LINEAR ALGEBRA – LECTURE 2

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Abstract. The aim of this lecture is to introduce the notion of linear independence and discuss many examples.

Definition 2.1. Let V be a real vector space. A collection of vectors $v_1, ..., v_n \in V$ is said to be *linearly independent* if

$$\lambda_1 v_1 + \cdots + \lambda_n v_n = 0$$

implies that $\lambda_j = 0$ for each $j \in \{1, ..., n\}$.

Remark. Linear independence expresses, in the appropriate way, what it means for two (or more) vectors in a vector space to be different. For example, one should think that colinear vectors, i.e., vectors which lie on the same line segment, are really the same vector. More generally, if I have a set of 3 vectors v_1, v_2, v_3, and I can write $v_3 = v_1 + v_2$, then one should convince themselves that $\{v_1, v_2, v_3\}$ does not contain more information than $\{v_1, v_2\}$.

Example 2.2.

(i) Let $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$ be two vectors in $\mathbb{R}^2$. We look at the equation

$$\begin{aligned} \lambda_1 v_1 + \lambda_2 v_2 = 0 \quad & \iff \quad \lambda_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ & \iff \quad \begin{bmatrix} \lambda_1 + 3\lambda_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \\ & \iff \quad \lambda_1 + 3\lambda_2 = 0. \end{aligned}$$

The solution set to $\lambda_1 + 3\lambda_2 = 0$ consists of more than just $\lambda_1 = \lambda_2 = 0$, since we can take $\lambda_1 = 1$ and $\lambda_2 = -\frac{1}{3}$. Hence, these vectors are **not** linearly independent.

(ii) The vectors $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are linearly independent.