# Chapter 1

# Sequences and Series.

We begin with some elementary notions and definitions pertaining to sequences. We define a sequence to be a function $s : \mathbb{N} \to \mathbb{R}$, elements of $\mathbb{N}$ are referred to as indices.

A sequence may be described in three ways:

† In an explicit manner, for example $\{1, 2, 3, 4, ...\}$.

† In a functional manner, for example $\{n\}_{n=1}^{\infty}$.

† In a recursive manner, for example $s_{n+1} = 1 + s_n$, where $s_1 = 0$.

We invite the reader to verify that $\{1, 2, 3, 4, ...\}$ is the same as $\{n\}_{n=1}^{\infty}$, which is the same as $s_{n+1} = 1 + s_n, s_1 = 0$.

**Definition 11.1.1**. We define an arithmetic sequence to be a sequence of the form

$$t_n = a + (n-1)d = (a-d) + nd,$$

where $n \in \mathbb{N}$ is the *index*, $a \in \mathbb{R}$ is the *initial term*, and $d \in \mathbb{R}$ is the *common difference*.

**Example 11.1.2**. Show that the sequence $\left\{-\frac{3}{8}, \frac{3}{8}, \frac{9}{8}, ...\right\}$ is arithmetic.

*Proof.* We need to show that $\left\{-\frac{3}{8}, \frac{3}{8}, \frac{9}{8}, ...\right\}$ is of the form $t_n = a + (n-1)d$, for some $a, d \in \mathbb{R}$. To this end, we observe that the initial term is $-\frac{3}{8}$, so $a = -\frac{3}{8}$. Moreover, we see that the difference between $-\frac{3}{8}$ and $\frac{3}{8}$ is

$$\frac{3}{8} - \left(-\frac{3}{8}\right) = \frac{3}{8} + \frac{3}{8} = \frac{3}{4},$$

and the difference between $\frac{9}{8}$ and $\frac{3}{8}$ is given by

$$\frac{9}{8} - \frac{3}{8} = \frac{6}{8} = \frac{3}{4}.$$

We may therefore assert that $d = \frac{3}{4}$ and write the sequence as

$$t_n = -\frac{3}{8} + \frac{3}{4}(n-1).$$

$\square$

**Example 11.1.3.** Let $\{4, 9, 14, ...\}$ be a sequence. By first showing that the sequence is arithmetic, determine the 16th term of the sequence.

*Proof.* It is clear that $a = 4$ is the initial term and the difference is $d = 5$. Moreover, we recall that the $n$th term of an arithmetic sequence is given by

$$t_n = a + (n-1)d = 4 + 5(n-1).$$

Therefore, we see that

$$t_{16} = 4 + 5(16-1) = 4 + 5 \cdot 15 = 4 + 60 = 64.$$

$\square$

We now turn our attention to series, which arise as the summation of a given number of terms of a sequence. A common notation for writing the sum of $n$ terms of a sequence is $S_n$. If we let $t_n = \{t_1, t_2, t_3, ...\}$ be some sequence, then we make explicit that

$$S_n = t_1 + t_2 + t_3 + \cdots + t_n.$$

It is often very common and convenient to write

$$S_n = \sum_{k=1}^{n} t_k,$$

which is equivalent to the above equation. In order to calculate such series, we have the following rather useful result.

**Theorem 11.1.4.** The sum of the first $n$ terms of an arithmetic sequence is given by

$$S_n = \frac{n}{2}[2a + (n-1)d].$$

*Proof.* (For those who have done induction): Consider first the case of $n = 1$, then

$$S_1 = \frac{1}{2}[2a] = a.$$

So the result holds for $n = 1$. Now suppose the result holds for $n$. That is, assume that

$$S_n = \frac{n}{2}[2a + (n-1)d],$$

and recall that the $n$th term of an arithmetic sequence is given by

$$t_n = a + (n-1)d.$$

Therefore, we see that

$$\begin{aligned} S_{n+1} = S_n + t_{n+1} &= S_n + (a + nd) \\ &= \frac{n}{2}[2a + (n-1)d] + a + nd \\ &= na + \frac{n}{2}(n-1)d + a + nd \\ &= \frac{(n+1)}{2}[2a + nd], \end{aligned}$$

which proves the result. □

*Proof.* (For those who have not seen induction): Let us recall that the $n$ term of an arithmetic sequence is given by

$$t_n = a + d(n-1).$$

Therefore,

$$\begin{aligned} S_1 &= t_1 = a + d(1-1) = a \\ S_2 &= t_1 + t_2 = a + d(1-1) + a + d(2-1) = 2a + d \\ S_3 &= t_1 + t_2 + t_3 = a + (a+d) + (a+2d) \\ &= 3a + 3d \\ S_4 &= t_1 + t_2 + t_3 = a + (a+d) + (a+2d) + (a+3d) + (a+4d) \\ &= 4a + 10d \\ &\vdots \\ S_n &= na + \frac{1}{2}n(n-1)d = \frac{n}{2}[2a + (n-1)d]. \end{aligned}$$

□

**Example 11.1.5.** Consider the sequence $\{1, 3, 5, ...\}$. Determine the sum of the first 100 terms.

*Proof.* We observe that $\{1, 3, 5, ...\}$ is given by

$$t_n = 1 + 2(n-1);$$

that is, $a = 1$ and $d = 2$. Using the formula just proved, we see that

$$\begin{aligned} S_{100} &= \left.\frac{n}{2}[2a + (n-1)d]\right|_{n=100}, \\ &= \frac{100}{2}[2a + 99d] \\ &= 50[2a + 99d] \\ &= 50[2 \cdot 1 + 99 \cdot 2] \\ &= 50(2 + 198) \\ &= 50 \cdot 200 \\ &= 10000. \end{aligned}$$

□

Let us now take a step back and see what we have done so far. So far we have studied arithmetic sequences, which can be thought of as sequences that are linear. Indeed, if we consider an arbitrary arithmetic sequence $t_n = a + (n-1)d$, we may think of $a$ as the '$y$-intercept' and $d$ as the gradient. Another important thing to notice however, is that a successive term in an arithmetic sequence is given by adding a fixed number, which we have denoted by $d$. We now consider the situation when we obtain the success term in a sequence by multiplying by a fixed number, which we shall denote by $r$. Such a sequence is referred to as a geometric sequence. The general form a geometric sequence is

$$t_n = ar^{n-1},$$

where $a \in \mathbb{R}$ is the initial term, and $r$ is the common ratio:

$$r := \frac{t_{n+1}}{t_n},$$

assuming $t_n \neq 0$.

**Example 11.1.6**. Consider the sequence $\{2\pi, 4\pi^2, 8\pi^3, ...\}$. Show that this sequence is geometric.

*Proof.* We begin by observing that $a = 2\pi$. To determine the common ratio, we simply observe that

$$\frac{4\pi^2}{2\pi} = 2\pi, \quad \text{and} \quad \frac{8\pi^3}{4\pi^2} = 2\pi.$$

This verifies that the given sequence is indeed geometric and we have

$$t_n = 2\pi(2\pi)^{n-1} = (2\pi)^n.$$

$\square$

We conclude this section by considering geometric series. Just as in the case of arithmetic sequences, we have the following useful results.

**Theorem 11.1.7**. The sum of the first $n$ terms of a geometric sequence is given by

$$S_n = \frac{a(r^n - 1)}{r - 1},$$

where $r \in \mathbb{R}\backslash\{1\}$.

*Proof.* (For those who have not seen induction): Let us recall that a geometric sequence has the form $t_n = ar^{n-1}$. Therefore,

$$\begin{aligned}
S_1 &= t_1 = ar^{1-1} = ar^0 = a = \frac{a(r-1)}{(r-1)} \\
S_2 &= t_1 + t_2 = a + ar = a(1+r) = \frac{a(1+r)(r-1)}{r-1} \\
S_3 &= t_1 + t_2 + t_3 = a + ar + ar^2 = a(r^2 + r + 1) = \frac{a(r^2+r+1)(r-1)}{r-1} \\
S_4 &= t_1 + t_2 + t_3 + t_4 = a + ar + ar^3 + ar^4 \\
&= \frac{a(r^4 + r^3 + r^2 + 1)(r-1)}{r-1} \\
&\vdots \\
S_n &= \frac{a(r^n - 1)}{r - 1}.
\end{aligned}$$

$\square$

## Exercises

Q1. For each of the following sequences, determine the next three terms.

a. $\{1, 4, 7, ...\}$.
b. $\{2, 7, 12, ...\}$.
c. $\{1, -3, -7, ...\}$.
d. $\{-4, -2, 0, ...\}$.
e. $\left\{\frac{1}{2}, 1, \frac{3}{2}, ...\right\}$.
f. $\{\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, ...\}$.

Q2. Determine the $n$th term of the following sequences.

a. $\{-2, -3, -4, ...\}$.
b. $\{1, 8, 15, ...\}$.
c. $\{-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, ...\}$.
d. $\{1, 1+\pi, 1+2\pi, ...\}$.
e. $\{2, 5, 8, 11, 14, ...\}$.
f. $\{2, 5, 9, 14, 20\}$.

Q3. Write each of the following sequences in recursive form.

a. $\{3, 6, 9, ...\}$.
b. $\{2, 7, 12, ...\}$.
c. $\{1, -3, -7, ...\}$.
d. $\{-4, -2, 0, ...\}$.
e. $\left\{\frac{1}{2}, 1, \frac{3}{2}, ...\right\}$.
f. $\{\sqrt{2}, 2\sqrt{2}, 3\sqrt{2}, ...\}$.
g. $\{-2, -3, -4, ...\}$.
h. $\{1, 8, 15, ...\}$.
i. $\{-\frac{3}{4}, -\frac{1}{2}, -\frac{1}{4}, ...\}$.
j. $\{1, 1+\pi, 1+2\pi, ...\}$.
k. $\{2, 5, 8, 11, 14, ...\}$.
l. $\{2, 5, 9, 14, 20\}$.

Q4. Determine which of the following sequences are arithmetic.

a. $\{4, 8, 16, ...\}$.
b. $\{3, 9, 14, ...\}$.
c. $\{1, -1, 1, -1, ...\}$.
d. $\{1, \pi, \pi^2, ...\}$.
e. $\{1, 4, 5, 9, ...\}$.
f. $\{-3, 6, 15, ...\}$.

Q5. Let $p_n$ denote the number cats in the abandoned houses in Braddon. The population of cats follows the logistic equation:

$$p_{n+1} = \frac{3}{10} p_n (1 - p_n).$$

In 2013, it is known that there are 15 cats in these houses. Should we worry about cat populations in the long run?

Q6. Determine the 10th term of the following arithmetic sequences:

a. The first term is 1 and the common difference is 4.
b. The first term is $-\frac{1}{2}$ and the common difference is $-9$.

c. The first term is $\sqrt{3}$ and the common difference is 1.

d. The first term is $\sqrt{2}$ and the common difference is $\pi$.

Q7. Find the $n$th term of the arithmetic sequence whose first term is 3 and third term is 10.

Q8. Find the $n$th term of the arithmetic sequence whose first term is $-4$ and 3rd term is 6.

Q9. By the hand of God, the monthly enrolments for students in guitar lessons follows an arithmetic sequence. If there were 3 students in the first month and 15 in the fourth month, determine the number of students in the 6th month.

Q10. Find the values of $m, n \in \mathbb{R}$ such that the sequence $\{16, m, 27, n\}$ is arithmetic.

Q11. Determine the values of $\lambda$ and $\mu$ such that the sequence $\{x-3y, \lambda, -3x+5y, \mu\}$ is an arithmetic sequence.

Q12. Evaluate the sum of the first 10 terms of the following sequences.

a. $\{1, 4, 7, ...\}$.

b. $\{2, 7, 12, ...\}$.

c. $\{1, -3, -7, ...\}$.

d. $\{-4, -2, 0, ...\}$.

e. $\left\{\frac{1}{2}, 1, \frac{3}{2}, ...\right\}$.

f. $\{\sqrt{5}, 2\sqrt{5}, 3\sqrt{5}, ...\}$.

Q13. Let $t_n = \frac{1}{2}n - 1$, where $n \in \mathbb{N}$. Determine $S_5, S_{10}$ and $S_{20}$.

Q14. Find the sum of the first 50 positive integers.

Q15. Evaluate the sum

$$\sum_{n=1}^{100} 2n + 1.$$

Q16. Evaluate the sum

$$\sum_{k=1}^{50} \frac{1}{2} - 3n.$$

Q17. a. Show that the sum of the first $N$ odd positive integers is equal to $N^2$.

b. Is the same true for the sum of the first $N$ even positive integers?

Q18. Determine whether the following sequences are geometric.

a. $\{3, 6, 9, ...\}$.
b. $\{4, 12, 36, ...\}$.
c. $\{-3, 1, -1/3, ...\}$.
d. $\{2, -6, 18, ...\}$.
e. $\{\sqrt{2}, 2, 2\sqrt{2}, ...\}$.
f. $\left\{\frac{3}{4}, \frac{3}{2}, \frac{9}{4}, ...\right\}$.

Q19. Determine the 10th term of the following sequences.

a. $\{5, 10, 20, ...\}$.
b. $\{2, -4, 8, ...\}$.
c. $\{x, 4x^4, 16x^7, ...\}$.
d. $\left\{\frac{1}{x}, \frac{\sqrt{5}}{x^2}, \frac{5}{x^3}, ...\right\}$.

Q20. Write the following sequences in recursive form.

a. $\{3, 6, 9, ...\}$.
b. $\{4, 12, 36, ...\}$.
c. $\{-3, 1, -1/3, ...\}$.
d. $\{2, -6, 18, ...\}$.
e. $\{\sqrt{2}, 2, 2\sqrt{2}, ...\}$.
f. $\left\{\frac{3}{4}, \frac{3}{2}, \frac{9}{4}, ...\right\}$.
g. $\{5, 10, 20, ...\}$.
h. $\{2, -4, 8, ...\}$.
i. $\{x, 4x^4, 16x^7, ...\}$.
j. $\left\{\frac{1}{x}, \frac{\sqrt{5}}{x^2}, \frac{5}{x^3}, ...\right\}$.

Q21. Let $\{16, p, 81, q, ...\}$ be a geometric sequence with $p, q \in \mathbb{R}_{>0}$. Determine the values of $p$ and $q$.

Q22. Let $t_n$ be a geometric sequence with $t_3 = 100$ and $t_5 = 400$. Determine the $n$th term of the sequence.

Q23. Determine the sum of the first 10 terms of the following geometric sequences.

a. $\{3, 6, 9, ...\}$.
b. $\{4, 12, 36, ...\}$.
c. $\{-3, 1, -1/3, ...\}$.
d. $\{2, -6, 18, ...\}$.
e. $\{\sqrt{2}, 2, 2\sqrt{2}, ...\}$.
f. $\left\{\frac{3}{4}, \frac{3}{2}, \frac{9}{4}, ...\right\}$.
g. $\{5, 10, 20, ...\}$.
h. $\{2, -4, 8, ...\}$.
i. $\{x, 4x^4, 16x^7, ...\}$.
j. $\left\{\frac{1}{x}, \frac{\sqrt{5}}{x^2}, \frac{5}{x^3}, ...\right\}$.

Q24. The first term of a geometric sequence is 440 and the 12th term is 880. Detemine $S_6$.

Q25. For $n \in \mathbb{N}$, define the geometric sequence $t_n$ by the formula

$$t_n = \frac{3}{2^{n-1}}.$$

Evaluate
$$\sum_{n=1}^{\infty} t_n.$$

Q26. Let $n \in \mathbb{N}$, define the sequence $t_n$ by the formula
$$t_n = 1 + \frac{1}{2}(n-1).$$
Evaluate the sum
$$\sum_{n=1}^{100} t_n.$$

Q27. The first term of a geometric sequence $t_n$ is 4 and $\sum_{n=1}^{\infty} t_n = 6$. Determine the common ratio of $t_n$.

Q28. A teacher began working at Narrabundah college 25 years ago. She began with a salary of $\$32,000$ and recieves an annual salary increase of $\$1,500$ per year.

a. What type of sequence does her annual income follow?
b. How much did she earn in her 20th year of employment?
c. What was her percentage increase at the end of her first year compared with her 15th year?

Q29. Joe has been working for the same grocery store for the page 10 years. He started working with a salary of $\$13,000$ and has recieved a 4% increase each year.

a. What type of sequence does his annual income follow?
b. How much did he earn in his 8th year of employment?
c. How much money has he earned from the company over the past 10 years?

Q30. Let $n \in \mathbb{N}$ and set $t_n := 2n - 3$ and $s_n = 3n$. Show that the pointwise product of the sequences $t_n$ and $s_n$ is an arithmetic sequence. That is, show that the sequence $r_n := t_n \cdot s_n$ is an arithmetic sequence.

Q31. Let $n \in \mathbb{N}$ and take $t_n$ and $s_n$ be arbitrary arithmetic sequences. Is it necessarily true that $r_n := t_n \cdots s_n$ is an arithmetic sequence?

Q32. Let $n \in \mathbb{N}$ and take $t_n$ and $s_n$ be arbitrary geometric sequences. Determine whether $r_n := t_n \cdot s_n$ is a geometric sequence.

Q33. The side lengths of a right-angled triangle form the successive terms of an arithmetic sequence. The perimeter of the triangle is 72cm. What are the side lengths of the triangle?

Q34. A hiker walks 36km on the first day and $\frac{2}{3}$ that distance on the second; this pattern continues. Will the hiker cover the distance of 100km to complete the walk and on what day will she complete the task?

Q35. Find the limits of the following sequences.

a. $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$.

b. $\left\{\frac{2n+1}{3n-2}\right\}_{n\in\mathbb{N}}$.

c. $\left\{\frac{3n^2-5n+1}{2n-3}\right\}_{n\in\mathbb{N}}$.

d. $\left\{\frac{4n^3+1}{6n^7-1}\right\}_{n\in\mathbb{N}}$.

Q36. Determine the limit

$$\lim_{n\to\infty}\left(\sqrt{n^2+n}-\sqrt{n^2-n}\right).$$