Calculus Practice Exam 5 Solutions

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In this practice exam, we attempt to explore an application of calculus. That is, finding maxima, minima and inflection points of functions $f : \mathbb{R} \to \mathbb{R}$.

Exercise 1. Determine the stationary points of the following functions.

a. $f(x) = x^2 - 5x + 6$.

Proof. The derivative is given by $f'(x) = 2x - 5$. So

$$f'(x) = 2x - 5 = 0 \implies x = \frac{5}{2}.$$

Inserting $x = \frac{5}{2}$ into $f(x)$, we see that the stationary point occurs at $\left(\frac{5}{2}, -\frac{1}{4}\right)$. $\square$

b. $f(x) = x^2 + 2x + 1$.

Proof. The derivative is given by $f'(x) = 2x + 2$. So

$$f'(x) = 2x + 2 = 0 \implies x = -1.$$

Inserting $x = -1$ into $f(x)$, we see that the stationary point occurs at $(-1, 0)$. $\square$

c. $f(x) = x^3 + 2x + 1$.

Proof. The derivative is given by $f'(x) = 3x^2 + 2$. So

$$f'(x) = 3x^2 + 2 = 0 \implies x \notin \mathbb{R}.$$

There are therefore no stationary points of f. $\square$

d. $f(x) = x^4 + 1$.

Proof. The derivative is given by $f'(x) = 4x^3$. So

$$f'(x) = 4x^3 = 0 \implies x = 0.$$

Inserting $x = 0$ into $f(x)$, we see that the stationary point occurs at $(0, 1)$. $\square$

e. $f(x) = x^3 - 2x^2 + x + 1$.

Proof. The derivative is given by $f'(x) = 3x^2 - 4x + 1$. So

$$\begin{aligned} f'(x) = 3x^2 - 4x + 1 = 0 \quad &\implies \quad x = \frac{4 \pm \sqrt{16 - 4(3)(1)}}{2(3)} \\ &\implies \quad \frac{4 \pm \sqrt{4}}{6} \\ &\implies \quad \frac{4 \pm 2}{6} \\ &\implies \quad x = 1, \text{ or } x = \frac{1}{3}. \end{aligned}$$

Inserting $x = 1$ into $f(x)$, we see that the stationary point occurs at $(1, 1)$ and inserting $x = \frac{1}{3}$ into $f(x)$, we see that the other stationary point is given by $\left(\frac{1}{3}, \frac{31}{27}\right)$. $\square$

f. $f(x) = 4x^3 - 12x$.

Proof. The derivative is given by $f'(x) = 12x^2 - 12$. So

$$f'(x) = 12x^2 - 12 = 0 \implies x^2 = 1 \implies x = \pm 1.$$

Hence we see that the stationary points of f occur at $(-1, 8)$ and $(1, -8)$. $\square$

g. $f(x) = 3x^3 - x^2 + 4x - 5$.

Proof. The derivative is given by $f'(x) = 9x^2 - 2x + 4$. So

$$f'(x) = 9x^2 - 2x + 4 = 0 \implies x = \frac{2 \pm \sqrt{4 - 4(9)(4)}}{2(9)} \notin \mathbb{R}.$$

There are therefore no stationary points of f. $\square$

h. $f(x) = 2x^3 + 5x + \sqrt{3}$.

Proof. The derivative of f is given by $f'(x) = 6x^2 + 5$. So

$$f'(x) = 6x^2 + 5 = 0 \implies x \notin \mathbb{R}.$$

Again we see that f has no stationary points. $\square$

Exercise 2. Determine the stationary points of the following functions and use the second derivative test to determine the nature of stationary points.

a. $f(x) = x^3 - 10x^2 + 31x - 30$.

Proof. The derivative is given by $f'(x) = 3x^2 - 20x + 31$. So

$$f'(x) = 3x^2 - 20x + 31 = 0 \implies x = \frac{20 \pm \sqrt{400 - 4(3)(31)}}{2(3)} = \frac{1}{3}\left(10 \pm \sqrt{7}\right).$$

The stationary points occur at

$$\left(\frac{1}{3}(10 - \sqrt{7}), \frac{2}{27}(-10 + 7\sqrt{7})\right), \quad \text{and} \quad \left(\frac{1}{3}(10 + \sqrt{7}), -\frac{2}{27}(10 + 7\sqrt{7})\right).$$

To determine the nature of these stationary points, we consider the second derivative, which is given by $f''(x) = 6x - 20$. At $x = \frac{1}{3}\left(10 - \sqrt{7}\right)$, we see that $f''(x) < 0$, so a local maximum occurs here. For $x = \frac{1}{3}\left(10 + \sqrt{7}\right)$, we see that $f''(x) > 0$, so a local minimum occurs here. $\square$

b. $f(x) = x^3 + 5x^2 - 8x - 12$.
c. $f(x) = x^3 - 4x^2 - 39x - 54$.
d. $f(x) = x^3 + 10x^2 - x - 10$.

Exercise 3. Determine the stationary points of the following functions.

a. $f(x) = \sin x$.

Proof. The derivative is given by $f'(x) = \cos x$. So

$$f'(x) = \cos x = 0 \implies x = \frac{k\pi}{2}, \quad k \in \mathbb{Z}.$$

$\square$

b. $f(x) = \cos x$.

Proof. The derivative is given by $f'(x) = -\sin x$. So

$$f'(x) = -\sin x = 0 \implies x = k\pi, \quad k \in \mathbb{Z}.$$

$\square$

c. $f(x) = 4\sin(3x - \pi)$.

Proof. The derivative is given by $f'(x) = 12\cos(3x - \pi)$. So

$$\begin{aligned} f'(x) = 12\cos(3x-\pi) = 0 \quad &\implies \quad 3x - \pi = \frac{k\pi}{2}, k \in \mathbb{Z} \\ &\implies \quad 3x = \frac{(k+2)\pi}{2}, k \in \mathbb{Z}, \\ &\implies \quad x = \frac{(k+2)\pi}{6}. \end{aligned}$$

$\square$

d. $f(x) = \frac{2}{5}\cos\left(\frac{1}{2} - x\right)$.

Proof. The derivative is given by $f'(x) = \frac{2}{5}\sin\left(\frac{1}{2} - x\right)$. So

$$\begin{aligned} f'(x) = \frac{2}{5}\sin\left(\frac{1}{2} - x\right) = 0 \quad &\implies \quad \frac{1}{2} - x = k\pi, k \in \mathbb{Z}, \\ &\implies \quad x = \frac{1}{2} - k\pi, \ k \in \mathbb{Z}. \end{aligned}$$

$\square$

Exercise 4. Determine the stationary points of the following functions.

a. $f(x) = x\sin x$.

Proof. The derivative is given by $f'(x) = x\cos x + \sin x$. So

$$\begin{aligned} f'(x) = x\cos x + \sin x = 0 \quad &\implies \quad x\cos x = -\sin x \\ &\implies \quad x = -\tan x. \end{aligned}$$

Solving this type of equation is rather difficult, so an acceptable solution for our mathematicial abilities here would be just to assert that $x = 0$ is a solution. $\square$

b. $f(x) = x^2\cos(-x)$.

Proof. The derivative is given by $f'(x) = x^2 \sin(-x) + 2x\cos(-x)$. So

$$f'(x) = x^2\sin(-x) + 2x\cos(-x) = 0 \quad \Longrightarrow \quad x^2\sin(-x) = -2x\cos(-x).$$

Just as what we did in the previous exercise, $x = 0$ is sufficient. □

c. $f(x) = e^x \sin x$.

Proof. The derivative is given by $f'(x) = e^x\cos x + e^x \sin x$. So

$$\begin{aligned} f'(x) = e^x(\cos x + \sin x) = 0 \quad &\Longrightarrow \quad \cos x + \sin x = 0 \\ &\Longrightarrow \quad \sin x = -\cos x \\ &\Longrightarrow \quad \tan x = -1 \\ &\Longrightarrow \quad x = \frac{3\pi}{4} + k\pi,\ \ k \in \mathbb{Z}. \end{aligned}$$

□

d. $f(x) = e^{-x}\cos(x^2)$.

Proof. The derivative is given by $f'(x) = -2xe^{-x}\sin(x^2) - e^{-x}\cos(x^2)$. So

$$f'(x) = e^{-x}(-2x\sin(x^2) - \cos(x^2)) = 0 \quad \Longrightarrow \quad -2x\sin(x^2) = \cos(x^2).$$

Taking $x = 0$ is a sufficient solution for our purposes here. □

Exercise 5.

a. Let $f(x) := \tan x$. Determine where f is differentiable.

Proof. The function $f(x) = \tan x$ is a composition of $\sin x$ and $\cos x$. Both of these functions are differentiable for all $x \in \mathbb{R}$ and so $\tan x$ is differentiable at all points where $\cos x \neq 0$. Therefore $f(x)$ is differentiable for all $x \in \mathbb{R} \backslash \{x \in \mathbb{R} : x = \frac{k\pi}{2}, k \in \mathbb{Z}\}$. □

b. Evaluate $f'(x)$ at all points that f was determined to be differentiable in part (a).

Proof. We simply observe that

$$\begin{aligned} \frac{d}{dx}\tan x \quad &= \quad \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{d}{dx}\sin x \cdot (\cos x)^{-1} \\ &= \quad \sin x \cdot (-1) \cdot (-\sin x) \cdot \frac{1}{\cos^2 x} + \cos x \cdot (\cos x)^{-1} \\ &= \quad \frac{\sin^2 x}{\cos^2 x} + 1 \\ &= \quad \frac{1}{\cos^2 x}\left(\sin^2 x + \cos^2 x\right) \\ &= \quad \frac{1}{\cos^2 x} =: \sec^2 x. \end{aligned}$$

□

c. Hence, or otherwise, determine the stationary points of $g(x) := \tan^2 x$.

Proof. The derivative is given by $f'(x) = 2\sec^2 x \tan x$. So

$$\begin{aligned} f'(x) = 2\sec^2 x \tan x = 0 &\implies \tan x = 0 \\ &\implies x = k\pi, k \in \mathbb{Z}. \end{aligned}$$

□

d. Using the second derivative test, or otherwise, determine the nature of the stationary points that were determined in part (c).

Proof. The second derivative of $\tan^2 x$ is given by

$$f''(x) = \frac{d}{dx}(2\sec^2 x \tan x) = -2(\cos(2x) - 2)\sec^4 x$$

Since we have a change of sign about $x = k\pi$, we have stationary points of inflection at $x = k\pi, k \in \mathbb{Z}$. □

Exercise 6. Investigate the consequence of assuming[1] that $f''(x) = 0$ implies that f has a points of inflection at x by considering the functions

$$f(x) = x^4, \quad \text{and} \quad f(x) = x^{\frac{1}{3}}.$$

[For more information on this, see Chapter 5.3 of *Introduction to Analysis*, Kyle Broder.]

Proof. A common misconception is that the $f''(x) = 0 \implies f(x)$ has a point of inflection at $x = 0$. These two examples show that this is monsterously falacious. Indeed, consider the function $f(x) = x^4$, which has first derivative $f'(x) = 4x^3$ and second derivative $12x^2$. Both $f'(x) = f''(x) = 0$ at $x = 0$. Consider a sketch of f is necessary however, we see that f has a minimum at $x = 0$, not an inflection point. This is because the second derivative does not change sign at $x = 0$.

Similarly, consider the function $f(x) = x^{\frac{1}{3}}$. The derivative is given by $f'(x) = \frac{1}{3x^{\frac{2}{3}}}$ and the second derivative is given by $f''(x) = -\frac{2}{9x^{\frac{5}{3}}}$. The second derivative is not even defined at $x = 0$, but f has an inflection point at $x = 0$! □

Exercise 7. Let $\psi : [0, 4\pi] \to \mathbb{R}$ be the function defined by $\psi : x \mapsto x - \sin x$. Determine whether ψ has any points of inflection. Are any of these points stationary?

Proof. The derivative is given by $\psi'(x) = 1 - \cos x$. So

$$\psi'(x) = 1 - \cos x = 0 \implies \cos x = 1 \implies x = 2k\pi, k \in \mathbb{Z}.$$

The second derivative is given by $\psi''(x) = \sin x$. The function $\sin x$ changes sign at $x = k\pi$ for $k \in \mathbb{Z}$. So ψ has a point of inflection at $x = k\pi, k \in \mathbb{Z}$ and stationary points of inflection at $x = 2k\pi, k \in \mathbb{Z}$. □

[1] incorrectly

Exercise 8. Let $\xi : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$\xi : x \longmapsto \frac{1}{x^2 + x + 1}.$$

a. Determine the coordinates of the inflection points of ξ.
b. Find the coordinates of the point of intersection of the tangents at the points of inflection.

Exercise 9. Show that for all $a, b, c \in \mathbb{R}$, the function $f(x) := ax^2 + bx + c$ has no points of inflection.

Exercise 10. Determine the points of inflection of the function

$$f(x) := e^{-x^2} \sin\left(\frac{1}{x}\right).$$

$\star$ Can you determine the limit: $\lim_{x \to 0} f(x)$?

Exericse 11. Determine the nature of the stationary points of

$$f(x) := x^2 \log_e(\log_e(x)) + 9 \log\left(\frac{3}{2}\right).$$

Proof. We simply observe that

$$f'(x) \quad = \quad \cdot\frac{x}{\log_e(x)} + 2x \log_e(\log_e(x)).$$

So f has a stationary point at $x = 0$, which will be the only stationary of f that we consider. $\square$

Exercise 12. The radius $r > 0$ and height $h > 0$ of a solid circular cylinder $\mathscr{C}$ vary in such a way that the volume of the cylinder is always 250π.

a. Show that the total surface area $\mathscr{A}$ of the cylinder is given by

$$\mathscr{A} = 2\pi r^2 + \frac{500\pi}{r}.$$

b. What is the minimum surface area of $\mathscr{C}$?