Chapter 1

Differential Equations.

In this chapter we use our new found understanding of differentiation and integration to solve differential equations. A differential equation is an equation of the form

$$a_n \frac{d^n y}{dx^n} + \cdots + a_1 \frac{dy}{dx} + a_0 y = c,$$

where $a_j, c \in \mathbb{R}$, for all $0 \leq j \leq n$.

We note that the positive integer n in the above equation is referred to as the degree of the differential equation.

1.1 Elementary Separable Differential Equations

In this section we look at the simplest of the differential equations. That is, we look at how to solve equations of the form

$$\frac{dy}{dx} = f(x), \ \frac{dy}{dx} = f(y), \ \frac{d^2 y}{dx^2} = f(x), \ \text{or} \ \frac{d^2 y}{dx^2} = f(y),$$

where $f : D \to \mathbb{R}$ is some integrable function[1] defined on some domain $D \subseteq \mathbb{R}$.

Example 10.1.1. Solve the differential equation

$$\frac{dy}{dx} = 2\cos(x).$$

[1]For almost every example we will consider here, we may simply suppose that f is smooth.

Proof. We simply observe that

$$\begin{aligned}
\frac{dy}{dx} = 2\cos(x) \quad &\implies \quad \int \frac{dy}{dx} dx = \int 2\cos(x) dx \\
&\implies \quad y = 2\int \cos(x) dx \\
&= \quad 2\sin(x) + \text{constant}.
\end{aligned}$$

Therefore the solutions of the given differential equation are of the form

$$y(x) = 2\sin(x) + c,$$

where $c \in \mathbb{R}$ is some constant. ☐

Notice that in the previous example, we did not obtain a unique solution, but rather an uncountably infinite number of solutions given by varying the constant c. If we wish to get a unique solution to a differential equation, we need to specify an initial condition that pins the solution down to a fixed curve.

Example 10.1.2. Solve the boundary-value problem

$$\begin{cases} \frac{dy}{dx} = e^x\sqrt{1+e^x}, \quad x \in \mathbb{R}, \\ y(0) = 1. \end{cases}$$

Proof. We simply observe that

$$\begin{aligned}
\frac{dy}{dx} = e^x\sqrt{1+e^x} \quad &\implies \quad y = \int e^x\sqrt{1+e^x} dx \\
&\implies \quad y = \int (u-1)\sqrt{u}\frac{1}{u} du \\
&\implies \quad y = \int (u-1)u^{-\frac{1}{2}} du \\
&\implies \quad y = \int u^{\frac{1}{2}} - u^{-\frac{1}{2}} du \\
&\implies \quad y = \frac{2}{3}u^{\frac{3}{2}} - 2\sqrt{u} + k, \quad k \in \mathbb{R}, \\
&\implies \quad y = \frac{2}{3}(1+e^x)^{\frac{3}{2}} - 2\sqrt{1+e^x} + k.
\end{aligned}$$

Now since $y(0) = 1$, we see that

$$\begin{aligned}
y(0) = 1 \quad &\Longrightarrow \quad \frac{3}{2}(1 + e^0)^{\frac{3}{2}} - 2\sqrt{1 + e^0} + k = 1 \\
&\Longrightarrow \quad \frac{3}{2} \cdot 2^{\frac{3}{2}} - 2\sqrt{2} + k = 1 \\
&\Longrightarrow \quad \frac{3}{2} \cdot 2\sqrt{2} - 2\sqrt{2} + k = 1 \\
&\Longrightarrow \quad k = 1 + \sqrt{2}.
\end{aligned}$$

So the solution to the boundary value problem is given by

$$y = \frac{2}{3}(1 + e^x)^{\frac{3}{2}} - 2\sqrt{1 + e^x} + 1 + \sqrt{2}.$$

□

Example 10.1.3. Solve the differential equation

$$(e^x + e^{-x})\frac{dy}{dx} = y^2.$$

Proof. We simply observe that

$$
\begin{aligned}
(e^x + e^{-x})\frac{dy}{dx} = y^2 \quad &\implies \quad \frac{dy}{dx} \cdot \frac{1}{y^2} = \frac{1}{e^x + e^{-x}} \\
&\implies \quad \frac{1}{y^2} dy = \frac{1}{e^x + e^{-x}} dx \\
&\implies \quad \int \frac{1}{y^2} dy = \int \frac{1}{e^x + e^{-x}} dx \\
&\implies \quad -\frac{1}{y} + k_1 = \int \frac{1}{e^x + e^{-x}} dx, \qquad k_1 \in \mathbb{R}, \\
&\implies \quad k_1 - \frac{1}{y} = \int \frac{e^x}{e^{2x} + 1} dx \\
&\implies \quad k_1 - \frac{1}{y} = \int \frac{u}{u^2 + 1} \frac{1}{u} du, \qquad e^x \mapsto u, \\
&\implies \quad k_1 - \frac{1}{y} = \int \frac{1}{u^2 + 1} du \\
&\implies \quad k_1 - \frac{1}{y} = \tan^{-1}(u) + k_2, \\
&\implies \quad k_1 - \frac{1}{y} = \tan^{-1}(e^x) + k_2, \\
&\implies \quad \frac{1}{y} = k_1 - k_2 - \tan^{-1}(e^x) \\
&\implies \quad y = \frac{1}{k - \tan^{-1}(e^x)}, \qquad k := k_1 - k_2.
\end{aligned}
$$

□

Exercises

Q1. Solve the following differential equations

a. $\frac{dy}{dx} = -3\sec^2(x)$.

b. $\frac{dy}{dx} = 2e^x + 4$.

c. $\frac{dy}{dx} = \sqrt{9 - x^2}$.

d. $\frac{dy}{dx} = \frac{\log_e(x)}{x}$.

e. $\frac{dy}{dx} = -4\sin(6x) - \tan(x)$.

f. $\frac{dy}{dx} = \frac{2x-1}{\sqrt{x^2-5x+6}}$.

Q2. Solve the following differential equations

a. $\frac{dy}{dx} = \cos(2-y)$.
b. $\frac{dy}{dx} = 1 - \sqrt{x}$.
c. $\frac{dy}{dx} = \frac{y^2-4}{x^2-4}$.
d. $\frac{dy}{dx} + 4y = -7$.
e. $\frac{dy}{dx} = x^2 \sin(y^2)$.
f. $\frac{dy}{dx} = y - y^3$.

Q3. Solve the following differential equations

a. $x^3 \frac{dy}{dx} = 2y$.
b. $\frac{1}{x^2} dy = \frac{1}{\log_e(y)} dx$.
c. $\sqrt{1-x^2}dy + \sqrt{1-y^2}dx = 0$.
d. $\frac{dy}{dx} = \frac{y}{e^x+4}$.
e. $x\frac{dy}{dx} = \frac{3y-4}{5x+1}$.
f. $2x\cos(x)\frac{dx}{dy} = 2e^y$.

Q4. Solve the boundary value problem

$$\begin{cases} e^x y \frac{dy}{dx} = e^{-y} + e^{-2x-y}, & x \in \mathbb{R}, \\ y(1) = 4. \end{cases}$$

Q5. Solve the boundary value problem

$$\begin{cases} \frac{dy}{dx} = \frac{3+4x}{\sqrt{5-y^2}}, & x \in \mathbb{R}, \\ y(2) = 1. \end{cases}$$

Q6. Solve the differential equation

$$\frac{dy}{dx}e^{-x} = \frac{1}{e^{2x} + 2e^x + 1}.$$

Q7. Solve the boundary value problem

$$\frac{dy}{dx} = x\sqrt{4+x},$$

where $y(0) = -4$.

Q8. Solve the boundary value problem

$$\frac{dy}{dx} = \frac{1}{3}\sin(x)\sqrt{1+\cos(x)},$$

where $y(\pi) = 2$.

Q9. Solve the boundary value problem

$$\frac{dy}{dx}\frac{1}{2x-7} = \frac{1}{\sqrt{x^2-7x+9}},$$

where $y(-2) = 0$.

Q10. Solve the differential equation

$$\frac{dy}{dx} = \sqrt{9-y^2}\tan^2(4x).$$

Q11. Solve the differential equation

$$\frac{dy}{dx} = \cos(y).$$

Q12. Solve the differential equation

$$\frac{dy}{dx} = \frac{1}{x\sqrt{x^2+3}}.$$

Q13. Solve the differential equation

$$\frac{dx}{dy} = \sin^{-1}(y) + 4.$$

Q14. Solve the differential equation

$$\frac{d^2y}{dx^2} = 4\cos(x).$$

Q15. Solve the differential equation

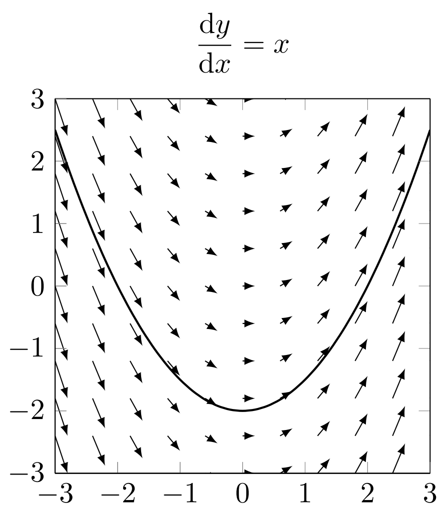
$$\frac{d^2y}{dx^2} = \sec^2(x) - x.$$

Q16. Solve the differential equation

$$\frac{d^2y}{dx^2} = x\sqrt{y}.$$

1.2 Direction Fields

In this section we look at how to graphically represent the behaviour of a differential equation. This is done through the method of direction fields. A direction field associates to every value of $(x, y) \in \mathbb{R}^2$ an arrow which describes the slope at that point. Below we have depicted the direction field associated to the differential equation $\frac{dy}{dx} = x$. The solid black line through the direction field denotes a particular solution corresponding to $y = \frac{1}{2}x^2 - 2$.



We do not wish to pursue the theory behind direction fields any further here.

Exercises

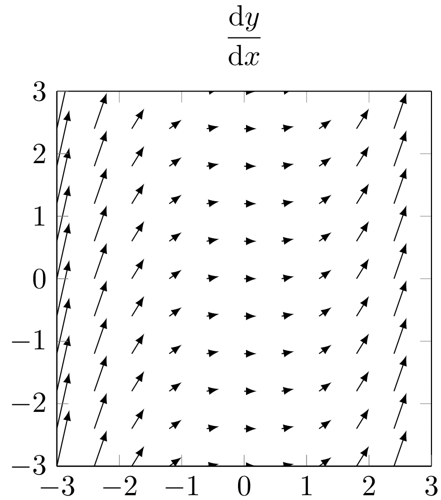
Q1. Draw the direction field associated to the differential equation $\frac{dy}{dx} = -x$.

Q2. Draw the direction field associated to the differential equation $\frac{dy}{dx} = x^2 - 1$.

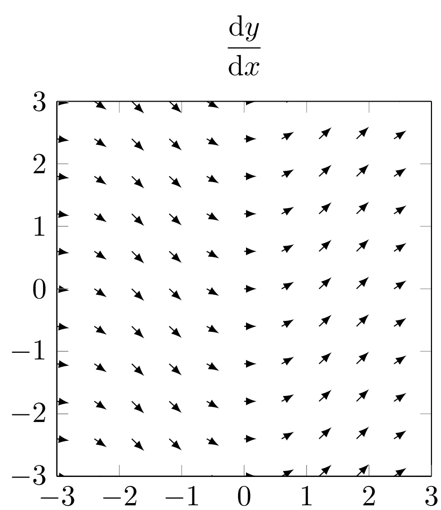
Q3. Draw the direction field associated to the differential equation $\frac{dy}{dx} = y$.

Q4. Draw the direction field associated to the differential equation $\frac{dy}{dx} = x - y$.

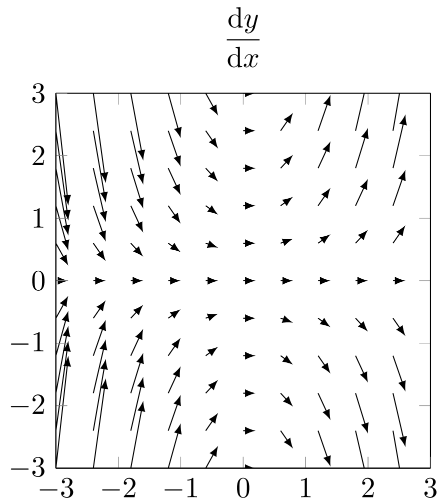
Q5. Determine the differential equation whose associated direction field is given by



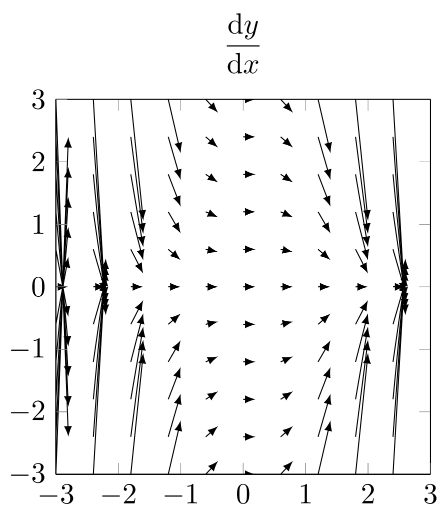
Q6. Determine the differential equation whose associated direction field is given by



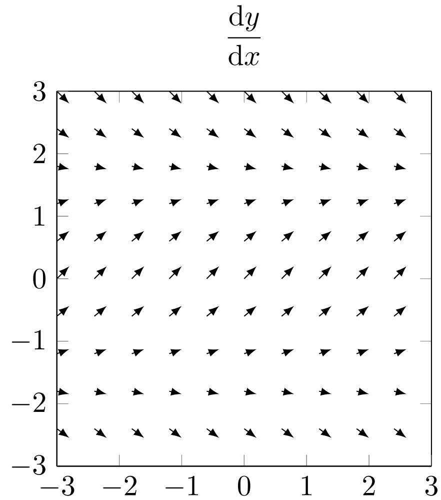
Q7. Determine the differential equation whose associated direction field is given by



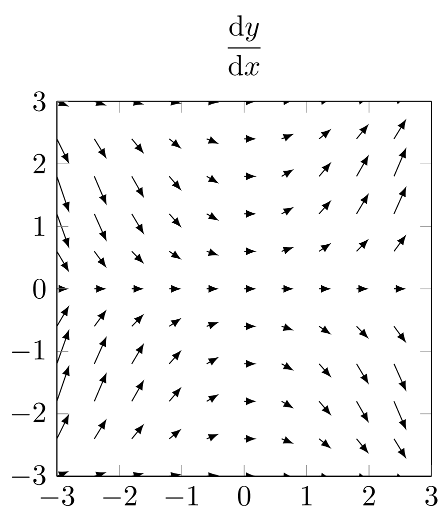
Q8. Determine the differential equation whose associated direction field is given by



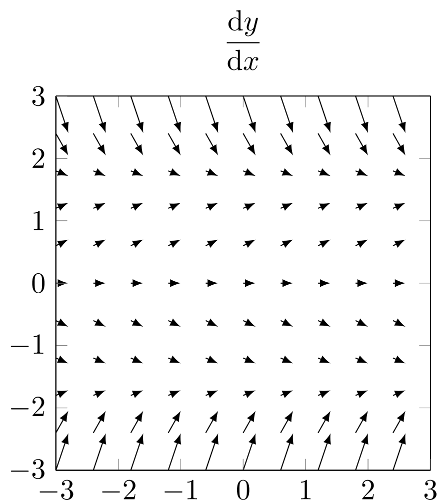
Q9. Determine the differential equation whose associated direction field is given by



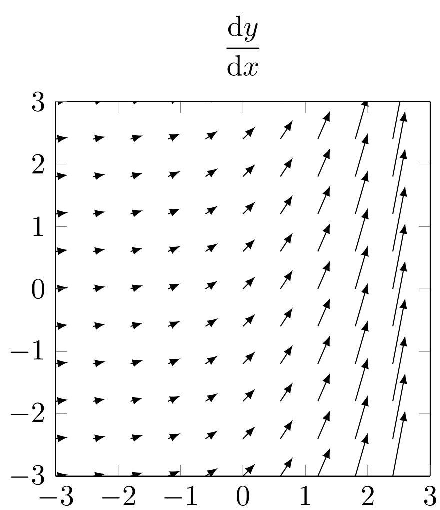
Q10. Determine the differential equation whose associated direction field is given by



Q11. Determine the differential equation whose associated direction field is given by



Q12. Determine the differential equation whose associated direction field is given by



1.3 Modelling with Differential Equations

"You better concentrate when you solve these problems, or you will get all mixed up." – Dr. Lloyd Gunatilake.

In this section we look at some differential equations that occur quite often in elementary applications. The first of these equations is the differential equation which models exponential growth and decay. If $\lambda \in \mathbb{R}$ is some fixed constant, then the general solution to the differential equation

$$\frac{dy}{dx} = \lambda y$$

is of the form

$$y = \mu e^{\lambda x},$$

where $\mu \in \mathbb{R}$ is some constant determined by the boundary conditions of the differential equation.

Example 10.3.1. The number of bacteria in the food you have purchased from the food court grows in proportion to the current number of bacteria present. If the number of bacteria $\mathcal{B}$ doubles every hour and t is measured in hours, determine the amount of bacteria in your food after 4 hours, assuming an initial amount of 3000 bacteria.

Proof. The differential equation that models the bacterial growth is given by

$$\frac{d\mathcal{B}}{dt} = 2\mathcal{B}.$$

Hence we see that

$$\begin{aligned}
\frac{d\mathcal{B}}{dt} = 2\mathcal{B} \quad &\Longrightarrow \quad \frac{1}{\mathcal{B}} d\mathcal{B} = 2dt \\
&\Longrightarrow \quad \int \frac{1}{\mathcal{B}} d\mathcal{B} = \int 2dt \\
&\Longrightarrow \quad \ln|\mathcal{B}| = 2t + k, \qquad k \in \mathbb{R} \\
&\Longrightarrow \quad \mathcal{B} = \pm e^{2t+k} \\
&\Longrightarrow \quad \mathcal{B} = Ae^{2t}, \qquad A := \pm e^{k}, \\
&\Longrightarrow \quad \mathcal{B} = 3000e^{2t},
\end{aligned}$$

where the last line follows from the fact that $\mathcal{B}(0) = 3000$. So by taking $t = 4$, we see that

$$\mathcal{B}(4) = 3000e^8 \approx 8.94 \times 10^6.$$

□

The next type of equation that we look at is Newton's law of cooling, which states that a body's temperature is proportional to the difference of the current temperature and the temperature of the surrounding. If T denotes the temperature of the body, T_s the temperature of the surrounding environment and t time, we have that

$$\frac{dT}{dt} = k(T - T_s),$$

for some $k \in \mathbb{R}$.

Example 10.3.2. Kuga likes his coffee quite hot in the morning, so he typically drinks his coffee at around 85° C. Unfortunately he has drink his coffee on the way to work during the cold mornings of Canberra. The outside temperature is given by 2° C and it takes only 4 minutes for Kuga's coffee to cool from 85° to 35°. Determine the temperature of Kuga's coffee after another 6 minutes.

Proof. We observe that T_s is given to be $T_s = 2$. Moreover, we see that

$$\begin{aligned}
\frac{dT}{dt} = k(T-2) \quad &\Longrightarrow \quad \frac{dt}{dT} = \frac{1}{k(T-2)} \\
&\Longrightarrow \quad dt = \frac{1}{k(T-2)} dT \\
&\Longrightarrow \quad \int dt = \int \frac{1}{k(T-2)} dT \\
&\Longrightarrow \quad t = \frac{1}{k} \ln|T-2| + T_0, \\
&\Longrightarrow \quad t - T_0 = \frac{1}{k} \ln|T-2| \\
&\Longrightarrow \quad \ln|T-2| = k(t - T_0) \\
&\Longrightarrow \quad T - 2 = \pm e^{k(t-T_0)} \\
&\Longrightarrow \quad T - 2 = Ae^{kt}, \qquad A := \pm e^{-kT_0}, \\
&\Longrightarrow \quad T = 2 + Ae^{kt} \\
&\Longrightarrow \quad T = 2 + 85e^{kt},
\end{aligned}$$

where the last equation follows from the fact that the initial temperature of the coffee is $85°$. Moreover, $T(4) = 35$; hence we see that

$$\begin{aligned} T(4) = 35 \quad &\Longrightarrow \quad 35 = 2 + 85e^{4k} \\ &\Longrightarrow \quad 85e^{4k} = 33 \\ &\Longrightarrow \quad e^{4k} = \frac{33}{85} \\ &\Longrightarrow \quad 4k = \ln\left(\frac{33}{85}\right) \\ &\Longrightarrow \quad k = \frac{1}{4}\ln\left(\frac{33}{85}\right). \end{aligned}$$

The temperature is therefore modelled by

$$T(t) = 2 + 85e^{\frac{t}{4}\ln\left(\frac{33}{85}\right)}.$$

Taking $t = 10$, we see that

$$T(10) \quad = \quad 2 + 85e^{\frac{10}{4}\ln\left(\frac{33}{85}\right)} \approx 9.98.$$

$\square$

The last type of equation that we look at here pertains to inflow/outflow mixing problems. These types of mixing problems allow us to describe concentration changes in a system. If A denotes the amount of a substance at time t, then

$$\frac{dA}{dt} = IR - OR,$$

where IR denotes the inflow rate and OR denotes the ourflow rate. We have the following relevant formulae,

$$\begin{aligned} IR \quad &= \quad \text{(inflow concentration)} \cdot \text{(inflow volume rate)}, \\ OR \quad &= \quad \text{(outflow concentration)} \cdot \text{(outflow volume rate)}. \end{aligned}$$

Example 10.3.3. A water tank holds 30 litres of water and has a salt solution of 2g/L flowing in at a rate of 3L per minute. Assume the mixture is uniform and leaks out at a rate of 4L per minute. Write out the associated differential equation that models this situation.

Proof. The initial amount of water in the tank is 30L, 3L flows in per minute and 4L flows out per minute. The volume of water contained in the tank at any time t is therefore given by

$$V(t) = 30 + 3t - 4t = 30 - t.$$

We next observe that the inflow rate is given by $IR = 2 \cdot 3 = 6$. The outflow rate is given by

$$OR = \frac{A}{30-t} \cdot 4 = \frac{4A}{30-t}.$$

So the differential equation is given by

$$\frac{dA}{dt} = 6 - \frac{4A}{30-t}.$$

□

Exercises

Q1. The half life of a radioactive isotope is given to be 4 hours. The initial amount of the isotope is 0.1 kg. Determine the amount of the isotope after 15 hours.

Q2. The number of antibodies released into the bloodstream during an allergic reaction grows exponentially. If the number of antibodies in the blood at the start of the reaction is 1000 and the number of antibodies doubles every 3 minutes. Determine the number of antibodies in the blood after 18 minutes.

Q3. Kate cooks her food in a large pot with her kitchen having a room temperature of 21° C. Upon taking the pot off the burner, she notes that the temperature of the pot is 95° and after 20 minutes it cools to 35° C. Determine the temperature of the pot 40 minutes after taking it off the burner.

Q4. Some Jelly is placed into a freezer which is set at a temperature of $-12°$ C. The initial temperature of the Jelly is 26° C and it cools to 21° C after being in the freezer for 10 minutes. How long does it take for the Jelly to reach 0° C?

Q5. Jamie is trying to make his own brand of soft drink in his bathtub. The bathtub has an initial 80L of carbonated water. Jamie adds 0.1 kg per litre of sugar at a rate of 4L per minute. The mixture is kept uniform and he allows the mixture to flow out at a rate of 6L per minute. Determine the differential equation that models the concentration rate and find the time it takes for the concentration of sugar in the bathtub to reach 0.6 kg/L.

Q6. A metal rod is heated to a temperature of 200° C and is placed in water which carries a temperatre of 1° C. Suppose the water warms to a temperature of 3° C in 1 minute.

a. Determine how long it will take for the temperature of the water to rise to 10°.

b. What temperature will the water stabilise to if the rod remains in the water for an indefinite amount of time?

Q7. A cylindrical vat holds 40L of water. A salt solution of 0.3 kg per litre flows into the vat a rate of 2L per minute. The mixture flows out with a concentration of 1kg/L. Determine the rate at which the mixture flows out.

Q8. Suppose we have 1kg of a particular isotope of plutonium and after 4 hours the mass of this isotope has decreased by 10 %.

a. Determine the half-life of the isotope. That is, find the time take for the isotope to reach half of its inital amount.

b. Determine the amount of the isotope remaining after 48 hours.

Q9. Cameron doesn't have a girlfriend, so he spends his Friday nights playing with the oven in his kitchen and a thermometer. Cameron observes that the thermometer initially reads 28° C. He then inserts the thermometer into his oven which has been preheated to a constant temperature. Through the oven door he notices that the temperature is 46° C after 10 minutes and 73° C after another 15 minutes. Determine the temperature that Cameron preheated the oven to.

Q10. A large vat is filled with 100 litres of water. Brine with with a salt concentration of 4g/L is pumped into the vat at a rate of 8 litres per minute. The mixture is pumped out at the same rate.

a. Determine how mcuh salt is in the vat at any given time t.

b. What is the concentration of salt in the vat in the long term?

1.4 ⋆ Euler's Method

So far we have discussed analytic solutions to ordinary differential equations. Euler's method will be our first introduction to numerical methods which are methods of obtaining approximations of solutions. Euler's method is an elementary numerical method which we will use to solve differential equations of the form

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0.$$

The ideal is simple and is given by

$$y_{n+1} = y_n + hf(x_n, y_n),$$

where $h > 0$ denotes the step size.

We note that the accuracy of the solution is increased as $h \to 0$.

Example 10.4.1. With a step size of $h = 0.2$ determine $y(1.6)$, where

$$\frac{dy}{dx} = x + y, \quad y(1) = e.$$

Proof. We observe that $x_0 = 1, x_1 = 1.2$, and $x_2 = 1.4$. Hence we see that

$$\begin{aligned}
y_1 &= y_0 + 0.2 \cdot f(x_0, y_0) \\
&= e + 0.2 \cdot (e + 1) \\
&= 1.2e + 0.2; \\
y_2 &= (1.2e + 0.2) + 0.2f(1.2, 1.2e + 0.2) \\
&= 1.2e + 0.2 + 0.2(1.2 + 1.2e + 0.2) \\
&= 0.48 + 1.64e; \\
y_3 &= (0.48 + 1.64e) + 0.2 \cdot f(1.4, 0.48 + 1.64e) \\
&= 0.48 + 1.64e + 0.2(1.4 + 0.48 + 1.64e) \\
&\approx 6.21.
\end{aligned}$$

□

Exercises

Q1. Let $h = 0.5$ be the step size for Euler's method. Determine the specified values of the approximate solutions for the given ordinary differential equations.

a. $\frac{dy}{dx} = x - 4$, $y(0) = 0$, $x = 1$.

b. $\frac{dy}{dx} = xy$, $y(1) = 1$, $x = 1.5$.

c. $\frac{dy}{dx} = \frac{1}{x+4y}$, $y(0) = 4$, $x = 2$.

d. $\frac{dy}{dx} = \sqrt{x^2 + y^2 + 1}$, $y(0) = 3$, $x = 2$.

e. $\frac{dy}{dx} = 1 - \sqrt{\cos(x - y)}$, $y(0) = 0$, $x = 1$.

Q2. Consider the differential equation

$$\frac{dy}{dx} = \frac{1}{\sqrt{9 - x^2}},$$

with $y(0) = 0$.

a. Solve the differential equation analytically.

b. Determine $y(2)$.

c. With a step size of 0.5 determine $\widetilde{y}(2)$, where $\widetilde{y}$ denotes the approximate soluton obtained using Euler's method.

1.5 Analysis Exercises

Q1. (The Logistic Equation). In the exercises throughout the main portion of this chapter, we saw population models that carried an exponential growth rate. True cases of this exponential growth however, over long periods of time, are rare since the resources in a given environment are often limited. Let us suppose that for a particular environment, the number of individuals in a population that may continue to live in this environment is fixed at some number K. This number K will be referred to as the *carrying capacity* of the environment. A logistic model of population growth factors in the carrying capacity of the environment and is given by

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right),$$

where $k > 0$ is a fixed constant with K denoting the carrying capacity. We note that if $P > K$, that is, the population has exceeded the carrying capacity, then $\frac{dP}{dt} < 0$ and so the population will decrease. If $P < K$, then $\frac{dP}{dt} > 0$ and so the population will increase until it reaches the carrying capacity. If $\frac{dP}{dt} = 0$, then $P = 0$ or $P = K$ and the system has reached an equillibrium.

a. Suppose that a population $P(t)$ is modelled by the Logistic equation and $P(0) = P_0$. Show that

$$P(t) = \frac{K}{1 + Ae^{-kt}},$$

where $A := \frac{1}{P_0}(K - P_0)$.

b. Evaluate the limit $\lim_{t\to\infty} P(t)$.

c. Suppose that the number of students who are exposed to a bacterial infection is governed by the logistic equation. The number of students in the school is 2000 and the initial number of infected students is 10. If the number of infected students $N(t)$ after t days is given by

$$N(t) = \frac{K}{1 + Ae^{-kt}},$$

determine the values of K, A and k if $N(2) = 150$. Assume the students are forced to attend school 7 days a week and are not allowed to leave.

i. Determine the maximum number of students who may be infected.
ii. Sketch the graph of $N(t)$ over a two week period.
iii. How many students are infected after 1 week?
iv. Determine the time t when the infection is spreading the fastest.

d. Suppose that the number of teenagers $P(t)$ who watch a particular video on the internet after t hours is modelled by the logistic equation. Suppose that within the first hour, the video has 500 views and after 4 hours, the video has reach 6000 views. Solve for $P(t)$ if the predicted limiting number of teenagers watching the video is 100,000.

Q2. (The Logistic Equation with Harvesting). (Dr. Lloyd Gunatilake). In this exercise we look at the logistic equation with a harvesting factor h,

$$\frac{dP}{dt} = kp\left(1 - \frac{P}{K}\right) - h,$$

where $k > 0, h > 0$ are constants and $K > 0$ is the carry capacity of the environment.

a. Every few years the International Whailing Commision meets to discuss the current ban on the hunting of certain species of Whales. Countries such as Norway, Japan and Iceland have proposed that harvesting a limited number of whales for a particular species will not affect the overall populations in the long term. Let us consider the harvesting of Blue Whales. Suppose that 30 Blue Whales are to be harvested each year. Scientists have devised the following model to predict the Global Blue Whale population P, with harvesting

$$\frac{dP}{dt} = 50P - 25P^2 - 30.$$

In this equation, each unit of time t equals 100 years. Today there are about 2500 Blue Whales in the oceans.

i. Show that

$$P(t) = \frac{1}{\sqrt{5}} \tan\left(-5^{\frac{3}{2}}t + \tan^{-1}(2499\sqrt{5})\right) + 1.$$

ii. For approximately how many years is this model valid?

iii. What is likely to happen to the Blue Whale population with continued harvesting? Explain this with a sketch of population versus time.

Q3. (Air Resistance). Suppose a particle of mass m falls with a velocity v. The air resistance that the particle is subject to is proportional to the square of the instantaneous velocity and some elementary mechanics tells us that

$$m\frac{dv}{dt} = mg - kv^2,$$

where g is gravity and $k > 0$ is some constant.

a. Solve the equation for the velocity of the particle if $m = 0.1$ kg, $k = \frac{1}{2}$ and $v(0) = 10$.

b. Determine the terminal velocity of the particle. That is, evaluate the limit $\lim_{t\to\infty} v(t)$.

Q4. (Predator-Prey Models). Suppose the number of foxes $F(t)$ and the number of chickens $C(t)$ at any given time t is modelled by the pair of differential equations

$$\frac{dC}{dt} = k_1 C\left(1 - \frac{F}{m}\right), \qquad \frac{dF}{dt} = k_2 F\left(\frac{C}{n} - 1\right),$$

where k_1, k_2, m, n are positive constants. Observe that if the number of foxes is zero, $F(t) = 0$, then

$$\frac{dC}{dt} = k_1 C,$$

and so the chicken population grows exponentially. Similarly, if the number of chickens is zero, $C(t) = 0$, then

$$\frac{dF}{dt} = -k_2 F,$$

and so the fox population decays exponentially.

a. Show that

$$\frac{dF}{dC} = \frac{k_2 F\left(\frac{C}{n} - 1\right)}{k_1 C\left(1 - \frac{F}{m}\right)}.$$

b. Hence, or otherwise, show that

$$k_1 \ln|F| - \frac{1}{m}k_1 F = \frac{1}{n}k_2 C - k_2 \ln|C| + \text{constant}.$$

Q5. (Beyond Separation of Variables). Consider the ordinary differential equation

$$x\frac{dy}{dx} - 2y = x^4 e^x.$$

a. Show that

$$\frac{dy}{dx} - \frac{2}{x}y = x^3 e^x.$$

b. Evaluate

$$u(x) = e^{\int -\frac{2}{x}dx},$$

where $u(1) = 0$.

c. Hence, or otherwise, show that by multiplying every term of $\frac{dy}{dx} - \frac{2}{x}y = x^5 e^x$ by $u(x)$, we obtain the ordinary differential equation

$$x^{-2}\frac{dy}{dx} - 2x^{-3}y = xe^x.$$

d. Show that

$$\frac{d}{dx}[x^{-2}y] = xe^x.$$

e. Hence, or otherwise, show that

$$y = x^5 e^x - x^2 e^x + kx^2,$$

for some $k \in \mathbb{R}$.

Q6. (Beyond Separation of Variables, continued). Using the procedured outlined in the previous exercise, solve the ordinary differential equation

$$x\frac{dy}{dx} - y = x^2\cos(x).$$