LINEAR ALGEBRA – LECTURE 1

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Linear algebra is the central subject of modern mathematics and has been shown to be essential tool in modern data science, computer science, physics, biology, etc. These lectures outline material I have taught at the Australian National University (ANU) and the University of Sydney (USYD). The emphasis is on the underlying unifying framework upon which the subject sits. This is often missed out in standard treatments of linear algebra.

Definition 1.1. A *real vector space* is a set V satisfying the following two properties:

(i) $u + v \in V$ for all $u, v \in V$.
(ii) $\lambda v \in V$ for all $v \in V$, $\lambda \in \mathbb{R}$.

Condition (i) is referred to as V being *closed under addition*, and condition (ii) is referred to as V being *closed under scalar multiplication*.

We also impose some additional conditions on a vector space:

(i) The set V forms a commutative group with respect to the operation of addition, and the identity element is denoted by $\mathbf{0}$, and is called the *zero vector*.
(ii) For all $\lambda, \mu \in \mathbb{R}$ and $u, v \in V$,
  (a) $\lambda(u + v) = \lambda u + \lambda v$.
  (b) $(\lambda + \mu)v = \lambda v + \mu v$.
  (c) $(\lambda\mu)v = \lambda(\mu v)$.
  (d) $0v = \mathbf{0}$ and $1v = v$.

Definition 1.2. Elements of a vector space V are called *vectors* and elements of the underlying field $\mathbb{R}$ are called *scalars*.

Example 1.3.

(i) $\mathbb{R}$ is a real vector space.
(ii) $\mathbb{C} = \{x + \sqrt{-1}y : x, y \in \mathbb{R}\}$ is a real vector space. It is also a complex vector space.
(iii) The set of quadratic polynomials $Q = \{ax^2 + bx + c : a, b, c \in \mathbb{R}\}$ is a real vector space.

Definition 1.4. Let V be a real vector space. A map $T : V \to W$ is said to be *linear* if

(i) $T(u + v) = T(u) + T(v)$ for all $u, v \in V$.
(ii) $T(\lambda v) = \lambda T(v)$ for all $v \in V$, $\lambda \in \mathbb{R}$.

Exercise 1.5. Determine which of the following are linear maps:

(i) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x$.
(ii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = 2x + 3$.
(iii) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 - 5x + 6$.
(iv) $T : \mathbb{R}^2 \to \mathbb{R}^2$, where T is defined to be rotation by 90° anti-clockwise.
(v) $T : \mathbb{R}^2 \to \mathbb{R}^2$, where T is defined to be rotation by 90° clockwise.