Trigonometry.

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The first thing we look at is converting degrees to radians. Radians are the primary unit of angles that mathematicians use, not degrees. The formula for converting degrees to radians is given by

$$x \longmapsto \frac{\pi}{180}x.$$

The corresponding formula that converts radians to degrees is given by

$$x \longmapsto \frac{180}{\pi}x.$$

Example 1. Convert $275°$ to radians.

Proof. Using the above formula, we simply observe that

$$275° \quad = \quad \frac{\pi}{180}(275) = \frac{55}{36}\pi.$$

□

Exercise 2. Convert the following to radians.

a. $360°$.
b. $200°$.
c. $160°$.
d. $670°$.
e. $120°$.
f. $165°$.
g. $180°$.
h. $720°$.

Exercise 3. Convert the following to degrees.

a. π.
b. $\frac{\pi}{2}$.
c. $\frac{3\pi}{2}$.
d. $\frac{4\pi}{7}$.
e. $\frac{3\pi}{8}$.
f. $\frac{4\pi}{3}$.

Exercise 4. Using the exact value triangles, determine the following.

a. $\sin\frac{\pi}{4}$.
b. $\cos\frac{3\pi}{4}$.
c. $\sin\frac{\pi}{3}$.
d. $\cos\frac{5\pi}{6}$.
e. $\sin\frac{7\pi}{6}$.
f. $\cos\frac{\pi}{4}$.

Exercise 5. By looking at the quadrants, determine the following.

a. $\sin\frac{\pi}{2}$.
b. $\cos 0$.
c. $\tan\frac{3\pi}{2}$.
d. $\cos\pi$.
e. $\sin\pi$.
f. $\tan 0$.
g. $\cos 2\pi$.
h. $\cos\frac{\pi}{2}$.
i. $\sin\frac{3\pi}{2}$.
j. $\tan 4\pi$.

Exercise 6. For $x \in [0, 2\pi]$, solve the following equations.

a. $\sin x = \frac{1}{\sqrt{2}}$.
b. $\cos x = -\frac{1}{2}$.
c. $\sin x = \frac{1}{2}$.
d. $\cos x = \frac{\sqrt{3}}{2}$.
e. $\sin x = -\frac{\sqrt{3}}{2}$.
f. $\tan x = 1$.
g. $\tan x = -\sqrt{3}$.
h. $\tan x = \frac{1}{\sqrt{3}}$.

Exercise 7. For $x \in [0, 2\pi]$, solve the following equations.

a. $\sin 2x = \frac{\sqrt{3}}{2}$.
b. $\cos 2x = \frac{\sqrt{2}}{2}$.
c. $\tan 3x = 1$.
d. $\sin \frac{x}{2} = -\frac{1}{2}$.
e. $\tan 2x = -\sqrt{3}$.
f. $\cos 4x = -1$.
g. $\tan x = 0$.
h. $\cos \frac{x}{3} = 0$.
i. $\sin x = 1$.
j. $\tan 2x = -1$.

Exercise 8. Solve the following equation for $x \in [0, 2\pi]$.

$$\sin x = \sqrt{3} \cos x.$$

Exercise 9. Solve the following equation for $x \in [0, 2\pi]$.

$$\sin 2x = \cos 2x.$$

Exercise 10. Suppose that $\cos(x) = \frac{1}{2}$ for $\frac{3\pi}{2} < x < 2\pi$. Find the exact values for

(a) $\sin(x)$
(b) $\tan(x)$

Compound and Double Angle Formulae

Compound Angle and Double Angle Formulae. The compound angle formulas enable us to calculate the exact values of expressions such as $\sin(\frac{\pi}{12})$ by breaking up $\frac{\pi}{12}$ into $\frac{\pi}{3} - \frac{\pi}{4}$. The double angle formula allow us to do a similar thing, with angles such as $\frac{\pi}{8}$, by expressing $\frac{\pi}{8}$ as $\frac{1}{2} \cdot \frac{\pi}{4}$.

The **compound angle** formulae are listed below:

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(x-y) &= \sin(x)\cos(y) - \cos(x)\sin(y) \\
\cos(x+y) &= \cos(x)\cos(y) - \sin(x)\sin(y) \\
\cos(x-y) &= \cos(x)\cos(y) + \sin(x)\sin(y) \\
\tan(x+y) &= \frac{\tan(x) + \tan(y)}{1 - \tan(x)\tan(y)} \\
\tan(x-y) &= \frac{\tan(x) + \tan(y)}{1 + \tan(x)\tan(y)}
\end{aligned}$$

The **double angle** formulae are listed below:

$$\begin{aligned}
\sin(2x) &= 2\sin(x)\cos(x) \\
\cos(2x) &= \cos^2(x) - \sin^2(x) \\
&= 2\cos^2(x) - 1 \\
&= 1 - 2\sin^2(x) \\
\tan(2x) &= \frac{2\tan(x)}{1-\tan^2(x)} \\
&= \frac{\sin(2x)}{\cos(2x)}
\end{aligned}$$

The double angle formula can be derived from the compound angle formulae. For example

Let $x = x$ and let $y = x$.

Consider

$$\begin{aligned}
\sin(x+y) &= \sin(x)\cos(y) + \cos(x)\sin(y) \\
\sin(x+x) &= \sin(x)\cos(x) + \cos(x)\sin(x) \\
\sin(2x) &= \sin(x)\cos(x) + \sin(x)\cos(x) \\
\sin(2x) &= 2\sin(x)\cos(x)
\end{aligned}$$

Example 2. Expand the following

(a) $\sin(x-2y)$

Proof.

$$\begin{aligned}
\sin(x-2y) &= \sin(x)\cos(2y) - \cos(x)\sin(2y) \\
&= \sin(x)(\cos^2(y) - \sin^2(y)) - \cos(x)(2\sin(y)\cos(y)) \\
&= \sin(x)\cos^2(y) - \sin(x)\sin^2(y) - 2\cos(x)\sin(y)\cos(y)
\end{aligned}$$

□

(b) $\cos(3x)$

Proof.

$$\begin{aligned}
\cos(3x) &= \cos(x+2x) \\
&= \cos(x)\cos(2x) - \sin(x)\sin(2x) \\
&= \cos(x)(1-2\sin^2(x)) - \sin(x)(2\sin(x)\cos(x)) \\
&= \cos(x) - 2\sin^2(x)\cos(x) - 2\sin^2(x)\cos(x) \\
&= \cos(x) - 4\sin^2(x)\cos(x)
\end{aligned}$$

□

(c) $\tan(2x+y)$

Proof.

$$\begin{aligned}
\tan(2x+y) &= \frac{\tan(2x)+\tan(y)}{1-\tan(2x)\tan(y)} \\
&= \frac{\frac{2\tan(x)}{1-\tan^2(x)}+\tan(y)}{1-\frac{2\tan(x)}{1-\tan^2(x)}\tan(y)} \\
&= \frac{2\tan(x)+\tan(y)(1-\tan^2(x))}{1(1-\tan^2(x))-2\tan(x)\tan(y)} \\
&= \frac{2\tan(x)+\tan(y)-\tan(x)\tan^2(y)}{1-\tan^2(x)-2\tan(x)\tan(y)}
\end{aligned}$$

□

Example 3. Find the following exact values using the compound angle formulae.

(a) $\cos(\frac{\pi}{12})$

Proof.

$$\begin{aligned}
\cos(\frac{\pi}{12}) &= \cos(\frac{\pi}{3}-\frac{\pi}{4}) \\
&= \cos(\frac{\pi}{3})\cos(\frac{\pi}{4})+\sin(\frac{\pi}{3})\sin(\frac{\pi}{4}) \\
&= (\frac{1}{2})\cdot(\frac{\sqrt{2}}{2})+(\frac{\sqrt{3}}{2})\cdot(\frac{\sqrt{2}}{2}) \\
&= \frac{\sqrt{2}+\sqrt{6}}{4}
\end{aligned}$$

□

(b) $\tan(\frac{7\pi}{12})$

Proof.

$$\begin{aligned}
\tan(\frac{7\pi}{12}) &= \tan(\frac{\pi}{3}+\frac{\pi}{4}) \\
&= \frac{\tan(\frac{\pi}{3})+\tan(\frac{\pi}{4})}{1-\tan(\frac{\pi}{3})\tan(\frac{\pi}{4})} \\
&= \frac{(\sqrt{3})+(1)}{1-(\sqrt{3})\cdot(1)} \\
&= \frac{1+\sqrt{3}}{1-\sqrt{3}}
\end{aligned}$$

□

Example 4. Find the exact value of $\cos(\frac{\pi}{8})$.

Proof. Using the double angle formula

$$\cos(2x) = 2\cos^2(x) - 1$$

yields

$$\begin{aligned}
\cos(\frac{\pi}{4}) &= 2\cos^2(\frac{\pi}{8}) - 1 \\
\frac{\sqrt{2}}{2} + 1 &= 2\cos^2(\frac{\pi}{8}) \\
\cos^2(\frac{\pi}{8}) &= \frac{\sqrt{2}+2}{4} \\
\cos(\frac{\pi}{8}) &= \frac{\pm\sqrt{\sqrt{2}+2}}{2}
\end{aligned}$$

□

Exercise 11. Find the exact values for the following.

a. $\cos\frac{\pi}{12}$.
b. $\sin\frac{7\pi}{12}$.
c. $\cos\frac{13\pi}{12}$.
d. $\sin -\frac{\pi}{12}$.
e. $\tan\frac{\pi}{8}$.
f. $\cos\frac{3\pi}{8}$.

Exercise 12. Simplify the following expression using double angle formulae.

$$2\tan\xi + \tan(\xi + \eta).$$

Exercise 13. Simplify the following expressions using double angle formulae.

(a)

$$\frac{2\tan\left(\frac{\beta}{2}\right)}{1 - \tan^2\left(\frac{\beta}{2}\right)}$$

(b)

$$\sin^2(\eta) - \cos^2(\eta)$$

(c)

$$\frac{\cos(2\delta)}{\sin^4(\delta) - \cos^4(\delta)}$$

Reciprocal Trigonometric Functions.

In this section we consider the function

$$\csc x := \frac{1}{\sin x}, \quad \sec x := \frac{1}{\cos x}, \quad \cot x := \frac{1}{\tan x}.$$

Exercise 14. Determine the following exact values

(a) $\cos(\frac{\pi}{3})$ (b) $\sin(\frac{2\pi}{3})$ (c) $\cos(\frac{5\pi}{6})$

Exercise 15. Evaluate the following expressions

(a) $\sec(\frac{\pi}{4})$ (b) $\csc(\frac{5\pi}{3})$ (c) $\cot(\frac{5\pi}{6})$

(d) $\csc(\frac{\pi}{2})$ (e) $\csc(\frac{7\pi}{6})$ (f) $\cot(\frac{2\pi}{3})$

(g) $\sec(\frac{5\pi}{6})$ (h) $\cot(\frac{9\pi}{4})$ (i) $\csc(\frac{5\pi}{2})$

Exercise 16. Sketch the following curves, on the domain $x \in [-2\pi, 2\pi]$

(a) $y = 2\cos(x)$ (b) $y = 3\sin(x)$ (c) $y = \frac{1}{2}\cos(2x)$

(d) $y = 2\tan(x)$ (e) $y = \tan(4x)$ (f) $y = \frac{3}{2}\sin(x - \pi)$

(g) $y = \cos(x + \frac{\pi}{3})$ (h) $y = 2\sin(2x - \frac{\pi}{3})$ (i) $y = 4\tan(3x + \frac{\pi}{2})$

Exercise 17. Sketch the reciprocal curves of the curves sketched in Exercise 17.

Exercise 18. Sketch the following curves, on the domain $x \in [-2\pi, 2\pi]$.

(a) $y = |\sec(x)|$ (b) $y = |2\cot(x)|$ (c) $y = \csc|x|$

Trigonometric Identities.

Recall the Pythagorean Identity

$$\sin^2(x) + \cos^2(x) = 1$$

From this expression, we can derive two more trigonometric identities that can help us in simplifying expressions involving trigonometric functions. Consider

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \frac{\sin^2(x)}{\cos^2(x)} + \frac{\cos^2(x)}{\cos^2(x)} &= \frac{1}{\cos^2(x)} \\ \tan^2(x) + 1 &= \sec^2(x) \end{aligned}$$

Therefore, we have derived the trigonometric identity $\tan^2(x) + 1 = \sec^2(x)$ by dividing every term in the Pythagorean identity by $\cos^2(x)$. If we divide every term in the Pythagorean identity by $sin^2(x)$, we can derive another trigonometric identity.

$$\begin{aligned} \sin^2(x) + \cos^2(x) &= 1 \\ \frac{\sin^2(x)}{\sin^2(x)} + \frac{\cos^2(x)}{\sin^2(x)} &= \frac{1}{\sin^2(x)} \\ 1 + \cot^2(x) &= \csc^2(x) \end{aligned}$$

There the trigonometric identities are

$$\begin{aligned}
\sin^2(x) + \cos^2(x) &= 1 \\
\tan^2(x) + 1 &= \sec^2(x) \\
1 + \cot^2(x) &= \csc^2(x)
\end{aligned}$$

Example 5. Suppose $\cos(x) = \frac{\sqrt{5}-1}{3}$, calculate the exact value of $\sin(x)$

Proof. Using the Pythagorean Identity

$$\sin^2(x) + \cos^2(x) = 1$$

yields

$$\begin{aligned}
\sin^2(x) + \left(\frac{\sqrt{5}-1}{3}\right)^2 &= 1 \\
\sin^2(x) &= 1 - \left(\frac{\sqrt{5}-1}{3}\right)^2 \\
&= 1 - \frac{5 - 2\sqrt{5} + 1}{9} \\
&= \frac{9 - (6 - 2\sqrt{5})}{9} \\
&= \frac{3 + 2\sqrt{5}}{9} \\
\therefore\ \sin(x) &= \frac{\pm\sqrt{3 + 2\sqrt{5}}}{3}.
\end{aligned}$$

□

Exercise 19. Prove the following results.

(a)

$$\left(1 + \cot^2(\alpha)\right)\left(1 - \cos^2(\alpha)\right) = 1$$

(b)

$$\csc^2(\eta) + \sec^2(\eta) = \csc^2(\eta)\sec^2(\eta)$$

Exercise 20. Simplify each of the following expressions using trigonometric identities.

(a)

$$\tan^2(x) - \sec^2(x)$$

(b)

$$\sin^2(x)\cot^2(x)$$

(c)

$$\frac{\tan^2(x)}{1 + \tan^2(x)}$$

(d)
$$\sin^4(x) - \cos^4(x)$$

(e)
$$\csc^2(x) - \cot^2(x)$$

(f)
$$\cot^2(x) + \sin^2(x)$$

Exercise 21. Show that
$$\frac{1}{\cos^4(x) - \sin^4(x)} = \sec(2x)$$
and *hence*, find the exact values of x which satisfy the equation
$$\frac{1}{\cos^4(x) - \sin^4(x)} = 2, \ x \in [0, 2\pi]$$

Inverse Trigonometric Functions.

Example 6. Simplify each of the following expressions

(a) $\cos^{-1}\left(\cos(\frac{\pi}{6})\right)$.

Proof.
$$\begin{aligned}\cos^{-1}\left(\cos(\frac{\pi}{6})\right) &= \cos^{-1}(\frac{\sqrt{3}}{2}) \\ &= \frac{\pi}{6}\end{aligned}$$
□

(b) $\tan\left(\tan^{-1}(\frac{\pi}{3})\right)$.

Proof.
$$\begin{aligned}\tan\left(\tan^{-1}(\sqrt{3})\right) &= \tan(\frac{\pi}{3}) \\ &= \sqrt{3}\end{aligned}$$
□

(c) $\sin^{-1}\left(\cos(\frac{7\pi}{3})\right)$.

Proof.
$$\begin{aligned}\sin^{-1}\left(\cos(\frac{7\pi}{3})\right) &= \sin^{-1}(\frac{1}{2}) \\ &= \frac{\pi}{6}\end{aligned}$$
□

Exercise 22. Determine the following exact values

(a) $\sin^{-1}(\frac{\sqrt{3}}{2})$ (b) $\tan^{-1}(\frac{1}{\sqrt{3}})$ (c) $\cos^{-1}(\frac{1}{2})$

(d) $\tan^{-1}(\sqrt{3})$ (e) $\cos^{-1}(\frac{\sqrt{2}}{2})$ (f) $\sin^{-1}(1)$

Exercise 23. Determine the following exact values

(a) $\sin^{-1}(\sin(\frac{\pi}{3}))$ (b) $\cos(\cos^{-1}(\frac{\sqrt{2}}{2}))$ (c) $\tan(\sin^{-1}(\frac{1}{\sqrt{3}}))$

(d) $\cos^{-1}(\tan^{-1}(\sqrt{3}))$ (e) $\sin(\cos^{-1}(\frac{1}{2}))$ (f) $\tan(\tan^{-1}(1))$

Exercise 24. Sketch the following inverse circular functions

(a) $y = \sin^{-1}(x-1)$ (b) $y = 3\cos^{-1}(x)$ (c) $y = \cos^{-1}(2x)$

(d) $y = \tan^{-1}(x) + \pi$ (e) $y = 2\tan^{-1}(2x)$ (f) $y = \sin^{-1}(2x+2)$

(g) $y = \cos^{-1}(\frac{x}{3}) + 1$ (h) $y = \tan^{-1}(x-2)$ (i) $y = \tan^{-1}(\frac{x}{2}) - \pi$

(j) $y = \sin^{-1}(\frac{x}{3} - 1)$ (k) $y = \cos^{-1}(\frac{x+1}{2})$ (l) $y = \sin^{-1}(\frac{x-2}{3})$

Exercise 25. Simplify each of the following expressions.

(a)

$$\tan^{-1}\left(\tan\left(\frac{\pi}{3}\right)\right)$$

(b)

$$\sin^{-1}\left(\cos\left(\frac{\pi}{6}\right)\right)$$

(c)

$$\tan^{-1}\left(\cos\left(\frac{2\pi}{3}\right)\right)$$

Exercise 26. Sketch the following curves, stating all relevant features. Include the domain and range.

(a)

$$f(x) = 2\sin^{-1}(x-1)$$

(b)

$$f(x) = \tan^{-1}(x+3) - 2$$

(c)

$$f(x) = \frac{1}{\cos^{-1}(x)}$$