Notes on Injectivity and Surjectivity

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Chapter 1

Linear Algebra – Linear Maps

Definition 1.1. A map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is said to be *linear* if

(i) (*Additivity*). $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$.

(ii) (*Homogeneity*). $T(\lambda \mathbf{v}) = \lambda T(\mathbf{v})$ for all $\mathbf{v} \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}$.

Examples of linear maps include:

(1) $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ defined by $\mathbf{v} \longmapsto A\mathbf{v}$, where A is an $m \times n$ matrix.
(2) The derivative operator.
(3) The integral operator.

The operator $T : \mathbb{R} \to \mathbb{R}$ given by $T(\mathbf{v}) = \mathbf{v}^2$ is an example of an operator which is *not* linear. Indeed,

$$T(\mathbf{u} + \mathbf{v}) \quad = \quad (\mathbf{u} + \mathbf{v})^2 = \mathbf{u}^2 + 2\mathbf{u}\mathbf{v} + \mathbf{v}^2,$$

while

$$T(\mathbf{u}) + T(\mathbf{v}) \quad = \quad \mathbf{u}^2 + \mathbf{v}^2.$$

This shows that $T(\mathbf{v}) = \mathbf{v}^2$ is not additive and is therefore not linear.

Theorem 1.2. A linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ satisfies $T(\mathbf{0}) = \mathbf{0}$.

Proof. Any linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ may be written in the form $T(\mathbf{v}) = A\mathbf{v}$ for some $m \times n$ matrix A. Elementary matrix multiplication then shows that $A \cdot \mathbf{0} = \mathbf{0}$ and this proves the result. □

Theorem 1.2 shows that

$$\boxed{T \text{ is linear} \implies T(\mathbf{0}) = \mathbf{0}.}$$

Therefore, if we want to show that an operator is not linear, it is sufficient to show that $T(\mathbf{0}) \neq \mathbf{0}$.

Warning 1.3. While linearity implies that the origin is fixed (Theorem 1.2), the converse is not true. That is, if $T(\mathbf{0}) = \mathbf{0}$ is not necessarily true that T is linear. The operator $T : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $T(\mathbf{v}) = \mathbf{v}^2$, which we showed earlier to be non-linear, satisfies $T(\mathbf{0}) = \mathbf{0}^2 = \mathbf{0}$.

Summary 1.3.

(a) If the question asks to determine whether an operator is linear and $T(\mathbf{0}) \neq \mathbf{0}$, then T is **not** linear.

(b) If the question asks to determine whether an operator is linear and $T(\mathbf{0}) = \mathbf{0}$, then you have to show that conditions (i) and (ii) of Definition 1.1. are satisfies.

Chapter 2

One-To-One Maps

Definition 2.1. A linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *one–to–one* if for every $\mathbf{b} \in \mathbb{R}^m$, there exists exactly one $\mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{v}) = \mathbf{b}$.

A better definition is that a map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is one–to–one if

$$T(\mathbf{v}) = \mathbf{0} \implies \mathbf{v} = \mathbf{0}.$$

Example 2.2. Consider the operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $\mathbf{v} \longmapsto A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}.$$

Let us check if T is one–to–one. Observe that

$$\begin{aligned} T(\mathbf{v}) = \mathbf{0} \quad &\implies \quad \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ &\implies \quad \begin{bmatrix} 1 & 3 & 0 \\ 2 & -1 & 0 \end{bmatrix} \qquad \text{(Form the augmented matrix)}, \\ &\implies \quad \begin{bmatrix} 1 & 3 & 0 \\ 0 & -7 & 0 \end{bmatrix} \\ &\implies \quad \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &\implies \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \\ &\implies \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies \mathbf{v} = \mathbf{0}, \end{aligned}$$

so T is one-to-one. (Insert confusing discussion about pivots).

Example 2.3. Consider the operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $\mathbf{v} \longmapsto A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

Proceeding as we did in the previous exercise. Observe that

$$T(\mathbf{v}) = \mathbf{0} \quad \implies \quad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_1 = 0,\ v_2 \text{ is free.}$$

So the set of all solutions to $T(\mathbf{v}) = \mathbf{0}$ is $\left\{ \mathbf{v} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} : \alpha \in \mathbb{R} \right\}$. For example $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \mathbf{0}$. It is clear that T is **not** one–to–one.

Let us consider a harder example.

Example 2.4. Consider the map $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ defined by $\mathbf{v} \longmapsto A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ 0 & 2 & 4 & 2 & 1 \\ 0 & 0 & -1 & 2 & 5 \\ 0 & 3 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that

$$\begin{aligned} T(\mathbf{v}) = \mathbf{0} \quad &\Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ 0 & 2 & 4 & 2 & 1 \\ 0 & 0 & -1 & 2 & 5 \\ 0 & 3 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ &\Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & 0 \\ 0 & 2 & 4 & 2 & 1 & 0 \\ 0 & 0 & -1 & 2 & 5 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \end{bmatrix} \\ &\Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 4 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 10 & 21 & 0 \end{bmatrix} \quad \text{(Row reduced with a calculator),} \\ &\Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & 10 & 21 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Observe that since v_5 is a free variable, $T(\mathbf{v}) = \mathbf{0}$ has an infinite number of solutions. It then follows that T is **not** one–to–one.

Definition 2.5. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear operator. The *null space* of T is the set

$$\mathrm{Nul}(T) = \{\mathbf{v} \in \mathbb{R}^n : T(\mathbf{v}) = \mathbf{0}\}.$$

Observations 2.6.

(a) The null space is a subset of the domain.
(b) T is one–to–one if and only if $\mathrm{Nul}(T) = \{\mathbf{0}\}$.

Theorem 2.7. Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear operator. The null space of T is a subspace of $\mathbb{R}^n$, i.e., $\mathrm{Nul}(T)$ is a subset of $\mathbb{R}^n$ that is also a vector space.

Proof. We need to check that $\mathrm{Nul}(T)$ is closed under vector addition and scalar multiplication. To this end, let $\mathbf{u}, \mathbf{v} \in \mathrm{Nul}(T)$. We claim that $\mathbf{u} + \mathbf{v} \in \mathrm{Nul}(T)$, i.e., $T(\mathbf{u} + \mathbf{v}) = \mathbf{0}$. So observe that by the linearity of T,

$$T(\mathbf{u} + \mathbf{v}) \quad = \quad T(\mathbf{u}) + T(\mathbf{v}) = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

Now let $\mathbf{v} \in \mathrm{Nul}(T)$ and $\lambda \in \mathbb{R}$. Then

$$T(\lambda \mathbf{v}) = \lambda T(\mathbf{v}) = \lambda \cdot \mathbf{0} = \mathbf{0},$$

so $\lambda \mathbf{v} \in \mathrm{Nul}(T)$. This proves the result. $\square$

Example 2.8. If we consider the operator given in Example 2.3, the null space was determined to be

$$\mathrm{Nul}(T) = \left\{ \mathbf{v} = \begin{bmatrix} 0 \\ \alpha \end{bmatrix} \ : \ \alpha \in \mathbb{R} \right\}.$$

Chapter 3

Onto Maps

Definition 3.1. A linear map $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *onto* if for every $\mathbf{b} \in \mathbb{R}^m$ there exists at least one $\mathbf{v} \in \mathbb{R}^n$ such that $T(\mathbf{v}) = \mathbf{b}$.

Remark 3.2. To check that an operator $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $\mathbf{v} \longmapsto A\mathbf{v}$, is onto, we form the augmented matrix $\begin{bmatrix} A \mid \mathbf{b} \end{bmatrix}$ and row reduce the system. If the system is consistent, then the operator is onto. If the system is inconsistent, then the matrix is not onto.

Example 3.3. Consider the operator $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by $\mathbf{v} \longmapsto A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}.$$

To check whether T is onto, let $\mathbf{b} \in \mathbb{R}^2$ be an arbitrary vector. Then

$$\begin{aligned}
T(\mathbf{v}) = \mathbf{b} \quad &\Longrightarrow \quad \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \\
&\Longrightarrow \quad \begin{bmatrix} 1 & 3 & b_1 \\ 2 & -1 & b_2 \end{bmatrix} \\
&\Longrightarrow \quad \begin{bmatrix} 1 & 3 & b_1 \\ 0 & -7 & b_2 - 2b_1 \end{bmatrix} \\
&\Longrightarrow \quad \begin{bmatrix} 1 & 3 & b_1 \\ 0 & 1 & \frac{1}{7}(2b_1 - b_2) \end{bmatrix} \\
&\Longrightarrow \quad \begin{bmatrix} 1 & 0 & b_1 - \frac{3}{7}(2b_1 - b_2) \\ 0 & 1 & \frac{1}{7}(2b_1 - b_2) \end{bmatrix} \\
&\Longrightarrow \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} b_1 - \dfrac{3}{7}(2b_1 - b_2) \\ \dfrac{1}{7}(2b_1 - b_2) \end{bmatrix} \\
&\Longrightarrow \quad \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \frac{1}{7} \begin{bmatrix} b_1 + 3b_2 \\ 2b_1 - b_2 \end{bmatrix}.
\end{aligned}$$

Since we may take $v_1 = \frac{1}{7}(b_1 + 3b_2)$ and $v_2 = \frac{1}{7}(2b_1 - b_2)$, it follows that T is onto.

Example 3.4. Consider the operator $T : \mathbb{R}^5 \longrightarrow \mathbb{R}^4$ given by $\mathbf{v} \longmapsto A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 5 & 1 & 8 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Let $\mathbf{b} \in \mathbb{R}^4$ be an arbitrary vector. Then

$$T(\mathbf{v}) = \mathbf{b} \quad \Longrightarrow \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 5 & 1 & 8 \\ 0 & 0 & -2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\Longrightarrow \quad \begin{bmatrix} v_1 + 2v_2 + 3v_3 + 4v_4 + 5v_5 = b_1 \\ 2v_2 + 5v_3 + v_4 + 8v_5 = b_2 \\ -2v_3 + v_5 = b_3 \\ 0 = b_4. \end{bmatrix}$$

Observe that any vector $\mathbf{b}$ with $b_4 \neq 0$ cannot be hit by T. That is, for any vector $\mathbf{b}$ with $b_4 \in \mathbb{R}\backslash\{0\}$ we cannot find a $\mathbf{v} \in \mathbb{R}^5$ such that $T(\mathbf{v}) = \mathbf{b}$. Hence we see that T is **not** onto.

Definition 3.5. Let $A = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}$ be an $m \times n$ matrix whose columns are given by the vectors $\mathbf{v}_j \in \mathbb{R}^m$ for $1 \leq j \leq n$. The vector space V given by

$$V = \operatorname{span}\{\mathbf{v}_1, ..., \mathbf{v}_n\}$$

is called the *column space*.

Intuitively, if $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is the operator given by $\mathbf{v} \longmapsto A\mathbf{v}$, then the column space is exactly the range of T. Note that this implies that the column space is a subspace of the codomain.

Example 3.6. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^4$ be the operator given by $\mathbf{v} \longmapsto A\mathbf{v}$, where

$$A = \begin{bmatrix} 1 & 4 & -2 \\ 2 & 1 & 7 \\ 3 & 0 & 0 \\ 4 & -2 & 7 \end{bmatrix}.$$

The column space consists of all vector $\mathbf{v} \in \mathbb{R}^4$ such that

$$\mathbf{v} = \lambda_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + \lambda_2 \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix} + \lambda_3 \begin{bmatrix} -2 \\ 7 \\ 0 \\ 7 \end{bmatrix}.$$

In other words, the column space is the vector space

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -2 \\ 7 \\ 0 \\ 7 \end{bmatrix} \right\}.$$

Since it is the span of the column vectors, one may also think of the column space as *everything that is hit by the operator*. This agrees with the interpretation of the column space being the range of the operator.

Definition 3.7. The *rank* of a matrix A is the dimension of the column space. In other words, the rank is the number of linearly independent vectors which span the range of the associated operator $T : \mathbf{v} \longmapsto A\mathbf{v}$.

A consequence of this is that we may assert that an operator $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is onto if and only if $\operatorname{rank}(A) = m$.

Remark 3.8. If we consider a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = ax + b$, where $a, b \in \mathbb{R}$ then the rank of f is either zero or one. Indeed, if $a \neq 0$, then $f(x) = ax + b$ has a range of $\mathbb{R}$ and $\dim \mathbb{R} = 1$. Hence,

$$\operatorname{rank}(f) = \dim \operatorname{Col}(f) = \dim \operatorname{Ran}(f) = \dim \mathbb{R} = 1.$$

If $a = 0$ however, then $f(x) = ax + b = b$, which is a constant, so the range is just $\{b\}$ and

$$\operatorname{rank}(f) = \dim \operatorname{Col}(f) = \dim \operatorname{Ran}(f) = \dim\{\text{point}\} = 0.$$

Note that we have not defined rank for non-linear operators such that $f(x) = x^2$.

Chapter 4

Row Space and the Transpose

Recall that for an $m \times n$ matrix A, the *transpose* of A is the $n \times m$ matrix A^T whose ijth entry is the jith entry of A. In other words, the transpose of the matrix

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 8 & 9 \end{bmatrix}$$

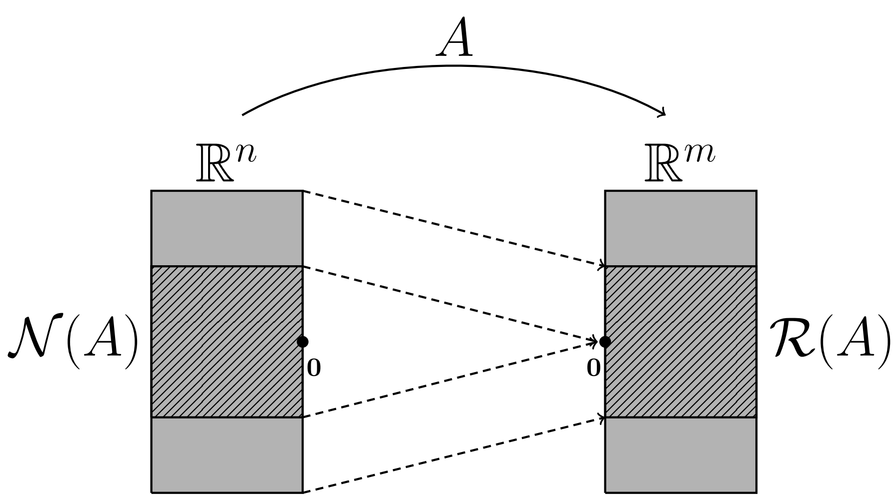
is the matrix

$$A^T = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 4 & 9 \end{bmatrix}.$$

The diagram below may be of some use.

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, bend right, "A"'] & \mathbb{R}^m \arrow[l, bend right, "A^T"'] \end{tikzcd}$$

So far, given our knowledge of the null space $\mathcal{N}(A)$ and column space $\mathcal{R}(A)$, we have the following.



Definition 4.1. The *row space* of an $m \times n$ matrix $A = \begin{bmatrix} \mathbf{w}_1 \\ \vdots \\ \mathbf{w}_n \end{bmatrix}$ is the vector space

$$V = \text{span}\{\mathbf{w}_1, ..., \mathbf{w}_n\}.$$

Intuitively, the row space is the column space of the transpose; or even better, the row space is the range of the transpose. In particular, we note if A is an $m \times n$ matrix, then the row space of A is the $\text{Range}(A^T)$ which is a subspace of $\mathbb{R}^n$. Compare this with the above diagram.

1. Summary

- ⋆ If T is a linear map, then $T(\mathbf{0}) = \mathbf{0}$.
- ⋆ If $T(\mathbf{0}) = \mathbf{0}$, it is **not necessarily** true that T is linear.
- ⋆ A map is injective or one-to-one if $T(\mathbf{v}) = \mathbf{0}$ implies that $\mathbf{v} = 0$. This is equivalent to the null space of T being trivial, i.e., $\mathcal{N}(T) = \{\mathbf{0}\}$.
- ⋆ The column space is just the range of the matrix. Remember that the range is a subspace of the codomain.
- ⋆ The rank of a matrix is just the dimension of the column space, which is the dimension of the range.
- ⋆ An operator $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ given by $\mathbf{v} \longmapsto A\mathbf{v}$ is onto if and only the rank of A is m. Equivalently, T is onto if and only if the column space $\mathcal{R}(A)$ is exactly $\mathbb{R}^m$.
- ⋆ The row space is the range of the transpose A^T and is a subspace of the domain of A.