SOME RESULTS RELATING TO INVERSE PROBLEMS ON RIEMANN SURFACES

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Abstract. In this short note, we prove that for a compact Riemann surface M with smooth boundary ∂M, there exists an open subset $\Gamma \subset \partial M$ and a conformal map $f : M \to \mathbb{C}$ such that $f(\Gamma) \subset \mathbb{R}$. We show that Γ cannot be arbitrarily chosen, and discuss results for bounded domains in $\mathbb{C}$. This work is motivated by attempts to extend results of Guillarmou–Tzou [**?**].

§1. Introduction

§2. Construction of Conformal maps with prescribed real parts

Simply Connected Case. Consider first the case when Ω is a bounded, simply connected domain in $\mathbb{C}$ with smooth boundary. The Riemann mapping theorem gives a conformal map to the upper-half space taking the boundary of Ω into the real line (the boundary of the upper-half space). Since the boundary of Ω is smooth, results of Bell–Krantz [1] (see also [3, 4]) ensure that this conformal map is smooth up to the boundary.

Finitely Connected Case. Let now Ω be a finitely connected planar domain with smooth boundary. Suppose Ω has k boundary components. By the continuity of a conformal map, the number of boundary components of Ω is preserved. By Koebe's extension of the Riemann mapping theorem (see, e.g., [5]) there is a conformal map φ from Ω onto a parallel slit domain Λ (with k parallel slits). Since Γ is connected, Γ lies in a single boundary component of Ω and is thus mapped to a single slit under the conformal map. It is then easy to map this slit to the real line. Concerning the regularity of this conformal map, we note that since Ω is a bounded domain with smooth boundary, we can apply the results of Bell–Krantz [1] (see also [3, 4]) again to conclude that $\varphi \in \mathcal{C}^\infty(\overline{\Omega})$.

Riemann Surface Case. For a general Riemann surface, the main ingredient is the following old result of Gunning–Narasimhan:

Theorem. ([2]). Let M be a connected open Riemann surface. There exists a holomorphic immersion $f : M \longrightarrow \mathbb{C}$.

The idea behind the proof of Theorem is as follows: Every holomorphic line bundle on M is trivial; in particular, the holomorphic cotangent bundle of M is trivial. So we can find a non-vanishing holomorphic 1–form ω on M which is closed. In general, such a 1–form will fail to be exact – the periods of ω may be non-trivial. The key insight of [2] is to consider a Runge exhaustion of M and construct a holomorphic function $g : M \longrightarrow \mathbb{C}$ to appropriately twist ω such that $e^g\omega$ is exact. The immersion $f : M \longrightarrow \mathbb{C}$ is then given by the primitive of this exact form.

Theorem. Let M be a compact (connected) Riemann surface with smooth boundary ∂M. Then there is an arc $\Gamma \subset \partial M$ and a holomorphic immersion $f : M \to \mathbb{C}$ such that $f(\Gamma) \subset \mathbb{R}$.

Proof. First, by compactness, M has only finitely many boundary components. We can close these off and assume that M has only one boundary component, which can be identified with an analytic circle S. The open set Γ is now an arc in S. The boundary of M is smooth, so we can find an open Riemann surface $\widetilde{M} \supset M$. By Gunning–Narasiman, we have a holomorphic immersion $f : \widetilde{M} \longrightarrow \mathbb{C}$. Now $f|(M \setminus \partial M)$ is open and so the boundary of this set lies in the image of the boundary. Let α be any component of the boundary of $f(M)$. Smoothness assures us that α separates the complex plane into simply connected regions (in the Riemann sphere). One of these contains $f(M)$ and we choose the Riemann mapping from this component to the upper-half space. The composition is now a conformal immersion one boundary component of which is the real line. The arc Γ can be any one of the finitely many components of the preimage of the real line. $\square$

Claim 2. The Γ in the previous claim cannot be arbitrarily chosen.

Argument. For the purposes of obtaining a contradiction, suppose the arc Γ in the above theorem can be chosen arbitrarily. Let S be the boundary circle as above, and parametrize S by some coordinate ϑ. Let Γ_j be the arc defined by $\Gamma_j := \left\{0 \leq \vartheta < (2\pi - \frac{1}{j})\right\}$, and let $f_j : M \to \mathbb{C}$ be holomorphic immersions with $f_j(\Gamma_j) \subset \mathbb{R}$. It suffices to show that $\{f_j\}$ is a normal family. Indeed, granted this, we can extract a subsequence which converges to some holomorphic immersion $f : M \to \mathbb{C}$ which takes the entire boundary circle to the real line, and this is not possible. To show that $\{f_j\}$ has a convergent subsequence, we note that since M is compact, this family is bounded. It suffices to show that the family is closed. $\square$

§3. Further Directions and Remarks on higher-dimensional analogues

The results we have included in this note are highly specific to the techniques of one variable complex function theory. Indeed, the Riemann mapping theorem permeates throughout the entire discussion. We mention, however, some potential further directions that may be of interest.

Let us start with a negative result in this direction due to Tumanov:

Theorem. Let $\Omega = \{z \in \mathbb{C}^{n>1} : \varphi(z) < 0\}$ be a domain with $\mathcal{C}^2$ boundary $\partial\Omega$. Let $\Gamma \subset \partial\Omega$ be a subset with positive $(2n-1)$–dimensional volume, on which the Levi form does not vanish:

$$L(\zeta, \varphi, v) \neq 0, \qquad \forall \zeta \in \Gamma,\ v \in T^{\mathbb{C}}_{\zeta}(\Gamma),\ v \neq 0.$$

If the function $f \in \mathcal{C}^2(\overline{\Omega}) \cap \mathcal{O}(\Omega)$ is real-valued on Γ, then f is constant.

The stronger rigidity in the higher-dimensional setting is, of course, due to the presence of the CR structure on the boundary. Indeed, the proof of the above theorem relies crucially on the tangently CR equations. Note also that the above theorem includes the (strictly) pseudoconvex case. We are not aware of any results of this type in the Levi-flat situation.

References

[1] Bell, S. R., Krantz, S. G., *Smoothness to the boundary of conformal maps*, The Rocky Mountain Journal of Mathematics, vol. 17, no. 1 (winter 1987), pp. 23–40.

[2] Gunning, R. C., Narasimhan, R., *Immersion of open Riemann surfaces*, Math. Ann., 174:103–108, (1967).

[3] Krantz, S. G., *Geometric Function Theory*. Cornerstones. Birkhäuser Boston, (2006)

[4] Krantz, S. G., *Partial Differential Equations and Complex Analysis*, CRC Press, Inc., (1992).

[5] Nehari, Z., *Conformal Mapping*, Dover Publications, Inc., New York, (1952).