Chapter 1

Elementary Function Theory.

1.1 Elementary Set Theory

In this section we recall some elementary notions that will appear throughout our study of functions $f : \mathbb{R} \to \mathbb{R}$.

Definition 1.1.1 A set is a collection of objects.

Examples of sets include the natural numbers $\mathbb{N} = \{1, 2, ...\}$ and the integers $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$. Other examples include the rationals

$$\mathbb{Q} = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N} \right\},$$

essentially, the rational numbers consist of the fractions. Examples of numbers that are not rational include $\sqrt{2}, \pi$ and e, where e denotes Euler's number. We also have as an example of a set, the real numbers $\mathbb{R}$, which, up until chapter 4 should be thought of as *every number you can think of.*

Definition 1.1.2. Let A and B be two sets. We define the intersection of A and B to be

$$A \cap B = \text{The elements that } A \text{ and } B \text{ have in common.}$$

For example,

$$\{1, 4, 6, 13, 15\} \cap \{1, 5, 12, 15\} = \{1, 15\}.$$

Definition 1.1.3. Let A and B be two sets. We define the union of A and B to be

$$A \cup B = \text{The set of all elements contained in } A \text{ and } B.$$

For example,

$$\{1,4,6,13,15\} \cup \{1,5,12,15\} = \{1,4,5,6,12,13,15\}.$$

Definition 1.1.4. Let A and B be sets. We say that A is a subset of B, and write $A \subset B$ if every element in A is a subset of B. Note that a set is always a subset of itself.

Definition 1.1.5. Suppose that $A \subset B$, we define the complement of A in B, and write $B \backslash A$ to be the set of elements in B that are not in A.

Definition 1.1.6 Let A and B be sets and let $f : A \to B$ be a function from A to B. We call A the domain and B the codomain. We define the range of f to be the set

$$\mathcal{R}(f) = \{y \in B : f(x) = y, x \in A\}.$$

Notice that $\mathcal{R}(f)$ is a subset of B.

We end this section by identifying the following set conventions:

† $(a,b) := \{x \in \mathbb{R} : a < x < b\}$.

† $[a,b) := \{x \in \mathbb{R} : a \leq x < b\}$.

† $(a,b] := \{x \in \mathbb{R} : a < x \leq b\}$.

† $[a,b] := \{x \in \mathbb{R} : a \leq x \leq b\}$.

Exercises

Q1. Determine the following sets.

a. $\{0,1,3\} \cap \{0,4\}$.
b. $\{1,3,5,6\} \cup \{2,7\}$.
c. $\{4,5\} \cup \{0,-3\}$.
d. $\{1,5,13\} \cap \{1,5,7,9\}$.
e. $\{1,3,5,7\} \cap \{2,4,6,8\}$.
f. $\{1,4,6\} \cup \{2,6,7\}$.

Q2. Determine the following sets.

a. $\mathbb{N} \cap \mathbb{Z}$.
b. $\mathbb{N} \cup \mathbb{Z}$.
c. $\mathbb{R} \cap \mathbb{Q}$.
d. $\mathbb{N} \cap \mathbb{Q}$.
e. $\mathbb{R} \cap \mathbb{R}$.
f. $\mathbb{Q} \cap (\mathbb{R} \backslash \mathbb{Q})$.

Q3. Determine the following sets.

a. $\mathbb{Z}\backslash\mathbb{N}$.

b. $\mathbb{R}\backslash\mathbb{Q}$.

c. $\mathbb{Q}\backslash\mathbb{Z}$.

d. $\mathbb{R}\backslash\{0\}$.

Q4. Determine which of the following assertions are true of false.

a. The integers $\mathbb{Z}$ are a subset of the naturals $\mathbb{N}$.

b. The rational numbers $\mathbb{Q}$ contain the number $\sqrt{5}$.

c. The empty set $\emptyset$, which contains no elements, is a set.

d. The empty set $\emptyset$ is a subset of every set.

e. The real numbers $\mathbb{R}$ contain the number $\sqrt{e}$, where e is Euler's number.

f. The rationals $\mathbb{Q}$ are a subset of the irrationals $\mathbb{R}\backslash\mathbb{Q}$.

g. The real numbers $\mathbb{R}$ are a subset of the integers $\mathbb{Z}$.

h. For every set X there exists a set Y such that $X \subset Y$.

i. There are more integers $\mathbb{Z}$ than there are natural numbers $\mathbb{N}$.

j. If $x, y \in \mathbb{Z}$, then $x + y \in \mathbb{Z}$.

k. If $x, y \in \mathbb{R}$, then $x + y \in \mathbb{R}$.

l. If $x \in \mathbb{Q}$ and $y \in \mathbb{Z}$, then $x + y \in \mathbb{Q}$.

m. If $x, y \in \mathbb{Q}$, then $x \cdot y \in \mathbb{Q}$.

o. $\infty \in \mathbb{R}$.

p. $\mathbb{R} = [-\infty, \infty]$.

q. $\mathbb{N}$ is an infinite set.

r. $(0, 1) \subset \mathbb{R}$ is bigger than $\mathbb{N}$.

Q5. Determine the following sets

a. $(1, 3) \cap [2, 4]$.

b. $(-3, 4) \cap (2, 4]$.

c. $(-\infty, 1) \cup (2, \infty)$.

d. $(-\infty, \sqrt{2}] \cup (-1, 3]$.

e. $(-10, \infty) \cap (-3, 2]$.

f. $[-4, 1] \cup [2, 19]$.

Q6. Determine the following sets.

a. $(0,1) \cap [1,3]$.

b. $[-3,2] \cap (-1,5]$.

c. $(13,18) \cap [-12,18]$.

d. $(-\infty,\infty) \cup (1,2)$.

e. $(-2,3) \cup (2,5)$.

f. $(-\infty,1] \cap [-4,\infty)$.

Q7. Write each of the sets in Q5. and Q6. in the form $\{x \in \mathbb{R} : a \leq x \leq b\}$.

Q8. Determine the following sets.

a. $(0,1) \cap ((1/2,1) \cap [1/2,3])$.

b. $[-3,2] \cup ([1,2] \cap (-3,4])$.

c. $([-1,1) \cap (-2,3)) \cap (-\infty,-10]$.

d. $(-\infty,\infty) \cap \mathbb{Z} \cap \{1,3\}$.

1.2 Linear Equations

Arguably the most important class of functions are the polynomials. We define a polynomial $p \in \mathscr{P}(\mathbb{R})$ to be a function of the form

$$p(x) = a_n x^n + \cdots + a_1 x + a_0,$$

where $n \in \mathbb{Z}_{\geq 0}$, $a_i \in \mathbb{R}$, for $0 \leq i \leq n$.

Examples of polynomials include $p(x) = x$, $p(x) = x^2 - 5x + 6$, $p(x) = x^7 + 2$. The functions $f(x) = \sqrt{x}$ and $f(x) = \frac{1}{x}$ are not polynomials however.

The simplest of the polynomials are the linear equations. Linear equations, also referred to as degree one polynomials, have the form

$$p(x) = ax + b,$$

where $a, b \in \mathbb{R}$.

The number a in the above equation corresponds to the gradient, often referred to as the slope. The number b in the above equation corresponds to a vertical translation, or y-intercept.

Example 1.2.1 Solve the equation $2x + 3 = 4$.

Proof. We simply observe that

$$\begin{aligned} 2x + 3 = 4 \quad &\Longrightarrow \quad 2x = 4 - 3 \\ &\Longrightarrow \quad 2x = 1 \\ &\Longrightarrow \quad x = \frac{1}{2}. \end{aligned}$$

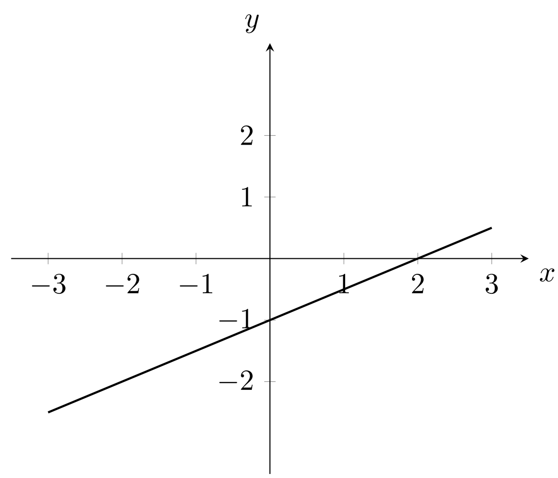
$\square$

Example 1.2.2. Graph the equation $p(x) = \frac{1}{2}x - 1$.

Proof. The gradient is given by $a = \frac{1}{2}$, while the y-intercept is given by $b = -1$. The x-intercept is given by setting $p(x) = 0$. Indeed, doing so, we see that

$$\frac{1}{2}x - 1 = 0 \implies \frac{1}{2}x = 1 \implies x = 2.$$

We then see that the graph is given by



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Example 1.2.3. Solve the simultaneous equations

$$\begin{cases} 2x + 3y = 5, \\ 4x + 7y = 1. \end{cases}$$

Proof. We may solve this system of equations by means of two methods, either elimination or substitution. The method of elimination involves the subtraction or addition of one equation to the other, hopefully removing one the variables, while the method of substitution involves the rearrangement of one of the equations for a particular variable, and inserting into the other equation. We proceed first to solve the system using the method of elimination.

Let [1] denote the equation $2x + 3y = 5$, and [2] denote the equation $4x+7y = 1$. We then see that $[2]-2\cdot[1]$ is the equation $y = -9$. Therefore, since $2x + 3y = 5$, it follows that $2x + 3 \cdot (-9) = 5$ and so, $x = 16$.

Using the method of substitution, solve the first equation for x. Indeed, we see that $x = \frac{1}{2}(5 - 3y)$. If we then insert this into the second equation, we see that

$$\begin{aligned}
4\left(\frac{1}{2}(5-3y)\right)+7y=1 \quad &\Longrightarrow \quad 4\cdot\frac{1}{2}\cdot(5-3y)+7y=1\\
&\Longrightarrow \quad 2(5-3y)+7y=1\\
&\Longrightarrow \quad 10-6y+7y=1\\
&\Longrightarrow \quad 10+y=1\\
&\Longrightarrow \quad y=-9.
\end{aligned}$$

Therefore, using the same argument as above, we have that $x=16$ and $y=-9$. □

Exercises.

Q1. Solve the following equations for $x \in \mathbb{R}$.

a. $3x+1=2x-3$.
b. $5x+2x=2-9x$.
c. $9x+18x=27x+1$.
d. $x+x=2x-3x+\frac{3}{5}$.
e. $x+4x=9x-\frac{8}{7}x$.
f. $\frac{1}{2}x+\frac{3}{5}x=2x-6$.

Q2. Solve the following equations for $x \in \mathbb{R}$.

a. $\frac{1}{4}x-\frac{3}{5}=\frac{1}{6}-x$.
b. $\frac{3}{7}x-2=1-\frac{6}{11}x$.
c. $x+\frac{4}{5}x=5x+6$.
d. $x+2=2-x$.
e. $x+3=\frac{3}{5}x-\frac{4}{7}x+1$.
f. $\frac{3}{7}x=\frac{3}{4}x+1$.

Q3. Solve the following linear equations.

a. $3x+1=0$.
b. $2x-3=4$.
c. $5-x=1$.
d. $7+3x=2x$.
e. $4x+2=3x-1$.
f. $9-2x=4x-4$.

Q4. Solve the following linear equations.

a. $5x+2x=3x-1$.
b. $8-2x+1=4x$.
c. $\frac{1}{2}x-4=\frac{1}{4}x-3$.
d. $\frac{1}{2}x-\frac{1}{6}x+1=\frac{3}{8}$.
e. $\frac{1}{3}+\frac{2-x}{5}=1$.
f. $\frac{4}{5}x+\frac{2}{3}x=-\frac{1}{3}x-7$.

Q5. Graph the following linear equations.

a. $f(x) = x - 3.$

b. $f(x) = 2x + 1.$

c. $f(x) = 3 - x.$

d. $f(x) = 4x - \frac{1}{2}.$

e. $f(x) = 3(x - 1).$

f. $f(x) = x + \frac{1}{2}.$

Q6. Solve the following equations.

a.
$$\frac{3-x}{4} = 1.$$

b.
$$\frac{4-x}{7} = 2x.$$

c.
$$\frac{1}{4}(7 - 6x) = 4x - 3.$$

d.
$$\frac{3x+9}{3-x} = 4.$$

Q7. Let $f(x) = 2x - 3$ and let $g(x) = 5 - x$. Determine the point of intersection between f and g.

Q8. Let $f(x) = 3 - 6x$ and let $g(x) = 4x + \frac{1}{3}$. Determine the point of intersection between f and g.

Q9. Determine the value of $k \in \mathbb{R}$ such that $f(x) = kx + 2$ never intersects $g(x) = 4x - 1$.

Q10. Determine the value of $k \in \mathbb{R}$ such that $f(x) = 4 - kx$ intersects $g(x) = 3 + x$ exactly once.

Q11. Determine the values of $x \in \mathbb{R}$ such that

a. $2x + 3 > 4x - 6.$

b. $5x + 1 > 7 - x.$

c. $4x - 4 < 1 + x.$

d. $3x + 2 > 5 - x.$

e. $\frac{1}{4}x + \frac{2}{7}x > 3 - \frac{3}{7}x.$

f. $\frac{1}{3}x + 4 < 5 - x.$

Q12. Suppose Dylan is running at a constant speed of 1/2 metres per second. Let $r(t) = mt$ denote his position in terms of time.

a. Given that Dylan runs at a constant speed of 1/2 metres per second, determine the value of m.

b. How far does Dylan travel in one hour?

c. How long does it take Dylan to travel 2 kilometres?

Q13. Graph the following linear equations.

a. $f(x) = 3x + 1.$

b. $f(x) = 2x - 3.$

c. $f(x) = 5x + 2.$

d. $f(x) = 2 - 9x.$

e. $f(x) = 1 + 4x.$

f. $f(x) = \frac{1}{2}x + \frac{3}{5}x.$

g. $f(x) = x + 3.$

h. $f(x) = \frac{3}{5}x - \frac{3}{7}.$

Q14. Sketch the following curves.

a. $y - 2x - 3 = 4x + 1$

b. $x + 1 - y = 2x - 1.$

c. $5x - y + 1 = x - 9.$

d. $3x - 2x + y = 9 - 10x.$

e. $13x - y + 7 = 12x + 9.$

f. $\frac{2x+4}{3} = 2y + x + 4.$

g. $x + 4 = \frac{1}{3}y - 4x + 3.$

h. $\frac{1}{2}y + x = \frac{\sqrt{3}}{2}x + 4y + 1.$

Q15. Solve the following systems of equations

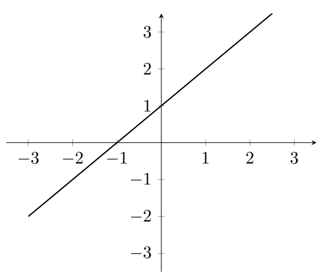
a.
$$\begin{cases} 2x - 3y = 1, \\ 4x + y = 5. \end{cases}$$

b.
$$\begin{cases} 6x - y = 4, \\ 2x + 9y = 1. \end{cases}$$

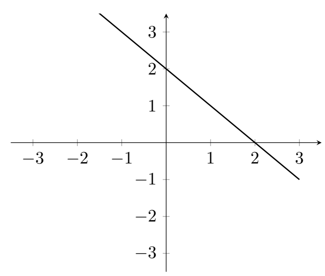
c.
$$\begin{cases} x - y = 4, \\ x + y = 3. \end{cases}$$

d.
$$\begin{cases} 7x - \frac{1}{2}y = \frac{\sqrt{2}}{3}, \\ x + 2y = 14. \end{cases}$$

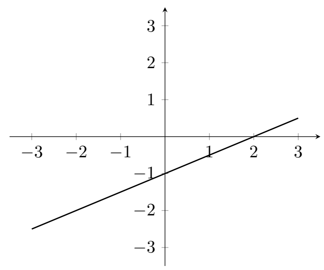
Q16. Determine the equation of the following graphs.



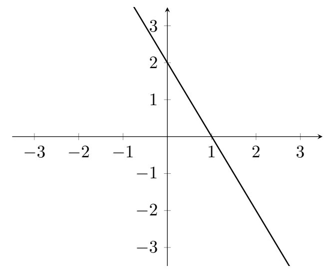
a.



b.



c.



d.

Q17. Determine the equation of the line which is parallel to $f(x) = 2x - 1$ and passes through $(-1, -3)$.

Q18. Determine the equation of the line which is parallel to $f(x) = 1 - 4x$ and passes through $(0, 0)$.

Q19. Determine the equation of the line which is perpendicular to

$$f(x) = 9 - 3x$$

and passes through $(1, 2)$.

Q20. Determine the equation of the line which is perpendicular to

$$f(x) = x$$

and passes through $\left(-\frac{1}{3}, 4\right)$.

Q21. Determine the angle made between the x-axis and the line

$$f(x) = 1 + \sqrt{3}x.$$

Q22. Determine the angle made between the x-axis and the line

$$f(x) = 2\sqrt{6} - 2x.$$

Q23. Determine the linear equation which passes through $(-3, 1)$ and $(2, 6)$ and write the expression in the form

$$ax + by + c = 0, \qquad a, b, c \in \mathbb{R}.$$

Q24. Consider the system of linear equations

$$\begin{cases} \lambda x = -8y, \\ 4x + (\lambda + 2)y = 0, \end{cases}$$

parameterised by $\lambda \in \mathbb{R}$. Determine the value(s) of λ such that the system has a unique solution.

Q25. Consider the system of linear equations

$$\begin{cases} 2 + \lambda x = 4 - y, \\ 5\lambda + y = 6y \end{cases}$$

parameterised by $\lambda \in \mathbb{R}$. Determine the value(s) of $\lambda \in \mathbb{R}$ such that the system has an infinite number of solutions.

Q26. Consider the system of linear equations

$$\begin{cases} 4\lambda - 6x = 2 - \lambda y, \\ (\lambda - 1)x + 2y = 0 \end{cases}$$

parameterised by $\lambda \in \mathbb{R}$. Determine the value(s) of $\lambda \in \mathbb{R}$ such that the system has no solutions.

1.3 Quadratic Equations

We now move on to the important class of polynomials, which are of degree two, the quadratic polynomials. A quadratic polynomial has the form

$$p(x) = ax^2 + bx + c.$$

An example of a quadratic polynomial is $f(x) = x^2 - 5x + 6$. There are three methods we can use to solve quadratic equations, which we outline in some time. We first exhibit the method of expanding quadratic expressions.

Example 1.3.1. Expand the following expression $(x-3)(x-1)$.

Proof. The method consists of multiplying each term involved and adding. For example,

$$\begin{aligned}(x-3)(x-1) &= x \cdot x + x \cdot (-1) + (-3) \cdot x + (-1) \cdot (-3) \\ &= x^2 - x - 3x + 3 \\ &= x^2 - 4x + 3.\end{aligned}$$

□

Example 1.3.2 Expand the following expression $(2x-3)(x+1)$.

Proof. Using the method that was given in Example 1.2.1, we see that

$$\begin{aligned}(2x-3)(x+1) &= 2x \cdot x + 2x \cdot 1 + (-3) \cdot x + (-3) \cdot 1 \\ &= 2x^2 + 2x - 3x - 3 \\ &= 2x^2 - x - 3.\end{aligned}$$

□

We now want to look at how to get from an expression of the form $x^2 - 4x + 3$ and write it as $(x-3)(x-1)$. That is, we want to look at the *opposite* procedure of Example 1.2.1 and Example 1.2.2. The first of these methods is by recognition, which is outlined in the following example.

Example 1.3.3. (Factorisation by Recognition). Determine the values of $a, b \in \mathbb{R}$ such that $x^2 + 2x + 1 = (x-a)(x-b)$.

Proof. The method is given by finding the numbers a and b such that $a+b = 2$ and $ab = 1$. We see that if $a = b = 1$, then $a + b = 2$ and $ab = 1$, so we can write

$$x^2 + 2x + 1 = (x + 1)^2.$$

□

The other method involves using the *quadratic formula*. If we let $f(x) = ax^2 + bx + c$, the solutions to $f(x) = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Example 1.3.4 Determine the roots of the equation $f(x) = x^2 - 5x + 6$.

Proof. Using the quadratic formula, we see that

$$\begin{aligned} x &= \frac{5 \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} \\ &= \frac{5 \pm \sqrt{25 - 24}}{2} \\ &= \frac{5 \pm \sqrt{1}}{2} \\ &= \frac{5 \pm 1}{2} = 2, 3. \end{aligned}$$

□

The methods that we have exhibited so far are excellent for determining the x-intercepts of a quadratic equation, but not so good for determining the turning point of the quadratic equation. This is because we have been writing the quadratic expressions in their cartesian form, $f(x) = x^2 + bx + c$. The following example shows us how to write f in a much more useful form. This form

$$f(x) = (x - \alpha)^2 + \beta,$$

where $\alpha, \beta \in \mathbb{R}$ is referred to as turning point form.

Example 1.3.5 (Completing the Square). Write the expression $f(x) = x^2 - 2x - 3$ in turning point form.

Proof. We simply observe that

$$\begin{aligned} x^2-2x-3 &= x^2-2x+1-1-3 \\ &= (x^2-2x+1)-4 \\ &= (x-1)^2-4. \end{aligned}$$

□

Example 1.3.6 Write the expression $f(x)=x^2-3x-18$ in turning point form.

Proof. We simply observe that

$$\begin{aligned} x^2-3x-18 &= x^2-3x-\frac{9}{4}+\frac{9}{4}-18 \\ &= \left(x^2-3x+\frac{9}{4}\right)-18-\frac{9}{4} \\ &= \left(x-\frac{3}{2}\right)^2-\frac{72}{4}-\frac{9}{4} \\ &= \left(x-\frac{3}{2}\right)^2-\frac{81}{4}. \end{aligned}$$

□

Example 1.3.7. Sketch the curve $f(x)=x^2-5x+6$.

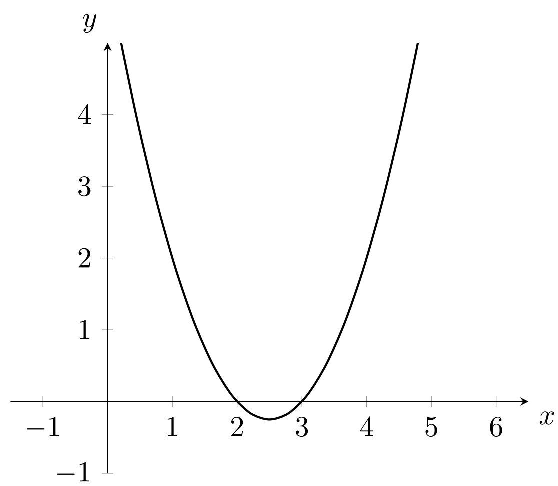
Proof. It is easy to see that the y-intercept is given by $y=6$. To determine the x-intercepts, we use the quadratic formula, which tells us that

$$x = \frac{-5\pm\sqrt{(-5)^2-4(1)(6)}}{2(1)}=\frac{5\pm 1}{2}=2,3.$$

So the x-intercepts occur at $x=2$ and $x=3$. To determine whether the turning point is, we write f in turning point form. Indeed, we see that

$$\begin{aligned} x^2-5x+6 &= x^2-5x-\frac{25}{4}+\frac{25}{4}+6 \\ &= \left(x-\frac{5}{2}\right)^2+\frac{24}{4}-\frac{25}{4} \\ &= \left(x-\frac{5}{2}\right)^2-\frac{1}{4}. \end{aligned}$$

So the turning point is given by $\left(\frac{5}{2},-\frac{1}{4}\right)$. Hence, the graph is given by



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The last topic we wish to discuss in this section is the descriminant, which allows us to determine the number of roots of a quadratic expression without having to compute the roots explicitly using the quadratic formula.

For a quadratic expression $f(x) = ax^2 + bx + c$, the descriminant Δ is given by

$$\Delta = b^2 - 4ac.$$

† If $\Delta > 0$, the quadratic $f(x)$ has two roots in $\mathbb{R}$.

† if $\Delta = 0$, the quadratic $f(x)$ has one root in $\mathbb{R}$.

† If $\Delta < 0$, the quadratic $f(x)$ has no roots in $\mathbb{R}$.

Exercises.

Q1. Expand the following expressions.

a. $(x-3)(x+1)$.
b. $(x+1)(x-1)$.
c. $(x+5)(x-2)$.
d. $(x+6)(x-7)$.
e. $(x+2)(x-8)$.
f. $(x-4)(x+1)$.
g. $(x-2)(x+6)$.
h. $(x-3)(x+8)$.

Q2. Expand the following expressions

a. $(x-3)^2$.
b. $(x+1)^2$.
c. $(x+6)^2$.
d. $(x-2)^2$.
e. $(x-1)^2$.
f. $(x-7)^2$.
g. $(x+8)^2$.
h. $(x-4)^2$.

Q3. Expand the following expressions.

a. $\left(x-\frac{1}{2}\right)\left(x-\frac{1}{5}\right)$.
b. $\left(x-\frac{1}{4}\right)\left(x-\frac{3}{4}\right)$.
c. $\left(x+\frac{1}{5}\right)\left(x+\frac{8}{9}\right)$.
d. $\left(\frac{3}{7}-x\right)(x+1)$.
e. $\left(\frac{1}{2}x+\frac{1}{5}\right)\left(x-\frac{1}{8}\right)$.
f. $(x+12)\left(2x-\frac{1}{5}\right)$.

Q4. Expand the following expressions.

a. $(2x-1)(x-\sqrt{3})$.
b. $(4x+3)(2x-\sqrt{7})$.
c. $\left(\frac{3}{5}x-\frac{\sqrt{3}}{2}\right)(x-1)$.
d. $\left(\frac{\sqrt{5}}{2}x-1\right)\left(x+\frac{1}{\sqrt{3}}\right)$.

Q5. Determine the values of $a, b \in \mathbb{R}$ such that $(x-3)(x+\sqrt{3}) = x^2+ax+b$.

Q6. Determine the values of $a, b \in \mathbb{R}$ such that $(x-a)(x-b) = x^2-5x+6$.

Q7. Factorise the following expressions.

a. x^2-2x-3.
b. x^2-5x-6.
c. x^2-x-2.
d. x^2+6x+9.
e. $x^2+9x+18$.
f. $x^2+3x-10$.
g. x^2-x-72.
h. x^2+2x+1.
i. x^2-1.
j. x^2-9.
k. x^2-16.
l. x^2-3.

Q8. Determine the roots of the following equations.

a. $f(x)=5x^2+3x+3$.
b. $f(x)=x^2-3x+1$.
c. $f(x)=6x^2+7x-3$.
d. $f(x)=10x^2-x-2$.
e. $f(x)=2x^2-5x+3$.
f. $f(x)=3-17x-6x^2$.

Q9. Calculate the descriminant of the following expressions.

a. $f(x) = x^2 + 3x + 1.$
b. $f(x) = 4x^2 - 5x + 2.$
c. $f(x) = \frac{1}{2}x^2 + 4x - 3.$
d. $f(x) = x^2 + 1.$
e. $f(x) = x^2 - 9x + 1.$
f. $f(x) = x^2.$
g. $f(x) = x.$
h. $f(x) = \frac{1}{7} - 6x + x^2.$

Q10. Write the following expressions in turning point form.

a. $f(x) = x^2 + 2x + 1.$
b. $f(x) = x^2 - 5x - 6.$
c. $f(x) = x^2 - 4x + 5.$
d. $f(x) = 5 - x + x^2.$
e. $f(x) = x^2 + 4x - 8.$
f. $f(x) = x^2 + 3.$
g. $f(x) = 3x^2 - 8x + 1.$
h. $f(x) = 2x^2 - 5x + \frac{1}{3}.$

Q11. Write the following expressions in cartesian form.

a. $f(x) = (x-3)^2 + 1.$
b. $f(x) = 2(x+1)^2 - \frac{2}{5}.$
c. $f(x) = (x+4)^2 - 3.$
d. $f(x) = (x+8)^2 - 2.$
e. $f(x) = \left(x - \frac{1}{3}\right)^2 + 4.$
f. $f(x) = \left(2x - \frac{5}{7}\right)^2 - \frac{2}{7}.$

Q12. Sketch the following quadratic equations.

a. $x^2 - 2x - 3.$
b. $x^2 - 5x - 6.$
c. $x^2 - x - 2.$
d. $x^2 + 6x + 9.$
e. $x^2 + 9x + 18.$
f. $x^2 + 3x - 10.$
g. $x^2 - x - 72.$
h. $x^2 + 2x + 1.$
i. $x^2 - 1.$
j. $x^2 - 9.$
k. $x^2 - 16.$
l. $x^2 - 3.$

Q13. Sketch the following quadratic equations.

a. $f(x) = x^2 + 2x + 1.$
b. $f(x) = x^2 - 5x - 6.$
c. $f(x) = x^2 - 4x + 5.$
d. $f(x) = 5 - x + x^2.$
e. $f(x) = x^2 + 4x - 8.$
f. $f(x) = x^2 + 3.$
g. $f(x) = 3x^2 - 8x + 1.$
h. $f(x) = 2x^2 - 5x + \frac{1}{3}.$

Q14. State the transformations necessary to map $f(x) = x^2$ to the following quadratic equations.

a. $f(x) = (x-3)^2 + 1$.

b. $f(x) = 2(x+1)^2 - \frac{2}{5}$.

c. $f(x) = (x+4)^2 - 3$.

d. $f(x) = (x+8)^2 - 2$.

e. $f(x) = \left(x - \frac{1}{3}\right)^2 + 4$.

f. $f(x) = \left(2x - \frac{5}{7}\right)^2 - \frac{2}{7}$.

Q15. Calculate the descriminant Δ for the following expressions.

a. $f(x) = x^2 + 4x - 5$.

b. $f(x) = 4x - 3x^2 + 1$.

c. $f(x) = 1 - x^2$.

d. $f(x) = x^2 + 2x + 1$.

e. $f(x) = x^2 - 5x - 7$.

f. $f(x) = -4 - x$.

Q16. Determine the value(s) of $k \in \mathbb{R}$ such that $f(x) = x^2 - kx$ intersects $g(x) = 2x - 3$ exactly once.

Q17. Determine the value(s) of $k \in \mathbb{R}$ such that $f(x) = 4x - k$ intersections $g(x) = kx^2 - 1$ two times.

Q18. Determine the value(s) of $k \in \mathbb{R}$ such that $f(x) = kx^2 - \frac{1}{6}$ does not intersect $g(x) = kx + \sqrt{2}$.

Q19. The volume of water in a tank, V m^3, over a 10 month period is given by the function $V(t) = 2t^2 - 16t + 40$, where t is the number of months, $0 \leq t \leq 10$.

a. Determine the initial volume of water in the tank.

b. Determine the maximum volume of water in the tank.

c. Determine the minimum volume of water in the tank.

Q20. A ball thrown upwards from a tower attains a height above the ground given by the model

$$h(t) = 12t - 3t^2 + 36,$$

where t is the time in seconds and h is in metres.

a. Find the maximum height above the ground that the ball reaches.

b. Determine the time taken for the ball to reach the ground.

c. Determine the domain and range of the function h.

Q21. A section of a roller-coaster at an amusement park follows the path of a parabola. The function $h(t) = t^2 - 12t + 48$, $0 \leq t \leq 11$, models the height above the ground of the front of one of the carriages, where t is the time in seconds and h is the height in metres.

a. Find the lower point of this section of the ride.

b. Find the time taken for the carriage to reach the lowest point.

c. Find the highest point above the ground.

d. Find the domain and range of the function.

e. Sketch the function.

Q22. Let $f(x) := 2x+1$ and $g(x) = x^2+k$. Determine the value(s) of $k \in \mathbb{R}$ such that f intersects g exactly once.

Q23. Let $f(x) := 4x + k$ and $g(x) = 2x^2 - 3$. Determine the value(s) of $k \in \mathbb{R}$ such that f intersects g exactly once.

Q24. Let $f(x) := kx^2 - x$ and $g(x) = x+1$. Determine the value(s) of $k \in \mathbb{R}$ such that f intersects g exactly once.

Q25. Let $f(x) := kx + \frac{2}{3}$ and $g(x) = 1 - kx^2$. Determine the value(s) of $k \in \mathbb{R}$ such that f intersects g twice.

Q26. Let $f(x) := kx+1$ and $g(x) = 2+x^2$. Determine the value(s) of $k \in \mathbb{R}$ such that f intersects g twice.

Q27. Let $f(x) := 1 - kx^2$ and $g(x) = 3kx$. Determine the value(s) of $k \in \mathbb{R}$ such that f intersects g twice.

Q28. Let $f(x) := 4 + 2x^2$ and $g(x) = \frac{1}{k}x + 1$. Determine the value(s) of $k \in \mathbb{R}$ such that f intersects g twice.

Q29. Let $f(x) := x^2 - kx + 6$ and $g(x) = x^2 + kx + 1$. Determine the value(s) of $k \in \mathbb{R}$ such that f and g do not intersect.

Q30. Let $f(x) := (x-3)(x+k)$ and $g(x) = 2x+1$. Determine the value(s) of $k \in \mathbb{R}$ such that f and g do not intersect.

Q31. Let $f(x) := (kx+1)^2$ and $g(x) = 2kx - 5$. Determine the value(s) of $k \in \mathbb{R}$ such that f and g do not intersect.

1.4 Cubic and Quartic Equations

In this section we study polynomials of degree three and four. A cubic equation has the form

$$f(x) = ax^3 + bx^2 + cx + d,$$

and a quartic equation has the form

$$f(x) = ax^4 + bx^3 + cx^2 + dx + e,$$

where $a, b, c, d, e \in \mathbb{R}$.

Example 1.4.1. Expand the following $(x-1)(x-2)(x+5)$.

Proof. We simply observe that

$$\begin{aligned}(x-1)(x-2)(x+5) &= (x-1)(x^2+5x-2x-10)\\ &= (x-1)(x^2+3x-10)\\ &= x^3+3x^2-10x-x^2-3x+10\\ &= x^3+2x^2-13x+10.\end{aligned}$$

□

Example 1.4.2. Expand the following $(x+3)(x-3)(x+1)(x-6)$.

Proof. We simply observe that

$$\begin{aligned}(x+3)(x-3)(x+1)(x-6) &= (x+3)(x-3)(x^2-6x+x-6)\\ &= (x+3)(x-3)(x^2-5x-6)\\ &= (x+3)(x^3-5x^2-6x-3x^2+15x+18)\\ &= (x+3)(x^3-8x^2+11x+18)\\ &= x^4-8x^3+11x^2+18x+3x^3-24x^2+33x+54\\ &= x^4-5x^3-10x^2+51x+54.\end{aligned}$$

□

Now, factorisation of cubics and quartics is a little more complicated than factorisation of quadratic expressions, and involves the method of polynomial long division.

Example 1.4.3. Factorise the expression $f(x) = x^3 - 3x^2 - 6x + 8$.

Proof. The first thing we need to do is *guess* the first root. We can do this however in an educated manner by looking at factors of the constant term, which in this case, is 8. Factors of 8 include $\pm 1, \pm 2$ and ± 4. If we try the simplest case, which $x = 1$, we see that

$$f(1) = 1^3 - 3 \cdot 1^2 - 6 \cdot 1 + 8 = 1 - 3 - 6 + 8 = 0.$$

So $x = 1$ is a root of $f(x) = x^3 - 3x^2 - 6x + 8$. We may therefore write f as

$$f(x) = (x-1)(x^2 + bx + c),$$

for some $b, c \in \mathbb{R}$. If we expand the above expression, we see that

$$\begin{aligned} f(x) &= (x-1)(x^2 + bx + c) \\ &= x^3 + bx^2 + cx - x^2 - bx - c \\ &= x^3 + (b-1)x^2 + (c-b)x - c. \end{aligned}$$

Since f is also given by $f(x) = x^3 - 3x^2 - 6x + 8$, we may equate coefficients and observe that $b - 1 = -3$, $c - b = -6$ and $-c = 8$. This tells us that $b = -2$ and $c = -8$. Therefore,

$$f(x) = (x-1)(x^2 - 2x - 8) = (x-1)(x+2)(x-4),$$

where the last equality follows from simply quadratic factorisation. $\square$

Example 1.4.4. Factorise the expression

$$f(x) = x^4 + 4x^3 - 31x^2 - 46x + 168.$$

Proof. Again, by looking at the factors of 168, we guess the first root. It is clear that ± 1 does not work, so let us try $x = 2$. Indeed, we see that

$$\begin{aligned} f(2) &= 2^4 + 4 \cdot 2^3 - 31 \cdot 2^2 - 46 \cdot 2 + 168 \\ &= 16 + 4 \cdot 8 - 31 \cdot 4 - 92 + 168 \\ &= 16 + 32 - 124 - 92 + 168 \\ &= 48 - 92 + 44 = 0, \end{aligned}$$

so $x = 2$ is a root of $f(x)$. We may therefore write

$$f(x) = (x-2)(x^3 + bx^2 + cx + d).$$

Expanding the above expression, we see that

$$\begin{aligned} f(x) &= (x-2)(x^3+bx^2+cx+d) \\ &= x^4+bx^3+cx^2+dx-2x^3-2bx^2-2cx-2d \\ &= x^4+(b-2)x^3+(c-2b)x^2+(d-2c)x-2d. \end{aligned}$$

Equating coefficients, we see that $b-2=4, c-2b=-31, d-2c=-46$ and $-2d=168$. So $b=6, c=-19$, and $d=-84$. Hence we see that

$$f(x)=(x-2)(x^3+6x^2-19x-84).$$

Now, we need repeat the process for the cubic expression

$$g(x)=x^3+6x^2-19x-84.$$

It is easy to see that $x=-3$ is a root of $g(x)$. We may therefore write

$$\begin{aligned} g(x) &= (x+3)(x^2+\alpha x+\gamma) \\ &= x^3+\alpha x^2+\gamma x+3x^2+3\alpha x+3\gamma \\ &= x^3+(\alpha+3)x^2+(\gamma+3\alpha)x+3\gamma. \end{aligned}$$

Equating coefficients, we see that $\alpha+3=6, \gamma+3\alpha=-19$ and $3\gamma=-84$. It follows that $\alpha=3$ and $\gamma=-28$. So,

$$g(x) = (x-3)(x^2+3x-28)=(x-3)(x+7)(x-4).$$

Thus, we may write $f(x)$ as

$$f(x)=(x-2)(x-3)(x+7)(x-4).$$

□

Example 1.4.5. Solve the equations $x^4-5x^2+6=0$.

Proof. Let $u=x^2$, then

$$\begin{aligned} x^4-5x^2+6=0 &\implies u^2-5u+6=0 \\ &\implies (u-2)(u-3)=0. \end{aligned}$$

Hence we see that $x^2=2$ and $x^2=3$. The solutions are therefore $x=\pm\sqrt{2}, \pm\sqrt{3}$. □

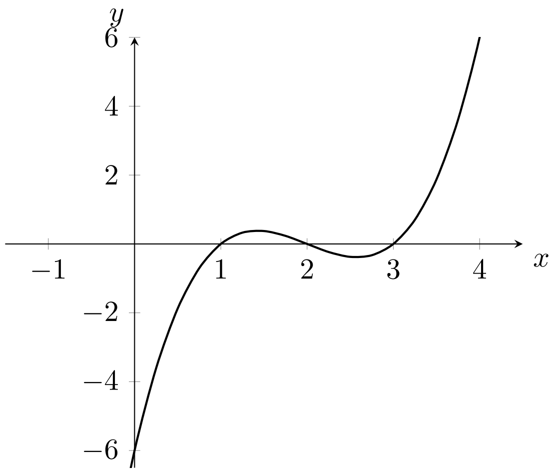
Example 1.4.6 Sketch the graph of the function

$$f(x) = x^3 - 6x^2 + 11x - 6.$$

Proof. We first observe that the y-intercept of $f(x)$ is given by $(0, -6)$. To determine the x-intercepts, we factorise $f(x)$. Indeed, we see that

$$f(x) = (x-1)(x-2)(x-3),$$

so f has roots at $x = 1, x = 2$ and $x = 3$, all of multiplicity 1. Hence we see that the graph is given by



□

Exercises

Q1. Expand the following expressions.

a. $(x-3)(x+1)(x-5)$.
b. $(x+5)(x-7)(x+8)$.
c. $(x+1)(x-4)(x-3)$.
d. $(x+4)(x+9)(x-5)$.
e. $(x-1)(x-2)(x-3)$.
f. $(x-4)(x-6)(x+7)$.

Q2. Expand the following expressions.

a. $\left(x-\frac{1}{3}\right)\left(x+\frac{1}{4}\right)\left(x-\frac{1}{8}\right)$.
b. $\left(x+\frac{1}{4}\right)\left(x-\frac{4}{5}\right)(x+1)$.
c. $\left(x+\frac{1}{3}\right)(2x-1)(4x-3)$.
d. $(x-4)(4x-6)(5x+2)$.
e. $(x+4)(3x-1)(9x-3)$.
f. $(2x+1)(2x-1)(x-4)$.

Q3. Expand the following expressions.

a. $(x+\sqrt{3})(x-\sqrt{2})(x-1)$.

b. $(x+\sqrt{5})(x+\sqrt{7})(2x-5)$.

c. $\left(x-\frac{\sqrt{3}}{2}\right)\left(x-\frac{1}{2}\right)\left(x+\sqrt{3}\right)$.

d. $\left(x-\sqrt[3]{2}\right)\left(x+\sqrt[3]{2}\right)\left(x+\sqrt[3]{2}\right)$.

Q4. Expand the following expressions.

a. $(x+1)^3$.

b. $(x+3)^3$.

c. $(x-6)^3$.

d. $(x+7)^3$.

e. $(x+\sqrt{3})^3$.

f. $\left(x-\frac{1}{2}\right)^3$.

Q5. Expand the following expressions.

a. $(x+1)(x-2)(x-3)(x+4)$.

b. $(x-3)(x-4)(x+2)(x-2)$.

c. $(x+1)(x-7)(x-8)(x-1)$.

d. $(x-3)(x+5)(x-8)(x+9)$.

Q6. Expand the following expressions.

a. $(x-1)^4$.

b. $(x+6)^4$.

c. $(x-3)^4$.

d. $(x+\sqrt{2})^4$.

Q7. Factorise the following expressions.

a. x^3+3x^2+3x+1.

b. $x^3-6x^2+5x+12$.

c. $2x^3+8x^2-38x+28$.

d. x^3-9x^2-x+9.

e. x^3+9x^2-108.

f. $x^3+x^2-56x+144$.

Q8. Factorise the following expressions.

a. $x^4-14x^3+73x^2-168x+144$.

b. $x^4+2x^3-3x^2-4x+4$.

c. $x^4+2x^3-13x^2-14x+24$.

d. $2x^4+7x^3-68x^2+32x$.

Q9. Solve the following equations.

a. $x^4 - 2x^2 - 3 = 0$.
b. $x^4 - 5x^2 - 6 = 0$.
c. $x^4 - x^2 - 2 = 0$.
d. $x^4 + 6x^2 + 9 = 0$.
e. $x^4 + 9x^2 + 18 = 0$.
f. $x^4 + 3x^2 - 10 = 0$.
g. $x^4 - x^2 - 72 = 0$.
h. $x^4 + 2x^2 + 1 = 0$.
i. $x^4 - 1 = 0$.
j. $x^4 - 9 = 0$.
k. $x^4 - 16 = 0$.
l. $x^4 - 3 = 0$.

Q10. Sketch the following curves, stating all relevant features.

a. $f(x) = (x+1)(x-2)(x-3)$.
b. $f(x) = (x-3)(x-4)(x-2)$.
c. $f(x) = (x+1)(x-8)(x-1)$.
d. $f(x) = (x-3)(x+5)(x+9)$.

Q11. Sketch the following curves, stating all relevant features.

a. $f(x) = x^3 + 3x^2 + 3x + 1$.
b. $f(x) = x^3 - 6x^2 + 5x + 12$.
c. $f(x) = 2x^3 + 8x^2 - 38x + 28$.
d. $f(x) = x^3 - 9x^2 - x + 9$.
e. $f(x) = x^3 + 9x^2 - 108$.
f. $f(x) = x^3 + x^2 - 56x + 144$.

Q12. Sketch the following curves, stating all relevant features.

a. $f(x) = (x-1)^4$.
b. $f(x) = (x-4)^4$.
c. $f(x) = (x+3)^4$.
d. $f(x) = (x+\sqrt{3})^4$.

Q13. Sketch the following curves, stating all relevant features.

a. $f(x) = x^4 - 14x^3 + 73x^2 - 168x + 144$.
b. $f(x) = x^4+2x^3-3x^2-4x+4$.
c. $f(x) = x^4+2x^3-13x^2-14x+24$.
d. $f(x) = 2x^4+7x^3-68x^2+32x$.

Q14. Sketch the following curves, stating all relevant features.

a. $f(x) = (x-1)^2(x-4)$.
b. $f(x) = (x-3)(x+1)(x-6)^2$.
c. $f(x) = (x-3)(x-6)$.
d. $f(x) = x(x-4)(x-4)(x+6)$.

Q15. Sketch the following curves, stating all relevant feautures.

a. $f(x) = (x-3)^3 + 1$.

b. $f(x) = (x-4)^3 - 5$.

c. $f(x) = (x+1)^3 + 2$.

d. $f(x) = x^3 + 2$.

Q16. Sketch the following curves, stating all relevant features.

a. $f(x) = (x+7)^4 - 4$.

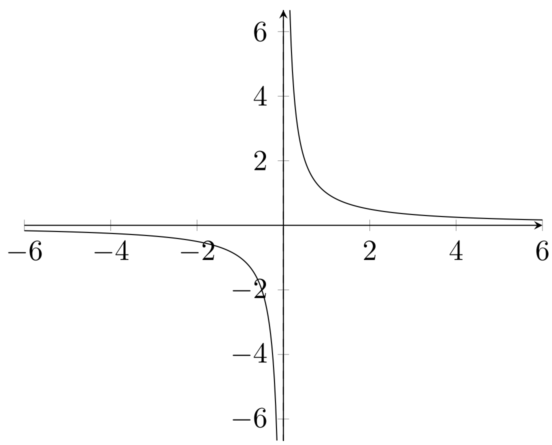
b. $f(x) = 2(x-3)^4 + \frac{1}{6}$.

c. $f(x) = (4-x)^4 + 1$.

d. $f(x) = x^2(x-4)^2 - \frac{1}{\sqrt{3}}$.

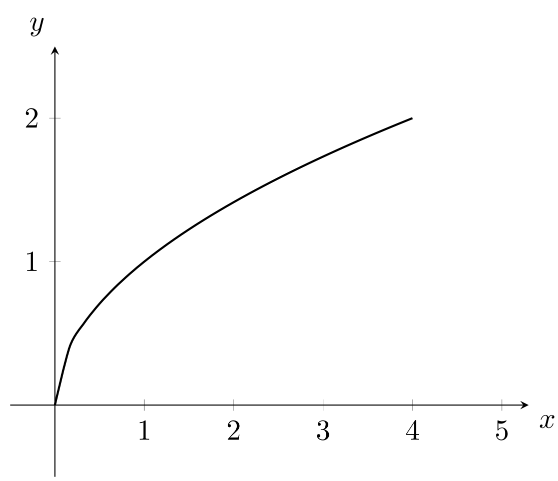
1.5 Hyperbolas, Square Roots and Truncated Curves

In this section we study the curves $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$ and $f(x) = \frac{1}{x^2}$. The function $f(x) = \frac{1}{x}$ is referred to as the hyperbola and has graph given by



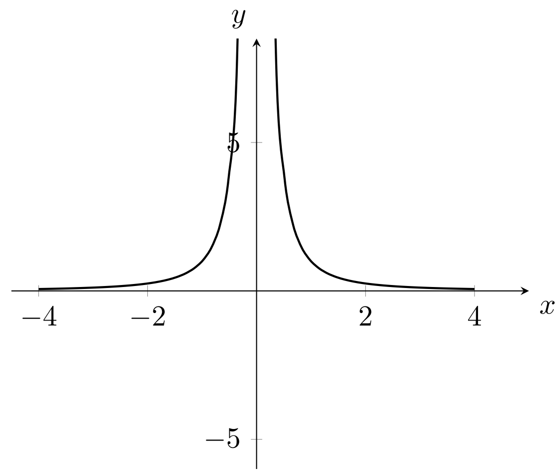
Notice that the domain of $f(x) = \frac{1}{x}$ is given by $x \in \mathbb{R}\backslash\{0\}$.

The function $f(x) = \sqrt{x}$ is referred to as the square root and has graph given by



Notice that the domain of $f(x) = \sqrt{x}$ is given by $x \in \mathbb{R}_{\geq 0}$.

The function $f(x) = \frac{1}{x^2}$ is referred to as the truncus and has graph given by



Notice that the domain of $f(x) = \frac{1}{x^2}$ is given by $x \in \mathbb{R}\backslash\{0\}$.

Example 1.5.1. State the domain and range of the function

$$f(x) = \frac{1}{4x-3} + 3.$$

Proof. The function $1/x$ is defined for all $x \in \mathbb{R}\backslash\{0\}$. Therefore, the function $f(x) = \frac{1}{4x-3} + 3$ is defined for all $x \in \mathbb{R}$ such that $4x - 3 \neq 0$. So the domain of $f(x)$ is given by

$$x \in \mathbb{R}\backslash\left\{\frac{3}{4}\right\}.$$

It is clear that the range of $f(x)$ is given by $f(x) \in (3, \infty)$. $\square$

Example 1.5.2. State the domain and range of the function

$$f(x) = 4 - 2\sqrt{x+6}.$$

Proof. The function $\sqrt{x}$ is defined for all $x \geq 0$. Therefore, the function $f(x) = 4 - 2\sqrt{x+6}$, is defined for all $x + 6 \geq 0$. So the domain of $f(x)$ is given by $x \geq 6$. An equivalent notation for this is $x \in [6, \infty)$ or $x \in \mathbb{R}_{\geq 6}$.

It is clear that the range of $f(x)$ is given by $f(x) \in (-\infty, 4]$. $\square$

Example 1.5.3. State the domain and range of the function

$$f(x) = \frac{6}{(x+7)^2} - 5.$$

Proof. The function $\frac{1}{x^2}$ is defined for all $x \in \mathbb{R}\backslash\{0\}$. Therefore, the function $f(x) = \frac{6}{(x+7)^2} - 5$ is defined for all $x + 7 \neq 0$. So the domain of $f(x)$ is given by $x \in \mathbb{R}\backslash\{-7\}$. An equivalent notation for this is $x \in (-\infty, -7) \cup (-7, \infty)$.

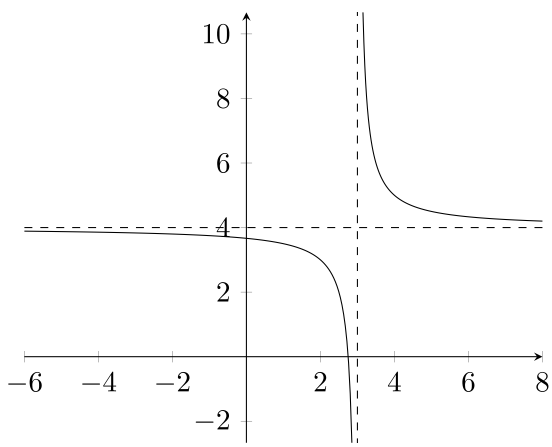
It is clear that the range of $f(x)$ is given by $f(x) \in (-5, \infty)$. An equivalent notation for this is $f(x) > -5$. □

Example 1.5.4. Sketch the curve

$$f(x) = \frac{1}{x-3} + 4,$$

stating all relevant features.

Proof. The curve f is given by shifting the curve $\frac{1}{x}$ to the right by 3 units and up 4 units. The resulting curve is given by



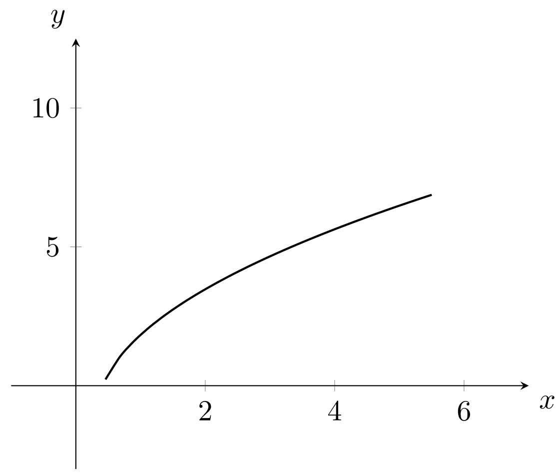
□

Example 1.5.5. Sketch the curves

$$f(x) = 2\sqrt{3x-1} - 1,$$

stating all relevant features.

Proof. The curve f is given by dilating the curve $\sqrt{x}$ by 2 from the x-axis, $\frac{1}{3}$ from the y-axis, translating by $\frac{1}{3}$ to the right, and 1 down. The resulting curve is given by



$\square$

Exercises.

Q1. Determine the domain and range of the following functions.

a. $f(x) = \sqrt{4x+1} - 1$.

b. $f(x) = 3 - \sqrt{x+5} - 3$.

c. $f(x) = \frac{1}{3} + \frac{2}{5}\sqrt{5x+7}$.

d. $f(x) = 9\sqrt{8x-3} - 3$.

Q2. Determine the domain and range of the following functions.

a. $f(x) = \frac{2}{3x+7} - 2$.

b. $f(x) = \frac{1}{2} + \frac{3}{5x+2}$.

c. $f(x) = \frac{4}{5-x} - 6$.

d. $f(x) = 1 - \frac{3}{x}$.

Q3. Determine the domain and range of the following functions.

a. $f(x) = \frac{1}{(x+3)^2} + 3$.

b. $f(x) = -\frac{1}{(x+4)^2}$.

c. $f(x) = \frac{6}{(3-x)^2} - \frac{1}{4}$.

d. $f(x) = 1 - \frac{4}{(x-8)^2}$.

Q4. Sketch the following curves, stating all relevant features.

a. $f(x) = \sqrt{4x+1} - 1$.

b. $f(x) = 3 - \sqrt{x+5} - 3$.

c. $f(x) = \frac{1}{3} + \frac{2}{5}\sqrt{5x+7}$.

d. $f(x) = 2\sqrt{8x-3} - 3$.

Q5. Sketch the following curves, stating all relevant features.

a. $f(x) = \frac{2}{3x+7} - 2$.

b. $f(x) = \frac{1}{2} + \frac{3}{5x+2}$.

c. $f(x) = \frac{4}{5-x} - 6$.

d. $f(x) = 1 - \frac{3}{x}$.

Q6. Sketch the following curves, stating all relevant features.

a. $f(x) = \frac{1}{(x+3)^2} + 3$.

b. $f(x) = -\frac{1}{(x+4)^2}$.

c. $f(x) = \frac{6}{(3-x)^2} - \frac{1}{4}$.

d. $f(x) = 1 - \frac{4}{(x-8)^2}$.

Q7. Sketch the curve f defined by

$$f(x) = \frac{3x-1}{2x+5},$$

stating all relevant features, including the domain and range.

Q8. Sketch the curve f defined by

$$f(x) = \frac{1-4x}{x+5},$$

stating all relevant features, including the domain and range.

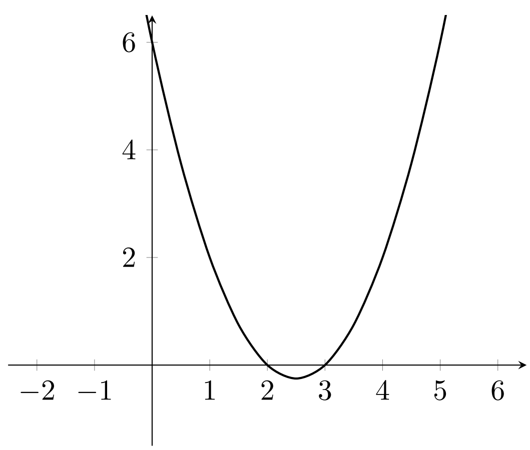
1.6 ⋆ Reciprocal Curves

In this section we discuss the methods which are used to graph reciprocal functions. That is, if we are given a function $f(x)$, we discuss how to sketch the curve $g(x) = \frac{1}{f(x)}$. Typical examples of choices for $f(x)$ are polynomials.

Example 1.6.1. Let $f(x) := x^2 - 5x + 6$.

a. Sketch the graph of $f(x)$.

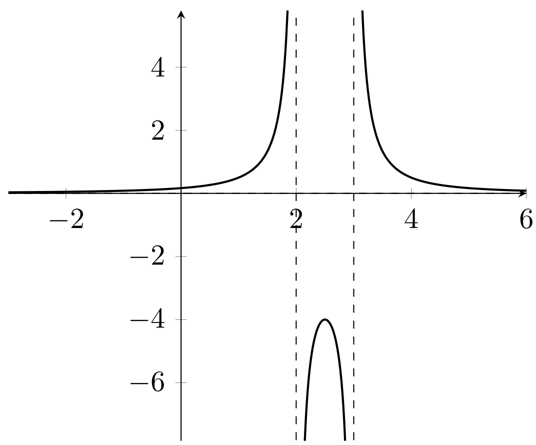
Proof. It is easy to see that the graph of f is given by



where the turning point is given by $\left(\frac{5}{2}, -\frac{1}{4}\right)$. □

b. Hence, sketch the graph of $\frac{1}{f(x)}$.

Proof. We see that since $f(x) = 0$ at $x = 2$ and $x = 3$, $\frac{1}{f(x)}$ is not defined at these points. We therefore draw asymptotes at these points. It is also clear that $\frac{1}{f(x)}$ has a horizontal asymptote at $y = 0$. Moreover, since the turning point of $f(x)$ occurs at $\left(\frac{5}{2}, -\frac{1}{4}\right)$, the turning point of $\frac{1}{f(x)}$ occurs at $\left(\frac{5}{2}, -4\right)$. We therefore see that



□

Exercises

Q1. Sketch the reciprocal curves of the following functions, stating all relevant features.

a. $x^2 - 2x - 3$.
b. $x^2 - 5x - 6$.
c. $x^2 - x - 2$.
d. $x^2 + 6x + 9$.
e. $x^2 + 9x + 18$.
f. $x^2 + 3x - 10$.
g. $x^2 - x - 72$.
h. $x^2 + 2x + 1$.
i. $x^2 - 1$.
j. $x^2 - 9$.
k. $x^2 - 16$.
l. $x^2 - 3$.

Q2. Sketch the reciprocal curves of the following functions, stating all relevant features.

a. $f(x) = x^3 + 3x^2 + 3x + 1$.
b. $f(x) = x^3 - 6x^2 + 5x + 12$.
c. $f(x) = 2x^3 + 8x^2 - 38x + 28$.
d. $f(x) = x^3 - 9x^2 - x + 9$.
e. $f(x) = x^3 + 9x^2 - 108$.
f. $f(x) = x^3 + x^2 - 56x + 144$.

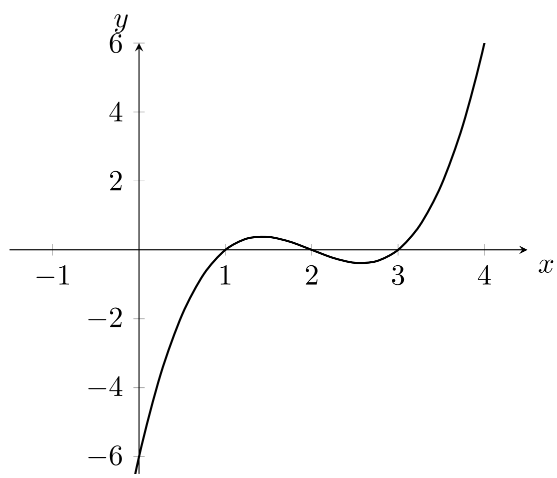
Q3. Sketch the following reciprocal curves, stating all relevant features.

a. $f(x) = x^4 - 14x^3 + 73x^2 - 168x + 144$.
b. $f(x) = x^4 + 2x^3 - 3x^2 - 4x + 4$.

c. $f(x) = x^4 + 2x^3 - 13x^2 - 14x + 24$.

d. $f(x) = 2x^4 + 7x^3 - 68x^2 + 32x$.

Q4. Consider the function $f(x)$ whose graph is given by



Sketch $\frac{1}{f(x)}$.

1.7 Inverse Functions

In this section we look at how to determine the conditions for when a function $f(x)$ has an inverse function $f^{-1}(x)$ and how to calculate $f^{-1}(x)$. That is, a function $f^{-1}(x)$ such that

$$f(f^{-1}(x)) = f^{-1}(f(x)) = x.$$

Recall that for a map of sets to be a function, it needs to satisfy the vertical line test. In other words, the map $y = x^2$ is a function, but the function $y^2 = x$ is not a function. This gives rise to the notion of a function being injective, or one-to-one.

Definition 1.7.1. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be injective if $f(x) = f(y)$ implies that $x = y$.

Example 1.7.2. The function $f(x) = x$ is injective. The function $f(x) = x^2$ is not injective however, since $f(1) = f(-1) = 1$.

A function $f : \mathbb{R} \to \mathbb{R}$ has an inverse function if it is injective.

Example 1.7.3. Let $f : (-\infty, \alpha] \to \mathbb{R}$ be the function defined by

$$f(x) = x^2 - 5x + 6.$$

Determine the value of $\alpha \in \mathbb{R}$ such that f is invertible.

Proof. We clearly need to restrict f to where f is injective. So α is the x-coordinate of the turning point of f. To determine the turning point of f, we complete the square. Indeed, we have

$$x^2 - 5x + 6 \quad = \quad \left(x - \frac{5}{2}\right)^2 - \frac{1}{4}.$$

Hence we see that $\alpha = \frac{5}{2}$. □

Example 1.7.4. Let $f : \left(-\infty, \frac{5}{2}\right] \to \mathbb{R}$ be the function defined by

$$f(x) = x^2 - 5x + 6.$$

Determine the inverse function $f^{-1}(x)$.

Proof. We begin by exchanging x and y, which gives us

$$x = y^2 - 5y + 6.$$

We then complete the square

$$x = \left(y - \frac{5}{2}\right)^2 - \frac{1}{4}.$$

Solving for y, we see that

$$\begin{aligned}
\left(y - \frac{5}{2}\right)^2 - \frac{1}{4} = x \quad &\Longrightarrow \quad \left(y - \frac{5}{2}\right) = \sqrt{x + \frac{1}{4}} \\
&\Longrightarrow \quad y - \frac{5}{2} = \frac{1}{2}\sqrt{4x+1} \\
&\Longrightarrow \quad y = \frac{5}{2} + \frac{1}{2}\sqrt{4x+1}.
\end{aligned}$$

So the inverse function is given by

$$f^{-1}(x) = \frac{5}{2} + \frac{1}{2}\sqrt{4x+1}.$$

□

Exercises

Q1. Determine the domain on which the following functions are injective and determine the inverse functions $f^{-1}(x)$ on these domains.

a. $f(x) = x + 3$.

b. $f(x) = 2x - \frac{1}{3}$.

c. $f(x) = x^2 + 5$.

d. $f(x) = (x - 3)^2$.

e. $f(x) = (x - 5)^2 + 9$.

f. $f(x) = \frac{3}{7}(x + 2)^2 + 11$.

Q2. Determine the domain on which the following functions are injective and determine the inverse functions $f^{-1}(x)$ on these domains.

a. $x^2 - 2x - 3$.

b. $x^2 - 5x - 6$.

c. $x^2 - x - 2$.

d. $x^2 + 6x + 9$.

e. $x^2 + 9x + 18$.

f. $x^2 + 3x - 10$.

g. $x^2 - x - 72$.

h. $x^2 + 2x + 1$.

i. $x^2 - 1$.

j. $x^2 - 9$.

k. $x^2 - 16$.

l. $x^2 - 3$.

1.8 Composite Functions

In this section we look at how to compose functions. We begin by recalling some elementary definitions.

For a function $f : X \to Y$, we refer to X as the domain and Y as the codomain. We often write $\mathcal{D}(f)$ for the domain of f. Moreover, the range of f, denoted by $\mathcal{R}(f)$, is defined as

$$\mathcal{R}(f) = \{y \in Y \mid y = f(x), \text{ for } x \in X\}.$$

It should be clear that the range is a subset of the codomain.

The general idea behind solving these problems is that if we have two functions f and g, then to define $f(g(x))$, we have

$$\mathcal{D}(f(g(x))) = \mathcal{D}(g) \text{ such that } \mathcal{R}(g) \subset \mathcal{D}(f).$$

So, the range of g has to fit into the domain of f. We illustrate this with some examples.

Example 1.9.1. Let $f(x) = \sqrt{3x+2} + 1$ and $g(x) = 9 - x^2$.

a. Determine the domains and ranges of f and g.

Proof. The domain of a square root function is given by ensuring that the inside of the square root is non-negative. We therefore simply have to solve the inequality

$$\begin{aligned} 3x + 2 \geq 0 \quad &\implies \quad 3x \geq -2 \\ &\implies \quad x \geq -\frac{2}{3}. \end{aligned}$$

The domain of f is therefore

$$\mathcal{D}(f) = [-2/3, \infty).$$

The range of f is given by

$$\mathcal{R}(f) = [1, \infty).$$

The function g is a polynomial. The domain of g is therefore

$$\mathcal{D}(g) = \mathbb{R}.$$

The range of g is
$$\mathcal{R}(g) = (-\infty, 9].$$
$\square$

b. Determine the maximal domain of f such that $g(f(x))$ is well defined.

Proof. For the function $g(f(x))$ to be well defined, we need to ensure that
$$\mathcal{R}(f) \subseteq \mathcal{D}(g).$$
Since $[1, \infty) \subseteq \mathbb{R}$, the function $g(f(x))$ is well defined with domain $[1, \infty)$. $\square$

c. Determine the equation of $g(f(x))$.

Proof. We simply observe that
$$g(f(x)) = 9 - (\sqrt{3x+2})^2 \quad = \quad 1 - 3x.$$
Note that if we simply computed the formula, it would seem that the function is defined for all $x \in \mathbb{R}$. In part (a) however, we showed that the domain of $g(f(x))$ was the domain of f, and this was $[-2/3, \infty)$. $\square$

d. Determine the range of $g(f(x))$.

Proof. The domain of $g(f(x))$ was determined to be $[-2/3, \infty)$. Inserting $-2/3$ into $g(f(x))$ yields
$$g(f(-2/3)) \quad = \quad 1 - 3\left(-\frac{2}{3}\right) = 1 + 2 = 3.$$
The range of $g(f(x))$ is therefore
$$\mathcal{R}(g \circ f) = (-\infty, 3].$$
$\square$

Exercises

Q1. Determine the domains and ranges of the following functions.

a. $f(x) = 3x + 5$.

b. $f(x) = x^2 - 5x + 6$.

c. $f(x) = \frac{1}{x}$.

d. $f(x) = \frac{2}{4x+1}$.

e. $f(x) = \frac{7}{9x-5} - 4$.

f. $f(x) = \frac{3x+1}{4x-6}$.

g. $f(x) = \frac{5x+2}{4x-5}$.

h. $f(x) = \frac{1-x}{x+1}$.

i. $f(x) = \sqrt{3x+1}$.

j. $f(x) = 3\sqrt{6x-4} + 2$.

k. $f(x) = 6\sqrt{3-x} + 4$.

l. $f(x) = \frac{2}{3}\sqrt{3 - \frac{1}{2}x} + 8$.

m. $f(x) = \frac{1}{x^2}$.

n. $f(x) = \frac{1}{(x-3)^2} + 5$.

o. $f(x) = \frac{2}{(x+1)^2} - 4$.

p. $f(x) = \frac{3}{x^2-5x+6} - 8$.

Q2. Determine the domains of the following functions.

a.
$$f(x) = \frac{1}{\sqrt{x^2 - 5x + 6}}.$$

b.
$$f(x) = \frac{1}{\sqrt{x^2 + 2x + 1}}.$$

c.
$$f(x) = \frac{7}{x^3 - 6x^2 + 11x - 6}.$$

d.
$$f(x) = \frac{3}{x^3 + 10x^2 + 17x - 28}.$$

e.
$$f(x) = \frac{5}{\sqrt{x^3 + 7x^2 - 17x + 9}}.$$

Q3. Let $f(x) = 3x + 5$ and $g(x) = x^2 - 5x + 6$.

a. Determine the maximal domain of f such that $g(f(x))$ is well defined.

b. Determine the equation of $g(f(x))$.

c. Determine the range of $g(f(x))$.

d. Determine the maximal domain of g such that $f(g(x))$ is well defined.

e. Determine the equation of $f(g(x))$.

f. Determine the range of $f(g(x))$.

Q4. Let $f(x) = 1/x$ and $g(x) = \frac{2}{4x+1}$.

a. Determine the maximal domain of f such that $g(f(x))$ is well defined.
b. Determine the equation of $g(f(x))$.
c. Determine the range of $g(f(x))$.
d. Determine the maximal domain of g such that $f(g(x))$ is well defined.
e. Determine the equation of $g(f(x))$.
f. Determine the range of $f(g(x))$.

Q5. Let $f(x) = \frac{7}{9x-5} - 4$ and $g(x) = \frac{3x+1}{4x+6}$.

a. Determine the maximal domain of f such that $g(f(x))$ is well defined.
b. Determine the equation of $g(f(x))$.
c. Determine the range of $g(f(x))$.
d. Determine the maximal domain of g such that $f(g(x))$ is well defined.
e. Determine the equation of $f(g(x))$.
f. Determine the range of $f(g(x))$.

Q6. Let $f(x) = \frac{1-x}{1+x}$ and $g(x) = \sqrt{3x+1}$.

a. Determine the maximal domain of f such that $g(f(x))$ is well defined.
b. Determine the equation of $g(f(x))$.
c. Determine the range of $g(f(x))$.
d. Determine the maximal domain of g such that $f(g(x))$ is well defined.
e. Determine the equation of $f(g(x))$.
f. Determine the range of $f(g(x))$.

Q7. If the function f is defined by $f(x) = \sqrt{x^2 - 9}$ and the function g is defined by $g(x) = x + 5$.

a. Determine the integers $c, d \in \mathbb{Z}$ such that $f(g(x)) = \sqrt{(x+c)(x+d)}$.
b. State the maximal domain for which $f(g(x))$ is defined.

Q8. Let $f(x) = \frac{1}{x}$ and $g(x) = \frac{2}{\sqrt{x+1}}$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.
c. State the rule for $f(g(x))$.
d. Determine the domain and range of $f(g(x))$.

Q9. Let $f(x) = x^2 + 1$ and $g(x) = \frac{1}{x}$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.
c. State the rule for $f(g(x))$.
d. Determine the domain and range of $f(g(x))$.

Q10. Let $f(x) = \frac{1}{\sqrt{x^2+1}}$ and $g(x) = 3x$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.
c. State the rule for $f(g(x))$.
d. Determine the domain and range of $f(g(x))$.

Q11. Let $f(x) = \frac{2x+1}{x-3}$ and $g(x) = x + \sqrt{x-3}$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.
c. State the rule for $f(g(x))$.
d. Determine the domain and range of $f(g(x))$.

Q12. Let $f(x) = \frac{1}{1+\sqrt{x}}$ and $g(x) = 2x + 1$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.
c. State the rule for $f(g(x))$.

d. Determine the domain and range of $f(g(x))$.

Q13. Let $f(x) = x^2 - 5$ and $g(x) = \frac{1}{(x-3)^2}$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.
c. State the rule for $f(g(x))$.
d. Determine the domain and range of $f(g(x))$.

Q14. Let $f(x) = 2x - \frac{1}{x}$ and $g(x) = \frac{1}{x}$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.
c. State the rule for $f(g(x))$.
d. Determine the domain and range of $f(g(x))$.

Q15. Let $f(x) = 4x - 2\sqrt{x^2 + 3}$ and $g(x) = \frac{4x+1}{3x-2}$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.
c. State the rule for $f(g(x))$.
d. Determine the domain and range of $f(g(x))$.

Q16. Let $f(x) = 2x + |x + 3|$ and $g(x) = 2x + \sqrt{x}$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.
c. State the rule for $f(g(x))$.
d. Determine the domain and range of $f(g(x))$.

Q17. Let $f(x) = \frac{1}{|x|}$ and $g(x) = 2x + \frac{1}{5+x}$.

a. State the domain and range of $f(x)$ and $g(x)$.
b. State the maximal domain of $g(x)$ such that $f(g(x))$ is well-defined.

c. State the rule for $f(g(x))$.

d. Determine the domain and range of $f(g(x))$.

Q18. Determine the domain of the function

$$f(x) := \frac{1}{\sqrt{x^2 - 5x + 6}}.$$

Q19. Determine the domain of the function

$$f(x) := \frac{1}{x^2 + 2x + 1}.$$

Q20. Determine the domain of the function

$$f(x) := \frac{1}{\sqrt{x^2 - 5x - 6}}.$$

Q21. Determine the domain of the function

$$f(x) = \frac{2x + 1}{\sqrt{x^2 + 6x + 9}}.$$

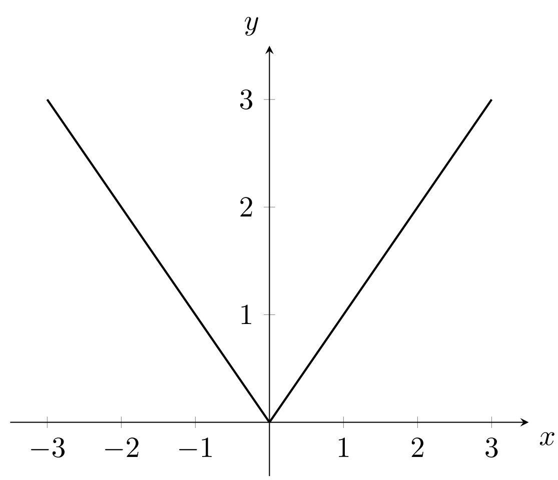
1.9 Piecewise Functions and the Absolute Value

In this section we study functions of the form

$$f(x) = \begin{cases} g(x), & x_1 \leq x \leq x_2, \\ h(x), & x_2 < x \leq x_3, \end{cases}$$

where $x_1, x_2, x_3 \in \mathbb{R}$. The most important example of such a piecewise function is the absolute value function

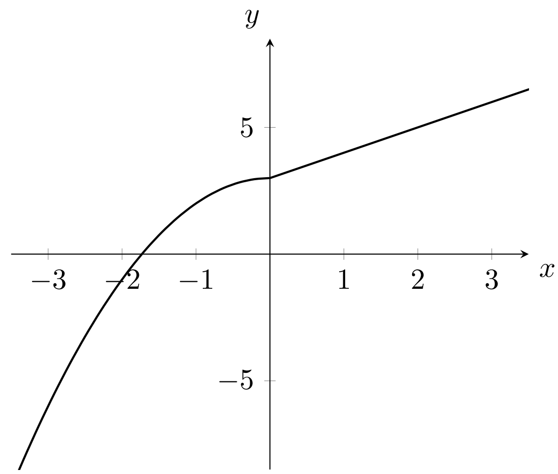
$$f(x) := \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$



Example 1.9.1. Sketch the function defined by

$$f(x) = \begin{cases} x + 3, & x \geq 0, \\ 3 - x^2, & x < 0. \end{cases}$$

Proof. We simply oberve that



$\square$

Example 1.9.2. Sketch the function

$$f(x) := |x+1| + |x-1|\,.$$

Proof. Let us begin by observing that

$$\begin{aligned} |x+1| &= \begin{cases} x+1, & x+1 \geq 0, \\ -x-1, & x+1 < 0, \end{cases} \\ &= \begin{cases} x+1, & x \geq -1, \\ -x-1, & x < -1. \end{cases} \end{aligned}$$

Similarly, we have

$$\begin{aligned} |x-1| &= \begin{cases} x-1, & x-1 \geq 0, \\ 1-x, & x-1 < 0, \end{cases} \\ &= \begin{cases} x-1, & x \geq 1, \\ 1-x, & x < 1. \end{cases} \end{aligned}$$

We therefore have three regions to consider: $x < -1$, $-1 < x < 1$, and $x > 1$.

† On the region $x < -1$, we see that $|x+1| = -x-1$ and $|x-1| = 1-x$. So
$$f(x) = |x+1| + |x-1| = -x-1+1-x = -2x.$$

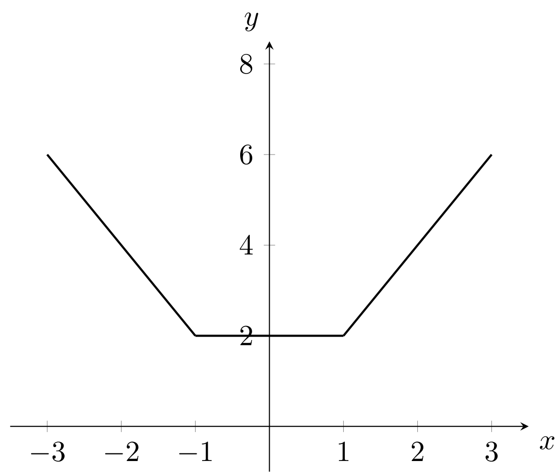
† On the region $-1 < x < 1$, we see that $|x+1| = x+1$ and $|x-1| = 1-x$. So

$$f(x) = x + 1 + 1 - x = 2.$$

† On the region $x > 1$, we see that $|x+1| = x+1$ and $|x-1| = x-1$. So

$$f(x) = x + 1 + x - 1 = 2x.$$

Hence we see that the graph is given by



□

Exercises

Q1. Sketch the following curves, stating all relevant features.

a. $f(x) = |x+3| - 1$.
b. $f(x) = 2|x-4| + 5$.
c. $f(x) = |x+9| + 1$.
d. $f(x) = \frac{1}{2}|2x-3|$.
e. $f(x) = |3-4x| + 1$.
f. $f(x) = 1 - |5-6x|$.

Q2. Sketch the following curves, stating all relevant features.

a. $f(x) = \left|x^2 - 9\right| - 3$.
b. $f(x) = \frac{1}{2}\left|x^2 + 3\right| + 1$.
c. $f(x) = 4\left|7 - x^2\right| + 5$.
d. $f(x) = 2\left|x^2 + 2x + 1\right| - 3$.
e. $f(x) = \frac{3}{2}\left|x^2 + 2x - 15\right| + 2$.
f. $f(x) = \left|x^2 - 5x - 6\right|$.

Q3. Sketch the following curves, stating all relevant features.

a. $f(x) = 2\left|x^3 + 4x^2 - 11x - 30\right| - 4$.
b. $f(x) = \frac{5}{7}\left|x^3 + 5x^2 - 18x - 72\right| - 4$.
c. $f(x) = \sqrt{3}\left|x^3 - 7x^2 - 6x + 72\right| + 6$.
d. $f(x) = \left|x^3 - 16x^2 + 55x + 72\right|$.

Q4. Sketch the curve defined by

$$f(x) = \left|\frac{3}{x+1} - 4\right| + 4,$$

stating all relevant features.

Q5. Sketch the following curves, stating all relevant features.

a. $f(x) = |x+1| - |x|$.
b. $f(x) = |x+3| + 2|x-1|$.
c. $f(x) = |x+4| - \frac{1}{2}|x-5|$.
d. $f(x) = \left|x^2 - 5x\right| - |x+1|$.

Q6. Sketch the function defined by

$$f(x) = \begin{cases} 2x - 3, & x > -1, \\ -x^2, & \text{otherwise.} \end{cases}$$

Q7. Sketch the function defined by

$$f(x) = \begin{cases} x + 6, & x < 4, \\ x, & 4 \leq x \leq 5, \\ -\sqrt{x}, & \text{otherwise.} \end{cases}$$

1.10 Revision Exercises

Q1. Solve the system of equations given by

$$\begin{cases} 2x + 4 = 2y, \\ 1 - y = -2x \end{cases}$$

Q2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 - 2x - 35.$$

a. Sketch the graph of f on a suitable domain, stating all relevant features.

b. Determine the value of $a \in \mathbb{R}$ such that $f : [a, \infty) \to \mathbb{R}$ is invertible and determine $f^{-1}(x)$.

c. State the domain and range of $f^{-1}(x)$.

Q3. Consider the function f defined by

$$f(x) := a + \sqrt{b + cx},$$

where $a, b, c \in \mathbb{R}$. Suppose that f passes through $\left(-\frac{1}{5}, \sqrt{3} - 3\right)$ and has a vertex at $y = -3$. Moreover, the y-intercept is given by $y = -1$. Determine the values of $a, b, c \in \mathbb{R}$.

Q4. Let f be the straight line described by

$$f(x) = 3 - x.$$

Determine the angle $0 \leq \vartheta \leq 2\pi$ made between f and the x-axis.

Q5. Let $f(x) = 2x+1$ and $g(x)$ be the line that is perpendicular to f which passes through $(1, 2)$. Determine the equation for g.

Q6. Let $f : \mathbb{R} \to \mathbb{R}$ be the curve defined by

$$f(x) = x^2 - 5x - 6.$$

a. By first sketching the curve of f if necessary, determine the value of $a \in \mathbb{R}$ such that $f : [a, \infty) \to \mathbb{R}$ is invertible.

b. Determine the inverse function of f, $f^{-1}(x)$.

c. On the same pair of axes, sketch the graph of $f^{-1}(x)$.

Q7. Let f be the function defined by

$$f(x) = \frac{3x + 1}{4x - 5}.$$

Sketch the curve of $f(x)$, stating all relevant features, including the domain and range.

Q8. Let $\widetilde{f}$ be the function defined by

$$\widetilde{f}(x) := \sqrt{\frac{3x + 1}{2x - 5}}.$$

Determine the domain and range of $\widetilde{f}$.

Q9. Let $f : \mathbb{R} \to \mathbb{R}$ be the curve defined by

$$f(x) = x^3 + x^2 - 4x - 4.$$

Determine the roots of $f(x)$, and hence, sketch the curve of f on an appropriate domain, stating all relevant features.

Q10. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x^{\frac{1}{3}}.$$

Sketch the curve of f on a suitable domain, stating all relevant features.

Q11. Sketch the curve of f, where f is defined by

$$f(x) = \frac{3x+1}{5x+9},$$

state all relevant features.

Q12. Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be the function defined by $f(x) = \sqrt{x}$. State the transformations necessary to map f to

$$\widetilde{f}(x) = \frac{1}{3}\sqrt{4x+1} - \frac{4}{7}.$$

Q13. Solve the inequality

$$\left|x^2 + 4x\right| \geq 1,$$

for $x \in \mathbb{R}$.

Q14. Consider the system of equations

$$\begin{cases} (\lambda - 2)x + 3y = 6, \\ 2x + (\lambda - 3)y = \lambda - 1, \end{cases}$$

parameterised by $\lambda \in \mathbb{R}$. Determine the value(s) of $\lambda \in \mathbb{R}$ such that the above system has only one solution.

Q15. Let $g : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$g(x) := \left|x^2 - 4\right| - 2.$$

Sketch the curve of g, stating all relevant features.

Q16. Let f be the function defined by

$$f(x) = \frac{1}{x} + 3,$$

and g be the function defined by

$$g(x) = |x + 2|.$$

Determine the maximal domain of g such that $f(g(x))$ is well-defined and state the equation for $f(g(x))$.

Q17. Let f be the function defined by

$$f(x) = -\frac{2}{(x-3)^2} - \frac{3}{5}.$$

Sketch the curve of f, stating all relevant features, including the domain and range.

Q18. Set $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ to be the function defined by

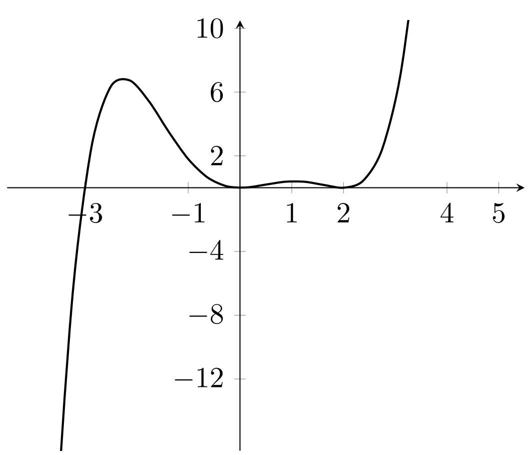
$$f(x) := \frac{4x+1}{x-3}.$$

a. Determine the domain on which f is invertible.

b. Determine the equation for the inverse function $f^{-1}(x)$ and verify that f^{-1} is at least a one-sided inverse for f.

c. Sketch the graph of $f^{-1}(x)$ on its maximal domain.

Q19. Let $f(x)$ be the function whose graph is given by



On the above set of axes, sketch the curve $f(|x|)$.

Q20. Let f be the function defined by $f(x) := \sqrt{x+1}$ and g be the function defined by $g(x) = \frac{1}{x-3} + 2$.

a. Determine the maximal domain of g such that $f(g(x))$ is well-defined.

b. State the equation of $f(g(x))$.

c. Determine the range of $f(g(x))$.

Q21. Determine the value(s) of $k \in \mathbb{R}$ such that $f(x) = x^2 - kx$ intersects $g(x) = x + 1$ twice.

Q22. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) := x^4 - x^3 - 9x^2 + 9x.$$

Sketch the curve of f on a reasonable domain.

Q23. State the domain of the function

$$g(x) := \sqrt{\frac{1-4x}{x+2}} + \frac{1}{\sqrt{x^2+2x+1}}.$$

Q24. Let $f(x) = (kx+1)^2$ and $g(x) = 2x - k$. Determine the value(s) of $k \in \mathbb{R}$ such that f intersects g twice.

Q25. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = 2\,|6x - 5| + \frac{1}{5}.$$

Sketch the curve of f, stating all relevant features.

Q26. Describe the transformations necessary to map

$$f(x) := -3(2x+1)^2 - 4$$

to $\widetilde{f}(x) = x^2$.

Q27. Let f be the function defined by $f(x) = x^2 + \frac{1}{2x}$ and g be the function defined by $g(x) = \frac{1}{x^2} + \frac{1}{\sqrt{x}}$.

a. State the domain and range of f and g.

b. State the maximal domain on which g may be defined such that $f(g(x))$ is well-defined.

c. Write an explicit rule for $f(g(x))$.

Q28. Let f be the function defined by

$$f(x) = \sqrt{2x+1} - 4.$$

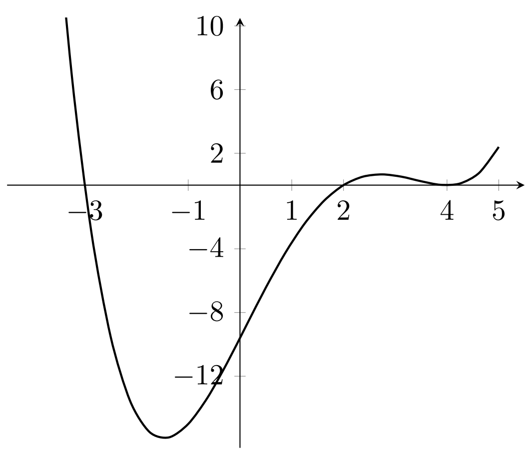
State the transformations necessary to map f to

$$\widetilde{f}(x) = \frac{1}{3}\sqrt{4x+1} - 4.$$

Q29. Let $f(x) = x^2$ and p be the point $(3,0)$. Determine the values of $a, b, c \in \mathbb{R}$ such that the distance between $f(x)$ and p is given by the function

$$d(x) = \sqrt{x^4 + ax^2 + bx + c}.$$

Q30. Determine the equation of the curve $f : \mathbb{R} \to \mathbb{R}$ whose graph is given by



Q31. Sketch the curve defined by

$$f(x) = -\frac{2}{(x-1)^2} - 1,$$

stating all relevant features.