

Elementary Function Theory – Lecture 2

We continue with some of the exercises that remained from last lecture.

Exercise 1. Let $f(x) := x^2 + bx + c$, where $b, c \in \mathbb{R}$. Show that the roots of f , i.e., the solutions of $f(x) = 0$ are determined by the

$$x = \frac{1}{2} \left(-b \pm \sqrt{b^2 - 4c} \right).$$

Exercise 2. Calculate the discriminant of the following quadratics. Hence, determine the number of roots of each quadratic (over \mathbb{R}).

(a) $x^2 + 4x - 1$.

(b) $x^2 - 5x + 6$.

(c) $1 - x^2$.

Exercise 3. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a quadratic with two roots, determine the sign of the discriminant.

Exercise 4. Complete the square for the following quadratic equations.

(a) $f(x) = x^2 + 2x + 1.$

(b) $f(x) = 3x^2 - 8x + 1.$

(c) $f(x) = 5 - x + x^2$.

Exercise 5. Determine the value(s) of $k \in \mathbb{R}$ such that $f(x) = x^2 - kx$ intersects $g(x) = 2x - 3$ exactly once.

Exercise 6. Determine the value(s) of $k \in \mathbb{R}$ such that $f(x) = 4x - k$ intersects $g(x) = kx^2 - 1$ two times.

Exercise 7. The volume of water in a tank, V m³, over a 10 month period is given by the function

$$V(t) = 2t^2 - 16t + 40,$$

where t is the number of months, $0 \leq t \leq 10$.

(a) Determine the initial volume of water in the tank.

(b) Determine the maximum volume of water in the tank.

(c) Determine the minimum volume of water in the tank.

Exercise 8. A section of a roller-coaster at an amusement park follows a parabolic path. The function

$$h(t) = t^2 - 12t + 48,$$

where $0 \leq t \leq 11$, models the height above the ground of the front of one of the carriages, where t is the time in seconds and h is the height in meters.

- (a) Find the lowest point of this section of the ride.

- (b) Find the time taken for the carriage to reach the lowest point.

- (c) Find the highest point above the ground.

(d) Find the domain and range of the function.

(e) Sketch the function.

Exercise 9. Sketch the graph of

$$f(x) = \frac{1}{x^2 + 3x - 10},$$

stating the domain and range.

Exercise 10. Sketch the graph of

$$f(x) = \frac{1}{x^2 + 9x + 18},$$

stating the domain and range.

Exercise 11. Sketch the graph of

$$f(x) = x^3 - 5x^2 + 11x - 6.$$

Exercise 12. Sketch the graph of

$$f(x) = x^3 + 3x^2 + 3x + 1.$$

Exercise 13. Sketch the graph of

$$f(x) = x^3 - 6x^2 + 5x + 12.$$

Exercise 14. Sketch the graph of

$$f(x) = \frac{1}{x^4 - 14x^3 + 73x^2 - 168x + 144}.$$