Calculus Exam 1 (Preparation)

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The contents of this examination require an understanding of the elementary calculus material that was covered in the calculus practice exams 1 and 2. An understanding of graphing techniques and function transformations are also required.

There are no permitted materials for this test. That is, you are not permitted any cheat notes, calculators or resources other than a pen/pencil, eraser, sharpener, ruler and water bottle.

There is to be no collaboration on this examination and any attempts of communication will result in a nullified score. You are permitted 10 minutes of reading time and 80 minutes of writing time. There is a total of 100 available marks. It is recommended that you use the reading time to ask the invigilator about any issues regarding the format of the test, the problems or other issues. No hints will be given. Best of luck!

Name: ___________________

Grade: __________/100

Question 1. [10 marks]. Evluate the following limit

$$\lim_{x\to -1} \frac{x^2+2x+1}{x+1}.$$

Proof. Write $x^2+2x+1=(x+1)^2$. Then we see that

$$\begin{aligned} \lim_{x\to -1} \frac{x^2+2x+1}{x+1} &= \lim_{x\to -1} \frac{(x+1)^2}{x+1} \\ &= \lim_{x\to -1} x+1 = 0. \end{aligned}$$

$\square$

Marking scheme: 5 marks for working, 3 marks for answer, 2 marks for writing limit at every step.

Question 2. [25 marks]. Consider the function $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ defined by

$$f(x) = \frac{5+x}{4-2x}.$$

Show that f is differentiable on $\mathbb{R}\backslash\{2\}$ and compute $f'(x)$.

Proof. We simply observe that

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{5+(x+\Delta x)}{4-2(x+\Delta x)} - \frac{5+x}{4-2x}\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{[5+x+\Delta x]\cdot[4-2x]}{[4-2(x+\Delta x)]\cdot[4-2x]} - \frac{[5+x]\cdot[4-2(x+\Delta x)]}{[4-2(x+\Delta x)]\cdot[4-2x]}\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{20+4x+4\Delta x-10x-2x^2-2x\Delta x}{(4-2x)(4-2(x+\Delta x))} - \frac{20-2x^2-2x\Delta x+4x-2x^2-2x\Delta x}{(4-2x)(4-2(x+\Delta x))}\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \frac{20+4x+4\Delta x-10x-2x^2-2x\Delta x-20+2x^2+2x\Delta x-4x+2x^2+2x\Delta x}{(4-2x)(4-2(x+\Delta x))} \\
&= \lim_{\Delta x \to 0} \frac{4\Delta x-10x+2x\Delta x}{(4-2x)(4-2x-2\Delta x)} \\
&= \frac{-10x}{(4-2x)^2}.
\end{aligned}$$

Therefore f is differentiable for all $x \in \mathbb{R}\backslash\{2\}$ with

$$f'(x) = \frac{-10x}{(4-2x)^2}.$$

□

Marking sheme: 20 marks for working, 5 marks for the derivative.

Question 3. [25 marks]. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{\sqrt{x-1}} + \frac{3}{(x-6)^{\frac{3}{7}}}.$$

Explain why f is differentiable on $(1, \infty)$ and compute $f'(x)$.

Proof. The function f is a composition of differentiable functions on $(1, \infty)$. Therefore f is differentiable on $(1, \infty)$. To evaluate $f'(x)$, we first evaluate the derivative of $\frac{1}{\sqrt{x-1}}$. Using the chain rule, we see that

$$\begin{aligned} \frac{d}{dx}\left(\frac{1}{\sqrt{x-1}}\right) &= \frac{d}{dx}\left((x-1)^{-\frac{1}{2}}\right) \\ &= -\frac{1}{2} \cdot (1) \cdot (x-1)^{-\frac{3}{2}} \\ &= -\frac{1}{2(x-1)^{\frac{3}{2}}}. \end{aligned}$$

Moreover, we see that

$$\begin{aligned} \frac{d}{dx}\left(\frac{3}{(x-6)^{\frac{3}{7}}}\right) &= \frac{d}{dx}\left(3(x-6)^{-\frac{3}{7}}\right) \\ &= 3 \cdot (1) \cdot -\frac{3}{7} \cdot (x-6)^{-\frac{10}{7}} \\ &= -\frac{6}{7(x-6)^{\frac{10}{7}}}. \end{aligned}$$

Hence we see that

$$f'(x) = -\frac{1}{2(x-1)^{\frac{3}{2}}} - \frac{6}{7(x-6)^{\frac{10}{7}}}.$$

□

Marking scheme: 10 marks for justification of f being differentiable on $(0, \infty)$, 10 marks for the computation of $f'(x)$ (getting to equation (1)), 5 marks for the algebraic simplification (getting to equation (2)).

Question 4. [20 marks]. Determine where the following functions are not differentiable.

a. [5 marks]. $f(x) = 2|x-3| + 1$.

Proof. f is not differentiable at $x = 3$. □

b. [5 marks]. $f(x) = 4|2x-5| - 7$.

Proof. f is not differentiable at $x = \frac{5}{2}$. □

c. [5 marks]. $f(x) = \frac{3}{2}|7-x|$.

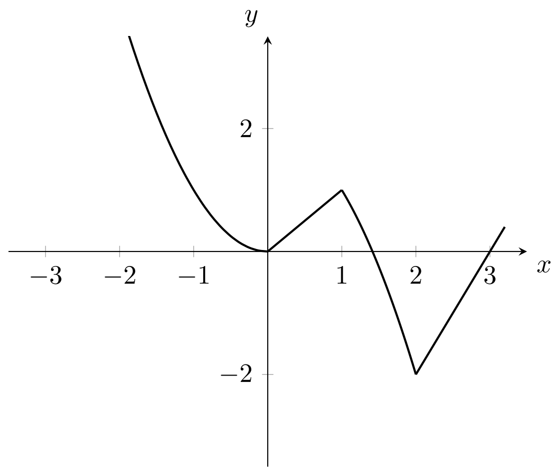
Proof. f is not differentiable at $x = 7$. □

d. [5 marks]. $f(x) = 1/|x|$.

Proof. f is not differentiable at $x = 0$. □

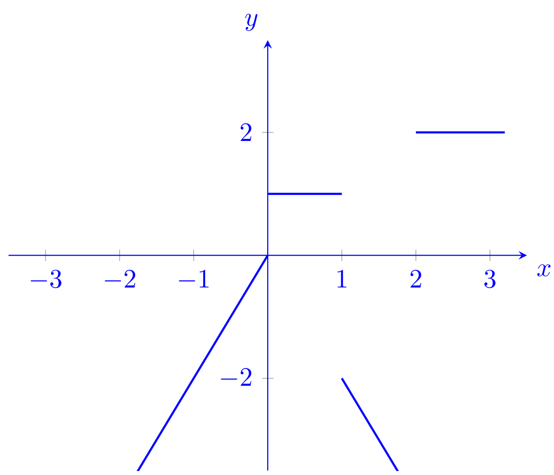
Marking scheme: Full marks for solution.

Question 5. [20 marks]. Consider the function f whose graph is given below.



Determine the domain of $f'(x)$ and sketch $f'(x)$ on this domain.

Proof. The function given in the graph is not differentiable at the points $x = 0, x = 1$ and $x = 2$. The domain of $f'(x)$ is therefore $\mathbb{R}\backslash\{0, 1, 2\}$. The graph of $f'(x)$ is given by



$\square$

Marking scheme: 10 marks for the domain of the $f'(x)$, 10 marks for the shape of the curve.