Chapter 1

Complex Numbers.

1.1 An Introduction to Complex Numbers.

In previous years you have learnt about different sets of numbers. For example, from a very early age we learn about the natural numbers

$$\mathbb{N} = \{1, 2, 3, ...\}$$

Then perhaps in high school, the integers

$$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$$

Then the rational numbers $\mathbb{Q}$, then the irrational numbers $\mathbb{R}\backslash\mathbb{Q}$ such as π, and then finally the real numbers $\mathbb{R}$.

Notice that the sets are listed in order of size. That is,

$$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}.$$

The question then immediately arises at to whether there is a set larger than the real numbers? The answer is an affirmative yes. The real numbers $\mathbb{R}$ are a subset of the complex numbers $\mathbb{C}$, where

$$\mathbb{C} := \{z = x + iy : x, y \in \mathbb{R}\}.$$

The complex numbers include both a real and imaginary component. Denoted by $z = \mathrm{Re}(z) + \mathrm{Im}(z)$. The imaginary component arises from the square root of a negative number.

In the real numbers $\mathbb{R}$, the square root of negative one, $\sqrt{-1}$ is not defined. However, in the complex numbers $\mathbb{C}$, we define $\sqrt{-1} = i$ and we build from there.

For example $\sqrt{-9} = \sqrt{-1 \times 9} = \sqrt{-1} \times \sqrt{9} = 3 \times i = 3i$

Using the definition that $\sqrt{-1} = i$, we can build some useful relations. Such as

$$\begin{aligned} i &= \sqrt{-1} \\ i^2 &= (\sqrt{-1})^2 = -1 \\ i^3 &= (\sqrt{-1}) \times (\sqrt{-1})^2 = -i \\ i^4 &= (\sqrt{-1})^2 \times (\sqrt{-1})^2 = 1 \end{aligned}$$

Example 4.1.1. Write the following expressions in terms of i

a. $\sqrt{-4}$.

Proof. We simply observe that

$$\begin{aligned} \sqrt{-4} &= \sqrt{4 \times -1} \\ &= \sqrt{4} \times \sqrt{-1} \\ &= 2i. \end{aligned}$$

□

b. $-\sqrt{-16}$.

Proof. We simply observe that

$$\begin{aligned} -\sqrt{-16} &= -\sqrt{16 \times -1} \\ &= -\sqrt{16} \times \sqrt{-1} \\ &= -4i. \end{aligned}$$

□

c. $\sqrt{-5}$.

Proof. We simply observe that

$$\begin{aligned} \sqrt{-5} &= \sqrt{5 \times -1} \\ &= \sqrt{5} \times \sqrt{-1} \\ &= \sqrt{5}i. \end{aligned}$$

□

Example 4.1.2. Write down the real and imaginary component for each of the following complex numbers.

a. $z = 1 - 2i$.

Proof. It is clear that $\mathrm{Re}(z) = 1$, $\mathrm{Im}(z) = -2$. □

b. $z = 2 - \frac{3}{5}i$.

Proof. It is clear that $\mathrm{Re}(z) = 2$, $\mathrm{Im}(z) = \frac{-3}{5}$. □

(c) $z = i$.

Proof. It is clear that $\mathrm{Re}(z) = 0$, $\mathrm{Im}(z) = 1$. □

Example 4.1.3. Simplify the expression

$$i^5 + 3i^2 - 4 + \frac{i}{2}.$$

Proof. It is obvious that

$$\begin{aligned} i^5 + 3i^2 - 4 + \frac{i}{2} &= i^4 \cdot i + 3i^2 - 4 + \frac{i}{2} \\ &= (1) \cdot i + 3 \cdot (-1) - 4 + \frac{i}{2} \\ &= i - 3 - 4 + \frac{i}{2} \\ &= \frac{3i}{2} - 7. \end{aligned}$$

□

Exercises

Q1. Simplify the following expressions in terms of i

a. $\sqrt{-9}$ b. $\sqrt{-4}$ c. $\sqrt{-16}$

d. $\sqrt{-8}$ e. $\sqrt{-18}$ f. $\sqrt{-17}$

Q2. Simplify the following expressions in terms of i

a. $i^3 - 4i^2$ b. $i^4 + i^3 - 2i$ c. $2i^2 - 3i + 1$

d. $i^7 - \frac{i}{3} + i^6$ e. $i^2(i - 3)$ f. $i^2 + i^3 + i^9$

Q3. Determine the real and imaginary components of the complex numbers simplified in question 2.

Q4. Determine
$$\mathrm{Re}(i^3 - 7i^2 + 2 + i^8)$$

Q5. Determine
$$\mathrm{Im}(i^4 + i^2 - 5 + i^8 + i^2(i^3))$$

Q6. Suppose that $z_1, z_2 \in \mathbb{C}$. Determine whether $z_1 \cdot z_2 \in \mathbb{C}$. What about $z_1 + z_2$? Prove or provide a counterexample.

1.2 Elementary Operations in $\mathbb{C}$

Consider two complex numbers $z_1 = a + bi$ and $z_2 = c + di$. Observe that $\mathrm{Re}(z_1) = a$, $\mathrm{Im}(z_1) = b$, $\mathrm{Re}(z_2) = c$ and $\mathrm{Im}(z_2) = d$.

The addition of two complex numbers is simply the addition of the real component with the real component and the addition of the imaginary component with the imaginary component. Therefore $z_1 + z_2 = a + bi + c + di = (a + c) + (b + d)i$.

Example 4.2.1. Let $z_1 = 1 + 2i$ and $z_2 = 3 - i$. Compute

a. $z_1 + z_2$.

Proof. We simply write

$$\begin{aligned} 1 + 2i + 3 - i &= (1 + 3) + (2 - 1)i \\ &= 4 + i. \end{aligned}$$

□

b. $z_1 - z_2$.

Proof. We simply write

$$\begin{aligned} (1 + 2i) - (3 - i) &= 1 + 2i - 3 + i \\ &= (1 - 3) + (2 + 1)i \\ &= -2 + 3i. \end{aligned}$$

□

c. $2z_1 - 3z_2$.

Proof. We simply write

$$\begin{aligned} 2(1 + 2i) - 3(3 - i) &= 2 + 4i - 9 + 3i \\ &= (2 - 9) + (3 + 4)i \\ &= -7 + 7i. \end{aligned}$$

□

To multiply two complex numbers, we simply expand the brackets, as we would if the numbers were real. Again, let $z_1 = a + bi$ and $z_2 = c + di$. Then

$$\begin{aligned} z_1 \cdot z_2 &= (a+bi)(c+di) \\ &= ac + adi + bci + bdi^2 \\ &= ac + adi + bci - bd \\ &= (ac - bd) + (ad + bc)i \end{aligned}$$

Example 4.2.2. Let $z_1 = 2 - i$ and $z_2 = 5 + i$. Compute

a. $z_1 \cdot z_2$.

Proof. We simply observe that

$$\begin{aligned} (2-i)\cdot(5+i) &= 10 + 2i - 5i - i^2 \\ &= 10 + 2i - 5i + 1 \\ &= 11 - 3i. \end{aligned}$$

□

b. $(z_1)^2$.

Proof. We simply observe that

$$\begin{aligned} (2-i)^2 &= (2-i)\times(2-i) \\ &= 4 - 2i - 2i + i^2 \\ &= 4 - 2i - 2i - 1 \\ &= 3 - 4i. \end{aligned}$$

□

Example 4.2.3. Solve the equation

$$(1+i)(2x - yi) = 11 - 3i,$$

for $x, y \in \mathbb{R}$.

Proof. We simply observe that

$$\begin{aligned}(1+i)(2x-yi) &= 11-3i\\ 2x-yi+2xi-yi^2 &= 11-3i\\ (2x+y)+(2x-y)i &= 11-3i.\end{aligned}$$

By then equating real and imaginary parts, we see that $2x+y=11$ and $2x-y=-3$. Solving these simultaneous equations, we have $x=2$ and $y=7$. □

To divide complex numbers, we need to introduce the notion of the complex conjugate. Consider the complex number $z=a+bi$. The complex conjugate of z, denoted by $\bar{z}$ is $\bar{z}=a-bi$.

Example 4.2.4. Write the complex conjugate for each of the following complex numbers

a. $z=5+i$.

Proof. It is obvious that $\bar{z}=5-i$. □

b. $z=2-i$.

Proof. It is obvious that $\bar{z}=2+i$. □

Now we can divide two complex numbers. With $z_1=a+bi$ and $z_2=c+di$, consider $\frac{z_1}{z_2}=\frac{a+bi}{c+di}$. We want to express this in terms of $x+iy$. Therefore, we multiply the numerator and denominator by the complex conjugate of z_2.

$$\begin{aligned}\frac{z_1}{z_2} &= \frac{a+bi}{c+di}\\ &= \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di}\\ &= \frac{ac-adi+bci+bd}{c^2-cdi+cdi+d^2}\\ &= \frac{ac+bd}{c^2+d^2}+\frac{bc-ad}{c^2+d^2}i\end{aligned}$$

Example 4.2.5. Let $z_1=2+3i$ and $z_2=1+i$. Evaluate the following

a. $\frac{z_1}{z_2}$.

Proof. We simply observe that

$$\begin{aligned}
\frac{2+3i}{1+i} &= \frac{2+3i}{1+i}\cdot\frac{1-i}{1-i} \\
&= \frac{2-2i+3i-3i^2}{1-i+i-i^2} \\
&= \frac{2-2i+3i+3}{1+1} \\
&= \frac{5+i}{2} \\
&= \frac{5}{2}+\frac{1}{2}i.
\end{aligned}$$

□

b. $\frac{z_2}{z_1}$.

Proof. We simply observe that

$$\begin{aligned}
\frac{1+i}{2+3i} &= \frac{1+i}{2+3i}\cdot\frac{2-3i}{2-3i} \\
&= \frac{2-3i+2i-3i^2}{3-6i+6i-9i^2} \\
&= \frac{2-3i+2i+3}{3+9} \\
&= \frac{5-i}{12} \\
&= \frac{5}{12}-\frac{1}{12}i.
\end{aligned}$$

□

Exercises

Q1. Evaluate the following expressions

a. $(1+2i)+(2-3i)$ b. $(2+i)+(1+i)$ c. $(3-i)+(2i+1)$

d. $(1-i)-(2+3i)$ e. $(2+i)-(2-i)$ f. $(7-2i)-(3+9i)$

Q2. Evaluate the following expressions

a. $(1+i)\cdot(1-2i)$ b. $(4-3i)\cdot(2+i)$ c. $(5-2i)\cdot(4+4i)$

d. $(3-3i)\cdot(3+2i)$ e. $(1+3i)\cdot(1-3i)$ f. $(2+2i)\cdot(4+3i)$

Q3. Determine the complex conjugates for each of the following

a. $1+i$ b. $2-5i$ c. $5+6i$

d. $2+3i$ e. $2i+1$ f. $4i-9$

Q4. Evaluate the following expressions

a. $\frac{3+i}{1-i}$ b. $\frac{2-i}{2+i}$ c. $\frac{1-3i}{1+5i}$

d. $\frac{2}{3-i}$ e. $\frac{1+2i}{4i+1}$ f. $\frac{3i}{2+i}$

Q5. Determine

$$\mathrm{Re}\left(\frac{2+i^3-7i^5+6}{i^2-1+2i^5}\right)$$

Q6. Determine

$$\mathrm{Im}\left(\frac{4i^4+7i^3-6i+1}{2-3i^3+i}\right)$$

Q7. Let $z_1 = 4-9i$ and $z_2 = 3+i$. Evaluate

$$\frac{z_1+z_2}{z_1\cdot z_1} - \frac{3z_2}{2z_1+z_2^2}.$$

1.3 The Argand Diagram and Polar Form

We are all familiar with the standard (x, y) cartesian plane, $\mathbb{R}^2$. With y being the vertical axis and x being the horizontal axis. An Argand diagram is essentially the same construction. However, the y-axis is now the imaginary axis, and the horizontal axis is now the real axis.

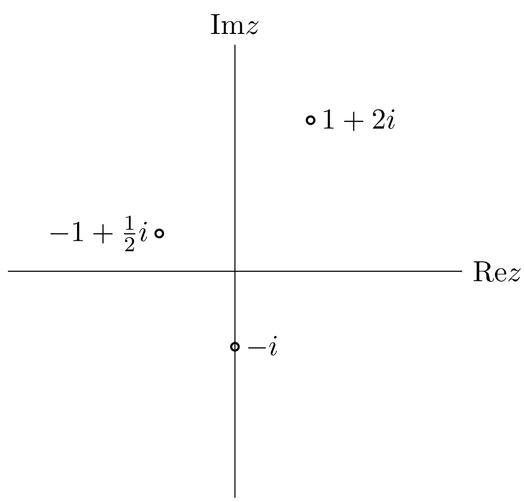
Example 4.3.1. Sketch the following on an Argand diagram.

a. $z = 1 + 2i$

b. $z = -i$

c. $z = -1 + \frac{1}{2}i$

Proof. We simply observe that



□

Previously, we expressed complex numbers in terms of $z = a + bi$, where a was the real component and b was the imaginary component. This is referred to as cartesian form. However, sometimes it is more convenient to express a complex number in terms of the distance from the origin and the angle the complex number makes with the horizontal (real) axis. When a complex number is written in this manner, it is said to be in **polar form**. Mathematically it is written

$$z = r\text{cis}(\theta)$$

Where r is the modulus and represents the distance from the origin, and θ is the argument and represents the angle made with the real axis. Cis is a way of condensing $\cos(\theta) + i\sin(\theta)$.

Therefore $z = r\text{cis}(\theta) = r\cos(\theta) + ri\sin(\theta)$.

Typically, the value of θ varies from $(-\pi, \pi]$. When $-\pi < \theta \leq \pi$ we say θ is the principal argument of the complex number.

To calculate the modulus r of a complex number, we simply take the square root of the square of the components. See the following example.

Example 4.3.2. Calculate the modulus r for each of the following complex numbers.

a. $z = 1 + 2i$.

Proof. We simply observe that

$$\begin{aligned} r &= \sqrt{1^2 + 2^2} \\ &= \sqrt{1+4} \\ \therefore r &= \sqrt{5}. \end{aligned}$$

□

b. $z = -2 - 3i$.

Proof. We simply observe that

$$\begin{aligned} r &= \sqrt{(-2)^2 + (-3)^2} \\ &= \sqrt{4+9} \\ \therefore r &= \sqrt{13}. \end{aligned}$$

□

To calculate the argument of a complex number, we take the inverse tangent of the imaginary component divided by the real component. See the following example.

Example 4.3.3. Calculate the argument θ for each of the following complex numbers, express in terms of the principal argument $\theta \in (\pi, \pi]$.

a. $z = 1 - \sqrt{3}i$.

Proof. We simply observe that

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{-\sqrt{3}}{1}\right) \\ &= \frac{-\pi}{3}, \end{aligned}$$

where the last line follows from the fact that z lies in the fourth quadrant. $\square$

(b) $z = -2 - 2i$.

Proof. We simply observe that

$$\begin{aligned} \theta &= \tan^{-1}\left(\frac{-2}{-2}\right) \\ &= \tan^{-1}(1) \\ &= \frac{-3\pi}{4}, \end{aligned}$$

where the last line follows from the fact that z lies in the third quadrant. $\square$

Example 4.3.4. Express the following complex numbers in polar form, with principal argument.

a. $z = 4 + 4i$.

Proof. We observe that

$$\begin{aligned} r &= \sqrt{4^2+4^2} \\ &= \sqrt{16+16} \\ &= \sqrt{32} \\ &= 4\sqrt{2} \\ \theta &= \tan^{-1}\left(\frac{4}{4}\right) \\ &= \tan^{-1}(1) \\ &= \frac{\pi}{4} \\ \therefore z &= 4\sqrt{2}\text{cis}\left(\frac{\pi}{4}\right). \end{aligned}$$

□

b. $z = 2\sqrt{3} + 2i$.

Proof. We observe that

$$\begin{aligned} r &= \sqrt{(2\sqrt{3})^2+2^2} \\ &= \sqrt{12+4} \\ &= \sqrt{16} \\ &= 4 \\ \theta &= \tan^{-1}\left(\frac{2}{2\sqrt{3}}\right) \\ &= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) \\ &= \frac{\pi}{6} \\ \therefore z &= 4\text{cis}\left(\frac{\pi}{6}\right). \end{aligned}$$

□

Example 4.3.5. Convert the following complex numbers from polar form to cartesian form.

a. $z = 2\text{cis}(\frac{\pi}{3})$.

Proof. We observe that

$$\begin{aligned} 2\text{cis}\left(\frac{\pi}{3}\right) &= 2\cos\left(\frac{\pi}{3}\right) + 2i\sin\left(\frac{\pi}{3}\right) \\ &= 2\cdot\left(\frac{1}{2}\right) + 2i\cdot\left(\frac{\sqrt{3}}{2}\right) \\ &= 1+\sqrt{3}i. \end{aligned}$$

□

b. $z = 3\text{cis}(\pi)$.

Proof. We observe that

$$\begin{aligned} \text{cis}(\pi) &= \cos(\pi) + i\sin(\pi) \\ &= -1+0i \\ &= -1. \end{aligned}$$

□

Exercises

Q1 Determine the modulus $|z|$ for each of the following

a. $z = 1+2i$ b. $z = 3+4i$ c. $1-i$

d. $2+2i$ e. $3-i$ f. $1-7i$

Q2. Determine the argument θ for each of the following

a. $z = 1+i$ b. $z = 1-\sqrt{3}i$ c. $z = 2-2i$

d. $z = \sqrt{6}+\sqrt{2}i$ e. $z = \sqrt{3}-i$ f. $z = \sqrt{7}+\sqrt{7}i$

Q3. Write each of the complex numbers from Q2 in polar form.

Q4. Convert each of the following to cartesian form.

a. $z = 2\text{cis}(\frac{\pi}{3})$ b. $z = 3\text{cis}(\frac{\pi}{6})$ c. $z = \text{cis}(\pi)$

d. $z = \frac{1}{3}\text{cis}(0)$ e. $z = \frac{2}{5}\text{cis}(\frac{2\pi}{3})$ f. $z = 2\text{cis}(\frac{5\pi}{6})$

Q5. Determine the modulus of the complex number

$$\frac{z}{|z|^2} + \frac{\overline{z}}{|z|^2},$$

where $z = 1 + 3i$.

Q6. Prove that if $z \in \mathbb{C}$ is a complex number, then

$$|z| = |\overline{z}|.$$

Q7. (Dr. Lloyd Gunatilake). Let $z = -3-3i$ and $w = -\sqrt{2}+i\sqrt{6}$. Express in polar form

a. zw. b. $\frac{z}{w}$.

Q8. Write the following complex numbers in cartesian form.

a. $z = 2\text{cis}\left(\frac{\pi}{4}\right)$. b. $z = 3\text{cis}\left(-\frac{\pi}{2}\right)$. c. $z = \sqrt{5}\text{cis}(2\pi)$.

1.4 Operations in Polar Form

Just as we can multiply and divide complex numbers in cartesian form, we can do the same in polar form. Often times it is much easier to multiply and divide complex numbers in polar form than cartesian form.

If we let $z_1 = r_1\text{cis}(\theta_1)$ and $z_2 = r_2\text{cis}(\theta_2)$. Then

$$z_1 \cdot z_2 = r_1 r_2 \text{cis}(\theta_1 + \theta_2), \text{ and } \frac{z_1}{z_2} = \frac{r_1\text{cis}(\theta_1)}{r_2\text{cis}(\theta_2)} = \frac{r_1}{r_2}\text{cis}(\theta_1 - \theta_2).$$

Example 4.4.1. Let $z_1 = 2\text{cis}(\frac{\pi}{3})$ and $z_2 = 3\text{cis}(\frac{\pi}{4})$. Compute the following in polar form.

a. $z_1 \cdot z_2$.

Proof. We simply observe that

$$\begin{aligned} z_1 \cdot z_2 &= 2\text{cis}\left(\frac{\pi}{3}\right) \times 3\text{cis}\left(\frac{\pi}{4}\right) \\ &= 2 \cdot 3\text{cis}\left(\frac{\pi}{3} + \frac{\pi}{4}\right) \\ \therefore z_1 \cdot z_2 &= 6\text{cis}\left(\frac{7\pi}{12}\right). \end{aligned}$$

□

b. $\frac{z_1}{z_2}$.

Proof. We simply observe that

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{2\text{cis}\left(\frac{\pi}{3}\right)}{3\text{cis}\left(\frac{\pi}{4}\right)} \\ &= \frac{2}{3}\text{cis}\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ \therefore \frac{z_1}{z_2} &= \frac{2}{3}\text{cis}\left(\frac{\pi}{12}\right). \end{aligned}$$

□

If we want to raise a complex number to a particular power, like multiplication and division, it is often much easier to do so in polar form. With the development of de Moivre's Theorem, this process is made remarkably simple.

de Moivre's theorem states that if we have a complex number $z = r\text{cis}(\theta)$, then $z^n = r^n\text{cis}(n\theta)$, where $n \in \mathbb{N}$.

The proof is quite simple, but is not so clear from the way in which we defined the polar form of a complex number, the proof is not illustrative. We therefore omit the proof, noting that a more conventional way to define the polar form a complex number is $z = re^{i\vartheta}$, where r is the modulus and ϑ is the argument.

Example 4.4.2. Let $z_1 = 2\text{cis}(\frac{\pi}{4})$. Compute the following in the polar form.

a. z^4.

Proof. We simply observe that

$$\begin{aligned} z^4 &= 2^4\text{cis}\left(4 \cdot \frac{\pi}{4}\right) \\ &= 16\text{cis}(\pi). \end{aligned}$$

□

b. z^5.

Proof. Similarly, we have

$$\begin{aligned} z^5 &= 2^5\text{cis}\left(5 \cdot \frac{\pi}{4}\right) \\ &= 32\text{cis}\left(\frac{5\pi}{4}\right) \\ &= 32\text{cis}\left(\frac{-3\pi}{4}\right). \end{aligned}$$

□

c. $z^{\frac{1}{2}}$.

Proof. It is easy to see that

$$\begin{aligned} z^{\frac{1}{2}} &= 2^{\frac{1}{2}}\text{cis}\left(\frac{1}{2}\cdot\frac{\pi}{4}\right) \\ &= \sqrt{2}\text{cis}\left(\frac{\pi}{8}\right). \end{aligned}$$

□

After introducing the notion of polar form, representing a complex number in terms of its distance from the origin and the angle it makes with the horizontal axis, it logically follows how we can rotate complex numbers.

† Multiplying any complex number by -1 or i^2 rotates the complex number by π in the anti-clockwise direction.

† Multiplying a complex number by $-i$ or i^3 rotates the complex number by $\frac{3\pi}{2}$ in the anti-clockwise direction or $\frac{\pi}{2}$ in the clockwise direction.

Exercises

Q1. Consider the complex numbers $z_1 = 2\text{cis}(\frac{\pi}{2})$ and $z_2 = \text{cis}(\frac{\pi}{3})$. Compute the following

a. $z_1 \times z_2$

b. $z_1 \div z_2$

c. $z_2 \div z_1$

Q2. Consider the complex numbers $z_3 = \frac{1}{3}\text{cis}(\frac{2\pi}{3})$ and $z_4 = \frac{3}{2}\text{cis}(\frac{\pi}{2})$. Compute the following

a. $z_3 \times z_4$

b. $\frac{z_3}{z_4}$

c. $\frac{z_4}{z_3}$

Q3. For the complex numbers z_1 and z_2 in Q1, compute the following

a. $(z_1)^2$

b. $(z_2)^3$

c. $(z_1)^2 \cdot (z_2)^3$

Q4. Express the following in cartesian form

$$z = \left(2\text{cis}\left(\frac{\pi}{9}\right)\right)^3$$

Q5. Express the following in cartesian form

$$\frac{(1-\sqrt{3}i)^3}{(2-2i)^2}$$

Q6. Consider the complex number $z = 4\text{cis}(\frac{\pi}{6})$. Compute the following in polar form

a. $\frac{1}{z}$

b. $\sqrt{z}$

Q7. Evaluate

$$z = (1+i\sqrt{3})^5.$$

Q8. Evaluate

$$z = (-2+2i)^4.$$

Q9. Evaluate

$$z = (\sqrt{2}-i\sqrt{6})^7.$$

Q10. (Dr. Lloyd Gunatilake). Let $z = 1 + \text{cis}(i\vartheta)$, where $\vartheta \in \left(0, \frac{\pi}{2}\right)$. Show that

a. $|z| = 2\cos\left(\frac{1}{2}\vartheta\right)$.

b. $\arg(z) = \frac{1}{2}\vartheta$.

Q11. Explain what happens to the argument of a complex number when you square the complex number.

Q12. Explain what happens to the argument of a complex number when you cube the complex number. Does this hold in general? What about square rooting a complex number?

Q13. Let $z_1, z_2, z_3 \in \mathbb{C}$. Show that these three complex numbers form an equilateral triangle in $\mathbb{C}$ if and only if

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_1z_3 + z_2z_3.$$

Q14. Let $z_1, ..., z_n$ be complex numbers. Prove that there exists a subset $J \subset \{1, 2, ..., n\}$ such that

$$\left| \sum_{j \in J} z_j \right| \geq \frac{1}{4\sqrt{2}} \sum_{j=1}^{n} |z_j| \, .$$

Q15. Let $x, y \in \mathbb{C}$ whose real parts are nonpositive. Prove that

$$|e^z - e^w| \leq |z - w| \, .$$

1.5 Polynomials over $\mathbb{C}$.

We begin by stating the *Fundamental Theorem of Algebra*.

Theorem 4.5.1. A polynomial $p(z)$ of degree n will have n roots in the complex plane $\mathbb{C}$, and at max n roots in the real plane $\mathbb{R}$.

For example, the polynomial $z^4 + 2z^3 + z^2 + 6z - 5$ has 4 roots in $\mathbb{C}$ or $(z-3)(z^2+8)$ has 1 root in $\mathbb{R}$ and 3 roots in $\mathbb{C}$.

For clarification, the root of a polynomial is the solution to the equation $p(z) = 0$.

The next big theorem in this section is the *Complex Conjugate Root Theorem*.

Theorem 4.5.2. Let $p(z)$ be a polynomial in $\mathbb{C}$ with real coefficients. If $p(z)$ has a root $z = a + bi$ then $\bar{z} = a - bi$ is also a root of $p(z)$.

Example 4.5.3. $p(z) = z^3 - 5z^2 + 4z + 10$ has a root $z = 3 + i$. Determine the other two roots using the appropriate theorems.

Proof. Given that $z = 3 + i$ is a root of the equation $p(z) = 0$, the conjugate root theorem tells us that $z = 3 - i$ is also a root of the equation $p(z) = 0$. We can therefore obtain the following quadratic equation

$$\begin{aligned}(z-3-i)\cdot(z-3+i) &= z^2 - 3z + zi - 3z + 9 - 3i - zi + 3i + 1 \\ &= z^2 - 6z + 10\end{aligned}$$

We can then use polynomial long division to express

$$\frac{z^3 - 5z^2 + 4z + 10}{z^2 - 6z + 10}$$

as a proper fraction with a remainder. The remainder is determined to be $z + 11$. Therefore the roots of the equation $p(z) = 0$ are $z = 3 + i, z = 3 - i$ and $z = -11$. Hence, by the fundamental theorem of algebra, we know that there are only 3 roots to this equation. $\square$

Example 4.5.4. Solve the following equation $z^2 + (-1 + i)z - i = 0$ over $\mathbb{R}$ and over $\mathbb{C}$.

Proof. We simply observe that

$$\begin{aligned} z &= \frac{(1-i) \pm \sqrt{(-1+i)^2 - 4(1)(-i)}}{2} \\ &= \frac{1-i \pm \sqrt{1-2i-1+4i}}{2} \\ &= \frac{1-i \pm \sqrt{2i}}{2}. \end{aligned}$$

So we have no solutions over $\mathbb{R}$ and two solutions over $\mathbb{C}$. $\square$

Example 4.5.5. Consider the polynomial $p(z) = z^3 + az^2 + bz + 1$, where a and b are real constants. If $z = i$ is a root, find the values of $a, b \in \mathbb{R}$.

Proof. If $z = i$ is a root, then we see that

$$\begin{aligned} p(i) = 0 &\implies i^3 + ai^2 + bi + 1 = 0 \\ &\implies -i - a + bi + 1 = 0 \\ &\implies (1-a) + i(b-1) = 0. \end{aligned}$$

So $a = 1$ and $b = 1$. $\square$

Now consider the complex number $z = r\text{cis}(\theta)$. If we wish to find the nth roots of this complex number then we write $z^n = r\text{cis}(\theta)$. De Moivre's theorem yields

$$z = r^{\frac{1}{n}} \text{cis}\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right), \ k \in \mathbb{Z}$$

The $\frac{2k\pi}{n}$ factor is present to allow us to generate the number of roots, since z^n will have n roots.

Note that the roots of the complex number z^n lie on a circle of radius $r^{\frac{1}{n}}$ and are separated by $\frac{2\pi}{n}$.

Example 4.5.6. Find the 4 roots of $z = 1 - i$.

Proof. We simply observe that

$$
\begin{aligned}
z^4 &= 1 - i \\
&= \sqrt{2}\text{cis}\left(\frac{-\pi}{4}\right) \\
\therefore z &= 2^{\frac{1}{8}}\text{cis}\left(\frac{-\pi}{16} + \frac{2k\pi}{4}\right), k \in \mathbb{Z} \\
z_1 &= 2^{\frac{1}{8}}\text{cis}\left(\frac{-\pi}{16}\right) \\
z_2 &= 2^{\frac{1}{8}}\text{cis}\left(\frac{-\pi}{16} + \frac{2\pi}{4}\right) \\
&= 2^{\frac{1}{8}}\text{cis}\left(\frac{7\pi}{16}\right) \\
z_3 &= 2^{\frac{1}{8}}\text{cis}\left(\frac{\pi}{16}\right) \\
z_4 &= 2^{\frac{1}{8}}\text{cis}\left(\frac{-7\pi}{16}\right).
\end{aligned}
$$

Where the last two lines follow from the conjugate root theorem. □

Example 4.5.7. Find the 5 roots of -1.

Proof. We simply observe that

$$\begin{aligned}
z^5 &= -1 \\
z^5 &= \operatorname{cis}\left(\frac{-\pi}{2}\right) \\
\therefore z &= \operatorname{cis}\left(\frac{-\pi}{10} + \frac{2k\pi}{5}\right), k \in \mathbb{Z} \\
z_1 &= \operatorname{cis}\left(\frac{-\pi}{10}\right) \\
z_2 &= \operatorname{cis}\left(\frac{-\pi}{10} + \frac{4\pi}{10}\right) \\
&= \operatorname{cis}\left(\frac{3\pi}{10}\right) \\
z_3 &= \operatorname{cis}\left(\frac{-\pi}{10} + \frac{8\pi}{10}\right) \\
&= \operatorname{cis}\left(\frac{7\pi}{10}\right) \\
z_4 &= \operatorname{cis}\left(\frac{-\pi}{10} - \frac{4\pi}{10}\right) \\
&= \operatorname{cis}\left(\frac{-\pi}{2}\right) \\
z_5 &= \operatorname{cis}\left(\frac{-\pi}{10} - \frac{-8\pi}{10}\right) \\
&= \operatorname{cis}\left(\frac{-9\pi}{10}\right).
\end{aligned}$$

□

Exercises

Q1. Solve the following equations for $z \in \mathbb{C}$.

a. $z^2 - 5z + 6 = 0.$

b. $z^2 + 2z + 1 = 0.$

c. $z^2 + 1 = 0.$

d. $z^2 + 2z + 3 = 0.$

e. $z^2 + 149 = 14z.$

f. $4z^2 - \sqrt{32}z = -4.$

Q2. Determine the three roots of the following complex numbers.

a. $z^3 = 1$.

b. $z^3 = -i$.

c. $z^3 = i$.

d. $z^3 = 2i$.

Q3. Determine the four roots of the complex number

$$z = \sqrt{2} - i\sqrt{6},$$

in polar form.

Q4. Determine the five roots of the complex number

$$z = \sqrt{3} - i,$$

in polar form.

Q5. Suppose that $p(z) = z^3 - z^2 - z + \lambda$, for some $\lambda \in \mathbb{C}$. Determine the value of λ if

$$p\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = 0.$$

Q6. Suppose that $p(z) = z^4 - 2z^3 + \lambda z - 9$, for some $\lambda \in \mathbb{C}$. Determine the value of λ is

$$p(1 + i\sqrt{2}) = 0.$$

Q7. Suppose that $p(z) = z^4 + az^3 + bz - 1$, for some $a, b \in \mathbb{C}$. Determine the values of a and b if $z = i$ and $z = 1 + \sqrt{2}$ are roots of p.

Q8. Suppose that $p(z) = z^4 - \lambda z^2 + 6z + \mu$, for some $\mu, \lambda \in \mathbb{C}$. Determine the value of μ and λ if $p(2) = 0$ and $p(1 - i) = 0$.

Q9. Let $1, \zeta_1, \zeta_2$ denote the three roots of unity. Show that

a. $\zeta_1 = \overline{\zeta_2} = \zeta_2^2$.

b. $\zeta_1 + \zeta_2 = -1$.

c. $\zeta_1\zeta_2 = 1$.

1.6 Revision Exercises

Q1. Let $z = 1 + i$ and $w = 1 - \sqrt{3}i$. Write $z \cdot w$ in polar form.

Q2. Suppose that $z = \sqrt{5}\text{cis}\left(-\frac{2\pi}{5}\right)$, $w = 2\text{cis}\left(\frac{3\pi}{8}\right)$ and $\zeta = \sqrt{10}\text{cis}\left(\frac{\pi}{12}\right)$. Write

$$\omega = \frac{z^2 \cdot w^3}{\zeta^4}$$

in polar form.

Q3. Determine the values of $\mu, \lambda \in \mathbb{R}$ if $z = 2i$ and $z = 3i$ are roots of

$$p(z) := z^3 + \lambda i z^2 + \mu z - 12i.$$

Does the conjugate root theorem apply here?

Q4. Determine the five roots of i.

Q5. (Dr. Lloyd Gunatilake). (30 marks). Take $w \in \mathbb{C}$ such that

$$w \cdot \overline{w} + 64i\sqrt{3} + 16iw = 0$$

and $\text{Im}(w) < 5$.

a. Write w in the form $x + iy$, where $x, y \in \mathbb{R}$.
b. Determine the values of $n \in \mathbb{N}$ such that w^n is real.

Q6. Suppose that $z \in \mathbb{C}$ is of the form

$$z = \frac{(\sqrt{6} + \sqrt{2}i)^3}{(1 - ki)^2},$$

for some $k \in \mathbb{R}$ with $k > 0$.

a. If $|z| = 4$, determine the value(s) of k.
b. Write z in polar form.

Q7. Let $p : \mathbb{C} \to \mathbb{C}$ be the complex polynomial defined by

$$p(z) = z^4 + az^3 + bz^2 + cz + 16,$$

where $a, b, c \in \mathbb{R}$.

a. Suppose that $z = 2\text{cis}\left(\frac{\pi}{4}\right)$ and $w = 2\text{cis}\left(-\frac{\pi}{3}\right)$ are roots of p. Determine the values of $a, b, c \in \mathbb{R}$.

b. Plot the roots of p on an Argand diagram.

Q8. Draw the region $S \cap T$ on an Argand diagram, where $S = \{z \in \mathbb{C} : |z-1| \leq 1\}$ and $T = \{z \in \mathbb{C} : \Re(z) + \Im(z) = 1\}$.

Q9. Determine the three roots of $1 + \sqrt{3}i$.

Q10. Let $p : \mathbb{C} \to \mathbb{C}$ be the complex polynomial defined by

$$p(z) = (z - w_1)^2 (z - w_2)^3.$$

a. What is the degree of p?

b. State the fundamental theorem of algebra.

c. Does the fundamental theorem of algebra apply to p? Justify your answer.

Q11. Let $p : \mathbb{C} \to \mathbb{C}$ be the complex polynomial defined by

$$p(z) = z^3 + az^2 + bz - 4.$$

a. If $z = 1$ and $z = 1 + \sqrt{3}i$ are roots of p, determine the third root of p.

b. Determine the values of $a, b \in \mathbb{R}$.

c. Now suppose q is another complex polynomial of the form $q(z) = z^3 + \lambda z^2 + \mu z + 1$, for some $\lambda, \mu \in \mathbb{R}$, with roots $z = x + iy$ and $z = x - iy$ as roots, with $x, y \in \mathbb{R}$. Determine the imaginary part of the third root of $q(z)$.

Q12. Suppose that $z \in \mathbb{C} \backslash \{\pm 1\}$ with $|z| = 1$. Show that

$$\arg\left(\frac{z-1}{z+1}\right) = \frac{\pi}{2}, \quad \text{or} \quad \arg\left(\frac{z-1}{z+1}\right) = \frac{3\pi}{2}.$$

Q13. Determine the four roots of $z = \sqrt{2} + i\sqrt{6}$.

Q14. a. Use de Moivre's theorem to show that

$$\cos 5\vartheta = \cos\vartheta (16\cos^4\vartheta - 20\cos^2\vartheta + 5).$$

b. Solve the equation $\cos 5\vartheta$ and deduce that

$$\cos^2\left(\frac{\pi}{10}\right) = \frac{5+\sqrt{5}}{8}.$$

c. Write explicit formulae for

$$\cos\left(\frac{3\pi}{10}\right), \cos\left(\frac{7\pi}{10}\right), \text{ and } \cos\left(\frac{9\pi}{10}\right).$$

Q15. Sketch the region $S \cap T$ on an Argand diagram, where $S = \{z \in \mathbb{C} : |z-2| \leq 1\}$ and $T = \left\{z \in \mathbb{C} : |z-1| \leq \frac{3}{2}\right\}$.

Q16. a. Solve the equation $z^5 = -1$.

b. Hence, or otherwise, show that

$$z^5 + 1 = (z+1)\left(z^2 - 2z\cos\left(\frac{\pi}{5}\right) + 1\right)\left(z^2 - 2z\cos\left(\frac{3\pi}{5}\right) + 1\right).$$

Q17. Determine the values of $a, b \in \mathbb{R}$ such that

$$p(z) = z^2 + az + b,$$

where $p(1+4i) = 0$.

Q18. Using de Moivre's theorem, show that

$$\cos(3\vartheta) = 4\cos^3\vartheta - 3\cos\vartheta.$$

Q19. a. State the fundamental theorem of algebra.

b. Provide an example to show that the fundamental theorem of algebra fails for real valued polynomials.

Q20. Prove that a polynomial $p : \mathbb{C} \to \mathbb{C}$ of the form

$$p(z) = z^3 + az^2 + bz + c,$$

with $a, b, c \in \mathbb{R}$ must necessarily have a root w with $\mathrm{Im}(w) = 0$.

Q21. Using de Moivre's theorem, show that

$$\sin^6\vartheta = \frac{1}{32}\left(10 - 15\cos 2\vartheta + 6\cos 4\vartheta - \cos 6\vartheta\right).$$

Q22. Suppose $z \in \mathbb{C}$ satisfies

$$z^2 = 5 - 12i.$$

Express z in the form $z = x + iy$, with $x, y \in \mathbb{R}$.

Q23. Let $n \in \mathbb{N}$ and suppose $z \in \mathbb{C}$ satisfies

$$\frac{1}{z^n}(z+i)^n = 1.$$

a. Show that $z = \frac{i}{\text{cis}\left(\frac{2k\pi}{n}\right)-1}$, where $1 \leq k \leq n-1$.

b. Hence, or otherwise, show that $z = \frac{1}{2}\cot\left(\frac{k\pi}{n}\right) - \frac{i}{2}$.

Q24. Let $z_1 = 1+2i$ and $z_2 = 1-\sqrt{3}i$. Evaluate

$$\text{Re}\left(\frac{z_1+z_2}{|z_1|^2} - \overline{z_2}\right).$$

Q25. Let $z \in \mathbb{C}$ satisfy

$$z^8 - 2\cos(\vartheta)z^4 + 1 = \left(z^4 - \text{cis}(\vartheta)\right)\left(z - \text{cis}(-\vartheta)\right).$$

Solve the equation

$$z^8 - z^4 + 1 = 0,$$

expressing your solution in polar form.

Q26. Solve the equation

$$z^4 - z^2 - 20 = 0.$$

Q27. Let $z = x+iy$, where $x, y \in \mathbb{R}$. Sketch the curve given by

$$3[\text{Re}(z)]^2 + 6[\text{Im}(z)] = 6$$

on an Argand diagram.

Q28. Prove that a complex polynomial of odd degree with real coefficients must necessarily have a real root.

Q29. Solve the equation

$$z^4 = \sqrt{2} - i\sqrt{6}.$$

1.7 Answers

4.1 - An Introduction to Complex Numbers

Q1. a. $3i$.
b. $2i$.
c. $4i$.
d. $2\sqrt{2}i$.
e. $3\sqrt{2}i$.
f. $\sqrt{17}i$.

Q2. a. $4-i$.
b. $1-3i$.
c. $-1-3i$.
d. $-1-\frac{4}{3}i$.
e. $3-i$.
f. -1.

Q3. a. $\Re(z)=4$, $\Im(z)=-1$.
b. $\Re(z)=1$, $\Im(z)=-3$.
c. $\Re(z)=-1$, $\Im(z)-3$.
d. $\Re(z)=-1$, $\Im(z)=-\frac{4}{3}$.
e. $\Re(z)=3$, $\Im(z)=-1$.
f. $\Re(z)=-1$, $\Im(z)=0$.

Q4. 10.

Q5. 1.

Q6. See Worked Solutions.

4.2 - Elementary Operations in $\mathbb{C}$.

Q1. a. $z=3-i$.
b. $z=3+2i$.
c. $z=4+i$.
d. $z=-1-4i$.
e. $z=2i$.
f. $z=4-11i$.

Q2. a. $z=3-i$.
b. $z=11-2i$.
c. $z=28+12i$.
d. $z=15-3i$.
e. $z=10$.
f. $z=2+14i$.

Q3. a. $\overline{z}=1-i$.
b. $\overline{z}=2+5i$.
c. $\overline{z}=5-6i$.
d. $\overline{z}=2-3i$.
e. $\overline{z}=1-2i$.
f. $\overline{z}=-9-4i$.

Q4. a. $z=1+2i$.
b. $z=\frac{3}{5}-\frac{4}{5}i$.
c. $z=-\frac{7}{13}-\frac{4}{13}i$.
d. $z=\frac{3}{5}+\frac{1}{5}i$.
e. $z=\frac{9}{17}-\frac{2}{17}i$.
f. $z=\frac{3}{5}+\frac{6}{5}i$.

Q5. -4.

Q6. $-\frac{23}{10}$.

Q7. $\frac{691}{9700}-\frac{3853}{9700}i$.

4.3 - The Argand Diagram and Polar Form

Q1.
a. $|z| = \sqrt{5}$.
b. $|z| = 5$.
c. $|z| = \sqrt{2}$.
d. $|z| = 2\sqrt{2}$.
e. $|z| = \sqrt{10}$.
f. $|z| = 5\sqrt{2}$.

Q2.
a. $\vartheta = \frac{\pi}{4}$.
b. $\vartheta = -\frac{\pi}{3}$.
c. $\vartheta = -\frac{\pi}{4}$.
d. $\vartheta = \frac{\pi}{6}$.
e. $\vartheta = -\frac{\pi}{6}$.
f. $\vartheta = \frac{\pi}{4}$.

Q3.
a. $z = \sqrt{2}\text{cis}\left(\frac{\pi}{4}\right)$.
b. $z = 2\text{cis}\left(-\frac{\pi}{3}\right)$.
c. $z = 2\sqrt{2}\text{cis}\left(-\frac{\pi}{4}\right)$.
d. $z = 2\sqrt{2}\text{cis}\left(\frac{\pi}{6}\right)$.
e. $z = 2\text{cis}\left(-\frac{\pi}{6}\right)$.
f. $z = \sqrt{14}\text{cis}\left(\frac{\pi}{4}\right)$.

Q4.
a. $z = 1 + \sqrt{3}i$.
b. $z = \frac{3\sqrt{3}}{2} + \frac{3}{2}i$.
c. $z = -1$.
d. $z = \frac{1}{3}$.
e. $z = -\frac{1}{5} + \frac{\sqrt{3}}{5}i$.
f. $z = -\sqrt{3} + i$.

Q5. $\frac{1}{5}$.

Q6. See Worked Solutions.

Q7.
a. $12\text{cis}\left(-\frac{\pi}{12}\right)$.
b. $\frac{3}{2}\text{cis}\left(\frac{7\pi}{12}\right)$.

Q8.
a. $\sqrt{2} + i\sqrt{2}$.
b. $-3i$.
c. $\sqrt{2}$.

4.4 - Operations in Polar Form

Q1.
a. $2\text{cis}\left(\frac{5\pi}{6}\right)$.
b. $2\text{cis}\left(\frac{\pi}{6}\right)$.
c. $\frac{1}{2}\text{cis}\left(-\frac{\pi}{6}\right)$.

Q2.
a. $\frac{1}{2}\text{cis}\left(-\frac{5\pi}{6}\right)$.
b. $2\text{cis}\left(\frac{\pi}{6}\right)$.
c. $\frac{9}{2}\text{cis}\left(-\frac{\pi}{6}\right)$.

Q3.
a. $4\text{cis}(\pi)$.
b. $\text{cis}(\pi)$.
c. $4\text{cis}(0)$.

Q4. $z = 4 + 4\sqrt{3}i$.

Q5. $-i$.

Q6.
a. $\frac{1}{4}\text{cis}\left(-\frac{\pi}{6}\right)$.
b. $2\text{cis}\left(\frac{\pi}{12}\right)$.

Q7. $16 - 16\sqrt{3}i$.

Q8. -64.

Q9. $512\sqrt{2} - 512\sqrt{6}i$.

Q10.
a. See Worked Solutions.
b. See Worked Solutions.

Q11. The argument doubles.

Q12. Triples. Yes. Halves.

4.5 - Polynomials over $\mathbb{C}$

Q1.
a. $z = 2, z = 3$.
b. $z = -1$.
c. $z = -i, z = i$.
d. $z = -1 - i\sqrt{2}, z = -1 + i\sqrt{2}$.
e. $z = 7 - 10i, z = 7 + 10i$.
f. $z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i, z = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$.

Q2.
a. $z = 1, z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$.
b. $z = \frac{\sqrt{3}}{2} - i\frac{1}{2}, z = -1, z = -\frac{\sqrt{3}}{2} - \frac{1}{2}i$.
c. $z = \frac{\sqrt{3}}{2} + i\frac{1}{2}, z = -\frac{\sqrt{3}}{2} + \frac{1}{2}i, z = -i$.
d. $z = 2^{\frac{1}{3}}\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right), z = 2^{\frac{1}{3}}\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right), z = -2^{\frac{1}{3}}i$.

Q3. $z = \sqrt[4]{2\sqrt{2}}\text{cis}\left(-\frac{\pi}{12}\right), z = \sqrt[4]{2\sqrt{2}}\text{cis}\left(\frac{5\pi}{12}\right), z = \sqrt[4]{2\sqrt{2}}\text{cis}\left(\frac{11\pi}{12}\right), z = \sqrt[4]{2\sqrt{2}}\text{cis}\left(-\frac{7\pi}{12}\right)$.

Q4. $z = 2^{\frac{1}{5}}\,\text{cis}\left(-\frac{\pi}{30}\right), z = 2^{\frac{1}{5}}\,\text{cis}\left(\frac{11\pi}{30}\right), z = 2^{\frac{1}{5}}\,\text{cis}\left(\frac{23\pi}{30}\right), z = 2^{\frac{1}{5}}\,\text{cis}\left(-\frac{5\pi}{6}\right), z = 2^{\frac{1}{5}}\,\text{cis}\left(-\frac{13\pi}{30}\right)$.

Q5. $\lambda = i\sqrt{3} - 1$.

Q6. $\lambda = 6$.

Q7. $a = -2, b = -2$.

Q8. $\lambda = 1 + i, \mu = 4i$.

Q9. See Worked Solutions.