Calculus Practice Exam 2

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This practice exam aims to emphasise the chain rule, which was proved in the notes. The chain rule allows us to differentiate composite functions. For your convenience, we begin with an illustrative example.

Example 1. Let $h(x) = \sqrt{x^2 - 5x + 6}$. Notice that h is given by the composition of $f(x)$ and $g(x)$, where $f(x) = x^2 - 5x + 6$ and $g(x) = \sqrt{x}$. The chain rule tells us that

$$h'(x) = f'(x)g'(f(x)).$$

Observe that

$$f(x) = x^2 - 5x + 6 \implies f'(x) = 2x - 5$$

and

$$g(x) = \sqrt{x} \implies g'(x) = \frac{1}{2\sqrt{x}}.$$

Hence we see that

$$h'(x) = (2x - 5) \cdot \frac{1}{2\sqrt{x^2 - 5x + 6}} = \frac{2x - 5}{2\sqrt{x^2 - 5x + 6}}.$$

Question 1. Using the chain rule, evaluate the following derivatives.

a. $f(x) = 2\sqrt{x - 3} + 1$.

Proof. We simply observe that

$$\begin{aligned} \frac{d}{dx}(2\sqrt{x-3} + 1) &= \frac{d}{dx}(2(x-3)^{\frac{1}{2}} + 1) \\ &= 2 \cdot \frac{d}{dx}(x-3)^{\frac{1}{2}} \\ &= 2 \cdot \frac{1}{2}(x-3)^{-\frac{1}{2}} \\ &= \frac{1}{\sqrt{x-3}}. \end{aligned}$$

□

b. $f(x) = 3\sqrt{x + 5} + 2x$.

Proof. We simply observe that

$$\begin{aligned} \frac{d}{dx}(3\sqrt{x+5} + 2x) &= \frac{d}{dx}(3(x+5)^{\frac{1}{2}} + 2x) \\ &= 3\frac{d}{dx}(x+5)^{\frac{1}{2}} + 2\frac{d}{dx}x \\ &= 3 \cdot \frac{1}{2} \cdot (x+5)^{-\frac{1}{2}} + 2 \\ &= \frac{3}{2\sqrt{x+5}} + 2. \end{aligned}$$

□

c. $f(x) = \sqrt{2x+3} + 4x^3$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}(\sqrt{2x+3} + 4x^3) &= \frac{d}{dx}((2x+3)^{\frac{1}{2}} + 4x^3) \\
&= \frac{d}{dx}(2x+3)^{\frac{1}{2}} + 4\frac{d}{dx}x^3 \\
&= \frac{1}{2} \cdot 2 \cdot (2x+3)^{-\frac{1}{2}} + 4 \cdot 3x^2 \\
&= \frac{1}{\sqrt{2x+3}} + 12x^2.
\end{aligned}$$

□

d. $f(x) = \frac{1}{2}\sqrt{3x-5} + 2x + 5$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{2}\sqrt{3x-5} + 2x + 5\right) &= \frac{d}{dx}\left(\frac{1}{2}(3x-5)^{\frac{1}{2}} + 2x + 5\right) \\
&= \frac{1}{2}\frac{d}{dx}(3x-5)^{\frac{1}{2}} + 2\frac{d}{dx}x + \frac{d}{dx}5 \\
&= \frac{1}{2} \cdot 3 \cdot \frac{1}{2} \cdot (3x-5)^{-\frac{1}{2}} + 2 \\
&= \frac{3}{4\sqrt{3x-5}} + 2.
\end{aligned}$$

□

e. $f(x) = \sqrt{4x-6} + \sqrt{x} + 3$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{4x-6} + \sqrt{x} + 3\right) &= \frac{d}{dx}\left((4x-6)^{\frac{1}{2}} + x^{\frac{1}{2}} + 3\right) \\
&= \frac{1}{2} \cdot 4 \cdot (4x-6)^{-\frac{1}{2}} + \frac{1}{2}x^{-\frac{1}{2}} \\
&= \frac{2}{\sqrt{4x-6}} + \frac{1}{2\sqrt{x}}.
\end{aligned}$$

□

f. $f(x) = \sqrt{3-7x} + \sqrt{4x-3} + \sqrt{x+10}$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}\left(\sqrt{3-7x} + \sqrt{4x-3} + \sqrt{x+10}\right) &= \frac{d}{dx}\left((3-7x)^{\frac{1}{2}} + (4x-3)^{\frac{1}{2}} + (x+10)^{\frac{1}{2}}\right) \\
&= \frac{1}{2} \cdot (-7) \cdot (3-7x)^{-\frac{1}{2}} + \frac{1}{2} \cdot (4) \cdot (4x-3)^{-\frac{1}{2}} + \frac{1}{2} \cdot (x+10)^{-\frac{1}{2}} \\
&= -\frac{7}{2\sqrt{3-7x}} + \frac{2}{\sqrt{4x-3}} + \frac{1}{2\sqrt{x+10}}.
\end{aligned}$$

□

Question 2. Using the chain rule, evaluate the following derivatives.

a. $f(x) = \frac{3}{x-5} + 1$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{3}{x-5}+1\right) &= \frac{d}{dx}\left(3(x-5)^{-1}+1\right) \\
&= 3\cdot(-1)\cdot(x-5)^{-2} \\
&= -\frac{3}{(x-5)^2}.
\end{aligned}$$

$\square$

b. $f(x) = \frac{1}{x+4} + \frac{3}{x-5}$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{1}{x+4}+\frac{3}{x-5}\right) &= \frac{d}{dx}\left((x+4)^{-1}+3(x-5)^{-1}\right) \\
&= (-1)\cdot(x+4)^{-2}+3\cdot(-1)\cdot(x-5)^{-2} \\
&= -\frac{1}{(x+4)^2}-\frac{3}{(x-5)^2}.
\end{aligned}$$

$\square$

c. $f(x) = \frac{5}{x+1} + \frac{2}{x-1}$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{5}{x+1}+\frac{2}{x-1}\right) &= \frac{d}{dx}\left(5(x+1)^{-1}+2(x-1)^{-1}\right) \\
&= 5\cdot(-1)\cdot(x+1)^{-2}+2\cdot(-1)\cdot(x-1)^{-2} \\
&= -\frac{5}{(x+1)^2}-\frac{2}{(x-1)^2}.
\end{aligned}$$

$\square$

d. $f(x) = \frac{3}{5x+4} + \frac{3}{4-5x}$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{3}{5x+4}+\frac{3}{4-5x}\right) &= \frac{d}{dx}\left(3(5x+4)^{-1}+3(4-5x)^{-1}\right) \\
&= 3\cdot(-1)\cdot 5\cdot(5x+4)^{-2}+3\cdot(-5)\cdot(-1)\cdot(4-5x)^{-2} \\
&= -\frac{15}{(5x+4)^2}+\frac{15}{(4-5x)^2}.
\end{aligned}$$

$\square$

e. $f(x) = \frac{7}{2x+1} - \frac{3}{5x-1}$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}\left(\frac{7}{2x+1} - \frac{3}{5x-1}\right) &= \frac{d}{dx}\left(7(2x+1)^{-1} - 3(5x-1)^{-1}\right) \\
&= 7 \cdot (-1) \cdot (2) \cdot (2x+1)^{-2} - 3 \cdot (-1) \cdot (5) \cdot (5x-1)^{-2} \\
&= -\frac{14}{(2x+1)^2} + \frac{15}{(5x-1)^2}.
\end{aligned}$$

$\square$

Question 3. Consider the function

$$f(x) = \frac{1}{x^2 - 8x + 15}.$$

a. Determine the values of A and B such that

$$f(x) = \frac{A}{x-3} + \frac{B}{x-5}.$$

Proof. We simply compute

$$\begin{aligned}
\frac{1}{x^2 - 8x + 15} = \frac{1}{(x-3)(x-5)} &= \frac{A}{x-3} + \frac{B}{x-5} \\
\therefore \ \frac{(x-3)}{(x-3)(x-5)} &= A + \frac{B(x-3)}{x-5} \\
\therefore \ 1 &= A(x-5) + B(x-3).
\end{aligned}$$

Inserting $x = 3$ into the above equation we see that $1 = -2A \implies A = -\frac{1}{2}$. Inserting $x = 5$ into the above equation we see that $1 = 2B \implies B = \frac{1}{2}$. Hence, we see that

$$\frac{1}{x^2 - 8x + 15} = -\frac{1}{2(x-3)} + \frac{1}{2(x-5)}.$$

$\square$

b. Hence, differentiate $f(x)$.

Proof. We simply observe that

$$\begin{aligned}
f'(x) &= \frac{d}{dx}\left(-\frac{1}{2}(x-3)^{-1} + \frac{1}{2}(x-5)^{-1}\right) \\
&= -\frac{1}{2} \cdot (-1) \cdot (x-3)^{-2} + \frac{1}{2} \cdot (-1) \cdot (x-5)^{-2} \\
&= \frac{1}{2(x-3)^2} - \frac{1}{2(x-5)^2}.
\end{aligned}$$

$\square$

Question 4. Using the chain rule, evaluate the following derivatives.

a. $f(x) = \sqrt{4x^3 + 2x + 1}$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= \frac{1}{2} \cdot (12x^2 + 2) \cdot \frac{1}{\sqrt{4x^3 + 2x + 1}} \\ &= \frac{6x^2 + 1}{\sqrt{4x^3 + 2x + 1}}. \end{aligned}$$

□

b. $f(x) = 2\sqrt{3 - x - x^2 - x^3 - x^4}$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= 2 \cdot (-1 - 2x - 3x^2 - 4x^3) \cdot \frac{1}{\sqrt{3 - x - x^2 - x^3 - x^4}} \\ &= -2\frac{1 + 2x + 3x^2 + 4x^3}{\sqrt{3 - x - x^2 - x^3 - x^4}}. \end{aligned}$$

□

c. $f(x) = 5\sqrt{2x + 5x^3 + 6x^7}$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= 5 \cdot (2 + 15x^2 + 42x^6) \cdot \frac{1}{\sqrt{2x + 5x^3 + 6x^7}} \\ &= 5\frac{2 + 15x^2 + 42x^6}{\sqrt{2x + 5x^3 + 6x^7}}. \end{aligned}$$

□

d. $h(x) = \sqrt{f(x) + 1} + g(x)$, where $f(x)$ and $g(x)$ are differentiable on the entire real line.

Proof. It is easy to see that

$$h'(x) = \frac{f'(x)}{2\sqrt{f(x) + 1}} + g'(x).$$

□

Question 5. Let

$$f(x) = \sqrt{1 + \sqrt{3x + 1}}.$$

Determine $f'(x)$.

Proof. Let $g(x) := 1 + \sqrt{3x+1}$ and $h(x) := \sqrt{x}$. It is clear that $g'(x) = \frac{3}{2\sqrt{3x+1}}$ and $h'(x) = \frac{1}{\sqrt{x}}$. We therefore see that

$$\begin{aligned} f'(x) &= g'(x)h'(g(x)) \\ &= \frac{3}{2\sqrt{3x+1}} \cdot \frac{1}{2\sqrt{g(x)}} \\ &= \frac{3}{4\sqrt{3x+1}} \cdot \frac{1}{\sqrt{\frac{3}{2\sqrt{3x+1}}}} \\ &= \frac{3}{4\sqrt{3x+1}} \cdot \frac{\sqrt{2\sqrt{3x+1}}}{\sqrt{3}} \\ &= \frac{3\sqrt{2\sqrt{3x+1}}}{4\sqrt{9x+3}} \end{aligned}$$

□

Question 6. Let

$$f(x) = \frac{1}{\sqrt{1+\sqrt{1+\sqrt{x}}}}.$$

Determine $f'(x)$.

Proof. Using a similar approach to Question 5, we see that

$$f'(x) = -\frac{1}{8\left(1+\sqrt{1+\sqrt{x}}\right)^{\frac{3}{2}}\sqrt{1+\sqrt{x}}\sqrt{x}}$$

□

Question 7. Using the chain rule, evaluate the following derivatives.

a. $f(x) = 2(x-3)^4$.

Proof. It is easy to see that

$$f'(x) = 2 \cdot 4 \cdot (x-3)^3 = 8(x-3)^3.$$

□

b. $f(x) = 4(x^2 - 5x + 6)^{13} + 2x + 5$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= 4 \cdot 13 \cdot (2x-5) \cdot (x^2 - 5x + 6)^{12} + 2 \\ &= 52(2x-5)(x^2-5x+6)^{12} + 2. \end{aligned}$$

□

c. $f(x) = (x^2 + 2x + 1)^6 - 3x - 13$.

Proof. It is easy to see that

$$f'(x) = 6(2x+2)(x^2+2x+1)^5 - 3.$$

□