Calculus Exam 2 (Preparation) (Solutions)

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The contents of this examination require an understanding of the calculus material that was covered in the calculus practice exams 3,4 and 5. An understanding of graphing techniques and function transformations may also be required.

There are no permitted materials for this test. That is, you are not permitted any cheat notes, calculators or resources other than a pen/pencil, eraser, sharpener, ruler and water bottle.

There is to be no collaboration on this examination and any attempts of communication will result in a nullified score. You are permitted 10 minutes of reading time and 105 minutes of writing time. There is a total of 100 available marks. It is recommended that you use the reading time to ask the invigilator about any issues regarding the format of the test, the problems or other issues. No hints will be given. Best of luck!

Name: __________________

Grade: __________/100

Question 1. [10 marks]. Let $f : (0, \pi) \to \mathbb{R}$ be the function defined by $f(x) := \cot x$. Show that $f'(x) = -\csc^2(x)$, where $\csc x := \frac{1}{\sin x}$.

Proof. It follows trivially from the fact that we may write

$$f(x) = \cot x = \frac{\cos x}{\sin x}.$$

$\square$

Question 2. [20 marks]. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \cos(x^3)\log_e(x+1).$$

Explain why f is differentiable on $(0, \infty)$ and evaluate $f'(x)$.

Proof. Since the functions x^3, $\cos x$, $x+1$ and $\log_e(x)$ are all differentiable functions on $(0,\infty)$, it follows that f is differentiable on $(0,\infty)$, since it is a composition of differentiable functions. It is clear that we need to apply the product rule and the chain rule. To this end, we observe that

$$\begin{aligned} f'(x) &= [\cos(x^3)] \cdot \frac{d}{dx}\log_e(x+1) + \log_e(x+1) \cdot \frac{d}{dx}\cos(x^3) \\ &= \cos(x^3) \cdot \frac{1}{x+1} + \log_e(x+1) \cdot (-3x^2 \sin(x^3)) \\ &= \frac{\cos(x^3)}{x+1} - 3x^2 \sin(x^3)\log_e(x+1). \end{aligned}$$

$\square$

Question 3. [20 marks]. Consider the function $f : (-\pi/2, \pi/2) \to \mathbb{R}$ defined by

$$f(x) = \frac{\sec x}{4 + \sqrt{3x+1}}.$$

Evaluate $f'(0)$.

Proof. It is clear that we need to use the quotient rule here. Before doing this however, let us first determine the derivative of $\sec x$. Indeed, we see that

$$\begin{aligned} \frac{d}{dx} \sec x &= \frac{d}{dx}(\cos x)^{-1} \\ &= (-1) \cdot (-\sin x) \cdot (\cos x)^{-2} \\ &= \frac{\sin x}{\cos^2 x} \\ &= \tan x \sec x. \end{aligned}$$

Moreover, the derivative of $4 + \sqrt{3x+1}$ is given by

$$\begin{aligned} \frac{d}{dx}\left(4 + \sqrt{3x+1}\right) &= \frac{d}{dx}(3x+1)^{\frac{1}{2}} \\ &= \frac{1}{2} \cdot 3 \cdot (3x+1)^{-\frac{1}{2}} \\ &= \frac{3}{2\sqrt{3x+1}}. \end{aligned}$$

By the quotient rule, we then see that

$$f'(x) = \frac{1}{(4+\sqrt{3x+1})^2}\left(\tan x \sec x(4 + \sqrt{3x+1}) - \sec x \cdot \frac{3}{2\sqrt{3x+1}}\right).$$

I do not want to simplify this any further. To evaluate $f'(0)$, we simply insert $x = 0$ into the above expression. In so doing, we obtain

$$\begin{aligned} f'(0) &= \frac{1}{(4+\sqrt{1})^2}\left(\tan 0 \sec 0(4 + \sqrt{1}) - \sec 0 \cdot \frac{3}{2\sqrt{1}}\right) \\ &= \frac{1}{25} \cdot \left(\frac{-3}{2}\right) \\ &= -\frac{3}{50}. \end{aligned}$$

$\square$

Question 4. [25 marks]. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = x^{\frac{1}{5}}.$$

Determine the point(s) of inflection of $f(x)$.

Proof. We need to investigate where the second derivative of f changes sign. To this end, we compute

$$\begin{aligned} f'(x) &= \frac{1}{5}x^{-\frac{4}{5}} \\ f''(x) &= -\frac{4}{25}x^{-\frac{9}{5}}. \end{aligned}$$

It is clear this the second derivative changes sign about the point $x = 0$. Note however that the second derivative is not even defined at $x = 0$! Nevertheless, we see that f has a point of inflection at $x = 0$. $\square$

Question 5. [25 marks]. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable $\forall x \in \mathbb{R}$.

a. Determine the maximal domain on which $\widetilde{f}(x) := 2\sqrt{f(x)+1}$ is differentiable.

Proof. The function $\widetilde{f}(x)$ is differentiable at all points $x \in \mathbb{R}$ such that $f(x) > -1$. $\square$

b. Evaluate the derivative of $\widetilde{f}(x)$ at all points where $\widetilde{f}$ was determined to be differentiable in part (a).

Proof. It is easy to see that

$$\widetilde{f}'(x) = \frac{f'(x)}{\sqrt{f(x)+1}}.$$

$\square$

c. $\star$ Let $g(x) := \log_e |f(x)| + \exp(f'(x))$. Is it necessarily true that g is differentiable? If not, are there any extra conditions you can put on f such that g is differentiable? Are there any conditions you can put on f such that g is differentiable for all $x \in \mathbb{R}$?

Proof. Not necessarily! Just because f is once differentiable, does not mean that its derivative is differentiable. If we assume that f is twice differentiable, then g is differentiable at all points where $f(x) \neq 0$. If $f(x) \neq 0$ for all $x \in \mathbb{R}$ and f is twice differentiable, then g is differentiable for all $x \in \mathbb{R}$. $\square$