MATH1013 Final Exam Preparation.

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This document is segregated into four sections.

LA1 The Linear Algebra Foundations Section.

C1 The Calculus Foundations Section.

LA2 The Linear Algebra Applications Section.

C2 The Calculus Applications Section.

These sections are then prioritised based on a $\star$ rating which indicates necessity of the skill. Every question which is labeled $\star$ is absolutely necessary. Questions labeled $\star\star$ are essential for a good mark. Questions labeled $\star\star\star$ are to ensure a perfect grade.

The questions labeled with $\star$ in the foundation sections (LA1) and (C1) should be completed prior to attempting any of the past examinations.

We encourage the reader to avoid doing the 2017 MATH1013 exam until the week before and also suggest completing all $\star\star$ labeled questions prior to the completion of this exam.

In the final 3-5 days leading up to the examination, we suggest the reader have completed all questions labeled $\star$ and $\star\star$, work now only on questions labeled $\star\star\star$.

All questions in this document are reasonable and may appear on the examination. Do not think that a problem is too hard.

Contents

LA1 – Linear Algebra – Foundations

1. Row Reductions

Q1.

(a) Consider the system of linear equations

$$\begin{aligned} x_1 - 2x_2 - x_3 + 3x_4 &= 0 \\ -2x_1 + 4x_2 + 5x_3 - 5x_4 &= 3 \\ 3x_1 - 6x_2 - 6x_3 + 8x_3 &= -3. \end{aligned}$$

(i) Form the associated augmented matrix and by using row operations find its reduced row echelon form. Explicitly show the sequence of row operations used to reduce the matrix.
(ii) How many solutions does the system have?
(iii) Which of the columns are pivot columns?
(iv) Which variables are free and which ones are fixed?

(b) Find the general solution of the system whose augmented matrix is

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Q2. Let

$$A = \begin{bmatrix} 1 & 2 & -3 \\ 2 & 1 & -3 \\ -1 & 1 & 0 \end{bmatrix}.$$

(a) Write the general solution of the homogeneous system $Ax = 0$ in parametric form.

(b) Let

$$\mathbf{b} = \begin{bmatrix} 5 \\ 13 \\ -8 \end{bmatrix} \qquad \mathbf{p} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}.$$

Show that $A\mathbf{p} = \mathbf{b}$. Use this information and your solution from part (a) to express all of the solutions to $Ax = b$ in parametric form. Provide a geometric comparison with the solution set from part (a). Justify your answer.

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2. Span and Linear Independence

Q1.

(a) Let

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -5 \\ -8 \\ 2 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 3 \\ -5 \\ h \end{bmatrix}.$$

For what value(s) of $h \in \mathbb{R}$ is $\mathbf{w}$ in the span of $\mathbf{v}_1$ and $\mathbf{v}_2$?

(b) Let

$$\mathbf{u} = \begin{bmatrix} 7 \\ 2 \\ 5 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 3 \\ 1 \\ 3 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

It can be shown that $2\mathbf{u} - 3\mathbf{v} - \mathbf{w} = \mathbf{0}$. Use this fact (and no row operations) to find particular values for x_1 and x_2 that satisfy the equation

$$\begin{bmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ 1 \end{bmatrix}.$$

Clearly justify your answer.

Q2.

(a) Determine whether the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ -8 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$$

are linearly independent.

(b) (i) Determine by inspection whether the following vectors are linearly independent.

$$\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 7 \end{bmatrix}.$$

(ii) Determine by inspection whether the following vectors are linearly independent.

$$\begin{bmatrix} 5 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} -7 \\ 2 \\ 1 \end{bmatrix}.$$

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4. Determinants

5. Subspaces

C1 – Calculus – Foundations

6. Domains and Ranges

Q1 ($\star$)**.** Determine the domain and range of the following functions.

a.
$$f(x) = \sqrt{4+3x}.$$

b.
$$f(x) = \frac{1}{7-9x}.$$

c.
$$f(x) = \frac{1}{3}\log_e(2+9x).$$

d.
$$f(x) = \exp\left(3 - \frac{1}{8}x\right) + x.$$

Q2 ($\star$)**.** Determine the domain and range of the following functions.

a.
$$f(x) = \sqrt{\frac{3-9x}{1+6x}}.$$

b.
$$f(x) = \sqrt{\frac{1}{3-x} + \frac{1}{4x+1}}.$$

c.
$$f(x) = \log_e\left(\sqrt{\frac{7+3x}{11-5x}}\right).$$

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7. Trigonometry

Q1 ($\star$)**.** Evaluate the following exact values.

a. $\sec\left(\frac{\pi}{4}\right)$.
b. $\csc\left(\frac{5\pi}{3}\right)$.
c. $\cot\left(\frac{5\pi}{6}\right)$.
d. $\csc\left(\frac{\pi}{2}\right)$.
e. $\csc\left(\frac{7\pi}{6}\right)$.
f. $\cot\left(\frac{2\pi}{3}\right)$.
g. $\sec\left(\frac{5\pi}{6}\right)$.
h. $\cot\left(\frac{9\pi}{4}\right)$.
i. $\csc\left(\frac{5\pi}{2}\right)$.

Q2 ($\star\star$)**.** Show that

$$\frac{\tan^2(x)}{1+\tan^2(x)} = \sin^2(x).$$

Q3 ($\star\star$)**.** Show that

$$\frac{\cos x}{1+\sin x} + \frac{1+\sin x}{\cos x} = 2\sec x.$$

Q4 ($\star\star$)**.** Show that

$$\tan^2 x = \csc^2 x \tan^2 x - 1.$$

Q5 ($\star\star$)**.** Suppose that $\sec x = -2$ for $\frac{\pi}{2} \le x \le \pi$. Determine the exact values for

a. $\cos x$.
b. $\tan x$.
c. $\cot x$.
d. $\csc x$.

Q6 ($\star\star$)**.** Determine the following exact values.

a.
$$\cos\left(\frac{\pi}{12}\right).$$

b.
$$\tan\left(\frac{7\pi}{12}\right).$$

c.
$$\csc\left(\frac{5\pi}{12}\right).$$

d.
$$\cos\left(\frac{\pi}{8}\right).$$

Q7($\star\star$)**.** Show that
$$\frac{\sin x + 1 - \cos x}{\sin x - 1 + \cos x} = \frac{1 + \tan\left(\frac{x}{2}\right)}{1 - \tan\left(\frac{x}{2}\right)}.$$

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8. Exponentials and Logarithms

Q1 ($\star$)**.** Solve the following logarithmic equations for $x \in \mathbb{R}$.

a. $\log_e(x) + \log_e(3) - \log_e(4) = \log_e(x+1)$.
b. $\log_e(2x) + \log_e(4) = 2\log_e(2)$.
c. $\log_e(7-x) + 4\log_e(1) = 2\log_e(x)$.

Q2 ($\star\star$)**.** Prove that

$$\log_a(b) = \frac{1}{\log_b(a)}.$$

Q3 ($\star\star$)**.** Solve the following equation

$$2\log_p 8 - \log_p 4 = 2.$$

Q4 ($\star\star$)**.** Solve the following equation for $x \in \mathbb{R}$,

$$5^{2x^2} - 5^{(x^2+1)} + 6 = 0.$$

Q5 ($\star\star\star$)**.** Show that

$$\log_a(N)\log_b(N) + \log_b(N)\log_c(N) + \log_c(N)\log_a(N) = \frac{\log_a(N)\log_b(N)\log_c(N)}{\log_{abc}(N)}.$$

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9. Differentiation and Limits

Q1 ($\star$)**.** Provide an example of a function which is continuous but not differentiable.

Q2 ($\star$)**.** Is every differentiable function continuous?

Q3 ($\star\star$)**.** Calculate the limit

$$\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right).$$

Q4 ($\star$)**.** Let

$$f(x) := \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

Evaluate $\lim_{x\to 0} f(x)$.

Q5 ($\star$)**.** Evaluate

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right).$$

Q6 ($\star$)**.** Evaluate

$$\lim_{x\to 0} x \cos\left(\frac{1}{x}\right).$$

Q7 ($\star$)**.** Evaluate

$$\lim_{x\to\infty} \frac{x+1}{x^2-1}.$$

Q8 ($\star$)**.** Evaluate

$$\lim_{x\to\infty} \frac{e^{-x}+1}{e^{-x}-1}.$$

Q9 ($\star$)**.** Provide an example of a function which is differentiable, but whose derivative is not differentiable.

Q10 ($\star$)**.** Show that the following functions are differentiable and calculate their derivatives.

a. $f(x) = 2x + 1$.
b. $f(x) = 3x - 5$.
c. $f(x) = 9 - x$.
d. $f(x) = x^2$.
e. $f(x) = 5 + 2x - x^2$.
f. $f(x) = x^2 - 5x + 6$.

Q11 ($\star\star$)**.** Let f be the function defined by

$$f(x) := \frac{3x - 1}{4x + 5}.$$

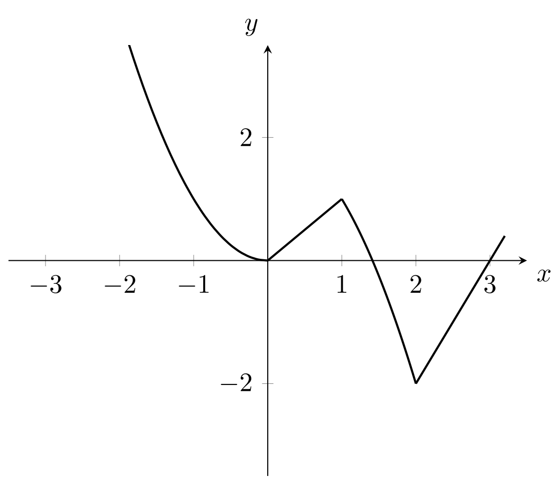
a. Determine the domain on which f is differentiable.
b. Show that f is differentiable on this domain and calculate the derivative of f for all points on this domain.

Q12 ($\star\star$)**.** Consider the function $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{3 - x}.$$

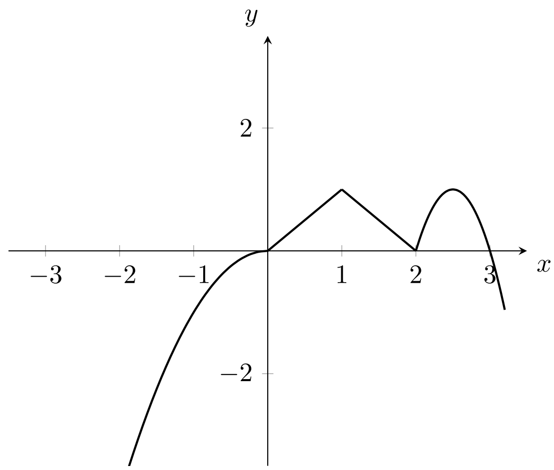
Show that f is differentiable on $\mathbb{R}\backslash\{3\}$ and compute $f'(x)$.

Q13 ($\star$)**.** Consider the function f whose graph is given by



Determine all points $x \in \mathbb{R}$ where f is not differentiable.

Q14 ($\star$)**.** Consider the function g whose graph is given by



Determine all points $x \in \mathbb{R}$ where g is not differentiable.

Q15 ($\star\star$)**.** Let

$$f(x) = \sqrt{1 + \sqrt{3x+1}}.$$

Determine $f'(x)$.

Q16 ($\star\star$)**.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on some set $\Omega_1 \subseteq \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is differentiable on some set $\Omega_2 \subseteq \mathbb{R}$.

a. Determine the domain on which $f + g$ and $f - g$ are differentiable.
b. Determine the domain on which $f \cdot g$ is differentiable.
c. Determine the domain on which f/g is differentiable.
d. Determine the domain on which $\sqrt{f} \cdot g$ is differentiable.
e. Determine the domain on which $|f|$ is differentiable.
f. Let $f(x) = \exp(x)$ and $g(x) = \sqrt{x}$. On what domain is $h(x) = f(x) \cdot g(x)$ differentiable?
g. Let $f(x) = |x|$ and $g(x) = \log_e(-x)$. On which domain in $f(x) + g(x)$ differentiable?

Q17 ($\star\star$)**.** Show that for all $a, b, c \in \mathbb{R}$, the function $f(x) := ax^2 + bx + c$ has no points of inflection.

Q18 ($\star$)**.** Describe the graph of $f(x)$ if $f'(x) = 0$ when $x = 3$ and $x = -2$, $f''(3) = 4$ and $f''(-2) = -5$.

Q19 ($\star$)**.** Suppose that $f(x)$ satisfies $f'(3) = 0$, and $f'(x) > 0$ for all $x \in (3, 5)$. Determine which of the following is true:

a. $0 > f''(3) > f''(5)$.
b. $0 < f''(3) < f''(5)$.
c. $f''(3) < 0 < f''(5)$.
d. $f''(5) < 0 < f''(3)$.

Q20 ($\star$)**.** Find two numbers $x, y \in \mathbb{R}$ whose difference is 10 and whose product is a minimum.

Q21 ($\star$)**.** Find two numbers $x, y \in \mathbb{R}$ whose product is 12 and whose sum is a minimum.

Q22 ($\star$)**.** Find the dimensions of a rectangle which has area 10 m^2 and whose perimeter is as small as possible.

Q23 ($\star$)**.** A cylinder is inscribed in a circle of radius $r = 4$. Determine the largest volume of the cylinder.

Q24 ($\star$)**.** A cylinder is inscribed in a cone of base radius $r = 4$ and height $h = 12$. Determine the largest volume of the cylinder.

Q25. Differentiate the following functions, with respect to x.

(a) $y = 2\sin^{-1}(x)$
(b) $y = 3\cos^{-1}(2x)$
(c) $y = \sin^{-1}(\frac{x}{2})$
(d) $y = \cos^{-1}(\frac{x}{6})$
(e) $y = \tan^{-1}(\frac{x}{3})$
(f) $y = 2\cos^{-1}(\frac{x}{3})$

Q26. Differentiate the function

$$f(x) = \sin^{-1}(x) \cdot \cos^{-1}(x).$$

Q27. Differentiate the function

$$f(x) = \tan^{-1}(\cos^{-1}(x)).$$

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10. Integration

Q1 ($\star$)**.** Evaluate the following integral

$$\int_0^{\pi} \tan x \cdot \cos x dx.$$

Q2 ($\star$)**.** By first differentiating the function $f(x) = \tan x$, evaluate the integral

$$\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} \frac{3}{7} \sec^2 x + \cos x dx.$$

Q3 ($\star\star$)**.** Determine the function $y = f(x)$ that satisfies the second order differential equations

$$\frac{d^2y}{dx^2} = -4x + \sqrt{x}.$$

Q4 ($\star\star$)**.** Let $f : \mathbb{R}_{>0} \to \mathbb{R}$ be the function which has derivative given by

$$f'(x) = \frac{kx + \sqrt{x}}{x^2},$$

where $k \in \mathbb{R}$ is some constant. Suppose that f has a stationary point at $(1, 2)$. Determine the value of k and consequently, a closed form expression for $f(x)$.

Q5 ($\star\star$)**.** Suppose that

$$\int_1^5 f(x) dx = 6.$$

Determine the value of

$$\int_1^5 3f(x) + 2x - \frac{1}{x} dx.$$

Q6 ($\star\star$)**.** Determine the value of $k \in \mathbb{R}$ such that

$$\int_1^k 2x - 3dx = 5.$$

Q7 ($\star$). Evaluate the integral

$$\int_0^1 |x-3|\,dx.$$

Q8 ($\star\star$). Caclulate the area bounded by $f(x) = \sin x$, $g(x) = \cos x$ and the vertical axis.

Q9 ($\star$). Evaluate the following expressions

(a) $\int \frac{2x}{x^2-5} dx$ (b) $\int \frac{x}{\sqrt{3-x^2}} dx$ (c) $\int \frac{2x+6}{(x^2+6x-5)^5} dx$

(d) $\int \frac{2x-9}{\sqrt{x^2-9x+6}} dx$ (e) $\int 3x^2 \cdot (x^3+2)^4 dx$ (f) $\int \frac{x}{(3-x^2)^{\frac{3}{2}}} dx$

Q10 ($\star$). Evaluate the following expressions

(a) $\int x^5 \cdot e^{x^6} dx$ (b) $\int \frac{2\log_e(x)}{x} dx$ (c) $\int \frac{-x^3}{e^{x^4}} dx$

(d) $\int \frac{\log_e(3x)}{7x} dx$ (e) $\int e^x \sqrt{1+e^x} dx$ (f) $\int \frac{\sin(2\log_e(x))}{3x} dx$

Q11 ($\star$). Evaluate the following expressions

(a) $\int \sin(x)\cos^2(x) dx$ (b) $\int 3\tan(x)\sec^2(x) dx$ (c) $\int 4\cos(x) e^{\sin(x)} dx$

(d) $\int 2\tan(x) dx$ (e) $\int \cot(x) dx$ (f) $\int \frac{\sin(x)}{1-\cos(x)} dx$

Q12 ($\star$). Evaluate the following expressions

(a) $\int 2\sin(x)\sqrt{1+\cos(x)} dx$ (b) $\int \frac{3x^2\sin(x^2)}{x} dx$

(c) $\int \frac{\log_e(2x+1)}{x} dx$

Q13 ($\star$). Evaluate the following expressions

(a) $\int \frac{x}{\sqrt{x-1}}dx$ (b) $\int x\sqrt{2+x}dx$ (c) $\int \frac{2-x}{(x+3)^4}dx$

(d) $\int \frac{x^2}{\sqrt{x+1}}dx$ (e) $\int \frac{e^{4x}}{1+e^x}dx$ (f) $\int \frac{(x+1)^2}{\sqrt{x-3}}dx$

Q14 ($\star\star$)**.** Let $f'(x) = \frac{2\tan^{-1}(x)}{1+x^2} + e^{\tan(x)} \cdot \sec^2(x)$.
Determine $f(x)$ if $f(0) = 1$.

Q15 ($\star$)**.** Evaluate the following expressions

a. $\int \sin^2(5x)dx$ c. $\int 3\sin^2(\frac{x}{2})dx$ e. $\int \frac{\tan(x)}{\sec^2(x)}dx$
b. $\int 2\cos^2(x+1)dx$ d. $\int \sin(2x)\cos(2x)dx$ f. $\int \sin(\frac{x+1}{2})\cos(\frac{x+1}{2})dx$

Q16 ($\star$)**.** Evaluate the following expressions

a. $\int 2\sin^3(2x)dx$ b. $\int 3\cos^3(5x)dx$ c. $\int \sin^5(x)dx$

d. $\int 2\cos^5(2x)dx$ e. $\int \frac{\tan^5(x)}{\sec^5(x)}dx$ f. $\int \sin^3(\frac{x}{3})dx$

Q17 ($\star$)**.** Evaluate the following expressions

a. $\int \tan(x)\sec^2(x)dx$ b. $\int \tan^2(5x)dx$ c. $\int \tan^3(x)dx$

d. $\int \sec^2(x)e^{\tan(x)}dx$ e. $\int \tan^2(x)\sec^4(x)dx$ f. $\int \tan^2(2x)\sec^2(2x)dx$

Q18 ($\star$)**.** Evaluate the following expressions

a. $\int 2\sin(x)\cos^4(x)dx$ b. $\int \sin(\frac{x}{2})\sin(x)dx$ c. $\int \cos(2x)\cos(4x)dx$

d. $\int (1+\tan^2(x))dx$ e. $\int \tan^5(x)\sec^6(x)dx$ f. $\int \tan^4(\frac{x}{7})\sec^4(\frac{x}{7})dx$

Q19 ($\star\star$)**.** Suppose $n \in \mathbb{N}$, evaluate the following integrals

a. $\int \sin^n(nx)\cos(nx)dx$
b. $\int \sec^2(x)\tan^n(x)dx$
c. $\int \cos^3(\frac{x}{2})\sin^n(\frac{x}{2})dx$

Q20 ($\star\star$)**.** Suppose $g'(x) = 1 + \tan^3(\frac{x}{5})$. Determine $g(x)$ if $g(\frac{\pi}{3}) = \frac{3}{2}$.

Q21 ($\star\star\star$)**.** Evaluate the following integral

$$\int \sec(x)dx$$

[Hint: Multiply the numerator and denominator by $\frac{\sec(x)+\tan(x)}{\sec(x)+\tan(x)}$ then make the substitution $u = \sec(x) + \tan(x)$].

Q22 ($\star\star$)**.** Let $k \in \mathbb{N}$ be a positive integer. Evaluate the integral

$$\int_0^{\frac{\pi}{3}} \sec^2(x)\tan^k(x)dx.$$

Q23 ($\star$)**.** Evaluate the integrals.

a.
$$\int x^2 e^x dx.$$

b.
$$\int x\cos(3x)dx.$$

c.
$$\int \frac{1}{2}x^2 e^{4-x}dx.$$

d.
$$\int 3\cos(2x) + e^x \sin x dx.$$

Q24 ($\star\star$)**.** Evaluate the integral

$$\int [\log_e(x)]^2 dx.$$

Q25 ($\star$)**.** Evaluate the integral

$$\int x\tan^{-1} x dx.$$

Q26 ($\star$)**.** Evaluate the integral

$$\int \sin(\ln x)dx.$$

Q27 ($\star\star\star$)**.** Show that

$$\int f(x)dx = xf(x) - \int xf'(x)dx,$$

where $f \in \mathscr{C}^1(\mathbb{R})$.

Q28 ($\star$)**.** Express the following as partial fractions

a. $\frac{4x-1}{(x+1)(x-2)}$. b. $\frac{x+2}{(x-1)(x+1)}$. c. $\frac{5x}{(x-3)(x-5)}$.

d. $\frac{2x}{x^2-5x+6}$. e. $\frac{x-3}{x^2-3x+2}$. f. $\frac{5x-4}{x^2+6x+8}$.

Q29. Evaluate the following using the method of partial fractions

(a) $\int \frac{3x+1}{x^2-5x+6}dx$.
(b) $\int \frac{5x-7}{x^2+2x+1}dx$.

Q30. Evaluate each of the following integrals

a. $\int \frac{3x-1}{(x+1)(x+2)}dx$. c. $\int \frac{1}{x^2-9}dx$. e. $\int \frac{7}{x^2+8x+7}dx$.
b. $\int \frac{x-2}{(x-5)(x+6)}dx$. d. $\int \frac{8x+1}{x(x-2)}dx$. f. $\int \frac{6x-5}{x^2-5x+6}dx$.

Q31. Evaluate the following integral

$$\int \frac{x^2-6x+8}{x^2-3x}dx.$$

Q32. Evaluate the following integral

$$\int \left(\frac{x}{\sqrt{x^2-1}}\right)^2 dx.$$

Q33. Evaluate the following integrals.

a.
$$\int \frac{1}{x^2\sqrt{x^2-4}}dx.$$

b.
$$\int x^3\sqrt{16-x^2}dx.$$

c.
$$\int \frac{x^3}{\sqrt{x^2+25}}dx.$$

d.
$$\int_1^{\sqrt{3}} \frac{x^3}{x^2\sqrt{4-x^2}}dx.$$

Q34. Evaluate the integral
$$\int \frac{1}{x\sqrt{x^2+5}}dx.$$

Q35. Evaluate the integral
$$\int \frac{1}{\sqrt{x^2-6x+13}}dx.$$

Q36. Evaluate the integral
$$\int \sqrt{e^{2x}-16}dx.$$

Q37. Evaluate the integral
$$\int_0^{\frac{1}{2}} \sqrt{x^2+1}dx.$$

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11. Riemann Sums

Q1 ($\star$). Consider the function

$$f(x) = x^2, \qquad 0 \le x \le 1.$$

(a) Sketch $f(x)$ and approximate the area under it by dividing the interval $[0, 1]$ into n equal partitions and drawing rectangles of height $f(x)$ using the right hand side of each partition. (Your sketch will only show a small number of these partitions).

(b) Find the height of the ith rectangle for $i \in \{1, 2, ..., n\}$.

(c) Write the area of the area under $f(x)$ as a limit of Riemann sums and evaluate this limit.

Q2 ($\star$). Consider the function

$$f(x) = \sqrt{x}, \qquad 0 \le x \le 1.$$

(a) Sketch $f(x)$ and approximate the area under it by dividing the interval $[0, 1]$ into n equal partitions and drawing rectangles of height $f(x)$ using the right hand side of each partition. (Your sketch will only show a small number of these partitions).

(b) Find the height of the ith rectangle for $i \in \{1, 2, ..., n\}$.

(c) Write the area of the area under $f(x)$ as a limit of Riemann sums and evaluate this limit.

Q3 ($\star$). Consider the function

$$f(x) = \log_e(x), \qquad 1 \le x \le e.$$

(a) Sketch $f(x)$ and approximate the area under it by dividing the interval $[1, e]$ into n equal partitions and drawing rectangles of height $f(x)$ using the right hand side of each partition. (Your sketch will only show a small number of these partitions).

(b) Find the height of the ith rectangle for $i \in \{1, 2, ..., n\}$.

(c) Write the area of the area under $f(x)$ as a limit of Riemann sums and evaluate this limit.

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12. Fundamental Theorem of Calculus and Exponentials

Q1 ($\star$)**.** Let

$$g(x) := \int_{2-x}^{1+x^2} s^2 ds.$$

Evaluate $g'(x)$.

Q2 ($\star\star$)**.** Let

$$g(x) := \int_{1+\sqrt{x}}^{2-\log_e(3x+1)} \tan^{-1}(t^2) dt.$$

Evaluate $g'(x)$.

Q3 ($\star$)**.** Evaluate

$$\frac{d}{dx}\left(\int_0^1 \exp\left(-\tan^{-1}(\zeta^2)\right) d\zeta\right).$$

Q4 ($\star\star$)**.** Evaluate

$$\frac{d}{dx}\left(\int_0^{2x+\sin^{-1}(x)} \tan^{-1}(\sqrt{\xi}) d\xi\right).$$

Q5($\star\star$)**.** Differentiate

$$f(x) = 3^x + 2^{3-x}.$$

Q6 ($\star$)**.**

(a) Differentiate

$$f(x) = x^{\cos x}.$$

(b) Differentiate

$$f(x) = x^{\tan x}.$$

(c) Differentiate

$$f(x) = x^{2\cos^2(x)-1}.$$

Q7 ($\star\star$)**.** Differentiate

$$f(x) = \log_x(3 - e).$$

13. Implicit Differentiation

Q1 ($\star$)**.** Evaluate $\frac{dy}{dx}$, where

$$xy + 2\sqrt{y} = 1.$$

Q2 ($\star\star$)**.** Evaluate $\frac{dy}{dx}$, where

$$\cos(xy) + \sin(xy) + \sqrt{y}\tan^{-1}(x) = 4x.$$

Q3 ($\star\star\star$)**.** Evaluate $\frac{dy}{dx}$, where

$$\tan^{-1}(\sqrt{y+x}) = \frac{2\cos^2(y) + 1}{\sin^2(x) + \cos^2(x)}.$$

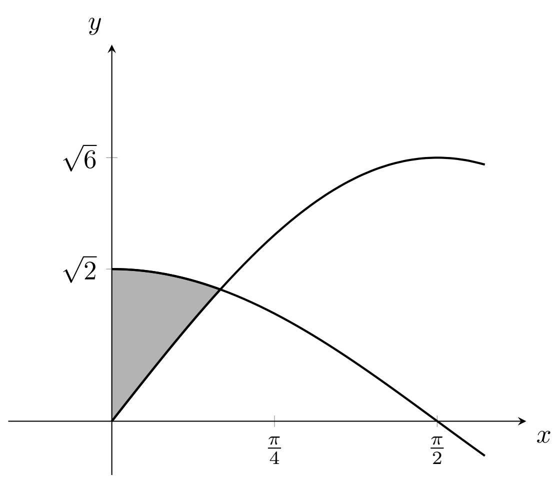
Calculus – Application Problems

Q1. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = 1 + 40x^3 - 3x^5.$$

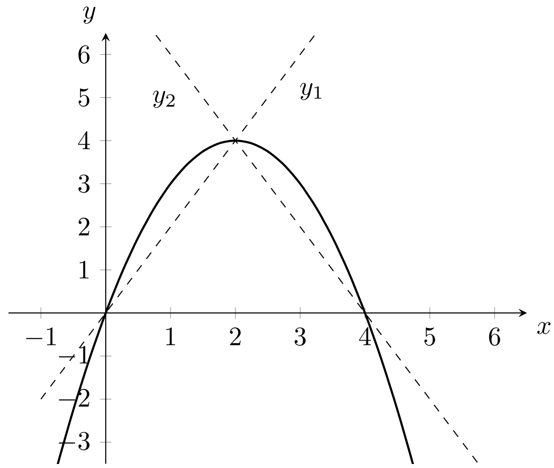
Determine the value of $x \in \mathbb{R}$ for which the tangent line to f at x have the largest slope.

Q2. The graph of $f(x) = \sqrt{6}\sin(x)$ and $g(x) = \sqrt{2}\cos(x)$ is given below



a. Determine the point $x \in [0, \pi]$ of intersection between $f(x)$ and $g(x)$.
b. Determine the area of the shaded region.

Q3. The graph of a parabola of the form $f(x) = ax + bx^2$ is given below.



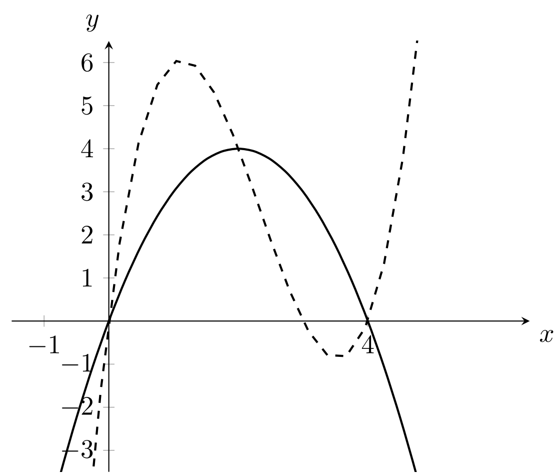
a. Determine the values of $a, b \in \mathbb{R}$.

b. Two secant lines, labelled y_1 and y_2, are given on the above graph.

  i. Determine the equation for y_1.

  ii. Determine the equation for y_2.

c. A cubic equation of the form $g(x) = (x-c)(x-d)(x-e)$ passes through the x-intercepts and the turning point of $f(x)$, as shown in the graph below.

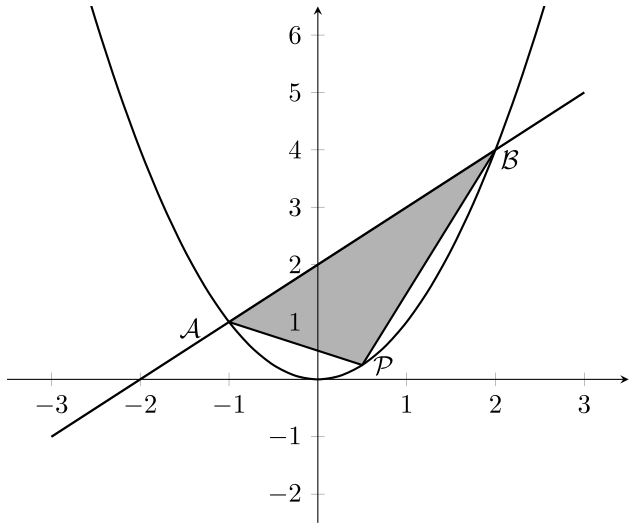


  i. Determine the values of $c, d, e \in \mathbb{R}$.

  ii. Determine the solution(s) of the equation $g(x) = y_1$.

  iii. Hence, or otherwise, determine the solution(s) of the $g(x) = 2g^{-1}(x)$, where g^{-1} denotes the inverse function of g.

Q4. The line $y = mx + c$ intersects the parabola $f(x) = x^2$ at the points $\mathcal{A} = (-1, 1)$ and $\mathcal{B} = (2, 4)$ as shown in the diagram below.



a. Determine the values of $m, c \in \mathbb{R}$.

b. Let $\mathcal{P} = (p, f(p))$ be a point on the parabola $f(x)$.

   i. Determine the equation of the line $\mathcal{AP}$ in terms of p.

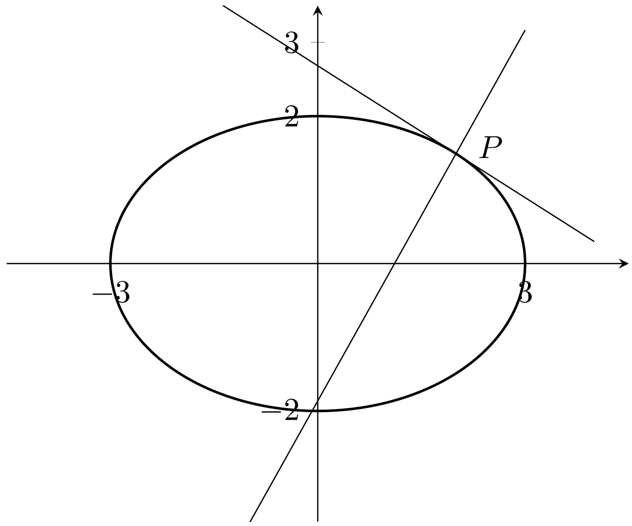
   ii. Determine the equation of the line $\mathcal{BP}$ in terms of p.

c. Determine an expression for the area of the triangle in terms of p.

d. Determine the value of p that maximises the area of the triangle.

e. Determine the maximum area of the triangle.

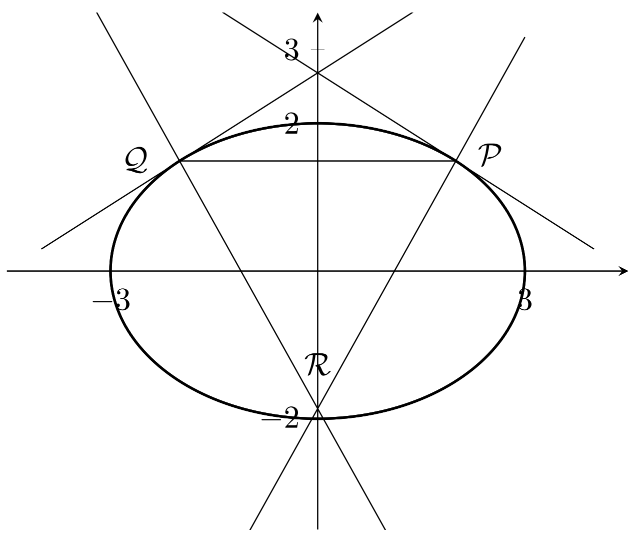
Q5. The graph below shows the graph of the ellipse $4x^2 + 9y^2 = 36$ with tangent and normal at the point P occuring at $x = p$.



a. i. Show that the gradient of the normal at P is given by $m_N = -\dfrac{p}{3\sqrt{9-p^2}}$.

ii. Hence, show that the equation of the normal at P is given by
$y = \dfrac{1}{3\sqrt{9-p^2}}\left(36 - 3p^2 - px\right)$.

b. Now consider the same graph, but with the addition of tangent and normal lines occuring at the point $\mathcal{Q}$, the point corresponding to a reflection of $\mathcal{P}$ about the y-axis.



Determine the equation of the normal at $\mathcal{Q}$.

c. Show that the normal at $\mathcal{P}$ and the normal at $\mathcal{Q}$ will intersect at $\mathcal{R} = \left(0, \frac{12-p^2}{\sqrt{9-p^2}}\right)$.

d. Find an expression for the area of the triangle $\mathcal{PQR}$.

e. Determine the value of p that maximises the area of the triangle $\mathcal{PQR}$.

f. Find the maximum area of the triangle.

Q6.

(a) Find the following limit if it exists. If it does not exist, explain why.

$$\lim_{x\to 0}\left(\frac{1}{x\sqrt{1+x}}-\frac{1}{x}\right).$$

(b) Is it possible to extend

$$f(x)=\frac{x^2+x-6}{x-2}$$

to a continuous function for all $x\in\mathbb{R}$? If so, what should be the extension of $f(x)$? What is the value of this new function at $x=2$?

(c) Show that

$$\lim_{x\to 0} x^4\sin\left(\frac{1}{x^{\frac{1}{3}}}\right)=0.$$

Q7.

(a) Suppose that the function g is defined by

$$g(x)=\frac{\sqrt{x^2-1}}{\sqrt{2-x^2}}.$$

(i) Find the domain of g. Justify why you have excluded certain points from the domain.

(ii) Show that

$$g'(x)=\frac{x}{(x^2-1)^{1/2}(2-x^2)^{3/2}}.$$

(b) Consider the function f defined by

$$f(x)=\begin{cases}1-x-x^2, & x\le 0,\\ ax^2+bx+c, & x>0.\end{cases}$$

(i) Show that f is a continuous function if and only if $c=1$.

(ii) Find both one-sided derivatives of f at $x=0$.

(iii) For what values of $a,b,c\in\mathbb{R}$ is f differentiable?

Q8.

(a) Find the following limit if it exists. If it does not exist, explain why.

$$\lim_{x\to 0}\left(\frac{\sqrt{1+x^2}-1}{x(\sqrt{1+x}-1)}\right).$$

(b) For which values of the constants r, d and e is the following function continuous on $\mathbb{R}$?

$$f(x)=\begin{cases} 2, & x<0,\\ r+(1+dx)\sin\left(\frac{\pi}{x-1}\right), & x\in[0,2]\backslash\{1\},\\ e, & x\geq 1.\end{cases}$$

Q9.

(a) Evaluate the limit $\lim_{x\to 0} f(x)$, where

$$f(x)=x\sqrt{\frac{1+x^2}{x^2}}.$$

(b) Evaluate the limit

$$\lim_{x\to 0}\frac{x}{\sqrt{1+x}-1}.$$

Q10. Suppose f is a continuous function on $D(f)\subset\mathbb{R}$ and $0<f(x)<1$ for all $x\in D(f)$. Define

$$g(x)=\sqrt{1-[f(x)]^2}.$$

Determine whether g is continuous.

Q11. Show that there exists an $x\in\mathbb{R}$ such that

$$x^7+\frac{3}{2+x^2+\sin x}=\pi.$$

Q12. Let

$$f(x)=\frac{\ln x}{1+\ln x}.$$

(a) Find the domain of f.

(b) Find possible intersections of the graph of f with the coordinate axes.

(c) Find vertical and horizontal asymptotes, if these exist.

(d) Determine whether the function f has any critical points, singular points and thus determine whether the function has local minima and/or maxima.

(e) Find the intervals over which the function f is increasing or decreasing.

(f) Determine whether the function f has any inflection points and find the intervals over which the function f is concave up or concave down.

(g) Sketch the graph of f.

Q13.

(a) Solve the inequality

$$\frac{x^2+2x}{x^2-1} \geq 1$$

and express the solution set as a union of intervals.

(b) Show that $\tan x > x$ for $0 < x < \frac{\pi}{2}$.

Q14. Find

$$\lim_{x\to\infty} \frac{2-x+\sin\left(\frac{1}{x}\right)}{x+\cos\left(\frac{1}{x}\right)}.$$

Q15. Let

$$f(x) = \frac{1}{3}(x+1) - x^{\frac{1}{3}}, \qquad -8 \leq x \leq 8.$$

(a) Find all local and global maximum and minimum points, and corresponding values, of f. Use the convention that local maxima and minima may occur at the endpoints. Justify your answers by applying an appropriate test at each such point.

(b) Where is f increasing and where is it decreasing?

(c) Where is f concave up and where is it concave down? Find any inflection points of f.

(d) Sketch the graph of f.

Q16.

(a) Find

$$\lim_{x\to\infty}\left(\frac{x^2}{x+1} - \frac{x^2}{x-1}\right).$$

(b) Find

$$\lim_{x\to 0}\frac{\sin(\sin x)}{x}.$$

Q17.

(a) Suppose that g and h are differentiable on the whole real line. Find the derivative of

$$f(x) = g(x + x^2 h(x^3)).$$

(b) Prove that the derivative of $f(x) = x^{1/3}$ is $f'(x) = \frac{1}{3}x^{-2/3}$ for $x \neq 0$.
Hint: Consider the function $g(x) = (x^{1/3})^3$ and use the chain rule.

Q18.

(a) Show that the function $f(x) = \cos^2 x + x^3 + 2x$ takes the value 0 exactly once.

(b) Suppose that f is a differentiable function which is zero at two distinct points a and b. Show that $f'(x)$ is zero at some point $a \leq x \leq b$.

Q19. Let

$$f(x) = x^{\frac{1}{3}}(x - 4).$$

(a) Find all local and global maximum and minimum points, and corresponding values, of f. Justify your answers by applying an appropriate test at each such point.

(b) Where is f increasing and where is it decreasing?

(c) Where is f concave up and where is it concave down? Find any inflection points of f.

(d) Sketch the graph of f.

Q20. Let $\lambda \in \mathbb{R}$ and show that

$$\int \frac{1}{\sqrt{x^2 + \lambda^2}} dx = \log_e \left(x + \sqrt{x^2 + \lambda^2} \right) + k,$$

where $k \in \mathbb{R}$ is some constant.

Exam Question Checkpoint

This section details the problems which are relevant to particular past examination problems.

14. 2016 Final - Calculus

Q1. (a) C1.6.11(a)-(f), C1.6.12(a), C1.6.15(a)-(f), C1.6.16(a)-(f), C1.6.17(a)-(f), C1.5.18(a)-(f), C1.5.19(a)-(c), C1.20, C1.21, C1.22.
 (b) C1.28(a)-(f).

Q2. (a) C1.8.1, C1.8.2, C1.8.3, C1.8.4.
 (b) Same as part (a).
 (c) C1.8.5, C1.8.6(a)-(c), C1.8.7.

Q3.

Q4. C1.5.1, C1.5.2, C1.5.13, C1.5.14.

Q5. (a) C1.5.3, C1.5.4, C1.5.5, C1.5.6, C1.5.6, C1.5.7, C1.5.8, C2.6(a), C2.6(c), C2.8(a), C2.9(a)-(b), C2.14, C2.16(a)-(b).
 (b) Same as part (a).

Q6. (a) C2.11
 (b)
 (c) C1.5.10(a)-(f), C1.5.11(a)-(b), C1.5.12, C1.5.16, C1.6.7.

Q7.

15. 2015 Final - Calculus

Q1. (a) Differentiation of Inverse Trig.
 (b) Inverse Functions.

Q2. Exponential Decay.

Q3. (a) C1.7.1, C1.7.2, C1.7.3.
 (b) Trigonometric Substitution.

Q4.

Q5. (a) C1.7