ADDITIONAL MATH1013 TUTORIAL PROBLEMS - WEEK 3

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Based on the results of the last tutorial quiz, students primarily struggled with definitions of linear independence, continuity and the distinction between span and linear independence. Students were also not able to correctly apply the squeeze theorem or intermediate value theorem.

Linear Algebra

1. Define what it means for a set of vectors $\{v_1, ..., v_n\}$ in $\mathbb{R}^n$ to be linearly independent.

2. Define what it means for a set of vectors $\{v_1, ..., v_n\}$ in $\mathbb{R}^n$ to span $\mathbb{R}^n$.

3. Provide an example of a set of vectors in $\mathbb{R}^3$ which are linearly independent and an example of a set of vectors in $\mathbb{R}^3$ which are linearly dependent.

4. Determine, with justification, whether the following are true or false:

a. A set of linearly independent vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$.
b. A set of vectors which span $\mathbb{R}^n$ are linearly independent.
c. Two vectors in $\mathbb{R}^n$ will always span a plane.

Calculus

1. Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous.

2. Define what it means for a function $f : X \to \mathbb{R}$ to be continuous, where $X \subseteq \mathbb{R}$ is some subset of $\mathbb{R}$.

3. Determine the point(s) where the function $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$, defined by $x \mapsto \frac{1}{x}$, is discontinuous.

4. Let $f, g \in \mathscr{C}(\mathbb{R})$, where $\mathscr{C}(\mathbb{R})$ denotes the space of continuous functions on the real line. Determine, with justification, whether

a. $f + g \in \mathscr{C}(\mathbb{R})$.
b. $f - g \in \mathscr{C}(\mathbb{R})$.
c. $f \cdot g \in \mathscr{C}(\mathbb{R})$.
d. $f/g \in \mathscr{C}(\mathbb{R})$.
e. $\sqrt{f} \in \mathscr{C}(\mathbb{R})$.

f. $\log_e |g| \in \mathscr{C}(\mathbb{R})$.

5. I make the claim that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if, for all choices of $\varepsilon > 0$, we may find some real number $\delta > 0$ such that

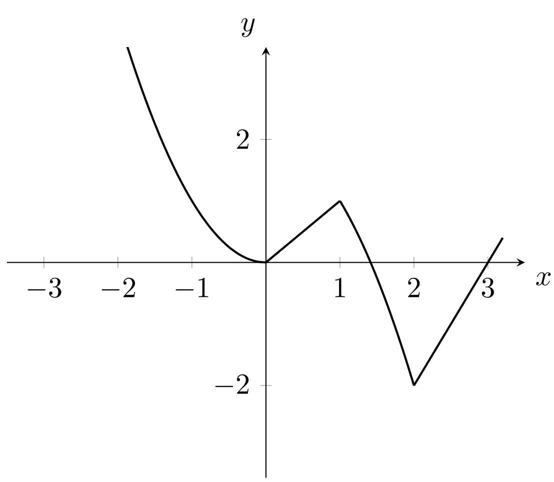
$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon, \tag{1}$$

for all $x \in \mathbb{R}$. Describe intuitively what equation (1) means and determine whether it agrees with the definition of continuity described in Question 1.

6. Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be differentiable.

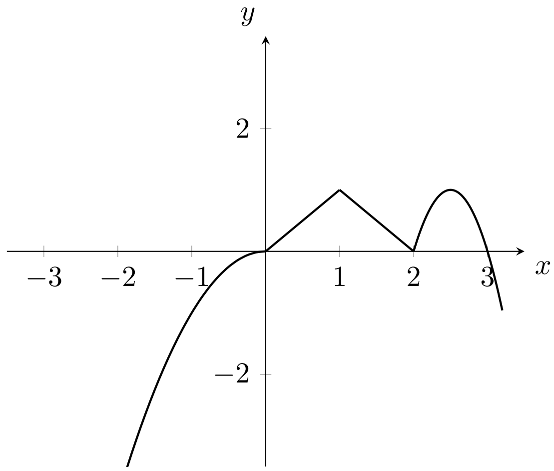
7. Suppose that f is a function which is differentiable for all $x \in \mathbb{R}$. Is it necessarily true that the derivative of f, $f'(x)$ is differentiable? Prove or provide a counterexample to justify your answer.

8. Consider the function whose graph is given by



Determine all points $x \in \mathbb{R}$ where f is not differentiable.

9. Consider the function g whose graph is given by



Determine all points $x \in \mathbb{R}$ where g is not differentiable.

10. Evaluate the following limits.

a.
$$\lim_{x\to 0} x^2 \exp\left(\sin\left(\frac{1}{x}\right)\right).$$

b.
$$\lim_{x\to 0} x^4 \cos\left(\frac{x^3+5}{x^7}\right).$$

c.
$$\lim_{x\to 0} x^3 \sin\left(\frac{1}{\sqrt[3]{x}}\right).$$

d.
$$\lim_{x\to \infty} \frac{2x^3+\sin(x^2)}{1+x^3}.$$

e.
$$\lim_{x\to \infty} e^{-x^2} \sin\left(\frac{2\pi x}{1+7x}\right).$$

11. Show that there exists a real number x such that $f(x)=0$, where
$$f(x) = 6x^3 + 4 + e^{-x^2} + \sin x.$$

12. Show that the intermediate value theorem does not hold for $\mathbb{Q}$.