Complex Numbers

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Introduction to Complex Numbers

The complex numbers $\mathbb{C}$ consist of elements of the form $x + iy$, where x and y are real numbers. In set notation, we would write

$$\mathbb{C} \quad = \quad \{x + iy : x, y \in \mathbb{R}\}.$$

For a complex number $z = x + iy$, we call x the **real part** and call y the **imaginary part**.

We think of $\mathbb{C}$ as a similar object to $\mathbb{R}^2$, where $\mathbb{R}^2 \cong \mathbb{R} \times \mathbb{R} = \{(x, y) : x, y \in \mathbb{R}\}$. On the face of it[1], $\mathbb{C}$ is equivalent to $\mathbb{R}^2$. What is different between $\mathbb{C}$ and $\mathbb{R}^2$ is the algebraic structure. That is, how do we add and multiply elements in $\mathbb{C}$ compared to how we add and subtract elements in $\mathbb{R}^2$. It is simple to add elements in $\mathbb{R}^2$. For example, let (x_1, y_1) and (x_2, y_2) be elements of $\mathbb{R}^2$. Then we may define $+$ on $\mathbb{R}^2$ to simply be

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2).$$

Multiplication however is not so simple. This is where the number $i = \sqrt{-1}$ saves us. With the use of i, we may define multiplication in a "reasonable" way.

Question 1.1. Determine the real and imaginary parts of the following complex numbers.

a. $z = 2 + 7i$.
b. $z = 4 - \frac{3}{2}i$.
c. $z = 2i$.

Example 1.2. Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Evaluate

a. $z_1 + z_2$.

Proof.

$$\begin{aligned} z_1 + z_2 &= (x_1 + iy_1) + (x_2 + iy_2) \\ &= (x_1 + x_2) + i(y_1 + y_2). \end{aligned}$$

$\square$

b. $z_1 \cdot z_2$.

Proof.

$$\begin{aligned} z_1 \cdot z_2 &= (x_1 + iy_1) \cdot (x_2 + iy_2) \\ &= x_1 \cdot x_2 + x_1 \cdot iy_2 + iy_1 \cdot x_2 + iy_1 \cdot iy_2 \\ &= x_1 \cdot x_2 - y_1 y_2 + i[x_1 y_2 + y_1 x_2] \end{aligned}$$

[1] topologically

$\square$

c. $z_1 - z_2$.

Proof.

$$\begin{aligned} z_1 - z_2 &= (x_1 + iy_1) - (x_2 - iy_2) \\ &= (x_1 - x_2) + i(y_1 + y_2). \end{aligned}$$

$\square$

d. z_1/z_2.

Proof.

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{x_1 + iy_1}{x_2 + iy_2} \\ &= \frac{x_1 + iy_1}{x_2 + iy_2} \cdot \frac{x_2 - iy_2}{x_2 - iy_2} \\ &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1x_2 + y_1y_2 + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}. \end{aligned}$$

$\square$

Question 1.3. Let $z_1 = 3 - 5i$ and $z_2 = 3 + \frac{2}{5}i$. Evaluate

a. $z_1 + z_2$.
b. $z_1 \cdot z_2$.
c. z_1/z_2.

Question 1.4. Let $z_1 = 4 - 9i$ and $z_2 = 3 + i$. Evaluate

$$\frac{z_1 + z_2}{z_1 \cdot z_1} - \frac{3z_2}{2z_1 + z_2^2}.$$

Question 1.5. We define the **complex conjugate** of $z = x+iy$ to be the complex number $\overline{z} = x-iy$. Let $z_1 = 4 - 5i$ and $z_2 = 3$. Evaluate

$$\frac{z_1}{\overline{z_2}} + \frac{z_2}{\overline{z_1}} + z_1\overline{z_1}.$$

Question 1.6. We define the **modulus** of a complex number z to be the real number $|z| = \sqrt{z \cdot \overline{z}}$. Determine the modulus of the following complex numbers.

a. $z = \frac{1}{3-2i}$.
b. $z = \frac{2}{1-5i}$.
c. $z = 2 - 8i$.
d. $z = i$.

Question 1.7. Determine the modulus of the complex number

$$\frac{z}{|z|^2} + \frac{\overline{z}}{|z|^2},$$

where $z = 1 + 3i$.

Question 1.8. Prove that if $z \in \mathbb{C}$ is a complex number, then

$$|z| = |\overline{z}| .$$

Plotting Complex Numbers on an Argand Diagram

Recall that, in some sense, we may think of $\mathbb{C}$ as just $\mathbb{R}^2$. When we think of $\mathbb{R}^2$ as $\mathbb{C}$, the plane is referred to as the **Argand Diagram**. The x-axis is referred to as the **real axis** and the y-axis is referred to as the **imaginary axis**.

Question 2.1. Plot the following complex numbers on an Argand diagram.

a. $z = 2 + 3i$.
b. $z = 3 - 10i$.
c. $z = \frac{1}{2} + \frac{1}{2}i$.
d. $z = 2$.
e. $z = 17i$.
f. $z = 0$.

Question 2.2. What does multiplication by i correspond to in terms of the geometry of the Argand diagram?

Question 2.3. What does the complex conjugate of complex number $z \mapsto \overline{z}$ correspond to in terms of the geometry of the Argand diagram?

Polar Form of a Complex Number

We have seen that a complex number may be written in the form $z = x + iy$, where $x, y \in \mathbb{R}$. We may also write a complex number in terms of its length, $|z|$, which we call the **modulus**, and the angle it forms with the real axis, we call this the **argument**. Explicitly, we may write a complex number in the form $z = re^{i\vartheta}$, where $r = |z|$ and $\vartheta = \tan^{-1}(y/x)$.

Question 3.1. Write the following complex numbers in polar form.

a. $z = 1 + i$.
b. $z = 2i$.
c. $z = 3 - 3i$.

Question 3.2.

a. $z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$.
b. $z = -1 - i\sqrt{3}$.
c. $z = 1 - i\sqrt{3}$.

Question 3.3. (Dr. Lloyd Gunatilake). Let $z = -3 - 3i$ and $w = -\sqrt{2} + i\sqrt{6}$. Express in polar form

a. zw.
b. $\frac{z}{w}$.

Question 3.4. Write the following complex numbers in cartesian form.

a. $z = 2e^{i\pi/4}$.
b. $z = e^{-i\pi/2}$.
c. $z = \sqrt{5}e^{2\pi i}$.

Theorem 3.5. (DeMoivre's theorem). Let $z = re^{i\vartheta}$. Then, for any $n \in \mathbb{N}$,

$$z^n = r^n e^{in\vartheta}.$$

Proof. We proceed by induction on n. In the case of $n = 1$, we see the result holds trivially. So suppose the result holds for n, we want to show the result holds for $n + 1$. To this end, we observe that

$$\begin{aligned} z^n = z^{n-1} \cdot z = \left(r^{n-1}e^{i(n-1)\vartheta}\right) \cdot \left(re^{i\vartheta}\right) &= r^{n-1} \cdot r \cdot e^{i(n-1)\vartheta} \cdot e^{i\vartheta} \\ &= r^n \cdot e^{in\vartheta}. \end{aligned}$$

This proves the result. $\square$

Question 3.6. Use Theorem 3.5 to evaluate

$$z = (1 + \sqrt{3})^5.$$

Question 3.7. Use Theorem 3.5 to evaluate

$$z = (-2 + 2i)^4.$$

Question 3.8. Use Theorem 3.5 to evaluate

$$z = (\sqrt{2} - i\sqrt{6})^7.$$

Question 3.9. (Dr. Lloyd Gunatilake). Let $z = 1 + e^{i\vartheta}$, where $\vartheta \in \left(0, \frac{\pi}{2}\right)$. Show that

a. $|z| = 2\cos\left(\frac{1}{2}\vartheta\right)$.
b. $\arg(z) = \frac{1}{2}\vartheta$.

Question 3.10. Explain what happens to the argument of a complex number when you square the complex number.

Question 3.11. Explain what happens to the argument of a complex number when you cuve the complex number. Does this hold in general? What about square rooting a complex number?

Question 3.12. (Dr. Lloyd Gunatilake). Let $ABCD$ denote a square with AC a diagonal. If A, C represent the complex numbers $-2 + 5i$ and $4 - 8i$ respectively, determine the complex numbers that represent B and D.

Solving Complex Polynomials

Theorem 4.1. (Fundamental Theorem of Algebra). Suppose that $f(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial in z of degree n. Then f has n roots in $\mathbb{C}$, not necessarily distinct, and can be written as

$$f(z) = a_n \prod_{k=1}^{n} (z - w_k),$$

where w_k are the roots of f.

Theorem 4.2. (Conjugate Root Theorem). Suppose that $f(z) = a_n z^n + \cdots + a_1 z + a_0$ is a polynomial in z with real coefficients. If $z = x + iy$ is a root of f, then $\overline{z} = x - iy$ is also a root of f.

Question 4.3. Solve the following equations for $z \in \mathbb{C}$.

a. $z^2 + 1 = 0$.
b. $z^2 + 2z = 5$.
c. $z^3 - 5z^2 + 8z - 4 = 0$.
d. $z^3 - 11z^2 + 36z = 36$.
e. $z^4 - 5z^2 + 6 = 0$.

Question 4.4. (Dr. Lloyd Gunatilake). Consider the polynomial $p(z) = 2z^3 + \lambda z^2 + \mu z - 12$, where $\lambda, \mu \in \mathbb{R}$. It is known that $z - 2i$ is a linear factor of $p(z)$. Determine the values of λ and μ and hence, write down all linear factors of $p(z)$.

Question 4.5. (Dr. Lloyd Gunatilake).

a. If $z = i$ is a solution of the equation $z^3 - kz^2 + 3iz + 1 - i = 0$, show that $k = 2 + 2i$.
b. Find the other two solutions of the equation $z^3 - kz^2 + 3iz + 1 - i = 0$.

Roots of Complex Numbers

Question 5.1. Find the five roots of i.

Question 5.2. Find the 3 roots of $1 + i$.

Question 5.3. Find the 7 roots of 1.

Question 5.4. Find the 4 roots of $\sqrt{2} - i\sqrt{6}$.

Question 5.5. (Dr. Lloyd Gunatilake).

a. Find the fourth roots of -16.
b. Hence, solve the equation

$$\left(\frac{z+1}{z-1}\right)^4 + 16 = 0.$$

Question 5.6. (Dr. Lloyd Gunatilake).

a. Find the solutions of the equation $z^5 = 1$ in polar form with principal argument.
b. Let w be a solution of the equation $z^5 = 1$ such that $w \neq 1$. By considering the factorization

$$z^5 - 1 = (z-1)(1 + z + z^2 + z^3 + z^4),$$

show that

i.
$$\cos\frac{2\pi}{5} + \cos\frac{4\pi}{5} + \cos\frac{6\pi}{5} + \cos\frac{8\pi}{5} = -1.$$

ii.
$$\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5} + \sin\frac{6\pi}{5} + \sin\frac{8\pi}{5} = 0.$$

Question 5.7. (Dr. Lloyd Gunatilake).

a. Show that the square roots of $1 + i$ are

$$\pm\left(\frac{\sqrt{1+\sqrt{2}}}{\sqrt{2}}\right) \pm i\left(\frac{1}{\sqrt{2}\sqrt{1+\sqrt{2}}}\right).$$

b. Find the square roots of $1 + i$ in polar form.
c. Hence, find the exact values of $\sin\frac{\pi}{8}$ and $\cos\frac{\pi}{8}$.

($\star$) **Question 5.8.** Determine the exact values of $\sin\frac{\pi}{5}$ and $\cos\frac{\pi}{5}$.