cdots < i_p \leq n} \eta_{i_1 \cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}.$$

The exterior derivative in local coordinates is then expressed as

$$d\eta \ = \ \sum_{k=1}^n \sum_I \frac{\partial \eta_I}{\partial x^k} dx^k \wedge dx^I.$$

In particular, the exterior derivative maps p–forms to $(p+1)$–forms and satisfies $d^2 = 0$. We say that a p–form ω is closed if $d\omega = 0$ and we say a p–form ω is exact if there exists a $(p-1)$–form η such that $\omega = d\eta$. It is clear that every exact form is closed, but the converse is not always true. The obstruction is measures by the de Rham cohomology groups:

$$H^p_{\mathrm{DR}}(M, \mathbb{R}) \ := \ \frac{\{\text{closed } p\text{–forms}\}}{\{\text{exact } p\text{–forms}\}}.$$

A beautiful result of the theory states that these are topological invariants of the manifolds – they do not depend on the smooth structure!

By the definition of the de Rham cohomology groups, two 2–forms ω, ω' lie in the same cohomology class is they differ by the exterior derivative of a 1–form:

$$\omega' = \omega + d\eta.$$

On compact Kähler manifolds, we can say more: If ω and ω' are two Kähler forms, which are $(1,1)$–forms, which are cohomologous, then we can write $\omega' = \omega + \sqrt{-1}\partial\bar{\partial}\varphi$ for $\varphi \in \mathscr{C}^\infty(M, \mathbb{R})$.