Calculus Practice Exam 3

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This practice exam aims to emphasise the differentiation techniques that allow us to differentiate exponential, logarithmic and trigonometric functions. The main technique at our disposal will be the chain rule. Recall that in the notes, we established that

$$\begin{aligned} f(x) = e^x &\implies f'(x) = e^x, \\ f(x) = \log_e x &\implies f'(x) = \frac{1}{x}, \quad x > 0, \\ f(x) = \sin x &\implies f'(x) = \cos x, \\ f(x) = \cos x &\implies f'(x) = -\sin x. \end{aligned}$$

Question 1. Using the chain rule, differentiate the following functions.

a. $f(x) = e^{3-x} + 1$.

Proof. It is immediate that

$$f'(x) = -e^{3-x}.$$

□

b. $f(x) = e^{2x-1} + 4x + 12$.

Proof. It is immediate that

$$f'(x) = 2e^{2x-1} + 4.$$

□

c. $f(x) = e^{7x+2} + 3$.

Proof. It is immediate that

$$f'(x) = 7e^{7x+2}.$$

□

d. $f(x) = \frac{1}{3}e^{4-x} + \frac{3}{7}$.

Proof. It is immediate that

$$f'(x) = -\frac{1}{3}e^{4-x}.$$

□

e. $f(x) = \sqrt{3}e^{4x+13} + 2$.

Proof. It is immediate that

$$f'(x) = 4\sqrt{3}e^{4x+13}.$$

□

Question 2. Using the chain rule, differentiate the following functions.

a. $f(x) = e^{\sqrt{x+1}}$.

Proof. We simply observe that

$$f'(x) \quad = \quad \frac{1}{2\sqrt{x+1}} e^{\sqrt{x+1}}.$$

□

b. $f(x) = e^{x^2-5x+6}$.

Proof. We simply observe that

$$f'(x) \quad = \quad (2x-5)e^{x^2-5x+6}.$$

□

c. $f(x) = 3e^{4x^2+\sqrt{x-3}+2} + 3x + 5$.

Proof. We simply observe that

$$f'(x) \quad = \quad 3 \cdot \left(8x + \frac{1}{2\sqrt{x-3}}\right) e^{4x^2+\sqrt{x-3}+2} + 3.$$

□

d. $f(x) = \frac{4}{3}e^{2(x-5)^3+1} + 4$.

Proof. We simply observe that

$$\begin{aligned} f'(x) \quad &= \quad \frac{4}{3} \cdot 2 \cdot 3 \cdot (x-5)^2 \cdot e^{2(x-5)^3+1} \\ &= \quad 8(x-5)^2 e^{2(x-5)^3+1}. \end{aligned}$$

□

e. $f(x) = \frac{\sqrt{2}}{5}e^{-x} + 4x^3$.

Proof. We simply observe that

$$f'(x) \quad = \quad -\frac{\sqrt{2}}{5}e^{-x} + 12x^2.$$

□

Question 3. Consider the function

$$f(x) = \frac{1}{\sqrt{3}\exp\left(\sqrt{5x+x^2-1}\right)}.$$

Evaluate $f'(x)$.

Proof. Let us begin by writing

$$f(x) = \frac{1}{\sqrt{3}} e^{-\sqrt{5x+x^2-1}}.$$

Then it is easy to see that

$$f'(x) = \frac{1}{\sqrt{3}} \cdot \left(-\frac{2x+5}{2\sqrt{x^2+5x-1}}\right) e^{-\sqrt{5x+x^2-1}}$$

□

Question 4. Using the chain rule, differentiate the following functions.

a. $f(x) = \log_e(x-3) + 1$.

Proof. It is easy to see that

$$f'(x) = \frac{1}{x-3}.$$

□

b. $f(x) = \frac{1}{3}\log_e(4-2x) + 4$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= \frac{1}{3} \cdot \frac{-2}{4-2x} \\ &= -\frac{1}{3(2-x)}. \end{aligned}$$

□

c. $f(x) = \frac{3}{5}\log_e(x+7) + 2x + 3$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= \frac{3}{5} \cdot \frac{1}{x+7} + 2 \\ &= \frac{3}{5(x+7)} + 2. \end{aligned}$$

□

d. $f(x) = \sqrt{3}\log_e(4x+5) + x^2$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= \sqrt{3} \cdot \frac{4}{4x+5} + 2x \\ &= \frac{4\sqrt{3}}{4x+5} + 2x. \end{aligned}$$

□

e. $f(x) = \log_e(x) + 1$.

Proof. It is easy to see that

$$f'(x) = \frac{1}{x}.$$

□

Question 5. Consider the function

$$f(x) = \frac{2}{3\log_e(x)}.$$

Evaluate $f'(x)$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{2}{3} \cdot (\log_e(x))^{-1}\right) \\ &= \frac{2}{3} \cdot (-1) \cdot \frac{1}{x} \cdot \frac{1}{\log_e^2(x)} \\ &= -\frac{2}{3x\log_e^2(x)}. \end{aligned}$$

□

Question 6. Let $|x|$ denote the absolute value function and consider the function

$$f(x) := \log_e |x|.$$

Evaluate $f'(x)$ and state the exact domain on which f is differentiable.

Proof. Recall the theory that was discussed in Practice Exam 1, we see that

$$f'(x) = \frac{1}{x}, \quad x \in \mathbb{R}\backslash\{0\}.$$

□

Question 7. Using the chain rule, differentiate the following functions.

a. $f(x) = \sin(x + \pi) - 3$.

Proof. It is easy to see that

$$f'(x) = \cos(x + \pi).$$

□

b. $f(x) = 2\cos(x - \pi/2) + 1$.

Proof. It is easy to see that

$$f'(x) = -2\sin(x - \pi/2).$$

□

c. $f(x) = 3\sin(2(x + \pi)) + x$.

Proof. It is easy to see that

$$f'(x) = 6\cos(2(x + \pi)) + 1.$$

□

d. $f(x) = \frac{1}{4}\cos(-x) + 3x + 5 + \frac{1}{x^2}$.

Proof. It is easy to see that

$$f'(x) = \frac{1}{4}\sin(-x) + 3 - \frac{3}{x^3}.$$

□

e. $f(x) = \sin^2(x) + \cos^2(x)$.

Proof. Using the Pythagorean identity, it is easy to see that

$$f'(x) = 0.$$

□

f. $f(x) = \cos^3(x) + \sin^2(x - \pi)$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= 3 \cdot (-\sin x) \cdot \cos^2(x) + 2 \cdot \cos(x - \pi) \cdot \sin(x - \pi) \\ &= -3\sin(x)\cos^2(x) + \sin(2(x - \pi)). \end{aligned}$$

□

g. $f(x) = 2\sin^3(x) + 5x - 3$.

Proof. It is easy to see that

$$f'(x) = 6\cos(x) \cdot \sin^2(x) + 5$$

□

Question 8. Consider the function

$$f(x) := \sec(x) := \frac{1}{\cos x}.$$

a. Determine the maximal domain on which $f(x)$ is defined.

Proof. The function $f(x) = \sec(x)$ is continuous everywhere that $\cos(x)$ is nonzero. We know from our studies of the $\cos(x)$ function that $\cos(x) = 0$ exactly when

$$x = k\pi, \ \ k \in \mathbb{Z}.$$

Therefore, $f(x)$ is continuous for all $x \in \mathbb{R} \backslash \mathcal{S}$, where $\mathcal{S} := \{y \in \mathbb{R} : y = k\pi, \ \ k \in \mathbb{Z}\}$. □

b. Evaluate $f'(x)$.

Proof. It is easy to see that

$$\begin{aligned} f'(x) &= \frac{d}{dx}(\cos(x))^{-1} \\ &= (-1) \cdot (-\sin x) \cdot (\cos(x))^{-2} \\ &= \frac{\sin x}{\cos^2 x} \\ &= \tan x \sec x. \end{aligned}$$

□

c. Let $g(x)$ be the function given by applying the following transformations to $f(x)$:

1. Dilate by factor 3 from the x-axis.
2. Reflect about the y-axis.
3. Translate by 2 units in the positive x-direction.

Write the equation for $g(x)$.

Proof. Step 1 takes $\sec(x) \longmapsto 3\sec(x)$. Step 2 takes $3\sec(x) \longmapsto 3\sec(-x)$. Step 3 takes $3\sec(-x) \longmapsto 3\sec(-(x-2) = 3\sec(2-x)$. Therefore,

$$g(x) = 3\sec(2-x).$$

□

d. Evaluate $g'(x)$.

Proof. It is easy to see that

$$g'(x) = -3\tan(2-x)\sec(2-x).$$

□

Question 9. Consider the function

$$f(x) := \csc(x) := \frac{1}{\sin x}.$$

a. Determine the maximal domain of $f(x)$.

Proof. The function $\csc(x)$ is defined for all $x \in \mathbb{R}$ such that $\sin(x) \neq 0$. From our studies of trigonometry, we know that $\sin(x) = 0$ exactly when $x = k\pi, k \in \mathbb{Z}$. Therefore, the maximal domain of $f(x)$ is $\mathbb{R} \backslash \mathcal{S}$, where $\mathcal{S} = \{x \in \mathbb{R} : x = k\pi, k \in \mathbb{Z}\}$. □

b. Evaluate $f'(x)$.

Proof. Begin by writing $f(x) = (\sin x)^{-1}$. Then by the chain rule it follows immediately that

$$f'(x) = -\cos x \sin x = -\frac{1}{2} \sin 2x.$$

□

Question 10. Consider the functions

$$\sinh(x) := \frac{e^x - e^{-x}}{2} \text{ and } \cosh(x) := \frac{e^x + e^{-x}}{2}.$$

a. Show that the derivative of $\sinh(x)$ is $\cosh(x)$.

Proof. We simply compute

$$\begin{aligned} \frac{d}{dx} \sinh(x) &= \frac{1}{2} \frac{d}{dx} (e^x - e^{-x}) \\ &= \frac{1}{2} (e^x + e^{-x}) \\ &= \cosh(x). \end{aligned}$$

□

b. Determine the derivative of $\cosh(x)$.

Proof. Using similar reasoning to that which was given in part (a), we see that

$$\begin{aligned} \frac{d}{dx} \cosh(x) &= \frac{1}{2} \frac{d}{dx} (e^x + e^{-x}) \\ &= \frac{1}{2} (e^x - e^{-x}). \end{aligned}$$

□

Question 11. Using the chain rule, differentiate the following functions.

a. $f(x) = \sinh(2x + 4) + 4x + 6$.

Proof. In the previous exercise we established that the derivative of $\sinh(x)$ was $\cosh(x)$, and the derivative of $\cosh(x)$ was $\sinh(x)$. Therefore, using the chain rule, we see that

$$f'(x) = 2\cosh(2x + 4) + 4.$$

□

b. $f(x) = 3\cosh(4x - 3) + 4x^2$.

Proof. Similarly,

$$f'(x) = 12\sinh(4x - 3) + 8x.$$

□

Question 12. Consider the function

$$f(x) = \sinh(\sqrt{x}) = \frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2}.$$

Evaluate $f'(x)$.

Proof. Using the chain rule, we simply observe that

$$f'(x) = \frac{1}{2\sqrt{x}}\cosh(\sqrt{x}).$$

□

Question 13. Consider the function

$$f(x) = \sin(x)\cos(x).$$

Evaluate $f'(x)$.

Proof. We may write $f(x) = \frac{1}{2}\sin(2x)$. Therefore,

$$f'(x) = \cos(2x).$$

□