Introduction to Calculus

Kyle Broder – ANU – MSI – 2017

A Panoramic View

Let us begin by defining what "Calculus" is. *Calculus*, in Latin, literally means *small pebble used for counting on an abacus*. In essence, calculus studies change in a continuous manner. Calculus is broken up into two branches, differential calculus and integral calculus. Differential calculus, which is the canonical starting point, is the study of very small, infinitesimal, change. Integral calculus is a way for us to convert finite sums to infinite sums and allow us to evaluate areas that would have previously been other too daunting or impossible.

Calculus was invented in the century in the 17th century independently by Isaac Newton and Gottfried Leibniz.

Limits

A limit allows us to encapsulate information about a function $f(x)$ at a point x_0, without concerning ourselves with the behaviour at the point x_0. That is, a limit describes the behaviour of a function very closed to x_0, but not at x_0. The associated notation for this phenomenon is typically

$$\lim_{x \to x_0} f(x).$$

Remark 1.1. To a mathematician, the above paragraph makes no sense. I have not made a precise definition, I have merely appealed to some general (imprecise) intuition. The formal definition of a limit is given by the following:

$$\forall \varepsilon > 0 \; \exists \delta > 0 \text{ such that } |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

As one may expect, this is above the scope of our course here. For more details on this however, see chapter 4 of my text *Introduction to Analysis*.

The following theorem describes the important properties of limits that we will use.

Theorem 1.2. Suppose that $\lim_{x \to x_0} f(x) = \ell_1$ and $\lim_{x \to x_0} g(x) = \ell_2$, then

(i) $\lim_{x \to x_0} (f + g)(x) = \ell_1 + \ell_2$.
(ii) $\lim_{x \to x_0} (f - g)(x) = \ell_1 - \ell_2$.
(iii) $\lim_{x \to x_0} (f \cdot g)(x) = \ell_1 \cdot \ell_2$.
(iv) $\lim_{x \to x_0} \frac{f}{g}(x) = \frac{\ell_1}{\ell_2}$, if $\ell_2 \neq 0$.

We now outline below some examples of how to compute limits.

Example 1.3. Calculate $\lim_{x\to 3} x^2 - 9$.

Proof. To compute this limit we simply need insert the value of $x = 3$ into the function $x^2 - 9$. Therefore,

$$\lim_{x\to 3} x^2 - 9 = (3)^2 - 9 = 0.$$

$\square$

Example 1.4. Calculate the limit $\lim_{x\to 2} \frac{x^2-4}{x-2}$.

Proof. Again using a similar process as before, we simply insert the value of $x = 2$ into the function $\frac{x^2-4}{x-2}$. Therefore,

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{(2)^2-4}{2-2} = \frac{0}{0}.$$

This is an example of an indeterminant form.[1] So we have to manipulate the equation first, prior to inserting the value of x. Observe that we can write

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x + 2 = 4.$$

$\square$

Example 1.5. Calculate the limit $\lim_{x\to\infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)$.

Proof.

$$\begin{aligned}
\lim_{x\to\infty} \left(\sqrt{x^2+x} - \sqrt{x^2-x}\right) &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x} - \sqrt{x^2-x}\right)\left(\sqrt{x^2+x} + \sqrt{x^2-x}\right)}{\sqrt{x^2+x} + \sqrt{x^2-x}} \\
&= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x} + \sqrt{x^2-x}} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{1}{x}} + \sqrt{1-\frac{1}{x}}} \\
&= 1.
\end{aligned}$$

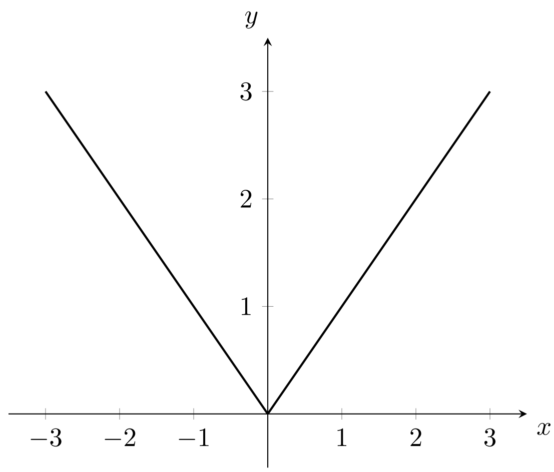
$\square$

By now the question has perhaps been raised as to when does a limit exist? We say that a limit exists as $x \to x_0$ if $\lim_{x\to x_0} f(x) = f(x_0)$ from both directions. That is, as x approaches x_0 from the left, $f(x)$ should approach $f(x_0)$, and if x approaches x_0 from the right, $f(x)$ should again approach $f(x_0)$. By letting x_0^- and x_0^+ denote the left and right of x_0 respectively, we can express that a limit exists if and only if

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x).$$

[1] An indeterminant form is a mathematical expression that is not definitively or precisely determined. Indeterminant form look like one of the following : $\dfrac{0}{0}, 0\cdot\infty, \infty-\infty, 0^0, \infty^0, 1^\infty$.

Example 1.6. If we consider the function $f(x) = |x|$,



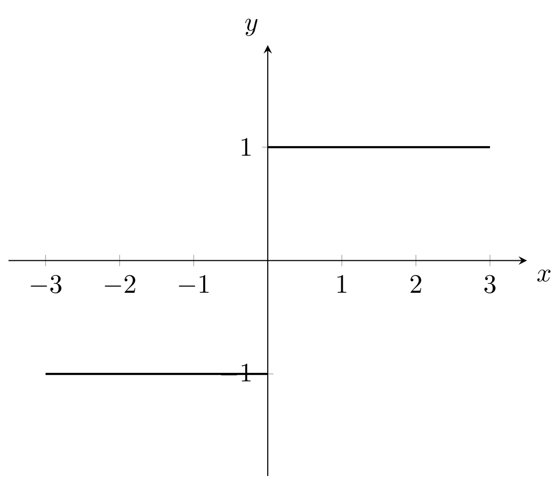
then

$$\lim_{x\to 0^-} |x| = 0, \text{ and } \lim_{x\to 0^+} |x| = 0 \text{ also.}$$

Therefore $\lim_{x\to x_0} |x|$ exists and is equal[2] to zero.

Example 1.7. If we consider the function

$$g(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$



[2] It is worth remarking that when we say that a limit is equal to something, the word 'equal' is quite distorting. The limit says nothing about what happens to $f(x)$ when $x = x_0$, just what happens when x gets arbitrarily close to x_0. So it would be better to write $\lim_{x\to 0} |x|$ approaches 0 on either side of x_0. This is rather cumbersome however, so we simply say that $\lim_{x\to 0} |x| = 0$.

Then as we approach $x = 0$ from the left, $g(x)$ approaches -1, and $\lim_{x\to 0^-} g(x) = -1$. As we approach $x = 0$ from the right however, $g(x)$ approaches 1, and $\lim_{x\to 0^+} g(x) = 1$. Therefore, since

$$\lim_{x\to 0^-} g(x) = -1 \neq 1 = \lim_{x\to 0^+} g(x),$$

the limit does not exist at $x = 0$.

Exercise 1.8. Evaluate the following limits.

a.
$$\lim_{x\to 1} \frac{x^2 - 2x - 3}{x - 3}.$$

b.
$$\lim_{x\to -1} \frac{2x^2 + 2x}{x + 1}.$$

c.
$$\lim_{x\to 3} \frac{x^2 + 3x}{x - 1}.$$

d.
$$\lim_{x\to -2} x^3 + x^2 - 6.$$

Exercise 1.9. (Dr. Lloyd Gunatilake). Evaluate the limit

$$\lim_{x\to 2} \frac{x - 2}{-2 + \sqrt{5x - 6}}.$$

Differentiation Theory

Definition 2.1. A function f is said to be differentiable at x if the limit

$$\lim_{\Delta x\to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

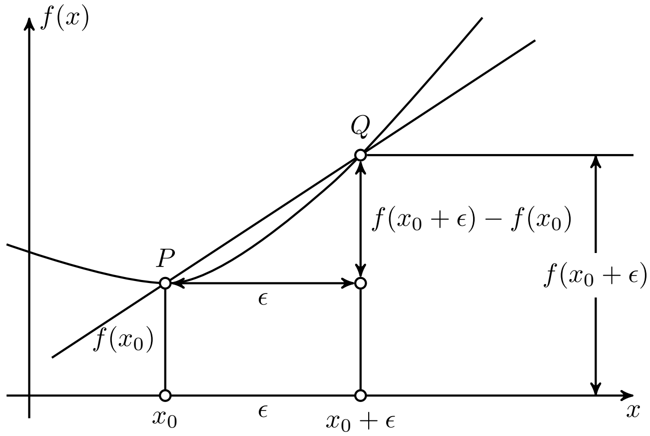
is defined. Moreover, the derivative of f at x is equal to the value of this limit.

Equivalently, we can say that f is differentiable at x_0 if the limit

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}$$

is defined. Moreover, the derivative of f at x_0 is equal to the value of the limit.

The associated picture is given by



Definition 2.2. We say that a function f is continuously differentiable, or $f \in C^1$, if the derivative of f, namely f' is continuous.

Theorem 2.3. For any positive integer n, the function $f(x) = x^n$ is differentiable.

Proof.

$$\begin{aligned}\lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} &= \lim_{x\to x_0} \frac{x^n - x_0^n}{x-x_0} \\ &= \lim_{x\to x_0} \frac{x-x_0}{x-x_0} \sum_{k=0}^{n-1} x^{n-k-1} x_0^k, \\ &= n x_0^{n-1}.\end{aligned}$$

Therefore,

$$f(x) = x^n \implies f'(x) = nx^{n-1} \quad \forall x \in \mathbb{R}.$$

□

Theorem 2.4. Prove that $\sin x$ is differentiable.

Proof.

$$\begin{aligned}\lim_{\Delta x\to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0} \frac{\sin(x+\Delta x) - \sin x}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\sin x\cos\Delta x + \sin\Delta x\cos x - \sin x}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\sin x(\cos\Delta x - 1) + \cos x\sin\Delta x}{\Delta x} \\ &= \lim_{\Delta x\to 0} \frac{\sin x(\cos\Delta x - 1)}{\Delta x} + \lim_{\Delta x\to 0} \frac{\cos x\sin\Delta x}{\Delta x} \\ &= \sin x\cdot 0 + 1\cdot\cos x \\ &= \cos x.\end{aligned}$$

Therefore,

$$f(x) = \sin x \implies f'(x) = \cos x \quad \forall x \in \mathbb{R}.$$

$\square$

Theorem 2.5. Suppose that f and g are differentiable at x, then the functions $f+g, f-g, f \cdot g$ and f/g are also differentiable at x_0, with

(i) $(f+g)'(x) = f'(x) + g'(x)$
(ii) $(f-g)'(x) = f'(x) - g'(x)$
(iii) $(f \cdot g)'(x) = f'(x)g(x) + f(x)g'(x)$
(iv) $(f/g)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, assuming for (iv) that $g'(x) \neq 0$.

We prove (iii) and leave the rest as an exercise.

Proof.

$$\begin{aligned}
\lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0)}{x - x_0} &= \lim_{x \to x_0} \frac{f(x)g(x) - f(x_0)g(x_0) + f(x_0)g(x) - f(x_0)g(x)}{x - x_0} \\
&= \lim_{x \to x_0} \frac{g(x)[f(x) - f(x_0)]}{x - x_0} + \lim_{x \to x_0} \frac{f(x_0)[g(x) - g(x_0)]}{x - x_0} \\
&= g(x_0)f'(x_0) + f(x_0)g'(x_0).
\end{aligned}$$

$\square$

Exercise 2.6. The previous theorem gives us the necessary formulae to differentiate sums, products and quotients of differentiable functions. The following theorem gives us the necessary tools to differentiate the composition of differentiable functions.

Theorem 2.7. (The Chain Rule) - Suppose that g is differentiable at x_0 and f is differentiable at $g(x_0)$. Then the composite function

$$f(g(x))$$

is differentiable at x_0 with

$$[f(g(x))]' = f'(g(x_0))g'(x_0).$$

Proof.

$$\begin{aligned}
[f(g(x))]' &= \lim_{\Delta x \to 0} \frac{f(g(x + \Delta x)) - f(g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f[g(x) + (g(x + \Delta x) - g(x))] - f(g(x))}{\Delta x}.
\end{aligned}$$

To simplify notation, let

$$\bar{g}_\Delta = g(x + \Delta x) - g(x),$$

and bear in mind that $\overline{g}_\Delta \to 0$ as $h \to 0$. Substituting this into our expression yields

$$\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{f(g(x)+\overline{g}_\Delta) - f(g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(g(x)+\overline{g}_\Delta) - f(g(x))}{\overline{g}_\Delta} \cdot \frac{\overline{g}_\Delta}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(g(x)+\overline{g}_\Delta) - f(g(x))}{\overline{g}_\Delta} \cdot \lim_{\Delta x \to 0} \frac{\overline{g}_\Delta}{\Delta x} \\
&= \lim_{\overline{g}_\Delta \to 0} \frac{f(g(x)+\overline{g}_\Delta) - f(g(x))}{\overline{g}_\Delta} \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(g(x)) \cdot g'(x).
\end{aligned}$$

□

Exercise 2.8. Using Definition 2.1, show that the following functions are differentiable and calculate their derivatives.

a. $f(x) = 2x + 1$.
b. $f(x) = 3x - 5$.
c. $f(x) = 9 - x$.
d. $f(x) = x^2$.
e. $f(x) = 5 + 2x - x^2$.
f. $f(x) = x^2 - 5x + 6$.

Exercise 2.9. Using Definition 2.1, show that the following functions are differentiable and calculate their derivatives.

a. $f(x) = \cos x$.
b. $f(x) = \tan x$.
c. $f(x) = \sin 2x$.
d. $f(x) = \frac{1}{3} \cos 2x$.

Exercise 2.10. Using Definition 2.1, show that

$$f(x) = e^x$$

is differentiable and calculate $f'(x)$.

Exercise 2.11. Using Definition 2.1, show that

$$f(x) = \log_e(x)$$

is differentiable and calculate $f'(x)$.

Exercise 2.12. Show that the function

$$f(x) = |x|$$

is not differentiable at $x = 0$.

Exercise 2.13. Using Theorem 2.3, calculate the derivatives of the following functions

a. $f(x) = 3x^2 + 5x + 1$.
b. $f(x) = 4 - 10x^{13} + 7x$.
c. $f(x) = 1 + x$.
d. $f(x) = 2x + 4x^2$.
e. $f(x) = \frac{1}{4}(x - 3)^2$.
f. $f(x) = (6 - x)^2 + 4$.