MOSTOW–SIU METRIC

KYLE BRODER

Let u, v be two real vectors in the complexified tangent space of a compact Kähler manifold M. We will compute the sectional curvature of the plane spanned by u and v. We may write $u = 2\mathrm{Re}(\xi)$, and $v = 2\mathrm{Re}(\eta)$, where $\xi = \xi^{\alpha}\partial_{z^{\alpha}}$, and $\eta = \eta^{\alpha}\partial_{z^{\alpha}}$. The sectional curvature of the plane spanned by u and v is

$$\frac{1}{\|u \wedge v\|^2} \sum_{i,j,k,\ell} R_{ijk\ell} u^i v^j u^k v^{\ell}.$$

We write $\xi^{\overline{\alpha}} := \overline{\xi^{\alpha}}$, and $\eta^{\overline{\alpha}} = \overline{\eta^{\alpha}}$. Then, since the only non-zero components of the curvature tensor are $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$, $R_{\overline{\alpha}\beta,\overline{\gamma}\delta}$, $R_{\alpha\overline{\beta}\overline{\gamma}\delta}$ and $R_{\overline{\alpha}\beta\gamma\overline{\delta}}$, we have

$$\begin{aligned}
\sum_{i,j,k,\ell} R_{ijk\ell} u^i v^j u^k v^{\ell} &= \sum_{\alpha,\beta,\gamma,\delta} \left(R_{\alpha\overline{\beta}\gamma\overline{\delta}} \xi^{\alpha}\eta^{\overline{\beta}}\xi^{\gamma}\eta^{\overline{\delta}} + R_{\alpha\overline{\beta}\overline{\gamma}\delta}\xi^{\alpha}\eta^{\overline{\beta}}\xi^{\overline{\gamma}}\eta^{\delta} + R_{\overline{\alpha}\beta\gamma\overline{\delta}}\xi^{\overline{\alpha}}\eta^{\beta}\xi^{\gamma}\eta^{\overline{\delta}} + R_{\overline{\alpha}\beta\overline{\gamma}\delta}\xi^{\overline{\alpha}}\eta^{\beta}\xi^{\overline{\gamma}}\eta^{\delta} \right) \\
&= \sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}} \left(\xi^{\alpha}\eta^{\overline{\beta}} - \eta^{\alpha}\xi^{\overline{\beta}} \right) \left(\eta^{\overline{\gamma}}\xi^{\delta} - \xi^{\overline{\gamma}}\eta^{\delta} \right) \\
&= - \sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}} \left(\xi^{\alpha}\eta^{\overline{\beta}} - \eta^{\alpha}\xi^{\overline{\beta}} \right) \overline{(\eta^{\overline{\gamma}}\xi^{\delta} - \xi^{\overline{\gamma}}\eta^{\delta})}.
\end{aligned}$$

Definition. A Kähler metric with curvature R is said to have negative sectional curvature if

$$\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}} \left(\xi^{\alpha}\eta^{\overline{\beta}} - \eta^{\alpha}\xi^{\overline{\beta}} \right) \overline{(\eta^{\overline{\gamma}}\xi^{\delta} - \xi^{\overline{\gamma}}\eta^{\delta})} \quad > \quad 0,$$

for all ξ, η with $\mathrm{Re}(\xi) \wedge \mathrm{Re}(\eta) \neq 0$.

Sectional curvature of Kähler surfaces. In the case of complex dimension $n = 2$, the sectional curvature affords the following description:

$$\begin{aligned}
&\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}} \left(\xi^{\alpha}\eta^{\overline{\beta}} - \eta^{\alpha}\xi^{\overline{\beta}} \right) \overline{(\eta^{\overline{\gamma}}\xi^{\delta} - \xi^{\overline{\gamma}}\eta^{\delta})} \\
= \quad & R_{1\overline{1}1\overline{1}} |\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}}|^2 + 4\mathrm{Re}\left(R_{1\overline{1}1\overline{2}} (\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}}) \overline{(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}})} \right) + R_{2\overline{2}2\overline{2}} |\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}}|^2 \\
& + 2R_{1\overline{1}2\overline{2}} \left(|\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}}|^2 + \mathrm{Re}(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}}) \overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})} \right) \\
& + 2\mathrm{Re}\left(R_{1\overline{2}1\overline{2}} \left(\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}} \right) \overline{\left(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}} \right)} \right) + 4\mathrm{Re}\left(R_{2\overline{2}1\overline{2}} \left(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}} \right) \overline{\left(\xi^2\eta^{\overline{1}} - \eta^2\xi^{\overline{1}} \right)} \right).
\end{aligned}$$

Moreover,

$$\|u \wedge v\|^2 \;=\; g_{1\overline{1}}^2|\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}}|^2 + 2g_{1\overline{1}}g_{2\overline{2}}\left(|\xi^1\eta^2 - \eta^1\xi^2|^2 + |\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}}|^2\right) + g_{2\overline{2}}|\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}}|^2.$$

Lemma. Suppose $g_{\alpha\overline{\beta}}$ is a Kähler metric in a neighbourhood of the origin in $\mathbb{C}^2$ with $g_{1\overline{2}}(0) = 0$. Suppose that all components of $R_{\alpha\overline{\beta}\gamma\overline{\delta}}$ are zero, except $R_{1\overline{1}1\overline{1}}$, $R_{1\overline{1}2\overline{2}}$, and $R_{2\overline{2}2\overline{2}}$. Then the sectional curvatures at the origin are negative if and only if $R_{1\overline{1}1\overline{1}} > 0$, $R_{1\overline{1}2\overline{2}} > 0$, $R_{2\overline{2}2\overline{2}} > 0$, and $(R_{1\overline{1}2\overline{2}})^2 < R_{1\overline{1}1\overline{1}}R_{2\overline{2}2\overline{2}}$.

Proof. The assumptions on the curvature tensor permit us to write

$$\begin{aligned}
&\sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}}\left(\xi^\alpha\eta^{\overline{\beta}} - \eta^\alpha\xi^{\overline{\beta}}\right)\overline{(\eta^{\overline{\gamma}}\xi^\delta - \xi^{\overline{\gamma}}\eta^\delta)} \\
= \;& R_{1\overline{1}1\overline{1}}|\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}}|^2 + R_{2\overline{2}2\overline{2}}|\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}}|^2 \\
& + 2R_{1\overline{1}2\overline{2}}\left(|\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}}|^2 + \mathrm{Re}(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})}\right),
\end{aligned}$$

which is negative if and only if this expression is some non-negative multiple of

$$|\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}}|^2 + |\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}}|^2 + |\xi^1\eta^2\eta^1\xi^2|^2 + |\xi^2\eta^2 - \eta^2\xi^2|^2.$$

Necessity is established from special choices of ξ^α and η^α:

- $\xi^2 = \eta^2 = 0 \implies R_{1\overline{1}1\overline{1}} > 0$,
- $\xi^1 = \eta^1 = 0 \implies R_{2\overline{2}2\overline{2}} > 0$,
- $\xi^1 = \eta^2 = 0 \implies R_{1\overline{1}2\overline{2}} > 0$,
- $\xi^1 = a\sqrt{-1}$, $\xi^2 = -\sqrt{-1}$, $\eta^1 = a$, $\eta^2 = 1$, for $a > 0$, implies

$$R_{1\overline{1}1\overline{1}}4a^4 - 2R_{1\overline{1}2\overline{2}}4a^2 + 4R_{2\overline{2}2\overline{2}} > 0.$$

  Hence, $(R_{1\overline{1}2\overline{2}})^2 < R_{1\overline{1}1\overline{1}}R_{2\overline{2}2\overline{2}}$.

To prove sufficiency: we simply note that

$$\begin{aligned}
|2R_{1\overline{1}2\overline{2}}(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})}| &\leq |2(R_{1\overline{1}1\overline{1}}R_{2\overline{2}2\overline{2}})^{\frac{1}{2}}(\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})}| \\
&\leq R_{1\overline{1}1\overline{1}}|\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}}|^2 + R_{2\overline{2}2\overline{2}}|\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}}|^2.
\end{aligned}$$

Moreover,

$$\begin{aligned}
|\xi^1\eta^2 - \eta^1\xi^2|^2 &= |\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}}|^2 - (\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}})\overline{(\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}})} \\
&\leq |\xi^1\eta^{\overline{2}} - \eta^1\xi^{\overline{2}}|^2 + \frac{1}{2}|\xi^1\eta^{\overline{1}} - \eta^1\xi^{\overline{1}}|^2 + \frac{1}{2}|\xi^2\eta^{\overline{2}} - \eta^2\xi^{\overline{2}}|^2.
\end{aligned}$$

$\square$

Bergman Kernel. Let D be the domain in $\mathbb{C}^2$ defined by $|z_1|^{2m} + |z_2|^2 < 1$. For $\alpha = 1, 2$, let $z_\alpha = r_\alpha e^{i\vartheta_\alpha}$. The Euclidean volume form dV on $\mathbb{C}^2$ is given by $(r_1 dr_1 d\vartheta_1)(r_2 dr_2 d\vartheta_2)$. The inner product of $z_1^k z_2^\ell$ and $z_1^p z_2^q$ is given by

$$\begin{aligned}(z_1^k z_2^\ell, z_1^p z_2^q) &= \int_D (z_1^k z_2^\ell)\overline{(z_1^p z_2^q)}dV \\ &= \int_{r_1^{2m}+r_2^2<1} \int_{0\le\vartheta_1\le 2\pi, 0\le\vartheta_2\le 2\pi} r_1^{k+p+1} r_2^{\ell+q+1} e^{i(k-p)\vartheta_1} e^{i(\ell-q)\vartheta_2} dr_1 dr_2 d\vartheta_1 d\vartheta_2,\end{aligned}$$

which is zero, unless $(k, \ell) = (p, q)$.
Consider therefore, the case $(k, \ell) = (p, q)$. We have

$$\begin{aligned}\frac{1}{4\pi^2}(z_1^p z_2^q, z_q^p, z_2^q) &= \int_{r_1^{2m}+r_2^2<1, r_1\ge 0, r_2\ge 0} r_1^{2p+1} r_2^{2q+1} dr_1 dr_2 \\ &= \int_{r_1=0}^1 r_1^{2p+1} \int_{r_2=0}^{\sqrt{1-r_1^{2m}}} r_2^{2q+1} dr_2 dr_1 \\ &= \frac{1}{2(q+1)} \int_0^1 r_1^{2p+1}(1 - r_1^{2m})^{q+1} dr_1.\end{aligned}$$

Set $u = r_1^2$; which yields

$$\frac{1}{4(q+1)} \int_0^1 u^p (1 - u^m)^{q+1} du = \frac{m^{q+1} q!}{4(p+1)(p+m+1)\cdots(p+(q+1)m+1)}.$$

The Bergman kernel is

$$\begin{aligned}\Phi(z_1, z_2) &= \sum_{p,q=0}^{\infty} \frac{1}{m^{p+1}\pi^2} \left(\frac{(p+1)(p+m+1)\cdots(p+(q+1)m+1)}{q!}\right) |z_1|^{2p}|z_2|^{2q} \\ &= \sum_{p=0}^{\infty} \left(\frac{(p+1)(p+m+1)}{m\pi^2}\right) \\ &\quad \cdot \left(\sum_{q=0}^{\infty} \frac{1}{q!} \left(\frac{p+2m+1}{m}\right)\left(\frac{p+2m+1}{m}+1\right)\cdots\left(\frac{p+2m+1}{m}+q-1\right)|z_2|^{2q}\right)|z_1|^{2p}.\end{aligned}$$

Claim.

$$\frac{1}{(1-|x|^2)^\alpha} = \sum_{k=0}^{\infty} \frac{1}{k!}\alpha(\alpha+1)\cdots(\alpha+q-1)x^{2k}.$$

Assuming the claim, and setting $\alpha = \frac{p+2m+1}{m}$, we see that

$$\begin{aligned}
\Phi(z_1, z_2) &= \sum_{p=0}^{\infty} \left(\frac{(p+1)(p+m+1)}{m\pi^2} \right) (1-|z_2|^2)^{-\frac{1}{m}(p+2m+1)} |z_1|^{2p} \\
&= \frac{1}{m\pi^2}(1-|z_2|^2)^{-\frac{1}{m}(2m+1)} \sum_{p=0}^{\infty} (p+1)(p+m+1)(1-|z_2|^2)^{-\frac{p}{m}} |z_1|^{2p} \\
&= \frac{1}{m\pi^2}(1-|z_2|^2)^{-\frac{1}{m}(2m+1)} \sum_{p=0}^{\infty} (p+1)(p+2) \left((1-|z_2|^2)^{-\frac{1}{m}} |z_1|^2 \right)^p \\
&\quad + \frac{1}{m\pi^2}(1-|z_2|^2)^{-\frac{1}{m}(2m+1)} \sum_{p=0}^{\infty} (p+1)(m-1) \left((1-|z_2|^2)^{-\frac{1}{m}} |z_1|^2 \right)^p .
\end{aligned}$$

Claim.

$$\left(1 - \frac{|z_1|^2}{(1-|z_2|^2)^{\frac{1}{m}}} \right)^{-\alpha} = \sum_{p=0}^{\infty} \frac{(-\alpha)\cdots(-\alpha-p+1)}{p!} \left(-(1-|z_2|^2)^{-\frac{1}{m}} |z_1|^2 \right)^p .$$

From the claim, it follows that

$$\Phi(z_1, z_2) = \frac{1}{m\pi^2} \frac{(m+1)(1-|z_2|^2)^{\frac{1}{m}} - (m-1)|z_1|^2}{(1-|z_2|^2)^{2-\frac{1}{m}} \left((1-|z_2|^2)^{\frac{1}{m}} - |z_1|^2 \right)^3}$$

Construction of the Mostow–Siu metric. Let us first remark on the construction of the surface. We first construct a subgroup $\Gamma \subset \mathrm{Aut}(\mathbb{B}^2)$ generated by three complex reflections. This subgroup Γ is not discrete, but is almost discrete in the sense that there is a complex surface Y and a holomorphic map $\sigma : Y \to \mathbb{B}^2$ such that Γ lifts to a discrete subgroup $\widetilde{\Gamma} \subset \mathrm{Aut}(Y)$, and the only kind of singularity of σ is simple winding singularity along an infinite number of disjoint complex curves whose images are complex lines in $\mathbb{B}^2$. The complex surface is the quotient of Y by $\widetilde{\Gamma}_0$, for some subgroup $\widetilde{\Gamma}_0 \subset \widetilde{\Gamma}$ of finite index, chosen solely for the purpose of making $Y/\widetilde{\Gamma}_0$ non-singular.

Let E be the set of all points of Y where the Jacobian determinant of $\sigma : Y \to \mathbb{B}^2$ is zero. Then E is the disjoint union of a countable number of non-singular complex curves E_i. Each E_i is biholomorphic onto its image $\sigma(E_i)$.

For each i, there is an open neighbourhood Q_i of E_i in Y, and an open neighbourhood U_i of $\sigma(E_i)$ in $\mathbb{B}^2$ such that we have the following commutative diagram

(see page 339 of Mostow–Siu).

where $\sigma(Q_i) = U_i$, the map ϑ is a biholomorphism, $\tau \in \mathrm{Aut}(\mathbb{B}^2)$ sends $\sigma(E_i)$ to the complex line $\{w_1 = 0\} \cap B$ with B realized as $\{(w_1, w_2) : |w_1|^2 + |w_2|^2 < 1\}$. The Thullen domain D is described by $D = \{(z_1, z_2) : |z_1|^{14} + |z_2|^2 < 1\}$, and $\pi(w_1, w_2) := (w_1^7, w_2)$.

Let $\kappa : Y \to M := Y/\widetilde{\Gamma_0}$ be the quotient map. Then $\kappa(E)$ is the disjoint union of a finite number of non-singular complex curves C_ν $(1 \le \nu \le \ell)$ in M. Let $\{E_i^{(\nu)}\}$ be the set of all E_i such that $\kappa(E_i) = C_\nu$. Each $E_i^{(\nu)}$ is a universal covering of C_ν. Let $\Gamma_i^{(\nu)} \subset \Gamma_0^*$ be the stabilizer of $E_i^{(\nu)}$; then $C_\nu = E_i^{(\nu)}/\Gamma_i^{(\nu)}$.

Extensions

Recall that (see, e.g., [**?**, Proposition 7.3]) that the holomorphic sectional curvature of a Kähler metric g is constant, equal to λ, if and only if

$$\begin{aligned}\frac{4}{\lambda}R(X,Y,Z,W) &= g(X,W)g(Y,Z) - g(X,Z)g(Y,W) + g(X,JW)g(Y,JZ)\\ &\quad -g(X,JZ)g(Y,JW) + 2g(X,JY)g(W,JZ).\end{aligned}$$

In terms of Siu's notation, the curvature of a Kähler metric $g_{\alpha\overline{\beta}}$ reads:

$$\begin{aligned}\sum_{i,j,k,\ell} R_{ijk\ell}u^i v^j p^k q^\ell &= \sum_{\alpha,\beta,\gamma,\delta}\left(R_{\alpha\overline{\beta}\gamma\overline{\delta}}\xi^\alpha\eta^{\overline{\beta}}\mu^\gamma\nu^{\overline{\delta}} + R_{\alpha\overline{\beta}\overline{\gamma}\delta}\xi^\alpha\eta^{\overline{\beta}}\mu^{\overline{\gamma}}\nu^\delta + R_{\overline{\alpha}\beta\gamma\overline{\delta}}\xi^{\overline{\alpha}}\eta^\beta\mu^\gamma\nu^{\overline{\delta}} + R_{\overline{\alpha}\beta\overline{\gamma}\delta}\xi^{\overline{\alpha}}\eta^\beta\mu^{\overline{\gamma}}\nu^\delta\right)\\ &= \sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}}\left(\xi^\alpha\eta^{\overline{\beta}}\mu^\gamma\nu^{\overline{\delta}} - \xi^\alpha\eta^{\overline{\beta}}\nu^\gamma\mu^{\overline{\delta}} - \eta^\alpha\xi^{\overline{\beta}}\mu^\gamma\nu^{\overline{\delta}} + \eta^\alpha\xi^{\overline{\beta}}\nu^\gamma\mu^{\overline{\delta}}\right)\\ &= \sum_{\alpha,\beta,\gamma,\delta} R_{\alpha\overline{\beta}\gamma\overline{\delta}}\left(\xi^\alpha\eta^{\overline{\beta}} - \eta^\alpha\xi^{\overline{\beta}}\right)\left(\mu^\gamma\nu^{\overline{\delta}} - \nu^\gamma\mu^{\overline{\delta}}\right)\end{aligned}$$

where $u = 2\mathrm{Re}(\xi)$, $v = 2\mathrm{Re}(\eta)$, $p = 2\mathrm{Re}(\mu)$, and $q = 2\mathrm{Re}(\nu)$.

Further work.

(i) Use Siu's computations to produce examples of complex surfaces with metrics of negative holomorphic sectional curvature.

Curvature of Fubini–Study metric

Let $[z_0 : \cdots : z_n]$ be homogeneous coordinates on $\mathbb{P}^n$. In the open affine chart $\mathcal{U}_0 = \{z_0 = 1\} \simeq \mathbb{C}^n$, the Fubini–Study metric affords the coordinate description

$$g_{i\overline{j}} = \frac{\partial^2}{\partial z_i \partial \overline{z}_j}\log\left(1 + \sum_{i=1}^n |z_i|^2\right).$$

We will compute the curvature of g in these coordinates at the origin in $\mathcal{U}_0$. Indeed, we have

$$\begin{aligned}-\partial_k\partial_{\overline{\ell}}g_{i\overline{j}} &= -\partial_k\partial_{\overline{\ell}}\partial_i\partial_{\overline{j}}\log\left(1 + \sum_{i=1}^n |z_i|^2\right)\\ &= -\partial_k\partial_{\overline{\ell}}\left[\frac{\left(1 + \sum_{i=1}^n |z_i|^2\right)\delta_{ij} - z_j\overline{z}_i}{\left(1 + \sum_{i=1}^n |z_i|^2\right)}\right]\\ &= \delta_{ij}\delta_{k\ell} + \delta_{i\ell}\delta_{jk}\end{aligned}$$

Cheung's theorem

Theorem. Let $f : X \longrightarrow Y$ be a holomorphic map of a compact complex manifold X into a complex manifold Y which has a Hermitian metric of negative holomorphic sectional curvature. Assume that f is everywhere of maximal rank, and that there exists a smooth family of Hermitian metrics on the fibers, which all have negative holomorphic sectional curvature. Then there is a Hermitian metric on X with negative holomorphic sectional curvature.

Let ω_Y be the Hermitian metric on Y with negative holomorphic sectional curvature. Let ω_t be the Hermitian metric on the fiber $f^{-1}(t)$ with negative holomorphic sectional curvature. Define Φ to be the Hermitian metric on X which restricts to ω_t on $f^{-1}(t)$ for each $t \in Y$. The desired metric is given by $\Psi_\lambda := \Phi + \lambda f^* \omega_Y$ for $\lambda > 0$ sufficiently large.
Let $p \in X$ be a point, and assume that p sits inside some fiber X_0. Since f is everywhere of maximal rank, we can choose a neighbourhood U of p and local product coordinates $(z^1, ..., z^s, z^{s+1}, ..., z^n) \in U = V \times W$, where $(V, (z^{s+1}, ..., z^n))$ is a coordinate neighbourhood of $f(p) \in Y$, and $(W, (z^1, ..., z^s))$ is a neighbourhood of p in the fiber X_0. Choose $\{z^{s+1}, ..., z^n\}$ such that $\partial_{z^{s+1}}, ..., \partial_{z^n}$ are ω_Y–orthonormal at $f(p)$.

Set $\Phi = \sqrt{-1} \sum_{i,j=1}^n g_{i\overline{j}} dz^i \wedge d\overline{z}^j$, and

$$\omega_Y = \sqrt{-1} \sum_{\alpha,\beta=s+1}^n \widetilde{g}_{\alpha\overline{\beta}} dz^\alpha \wedge d\overline{z}^\beta, \qquad \widetilde{g}_{\alpha\overline{\beta}}(p) = \delta_{\alpha\beta}.$$

If $\Psi_\lambda = \sqrt{-1} \sum_{i,j=1}^n h_{i\overline{j}} dz^i \wedge d\overline{z}^j$, then

$$\begin{aligned} \sqrt{-1} \sum_{i,j=1}^n h_{i\overline{j}} dz^i \wedge d\overline{z}^j &= \sqrt{-1} \sum_{i,j=1}^s g_{i\overline{j}} dz^i \wedge d\overline{z}^j + \sqrt{-1} \sum_{i=1}^s \sum_{\alpha=s+1}^n g_{i\overline{\alpha}} dz^i \wedge d\overline{z}^\alpha \\ &\quad + \sqrt{-1} \sum_{j=1}^s \sum_{\beta=s+1}^n g_{\beta\overline{j}} dz^\beta \wedge d\overline{z}^j + \sqrt{-1} \sum_{\alpha,\beta=s+1}^n (g_{\alpha\overline{\beta}} + \lambda \widetilde{g}_{\alpha\overline{\beta}}) dz^\alpha \wedge d\overline{z}^\beta. \end{aligned}$$

Write A for the $s \times s$ matrix with coefficients $(g_{a\overline{b}}(p))$, with $a, b \leq s$, and let A_{ab} denote the (a, b)th cofactor of A. Direct calculation yields

$$\begin{aligned} h^{a\overline{b}}(p) &= \frac{\lambda^{n-s} \det(A_{ab}) + O(\lambda^{n-s-1})}{\lambda^{n-s} \det(A) + O(\lambda^{n-s-1})}, \\ h^{\eta\overline{\eta}}(p) &= \frac{\lambda^{n-s-1} \det(A) + O(\lambda^{n-s-2})}{\lambda^{n-s} \det(A) + O(\lambda^{n-s-1})}, \end{aligned}$$

$h^{a\overline{\eta}}(p) = O(\lambda^{-1})$, $h^{\eta\overline{b}}(p) = O(\lambda^{-1})$, and $h^{\mu\overline{\eta}}(p) = O(\lambda^{-2})$, where $a, b \leq s$, and $\mu \neq \eta \geq s + 1$.
The main computation is guided by the following lemma:

Lemma. Let (M^n, g) be a Hermitian manifold. Suppose that at a point p,

(i)
$$\sum_{i,j,k,\ell=1}^{s} R_{i\bar{j}k\bar{\ell}}(p)\zeta^i\overline{\zeta}^j\zeta^k\overline{\zeta}^\ell \;\le\; -K_0\sum_{i,j=1}^{s}\zeta^i\overline{\zeta}^i\zeta^j\overline{\zeta}^j,$$
for all $\zeta^i \in \mathbb{C}$, where $i = 1, ..., s$.

(ii) For $\min(i, j, k, \ell) \le s$, we have
$$|R_{i\bar{j}k\bar{\ell}}(p)| < K_1.$$

(iii)
$$\sum_{\alpha,\beta,\gamma,\delta=s+1}^{n} R_{\alpha\bar{\beta}\gamma\bar{\delta}}(p)\zeta^\alpha\overline{\zeta}^\beta\zeta^\gamma\overline{\zeta}^\delta \;\le\; -K_2\sum_{\alpha,\beta=s+1}^{n}\zeta^\alpha\overline{\zeta}^\alpha\zeta^\beta\overline{\zeta}^\beta,$$
for any $\zeta^\alpha \in \mathbb{C}$, where $\alpha = 1, ..., n$.

Then there exists a positive constant K, depending only on K_0/K_1, such that if $K_2/K_1 \ge K$, the metric g has negative holomorphic sectional curvature at p.

Explicit formula for the curvature tensor. Recall that

$$\begin{aligned}\sqrt{-1}\sum_{i,j=1}^{n} h_{i\bar{j}}dz^i\wedge d\overline{z}^j \;&=\; \sqrt{-1}\sum_{i,j=1}^{s} g_{i\bar{j}}dz^i\wedge d\overline{z}^j + \sqrt{-1}\sum_{i=1}^{s}\sum_{\alpha=s+1}^{n} g_{i\overline{\alpha}}dz^i\wedge d\overline{z}^\alpha\\ &\quad+\sqrt{-1}\sum_{j=1}^{s}\sum_{\beta=s+1}^{n} g_{\beta\bar{j}}dz^\beta\wedge d\overline{z}^j + \sqrt{-1}\sum_{\alpha,\beta=s+1}^{n}(g_{\alpha\overline{\beta}} + \lambda\widetilde{g}_{\alpha\overline{\beta}})dz^\alpha\wedge d\overline{z}^\beta.\end{aligned}$$

The curvature tensor of h is
$$\sum_{i,j,k,\ell=1}^{n} R_{i\bar{j}k\bar{\ell}} \;=\; -\sum_{i,j,k,\ell=1}^{n}\frac{\partial^2 h_{i\bar{j}}}{\partial z_k\partial\overline{z}_\ell} + \sum_{i,j,k,\ell,p,q=1}^{n} h^{p\bar{\ell}}h^{k\bar{q}}\frac{\partial h_{i\bar{q}}}{\partial z_k}\frac{\partial h_{p\bar{j}}}{\partial\overline{z}_\ell}.$$

The second-order term expands to
$$\begin{aligned}\sum_{i,j=1}^{n}\frac{\partial^2 h_{i\bar{j}}}{\partial z_k\partial\overline{z}_\ell} \;&=\; \sum_{i,j=1}^{s}\frac{\partial^2 g_{i\bar{j}}}{\partial z_k\partial\overline{z}_\ell} + \sum_{i=1}^{s}\sum_{\alpha=s+1}^{n}\frac{\partial^2 g_{i\overline{\alpha}}}{\partial z_k\partial\overline{z}_\ell} + \sum_{j=1}^{s}\sum_{\beta=s+1}^{n}\frac{\partial^2 g_{\beta\bar{j}}}{\partial z_k\partial\overline{z}_\ell}\\ &\quad+\sum_{\alpha,\beta=s+1}^{n}\frac{\partial^2 g_{\alpha\overline{\beta}}}{\partial z_k\partial\overline{z}_\ell} + \lambda\sum_{\alpha,\beta=s+1}^{n}\frac{\partial^2\widetilde{g}_{\alpha\overline{\beta}}}{\partial z_k\partial\overline{z}_\ell}.\end{aligned}$$

If $\min(k, \ell) \le s$, then this last term involving λ vanishes.