Calculus Practice Exam 4 Solutions

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Recall that in the notes we proved the following result: if f and g are both differentiable, then the derivative of the product $f \cdot g$ is given by

$$\frac{d}{dx}(f(x) \cdot g(x)) = f(x) \cdot g'(x) + f'(x) \cdot g(x).$$

This is known as the *product rule*, or more formally, *Leibniz rule*.

This practice exam aims at illustrating the use of this rule.

Question 1. Using the product rule, differentiate the following functions.

a. $f(x) = xe^x$.

Proof. It is easy to see that $f'(x) = e^x + xe^x = e^x(1 + x)$. $\square$

b. $f(x) = (x - 1)e^{3x}$.

Proof. It is easy to see that $f'(x) = 3(x - 1)e^{3x} + e^{3x} = e^{3x}(3x - 3 + 1) = e^{3x}(3x - 2)$. $\square$

c. $f(x) = x^2 e^{4-6x}$.

Proof. The derivative of x^2 is $2x$ and the derivative of e^{4-6x} is $-6e^{4-6x}$. The product rule then gives us that

$$f'(x) = 2xe^{4-6x} - 6x^2 e^{4-6x} = e^{4-6x}(2x - 6x^2).$$

$\square$

d. $f(x) = \sqrt{x}e^{\sqrt{x}}$.

Proof. The derivative of $\sqrt{x}$ is $\frac{1}{2\sqrt{x}}$ and the derivative of $e^{\sqrt{x}}$ is $\frac{1}{2\sqrt{x}}e^{\sqrt{x}}$ by the chain rule. We therefore see that

$$\begin{aligned} f'(x) &= \sqrt{x} \cdot \frac{1}{2\sqrt{x}}e^{\sqrt{x}} + \frac{1}{2\sqrt{x}}e^{\sqrt{x}} \\ &= \frac{1}{2}e^{\sqrt{x}}\left(1 + \frac{1}{\sqrt{x}}\right) \end{aligned}$$

$\square$

Question 2. Using the product rule, differentiate the following functions.

a. $f(x) = x \sin x$.

Proof. It is immediate to see that $f'(x) = \sin x + x \cos x$. $\square$

b. $f(x) = x \cos x$.

Proof. Similarly, $f'(x) = \cos x - x \sin x$. $\square$

c. $f(x) = 3x \cos(2x)$.

Proof. It is easy to see that $f'(x) = (3x) \cdot (-2\sin(2x)) + 3\cos 2x = -6x \sin 2x + 3\cos 2x$. $\square$

d. $f(x) = (x^2 - 5x + 6)\sin(x)$.

Proof. The derivative of $x^2 - 5x + 6$ is $2x - 5$ and the derivative of $\sin x$ is $\cos x$. Therefore, we see that

$$f'(x) = (2x - 5)\sin x + (x^2 - 5x + 6)\cos x.$$

$\square$

Question 3. Using the product rule, differentiate the following functions.

a. $f(x) = e^x \log_e(x)$.

Proof. It is easy to see that $f'(x) = e^x \log_e(x) + \frac{e^x}{x} = e^x \left(\log_e(x) + \frac{1}{x}\right)$. $\square$

b. $f(x) = e^x \sin x$.

Proof. It is easy to see that $f'(x) = e^x \cos x + e^x \sin x = e^x(\cos x + \sin x)$. $\square$

c. $f(x) = \sqrt{x} \log_e(x)$.

Proof. It is easy to see that

$$f'(x) = \sqrt{x}\frac{1}{x} + \frac{1}{2\sqrt{x}} \log_e(x).$$

$\square$

d. $f(x) = 3x^{\frac{1}{3}} \log_e(x - 5) + 1$.

Proof. The derivative of $3x^{\frac{1}{3}}$ is $x^{-\frac{2}{3}}$ and the derivative of $\log_e(x - 5)$ is $\frac{1}{x-5}$. The product rule then gives us that

$$f'(x) = x^{-\frac{2}{3}} \log_e(x - 5) + 3x^{\frac{1}{3}} \frac{1}{x - 5}.$$

$\square$

e. $f(x) = \sin(x)\cos(3x)$.

Proof. The derivative of $\sin x$ is $\cos x$ and the derivative of $\cos 3x = -3\sin 3x$. Hence, by the product rule we see that

$$f'(x) = \cos x \cos(3x) - 3\sin(3x)\sin x.$$

$\square$

f. $f(x) = \sin(x + \pi)\log_e(x + \pi)$.

Proof. The derivative of $\sin(x + \pi)$ is $\cos(x + \pi)$, and the derivative of $\log_e(x + \pi)$ is $\frac{1}{x+\pi}$. Hence, by the product rule we see that

$$f'(x) = \cos(x + \pi)\log_e(x + \pi) + \frac{\sin(x + \pi)}{x + \pi}.$$

$\square$

Question 4. Using the product rule, prove that

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

Proof. Write $\tan x = (\sin x)(\cos x)^{-1}$. The derivative of $\sin x$ is $\cos x$ and the derivative of $(\cos x)^{-1}$ is $\sin x(\cos x)^{-2} = \tan x \sec x$. Therefore, we see that

$$\begin{aligned} f'(x) &= \sin x \tan x \sec x + \cos x \cdot (\cos x)^{-1} \\ &= \tan^2 x + 1 \\ &= \sec^2 x. \end{aligned}$$

□

Question 5. Consider the function

$$f(x) = \cot(x) := \frac{1}{\tan(x)}.$$

Using a similar method to that considered in Question 4, determine the derivative of $f(x)$.

Proof. Write $\cot x = \cos x(\sin x)^{-1}$. The derivative of $\cos x = -\sin x$ and the derivative of $(\sin x)^{-1}$ is $-\cos x(\sin x)^{-2}$. The product rule then gives us that

$$\begin{aligned} f'(x) &= \cos x \cdot (-\cos x)\frac{1}{\sin^2 x} - \sin x \frac{1}{\sin x} \\ &= -\cot^2 x - 1 = -\csc^2(x). \end{aligned}$$

□

Question 6. Using the product rule, differentiate the following functions.

a.

$$f(x) = \frac{x-3}{x+2}.$$

Proof. Write

$$f(x) = \frac{x-3}{x+2} = (x-3)(x+2)^{-1}.$$

The derivative of $x-3$ is 1 and the derivative of $(x+2)^{-1}$ is $-\frac{1}{(x+2)^2}$. The product rule then gives us that

$$\begin{aligned} f'(x) &= (x-3) \cdot -\frac{1}{(x+2)^2} + \frac{1}{x+2} \\ &= \frac{1}{(x+2)^2}(3 - x + x - 2) \\ &= \frac{1}{(x+2)^2}. \end{aligned}$$

□

b.

$$f(x) = \frac{3x+1}{4x+7}.$$

Proof. Write
$$f(x) = \frac{3x+1}{4x+7} = (3x+1)(4x+7)^{-1}.$$
The derivative of $3x+1$ is 3 and the derivative of $(4x+7)^{-1}$ is $-\frac{4}{(4x+7)^2}$. The product rule then gives us that
$$\begin{aligned} f'(x) &= (3x+1)\cdot -\frac{4}{(4x+7)^2} + 3\frac{1}{4x+7} \\ &= \frac{1}{(4x+7)^2}(-1-3x+3(4x+7)) \\ &= \frac{1}{(4x+7)^2}(20+9x). \end{aligned}$$
□

c.
$$f(x) = \frac{4x+6}{3-x}.$$

Proof. Write
$$f(x) = (4x+6)(3-x)^{-1}.$$
The derivative of $4x+6$ is 4 and the derivative of $(3-x)^{-1}$ is $(-1)\cdot(-1)\frac{1}{(3-x)^2} = \frac{1}{(3-x)^2}$. The product rule then gives us that
$$\begin{aligned} f'(x) &= \frac{4x+6}{(3-x)^2} + \frac{4}{3-x} \\ &= \frac{1}{(3-x)^2}(4x+6+4(3-x)) \\ &= \frac{18}{(3-x)^2}. \end{aligned}$$
□

d.
$$f(x) = \frac{3-6x}{5-x}.$$

Proof. Write
$$f(x) = (3-6x)(5-x)^{-1}.$$
The derivative of $3-6x$ is -6 and the derivative of $(5-x)^{-1}$ is $\frac{1}{(5-x)^2}$. The product rule then gives us that
$$\begin{aligned} f'(x) &= \frac{3-6x}{(5-x)^2} - \frac{6}{5-x} \\ &= \frac{1}{(5-x)^2}(3-6x-6(5-x)) \\ &= -\frac{27}{(5-x)^2}. \end{aligned}$$
□

e.

$$f(x) = \frac{2}{x^2 - 5x + 6}.$$

Proof. Write

$$f(x) = 2(x^2 - 5x + 6)^{-1}.$$

Then the chain rule gives us that

$$\begin{aligned} f'(x) &= 2 \cdot (-1) \cdot (2x - 5) \cdot (x^2 - 5x + 6)^{-2} \\ &= \frac{10 - 4x}{(x^2 - 5x + 6)^2}. \end{aligned}$$

□

f.

$$f(x) = \frac{2x - 6}{x^2 + 12x + 1}.$$

Proof. Write

$$f(x) = (2x - 6)(x^2 + 12x + 1)^{-1}.$$

The derivative of $2x - 6$ is 2 and the derivative of $(x^2 + 12x + 1)^{-1}$ is $-\frac{2x+12}{(x^2+12x+1)^2}$. The product rule then gives us that

$$f'(x) = -(2x - 6) \cdot \frac{(2x + 12)}{(x^2 + 12x + 1)^2} + \frac{2}{(x^2 + 12x + 1)}.$$

I do not want to simplify this any further. □

g.

$$f(x) = \frac{x^2 + 2x + 1}{x^2 + 4x + 1}.$$

Proof. The derivative of $x^2 + 2x + 1$ is $2x + 2$ and the derivative of $x^2 + 4x + 1$ is $2x + 4$. The product rule then gives us that

$$f'(x) = \frac{2(x^2 - 1)}{(x^2 + 4x + 1)^2}.$$

□

h.

$$f(x) = \frac{x + 3}{x^2 - 9}.$$

Proof. The derivative of $x + 3$ is 1 and the derivative of $x^2 - 9$ is $2x$. The product rule then gives us that

$$f'(x) = -\frac{1}{(x - 3)^2}.$$

□

i.

$$f(x) = \frac{4x^3 + 5x + 1}{x^3 + 2x + 1}.$$

Proof. The derivative of $4x^3+5x+1$ is $12x^2+5$ and the derivative of x^3+2x+1 is $3x^2+2$. The product rule then gives us that

$$f'(x) \quad = \quad \frac{6x^3+9x^2+3}{(x^3+2x+1)^2}.$$

□

Question 7. Using the product rule, differentiate the following functions.

a.

$$f(x) = \frac{\sqrt{x}+\sqrt{-x}}{2x+3}.$$

Proof.

$$f'(x) \quad = \quad \frac{1}{2x+3}\left(\frac{1}{2\sqrt{x}} - \frac{1}{2\sqrt{-x}}\right) - \frac{2}{(2x+3)^2}\left(\sqrt{-x}+\sqrt{x}\right).$$

□

b.

$$f(x) = \frac{\sin x + \cos x}{\tan x}.$$

Proof.

$$f'(x) \quad = \quad \cot(x)[\cos x - \sin x] - \csc^2 x[\sin x + \cos x].$$

□

c.

$$f(x) = \frac{3x+\cos(x^2)}{2x-4}.$$

Proof.

$$f'(x) \quad = \quad -\frac{2x\sin x^2}{2x-4} - \frac{2\cos x^2+6}{(2x-4)^2}.$$

□

d.

$$f(x) = \frac{2e^{x-3}\cos x + \sin(\cos(x))}{e^x - e^{-x}}.$$

Proof.

$$\begin{aligned} f'(x) \quad &= \quad \frac{1}{e^x-e^{-x}}\left(-2e^{x-3}\sin x + 2e^{x-3}\cos x - \sin x\cos(\cos x)\right) \\ &\quad -\frac{1}{(e^x-e^{-x})^2}\left(e^{-x}+e^x\right)\left(2e^{x-3}\cos x + \sin(\cos x)\right) \end{aligned}$$

□