COMPANION NOTES TO LINEAR ALGEBRA

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Unless otherwise stated, V, W denote vector spaces and k denotes a field.

1. Basic Definitions and Elementary Notions.

In this section we offer a review of much of the material that was covered in MATH1115. In the topic of linear algebra we are concerned with a particular object, called a **vector space** and the maps between them, which are called **linear maps**.

Definition 1.1. A **field** is a set k with two binary operations $+ : k \times k \to k$ and $\cdot : k \times k \to k$, defined by

$$\begin{aligned} +(x,y) &= x+y, \quad \forall x,y \in k, \\ \cdot(x,y) &= x \cdot y, \quad \forall x,y \in k. \end{aligned}$$

These operations are *associative*, meaning that

$$\begin{aligned} (x+y)+z &= x+(y+z), \quad \forall x,y,z \in k, \\ (x \cdot y) \cdot z &= x \cdot (y \cdot z), \quad \forall x,y,z \in k. \end{aligned}$$

We assume the field is *commutative*, that is, with these operations of $+$ and $\cdot$, we have

$$\begin{aligned} x+y &= y+x, \quad \forall x,y \in k, \\ x \cdot y &= y \cdot x, \quad \forall x,y \in k. \end{aligned}$$

There exists an additive identity, $0 \in k$. That is, an element 0 such that for all $x \in k$,

$$x + 0 = 0 + x = x.$$

There exists a multiplicative identity, $1 \in k$. That is, an element 1 such that for all $x \in k$,

$$x \cdot 1 = 1 \cdot x = x.$$

Every element in a field is necessarily has an *additive inverse*, meaning that for all $x \in k$, there exists some $y \in k$ such that $x + y = y + x = 0$. Every element in a field, excluding the additive identity, has a *multiplicative inverse*, meaning that for all $x \in k \backslash \{0\}$, there exists some $y \in k \backslash \{0\}$ such that $x \cdot y = y \cdot x = 1$. The last thing that is satisfied is the relationship between the operations $+$ and $\cdot$, that is, the distributive property,

$$x \cdot (y+z) = x \cdot y + x \cdot z, \quad \forall x,y,z \in k.$$

Exercise 1.2. Verify that the real numbers $\mathbb{R}$ and the complex numbers $\mathbb{C}$ form a fields.

Exercise 1.3. Verify that the integers modulo p, $\mathbb{Z}_p$, (see Exercise 2.9.5 of Introduction to Analysis), form a field. These form examples of finite fields.

Note that most fields that we will consider here will be either $\mathbb{R}$ or $\mathbb{C}$.

Definition 1.4. A k-**vector space**, for some field k, is a set V such that

i. $v_1 + v_2 \in V$ for all $v_1, v_2 \in V$,
ii. $\lambda v \in V$ for all $\lambda \in k,\ v \in V$.

To elaborate on this definition a little more, condition (i) is says that a vector space is closed under vector addition, while condition (ii) says that a k-vector space is closed under scalar multiplication by any element in k. Recall that k is often just $\mathbb{R}$ or $\mathbb{C}$. If $k = \mathbb{R}$, we say that V is a real vector space or $\mathbb{R}$-vector space. If $k = \mathbb{C}$, we say that V is a complex vector space or $\mathbb{C}$-vector space.

Exercise 1.5. Prove that every vector space contains the *zero vector*. That is, for every vector space V there exists an element 0 such that

$$0 + v = v + 0 = v, \quad \forall v \in V.$$

Exercise 1.6. Show that the real numbers $\mathbb{R}$ form a vector space over $\mathbb{R}$ and over $\mathbb{C}$.

Exercise 1.7. Show that the space of continuous functions $\mathscr{C}[0,1]$ on the interval $[0,1]$ form a vector space over $\mathbb{R}$ and over $\mathbb{C}$.

Exercise 1.8. Determine whether the space of continuously differentiable functions $\mathscr{C}^1[0,1]$ on the interval $[0,1]$ forms a vector space over $\mathbb{R}$ or over $\mathbb{C}$.

Exercise 1.9. Determine whether the space of polynomials $\mathcal{P}[0,1]$ on the interval $[0,1]$ forms a vector space over $\mathbb{R}$ or over $\mathbb{C}$.

Exercise 1.10. Determine whether the space of Riemann integrable functions $\mathcal{R}[0,1]$ on the interval $[0,1]$ forms a vector space over $\mathbb{R}$ or over $\mathbb{C}$.

Exercise 1.11. Let $GL_n(\mathbb{R})$ be the space of all $n \times n$ matrices with real entries. Show that $GL_n(\mathbb{R})$ forms a real vector space.

In mathematics, we not only care about objects and categories; we also care about the maps between these objects. Such maps, in the context of elementary linear algebra, form the content of the next definition.

Definition 1.12. Let V and W be k-vector spaces. We say that a map $T : V \to W$ is said to be **linear** if the following conditions are satisfied:

i. $T(v_1 + v_2) = T(v_1) + T(v_2), \quad \forall v_1, v_2 \in V$, (Additivity).
ii. $T(\lambda v) = \lambda T(v), \quad \forall \lambda \in k,\ v \in V$. (Homogeneity).

Remark 1.13. It is important to note how natural the definition of a linear map is in terms of vector spaces. Notice that the additivity condition says that we may add vectors in V then apply the map, or equivalently, apply the map to each vector, then add them. Similarly, the homogeneity condition provides us with the option of multiplying by a scalar before or after applying the map. In essense, a map being linear preserves a vector space structure. That is, the image of a vector space, under a linear map, is itself a vector space!

Exercise 1.13. Show that the map $\frac{d}{dx} : \mathscr{C}^1[0,1] \to \mathscr{C}[0,1]$ defined by

$$\frac{d}{dx}(f)(x) \longmapsto f'(x),$$

where $f'(x)$ denotes the derivative of f with respect to x, is a linear map.

Exercise 1.14. Adopting the notation as in Exercise 1.10., show that the map $\int : \mathcal{R}[0,1] \to \mathcal{R}[0,1]$ defined by

$$\int dx(f)(x) \longmapsto F(x),$$

where F denotes the Riemann integral of f with respect to x, is a linear map.

Exercise 1.15. Let $V = \mathbb{R}^2$ and let $T : V \to V$ be the map given by the matrix

$$T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Determine whether T is a linear map.

Exercise 1.16. Let $V = \mathbb{R}^2$.

i. Determine the matrix that corresponds to a rotation by ϑ anticlockwise.
ii. Let $T : V \to V$ denote the map given by the matrix that was determined in part (i). Is T a linear map?

Exercise 1.17. Let V be a vector space over $\mathbb{R}$ and consider the space of all linear maps $T : V \to V$ which we will denote by $\mathcal{B}(V)$. Determine whether $\mathcal{B}(V)$ forms a real vector space.

Definition 1.18. We say that a linear map $T : V \to W$ is **injective** if

$$T(v_1) = T(v_2) \implies v_1 = v_2.$$

We say that a linear map $T : V \to W$ is **surjective** if

$$\forall w \in W \ \exists v \in V \text{ such that } T(v) = w.$$

Intuitively, one should think of injectivity as the image of V being inserted into the space without any folds, while one should think of surjectivity as the image of V encompassing all of W.

Definition 1.19. Let $T : V \to W$ be a linear map between two vector spaces V and W. We define the **kernel** or **nullspace** of T to be the set

$$\ker T = \{v \in V : T(v) = 0\}.$$

We define the **range** of T to be the set

$$\text{ran } T = \{w \in W : T(v) = w, \text{ for some } v \in V\}.$$

Exercise 1.20. Prove or provide a counterexample to the following.

a. The kernel of a linear map is a subspace of V.
b. The image of a linear map is a subspace of W.

Exercise 1.21. Prove that a linear map is injective if and only if the kernel is trivial. That is, $\ker T = \{0\}$.

2. The Notion of a Basis

One of the most beautiful notions in linear algebra which makes it so useful is that the objects which form the study of linear algebra have information that may be presented in a compact form. That is, even though a vector space may have an infinite number of elements, it is possible to describe all of these elements in terms of a (often) finite set. This idea forms the content of the following definitions.

Definition 2.1. We say that a set of vectors $\{v_1, ..., v_n\}$ is **linearly independent** if for $\alpha_1, ..., \alpha_n \in k$,

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0 \iff \alpha_1, ..., \alpha_n = 0. \tag{1}$$

That is, the sum

$$\sum_{j=1}^{n} \alpha_j v_j$$

is called a **linear combination**, and a linear combination represents *no new information* in the sense of linear algebra. Equation (1) then asserts that a set of vectors is linearly independent, meaning they all represent new pieces of linear information, if and only if they cannot be written as a linear combination of one another.

We say that a set of vectors is **linearly dependent** if it not linearly independent.

Exercise 2.2. Show that any set of vectors that contains the zero vector is necessarily linearly dependent.

Exercise 2.3. Show that any set of 3 distinct vectors in $\mathbb{R}^2$ is necessarily linearly dependent.

Exercise 2.4. Find two sets of vectors in $\mathbb{R}^3$ that are linearly independent.

Definition 2.5. Let V be a k-vector space. We say that a set of vectors $\{v_1, ..., v_n\}$ **spans** V if for every $v \in V$, we may write

$$v = \sum_{j=1}^{n} \alpha_j v_j.$$

Exercise 2.6. Show that every vector in $\mathbb{R}^2$ may be written as a linear combination of the vectors

$$e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad e_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Exercise 2.7. Let $\mathcal{P}[0,1]$ denote the space of polynomials on the interval $[0,1]$. Show that every polynomial can be written as a linear combination of the vectors $\{1, x, x^2, ...\}$.

Definition 2.8. Let V be a basis of a vector space. A **basis** for V is a set of linearly independent vectors $\{v_1, ..., v_n\}$ that spans the vector space.

Notice that in the above definition we assumed that the basis consisted of a finite number of elements. This is not necessarily true in general, but many of the vector spaces we will be considering will have a finite number of basis elements.

Exercise 2.9. Adopting the notation as in Exercise 2.6., show that $\{e_1, e_2\}$ forms a basis of $\mathbb{R}^2$.

Exercise 2.10. Let $\mathcal{P}^n[0,1]$ denote the space of polynomials of degree n on the interval $[0,1]$. Show that $\{1, ..., x^n\}$ forms a basis for $\mathcal{P}^n[0,1]$.

Exercise 2.11. Find a basis for $\mathbb{C}$.

Exercise 2.12. Let $M_n(\mathbb{R})$ denote the space of $n \times n$ matrices with real entries. Find a basis for $M_n(\mathbb{R})$.

Definition 2.13. We define the **dimension** of a vector space V to be the number of basis elements for V.

Exercise 2.14. Let V and W be two vector spaces of the same dimension. Prove that a linear map $T : V \to W$ is injective if and only if it is surjective.

Exercise 2.15. Provide an example to show that a linear map between two vector spaces of the same dimension is not necessarily surjective.

Exercise 2.16. Let $A \in M_n(\mathbb{R})$ be an $n \times n$ matrix with real entries. Show that the columns of A form a basis for $\mathbb{R}^n$ if the determinant of A is nonzero.

Exercise 2.17. Determine whether the determinant map $\det : M_n(\mathbb{R}) \to \mathbb{R}$, defined by $A \mapsto \det A$ is a linear map.

Exercise 2.18. Determine whether the map that sends a matrix $A \in M_n(\mathbb{R})$ to its transpose $A^T \in M_n(\mathbb{R})$ is linear.

3. Inner Product Spaces

Definition 3.1. We may equip a vector space V with an **inner product** $\langle \cdot, \cdot \rangle$. That is, a symmetric, bilinear form $\langle \cdot, \cdot \rangle : V \times V \to k$ satisfying the following conditions:

i. $\langle \lambda_1 x + \lambda_2 y, z \rangle = \lambda_1 \langle x, z \rangle + \lambda_2 \langle y, z \rangle, \quad \forall \lambda_1, \lambda_2 \in k, \ x, y, z \in V$. (Linearity in the first entry).
ii. $\lambda x, \lambda_1 y + \lambda_2 z \rangle = \lambda_1 \langle x, y \rangle + \lambda_2 \langle x, z \rangle, \quad \forall \lambda_1, \lambda_2 \in k, \ x, y, z \in V$. (Linearity in the second entry).
iii. $\langle x, y \rangle = \langle y, x \rangle, \quad \forall x, y \in V$. (Symmetry).

Definition 3.2. An **inner product space** is a vector V equipped with an inner product $\langle \cdot, \cdot \rangle$.

Exercise 3.3. An elementary example of an inner product space is given by $\mathbb{R}^n$ equipped with the **dot product**. Recall that for $x = (x_1, ..., x_n), y = (y_1, ..., y_n)$, we define the dot product by

$$\langle x, y \rangle \quad := \quad \sum_{j=1}^{n} x_j y_j.$$

Verify that the dot product is indeed an example of an inner product.

Exercise 3.4. Adopting the notation as in Exercise 1.10., verify that

$$\langle f, g \rangle \quad := \quad \int_0^1 f(x)\overline{g(x)} dx$$

is an inner product on $\mathcal{R}[0, 1]$.

Exercise 3.5.

Definition 3.6. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. We say that two vectors $v_1, v_2 \in V$ are orthogonal if

$$\langle v_1, v_2 \rangle = 0.$$

Exercise 3.7. Prove that if two vectors $v_1, v_2 \in V$ are orthogonal, they are linearly independent.

Exercise 3.8. Generalise the above exercise to an arbitrary set of vectors. That is, suppose that $\{v_1, ..., v_n\}$ are pairwise orthogonal, meaning that

$$\langle v_i, v_j \rangle = 0, \quad \forall\ 1 \leq i \neq j \leq n,$$

then the vectors $\{v_1, ..., v_n\}$ are linearly independent.

Exercise 3.9. Prove or provide a counterexample to establish whether a linear map preserves orthogonality. That is, suppose $T : V \to W$ is a linear map between the inner product spaces $(V, \langle \cdot, \cdot \rangle_V)$ and $(W, \langle \cdot, \cdot \rangle_W)$. If $\langle v_1, v_2 \rangle_V = 0$, does it necessarily follows that $\langle T(v_1), T(v_2) \rangle_W = 0$?

Definition 3.10. Let V be a vector space, not necessarily an innner product space. We define a **norm** to be function $\| \cdot \| : V \times V \to [0, \infty)$ that satisfies

(i) $\|v\| = 0 \iff v = 0$.
(ii) $\|v\| \geq 0$ for all $v \in V$. (Positivity).
(iii) $\|\lambda v\| = |\lambda|\,\|v\|$ for all $\lambda \in \mathbb{C},\ v \in V$. (Homogeneity).
(iv) $\|u + v\| \leq \|u\| + \|v\|$ for all $u, v \in V$. (Triangle inequality).

Exercise 3.11. Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space. Verify that $\sqrt{\langle \cdot, \cdot \rangle}$ is a norm on V. This exercise establishes that

$$\boxed{\text{Every inner product gives rise to a norm.}}$$

Exercise 3.12. Let $\mathscr{C}[0, 1]$ denote the vector space of continuous functions on the interval $[0, 1]$. Verify that

$$\|f - g\|_\infty := \sup_{x \in [0,1]} |f(x) - g(x)|$$

is a norm on $\mathscr{C}[0, 1]$.

Exercise 3.13. Show that the square of the norm given in Exercise 2.15. is not an inner product. This exercise establishes that

$$\boxed{\text{Not every norm comes from an inner product.}}$$

Exercise 3.14. Equip the vector space of continuously differentiable functions $\mathscr{C}^1[0,1]$ on the interval $[0,1]$ with the norm

$$\|f\|_{\mathscr{C}^1} := \sup_{x\in[0,1]} |f(x)| + \sup_{x\in[0,1]} \left|f'(x)\right|.$$

Show that $\|\cdot\|_{\mathscr{C}^1}$ is indeed a norm. Does this norm come from an inner product?

Exercise 3.15. Equip the vector space of n-times continuously differentiable functions $\mathscr{C}^n[0,1]$ on the interval $[0,1]$ with the norm

$$\|f\|_{\mathscr{C}^n} := \sum_{j=0}^{n} \sup_{x\in[0,1]} \left|f^{(j)}(x)\right|,$$

where $f^{(j)}(x)$ denotes the jth derivative of f and we adopt the covention that $f^{(0)}(x) = f(x)$. Show that $\|\cdot\|_{\mathscr{C}^n}$ is indeed a norm.