Calculus Exam 2 (Preparation) (Extended) Solutions

Kyle Broder – ANU – MSI – 2017

The contents of this examination require an understanding of the elementary calculus material that was covered in the calculus practice exams 1-5. An understanding of graphing techniques and function transformations may also be required.

There are no permitted materials for this test. That is, you are not permitted any cheat notes, calculators or resources other than a pen/pencil, eraser, sharpener, ruler and water bottle.

There is to be no collaboration on this examination and any attempts of communication will result in a nullified score. You are permitted 10 minutes of reading time and 105 minutes of writing time. There is a total of 100 available marks. It is recommended that you use the reading time to ask the invigilator about any issues regarding the format of the test, the problems or other issues. No hints will be given. Best of luck!

Name: ____________________

Grade: __________/100

Question 1. Evaluate the derivatives of the following functions.

a. $f(x) = \sin 2x$.

Proof. We simply observe that $f'(x) = 2\cos 2x$. $\square$

b. $f(x) = \sec x$.

Proof. We simply observe that $f'(x) = \tan x \sec x$. $\square$

c. $f(x) = \cos^2 x$.

Proof. We simply observe that $f'(x) = -2\sin x \cos x$. $\square$

d. $f(x) = \csc(3x)$.

Proof. We simply observe that $f'(x) = -3\cot(3x)\csc(3x)$. $\square$

e. $f(x) = \sec x \tan x + \cos^2(x + \pi)$.

Proof. The derivative of $\sec x$ is $\tan x \sec x$ and the derivative of $\tan x$ is $\sec^2 x$. So it is easy to see that

$$f'(x) = \sec^3(x) - 2\sin x \cos x + \tan^2 \sec(x).$$

$\square$

f. $f(x) = \tan^4 x + \sec^3 x + \frac{3}{5}\cot x$.

Proof. We simply observe that

$$f'(x) = -\frac{3}{6}\csc^2(x) + 3\tan x \sec^3(x) + 4\tan^3(x)\sec^2(x).$$

$\square$

Question 2. Determine the domain on which the following functions are differentiable and evaluate their derivatives.

a. $f(x) = \log_e(x)$.

Proof. The function $f(x) := \log_e(x)$ is differentiable $\forall x \in \mathbb{R}_+$,
where $\mathbb{R}_+ := \{x \in \mathbb{R} : x > 0\}$. $\square$

b. $f(x) = \exp(x)$.

Proof. The function $f(x) = e^x$ is differentiable $\forall x \in \mathbb{R}$. $\square$

c. $f(x) = \sin x$.

Proof. The function $f(x) = \sin x$ is differentiable $\forall x \in \mathbb{R}$. $\square$

d. $f(x) = |x|$.

Proof. The function $f(x) := |x|$ is differentiable $\forall x \in \mathbb{R}\backslash\{0\}$. $\square$

e. $f(x) = \tan^3(x)$.

Proof. The function $f(x) = \tan^3(x)$ is differentiable for all $x \in \mathbb{R}\backslash\mathscr{S}$,
where $\mathscr{S} := \{x \in \mathbb{R} := \frac{k\pi}{2}, k \in \mathbb{Z}\}$. $\square$

f. $f(x) = \log_e |x|$.

Proof. The function $f(x) := \log_e |x|$ is differentiable $\forall x \in \mathbb{R}\backslash\{0\}$. $\square$

g. $f(x) = 2\exp(-x^2) + \cos x$.

Proof. The function $f(x) := 2\exp(-x^2) + \cos x$ is differentiable $\forall x \in \mathbb{R}$ since x^2, $\exp(-x)$ and $\cos x$ are differentiable $\forall x \in \mathbb{R}$. $\square$

h. $f(x) = \log_e |x| + \cot(x)$.

Proof. The function $f(x) := \log |x| + \cot(x)$ is differentiable for all $x \in \mathbb{R}\backslash\{0\} \cap \mathbb{R}\backslash\mathscr{A}$, where $\mathscr{A} := \{x \in \mathbb{R} : x = k\pi, k \in \mathbb{Z}\}$. So f is differentiable $\forall x \in \mathbb{R}\backslash\mathscr{A}$ since $0 \in \mathscr{A}$. $\square$

Question 3. Evaluate the derivatives of the following functions.

a. $f(x) = \sec^2 x + (x^2+1)e^{-x}$.

Proof. We simply observe that

$$f'(x) = e^{-x}(2x - x^2 - 1) + 2\tan(x)\sec^2(x).$$

□

b. $f(x) = \log_e(x)\sqrt{x^2+5}$.

Proof. We simply observe that

$$f'(x) = \frac{x^2 + x^2\log_e(x) + 5}{x\sqrt{x^2+5}}.$$

□

c. $f(x) = \frac{1}{x^3+x}$.

Proof. We simply observe that

$$f'(x) = -\frac{3x^2+1}{(x^3+x)^2}.$$

□

d. $f(x) = e^{-\sqrt{x}} + x\csc(x^3)$.

Proof. We simply observe that

$$f'(x) = \csc(x^3) - 3x^3\cot(x^3)\csc(x^3) - \frac{1}{2\sqrt{x}}e^{-\sqrt{x}}.$$

□

e. $f(x) = x^2 + x^3 + \cot(x)$.

Proof. We simply observe that

$$f'(x) = 2x + 3x^2 - \csc^2(x).$$

□

f. $f(x) = \frac{\sec^2(x)}{x^2-5x+6}$.

Proof. We simply observe that

$$f'(x) = \frac{2\tan(x)\sec^2(x)}{x^2-5x+6} - \frac{(2x-5)\sec^2(x)}{(x^2-5x+6)^2}.$$

□

Question 4. Determine the stationary point(s), and their nature for the following functions.

a. $f(x) = xe^{-x}$.

Proof. Local maximum at $x = 1$. $\square$

b. $f(x) = e^{-x^2}$.

Proof. Local maximum at $x = 0$. $\square$

c. $f(x) = x^4 + 2x + 1$.

Proof. Local minimum at $x = -\frac{1}{\sqrt[3]{2}}$. $\square$

d. $f(x) = x^{\frac{1}{5}}$.

Proof. No stationary points, but there is a non-stationary point of inflection at $x = 0$. $\square$

e. $f(x) = x^{\frac{2}{5}}$.

Proof. No stationary points. $\square$

f. $f(x) = e^x$.

Proof. No stationary points. $\square$

g. $f(x) = x^2 \log_e(x)$.

Proof. Local minimum at $x = \frac{1}{\sqrt{e}}$. $\square$

Question 5. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on some set $\Omega_1 \subseteq \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is differentiable on some set $\Omega_2 \subseteq \mathbb{R}$.

a. Determine the domain on which $f + g$ and $f - g$ are differentiable.

Proof. The functions $f + g$ and $f - g$ are differentiable on $\Omega_1 \cup \Omega_2$. □

b. Determine the domain on which $f \cdot g$ is differentiable.

Proof. The function $f \cdot g$ is differentiable on $\Omega_1 \cap \Omega_2$. □

c. Determine the domain on which f/g is differentiable.

Proof. The function f/g is differentiable on $\Omega_1 \cap \Omega_2 \backslash \{x \in \mathbb{R} : g(x) = 0\}$. □

d. Determine the domain on which $\sqrt{f} \cdot g$ is differentiable.

Proof. The function $\sqrt{f} \cdot g$ is differentiable on $\Omega_1 \cap \Omega_2 \backslash \{x \in \mathbb{R} : f(x) \leq 0\}$. □

e. Determine the domain on which $|f|$ is differentiable.

Proof. The function $|f|$ is differentiable on $\Omega_1 \backslash \{x \in \mathbb{R} : f(x) = 0\}$. □

f. Let $f(x) = \exp(x)$ and $g(x) = \sqrt{x}$. On what domain is $h(x) = f(x) \cdot g(x)$ differentiable?

Proof. The function $f(x) = \exp(x)$ is differentiable $\forall x \in \mathbb{R}$. The function $g(x) := \sqrt{x}$ is differentiable $\forall x > 0$. Therefore, using part (b) of this question, we see that $h(x)$ is differentiable $\forall x > 0$. □

g. Let $f(x) = |x|$ and $g(x) = \log_e(-x)$. On which domain in $f(x) + g(x)$ differentiable?

Proof. The function $f(x) = |x|$ is differentiable $\forall x \in \mathbb{R} \backslash \{0\}$ and the function $g(x) := \log_e(-x)$ is differentiable $\forall x < 0$. So the function $f(x) + g(x)$ is differentiable $\forall x < 0$. □