Chapter 1

An Introduction to Pure Mathematics.

The interested young student has embarked on her journey in pure mathematics, a mathematics quite foreign and unnerving when compared to mathematics seen previously. Pure mathematics often moves away from a structure of question-answer to a more proposition-argument setting. While there may be many cases where we need to actually compute something, such as a derivative or the determinant of a matrix, much of pure mathematics is characterised by the proving of ideas in the form of theorems, propositions, lemmas or corollaries. In order for a young mathematician to begin proving these mathematical truths however, she needs to be familiar with the underlying language of argument, that is, logic.

1.1 Preliminary Definitions

In this section we define all the necessary terminology required in order to speak about logic, and subsequently mathematics.

Definition 0.1.1. A proposition is a statement that is either true or false. Examples of statements include : *The Earth is spherical, Simon does biology* or *An object falls due to the force of gravity.*

Definition 0.1.2. A premise is a statement (or proposition) from which another statement or proposition is then inferred.

Definition 0.1.3. Deductive reasoning is the process of obtaining logically certain conclusions from a particular number of premises.

Example 0.1.4. Suppose that all humans breath in oxygen and exhale carbon dioxide. Suppose further that Pete is a human. We can then conclude that Pete breaths in oxygen and exhales carbon dioxide.

Note that the premises are not required to be true, but the conclusions will always be true with respect to such premises. Conversely, inductive reasoning demands true premises, while the conclusions may be not be as certain. We will add to this discussion shortly, for now, consider one more example.

Example 0.1.5. Suppose that every mammal is indeed a reptile. We know that dogs are mammals. Therefore, dogs are also reptiles.

1.2 Quantifiers

In order to understand basic logical statments, we need to understand what are referred to as *quantifiers*. There are two types of quantifiers that we concern ourselves with, the universal quanitifier and the existential quantifier.

The universal quantifier is observed in statements such as *for all, for any, for every*, and so on. The symbol used to represent the universal quantifer is $\forall$. For example, if one wished to say that *for all elements in the real numbers, the square of that number is greater than or equal to zero*, we can simply write

$$\forall x \in \mathbb{R},\ x^2 \geq 0.$$

If we want to say something about two distinct objects, such as *for all elements in the real numbers, and for all elements in the rational numbers, the square of the elements is greater than equal to zero*, we can write in a similar manner

$$\forall x \in \mathbb{R}, \forall y \in \mathbb{Q},\ x^2 \geq 0 \text{ and } y^2 \geq 0.$$

Moreover, the existential quantifier is observed in statements such as *there exists, there is, we can find*, and so on. The symbol used to represent the existential quantifier is $\exists$. For example, if one wished to say that *in the set of natural numbers, there exist numbers which are odd*, we can simply write

$$\exists n \in \mathbb{N} \text{ such that } n = 2m + 1,\ m \in \mathbb{N}.$$

Now it is important to stress the order of such quantifiers. I claim that the universal quantifer ($\forall$) always preceeds the existential quantifier ($\exists$). This is best understood with an example. Hence, take our proposition to be that for a pair (x, y), y is the parent of x. Then,

$$\forall x \ \exists y \text{ such that } P(x, y)$$

means that for every x there is a parent y, or equivalently, everyone has a parent. This seems reasonable[1]. On the other hand however, consider

$$\exists y \ \text{ such that } \forall x \ P(x, y).$$

This means that there is some parent y that is the parent of every child. This is clearly not true.

1.3 Connectives

In this section we look at how propositions can be connected or related. In this section we let p and q denote distinct propositions.

We begin the notion of a negation. If we suppose that p is some true statement, then $\neg p$, said *not* p is false. For example, suppose p is the statement that *I can read*. We assume here that p is true and therefore $\neg p$ is the statement *I can not read* which is subsequently false if p is true. Similarly, if p is false, $\neg p$ is true, and $\neg(\neg p) = p$.

Let us now consider the negation of the quantifiers we saw in the last section. So if we let p be the statement that *every book has more than three pages*, then this can be written as $\forall x \ p(x)$, where x denotes the category of books. We want to consider the negation of this statement. It should be clear that if p holds for every book, we can constradict p by simply finding one book with less than three pages. Therefore, the negation of $\forall x \ p(x)$ would be $\exists x$ such that $\neg p(x)$. We can therefore write that

$$\neg\left(\forall x \ p(x)\right) = \exists x \ \left(\neg p(x)\right).$$

Moreover, if we let p be the proposition that there exists a chair with only two legs. A way to contradict or negate the statement, is to show that

[1] I hope.

every chair has more or less than two legs. That is, no chair has two legs. So in a similar manner to the previous example,

$$\neg\left(\exists x\ p(x)\right) = \forall x\ \left(\neg p(x)\right).$$

We are ready to move on to a very important connective, which is implication,

the symbol for which is $\implies$. Hence, if p and q are logical statement, and q follows from p, we can write

$$p \implies q.$$

This can also be read as q holding only if p holds.

For example, consider the statement that Jack is bigger than Fred. Then if Fred is bigger than Tony, then it should be clear that Jack is bigger than Tony. Therefore, if we let p be the proposition that Jack is bigger than Fred and Fred is bigger than Tony; and let q be the proposition that Jack is bigger than Tony, then we see that $p \implies q$.

In a more mathematical example, we can consider the statements $x > 2$ and $x > 1$. Clearly

$$x > 2 \implies x > 1,$$

but it is not true that

$$x > 1 \implies x > 3,$$

since we can simpy take $x = 2$ as a counterexample.

This motivates the following assertions. If $p \implies q$, then $\neg q \implies \neg p$, but $\neg p \not\Longrightarrow \neg q$. In the case of $\neg q \implies \neg p$, we refer to this as the *contrapositon* of $p \implies q$.

If it is true that $p \implies q$ and $q \implies p$, then we have what is referred to as an *if and only if statement*, we often write $p \Leftrightarrow q$ in this case. An example of an if and only if statement[2] is *a polygon is a triangle if and only if it has three edges.* We leave it to the reader to verify that the implication is indeed in both directions.

1.4 Methods of Proof

In order to prove propositions, theorems, lemmas or corollaries, we use all the above ideas in order to conclude the desired result from the necessary

[2] We can also write iff for if and only if.

premises. That is, if we start with some premises or some assumptions, and use logical reasoning to obtain a particular result, that result is what we call a theorem, proposition, lemma or corollary[3]. Methods of proof are often of one of the following forms.

† **Direct Proof** - This is where we simply show the result deductively.

For example, let us prove the statement that, for every integer n, there is an integer m such that $m > n$.

Proof. Let $n \in \mathbb{Z}$ be some integer. Then, $n + 1$ is also an integer, and we can assert that $n+1 > n$ for all $n \in \mathbb{Z}$. Therefore, taking $m = n+1$ proves the result. □

† **Proof by Contradiction** - This is where we suppose that the negation is true and obtain a contradiction. The contradiction indicates that the assumption was false and so proves the converse of such an assumption. We illustrate this by proving that the square root of 2 is irrational.

Proof. Suppose that $\sqrt{2}$ is rational.

Then we can write

$$\sqrt{2} = \frac{m}{n}, \quad \text{for } m \in \mathbb{Z}, n \in \mathbb{N},$$

and m, n are coprime[4]. Therefore,

$$\sqrt{2} = \frac{m}{n} \implies m^2 = 2n^2$$

and so m^2 is even[5], and therefore m is even[6]. We can therefore write $m = 2p$ for some $p \in \mathbb{Z}$ and therefore

$$4p^2 = 2n^2 \implies n^2 = 2p^2.$$

[3]A lemma refers to something we prove before proving a more important or bigger result, the title of such a result is usually called a theorem. Moreover, a corollary refers to something we deduce from the main theorem, an easy consequence of the main theorem if you will.

[4]The greatest common divisor of m and n is 1.

[5]Since m is a multiple of 2.

[6]Since the square of an odd integer is odd.

Therefore, since n^2 is even, we can write $n = 2q$ for some $q \in \mathbb{Z}$. This gives us that

$$\sqrt{2} = \frac{2p}{2q},$$

which implies that m, n are no longer coprime, which contradicts our assumption. So the square root of 2 is irrational. $\square$

† **Contrapositive Proof** - This method makes use of the fact that if $p \implies q$ then $\neg q \implies \neg p$. Therefore, let us prove that if $x^2 - 6x + 5$ is even, for $x \in \mathbb{Z}$, then x must be odd, using the method of contraposition.

Proof. Suppose $x \in \mathbb{Z}$ is even and consider the quantity $x^2 - 6x + 5$. Since x is assumed even, we can write $x = 2y$ for some $y \in \mathbb{Z}$, and therefore

$$\begin{aligned} x^2 - 6x + 5 &= (2y)^2 - 6(2y) + 5 \\ &= 4y^2 - 12y + 5 \\ &= 2(2y^2 - 6y + 2) + 1. \end{aligned}$$

This says that $x^2 - 6x + 5$ is odd and this proves the proposition. $\square$

† **Induction** - This method of proof allows us to prove that a particular result holds for all n finite. The proof consists of two steps. The first of which is referred to as the base step. This is where we prove that the statement holds for the first number in our set. This number is typically $n = -1, 0, 1$. Then we proceed to the second step, known as the inductive step. This is where we assume the result holds for n and then prove that it holds for $n + 1$. This is best illustrated with an example. So we will prove that $n! \leq n^n$ for all $n \in \mathbb{N}$.

Proof. We first prove the base case, that is for $n = 1$. This is often trivial, and in our case it is, since $1! = 1 \leq 1 = 1$.

Now we assume that $n! \leq n^n$ and we want to show that $(n + 1)! \leq (n + 1)^{n+1}$. To do this, we observe that

$$(n + 1)! = (n + 1)n! \leq (n + 1)n^n < (n + 1)(n + 1)^n = (n + 1)^{n+1}.$$

So $(n + 1)! \leq (n + 1)^{n+1}$ and we are done. $\square$

⋆ **How do we do proofs?**

If we wish to prove anything, often the first, and biggest, point of confusion is simply where to begin. On this matter, there is really only one answer: Guess! If you guess incorrectly, then so what? An incorrect guess yields no harm, it is better than sitting there with nothing. Over time the guesses get better as the number of approaches that a mathematician has seen increases.

A particularly illustrative example of this is the proof that $\sqrt{2}$ is irrational. When first attempting to prove something like this, a young mathematician may wish to prove this using direct means. So she starts with something such as *...Suppose we consider the quantity $\sqrt{2}$, we wish to show that this cannot be written as the ratio of two coprime integers m and n.* The young mathematician then finds herself with nowhere else to go. What she then realises however is that she cannot find a nice way to prove something directly when the thing she wishes to prove is something that is not direct. What we mean by this is that an irrational number is defined as a number that is *not rational*. Therefore, it seems only fitting to try some indirect proof, such as a contradiction. After some algebraic manipulation and playing around, hopefully she ends up obtaining the proof seen previously, or something akin to it.

Another thing to keep in mind is that a proof is rarely formulated in such a structured way to begin. Often the ideas are scribbled on a page, with a guess followed by a string of implications. Once the mathematician has reached her desired conclusion then she may attempt to write the product up in a polished manner. Let us look at an example proof that will be seen in Chapter 2. Namely, proving that the natural numbers $\mathbb{N}$ are unbounded. Like the case of proving that $\sqrt{2}$ is irrational, we want to prove that something is not something we can clearly defined. In other words, we know how to define concretely what it means for a set to be bounded; that is a set X is bounded (above) if for every $x \in X$, $x \leq \lambda$ for some λ. Therefore, we want to show that there is some $y \in X$ such that $y > \lambda$. So we take a stab and suppose that $\mathbb{N}$ is bounded. Then we recall that we learnt something about the least upper bound axiom[7], so we apply this to $\mathbb{N}$. Then $n \leq \lambda$ for all $n \in \mathbb{N}$. We also know that if $n \in \mathbb{N} \implies n+1 \in \mathbb{N}$ also. So $n+1 \leq \lambda \; \forall n \in \mathbb{N}$. But this implies that $n \leq \lambda - 1 \; \forall n \in \mathbb{N}$, which contradicts the fact that λ was the least upper bound, and we are done. See Chapter 2 for a proper write up of this proof.

[7]See Chapter 2

The last thing that should be mentioned here is in regard to proving certain propositions that hold for a particularly large number. For example, let us consider the problem of determining the number of pieces that six nonparallel planes separate in $\mathbb{R}^3$. That is, we want to determine the number of distinct reagions given by the complement of the intersection of six nonparallel planes embedded in $\mathbb{R}^3$. At first, this seems quite challenging, six planes are hard to visualise. Let us therefore reduce the problem to something easier. If we first consider one plane in $\mathbb{R}^3$, then the problem is easy; there are two distinct regions. Similarly, if there are two planes, then the number of regions is four. It is perhaps better to draw this out, but it is also easy to see that three planes give us eight regions. We can then guess (intelligently) and suppose that the number of regions is given by the relation 2^n. This would then tell us that there are $2^6 = 64$ regions. We would have to prove this formula however, what method seems most fitting?

Exercises

Q1. Suppose that $m, n \in \mathbb{Z}$ are odd. Prove that $m + n$ is even.

Q2. Suppose that $m, n \in \mathbb{Z}$ are even. Prove that $m - n$ is even also.

Q3. Suppose that $m, n \in \mathbb{Z}$ are even. Then the product mn is divisible by 4.

Q4. Suppose that η is an positive integer. Then η is even if and only if $3\eta^2 + 8$ is even also.

Q5. Prove that $\sqrt{3}$ is irrational.

Q6. Prove that $\sqrt{6}$ is irrational.

Q7. Suppose that $x, y \in \mathbb{R}\backslash\mathbb{Q}$, that is, x and y are irrational. Is $x + y \in \mathbb{R}\backslash\mathbb{Q}$, prove or disprove by providing a simple counterexample.

Q8. Prove that $\sqrt{2} + \sqrt{3}$ is irrational.

Q9. Prove that $\sqrt{2} + \sqrt{5}$ is irrational.

Q10. Prove the π is irrational.

Q11. Prove that there are an infinite number of primes.

Q12. Prove that $n < 4^n$ for all $n \in \mathbb{N}$.

Q13. Prove that $2^n < n!$ for all $n \geq 4$.

Q14. Prove that $n^2 < n!$ for all $n \geq 4$.

Q15. Show that for all $x \in \mathbb{Z}_{\geq 2}$, 3 divides $x^3 - x$.

Q16. Determine for which positive real numbers $\mu, \lambda \in \mathbb{R}$, the following inequality holds true

$$(\mu + \lambda)\left(\frac{1}{\mu} + \frac{4}{\lambda}\right) \geq 9.$$

Q17. Prove the Pythagorean theorem. That is, if a and b are orthogonal, in the sense of the dot product $\cdot$, then

$$|x|^2 + |y|^2 = |x + y|^2 .$$

Q18. Prove that

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Q19. Prove that the natural numbers $\mathbb{N}$ are unbounded. That is, for all $n \in \mathbb{N}$ we cannot find an $M \in \mathbb{N}$ such that $n \leq M$.

Q20. Prove that for all $x, y \in \mathbb{R}$,

$$|x + y| \leq |x| + |y| .$$

This a very important type of inequality referred to as the *triangle inequality.*

Q21. Use the triangle inequality from the previous exercise to show that if

$$|x - z| < \frac{\varepsilon}{2} \quad \text{and} \quad |y - z| < \frac{\varepsilon}{2},$$

then $|x - y| < \varepsilon$ for all choices of $\varepsilon > 0$.

Q22. a. Prove that for $x, y \in \mathbb{R}$,

$$xy \leq \frac{1}{2}(x^2 + y^2).$$

b. Hence, or otherwise, show that, for all $\varepsilon > 0$,

$$xy \leq \frac{1}{4\varepsilon}(4\varepsilon^2 x^2 + y^2).$$

Q23. Show that
$$\sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2.$$

Q24. Show that
$$\sum_{k=1}^{n} \lambda = \lambda n,$$
where $\lambda \in \mathbb{R}$.

Q25. Show that
$$\sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1).$$

Q26. Show that
$$\left|\sum_{k=1}^{n} \lambda_k\right| \leq \sum_{k=1}^{n} |\lambda_k|.$$

Q27. Show that for all $x, y \in \mathbb{R}_{>0}$,
$$\frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2}.$$
When does equality hold?

Q28. Set
$$\begin{aligned} \mathcal{A} &:= \{x \in \mathbb{Z} : x = 2k+1, \ \ k \in \mathbb{Z}\}, \\ \mathcal{B} &:= \{x \in \mathbb{Z} : x = 2k-1, \ \ k \in \mathbb{Z}\} \end{aligned}$$
Show that $\mathcal{A} = \mathcal{B}$, where the equality is taken as an equality of sets. That is, $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.

Q29. Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ are continuous functions defined on the real line. We say that a continuous function $h : \mathbb{R} \to \mathbb{R}$ is bounded if there exists an $M \geq 0$ such that
$$|h(x)| \leq M,$$
for all $x \in \mathbb{R}$. Determine which of the following are true or false. If you claim the result is true, provide a proof, if you claim the result is false, provide a counterexample.

a. If f and g are bounded, then $f+g$ is bounded.

b. If f and g are bounded, then $f-g$ is bounded.

c. If f and g are bounded, then $f \cdot g$ is bounded.

d. If f and g are bounded, then f/g is bounded.

e. If $f+g$ is bounded, then f and g is bounded.

f. If f/g is bounded, then f and g is bounded.

h. If $f \cdot g$ is bounded, then f or g is bounded.

Q30. Let $A, B \subseteq \mathbb{R}$ be two sets with $f : A \to B$ a continuous map between them. Denote the boundary of a set $X = [a, b]$ to be the set $\partial X = \{a, b\}$. Is it necessarily true that f maps ∂A to ∂B? Prove or provide a counterexample. If it is not true, can you determine any conditions on f such that ∂A is mapped to ∂B?

Q31. Let $A, B \subseteq \mathbb{R}$ be two sets. Prove or provide a counterexample to establish whether

$$f(A \cap B) = f(A) \cap f(B).$$

Q32. Let $A, B \subseteq \mathbb{R}$ be two sets. Prove or provide a counterexample to establish whether

$$f(A \cup B) = f(A) \cup f(B).$$

Q33. Let $A, B \subseteq \mathbb{R}$ be two sets. Prove or provide a counterexample to establish whether

$$f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B),$$

where $f^{-1}(X)$ denotes the preimage of X.

Q34. Let $A, B \subseteq \mathbb{R}$ be two sets. Prove or provide a counterexample to establish whether

$$f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B),$$

where $f^{-1}(X)$ denotes the preimage of X.

Q35. Write out the negations of the following statements.

a. $\forall x, y \in \mathbb{R}$, $x + y < 4$.

b. $\forall \varepsilon > 0$ $\exists \delta > 0$ such that $|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon$.

c. $\forall m \in \mathbb{Z}$ we may find an $n \in \mathbb{N}$ such that $m > n$.

d. For all $y \in Y$ and functions $f : X \to Y$, there exists an $x \in X$ such that $f(x) = y$.

e. There exists an $m \in \mathbb{Z}$ such that for all $x \in \mathbb{R}$, $xm \in \mathbb{Q}$.

Q36. Let $\mathcal{P}$ and $\mathcal{Q}$ be two statements. Prove that

a. $\mathcal{Q} \wedge \neg\mathcal{Q} \implies \mathcal{P}$.

b. $\mathcal{P} \wedge \mathcal{Q} \implies \mathcal{P}$.

c. $\mathcal{P} \implies \mathcal{P} \vee \mathcal{Q}$.

Q37. Suppose that $x, y \in \mathbb{R}$ and $n \in \mathbb{N}$. Show that

$$x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1}).$$

Q38. Let $j \in \mathbb{R}_{\geq 2}$. Prove that for all $n \in \mathbb{R}$, $n < j^n$.

Q39. Show that for each $n \in \mathbb{N}$, the sum of the first n odd integers is equal to $2n + n^2$.

Q40. Prove that

$$10n < n^2 + 25,$$

for all $n \in \mathbb{N}$.

Q41. Let $\mathbb{R}[x]$ be the set of polynomials with coefficients in $\mathbb{R}$. Show that $f, g \in \mathbb{R}[x] \implies f \cdot g \in \mathbb{R}[x]$.