CONFORMAL MAPPING PROBLEM

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Claim 1. Let M be a compact (connected) Riemann surface with smooth boundary ∂M. Then there is an arc $\Gamma \subset \partial M$ and a holomorphic immersion $f : M \to \mathbb{C}$ such that $f(\Gamma) \subset \mathbb{R}$.

Argument. First, by compactness, M has only finitely many boundary components. We can close these off and assume that M has only one boundary component, which can be identified with an analytic circle S. The open set Γ is now an arc in S. The boundary of M is smooth, so we can find an open Riemann surface $\widetilde{M} \supset M$. An old result of Gunning–Narasiman [1] yields a holomorphic immersion $f : \widetilde{M} \longrightarrow \mathbb{C}$. Now $f|(M \setminus \partial M)$ is open and so the boundary of this set lies in the image of the boundary. Let α be any component of the boundary of $f(M)$. Smoothness assures us that α separates the complex plane into simply connected regions (in the Riemann sphere). One of these contains $f(M)$ and we choose the Riemann mapping from this component to the upper-half space. The composition is now a conformal immersion one boundary component of which is the real line. The arc α can be any one of the finitely many components of the preimage of the real line.
Joint work with Yilin Ma.

Problem 1: If a compact Riemann surface M with smooth boundary is given (so one could interpret this as a bounded domain on a open Riemann surface), then we want to construct a holomorphic function without critical point which is purely real on a prescribed subset of the boundary.

We will consider the following (slight simpler problem):

Problem 2: Let Ω be a bounded domain in $\mathbb{C}$ and $\Gamma \subset \partial\Omega$ some non-empty open subset of the boundary of Ω. Construct a holomorphic function $f : \Omega \to \mathbb{C}$ without critical points with $f(\Gamma) \subset \mathbb{R}$.

Proof in the simply connected case: Assume Ω is simply connected. By the Riemann mapping theorem, we can assume that Ω is the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and that Γ is an arc $\Gamma_{\alpha\beta} = \{z = e^{i\vartheta} : \alpha < \vartheta < \beta\}$. The unit disk can be conformally mapped to upper half-plane $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$, taking the boundary of $\mathbb{D}$ to the boundary of $\mathbb{H}$, i.e., the real line.[1]

Proof in the case of annuli (and disjoint unions of concentric annuli). Let

$$A_{r_1,r_2} = \{z \in \mathbb{C} : r_1 < |z| < r_2\}.$$

Invert the exponential map $z \mapsto \exp(z)$ to get a vertical strip

$$\{z \in \mathbb{C} : e^{r_1} < \mathrm{Re}(z) < e^{r_2}\}.$$

Rotate by $\sqrt{-1}$, to get a horizontal strip

$$\{z \in \mathbb{C} : e^{r_1} < \mathrm{Im}(z) < e^{r_2}\}$$

and translate down accordingly to map the boundary portion to the real line.
The same argument applies to disjoint unions of concentric annuli.

Comments from Forstneric: If Ω is a finitely connected planar domain, you can map it conformally onto a slit domain whose ends (boundary) are straight line segments; hence, you can map (at least) one boundary components to the real line. However, this map will not be smooth or an immersion up to the boundary.
Perhaps one can use some ad hoc method if the set E is short enough and you start by applying the Gunning-Narasimhan theorem on a somewhat bigger Riemann surface. For example, if the image of E by such a map is an arc then it seems somewhat hopeful, but it seems delicate.

Gunning–Narasimhan statement: Any (connected) open Riemann surface X admits a holomorphic immersion into the complex plane.[2]

The argument is simple enough: On X there is a non-vanishing holomorphic 1–form ω. Indeed, the vanishing of $H^2(X, \mathbb{Z})$ and the fact that X is Stein implies that the analytic tangent bundle is trivial. In fact, every vector bundle on an open Riemann surface is trivial. This form is closed since X is a surface and you want to show that ω is exact. So they produce a holomorphic function g such that e^g kills the period of ω and so $e^g\omega = df$. So the desired function is just $f = \int_{x_0}^{x} e^g\omega$.

[1] Lift the unit disk onto the Riemann sphere, rotate it to be on centre of the back end of the Riemann sphere. Enlarge the disk to cover the entire back end of the Riemann sphere, and stereographically project forward.

[2] Interestingly, it remains unknown whether an open Riemann surface can be holomorphically embedding into $\mathbb{C}^2$.

Conformal maps of multiply-connected domains. Let us briefly remind ourselves of the ideas behind the proof of the Riemann mapping theorem. Let D be a simply connected domain in the Riemann sphere which is not all of $\mathbb{C}$. The Riemann mapping theorem gives us a conformal map $f : D \to \mathbb{D}$ onto the unit disk.
Here we will prove that any multiply connected domain (including simply connected domains) can be conformally mapped onto a parallel slit domain of inclination ϑ. We first review some well-known facts concerning univalent functions.

Definition. A holomorphic function (or more generally, meromorphic function) is said to be *univalent* if it is injective.

Remark. A univalent function can have at worst a single simple pole singularity, otherwise it would attain ∞ more than once, violating univalency.

Using the Koebe distortion theorems, one can prove the following:

Lemma.

(i) The family of holomorphic univalent functions in the unit disk is normal and compact.
(ii) A convergent sequence of univalent functions converges to a univalent function or a constant.

One may normalise univalent functions in $\mathbb{D}$ by the requirement that $f'(0) = 1$, which in particular, excludes the possibility of constant limits.

Definition. A domain is said to be *Schlicht* if it not self-overlapping and has no branch points.
The image of a univalent function is a Schlicht domain.

Theorem. Any multiply connected domain can be conformally mapped onto a parallel slit domain of inclination ϑ.

In the simply connected case, this is easy enough. Indeed, take D to be simply connected and map it to the interior of the unit disk. Then the function

$$\varphi(z) = \frac{1}{1-|\zeta|^2}\left[\frac{1-\overline{\zeta}z}{z-\zeta} + e^{2i\vartheta}\left(\frac{z-\zeta}{1-\zeta z}\right)\right]$$

maps $\mathbb{D}$ conformally onto the parallel slit domain of inclination ϑ.

We will establish the existence of a conformal map from a multiply connected domain onto a parallel slit domain of inclination ϑ by establishing the existence of a maximiser of a particular variational problem. Let S_ζ be the class of functions f which are univalent in D and have a simple pole of residue 1 at $z = \zeta$ in D. Hence, near $z = \zeta$, we can write

$$f(z) = \frac{1}{z-\zeta} + a_0 + a_1(z-\zeta) + \cdots .$$

If $\zeta = \infty$, we write

$$f(z) = z + a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots .$$

Variational problem: Within the class S_ζ, consider the problem of maximising $\mathrm{Re}\{e^{-2i\vartheta}a_1\}$, where a_1 is as above.

By the normalisations that we have taken, constant limits of univalent functions are exclused and consequently, the class S_ζ is compact. Hence, we have a univalent function

$$F(z) = z + A_0 + \frac{A_1}{z} + \cdots$$

such that $\mathrm{Re}\{e^{-2i\vartheta}a_1\} \leq \mathrm{Re}\{e^{-2i\vartheta}A_1\}$. We will show that F maps D onto a parallel slit domain. Indeed, suppose this is not the case. Then the image $F(D)$ has at least one boundary component E_1 which is not a rectilinear slit of inclination ϑ. The complement of E_1 is a simply connected region which can be mapped to the exterior of a rectlinear slit of inclination ϑ. Let

$$p(w) = w + \frac{b_1}{w} + \cdots$$

by the function effecting this map. The composite map

$$p(F(z)) = z + A_0 + \frac{A_1 + b_1}{z} + \cdots$$

takes D onto another Schlitcht domain, and so $p(F)$ belongs to S. Hence,

$$\mathrm{Re}\{e^{-2i\vartheta}(A_1 + b_1)\} \leq \mathrm{Re}\{e^{-2i\vartheta}A_1\},$$

which implies that

$$\mathrm{Re}\{e^{-2i\vartheta}b_1\} \leq 0.$$

Theorem. Let Ω be a domain in $\mathbb{C}$ and $\Gamma \subset \partial\Omega$ some non-empty connected open subset of the boundary. Then there exists a univalent conformal map $f : \Omega \to \mathbb{C}$ with $f(\Gamma) \subset \mathbb{R}$.

Proof. If Ω is simply connected, then the Riemann mapping theorem allows us to map Ω onto the unit disk and Γ to some open arc on the unit circle. Conformally mapping the unit disk to the upper half-plane takes the unit circle to the real axis therefore maps Γ to the real axis along with it.

Assume now that Ω is a multiply connected domain with n boundary components. Since Γ is connected, it is clear that Γ is contained in only one of the boundary components. For a fixed $\zeta \in \Omega$, we let S_ζ be the set of univalent functions on Ω with a residue of 1 at ζ. Let φ be the univalent function which solves the variational problem:

$$\max_{\psi \in S_\zeta} \operatorname{Re}\{e^{-2i\vartheta} a_1\},$$

where a_1 is the coefficient of $(z-\zeta)^{-1}$ in the Laurent expansion. The existence of the solution φ is given by standard compactness results for univalent functions and maps Ω conformally onto a parallel slit domain of inclination ϑ. Since Ω has n boundary components, the boundary of $\varphi(\Omega)$ consists of n parallel slits which we take to be parallel to the real axis. Since Γ is contained in one boundary component of Ω, it is mapped onto a single slit. By translating accordingly, we can ensure that Γ is mapped to the real line. □

Remarks on the general case: Let M be a compact Riemann surface with smooth boundary ∂M. Let $\Gamma \subset \partial M$ be a non-empty connected open subset. Let $\widetilde{M}$ be an open Riemann surface containing M with $\Gamma \subset \widetilde{M}$. The boundary components of M can be assumed, without loss of generality, to be analytic circles, and Γ is an open arc in one of these analytic circles. The holomorphic cotangent bundle of $\widetilde{M}$ is trivial, and so $\widetilde{M}$ admits a non-vanishing holomorphic 1–form ω. We will construct a holomorphic immersion $f : \widetilde{M} \to \mathbb{C}$ with $f(\Gamma) \subset \mathbb{R}$ as the primitive of an appropriate twisting of ω. That is, following Gunning–Narasimhan, we will consider a Runge exhaustion (R_ν) of $\widetilde{M}$ such that, for each $\nu \geq 1$, ω is exact on R_ν, and there exists a holomorphic function f_ν satisfying:

(i) $\|f_{\nu+1}\|_{\overline{R}_\nu} < 2^{-\nu}$;
(ii) $\int_\gamma \exp(f_{\nu+1})\omega = 0$ for all loops $\gamma \subset R_{\nu+1}$;
(iii) $\operatorname{Re}(f_\nu)\operatorname{Im}(\omega) + \operatorname{Re}(\omega)\operatorname{Im}(f_\nu) \to 0$ as $\operatorname{dist}(R_\nu, \Gamma) \to 0$.

To exhaust a domain in Ω in $\mathbb{C}^n$ by compact sets K_ν, we set

$$K_\nu = \left\{ z \in \Omega : |z| \leq \nu,\ \operatorname{dist}(\mathbb{C}^n \backslash \Omega, z) \geq \frac{1}{\nu} \right\}.$$

In this section, we will prove the following theorem:
What remains is to discuss the boundary regularity of this conformal map. To this end, we recall the following classical result concerning boundary regularity of the Riemann map from a simply-connected region to the disk:

Theorem. ([**?**, Theorem 5.3.8]). If $\Omega = \{z \in \mathbb{C} : \rho(z) < 0\}$ is a bounded, simply connected domain with smooth boundary and $f : \Delta(0,1) \to \Omega$ is a conformal map, then $f \in \mathcal{C}^{\infty}(\overline{\Delta(0,1)})$ and $F^{-1} \in \mathcal{C}^{\infty}(\overline{\Omega})$.

□

Claim 2. The Γ in the previous claim cannot be arbitrarily chosen.

Argument. For the purposes of obtaining a contradiction, suppose the arc Γ in the above theorem can be chosen arbitrarily. Let S be the boundary circle as above, and parametrize S by some coordinate ϑ. Let Γ_j be the arc defined by $\Gamma_j := \left\{ 0 \leq \vartheta < (2\pi - \frac{1}{j}) \right\}$, and let $f_j : M \to \mathbb{C}$ be holomorphic immersions with $f_j(\Gamma_j) \subset \mathbb{R}$. It suffices to show that $\{f_j\}$ is a normal family. Indeed, granted this, we can extract a subsequence which converges to some holomorphic immersion $f : M \to \mathbb{C}$ which takes the entire boundary circle to the real line, and this is not possible. To show that $\{f_j\}$ has a convergent subsequence, we note that since M is compact, this family is bounded. It suffices to show that the family is closed. □

References

[1] Gunning, R. C., Narasimhan, R., *Immersion of open Riemann surfaces*, Math. Ann., 174:103–108, 1967.