Chapter 1

Integration Theory.

"*The reader will probably observe the conspicuous absence of a time-honored topic in calculus courses, the "Riemann integral". It may well be suspected that, had it not been for its prestiguous name, this would have been dropped long ago, for (with due respect to Riemann's genius) it is certainly quite clear for any working mathematician that nowadays such a "theory" has at best the importance of a mildly interesting exercise [...]. Only the stubborn conservatism of academic tradition could freeze it into a regular part of the curriculum, long after it had outlived its historical importance.*" - Dieudonné.

"*It is frequently claimed that Lebesgue integration is as easy to teach as Riemann integration. This is probably true, but I have yet to be convinced that it is as easy to learn.*" - T.W. Körner.

We now consider the other main concept in calculus which parallels differentiation, that is, integration. Integration allows us to calculate areas under curves and evaluate infinite sums.

Integration is a method of find the areas under curves, or volumes in three dimensions or even hyper-volumes in more than three dimensions.

Integration is often referred to as 'anti-differentiation'. However, I feel that this terminology is misguided, as it is not by construction that the integral is the opposite of the derivative.

Intuitively, it doesn't make much sense for the opposite of the slope of a tangent line at a given point (the derivative), gives the area under a curve...

The link between differentiation and integration is based heavily on the notion of a limit. The links between the two are seen more precisely in measure theory, advanced probability and classical calculus, i.e. the fundamental theorem of calculus (seen later).

The **indefinite integral** takes the form

$$\int f(x)dx$$

The **definite integral** takes the form

$$\int_a^b f(x)dx$$

The **double integral** takes the form

$$\iint f(x,y)dxdy$$

The **triple integral** etc...

$$\iiint f(x,y,z)dxdydz$$

There other types of integrals, such as surface integrals, iterated integrals, Lebesgue integrals, Itô integrals, Cauchy Integrals, Martinelli-Bochner integrals and much more. For our purposes here, we are only concerned with the indefinite and the definite integral.

1.1 Elementary Computations

We have the following list of integrals, which we invite the reader to parallel with the relevant differentiation formulae.

† $\int x^n dx = \frac{x^{n+1}}{n+1} + \text{constant}$.

† $\int e^{ax+b} = \frac{1}{a}^{ax+b} + \text{constant}$.

† $\int \sin(f(x))dx = -\frac{1}{f'(x)} \cos(f(x)) + \text{constant}$.

† $\int \cos(f(x))dx = \frac{1}{f'(x)} \sin(f(x)) + \text{constant}$.

Let us proceed with some elementary examples.

Example 6.1.1. Evaluate the integral

$$\int \sin\left(\frac{\pi}{3} + x\right) + 2x^5 dx.$$

Proof. We simply observe that

$$\begin{aligned}
\int \sin\left(\frac{\pi}{3} + x\right) + 2x^5 dx &= \int \sin\left(\frac{\pi}{3} + x\right) dx + 2\int x^5 dx \\
&= -\cos\left(\frac{\pi}{3} + x\right) + 2 \cdot \frac{1}{6}x^6 + k \\
&= -\cos\left(\frac{\pi}{3} + x\right) + \frac{1}{3}x^6 + \text{constant}.
\end{aligned}$$

□

Example 6.1.2 Evaluate the integral

$$\int \cos(3x) + e^{-x} dx.$$

Proof. It is easy to see that

$$\begin{aligned}
\int \cos(3x) + e^{-x} dx &= \int \cos(3x) dx + \int e^{-x} dx \\
&= \frac{1}{3}\sin(3x) - e^{-x} + \text{constant}.
\end{aligned}$$

□

To evaluate a definite integral, we simply have a lower bound and an upper bound.

Example 6.1.3. Evaluate the integral

$$\int_0^1 x^2 - 5x + 6dx.$$

Proof. It is obvious that

$$\begin{aligned} \int_0^1 x^2 - 5x + 6dx &= \frac{1}{3}x^3 - \frac{5}{2}x^2 + 6x\Big|_0^1 \\ &= \frac{1}{3} - \frac{5}{2} + 6 \\ &= \frac{23}{6}. \end{aligned}$$

□

Exercises.

Q1. Evaluate the following integrals.

a. $\int xdx$.

b. $\int x^3 - 5x + 1dx$.

c. $\int 4x^3 + 6xdx$.

d. $\int \frac{1}{4x^3}dx$.

e. $\int 4 - x^9 dx$.

f. $\int dx$.

g. $\int 5x - \frac{1}{5x^8} + 2xdx$.

h. $\int 4x^3 - 5x + 1dx$.

Q2. Evaluate the following integrals.

a. $\int \sin(x + \pi)dx$.

b. $\int 4x - 3\cos(x)dx$.

c. $\int 7x + 5x^2 + \cos(-x)dx$.

d. $\int x + \cos(3 - 6x)dx$.

e. $\int \cos(1 + 10x)dx$.

f. $\int 4x^{-5} + \sin(9x + 2)dx$.

Q3. Evaluate the following integrals.

a. $\int \sqrt{x + 3}dx$.

b. $\int x + (x - 3)^{\frac{1}{5}}dx$.

c. $\int x^4 + \frac{1}{4(x-3)^{\frac{3}{2}}}dx$.

d. $\int \sqrt{1 - 6x}dx$.

Q4. Evaluate the following definite integrals.

a. $\int_0^1 x^2 - 4x dx$.

b. $\int_{-\pi}^{\pi} \sin x dx$.

c. $\int_0^{\frac{\pi}{2}} x - \cos(x) dx$.

d. $\int_0^1 4x - e^{-x} + \sqrt{x} dx$.

e. $\int_{-3}^{2} x^2 + \sqrt[3]{x+4} dx$.

f. $\int_{-1}^{1} \frac{1}{e^x} dx$.

Q5. Evaluate the following integral

$$\int_0^{\pi} \tan x \cdot \cos x dx.$$

Q6. By first differentiating the function $f(x) = \tan x$, evaluate the integral

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{3}{7} \sec^2 x + \cos x dx.$$

Q7. Determine the function $f(x)$ that satisfies the following data,

$$\begin{cases} f'(x) = 4x + e^{-x}, \quad \forall x > 0, \\ f(0) = 2. \end{cases}$$

Q8. Determine the function $f(x)$ that satisfies the following data,

$$\begin{cases} f'(x) = \cos\left(\frac{\pi}{3}(1-x)\right) + \sin(\pi x), \quad \forall x > 0, \\ f(4) = -1. \end{cases}$$

Q9. Determine the function $f(x)$ that satisfies the following data,

$$\begin{cases} f''(x) = 2x - e^{-x} + \cos(2x - \pi), \forall x > 0, \\ f'(0) = 1, \\ f(5) = -6. \end{cases}$$

Q10. Determine the function $y = f(x)$ that satisfies the second order differential equations

$$\frac{d^2y}{dx^2} = -4x + \sqrt{x}.$$

Q11. Evaluate the following integrals.

a. $\int \frac{1}{2x-3} dx$.

b. $\int \frac{2}{4-x} dx$.

c. $\int \frac{3}{5(x-6)} dx$.

d. $\int \frac{3}{7-9x} dx$.

e. $\int \frac{2x-3}{2x} dx$.

f. $\int \frac{5}{7-5x} dx$.

Q12. Evaluate the following integrals.

a. $\int \frac{4x-3}{6x+1} dx$.

b. $\int \frac{x-5}{x+7} dx$.

c. $\int \frac{6x+5}{6x+8} dx$.

d. $\int \frac{4x+7}{2-x} dx$.

e. $\int \frac{9-x}{9+x} dx$.

f. $\int \frac{2x+3}{5x-1} dx$.

Q13. Evaluate the following integrals.

a. $\int \frac{x^2-5x+6}{\sqrt{2x}} dx$.

b. $\int \frac{4x^3-5x^2+x}{\sqrt{x}} dx$.

Q14. Find the equation of the curve $f(x)$ with derivative given by

$$f'(x) = \frac{1}{3} \cos\left(\frac{1}{2}(x-\pi)\right),$$

and passes through the point $(2, 3)$.

Q15. Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be the function which has derivative given by

$$f'(x) = \frac{kx + \sqrt{x}}{x^2},$$

where $k \in \mathbb{R}$ is some constant. Suppose that f has a stationary point at $(1, 2)$. Determine the value of k and consequently, a closed form expression for $f(x)$.

Q16. Let $g : \mathbb{R} \to \mathbb{R}$ be the function with derivative given by

$$g'(x) := 4\cos(2x) + ke^{-x},$$

where $k \in \mathbb{R}$ is some constant. Suppose that g has a stationary point at $(0, -1)$. Determine the value f at $x = \frac{\pi}{2}$.

Q17. Suppose that

$$\int_1^5 f(x) dx = 6.$$

Determine the value of

$$\int_1^5 3f(x) + 2x - \frac{1}{x} dx.$$

Q18. Determine the value of $k \in \mathbb{R}$ such that

$$\int_1^k 2x - 3dx = 5.$$

Q19. Determine the value of $k \in \mathbb{R}$ such that

$$\int_{-k}^k e^{-x}dx = 1.$$

Q20. Evaluate the integral

$$\int_0^1 |x-3|\, dx.$$

Q21. Evaluate the integral

$$\int_0^\pi |\sin(x)|\, dx.$$

Q22. Caclulate the area bounded by $f(x) = \sin x$, $g(x) = \cos x$ and the vertical axis.

Q23. Calculate the area bounded by $f(x) = e^x$, $g(x) = \sqrt{x}$ and the vertical axis.

Q24. Thusha is quite frugal in the way he spends his money. Recently however, Thusha started dating Felicity. Upon doing some mathematics one Thursday afternoon, Thusha determines that the cost of maintaining felicity as a girlfriend is given by

$$\frac{d\mathscr{C}}{dt} = 75t^3 + 60t + 300,$$

where $\mathscr{C}$ is the accumulated cost in dollars and t is the time in months since they started dating.

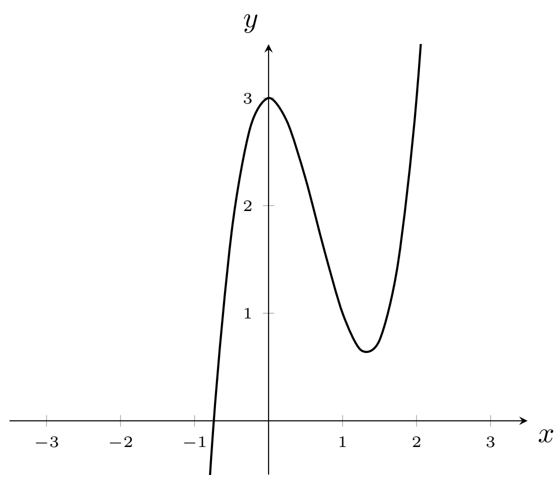
a. Determine how much Felicity took out of Thusha's pocket on the first date.

b. Determine the cost as a function of time.

c. Determine the total cost during the first 6 months of dating.

d. Thusha tells himself that he will break up with Felicity if she takes more than 15,000 dollars from him. At what time does Thusha break up with Felicity, assuming she does not break up with him beforehand?

Q25. Consider the function f whose derivative is given by



Sketch the graph of $f(x)$.

Q26. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) := \begin{cases} 1 - 2x, & 0 < x < 4, \\ 0, & \text{otherwise.} \end{cases}$$

Evaluate the integral

$$\int_{-\infty}^{\infty} f(x)dx.$$

Q27. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} e^{-x}, & -3 \leq x \leq 1, \\ x^2, & 1 < x \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

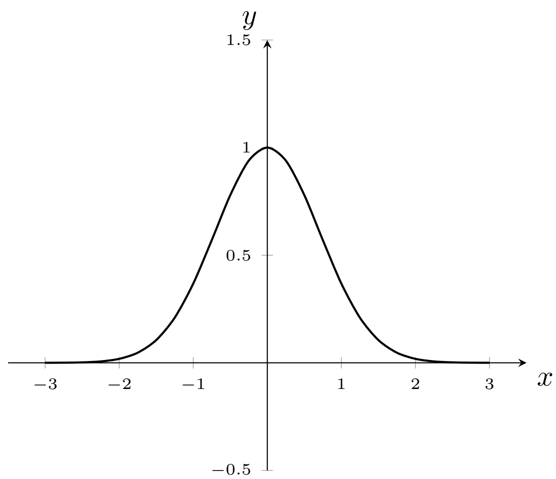
Determine the value of $k \in \mathbb{R}$ such that

$$\int_1^k f(x)dx = \frac{4}{3}.$$

Q28. Determine whether the following is true,

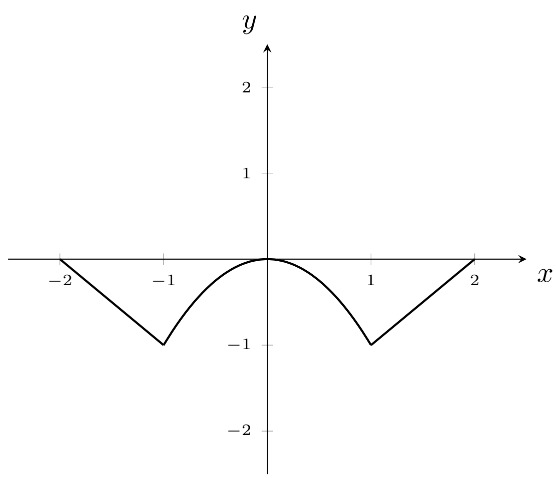
$$\int f(x) \cdot g(x)dx = \int f(x)dx \cdot \int g(x)dx.$$

Q29. Consider the function f whose derivative is given by



Sketch the graph of $f(x)$ and $f''(x)$.

Q30. Consider the function f whose derivative is given by



Sketch the graph of $f(x)$ and evaluate $\int_{-2}^{2} f(x)dx$.

1.2 Integration by Substitution

In this section we look at an integration method that can be used when the derivative of an expression is present in the integrand.

Example 1.10.1. Evaluate the expression

$$\int \frac{2x-5}{\sqrt{x^2-5x+6}}dx.$$

Proof. Notice that the derivative of x^2-5x+6 is $2x-5$, which is the numerator. So, let $u = x^2-5x+6$ such that $du = (2x-5)dx$. We then see that $dx = \frac{du}{2x-5}$ and therefore,

$$\begin{aligned}
\int \frac{2x-5}{\sqrt{x^2-5x+6}}dx &= \int \frac{2x-5}{\sqrt{u}}\frac{du}{2x-5} \\
&= \int \frac{1}{\sqrt{u}}du \\
&= \int u^{-\frac{1}{2}}du \\
&= 2\sqrt{u}+k \\
&= 2\sqrt{x^2-5x+6}+k,
\end{aligned}$$

where of course k is a constant. □

Example 1.10.2. Evaluate the expression

$$\int \frac{\log_e(3x)}{2x}dx.$$

Proof. Notice that the derivative of $\log_e(3x)$ is $\frac{1}{x}$, which is in the above expression. So let $u = \log_e(3x)$ such that $du = \frac{1}{x}dx$. We then see that

$$\begin{aligned}
\int \frac{\log_e(3x)}{2x}dx &= \frac{1}{2}\int \frac{\log_e(3x)}{x}dx \\
&= \frac{1}{2}\int \frac{u}{x}xdu \\
&= \frac{1}{2}\int udu \\
&= \frac{1}{2}\frac{u^2}{2} + k, \\
&= \frac{1}{4}u^2 + k, \\
&= \frac{1}{4}[\log_e(3x)]^2 + k,
\end{aligned}$$

where of course k is a constant. □

Exercises

Q1. Evaluate the following expressions

(a) $\int \frac{2x}{x^2-5}dx$ (b) $\int \frac{x}{\sqrt{3-x^2}}dx$ (c) $\int \frac{2x+6}{(x^2+6x-5)^5}dx$

(d) $\int \frac{2x-9}{\sqrt{x^2-9x+6}}dx$ (e) $\int 3x^2 \cdot (x^3+2)^4 dx$ (f) $\int \frac{x}{(3-x^2)^{\frac{3}{2}}}dx$

Q2. Evaluate the following expressions

(a) $\int x^5 \cdot e^{x^6}dx$ (b) $\int \frac{2\log_e(x)}{x}dx$ (c) $\int \frac{-x^3}{e^{x^4}}dx$

(d) $\int \frac{\log_e(3x)}{7x}dx$ (e) $\int e^x\sqrt{1+e^x}dx$ (f) $\int \frac{\sin(2\log_e(x))}{3x}dx$

Q3. Evaluate the following expressions

(a) $\int \sin(x)\cos^2(x)dx$ (b) $\int 3\tan(x)\sec^2(x)dx$ (c) $\int 4\cos(x)e^{\sin(x)}dx$

(d) $\int 2\tan(x)dx$ (e) $\int \cot(x)dx$ (f) $\int \frac{\sin(x)}{1-\cos(x)}dx$

Q4. Evaluate the following expressions

(a) $\int 2\sin(x)\sqrt{1+\cos(x)}dx$ (b) $\int \frac{3x^2\sin(x^2)}{x}dx$

(c) $\int \frac{\log_e(2x+1)}{x}dx$

Q5. Evaluate the following expressions

(a) $\int \frac{x}{\sqrt{x-1}}dx$ (b) $\int x\sqrt{2+x}dx$ (c) $\int \frac{2-x}{(x+3)^4}dx$

(d) $\int \frac{x^2}{\sqrt{x+1}}dx$ (e) $\int \frac{e^{4x}}{1+e^x}dx$ (f) $\int \frac{(x+1)^2}{\sqrt{x-3}}dx$

Q6. Evaluate the following definite integrals

(a) $\int_0^1 \frac{x^2}{(x-3)^3}dx$ (b) $\int_1^2 \frac{e^{2x}}{1-e^x}dx$ (c) $\int_0^4 x\sqrt{4+x}dx$

Q7. Let $f'(x) = \frac{2\tan^{-1}(x)}{1+x^2} + e^{\tan(x)} \cdot \sec^2(x)$.
Determine $f(x)$ if $f(0) = 1$.

Q8. Evaluate the following expression

$$\int \frac{e^x}{e^{2x} - 5e^x + 6}dx$$

Q9. Evaluate the following definite integral

$$\int (3x-3)\sin(x^2-2x)dx$$

Q10. Find the antiderivative of

$$\frac{2+6x}{\sqrt{4-x^2}}$$

1.3 Integration using Trigonometric Identities

The following trigonometric identities, derived from those seen in the circular functions chapter, can be very useful for simplifying integral involving trigonometric functions.

† $\sin(ax)\cos(ax) = \frac{1}{2}\sin(2ax)$.

† $\sin^2(ax) = \frac{1}{2}(1 - \cos(2ax))$.

† $\cos^2(ax) = \frac{1}{2}(1 + \cos(2ax))$.

Example 6.2.1. Evaluate the following integrals using the above trigonometric identities

a. $\int 5\sin^2(2x)dx$.

Proof. We simply observe that

$$\begin{aligned}
\int 5\sin^2(2x)dx &= 5\int \frac{1}{2}(1-\cos(4x))dx \\
&= \frac{5}{2}\int 1-\cos(4x)\ dx \\
&= \frac{5}{2}\left(x - \frac{1}{4}\sin(4x)\right) + C \\
&= \frac{5x}{2} - \frac{5}{8}\sin(4x) + C.
\end{aligned}$$

□

b. $\int 2\cos^2(\frac{x}{6})dx$.

Proof. We simply observe that

$$\begin{aligned}
\int 2\cos^2(\frac{x}{6})dx &= 2\int \frac{1}{2}(1+\cos(\frac{x}{3}))dx \\
&= \int 1+\cos(\frac{x}{3})\ dx \\
&= x + 3\sin(\frac{x}{3}) + C.
\end{aligned}$$

□

c. $\int \cos(2x)\sin(2x)dx$.

Proof. We simply observe that

$$\begin{aligned} \int \cos(2x)\sin(2x)dx &= \int \frac{1}{2}\sin(4x)dx \\ &= \frac{-1}{8}\cos(4x) + C. \end{aligned}$$

□

Integrals with odd powers of sine and cosine can be simplified using the Pythagorean identity

$$\sin^2(ax) + \cos^2(ax) = 1.$$

By integrals with odd powers of sine and cosine, we refer to integrals such as $\int \sin^3(2x)dx$, $\int \cos^3(ax)\sin(ax)$ or $\int \sin^3(ax)\cos^3(ax)dx$.

Example 6.2.2 Evaluate the following integrals.

a. $\int \sin^3(2x)dx$.

Proof. We simply observe that

$$\begin{aligned} \int \sin^3(2x)dx &= \int \sin(2x)(1-\cos^2(2x))dx \\ \text{let } u &= \cos(2x) \\ \frac{du}{dx} &= \frac{-1}{2}\sin(2x) \\ \therefore dx &= \frac{-2\cdot du}{\sin(2x)} \\ \therefore \int \sin(2x)(1-\cos^2(2x))dx &= \int \sin(2x)(1-u^2)\cdot\frac{-2\cdot du}{\sin(2x)} \\ &= 2\int u^2 - 1\ du \\ &= 2\left(\frac{u^3}{3} - u\right) + C \\ &= \frac{2}{3}u^3 - 2u + C \\ &= \frac{2}{3}\cos^3(2x) - 2\cos(2x) + C. \end{aligned}$$

□

b. $\int \cos^5(x)dx$.

Proof. We simply observe that

$$\begin{aligned}
\int \cos^5(x)dx &= \int \cos(x)\cdot(1-\sin^2(x))\cdot(1-\sin^2(x))dx \\
&= \int \cos(x)\cdot(1-2\sin^2(x)+\sin^4(x))dx \\
&= \int \cos(x)dx - 2\int \cos(x)\cdot\sin^2(x)dx + \int \cos(x)\cdot\sin^4(x)dx
\end{aligned}$$

The first integral is easily evaluated, the other two integrals require a substitution as follows

$$\begin{aligned}
\text{let } u &= \sin(x) \\
\therefore \frac{du}{dx} &= \cos(x) \\
\text{Therefore we have } &= \sin(x) - 2\int \cos(x)\cdot u^2\cdot\frac{du}{\cos(x)} + \int \cos(x)\cdot u^4\cdot\frac{du}{\cos(x)} \\
&= \sin(x) - 2\int u^2\cdot du + \int u^4\cdot du \\
&= \sin(x) - 2\cdot\frac{1}{3}u^3 + \frac{1}{5}\cdot u^5 + C \\
&= \sin(x) - \frac{2}{3}\cdot\sin^3(x) + \frac{1}{5}\cdot\sin^5(x) + C.
\end{aligned}$$

$\square$

c. $\int \cos^3(x)\sin(x)dx$.

Proof. We simply observe that

$$\begin{aligned}
\text{let } u &= \cos(x) \\
\frac{du}{dx} &= -\sin(x) \\
\therefore \int \cos^3(x)\sin(x)dx &= \int u^3\sin(x)\cdot\frac{du}{-\sin(x)} \\
&= \int -u^3du \\
&= \frac{-1}{4}u^4 + C \\
&= \frac{-1}{4}\cdot\cos^4(x) + C
\end{aligned}$$

$\square$

Integrals with powers of sec and tan can simplified using the trigonometric identity

$$1 + \tan^2(ax) = \sec^2(ax).$$

Examples of such integrals include $\int \tan^2(ax)dx$ and $\int \tan^3(x)\sec^2(x)dx$.

Example 6.2.3. Evaluate the following integrals

a. $\int \tan^2(2x)dx$.

Proof. We simply observe that

$$\begin{aligned}\int \tan^2(2x)dx &= \int (\sec^2(2x) - 1)dx \\ &= \frac{1}{2}\tan(2x) - x + C.\end{aligned}$$

$\square$

b. $\int 2\tan(x)\sec^2(x)dx$.

Proof. We simply observe that

$$\begin{aligned}\text{let } u &= \tan(x) \\ \therefore \frac{du}{dx} &= \sec^2(x) \\ \therefore \int 2\tan(x)\sec^2(x)dx &= 2\int u \cdot \sec^2(x) \cdot \frac{du}{\sec^2(x)} \\ &= 2\int u \cdot du \\ &= 2 \cdot \frac{u^2}{2} + C \\ &= u^2 + C \\ &= \tan^2(x) + C\end{aligned}$$

$\square$

Example 6.2.4. Evaluate

$$\int \tan^4(x)dx.$$

Proof. Let us simply observe that

$$\begin{aligned}
\int \tan^4(x)dx &= \int \tan^2(x)\cdot\tan^2(x)dx \\
&= \int \tan^2(x)\cdot(\sec^2(x)-1)dx \\
&= \int \tan^2(x)\cdot\sec^2(x)-\tan^2(x)dx \\
&= \int \left(\tan^2(x)\cdot\sec^2(x)-(\sec^2(x)-1)\right)dx \\
&= \int \tan^2(x)\sec^2(x)-\sec^2(x)+1dx \\
&= \int \tan^2(x)\sec^2(x)dx-\int\sec^2(x)dx+\int 1dx
\end{aligned}$$

The first integral requires a substitution, the other two integrals are standard integrals that we know how to compute.

$$\begin{aligned}
\text{let } u &= \tan(x) \\
\frac{du}{dx} &= \sec^2(x) \\
&= \int u^2\cdot\sec^2(x)\cdot\frac{du}{\sec^2(x)}-\int\sec^2(x)dx+\int 1dx \\
&= \int u^2du-\int\sec^2(x)dx+\int 1dx \\
&= \frac{1}{3}\cdot u^3-\tan(x)+x+C \\
&= \frac{1}{3}\cdot\tan^3(x)-\tan(x)+x+C.
\end{aligned}$$

□

Exercises.

Q1. Evaluate the following expressions

a. $\int \sin^2(5x)dx$ b. $\int 2\cos^2(x+1)dx$ c. $\int 3\sin^2(\frac{x}{2})dx$

d. $\int \sin(2x)\cos(2x)dx$ e. $\int \frac{\tan(x)}{\sec^2(x)}dx$ f. $\int \sin(\frac{x+1}{2})\cos(\frac{x+1}{2})dx$

Q2. Evaluate the following expressions

a. $\int 2\sin^3(2x)dx$ b. $\int 3\cos^3(5x)dx$ c. $\int \sin^5(x)dx$

d. $\int 2\cos^5(2x)dx$ e. $\int \frac{\tan^5(x)}{\sec^5(x)}dx$ f. $\int \sin^3(\frac{x}{3})dx$

Q3. Evaluate the following expressions

a. $\int \tan(x)\sec^2(x)dx$ b. $\int \tan^2(5x)dx$ c. $\int \tan^3(x)dx$

d. $\int \sec^2(x)e^{\tan(x)}dx$ e. $\int \tan^2(x)\sec^4(x)dx$ f. $\int \tan^2(2x)\sec^2(2x)dx$

Q4. Evaluate the following expressions

a. $\int 2\sin(x)\cos^4(x)dx$ b. $\int \sin(\frac{x}{2})\sin(x)dx$ c. $\int \cos(2x)\cos(4x)dx$

d. $\int (1+\tan^2(x))dx$ e. $\int \tan^5(x)\sec^6(x)dx$ f. $\int \tan^4(\frac{x}{7})\sec^4(\frac{x}{7})dx$

Q5. Suppose $n \in \mathbb{N}$, evaluate the following integrals

a. $\int \sin^n(nx)\cos(nx)dx$
b. $\int \sec^2(x)\tan^n(x)dx$
c. $\int \cos^3(\frac{x}{2})\sin^n(\frac{x}{2})dx$

Q6. Suppose $g'(x) = 1 + \tan^3(\frac{x}{5})$. Determine $g(x)$ if $g(\frac{\pi}{3}) = \frac{3}{2}$.

Q7. Evaluate the following integral

$$\int \sec(x)dx$$

[Hint: Multiply the numerator and denominator by $\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$ then make the substitution $u = \sec(x) + \tan(x)$]

Q8. Evaluate the following integral

$$\int \tan^6(x)dx$$

Q9. Suppose that $f'(x) = \sin^3(x)\cos(x)$ and $f\left(\frac{\pi}{3}\right) = 1$. Determine $f(x)$.

Q10. Suppose that $f'(x) = \tan^2(x)$ and $f\left(\frac{\pi}{6}\right) = -\frac{1}{2}$. Determine $f(x)$.

Q11. Let $k \in \mathbb{N}$ be a positive integer. Evaluate the integral

$$\int_0^{\frac{\pi}{3}} \sec^2(x)\tan^k(x)dx.$$

1.4 Integration of Inverse Circular Functions

Listed below are the standard integrals involving the inverse trigonometric functions.

- †
$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\left(\frac{x}{a}\right) + C.$$

- †
$$\int \frac{-1}{\sqrt{a^2 - x^2}} dx = \cos^{-1}\left(\frac{x}{a}\right) + C.$$

- †
$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C.$$

Example 6.3.1. Evaluate the following expressions

a. $\int \frac{1}{\sqrt{9-x^2}} dx$.

Proof. Using the above list of standard integrals, we see that this is an example of the first type of integral, yielding an inverse sine function.

$$\int \frac{1}{\sqrt{9 - x^2}} dx \quad = \quad \sin^{-1}\left(\frac{x}{3}\right) + C.$$

□

b. $\int \frac{2}{4+x^2} dx$.

Proof. Using the above list of standard integrals, we that see that this is an example of the third type of integral, yielding an inverse tan function.

$$\begin{aligned} \int \frac{2}{4 + x^2} dx &= 2\int \frac{1}{4 + x^2} dx \\ &= 2\tan^{-1}\left(\frac{x}{2}\right) + C. \end{aligned}$$

□

Example 6.3.2. Evaluate the following expressions

a. $\int \frac{-2}{\sqrt{3-4x^2}}dx$.

Proof. These integrals require us to manipulate the expression, such that it is in the form of one of the standard integrals we have learned previously.

$$\begin{aligned}
\int \frac{-2}{\sqrt{3-4x^2}}dx &= -2\int \frac{1}{\sqrt{3-4x^2}}dx \\
&= -2\int \frac{1}{\sqrt{4(\frac{3}{4}-x^2}}dx \\
&= -2\int \frac{1}{2\sqrt{\frac{3}{4}-x^2}}dx \\
&= \int \frac{-1}{\sqrt{\frac{3}{4}-x^2}}dx \\
&= -\sin^{-1}\left(\frac{2\sqrt{3}x}{3}\right) + C.
\end{aligned}$$

□

b. $\int \frac{1}{5x^2+9}dx$.

Proof.

$$\begin{aligned}
\int \frac{1}{5x^2+9}dx &= \int \frac{1}{5(x^2+\frac{9}{5})}dx \\
&= \frac{1}{5}\int \frac{1}{x^2+\frac{9}{5}}dx \\
&= \frac{\sqrt{5}}{15}\tan^{-1}\left(\frac{\sqrt{5}x}{3}\right) + C
\end{aligned}$$

□

Example 6.3.3. Evaluate the following expressions

a. $\int \frac{3}{\sqrt{4-(x+1)^2}}dx$.

Proof. We simply observe that

$$\begin{aligned}\int \frac{3}{\sqrt{4-(x+1)^2}}dx &= 3\int \frac{1}{\sqrt{4-(x+1)^2}}dx \\ &= 3\sin^{-1}\left(\frac{x+1}{\sqrt{4}}\right)+C \\ &= 3\sin^{-1}\left(\frac{x+1}{2}\right)+C.\end{aligned}$$

□

b. $\int \frac{1}{\sqrt{-x^2-4x+7}}dx.$

Proof. Before attempting to integrate, we want to express this integral like the integral seen in part (a) of this example. By completing the square, we can write $-x^2-4x+7$ as $11-(x-2)^2$.

$$\begin{aligned}\int \frac{1}{\sqrt{-x^2-4x+7}}dx &= \int \frac{1}{\sqrt{11-(x-2)^2}}dx \\ &= \sin^{-1}\left(\frac{x-2}{\sqrt{11}}\right)+C \\ &= \sin^{-1}\left(\frac{\sqrt{11}(x-2)}{11}\right)+C.\end{aligned}$$

□

Exercises

Q1 Evaluate the following expressions

(a) $\int \frac{2}{\sqrt{4-x^2}}dx$ (b) $\int \frac{-3}{\sqrt{9-x^2}}dx$ (c) $\int \frac{1}{8+x^2}dx$

(d) $\int \frac{1}{\sqrt{5-6x^2}}dx$ (e) $\int \frac{2}{\sqrt{9-2x^2}}dx$ (f) $\int \frac{-5}{7+2x^2}dx$

Q2 Evaluate the following expressions

(a) $\int \frac{x-3}{\sqrt{1-x^2}} dx$ (b) $\int \frac{x+1}{9+x^2} dx$ (c) $\int \frac{1-7x}{\sqrt{2-x^2}} dx$

Q3 Evaluate the following expressions

(a) $\int \frac{2}{\sqrt{9-(x+2)^2}} dx$ (b) $\int \frac{1}{1-(x+8)^2} dx$ (c) $\int \frac{-4}{\sqrt{1-(x-3)^2}} dx$

Q4. Evaluate the following expressions

a. $\int \frac{1}{\sqrt{2-2x-x^2}} dx$ b. $\int \frac{5}{x^2-6x+10} dx$ c. $\int \frac{4}{\sqrt{6-2x-x^2}} dx$

Q5. Evaluate the following expressions

a. $\int \frac{x^2-6}{x^2+1} dx$ b. $\int \frac{x^2+5x}{x^2+2} dx$ c. $\int \frac{x^2-5}{7+x^2} dx$

Q6. Antidifferentiate

$$f(x) = \frac{2}{\sqrt{3-(x-2)^2}} - \frac{5}{2+7x^2}$$

Q7. Determine $f(x)$ if $f'(x) = \frac{3}{16+(x-1)^2}$ and $f(1) = 0$.

Q8. Find the antiderivative of

$$\frac{6+x}{x^2+4}.$$

Q9. Suppose that $f'(x) = \frac{5}{3\sqrt{12-x^2}}$ and $f'(0) = 1$. Determine $f(x)$.

Q10. Evaluate the following integrals.

a. $\int \frac{3+x^2}{1+x^2} dx$.

b. $\int \frac{4x^3+12x-8}{x^2+9} dx$.

c. $\int \frac{x^3-x-5}{x^2+4} dx$.

d. $\int \frac{x^2+7}{x^2+9} dx$.

Q11. By first differentiating the function $f(x) = \sin^{-1}(x^2)$, evaluate the integral

$$\int \frac{3x}{\sqrt{1-x^4}} dx.$$

Q12. By first differentiating the function $f(x) = \cos^{-1}(\sqrt{x})$, evaluate the integral

$$\int \frac{5}{7\sqrt{x-x^2}} dx.$$

Q13. By first differentiating the function $f(x) = \tan^{-1}(\sqrt{x})$, evaluate the integral

$$\int \frac{-\pi}{\sqrt{x} + x^{\frac{3}{2}}} dx.$$

Q14. Let f be the function defined by

$$f(x) := \frac{1}{x} \sin^{-1}(\ln x).$$

a. Determine the values of a and b such that $f : (a, b) \to \mathbb{R}$ is defined on (a, b).

b. Hence, or otherwise, evaluate the integral

$$\int_a^b f(x) dx.$$

1.5 Integration using Partial Fractions

As we saw in previous sections, partial fractions allowed us to simply expressions. While previously they were simplified such that they could be sketched, now we will simplify functions using partial fractions such that they can be integrated.

Example 6.4.1. Evaluate the following using the method of partial fractions

(a) $\int \frac{3x+1}{x^2-5x+6} dx$.

Proof. We simply observe that

$$\begin{aligned} \int \frac{3x+1}{x^2-5x+6} dx &= \int \frac{3x+1}{(x-3)(x-2)} dx \\ &= \int \frac{A}{(x-3)} + \frac{B}{(x-2)} dx \end{aligned}$$

Using the methods discussed in chapter 1, the values of A and B are 10 and -7, respectively. Therefore,

$$\begin{aligned} \int \frac{3x+1}{x^2-5x+6} &= \int \frac{10}{x-3} - \frac{7}{x-2} dx \\ &= 10 \log_e |x-3| - 7 \log_e |x-2| + C. \end{aligned}$$

□

(b) $\int \frac{5x-7}{x^2+2x+1} dx$.

Proof. We simply observe that

$$\begin{aligned} \int \frac{5x-7}{x^2+2x+1} dx &= \int \frac{5x-7}{(x+1)^2} dx \\ &= \int \frac{A}{(x+1)} + \frac{B}{(x+1)^2} dx \\ &= \int \frac{5}{(x+1)} - \frac{12}{(x+1)^2} dx \\ &= 5 \log_e |x+1| + \frac{12}{x+1} + C. \end{aligned}$$

□

Exercises

Q1. Express the following as partial fractions

a. $\frac{4x-1}{(x+1)(x-2)}$. b. $\frac{x+2}{(x-1)(x+1)}$. c. $\frac{5x}{(x-3)(x-5)}$.

d. $\frac{2x}{x^2-5x+6}$. e. $\frac{x-3}{x^2-3x+2}$. f. $\frac{5x-4}{x^2+6x+8}$.

Q2. Express the following as partial fractions

a. $\frac{x+1}{(x+3)^2}$. b. $\frac{5x+2}{(x-5)^2}$. c. $\frac{4x-9}{x^2+2x+1}$.

Q3. Determine the antiderivative of each rational expression in Q1.

Q4. Determine the antiderivative of each rational expression in Q2.

Q5. Evaluate each of the following integrals

a. $\int \frac{3x-1}{(x+1)(x+2)} dx$. b. $\int \frac{x-2}{(x-5)(x+6)} dx$. c. $\int \frac{1}{x^2-9} dx$.

d. $\int \frac{8x+1}{x(x-2)} dx$. e. $\int \frac{7}{x^2+8x+7} dx$. f. $\int \frac{6x-5}{x^2-5x+6} dx$.

Q6. Evaluate the following integral

$$\int \frac{x^2-6x+8}{x^2-3x} dx.$$

Q7. Evaluate the following integral

$$\int \left(\frac{x}{\sqrt{x^2-1}}\right)^2 dx.$$

1.6 ⋆ Integration using Trigonometric Substitutions

The following method will allow us to evaluate integrals of the form

$$\int \sqrt{a^2 - x^2}dx \qquad \int \sqrt{a^2 + x^2}dx \qquad \int \sqrt{x^2 - a^2}dx.$$

To evaluate the integral $\int \sqrt{a^2 - x^2}dx$, we let $x = a\sin(\theta)$. To see why we could make such a substitution, consider that

$$x = a\sin(\theta) \implies dx = a\cos(\theta)d\theta.$$

Therefore, we have that

$$\int \sqrt{a^2 - x^2}dx = \int \sqrt{a^2 - a^2\sin^2(\theta)}(a\cos(\theta))d\theta.$$

Given the Pythagorean identity,

$$\sin^2(\theta) + \cos^2(\theta) = 1,$$

this implies that

$$a^2 - a^2\sin^2(\theta) = a^2\cos^2(\theta).$$

Therefore,

$$\begin{aligned} \int \sqrt{a^2 - a^2\sin^2(\theta)}(a\cos(\theta)d\theta &= \int \sqrt{a^2\cos^2(\theta)}(a\cos(\theta)d\theta \\ &= \int a\,|a| \cdot |\cos(\theta)| \cdot \cos(\theta)d\theta. \end{aligned}$$

Since $\cos^2(\theta) \geq 0$, for all θ and assuming $a \geq 0$, we can simplify the expression to be

$$\int a^2\cos^2(\theta)d\theta.$$

Using the identity

$$\cos(2\theta) = 2\cos^2(\theta) - 1,$$

the intergral simplifies to

$$\begin{aligned} \int a^2\cos^2(\theta)d\theta &= a^2 \int \frac{1}{2}(1 + \cos(2\theta))d\theta \\ &= \frac{a^2}{2}\left(\theta + \frac{1}{2}\sin(2\theta)\right) + C. \end{aligned}$$

Since we made the substitution $x = a\sin(\theta)$, we can see that $\theta = \sin^{-1}(\frac{x}{a})$. Substituting this into the above expression yields that

$$\int \sqrt{a^2 - x^2}dx = \frac{a^2}{2}\left(\sin^{-1}\left(\frac{x}{a}\right) + \frac{1}{2}\sin\left(2\sin^{-1}\left(\frac{x}{a}\right)\right)\right) + C.$$

A table summarising the necessary substitutions for a given intergrand is included below.

Form of Integrand	Substitution	Trigonometric Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\vartheta$	$1 - \sin^2\vartheta = \cos^2\vartheta$
$\sqrt{a^2 + x^2}$	$x = a\tan\vartheta$	$1 + \tan^2\vartheta = \sec^2\vartheta$
$\sqrt{x^2 - a^2}$	$x = a\sec\vartheta$	$\sec^2\vartheta - 1 = \tan^2\vartheta$

Example 6.5.1. Evaluate the integral

$$\int \sqrt{x^2 + 9}\ dx.$$

Proof. Using the table above, it is necessary to make the substitution $x = 3\tan(\theta)$. Therefore, $dx = 3\sec^2(\theta)d\theta$. Substituting this into the integral gives us

$$\int \sqrt{x^2 + 9}\ dx = \int \sqrt{9 + 9\tan^2(\theta)}\ (3\sec^2(\theta))d\theta$$

The following identity $\tan^2(\theta) + 1 = \sec^2(\theta)$, gives us that $9 + 9\tan^2(\theta) = 9\sec^2(\theta)$. Therefore, the above integral simplifies to

$$\begin{aligned}
\int \sqrt{9\sec^2(\theta)}(3\sec^2(\theta))d\theta &= \int 9\sec^3(\theta)d\theta \\
&= 9\int \frac{1}{\cos^3(\theta)}dx \\
&= 9\int \frac{1}{\cos^3(\theta)} \cdot \frac{\cos(\theta)}{\cos(\theta)}d\theta = 9\int \frac{\cos(\theta)}{\cos^4(\theta)}dx \\
&= 9\int \frac{\cos(\theta)}{(1 - \sin^2(\theta))^2}dx
\end{aligned}$$

Let $u = \sin(\theta) \implies du = \cos(\theta)d\theta$. The integral therefore simplifies to

$$\begin{aligned}
9\int \frac{\cos(\theta)}{(1-\sin^2(\theta))^2}dx &= 9\int \frac{\cos(\theta)}{(1-u^2)^2}\frac{du}{\cos(\theta)} \\
&= 9\int \frac{1}{(1-u^2)^2}du \\
&= 9\int \frac{1}{4(1+u)} + \frac{1}{4(1+u)^2} + \frac{1}{4(1-u)} + \frac{1}{4(1-u)^2}du \quad (\star) \\
&= \frac{9}{4}\ln|u+1| - \frac{9}{4(u+1)} - \frac{9}{4}\ln|1-u| + \frac{9}{4(1-u)} + C \\
&= \frac{9}{4}\ln\left|\frac{1+u}{1-u}\right| - \frac{9u}{2(u^2-1)} + C
\end{aligned}$$

We made the substitution $u = \sin(\theta)$. However, we want the final expression in terms of x, therefore, recall that the first substitution was $x = 3\tan(\theta) \implies \theta = \tan^{-1}(\frac{x}{3}) \implies u = \sin(\tan^{-1}(\frac{x}{3})) = \frac{x}{\sqrt{x^2+9}}$. To see that $\sin(\tan^{-1}(\frac{x}{3})) = \frac{x}{\sqrt{x^2+9}}$, consider the triangle with side lengths x and 3.

The final answer is therefore given to be

$$\frac{9}{4}\ln\left|\frac{\sqrt{x^2+9}+x}{\sqrt{x^2+9}-x}\right| + \frac{9x\sqrt{x^2+9}}{2} + C.$$

Remark: The equation $(\star)$ was obtained using the method of partial fractions. □

Exercises

Q1. Evaluate the following integrals.

a.
$$\int \frac{1}{x^2\sqrt{x^2-4}}dx.$$

b.
$$\int x^3\sqrt{16-x^2}dx.$$

c.
$$\int \frac{x^3}{\sqrt{x^2+25}}dx.$$

d.
$$\int_1^{\sqrt{3}} \frac{x^3}{x^2\sqrt{4-x^2}}dx.$$

Q2. Evaluate the integral
$$\int \frac{1}{x\sqrt{x^2+5}}dx.$$

Q3. Evaluate the integral

$$\int \frac{1}{\sqrt{x^2 - 6x + 13}} dx.$$

Q4. Evaluate the integral

$$\int \sqrt{e^{2x} - 16} dx.$$

Q5. Evaluate the integral

$$\int_0^{\frac{1}{2}} \sqrt{x^2 + 1} dx.$$

Q6. Let $\lambda \in \mathbb{R}$ and show that

$$\int \frac{1}{\sqrt{x^2 + \lambda^2}} dx = \log_e \left(x + \sqrt{x^2 + \lambda^2} \right) + k,$$

where $k \in \mathbb{R}$ is some constant.

1.7 Integration by Parts

This method of integration allows us to integrate functions of the form $f(x)\cdot g'(x)$, and is derived from the product rule, which has been seen in the differential calculus portion of the course. We know that the product rule is given by

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot g'(x) + f'(x)\cdot g(x).$$

Therefore, consider that

$$\begin{aligned}\int \frac{d}{dx}[f(x)\cdot g(x)]dx &= \int f(x)\cdot g'(x) + f'(x)\cdot g(x)dx \\ \therefore f(x)\cdot g(x) &= \int f(x)\cdot g'(x) + f'(x)\cdot g(x)dx \\ &= \int f(x)\cdot g'(x)dx + \int f'(x)\cdot g(x)dx\end{aligned}$$

Therefore, we have the equation

$$\int f(x)\cdot g'(x)dx = f(x)\cdot g(x) - \int f'(x)\cdot g(x)dx.$$

The above equation is the formula for the method of integration by parts. If we let $u = f(x)$ and $v = g(x)$, the equation simplifies to

$$\int udv = uv - \int vdu.$$

Example 6.6.1 Integrate the following function by the method of integration by parts

$$\int xe^x dx.$$

Proof. We need to choose a suitable function for u and a suitable function for dv. The general approach is to look for the function that integrates easily, and set that equal to dv. Conversely, loook for the function that differentiates to something simple, perhaps a constant, and set that equal to u.

In this case, observe that $\int e^x dx = e^x$ and $\frac{d}{dx}(x) = 1$. Therefore, in this case we have a function, e^x, that integrates to something simple, and a function x, that differentiates to a constant. It therefore seems reasonable to let $u = x$ and $dv = e^x$. Consequentially, we can generate the table

$$\begin{array}{ll} u = x & v = e^x \\ du = dx & dv = e^x \end{array}$$

Substituting these values into the integration by parts formula yields that

$$\begin{aligned}\int xe^x dx &= xe^x - \int e^x dx \\ &= xe^x - e^x + C \\ &= e^x(x-1) + C.\end{aligned}$$

□

Example 6.6.2. Integrate the following function by the method of integration by parts

$$\int \log_e(x) dx.$$

Proof. The trick to integrating $\log_e(x)$ with the method of integration by parts, is to realise that $\log_e(x) = 1 \cdot \log_e(x)$. We can therefore let $u = \log_e(x)$ and $dv = 1$. Thus

$$\begin{aligned} u &= \log_e(x) & v &= x \\ du &= \frac{1}{x} dx & dv &= 1 \end{aligned}$$

Substituting these values into the integration by parts formula yields

$$\begin{aligned}\int \log_e(x) dx &= x\log_e(x) - \int x \cdot \frac{1}{x} dx \\ &= x\log_e(x) - x + C \\ &= x(\log_e(x) - 1) + C.\end{aligned}$$

□

Exercises

Q1. Evaluate the integrals.

a.
$$\int x^2 e^x dx.$$

b.
$$\int x\cos(3x) dx.$$

c.
$$\int \frac{1}{2} x^2 e^{4-x} dx.$$

d.
$$\int 3\cos(2x) + e^x \sin x dx.$$

Q2. Evaluate the integral

$$\int [\log_e(x)]^2 dx.$$

Q3. Evaluate the integral

$$\int x \tan^{-1} x dx.$$

Q4. Evaluate the integral

$$\int \sin(\ln x) dx.$$

Q5. Evaluate the integral

$$\int \cos(\ln x) dx.$$

Q6. Evaluate the integral

$$\int \cos x \ln(\sin x) dx.$$

Q7. Show that

$$\int f(x) dx = x f(x) - \int x f'(x) dx,$$

where $f \in \mathscr{C}^1(\mathbb{R})$.

Q8. Evaluate the integral

$$\int_2^e [\ln x]^3 dx.$$

1.8 Integration by Recognition

This method of integration involves using the derivative of a more complex that contains the expression that we wish to integrate.

Example 6.7.1

a. Differentiate $2x\sin^{-1}(x)$.

Proof. It is easy to see that

$$\begin{aligned}\frac{d}{dx}(2x\sin^{-1}(x)) &= 2\cdot\sin^{-1}(x) + 2x\cdot\frac{1}{\sqrt{1-x^2}} \\ &= 2\sin^{-1}(x) + \frac{2x}{\sqrt{1-x^2}}.\end{aligned}$$

□

b. Hence, evaluate $\int \sin^{-1}(x)dx$.

Proof. Begin by setting up the expression as follows

$$\begin{aligned}\frac{d}{dx}(2x\sin^{-1}(x)) &= 2\sin^{-1}(x) + \frac{2x}{\sqrt{1-x^2}} \\ \int \frac{d}{dx}(2x\sin^{-1}(x))dx &= \int 2\sin^{-1}(x) + \frac{2x}{\sqrt{1-x^2}}dx \\ 2x\sin^{-1}(x) &= \int 2\sin^{-1}(x)dx + \int \frac{2x}{\sqrt{1-x^2}}dx\end{aligned}$$

Now rearrange the expression as following

$$\begin{aligned}2\int \sin^{-1}(x)dx &= 2x\sin^{-1}(x) - \int \frac{2x}{\sqrt{1-x^2}}dx \\ \therefore \int \sin^{-1}(x)dx &= x\sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx\end{aligned}$$

Now we simply need to integrate the expression $\int \frac{x}{\sqrt{1-x^2}}dx$, which can be done with a simple substitution. Let $u = 1 - x^2$. The result is

$$\int \sin^{-1}(x)dx = x\sin^{-1}(x) + \sqrt{1-x^2} + C.$$

□

Example 6.7.2.

a. Differentiate $x\log_e(x-3)$.

Proof. It is easy to see that

$$\begin{aligned}\frac{d}{dx}(x\log_e(x-3)) &= 1\cdot\log_e(x-3)+x\cdot\frac{1}{x-3}\\ &= \log_e(x-3)+\frac{x}{x-3}.\end{aligned}$$

□

b. Hence, evaluate $\int \log_e(x-3)dx$.

Proof. Begin by setting up the expression as follows

$$\begin{aligned}\frac{d}{dx}(x\log_e(x-3)) &= \log_e(x-3)+\frac{x}{x-3}\\ \int\frac{d}{dx}(x\log_e(x-3))dx &= \int \log_e(x-3)+\frac{x}{x-3}dx\\ x\log_e(x-3) &= \int\log_e(x-3)dx+\int\frac{x}{x-3}dx\\ \therefore \int\log_e(x-3)dx &= x\log_e(x-3)-\int\frac{x}{x-3}dx\\ &= x\log_e(x-3)-3\log_e|x-3|+x+C.\end{aligned}$$

□

Exercises

Q1. By first differentiating $x\log_e(x)$, evaluate $\int\log_e(x)dx$.

Q2. By first differentiating $3x\sin(x)$, evaluate $\int x\cos(x)dx$.

Q3. By first differentiating $x\cos^{-1}(x)$, evaluate $\int\cos^{-1}(x)dx$

Q4. By first differentiating $5x\tan^{-1}(x)$, evaluate $\int\tan^{-1}(x)dx$.

Q5. Evaluate $\int\log_e(x+7)dx$ by first differentiating $x\log_e(x+7)$.

Q6. Evaluate $\int 5xe^x dx$ by first differentiating xe^x.

Q7. i. Differentiate $\log_e(x+\sqrt{x^2-1})$.

ii. Hence, evaluate

$$\int_1^2 \frac{1}{\sqrt{1+x^2}}dx$$

1.9 Volumes of Solids of Revolution

In this section, we find volumes by rotating areas about an axis.

If we rotate a region $\int_a^b f(x)dx$ about the x-axis, the general formula to find the volume of revolution is

$$V = \pi \int_a^b [f(x)]^2 dx.$$

Example 6.8.1. Find the volume of the solid of revolution given by the region bounded by $y = x^2$ and the x-axis, for $0 \leq x \leq 3$, rotated about the x-axis.

Proof. Using the above formula, the volume is given by

$$\begin{aligned} V &= \pi \int_0^3 (x^2)^2 dx \\ &= \pi \int_0^3 x^4 dx \\ &= \frac{1}{5}x^5 \Big|_0^3 \\ &= \frac{729}{5}. \end{aligned}$$

□

If we rotate a region $\int_a^b f(y)dx$ about the y-axis, the general formula to find the volume of revolution is

$$V = \pi \int_{f(a)}^{f(b)} [f^{-1}(x)]^2 dx.$$

Example 6.8.2. Find the volume of the solid of revolution given by the region bounded by $y = \sqrt{x}$ and the y-axis, for $0 \leq x \leq 4$, rotated about the y-axis.

Proof. Using the above formula, the volume is given by

$$\begin{aligned} V &= \pi \int_{\sqrt{0}}^{\sqrt{4}} (x^2)^2 dx \\ &= \pi \int_0^2 x^4 dx \\ &= \frac{1}{5} x^5 \bigg|_0^2 \\ &= \frac{32}{5}. \end{aligned}$$

□

Exercises

Q1. Find the volume of $f(x) = \frac{1}{\sqrt{1-x^2}}$ revolved around the x-axis, between $-1 \leq x \leq 1$.

Q2. Find the volume of $f(x) = \frac{2}{\sqrt{1+x^2}}$ revolved around the x-axis, between $\frac{-1}{\sqrt{3}} \leq x \leq \frac{1}{\sqrt{3}}$.

Q3. Find the volume of generated by rotating the ellipse $\frac{(x-1)^2}{4} + \frac{y^2}{9} = 1$ about the x-axis.

Q4. Find the volume of the solid of revolution for $y = \sin(2x)$ for $0 \leq x \leq \frac{\pi}{2}$ when

(i) rotated about the x-axis

(ii) rotated about the y-axis

Q5. Find the volume generated by rotating the curve $y = \tan^{-1}(x)$ about the y-axis, for $-\sqrt{3} \leq x \leq \sqrt{3}$.

Q6. Find the volume generated by rotating the curve $g(x) = \frac{1}{x^2}$ about the x-axis, for $1 \leq x \leq 2$.

Q7. Find the volume generated by rotating the curve $y = \log_e(x)$ about the y-axis for $\log_e(2) \leq x \leq \log_e(3)$.

Q8. Consider the curve $y = x^2 + 1$ and $y = -2x$.

(i) Find the intersection between these curves.

(ii) Find the area between these curves.

(iii) Find the volume generated by rotating this area about the x-axis.

(iv) Find the volume generated by rotating this area about the y-axis.

Q9. A hole of radius $\frac{\lambda}{3}$ is bored through the centre of a sphere of radius λ. Determine the volume of the remaining solid.

1.10 Analysis Exercises

Q1. Consider the function
$$f(x) = \frac{2x+4}{3-7x}.$$

a. Determine the largest domain on which f is integrable.

b. Evaluate $\int f(x)dx$.

Q2. Evaluate the integral
$$\int_0^4 2\,|x-2|\,dx.$$

Q3. Evaluate the integral
$$\int_0^4 \frac{1}{x}\ln(x+1) + 2x dx.$$

Q4. Set $f'(x) = \frac{1}{x} + \cos^3(x)\sin^2(x)$.

a. Determine the domain of $f'(x)$.

b. Evaluate $\int f'(x)dx$.

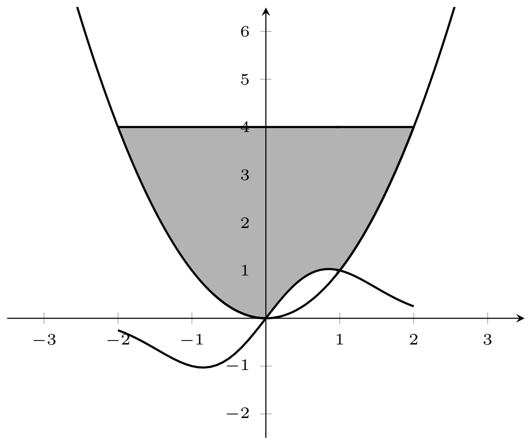
c. Determine $f(x)$ is $f\left(\frac{\pi}{4}\right) = -3$.

Q5. Evaluate the integral
$$\int \frac{1}{x^2+x-12}dx.$$

Q6. Evaluate the integral
$$\int \frac{4x+1}{(x+2)(x+5)}dx.$$

Q7. Evaluate
$$\int_0^{\frac{\sqrt{3}}{2}} x\sin^{-1}(x)dx.$$

Q8. Below is the graph of the functions $f(x) = xe^{-x^2}+1$ and $g(x) = x^2$.



a. Determine the points of intersection between $f(x)$ and $g(x)$.

b. Determine the area of the shaded region above.

Q9. Let $f(x) = 2(x+1)e^x$.

a. Evaluate $f'(x)$.

b. Hence, or otherwise, evaluate $\int xe^x dx$.

Q10. Evaluate the integral

$$\int \frac{4}{\sqrt{5-(x+4)^2}} + \frac{3}{8-x^2}.$$

Q11. Evaluate the integral

$$\int \tan^3\left(\frac{1}{2}x\right) + \sqrt{4-x}dx.$$

Q12. Evaluate the integral

$$\int \frac{e^x}{e^{2x}+2e^x+1}dx.$$

Q13. (Dr. Lloyd Gunatilake). Show that

$$\int_0^{\frac{\pi}{2}} \frac{1}{2+4\cos\vartheta}d\vartheta = \frac{1}{\sqrt{3}}\log_e(2+\sqrt{3}).$$

[Hint: make a substitution $u = \tan\frac{1}{2}\vartheta$].

Q14. Evaluate the integral

$$\int x^2 \cos(x) dx.$$

Q15. (Dr. Lloyd Gunatilake). The graph of the curve $x(x^2 + y^2) = a(x^2 - y^2)$, where $a > 0$, is given below

INSERT GRAPH

a. Show that the area of the shaded region is given by

$$\mathcal{A}_1 = 2\int_0^a x\sqrt{\frac{a-x}{a+x}}dx.$$

b. Hence, or otherwise, show that

$$A_1 = \frac{4-\pi}{2}a^2.$$

Q16. (Dr. Lloyd Gunatilake). We know that by using the substitution $x = \tan\vartheta$ that

$$\int \frac{1}{1+x^2}dx = \tan^{-1}(x) + \text{constant}.$$

Trigonometric substitutions are not very useful for evaluating integrals of the form

$$\int \frac{1}{x^n + \lambda^n}, \quad n \geq 3, \ \lambda \in \mathbb{R}.$$

What does become useful however are the methods of partial fractions and complex numbers.

a. i. Show that the cube roots of -1 are

$$\operatorname{cis}\left(\pm\frac{\pi}{3}\right), \operatorname{cis}(\pi).$$

ii. Hence, or otherwise, show that

$$x^3 + 1 = (x+1)(x^2 - x + 1).$$

iii. By writing

$$\frac{1}{x^3+1} = \frac{A}{x+1} + \frac{Bx + C}{x^2 - x + 1},$$

show that $A = \frac{1}{3}, B = -\frac{1}{3}$ and $C = \frac{2}{3}$.

iv. Hence, or otherwise, show that

$$\begin{aligned}\int \frac{1}{x^3+1}dx &= \frac{1}{3}\log_e(x+1) - \frac{1}{6}\log_e(x^2-x+1)\\ &\quad + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2x-3}{\sqrt{3}}\right) + \text{constant.}\end{aligned}$$

b. i. Show that the fourth roots of -1 are

$$\operatorname{cis}\left(\pm\frac{\pi}{4}\right), \operatorname{cis}\left(\pm\frac{3\pi}{4}\right).$$

ii. Hence, or otherwise, show that

$$x^4+1 = (x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1).$$

iii. By writing

$$\frac{1}{x^4+1} = \frac{Ax+B}{x^2-\sqrt{2}x+1} + \frac{Cx+D}{x^2+\sqrt{2}x+1},$$

show that $A = -\frac{1}{2\sqrt{2}} = -C$, and $B = D = \frac{1}{2}$.

iv. Show that

$$\tan^{-1}\vartheta_1 + \tan^{-1}\vartheta_2 = \tan^{-1}\left(\frac{\vartheta_1+\vartheta_2}{1-\vartheta_1\vartheta_2}\right).$$

v. Hence, or otherwise, show that

$$\begin{aligned}\int \frac{1}{x^4+1}dx &= \frac{1}{4\sqrt{2}}\log_e\left(\frac{x^2+\sqrt{2}x+1}{x^2-\sqrt{2}x+1}\right)\\ &\quad + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{1-x^2}\right) + \text{constant.}\end{aligned}$$

c. i. Show that the fifth roots of -1 are

$$\operatorname{cis}\left(\pm\frac{\pi}{5}\right), \operatorname{cis}\left(\pm\frac{3\pi}{5}\right), \operatorname{cis}(\pi).$$

ii. Hence, show that

$$x^5+1 = (x+1)\left(x^2-2x\cos\left(\frac{\pi}{5}\right)+1\right)\left(x^2-2x\cos\left(\frac{3\pi}{5}\right)+1\right).$$

iii. By writing

$$\frac{1}{x^5+1} = \frac{A}{x+1} + \frac{B\left(1 - x\cos\left(\frac{\pi}{5}\right)\right)}{x^2 - 2x\cos\left(\frac{\pi}{5}\right)+1} + \frac{C\left(1 - x\cos\left(\frac{3\pi}{5}\right)\right)}{x^2 - 2x\cos\left(\frac{3\pi}{5}\right)+1},$$

show that $A = \frac{1}{5} = 2B = 2C$.

iv. Since

$$\begin{aligned}\int \frac{1}{x^5+1}dx &= \frac{\sqrt{5}-1}{20}\ln\left(\frac{x^2+\frac{1}{2}(\sqrt{5}-1)x+1}{x^2-\frac{1}{2}(\sqrt{5}+1)x+1}\right) \\ &\quad + \frac{1}{5}\ln(x+1) - \frac{1}{10}\sqrt{10-2\sqrt{5}}\tan^{-1}\left(\frac{1+\sqrt{5}-4x}{\sqrt{10-2\sqrt{5}}}\right) \\ &\quad + \frac{1}{10}\sqrt{10+2\sqrt{5})}\tan^{-1}\left(\frac{4x+\sqrt{5}-1}{\sqrt{10+2\sqrt{5}}}\right) + \text{constant}\end{aligned}$$

we leave it only to the very concerned reader to solve this exercise.