# MATH1013 Final Exam Preparation

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Many of these questions were written years ago. Hence, there are likely to be typos. If there are any issues or typos please let me know. For more foundational material, please see *An Invitation to Analysis* which is available on the one drive. If you would like solutions to any of the exericises contained in *An Invitation to Analysis* please email kylebroder@gmail.com.

# Linear Algebra – Foundations

## 1. Linear Operators

**Q1.** Let $\mathscr{C}([0,1])$ denote the space of continuous functions on $[0,1]$. The operator $T : \mathscr{C}([0,1]) \longrightarrow \mathbb{R}$ defined by

$$f \longmapsto \int_0^1 f(x)dx$$

provides an example of a linear operator.

a. Define *linear transformation.*

*Proof.* A linear transformation is a map between $T : V \longrightarrow W$ between vectors $V$ and $W$ such that

(i) $T(u+v) = T(u) + T(v)$ for all $u, v \in V$.

(ii) $T(\lambda v) = \lambda T(v)$ for all $v \in V$, $\lambda \in \mathbb{R}$.

Condition (i) is called additivity and condition (ii) is called homogeneity. □

b. Consider the following computation.

$$\begin{aligned} \int_0^1 x + 4x^2 dx &= \int_0^1 xdx + \int_0^1 4x^2 dx \\ &= \int_0^1 xdx + 4\int_0^1 x^2 dx = \frac{11}{6}. \end{aligned}$$

Detail exactly where the linearity of $T$ is used in the above computation.

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**Q2.** Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be the linear operator defined by

$$T(x_1, x_2, x_3) = (x_1 + 4x_2 + 14x_3, 2x_1 + 9x_2 + 31x_3, x_1 + 5x_2 + 17x_3).$$

Write down the standard matrix for this transformation.

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**Q3.** Recall that an operator $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is *linear* if

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$,

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for all $\mathbf{u} \in \mathbb{R}^n$ and $c \in \mathbb{R}$.

(a) Use the definition to determine whether $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is linear, where

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 3x+4 \\ x+6-2y \end{bmatrix}.$$

(b) Suppose that $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a linear transformation, such that

$$S\left(\begin{bmatrix}0\\3\end{bmatrix}\right) = \begin{bmatrix}1\\4\end{bmatrix} \quad \text{and} \quad S\left(\begin{bmatrix}1\\2\end{bmatrix}\right) = \begin{bmatrix}3\\5\end{bmatrix}.$$

(i) Find $S\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$. (Show working).

(ii) Find $\mathbf{A}$, the standard matrix of $S$.

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**Q4.** Determine whether the map $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$T\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \begin{bmatrix}x+1\\2y-4\end{bmatrix}$$

is linear.

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**Q5.** Suppose $S : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ a linear operator such that

$$S\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3\\5\end{bmatrix} \quad \text{and} \quad S\left(\begin{bmatrix}0\\4\end{bmatrix}\right) = \begin{bmatrix}1\\4\end{bmatrix}.$$

Evaluate $S\left(\begin{bmatrix}1\\-2\end{bmatrix}\right)$.

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## 2. Differentiation and Limits

**Q1.** Provide an example of a function which is continuous but not differentiable.

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**Q2.** Is every differentiable function continuous?

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**Q3.** Let

$$f(x) := \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

Evaluate $\lim_{x \to 0} f(x)$.

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**Q4.** Evaluate

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right).$$

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**Q5.** Evaluate

$$\lim_{x \to 0} x \cos\left(\frac{1}{x}\right).$$

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**Q6.** Evaluate

$$\lim_{x \to \infty} \frac{x+1}{x^2-1}.$$

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**Q7.** Evaluate

$$\lim_{x \to \infty} \frac{e^{-x}+1}{e^{-x}-1}.$$

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**Q8.** Provide an example of a function which is differentiable, but whose derivative is not differentiable.

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**Q9.** Show that the following functions are differentiable and calculate their derivatives.

a. $f(x) = 2x + 1$.
b. $f(x) = 3x - 5$.
c. $f(x) = 9 - x$.
d. $f(x) = x^2$.
e. $f(x) = 5 + 2x - x^2$.
f. $f(x) = x^2 - 5x + 6$.

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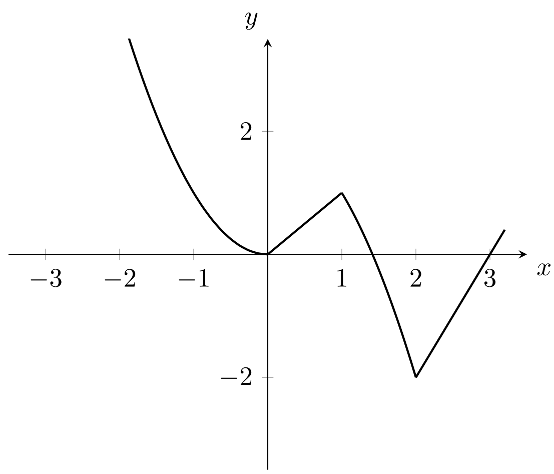
**Q10.** Consider the function $f$ whose graph is given by




Determine all points $x \in \mathbb{R}$ where $f$ is not differentiable.

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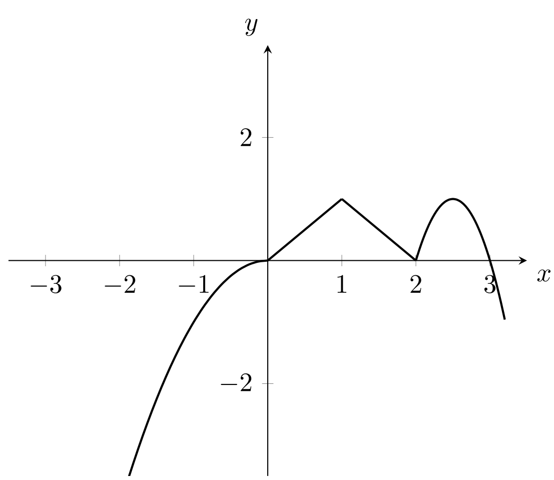
**Q11.** Consider the function $g$ whose graph is given by



Determine all points $x \in \mathbb{R}$ where $g$ is not differentiable.

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**Q12.** Describe the graph of $f(x)$ if $f'(x) = 0$ when $x = 3$ and $x = -2$, $f''(3) = 4$ and $f''(-2) = -5$.

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**Q13.** Suppose that $f(x)$ satisfies $f'(3) = 0$, and $f'(x) > 0$ for all $x \in (3, 5)$. Determine which of the following is true:

a. $0 > f''(3) > f''(5)$.
b. $0 < f''(3) < f''(5)$.
c. $f''(3) < 0 < f''(5)$.
d. $f''(5) < 0 < f''(3)$.

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**Q14.** Find two numbers $x, y \in \mathbb{R}$ whose difference is 10 and whose product is a minimum.

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**Q15.** Find two numbers $x, y \in \mathbb{R}$ whose product is 12 and whose sum is a minimum.

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**Q16.** Find the dimensions of a rectangle which has area 10 $\mathrm{m}^2$ and whose perimeter is as small as possible.

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**Q17.** A cylinder is inscribed in a circle of radius $r = 4$. Determine the largest volume of the cylinder.

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**Q18.** A cylinder is inscribed in a cone of base radius $r = 4$ and height $h = 12$. Determine the largest volume of the cylinder.

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**Q19.** Differentiate the function

$$f(x) = \sin^{-1}(x) \cdot \cos^{-1}(x).$$

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**Q20.** Differentiate the function

$$f(x) = \tan^{-1}(\cos^{-1}(x)).$$

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**Q21.** Differentiate the function

$$g(x) = \left|\tan^{-1}(x)\right|.$$

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**Q22.** Let $f$ be the function defined by

$$f(x) = \tan^{-1}(\sqrt{x+4}) + \sec(x).$$

Evaluate $f'(x)$.

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## 3. Integration

**Q1.** Evaluate the following integral

$$\int_0^{\pi} \tan x \cdot \cos x dx.$$

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**Q2.** By first differentiating the function $f(x) = \tan x$, evaluate the integral

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{3}{7} \sec^2 x + \cos x dx.$$

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**Q3.** Evaluate the integral

$$\int_0^1 |x-3|\, dx.$$

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**Q4.** Evaluate the following expressions

(a) $\int \frac{2x}{x^2-5} dx.$
(b) $\int \frac{x}{\sqrt{3-x^2}} dx.$
(c) $\int \frac{2x+6}{(x^2+6x-5)^5} dx.$
(d) $\int \frac{2x-9}{\sqrt{x^2-9x+6}} dx.$
(e) $\int 3x^2 \cdot (x^3+2)^4 dx.$
(f) $\int \frac{x}{(3-x^2)^{\frac{3}{2}}} dx.$

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**Q5.** Evaluate the following expressions

(a) $\int x^5 \cdot e^{x^6} dx.$
(b) $\int \frac{2\log_e(x)}{x} dx.$
(c) $\int \frac{-x^3}{e^{x^4}} dx.$
(d) $\int \frac{\log_e(3x)}{7x} dx.$
(e) $\int e^x \sqrt{1+e^x} dx.$
(f) $\int \frac{\sin(2\log_e(x))}{3x} dx.$

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**Q6.** Evaluate the following expressions

(a) $\int \sin(x)\cos^2(x)dx$.

(b) $\int 3\tan(x)\sec^2(x)dx$.

(c) $\int 4\cos(x)e^{\sin(x)}dx$.

(d) $\int 2\tan(x)dx$.

(e) $\int \cot(x)dx$.

(f) $\int \frac{\sin(x)}{1-\cos(x)}dx$.

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**Q7.** Evaluate the following expressions

(a) $\int 2\sin(x)\sqrt{1+\cos(x)}dx$.
(b) $\int \frac{3x^2\sin(x^2)}{x}dx$.

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**Q8.** Evaluate the following expressions

(a) $\int \frac{x}{\sqrt{x-1}}dx$.
(b) $\int x\sqrt{2+x}dx$.
(c) $\int \frac{2-x}{(x+3)^4}dx$.
(d) $\int \frac{x^2}{\sqrt{x+1}}dx$.
(e) $\int \frac{e^{4x}}{1+e^x}dx$.
(f) $\int \frac{(x+1)^2}{\sqrt{x-3}}dx$.

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**Q9.** Evaluate the following expressions

a. $\int \sin^2(5x)dx$.
b. $\int 2\cos^2(x+1)dx$.
c. $\int 3\sin^2(\frac{x}{2})dx$.
d. $\int \sin(2x)\cos(2x)dx$.
e. $\int \frac{\tan(x)}{\sec^2(x)}dx$.
f. $\int \sin(\frac{x+1}{2})\cos(\frac{x+1}{2})dx$.

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**Q10.** Evaluate the following expressions

a. $\int 2\sin^3(2x)dx$.
b. $\int 3\cos^3(5x)dx$.
c. $\int \sin^5(x)dx$.
d. $\int 2\cos^5(2x)dx$.
e. $\int \frac{\tan^5(x)}{\sec^5(x)}dx$.
f. $\int \sin^3(\frac{x}{3})dx$.

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**Q11.** Evaluate the following expressions

a. $\int \tan(x)\sec^2(x)dx$.
b. $\int \tan^2(5x)dx$.
c. $\int \tan^3(x)dx$.
d. $\int \sec^2(x)e^{\tan(x)}dx$.
e. $\int \tan^2(x)\sec^4(x)dx$.

f. $\int \tan^2(2x)\sec^2(2x)dx$.

**Q12.** Evaluate the following expressions

a. $\int 2\sin(x)\cos^4(x)dx$.
b. $\int \sin(\frac{x}{2})\sin(x)dx$.
c. $\int \cos(2x)\cos(4x)dx$.
d. $\int (1+\tan^2(x))dx$.
e. $\int \tan^5(x)\sec^6(x)dx$.
f. $\int \tan^4(\frac{x}{7})\sec^4(\frac{x}{7})dx$.

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**Q13.** Evaluate the integrals.

a.
$$\int x^2 e^x dx.$$
b.
$$\int x\cos(3x)dx.$$
c.
$$\int \frac{1}{2}x^2 e^{4-x} dx.$$
d.
$$\int 3\cos(2x) + e^x \sin x dx.$$

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**Q14.** Evaluate the integral
$$\int x\tan^{-1} x dx.$$

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**Q15.** Express the following as partial fractions.

a. $\frac{4x-1}{(x+1)(x-2)}$.
b. $\frac{x+2}{(x-1)(x+1)}$.
c. $\frac{5x}{(x-3)(x-5)}$.
d. $\frac{2x}{x^2-5x+6}$.
e. $\frac{x-3}{x^2-3x+2}$.
f. $\frac{5x-4}{x^2+6x+8}$.

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**Q16.** Evaluate each of the following integrals

a. $\int \frac{3x-1}{(x+1)(x+2)}dx.$
b. $\int \frac{x-2}{(x-5)(x+6)}dx.$
c. $\int \frac{1}{x^2-9}dx.$
d. $\int \frac{8x+1}{x(x-2)}dx.$
e. $\int \frac{7}{x^2+8x+7}dx.$
f. $\int \frac{6x-5}{x^2-5x+6}dx.$

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**Q17.** Evaluate the following integral

$$\int \frac{x^2-6x+8}{x^2-3x}dx.$$

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**Q18.** Evaluate the following integral

$$\int \left(\frac{x}{\sqrt{x^2-1}}\right)^2 dx.$$

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## 4. Fundamental Theorem of Calculus and Exponentials

**Q1.** Let

$$g(x) := \int_{2-x}^{1+x^2} s^2 ds.$$

Evaluate $g'(x)$.

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**Q2.** Let

$$g(x) := \int_{1+\sqrt{x}}^{2-\log_e(3x+1)} \tan^{-1}(t^2) dt.$$

Evaluate $g'(x)$.

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**Q3.** Evaluate

$$\frac{d}{dx}\left(\int_0^1 \exp\left(-\tan^{-1}(\zeta^2)\right) d\zeta\right).$$

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**Q4.** Evaluate

$$\frac{d}{dx}\left(\int_0^{2x+\sin^{-1}(x)} \tan^{-1}(\sqrt{\xi}) d\xi\right).$$

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**Q5.** Differentiate

$$f(x) = 3^x + 2^{3-x}.$$

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**Q6.**

(a) Differentiate

$$f(x) = x^{\cos x}.$$

(b) Differentiate

$$f(x) = x^{\tan x}.$$

(c) Differentiate

$$f(x) = x^{2\cos^2(x)-1}.$$

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