Calculus Exam 1 Solutions

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The contents of this examination require an understanding of the elementary calculus material that was covered in the calculus practice exams 1 and 2. An understanding of graphing techniques and function transformations are also required.

There are no permitted materials for this test. That is, you are not permitted any cheat notes, calculators or resources other than a pen/pencil, eraser, sharpener, ruler and water bottle.

There is to be no collaboration on this examination and any attempts of communication will result in a nullified score. You are permitted 10 minutes of reading time and 80 minutes of writing time. There is a total of 100 available marks. It is recommended that you use the reading time to ask the invigilator about any issues regarding the format of the test, the problems or other issues. No hints will be given. Best of luck!

Name: ____________________

Grade: ___________/100

Question 1. [10 marks]. Evluate the following limit

$$\lim_{x\to 3} \frac{x^2 - 5x + 6}{x - 3}.$$

Proof. Begin by writing $x^2 - 5x + 6 = (x-3)(x-2)$. Then

$$\begin{aligned} \lim_{x\to 3} \frac{x^2 - 5x + 6}{x - 3} &= \lim_{x\to 3} \frac{(x-3)(x-2)}{x-3} \\ &= \lim_{x\to 3} x - 2 = 3 - 2 = 1. \end{aligned}$$

□

Marking scheme: 5 marks for working, 3 marks for answer, 2 marks for writing limit at every step.

Question 2. [25 marks]. Consider the function $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{3-x}.$$

Show that f is differentiable on $\mathbb{R}\backslash\{3\}$ and compute $f'(x)$.

Proof. To show that f is differentiable on $\mathbb{R}\backslash\{3\}$, we need to show that

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} < \infty,$$

for all $x \in \mathbb{R}\backslash\{3\}$. To this end, we observe that

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{1}{3-(x+\Delta x)} - \frac{1}{3-x}\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{3-x}{(3-x-\Delta x)(3-x)} - \frac{(3-x-\Delta x)}{(3-x)(3-x-\Delta x)}\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left(\frac{3-x-3+x+\Delta x}{(3-x-\Delta x)(3-x)}\right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\frac{\Delta x}{(3-x-\Delta x)(3-x)} \\
&= \lim_{\Delta x \to 0} \frac{1}{(3-x-\Delta x)(3-x)} \\
&= \frac{1}{(3-x)^2},
\end{aligned}$$

which is bounded for all $x \in \mathbb{R}\backslash\{3\}$. Hence we have shown that f is differentiable on $\mathbb{R}\backslash\{3\}$ and

$$f'(x) = \frac{1}{(3-x)^2}.$$

$\square$

Marking sheme: 20 marks for working, 5 marks for the derivative.

Question 3. [25 marks]. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{4}{x^{\frac{3}{2}} + 1} + \sqrt{x}.$$

Explain why f is differentiable on $(0, \infty)$ and compute $f'(x)$.

Proof. The functions $x \mapsto 4$, $x \mapsto x^{\frac{3}{2}} + 1$, $x \mapsto \frac{1}{x}$ and $x \mapsto \sqrt{x}$ are all differentiable on $(0, \infty)$. Since the composition of differentiable functions is differentiable, it follows that $f(x)$ is differentiable on $(0, \infty)$. To compute $f'(x)$, let us begin by writing

$$f(x) = 4(x^{\frac{3}{2}} + 1)^{-1} + \sqrt{x}.$$

The derivative of $x^{\frac{3}{2}} + 1$ is $\frac{3}{2}\sqrt{x}$ and so, using the chain rule, we see that

$$\begin{aligned} f'(x) &= 4 \cdot \frac{3}{2}\sqrt{x} \cdot (x^{\frac{3}{2}} + 1)^{-2} + \frac{1}{2\sqrt{x}} && (1) \\ &= \frac{6\sqrt{x}}{(x^{\frac{3}{2}} + 1)^2} + \frac{1}{2\sqrt{x}}. && (2) \end{aligned}$$

□

Marking scheme: 10 marks for justification of f being differentiable on $(0, \infty)$, 10 marks for the computation of $f'(x)$ (getting to equation (1)), 5 marks for the algebraic simplification (getting to equation (2)).

Question 4. [20 marks]. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

This function forms the prototypical example of a function that is continuous but *not* differentiable.

a. [10 marks]. Explain exactly what is meant by a function not being differentiable.

Proof. For a function to be differentiable the left derivative has to equal the right derivative. Equivalently, a function f is differentiable at x if the limit

$$\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} < \infty,$$

and exits. Intuitively, a function is not differentiable at sharp points. □

Marking scheme: 5 marks for the first and third definitions, 10 marks for the second definition.

b. [5 marks]. Determine the point(s) where f is not differentiable.

Proof. f is not differentiable at $x = 0$ since the left derivative is -1 and the right derivative is 1. □

Marking scheme: 3 marks for $x = 0$, 2 marks for justification.

c. [5 marks]. Compute the derivative of f at the points where f is differentiable.

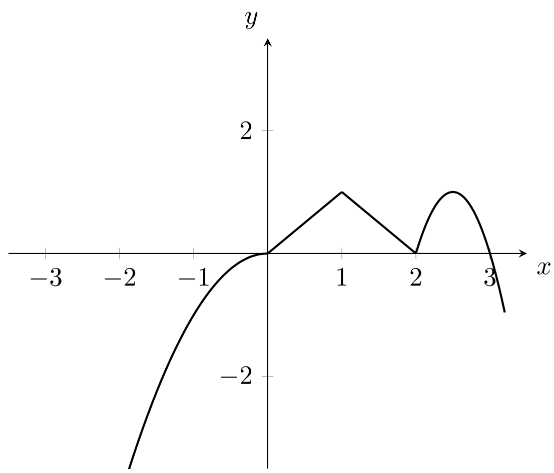
Proof. The derivative is simply given by

$$f'(x) = \begin{cases} 1, & x > 0, \\ -1, & x < 0. \end{cases}$$

□

Marking scheme: 4 marks for computation, 1 mark for getting the inequalities correct.

Question 5. [20 marks]. Consider the function f whose graph is given below.



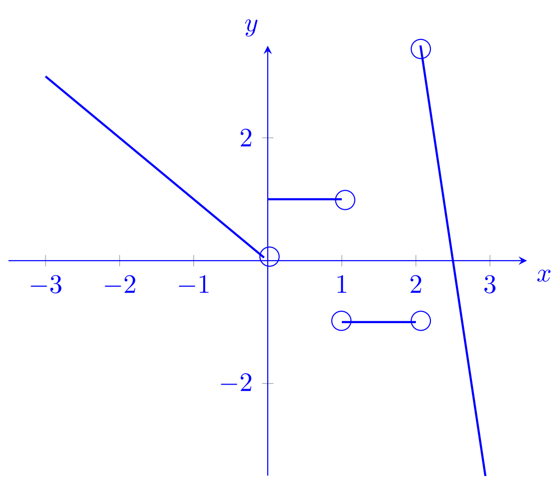
Determine the domain of $f'(x)$ and sketch $f'(x)$ on this domain.

Proof. The function $f(x)$ has sharp points at $x = 0, x = 1, x = 2$ and $x = 3$.

Let us now collect some information about f' before attempting to sketch $f'(x)$.

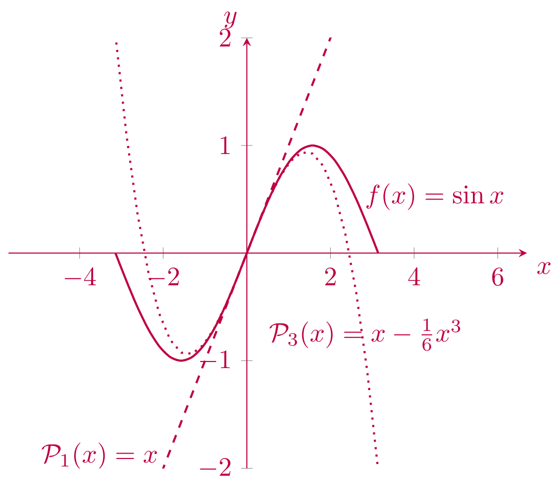
† For $-\infty < x < 0$, the slope of f is decreasing to 0.
† For $0 < x < 1$, the slope of f is constantly 1.
† For $1 < x < 2$, the slope of f is constantly -1.
† For $2 < x < \frac{5}{2}$, the slope is decreasing to zero.
† For $\frac{5}{2} < x < \infty$, the slope is becoming more negative.

Hence we see that the graph of $f'(x)$ is given by



□

Marking scheme: 10 marks for the domain of the $f'(x)$, 10 marks for the shape of the curve.



y
2
1
$f(x) = \sin x$
-4
-2
2
4
6
x
$\mathcal{P}_3(x) = x - \frac{1}{6}x^3$
-1
$\mathcal{P}_1(x) = x$
-2