LECTURE NOTES ON VECTOR CALCULUS

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Abstract. The purpose of these notes is to present a unified geometric picture of vector calculus. The subject has many theorems which are intimately related, but this aspect is often not conveyed clearly. We hope to offer a clear perspective of this beautiful subject in these notes.

1. Calculus on $\mathbb{R}^n$

In this section, we will look at double and triple integrals.

Video Reference: A discussion of double integrals over general regions is given: https://www.youtube.com/watch?v=KUxCzXpuKj0&list=PL912tg7wFUfXIwSKeTyxU5LTqdRZ8Urza&index=5&t=0s

Video Reference: A discussion of double integrals in polar coordinates is given: https://www.youtube.com/watch?v=gC9ivl5hMQU&list=PL912tg7wFUfXIwSKeTyxU5LTqdRZ8Urza&index=6&t=0s

2. Vector Fields

2.1. Definition and Examples.

Definition 2.1.1. A vector field on $\mathbb{R}^n$ is a function $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ which assigns to each point $x \in \mathbb{R}^n$ a vector $\mathbf{F}(x) \in \mathbb{R}^n$.

For the most part, we will be concerned with vector fields on $\mathbb{R}^2$ and $\mathbb{R}^3$. If $\mathbf{F}$ is a vector field on $\mathbb{R}^2$, we often write

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j},$$

where (x, y) are the coordinates on $\mathbb{R}^2$. If $\mathbf{F}$ is a vector field on $\mathbb{R}^3$, we often write

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k},$$

where (x, y, z) are the coordinates on $\mathbb{R}^3$.

Example 2.1.2. Consider the vector field $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$.

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Example 2.1.3. Consider the vector field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$.

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Example 2.1.4. Consider the vector field $\mathbf{F}(x, y) = y\mathbf{i} + \cos(x)\mathbf{j}$.

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Example 2.1.5. Consider the vector field $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

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Example 2.1.6. Consider the vector field $\mathbf{F}(x, y, z) = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

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2.2. Gradient and Conservative Vector Fields.

Definition 2.2.1. Let $(x_1, ..., x_n)$ denote the coordinates on $\mathbb{R}^n$. The **grad vector** is the vector $\nabla \in \mathbb{R}^n$ given by

$$\nabla = \left\langle \frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n} \right\rangle.$$

If we let (x, y, z) denote the coordinates on $\mathbb{R}^3$, then

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}.$$

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with continuous partial derivatives. For example, if f is continuously differentiable, then f has continuous partial derivatives.

Definition 2.2.2. The **gradient vector field** associated to a function $f : \mathbb{R}^n \to \mathbb{R}$ with continuous partial derivatives is the vector field

$$\nabla f = \left\langle \frac{\partial f}{\partial x_1}, ..., \frac{\partial f}{\partial x_n} \right\rangle.$$

If we let (x, y, z) denote the coordinates on $\mathbb{R}^3$, then

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

Example 2.2.3. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = x + y$. Then

$$\nabla f(x, y) = \mathbf{i} + \mathbf{j}.$$

Example 2.2.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x, y) = \sin(x) + \cos(y)$. Then

$$\nabla f(x, y) = \cos(x)\mathbf{i} - \sin(y)\mathbf{j}.$$

Remark 2.2.5. It turns out that gradient vector fields have many useful properties that make calculations much simpler. In general, we would much rather work with gradient vector fields than arbitrary vector fields.

Definition 2.2.6. A vector field $\mathbf{F}$ on $\mathbb{R}^n$ is said to be **conservative** if there exists a function $f : \mathbb{R}^n \to \mathbb{R}$ such that

$$\mathbf{F} = \nabla f.$$

Remark 2.2.6. In some sense, conservative vector fields are exactly those vector fields which possess an "*anti-derivative*". Note that we have not yet made precise what this means. To do this, we need to introduce the concept of a *line integral* or *path integral*.

3. Line Integrals and Green's Theorem

3.1. Computing Line Integrals.

Definition 3.1.1. Let $\mathbf{F}$ be a continuous vector field on a smooth curve C given by the vector-valued function $\mathbf{r}(t)$, $a \leq t \leq b$. Then the **line integral of F along C** is

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt.$$

3.2. Green's Theorem.

Theorem 3.2.1. Let C be a positively oriented, piecewise smooth, simple closed curve in $\mathbb{R}^2$, and let D be the region bounded by C. If P, Q have continuous partial derivatives on an open region that contain D, then

$$\int_C P dx + Q dy \;=\; \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA.$$

4. More on Conservative Vector Fields and the Fundamental Theorem of Line Integrals

Recall that in §1 we mentioned the fundamental theorem of calculus: If $f : [a, b] \to \mathbb{R}$ is a continuous function, there exists a continuously differentiable function $F : [a, b] \to \mathbb{R}$ such that

$$\int_a^b f(x) dx = F(b) - F(a).$$

In this section, we want to look at the analogue of this theorem for line integrals:

4.1. Fundamental Theorem of Line Integrals.

Theorem 4.1.1. Let $\mathcal{C}$ be a smooth curve described by the vector-valued function $\mathbf{r}(t)$, where $t \in [a, b]$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with gradient vector field ∇f. Then

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} \;=\; f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

Remark 4.1.2. In particular, if $\mathbf{F}$ is a conservative vector field with potential f, the line integral of $\mathbf{F}$ is given by the difference of the values of the potential at the endpoints.

Remark 4.1.3. Although we assume in Theorem 4.1.1 that $\mathcal{C}$ is smooth, it is enough to assume that $\mathcal{C}$ is piecewise smooth. Indeed, we simply subdivide $\mathcal{C}$ into a finite number of smooth curves, and then add up the resulting integrals.

4.2. **Independence of Path.** Let us now consider the following natural question: Suppose that $\mathcal{C}_1$ and $\mathcal{C}_2$ are two smooth paths which have the same endpoints:

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What is the relation between the line integrals $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$ and $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$?

Definition 4.2.1. Let $\mathbf{F}$ be a continuous vector field on $\mathbb{R}^n$. We say that the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

is **independent of path** if for all smooth paths $\mathcal{C}_1$, $\mathcal{C}_2$ which share endpoints, we have

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}.$$

An immediate consequence of Theorem 4.1.1 is the following:

Corollary 4.2.2. Let $\mathbf{F}$ be a conservative vector field on $\mathbb{R}^n$. Then the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path.

Question 4.2.3. Are conservative vector fields the only vector fields for which the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path?

To address this question, we introduce some further terminology:

Definition 4.2.4. A curve $\mathcal{C}$ in $\mathbb{R}^n$ is said to be **closed** if the endpoints coincide. That is, if $\mathcal{C}$ is described by the vector-valued function $\mathbf{r} : [a, b] \to \mathbb{R}^n$, then the curve is closed if $\mathbf{r}(a) = \mathbf{r}(b)$.

Theorem 4.2.5. Let D be a domain[1] in $\mathbb{R}^n$ and $\mathbf{F}$ a continuous vector field in D. The line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths $\mathcal{C}$ in D.

The following answers Question 4.2.3:

Theorem 4.2.6. Let $\mathbf{F}$ be a vector field which is continuous on the domain D. If $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path in D, then $\mathbf{F}$ is a conservative vector field.

[1]A domain is understood to be a (path-)connected open subset of $\mathbb{R}^n$.

4.3. **Criteria for a vector field to be conservative.** So far, we have seen that a vector field $\mathbf{F}$ is conservative if and only if its line integral is independent of path. Although this is a very nice theorem, it's difficult to use as a test for conservativeness of a given vector field. We want a more readily available test to know whether a vector field is conservative or not.

Video Reference: Conservative vector fields are introduced in the video: https://www.youtube.com/watch?v=eQhj4m8pt9E&t=0s

We can get a necessary criterion for a vector field to be conservative rather easily. Let us first consider a vector field $\mathbf{F}$ on $\mathbb{R}^2$ given by

$$\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}.$$

Suppose $\mathbf{F}$ is conservative, i.e., there exists a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. This implies that

$$P(x,y) = \frac{\partial f}{\partial x} \quad \text{and} \quad Q(x,y) = \frac{\partial f}{\partial y}.$$

By Clairaut's theorem, we know that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

Therefore,

$$\frac{\partial P}{\partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial Q}{\partial x}.$$

Theorem 4.3.1. Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i}+Q(x,y)\mathbf{j}$ be a conservative vector field on a domain D, where P, Q have continuous first-order partial derivatives on D. Then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}.$$

Remark 4.3.2. The above theorem is useless for showing that a vector field is conservative. But it provides a very useful test for showing that a vector field is **not conservative**.

Example 4.3.3. The vector field $\mathbf{F}(x,y) = (y-2)\mathbf{i} + (3x-2y)\mathbf{j}$ is not conservative. Indeed,

$$\frac{\partial P}{\partial y} = 1, \qquad \frac{\partial Q}{\partial x} = 3.$$

By Theorem 4.3.1, $\mathbf{F}$ is not conservative.

It turns out that the converse of Theorem 4.3.1 is also valid, so long as we restrict to simply-connected domains. For the moment, we understand *simply-connected domains* to be those domains which have no holes and cannot consists of two separate pieces. We will provide more detail of simply-connectedness in the next section.

Theorem 4.3.4. Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ be a vector field on a simply-connected domain D, where P, Q have continuous first-order partial derivatives on D. Then if

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

at every point of D, then $\mathbf{F}$ is conservative.

Example 4.3.5. Consider the vector field $\mathbf{F}(x,y) = xe^y\mathbf{i} + ye^x\mathbf{j}$. We will check whether $\mathbf{F}$ is conservative on $\mathbb{R}^2$, which is a simply connected region. Indeed, we set

$$P(x,y) = xe^y, \qquad Q(x,y) = ye^x,$$

which implies that

$$\frac{\partial P}{\partial y} = xe^y, \qquad \frac{\partial Q}{\partial x} = ye^x,$$

so $\mathbf{F}$ is not conservative away from $\{x = y\}$.

Example 4.3.6. Consider the vector field

$$\mathbf{F}(x,y) = (2x\cos(y) - y\cos(x))\mathbf{i} + (-x^2\sin(y) - \sin(x))\mathbf{j}.$$

We will check whether $\mathbf{F}$ is conservative on $\mathbb{R}^2$. Here we set

$$P(x,y) = 2x\cos(y) - y\cos(x), \qquad Q(x,y) = -x^2\sin(y) - \sin(x),$$

which implies that

$$\begin{aligned}\frac{\partial P}{\partial y} &= -2x\sin(y) - \cos(x), \\ \frac{\partial Q}{\partial x} &= -2x\sin(y) - \cos(x).\end{aligned}$$

So $\mathbf{F}$ is conservative on $\mathbb{R}^2$ by Theorem 4.3.4. Let us now find the potential f such that $\mathbf{F} = \nabla f$. The function $f : \mathbb{R}^2 \to \mathbb{R}$ necessarily satisfies

$$\begin{aligned}\frac{\partial f}{\partial x} &= 2x\cos(y) - y\cos(x) \\ \frac{\partial f}{\partial y} &= -x^2\sin(y) - \sin(x).\end{aligned}$$

The first equation tells us that

$$f(x,y) = x^2\cos(y) - y\sin(x) + h(y),$$

by simply integrating with respect to x. Differentiating this expression for f with respect to y, we see that

$$-x^2\sin(y) - \sin(x) + h'(y) = -x^2\sin(y) - \sin(x),$$

which implies that $h'(y) = 0$, i.e., $h(y) = c$ for some $c \in \mathbb{R}$. Therefore,

$$f(x, y) = x^2 \cos(y) - y \sin(x) + c.$$

Remark 4.3.7. From what we have seen in this section, the obstruction to a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ being conservative is the difference of $\frac{\partial P}{\partial y}$ and $\frac{\partial Q}{\partial x}$. For a vector field in $\mathbb{R}^2$, we set

$$\operatorname{curl}(\mathbf{F}) = \frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}.$$

In §5, we will give a much more natural definition for the curl of a vector field which is possible only for vector fields on $\mathbb{R}^3$.

Remark 4.3.8. Note that as a consequence of Green's theorem, the line integral of a vector field around a closed loop $\int_C \mathbf{F} \cdot d\mathbf{r}$ is zero if and only if $\operatorname{curl}(\mathbf{F}) = 0$.

4.4. **Simply connected regions.** Intuitively, a simply connected region in $\mathbb{R}^n$ is one without holes. Before providing any formal definitions of simply connectedness, we give some examples:

Example 4.4.1.

(i) $\mathbb{R}^n$ is simply connected for all $n \in \mathbb{N}$.
(ii) The circle $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is not simply connected.
(iii) The sphere $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ is simply connected.
(iv) The punctured disk $\Delta^* = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$ is not simply connected.
(v) Convex subsets of $\mathbb{R}^n$ are simply connected.

5. Properties of Vector Fields: Curl and Divergence

The main theme of this section will continue to be on determining whether a given vector field is conservative. It turns out that the obstruction to a vector field being conservative is the presence of a rotational component. This rotational component is made precise by the notion of the curl of a vector field.

5.1. Curl of a Vector Field in $\mathbb{R}^3$.

Definition 5.1.1. Let $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$ be a vector field on $\mathbb{R}^3$, with P, Q, R having continuous partial derivatives everywhere. The **curl** of $\mathbf{F}$ is

$$\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F},$$

where $\times$ denotes the cross product.

In particular, we observe that

$$\begin{aligned}\operatorname{curl}(\mathbf{F}) &= \det\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial_x & \partial_y & \partial_z\\ P & Q & R\end{vmatrix}\\ &= \mathbf{i}\begin{vmatrix}\partial_y & \partial_z\\ Q & R\end{vmatrix} - \mathbf{j}\begin{vmatrix}\partial_x & \partial_z\\ P & R\end{vmatrix} + \mathbf{k}\begin{vmatrix}\partial_x & \partial_y\\ P & Q\end{vmatrix}\\ &= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}.\end{aligned}$$

Theorem 5.1.2. Let $\mathbf{F}$ be a conservative vector field. Then $\operatorname{curl}(\mathbf{F}) = 0$.

Proof. If $\mathbf{F}$ is conservative there exists a function f such that $\mathbf{F} = \nabla f$. In particular,

$$\mathbf{F} = f_x\mathbf{i} + f_y\mathbf{j} + f_z\mathbf{k},$$

where f_x denotes the partial derivative of f with respect to x and similarly for f_y and f_z. That is, $P = f_x$, $Q = f_y$ and $R = f_z$. Inserting this into the above formula for the curl of $\mathbf{F}$, we see that

$$\begin{aligned}\operatorname{curl}(\mathbf{F}) &= \left(\frac{\partial f_z}{\partial y} - \frac{\partial f_y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_x}{\partial z} - \frac{\partial f_z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_y}{\partial x} - \frac{\partial f_x}{\partial y}\right)\mathbf{k}\\ &= (f_{zy} - f_{yz})\,\mathbf{i} + (f_{xz} - f_{zx})\,\mathbf{j} + (f_{yx} - f_{xy})\,\mathbf{k},\end{aligned}$$

but since f is smooth, the partial derivatives commute and we see that $\operatorname{curl}(\mathbf{F}) = 0$, as claimed. □

Example 5.1.3. Consider the vector field $\mathbf{F}(x,y,z) = e^x\sin(y)\mathbf{i} + e^x\cos(y)\mathbf{j} + z\mathbf{k}$.

Theorem 5.1.4. Let $\mathbf{F}$ be a vector field on a simply connected region $D \subset \mathbb{R}^3$. If $\operatorname{curl}(\mathbf{F}) = 0$, then $\mathbf{F}$ is a conservative vector field.

Example 5.1.5. Let

$$\mathbf{F}(x,y) = \left(-\frac{y}{x^2+y^2}\right)\mathbf{i} + \left(\frac{x}{x^2+y^2}\right)\mathbf{j}$$

be a vector field on $\mathbb{R}^2\backslash\{(0,0)\}$.

Video Reference. Example computations of the curl of a vector field are given in the video: https://www.youtube.com/watch?v=-53D3UjXQkQ

5.2. Divergence of a vector field.

Definition 5.2.1. The divergence of a vector field $\mathbf{F}$ on $\mathbb{R}^3$ is given by

$$\operatorname{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}.$$

Proposition 5.2.2. Let $\mathbf{F}$ be a vector field on $\mathbb{R}^3$ whose component functions have continuous second-order partial derivatives. Then

$$\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0.$$

Definition 5.2.3. A vector field $\mathbf{F}$ is said to be **incompressible** if $\operatorname{div}(\mathbf{F}) = 0$.

6. Surface Integrals

7. Stokes' Theorem

7.1. Stokes' Theorem Statement and Proof.

Theorem 7.1.1. Let S be an oriented piecewise smooth surface which is bounded by a simple, closed, piecewise smooth boundary curve C with positive orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region containing S. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_S \operatorname{curl}(\mathbf{F}) d\mathbf{S}.$$

Stokes' Theorem and Path Independence. Of course, we have seen previously that a vector field $\mathbf{F}$ (on a simply connected region) is conservative if and only if $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$. Let us give a slick proof of the path independence of $\int_C \mathbf{F} \cdot d\mathbf{r}$ for conservative vector fields using Stokes' theorem:

Let C_1 and C_2 be two curves which both start at a point p and both terminate at a point q. The line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of path if

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}.$$

Or equivalently,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} = 0.$$

But the left-hand side of the above equation is integral over the closed loop $C = C_1 - C_2$, and by Stokes' theorem we have

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} - \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_C \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_S \operatorname{curl}(\mathbf{F}) d\mathbf{S},$$

but this last integral is zero if and only if $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$, i.e., the line integral is independent of the choice of path if and only if $\mathbf{F}$ is conservative.

Video Reference. The above argument is presented in the video: https://www.youtube.com/watch?v=2FGKxFG5lSI

8. The Divergence Theorem

Divergence Theorem Statement and Proof.

Theorem 8.1.1. Let E be a simple solid region and let S be the boundary of the surface of E, given with positive (outward) orientation. Let $\mathbf{F}$ be a vector field whose components have continuous partial derivatives on an open region that contains E. Then

$$\iint_S \mathbf{F} \cdot d\mathbf{S} = \iiint_E \operatorname{div}(\mathbf{F}) dV.$$