THE SCALAR MAXIMUM PRINCIPLE

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Proposition. Let $g(t)$, for $t \in [0,T)$, be a family of Riemannian metrics on a compact manifold M. Suppose $u : M \times [0,T) \to \mathbb{R}$ satisfies

$$\frac{\partial u}{\partial t} - \Delta_{g(t)} u \quad \geq \quad 0.$$

If $u(\bullet, 0) \geq C$ for some $C \in \mathbb{R}$, then $u(\bullet, t) \geq C$ for all $t \in [0,T)$.

Proof. Fix $\varepsilon > 0$, and set $u_\varepsilon := u + \varepsilon(1+t)$; by assumption, $u_\varepsilon(\bullet, 0) > C$. Proceed by contradiction, supposing that one may choose $\varepsilon > 0$ such that $u_\varepsilon(x,t) \leq C$ for some $(x,t) \in M \times [0,T)$. Since M is compact, there is a point $(x_0, t_0) \in M \times [0,T)$ such that $u_\varepsilon(x_0, t_0) = C$ and $u_\varepsilon(x,t) \geq C$ for all $x \in M$ and $0 \leq t \leq t_0$. Of course, at (x_0, t_0), we have $\frac{\partial u_\varepsilon}{\partial t} \leq 0$ and $\Delta_{g(t)} u_\varepsilon \geq 0$. Hence,

$$0 \ \geq \ \frac{\partial u_\varepsilon}{\partial t} \quad \geq \quad \Delta_{g(t)} u_\varepsilon + \varepsilon \ > \ 0,$$

furnishing the desired contradiction. It follows that $u_\varepsilon(x,t) > C$ for all $(x,t) \in M \times [0,T)$, and since $\varepsilon > 0$ was arbitrary, we have $u(x,t) \geq C$ for all $(x,t) \in M \times [0,T)$. □