Chapter 1

Probability Theory and Combinatorics.

1.1 Elementary Probability Theory

In this section we begin with an elementary introduction of probability theory and define some necessary terms that we shall use throughout the chapter.

Definition 7.1.1. We define a probability space to be the triple $(\Omega, \Sigma, \mathbb{P})$, where Ω is a set which consists of all outcomes, Σ is the the set of all subsets of Ω and $\mathbb{P} : \Omega \to \mathbb{R}$ is a function which assigns to each outcome in Ω, a real number, referred to as the probability of that outcome.

If $A \in \Omega$ is an event, we denote the probability of A occuring by $\mathbb{P}(A)$. If $A, B \in \Omega$ are two events, we denote the probability of A and B occuring by $\mathbb{P}(A \cap B)$, and the probability of A or B occuring by $\mathbb{P}(A \cup B)$.

Definition 7.1.2. We say that two events $A, B \in \Omega$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

Definition 7.1.3. If $A, B \in \Omega$ are two events and the occurrence of B has an effect on A, then the probability is conditional. The probability of A given by B is denoted by $\mathbb{P}(A|B)$ and is given by the formula

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Example 7.1.4. Suppose that $\mathbb{P}(A) = \frac{1}{5}$, $\mathbb{P}(B) = \frac{1}{10}$ and $\mathbb{P}(A \cap B) = \frac{1}{20}$. Determine the following.

a. $\mathbb{P}(A \cup B)$.

Proof. Using a Venn diagram if necessary, we observe that

$$\begin{aligned}
\mathbb{P}(A \cup B) &= \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B) \\
&= \frac{1}{5} + \frac{1}{10} - \frac{1}{20} \\
&= \frac{1}{20}(4 + 2 - 1) \\
&= \frac{5}{20} = \frac{1}{4}.
\end{aligned}$$

□

b. $\mathbb{P}(A|B)$.

Proof. We simply observe that

$$\begin{aligned}
\mathbb{P}(A|B) &= \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)} \\
&= \frac{1}{\mathbb{P}(B)} \cdot \mathbb{P}(A \cap B) \\
&= \frac{1}{1/10} \cdot \frac{1}{20} \\
&= 10 \cdot \frac{1}{20} = \frac{1}{2}.
\end{aligned}$$

□

c. $\mathbb{P}(B|A)$.

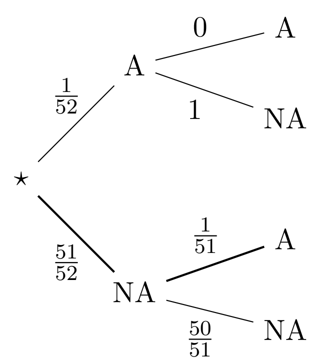
Proof. We simply observe that

$$\begin{aligned}
\mathbb{P}(B|A) &= \frac{\mathbb{P}(B \cap A)}{\mathbb{P}(A)} \\
&= 5 \cdot \frac{1}{20} \\
&= \frac{1}{4}.
\end{aligned}$$

□

Example 7.1.5. (Tree Diagram). Consider a pack of 52 cards and suppose that one draws a card from the top of the deck and places it in their pocket. What is the probability that the next card drawn from the top of the deck is the ace of spades?

Proof. We consider that on the first draw, there is a $\frac{1}{52}$ probability that the first card is an ace of spades, and a $\frac{51}{52}$ probability that it is not. On the second draw, if the first card was an ace of spades, the probability that the second card is an ace of spades is 0 and the probability that it is not an ace of spades is 1. If the first card was not an ace of spades, the probability that the second card is an ace of spades is $\frac{1}{51}$ and the probability that it is not an ace of spades is $\frac{50}{51}$. The most effective way of encapsulating all of this information is through a tree diagram. The tree diagram associated to this problem is given by



Where A denotes the event of an ace occurring and NA denotes the event of an ace not occurring. Hence we see by multiplying probabilities that

$$\mathbb{P}(\text{Ace on Second Draw}) = \frac{51}{52} \cdot \frac{1}{51} = \frac{1}{52}.$$

□

Exercises

Q1. Suppose we have two dice which are rolled simultaneously and suppose that we record the sum of the numbers which lie on the top face of each die. Determine the probability of the sum of the rolls being

a. 6.
b. 4.
c. Less than 7.
d. Greater than 10.
e. Less than 9 but greater than 3.
f. Greater than 5 but less than 11.
g. An even number.

Q2. Suppose that A and B are events with $\mathbb{P}(A) = \frac{1}{2}, \mathbb{P}(B) = \frac{2}{5}$ and $\mathbb{P}(A \cup B) = \frac{4}{5}$. Determine the following.

a. $\mathbb{P}(A \cap B)$.
b. Whether A and B are mutually exclusive events.
c. Whether A and B are independent events.

Q3. Suppose that A and B are events with $\mathbb{P}(A) = \frac{3}{10}, \mathbb{P}(B) = \frac{3}{5}$ and $\mathbb{P}(A \cap B) = \frac{1}{5}$. Determine the following.

a. $\mathbb{P}(A \cup B)$.
b. Whether A and B are mutually exclusive events.
c. Whether A and B are independent events.

Q4. Suppose that A and B are two events such that $\mathbb{P}(A \cap B) = 0.22$, $\mathbb{P}(A' \cap B') = 0.5$ and $\mathbb{P}(A') = 0.72$. Determine the probabilities of the following events.

a. $\mathbb{P}(A' \cap B)$.
b. $\mathbb{P}(A \cap B')$.
c. $\mathbb{P}(A)$.
d. $\mathbb{P}(B)$.

Q5. The probability that Solbee goes to school on any particular day is determined entirely by the weather. If it is cold, Solbee has a $\frac{2}{7}$ chance of going to school, while if it warm, she has a $\frac{8}{9}$ chance of going to school. The probability of it being cold tomorrow is $\frac{3}{8}$. What is the probability that Solbee goes to school?

Q6. The probability that Flint studies on any particular day is determined entirely by his hunting activities. If Flint is hunting, he has a $\frac{1}{50}$ chance of studying. If he does not hunt however, he has $\frac{1}{20}$ chance of studying. Either way, it is clear that his study habits are terrible. The probability of Flint hunting tomorrow is $\frac{7}{8}$. What is the probability that Flint studies?

Q7. Joe has recently been purchasing Crypto-currencies such as Bitcoin and has developed an approach for making lots of money. If the price of Bitcoin goes down, he has a $\frac{1}{3}$ chance of buying more Bitcoins. If the price of Bitcoin goes up however, we have $\frac{5}{6}$ chance of buying Bitcoin. If Joe has a $\frac{1}{2}$ chance of buying Bitcoins tomorrow, what is the probability that Bitcoin will increase in price?

Q8. The amount of water Billy drinks on any given day is determined by the temperature outside. If the temperature is above 26° C, Billy has a $\frac{1}{2}$ chance of drinking over 2 litres water. If the temperature is below 26° C however, Billy has a $\frac{2}{15}$ chance of drinking over 2 litres of water. If Billy has a $\frac{1}{4}$ chance of drinking over 2 litres of water tomorrow, determine the probability of the temperature being above 26° C outside.

Q9. Suppose that $\mathcal{A}$ and $\mathcal{B}$ are two events such that $\mathbb{P}(\mathcal{A} \cup \mathcal{B}) = 0.7$, $\mathbb{P}(\mathcal{A}' \cap \mathcal{B}') = 0.2$ and $\mathbb{P}(\mathcal{B}) = 0.4$. Determine the probabilities of the following events.

a. $\mathbb{P}(\mathcal{A} \cap \mathcal{B})$.

b. $\mathbb{P}(\mathcal{A}' \cap \mathcal{B})$.

c. $\mathbb{P}(\mathcal{A})$.

d. $\mathbb{P}(\mathcal{B}')$.

Q10. Let A and B be two events in an event space with $\mathbb{P}(A) = \frac{1}{4}$, $\mathbb{P}(B) = \frac{1}{8}$ and $\mathbb{P}(A \cup B) = \frac{4}{9}$.

a. Determine whether A and B are mutually exclusive events.

b. Determine whether A and B are independent events.

1.2 Combinatorics

Combinatorics is the primarily concerned with counting, both as a means and as an end. Not only does combinatorics help us count the number of ways in which something can be arranged, but it tells us how to count 'properly'. To see what is meant by this, let us consider the following very elementary example. Suppose you head to Rosanne's chicken store for dinner and observe that there are 5 different chicken wings, 4 different chicken snack packs and 6 different deserts. Although rather obvious, it is clear that we identify the chicken wings in a particular class, snack packs in another and deserts in another. That is, even though 2 deserts may be different, they are identified as the same objects. The emphasis on this idea will become apparent throughout the chapter.

Like many other branches of mathematics, combinatorics has a very intimate relationship with set theory. Indeed, sets provide the perfect housing place for events upon which we will hope to count. So for the moment, let A be a set of size m and B a set of size n for some $m, n \in \mathbb{N}$. We invite the reader to convince herself that

$$|A \cup B| = |A| + |B| - |A \cap B|,$$

where $|A|$ denotes the size of A. In particular, if A and B correspond to independent events, such as the rolling of dice, then we see that $|A \cap B| = 0$ and

$$|A \cup B| = |A| + |B| = m + n.$$

In the case that $|A \cap B| \neq 0$, we may rearrange the above formula to obtain

$$|A \cap B| = |A| + |B| - |A \cup B|.$$

We would also like to invite the reader at this point to recall that the event A or B is described by the set $A \cup B$, while the event A and B is described by the event $A \cap B$.

If we now generalise the above discussion to any number of sets, say n of them, then we obtain the following so-called *Addition Principle.*

Proposition 7.2.1. (The Addition Principle). Let $A_1, ..., A_n$ be the sets corresponding to independent events. That is, $|A_i \cap A_j| = 0$ for all $i \neq j$. Then

$$|A_1 \cup \cdots \cup A_n| = |A_1| + \cdots + |A_n|.$$

Example 7.2.2. Habib's Kebabs sell 4 varieties of chicken kebabs, 6 varieties of beef kebabs and 1 type of vegetarian kebab. How many different kebabs are sold by Habib's Kebabs.

Proof. Let A_1 be the set corresponding to the chicken kebab, A_2 correspond to the beef kebab and A_3 correspond to the vegetarian option. We see that $|A_1 \cap A_2| = |A_1 \cap A_3| = |A_2 \cap A_3| = 0$, so the sets are disjoint. We may therefore use the addition principle and conclude that there are 11 different kebabs. □

Example 7.2.3. Determine the number of ways in which we can roll two dice so that we don't get pairs of $(1,1),(2,2),...,(6,6)$.

Proof. We may completely characterise all events of rolling two dice by the pairs $A_1 = (1,x), A_2 = (2,x), ..., A_6 = (6,x)$, where x is permitted to range from 1 to 6. Each set has size 6, so the addition principle tells us that the total number of arrangements $\mathcal{T}$ is given by

$$\mathcal{T} = |A_1| + |A_2| + \cdots + |A_6| = 6 + 6 + \cdots + 6 = 36.$$

We want to exclude $(1,1),(2,2),...,(6,6)$ however, which means we reject 6 possible arrangements. We are therefore left with 30 possible arrangements. □

Another way of solving the above problem was to realise that we have six events $A_1,...,A_6$, each with 6 possible outcomes. Then the total number of possible arrangements is $6 \cdot 6 = 36$, then simply reject the 6 arrangements of the form (x,x) with $1 \leq x \leq 6$. This is formalised by the following proposition.

Proposition 7.2.4. (The Multiplication Principle). Let $E_1,...,E_n$ be a sequence of independent events and suppose that each event E_j has m_j possible outcomes. Then the total number of possible outcomes $\mathcal{T}$ is given by

$$\mathcal{T} = m_1 \cdot m_2 \cdots m_n.$$

Example 7.2.5. Suppose we have 10 whiteboard markers in our bag and we pull out 4 of them and place them on the table. How many arrangements are possible?

Proof. We have 10 markers and 4 spots to fill. The first position is given by the pair $(1, x_1)$, where x_1 ranges from 1 through 10. The second position is given by the pair $(2, x_2)$, where x_2 ranges from 1 to $10 - 1 = 9$, and so on. We stop with position 4 given by $(4, x_4)$, where x_4 ranges from 1 to $10 + 1 - 4 = 7$. The multiplication principle then tells us that the number of outcomes is given by

$$10 \cdot 9 \cdot 8 \cdot 7 = 5040.$$

□

In light of this example, we have the following corollary of the multiplication principle.

Corollary 7.2.6. The number of ways $\mathcal{N}_{\text{ord}}(n, k)$ in which n objects, taken k at a time, can be arranged is given by

$$\mathcal{N}_{\text{ord}}(n, k) = n(n-1)(n-2)\cdots(n+1-k) = \frac{n!}{(n-k)!}.$$

The subscript ord is to emphasise the fact that order is important here. This emphasis on order therefore begs the question of how do we determine the number of ways in which n objects, taken k at a time can be arranged if we do not care about the order. We consider this in the following example.

Example 7.2.7. Suppose we roll three dice and want to determine the number of possible outcomes, but consider the event $(1, 2, 3)$ equivalent to the events $(1, 3, 2), (3, 1, 2)$ and so on. Determine the number of total of possible arrangements under these constraints.

Proof. The total number of ways to roll three dice is, by the multiplication principle $6 \cdot 6 \cdot 6 = 216$. Now for any triple (x, y, z), there are $3! = 6$ ways of permuting this triple. So we essentially get 6 of every element, which we want to normalise to 1. The natural thing to do therefore is to observe that there are

$$\frac{6 \cdot 6 \cdot 6}{6} = 36,$$

possible arrangements under the given constraints. □

The above example provides us with the following useful expression.

Proposition 7.2.8. Suppose we n objects. The numbers of ways $\mathcal{N}(n,k)$ in which we may choose k of these objects is given by

$$\mathcal{N}(n,k) = \frac{1}{k!}\mathcal{N}_{\text{ord}}(n,k) = \frac{n!}{k!(n-k)!}.$$

As a direct consequence of this, we see that the number of subsets of size k in a set of size n is given by $\mathcal{N}(n,k)$ also. Some common notation for the function $\mathcal{N}(n,k)$ is

$$\binom{n}{k} := \mathcal{N}(n,k),$$

pronounced n *choose* k.

The last type of problem we would like to consider in this section is the problem of determining arrangements of objects in circular arrangements.

Suppose we want n people to sit at a round table and the choice of seat does not matter. We claim that the number of possible arrangments is given by

$$\mathcal{R}(n) = (n-1)!.$$

To see this, let $n = 4$ and denote each person by person A, person B, person C or person D. Person A steps into the room where the table is placed and there are 4 available chairs. All choices of seating however are equivalent for person A, so the seating of person A is only counted as 1 arrangement. Person B then walks in, and now there are 3 possibilities, but each choice is counted since the position of person B relative to person B is dependent on the seat. Person C then walks in and similar reasoning gives us 2 arrangements and then finally person D yields only 1 arrangments. The multiplication principle then tells us that

$$\mathcal{R}(4) = 1 \cdot 3 \cdot 2 \cdot 1 = 6 = 3!.$$

For our purposes here, this will serve as a good enough hand-wave so that we may formalise this into a proposition.

Proposition 7.2.9. The number of ways in which n people may be situated around a circular table, where their chair does not matter is given by

$$\mathcal{R}(n) = (n-1)!.$$

Exercises

Q1. A standard deck of cards has 52 cards and 4 suits (spades, clubs, hearts and diamonds). A standard hand consists of 5 cards and a full house consists of three cards of a particular rank and two cards of another rank. Determine the number of ways in which we may draw a full house.

Q2. A standard deck of cards has 52 cards and 4 suits (spades, clubs, hearts and diamonds). A standard hand consists of 5 cards and a flush consists of 5 cards from the same suit. Determine the number of ways in which we may draw a flush. We permit straight flushes in the definition of a flush.

Q3. A standard deck of cards has 52 cards and 4 suits (spades, clubs, hearts and diamonds). A standard hand consists of 5 cards and a straight consists of any consecutive ordering of 5 cards (the suit may vary). Determine the number of ways in which we draw a straight. We order a suit in the following manner: $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < J < Q < K < A$, where A denotes the ace, J the Jack, Q the Queen and K the King.

Q4. A standard deck of cards has 52 cards and 4 suits (spades, clubs, hearts and diamonds). A standard hand consists of 5 cards and a straight flush consists of 5 cards from the same suit in a row. An example of a straight flush is the hand $(2, 3, 4, 5, 6)$, while $(1, 3, 4, 5, 6)$ is not a straight flush.

Q5. A standard deck of cards has 52 cards and 4 suits (spades, clubs, hearts and diamonds). A standard hand consists of 5 cards and we define a royal flush to be a hand consisting of $10, J, Q, K, A$, where J denotes the Jack, Q denotes the Queen, K denotes the King, A denotes the Ace, and we require them all to be in the same house.

Q6. You have been studying combinatorics all day and are now bragging to your friend Peter claiming that you could beat him in any game of poker based on your new found combinatorics skills. Given any deck of 52 cards with 4 suits and 13 cards from each suit, you know all the numbers pertaining the different ways in which each type of hand may drawn. Peter does not care however and challenges you to a game. Peter has a trick up his sleeve however, he does not use a standard deck. Peter's deck consists of 55 cards, 11 ranks for each suit $(1, 2, 3, 4, 5, 6, 7, 8, 9, 10, A)$, and there are 5 different suits (spades, clubs, diamonds, hearts, swords). How many four-of-a-kind hands exist in Peter's deck?

Q7. Let Peter's deck be as defined in the previous exercise. Define a flush as we did in Question 2. How many possible flushes can we draw if we play with Peter's deck?

Q8. Consider a table with 8 chairs around it. There are 4 men and 4 women who will be sitting at this table. The ordering does not matter, but it is required that to the left and right of each man, there is a woman. Determine the number of possible seating arrangements.

Q9. Determine the number of ways in which 3 women and 4 men can be seated in a round table if

a. the women all sit together.

b. the women do not sit together.

Q10. Suppose 5 math students and 3 psychology students sit around a table. Determine the number of ways in which the students can be seated if no psychology students wants to sit next to the other psychology students.

Q11. Suppose we have three married couples sitting around a table, where the husband and wife necessarily sit together.

1.3 Discrete Random Variables

In this section we look at discrete random variables. For our purposes here, we define discrete rather loosely to mean 'whole' and 'not-continuous'. For example the number of cars in a parking lot is discrete, but the heights of students in a classroom is not discrete.

Consider the following distribution of a discrete random variable.

x	x_1	x_2	x_3	x_4	x_5
$\mathbb{P}(X=x)$	y_1	y_2	y_3	y_4	y_5

The expected value, or average, or mean, of a random variable X is the average value of X, it often denoted by $\mathbb{E}(X)$ or μ. In the discrete case, it is calculated by using the formula

$$\mathbb{E}(X) = \sum_{j=1}^{5} x_j y_j = x_1 y_1 + \cdots + x_5 y_5.$$

It is worth noting that the expected value $\mathbb{E}(\cdot)$ is linear. That is, if X, Y are discrete random variables and $a, b \in \mathbb{R}$, then

$$\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y).$$

Notice that since $\mathbb{E}(1) = 1$, it follows that $\mathbb{E}(a) = a$ for all $a \in \mathbb{R}$.

Example 7.3.1. Determine the expected value of the following probability distribution.

x	1	2	3	4	5
$\mathbb{P}(X=x)$	2/5	3/10	1/10	1/10	1/10

Proof. We simply observe that

$$\begin{aligned} \mathbb{E}(X) &= 1 \cdot \frac{2}{5} + 2 \cdot \frac{3}{10} + 3 \cdot \frac{1}{10} + 4 \cdot \frac{1}{10} + 5 \cdot \frac{1}{10} \\ &= \frac{2}{5} + \frac{3}{5} + \frac{3}{10} + \frac{2}{5} + \frac{1}{2} \\ &= \frac{11}{5}. \end{aligned}$$

□

A measure of the spread of the spread of the distribution is given by the variance. To calculate the variance, we use the formula :

$$\mathbb{V}\mathrm{ar}(X) := \mathbb{E}(X^2) - [\mathbb{E}(X)]^2.$$

Subsequently, the standard deviation, measuring how far a random variable is from the mean is given

$$\sigma(X) := \sqrt{\mathbb{V}\mathrm{ar}(X)}.$$

Example 7.3.2. Consider the following discrete probability distribution

x	0	1	2	3	4	5	6
$\mathbb{P}(X = x)$	1/3	2/9	1/18	1/18	1/9	1/6	1/18

Determine the expected value, variance and standard deviation.

Proof. The expected value is simply given by

$$\begin{aligned}
\mathbb{E}(X) &= 0 \cdot \frac{1}{3} + 1 \cdot \frac{2}{9} + 2 \cdot \frac{1}{18} + 3 \cdot \frac{1}{18} + 4 \cdot \frac{1}{9} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{18} \\
&= \frac{2}{9} + \frac{1}{9} + \frac{1}{6} + \frac{4}{9} + \frac{5}{6} + \frac{1}{3} \\
&= \frac{19}{9}.
\end{aligned}$$

Before attempting to calculate the variance, we must calculate $\mathbb{E}(X^2)$. To this end, we observe that

$$\begin{aligned}
\mathbb{E}(X^2) &= 0^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{2}{9} + 2^2 \cdot \frac{1}{18} + 3^2 \cdot \frac{1}{18} + 4^2 \cdot \frac{1}{9} + 5^2 \cdot \frac{1}{6} + 6^2 \cdot \frac{1}{18} \\
&= \frac{2}{9} + 4 \cdot \frac{1}{18} + 9 \cdot \frac{1}{18} + 16 \cdot \frac{1}{9} + 25 \cdot \frac{1}{6} + 36 \cdot \frac{1}{18} \\
&= \frac{2}{9} + \frac{2}{9} + \frac{1}{2} + \frac{16}{9} + \frac{25}{6} + 2 \\
&= \frac{80}{9}.
\end{aligned}$$

Hence we see that the variance is given by

$$\mathbb{V}\mathrm{ar}(X) = \frac{80}{9} - \frac{19}{9} = \frac{61}{9}.$$

It is then an easy consequence that

$$\sigma(X) = \sqrt{\frac{61}{9}} = \frac{\sqrt{61}}{3}.$$

$\square$

We now take a moment to look at two specific examples of discrete probability distributions: the Poisson distribution and the Hypergeometric distribution.

The Poisson Distribution

Exercises

Q1. Which of the following are discrete random variables?

a. The number of people that participate in a bank robbery?
b. The time taken to wake up in the morning?
c. The shoe size of 30 people.
d. The height of class of students.
e. The weights of babies in a maternity ward.
f. The number people who have the gene monoamine oxadise A expressed in their genome.

Q2. Calculate the expected value, variance and standard deviation of the following probability distributions.

a.

x	1	2	3	4	5
$\mathbb{P}(X = x)$	0.05	0.2	0.5	0.2	0.05

b.

x	2	4	6	8	10
$\mathbb{P}(X = x)$	0.1	0.2	0.4	0.2	0.1

x	1	3	5	7	9
$\mathbb{P}(X = x)$	0.2	0.3	0.2	0.2	0.1

c.

.

d.

x	-4	-3	-1	1	2
$\mathbb{P}(X = x)$	0.1	0.1	0.4	0.2	0.2

Q3. Determine the values of $\lambda \in \mathbb{R}$ such that the distribution given below is a probability distribution.

x	2	4	6	8	10
$\mathbb{P}(X = x)$	0.1	0.1	0.4	0.2	λ

Q4. Determine the value of $\lambda \in \mathbb{R}$ such that the distribution given below is a probability distribution.

x	-1	2	4	5	8
$\mathbb{P}(X = x)$	λ	0.25	0.5λ	0.1	λ

Q5. Determine the value(s) of $\lambda \in \mathbb{R}$ such that the distribution given below is a probability distribution.

x	1	2	3	4	5
$\mathbb{P}(X = x)$	$\frac{3\lambda}{13}$	$\frac{\lambda^2}{13}$	$\frac{5\lambda-4}{13}$	$\frac{\lambda^2}{13}$	$\frac{7}{13}$

Q6. Let $p : \{0, 1, 2, 3, 4\} \to \mathbb{R}$ be the function defined by

$$p(x) := \frac{1}{90}(8x + 2).$$

Show that p describes a probability distribution.

Q7. Let $p : \{1, 2, 5, 6\} \to \mathbb{R}$ be the function defined by

$$p(x) = \left| x - \frac{1}{k} \right|.$$

Determine the value of $k \in \mathbb{R}$ such that p describes a probability distribution.

Q8. Let $g : \{0, 1, 2, 3, 4\} \to \mathbb{R}$ be the function defined by

$$g(x) = \frac{1}{66}(3x + 7).$$

Determine whether g describes a probability distribution.

Q9. A fair coin is tossed 3 times. If X represents the numer of tails obtained, determine the following.

a. The probability distribution of each outcome.
b. $\mathbb{E}(X)$.
c. $\mathbb{V}\text{ar}(X)$.
d. $\sigma(X)$.

Q10. Let X be a random variable with the following probability distribution. Determine the value of $\lambda \in \mathbb{R}$ such that the distribution given below is a probability distribution.

x	-2	3	8	10	14	k
$\mathbb{P}(X = x)$	0.1	0.08	0.07	0.27	0.16	0.32

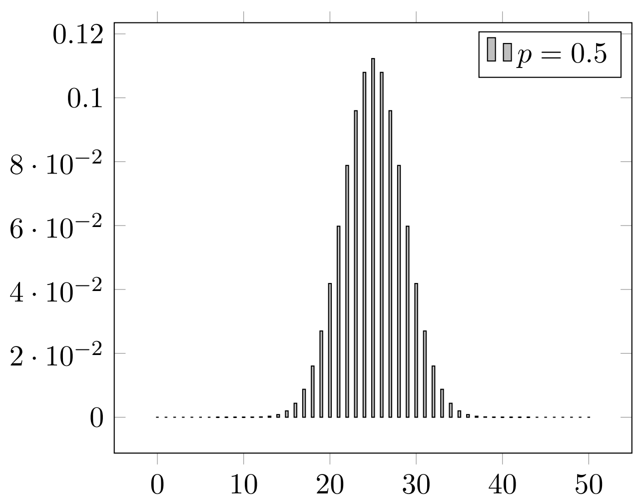
Find the value for k if the mean of X is 10.98.

1.4 The Binomial Distribution

Binomial probability is the probability theory behind independent events where there are only two possible outcomes. Since the events must be independent, the order of such events must therefore be of no influence if we are to use binomial probability theory. Studying coin flips is a perfect example for the applications of binomial probability theory.

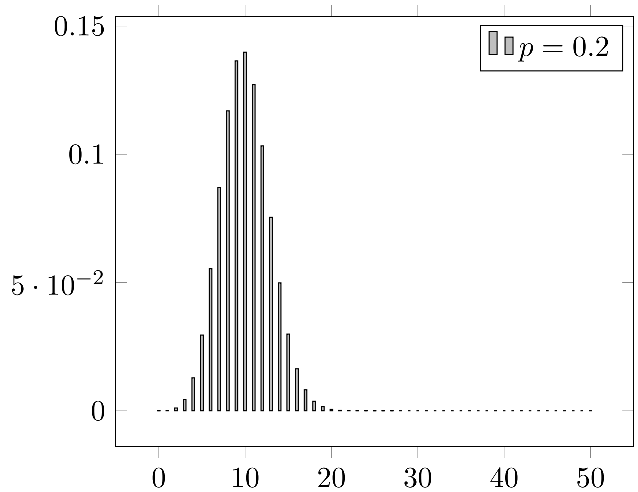
If we consider however the question of determining the probability that Alex does not shower on a Friday, given that he has a probability of 0.6 of showering on each day, then this would not be binomial, since are considering an order.

The binomial distribution is a discrete probability distribution and can be placed into three classes. Suppose there are 50 students in a class and the probability of a studnt turning up to the class is 0.5. The associated binomial distribution will take the form



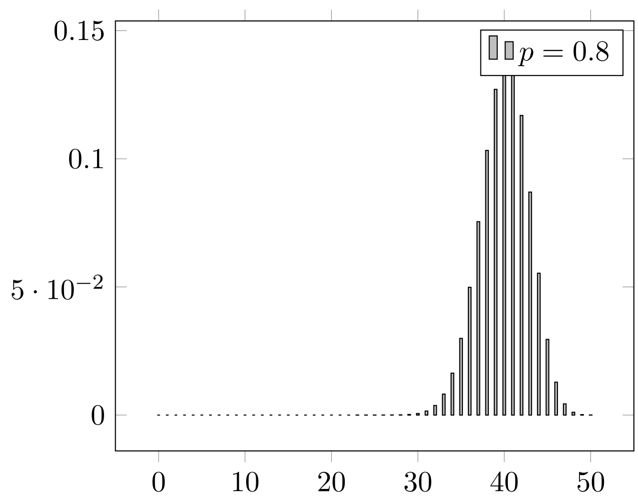
A binomial distribution of this form is said to be *symmetric*.

If the probability of a student showing up to the class is 0.2, the associated binomial distribution takes the form



A binomial distribution of this form is said to be *positively skewed*.

If the probability of a student showing up to the class is 0.8, the associated binomial distribution takes the form



A binomial distribution of this form is said to be *negatively skewed*.

Now, in the study of binomial distributions, there are two parameters involved, n, which denotes the total number of outcomes, and p, which denotes the probability of the favourable event occurring. Note that p must be fixed for each event. The probability that a particular binomially distributed event occurs is given by the formula

$$\mathbb{P}(X = x) = \binom{n}{x} p^x (1-p)^{n-x},$$

where $\binom{n}{x} = {}^nC_x := \frac{n!}{x!(n-x)!}$.

For the concerned reader, we note that the factorial ! arises from the *gamma function*, which is defined by the integral

$$\Gamma(x) = (x-1)! := \int_0^\infty s^{x-1} e^{-s} ds.$$

Notice that if we take $x = 1$, then

$$\begin{aligned} \Gamma(1) = 0! &= \int_0^\infty s^{1-1} e^{-s} ds \\ &= \int_0^\infty e^{-s} ds \\ &= -e^{-s} \Big|_0^\infty \\ &= \lim_{k\to\infty} -e^{-k} + 1 \\ &= 0 + 1 = 1. \end{aligned}$$

This is the reason why $0! = 1$.

Let us now take a look at some examples of this binomial theory.

Example 7.4.1. A fair die is rolled 4 times and the number of 3s is recorded. Determine the probability of obtaining

a. Four 3s.

Proof. We observe that $p = \frac{1}{6}, n = 4$ and $x = 4$. Therefore, using the binomial probability formula, we see that

$$\begin{aligned} \mathbb{P}(X = 4) &= \binom{4}{4} \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^{4-4} \\ &= \binom{4}{4} \cdot \frac{1}{6^4} \cdot 1 \\ &= \frac{1}{6^4} = \frac{1}{1296}. \end{aligned}$$

□

b. Less than three 3s.

Proof. Again we see that $p = \frac{1}{6}$ and $n = 4$, while $x = 1$ and $x = 2$. Hence we see that

$$\begin{aligned}
\mathbb{P}(X < 3) &= \binom{4}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{4-1} + \binom{4}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{4-2} \\
&= \binom{4}{1}\cdot\frac{1}{6}\cdot\left(\frac{5}{6}\right)^3 + \binom{4}{2}\cdot\frac{1}{36}\cdot\left(\frac{5}{6}\right)^2 \\
&= 4\cdot\frac{1}{6}\cdot\frac{125}{216} + 6\cdot\frac{1}{36}\cdot\frac{25}{36} \\
&= \frac{325}{648}.
\end{aligned}$$

□

In the previous section we looked at calculations of the expected value, variance and standard deviation for discrete random variables. In the case of binomial random variables, we have some rather useful formulae for such statistics that we present below.

Theorem 7.4.2. The expected value of a binomially distributed random variable is given by

$$\mathbb{E}(X) = np,$$

where n and p are as above.

Proof. Let us observe that for a discrete random variable X, the expected value is given by

$$\mathbb{E}(X) = \sum_{k=0}^{n} x_k p_k,$$

where p_k denotes the probability of the event x_k. Since each x_k is binomial, the probability of x_k is given by

$$p_k = \binom{n}{k} p^k (1-p)^{n-k}$$

for some fixed n and fixed p. Hence we see that

$$
\begin{aligned}
\mathbb{E}(X) &= \sum_{k=0}^{n} k\binom{n}{k}p^k(1-p)^{n-k} \\
&= \sum_{k=0}^{n} k\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k} \\
&= np\sum_{k=0}^{n} k\frac{(n-1)!}{k!(n-k)!}p^{k-1}(1-p)^{(n-1)-(k-1)} \\
&= np\sum_{k=1}^{n} k\frac{(n-1)!}{(k-1)!((n-1)-(k-1))!}p^{k-1}(1-p)^{(n-1)-(k-1)} \\
&= np\sum_{k=1}^{n} \binom{n-1}{k-1}p^{k-1}(1-p)^{(n-1)-(k-1)} \\
&= np\sum_{j=0}^{n-1} \binom{n-1}{j}p^{j}(1-p)^{n-1-j}, \quad \text{where } j=k-1, \\
&= np\sum_{j=0}^{m} \binom{m}{j}p^{j}(1-p)^{m-j}, \quad \text{where } m=n-1, \\
&= np(p+(1-p))^m \\
&= np.
\end{aligned}
$$

□

Theorem 7.4.3. The variance of a binomially distributed random variable is given by

$$\mathbb{V}\mathrm{ar}(X) = np(1-p).$$

Proof. The variance of a discrete random variable is given by $\mathbb{V}\mathrm{ar}(X) = \mathbb{E}(X^2) - [\mathbb{E}(X)]^2$. Using the previous theorem, we know that $\mathbb{E}(X) = np$ and a simple computation shows us that $\mathbb{E}(X(X-1)) = n(n-1)p^2$. Therefore,

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}(X) &= \mathbb{E}(X^2) - [\mathbb{E}(X)]^2 \\
&= \mathbb{E}(X(X-1)) + \mathbb{E}(X) - [\mathbb{E}(X)]^2 \\
&= n(n-1)p^2 + np - n^2p^2 \\
&= n^2p^2 - np^2 + np - n^2p^2 \\
&= np - np^2 \\
&= np(1-p).
\end{aligned}
$$

□

Theorem 7.4.4. The standard deviation of a binomially distributed random variable is given by

$$\sigma(X) = \sqrt{np(1-p)}.$$

Proof. Obvious. □

Exercises

Q1. Which of the following are binomially distributed random variables?

a. The weights of 25 students in a class.
b. The outcomes of rolling a die.
c. The outcome of tossing a coin.
d. The probability that Ishan shows up to class.

Q2. A coin is tossed five times. Determine the probability of obtaining

a. three heads.
b. no heads.
c. at least one tail.

Q3. Solbee has recently taken up archery and finds that she is able to get a bullseye every one out of seven occassions. Twelve rounds are fired at the target. Find the probability that Solbee gets

a. at least 9 bullseyes.
b. No bullseyes.

Q4. To keep his wife happy, Archie starts a veggie patch in his backyard. Archie plants 6 seeds in the beginning of Spring. The probability that a seed will germinate is 0.8. What is that the probability that at least 4 of the seeds will germinate?

Q5. Martyna finds that the probability of her accepting a date with Lachlan on any given day is $\frac{1}{13}$. How many days should Lachlan ask Martyna out if he wants the probability of Martyna accepting to be greater than 0.8?

Q6. A fair coin is tossed 8 times. What is the expected number of heads and the probability of obtaining this number of heads?

Q7. Let X be a binomially distributed random variable with $p = \frac{1}{3}$ and $n = 5$.

a. Find the values of $\mathbb{P}(X = x)$ for $0 \leq x \leq 5$.

b. Caclulate the mean of this distribution.

Q8. Let X be a binomially distributed random variable with mean 4 and variance 2. Find $\mathbb{P}(X = 3)$.

Q9. Let X be a binomially distributed random variable with mean 6 and standard deviation 1. Find $\mathbb{P}(X = 2)$.

1.5 Markov Chains

Markov chains allow us to use matrices to study the transitions between particular events and study long term probabilities. For example, suppose that 0.2 of the number of customers who shop at SuperMart on one day, shop at LowMart the next day; and the number 0.6 of the number of customers who shop at LowMart one day, shop at SuperMart the next day. We could capture this information in the matrix

$$T = \begin{pmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{pmatrix}.$$

This matrix is referred to as the *transition matrix.* If we wanted to look at the probabilities of moving from one shop to another after n days, then we could simply consider the nth transition matrix which is given by

$$T^n = \begin{pmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{pmatrix}^n.$$

Now suppose that there are initially 320 people who shop at SuperMart and 250 who shop at LowMart. We can represent this information in the vector

$$v_0 = \begin{pmatrix} 320 \\ 250 \end{pmatrix}.$$

The number of people shopping at each store after one day is then given by

$$Tv_0 = \begin{pmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{pmatrix} \begin{pmatrix} 320 \\ 250 \end{pmatrix} = \begin{pmatrix} 0.2 \cdot 320 + 0.4 \cdot 250 \\ 0.8 \cdot 320 + 0.6 \cdot 250 \end{pmatrix} = \begin{pmatrix} 164 \\ 406 \end{pmatrix}.$$

The number of people shopping at each store after 3 days is given by

$$\begin{aligned} T^3 v_0 &= \begin{pmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{pmatrix}^3 \begin{pmatrix} 320 \\ 250 \end{pmatrix} \\ &= \begin{pmatrix} 0.328 & 0.336 \\ 0.672 & 0.664 \end{pmatrix} \begin{pmatrix} 320 \\ 250 \end{pmatrix} \\ &= \begin{pmatrix} 188.96 \\ 381.04 \end{pmatrix} \end{aligned}$$

If we then want to determine the steady state of this transition. That is, we want to determine the number of customers shopping at each store in the long run, then we take the limit as $n \to \infty$. Computationally, ∞ is nonsense, so we take simply a large number and see if the matrix T^n differs at all from T^{n+1}. Typicaly choices for such an n are $n = 30$ and $n = 50$. If we look at our matrix here and exponentiate it to the 30th power, we see that

$$T^{30} = \begin{pmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{pmatrix}^{30} = \begin{pmatrix} 0.3333333 & 0.3333333 \\ 0.6666667 & 0.6666667 \end{pmatrix}.$$

Now by exponentiating T to the 31st power, we see that

$$T^{31} = \begin{pmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{pmatrix}^{31} = \begin{pmatrix} 0.3333333 & 0.3333333 \\ 0.6666667 & 0.6666667 \end{pmatrix},$$

so we have reached a steady state. Therefore, if we started with initial vector $v_0 = \begin{pmatrix} 320 \\ 250 \end{pmatrix}$, in the long run, we would have

$$\begin{pmatrix} 0.2 & 0.4 \\ 0.8 & 0.6 \end{pmatrix}^{30} \begin{pmatrix} 320 \\ 250 \end{pmatrix} = \begin{pmatrix} 0.3333333 & 0.3333333 \\ 0.6666667 & 0.6666667 \end{pmatrix} \begin{pmatrix} 320 \\ 250 \end{pmatrix} = \begin{pmatrix} 190 \\ 380 \end{pmatrix}.$$

Exercises

Q1. There are two Kebab shops in Canberra, Yarralumla Kebabs and Habib's Kebabs. It is known that 20% of customers who go to Yarralumla Kebabs go to Habib's Kebabs the following week, while 70% of the customers who go to Habib's Kebabs go to Yarralumla kebabs the following week.

   a. Write out the transition matrix for the Kebab shops.
   b. If there are initially 300 people who go to Yarralumla kebabs and 180 people who go to Habib's kebabs, determine the number of customers who go to Yarralumla kebabs after 4 weeks.
   c. Determine the number of customers that Yarralumla kebabs will have in the long run.

Q2. (Dr. Lloyd Gunatilake). If it rains on a particular day this week, then the chance that it rains the following day is 0.7. If it is sunny on a particular day this, then the chance that it is sunny the following day is 0.5. It rains on Tuesday.

   a. Determine the probability that it rains on Friday.
   b. Find the probability that it rains on either Wednesday, Thursday or Friday.

c. Can Markov Chains be used to find the above probability?

Q3. Mahakaran really enjoys chocolate milk. If Mahakaran drinks a chocolate milk on a particular day, he has a 70 percent chance of drinking it the next day. If he does not drink a chocolate milk on a particular day, he has a 40 percent chance of having it the next day.

a. Determine the probability that Mahakaran has a chocolate milk on Wednesday if he has a chocolate milk on Sunday.

b. Determine the probability that Mahakaran has a chocolate milk on any one of the days between Monday and Wednesday if he has a chocolate milk on Sunday.

Q4. Adrian has some terrible spending habits, but his spending is predictable enough for us to model it with a Markov chain. We know that if Adrian buys useless things one week, he has a probability of 0.2 of affording his rent the next week. Conversely, if Adrian can afford his rent this week, he has a probability of 0.9 of buying useless things the next week. Determine the steady state of Adrian's spending habits.

Q5. Jane travels between Melbourne and Sydney either by train or by flying with *HopeWeDon'tCrash* airways. If it is expected to rain during the week of her travel, Jane will have a probability of 0.8 of flying. If it it sunny during the week of her travel, Jane will take the train with a probability of 0.4. If there is a 0.6 probability that it will rain during the week of her travel, determine the probability that Jane will fly.

Q6. On any given day Nat forces her boyfriend to shop with her at either ValleyGirl or SportsGirl. Nat makes this choice based on what time she wakes up. If Nat wakes up before 1pm, they have a 0.6 chance of shopping at ValleyGirl. If Nat wakes up after 1pm however, they have a 0.2 chance of shopping at ValleyGirl.

a. If Nat has a 0.1 chance of waking up before 1pm tomorrow, what is the probability that they shop at ValleyGirl?

b. If Nat has a 0.2 chance of waking up before 1pm tomorrow, what is the probability that they shop at SportsGirl.

c. If Nat wakes up before 1pm tomorrow, what is the probability that they will shop at ValleyGirl every day for the next 4 days?

d. If Nat wakes up after 1pm on Monday, what is the probability that they shop at SportsGirl Wednesday and Thursday, given that shopped at Valleygirl on Monday and Tuesday?

1.6 Continuous Random Variables

In this section we begin to look at the continuous analogue of the probability theory that we saw in the discrete case. Our method for evaluating probabilities will be to evaluate the area under a probability distribution function. For example, if a random variable X has a probability density function given by $p(x)$, then

$$\mathbb{P}(a \leq X \leq b) = \int_a^b p(x)dx.$$

Note that as a consequence of this, a function $p : \mathbb{R} \to \mathbb{R}$ is a probability density function only if

$$\int_{-\infty}^{\infty} p(x)dx = 1,$$

that is, the sum of the probabilities is one.

Example 7.6.1. Determine the value of $k \in \mathbb{R}$ such that

$$p(x) := \begin{cases} 1 - kx^2, & 0 \leq x \leq \frac{1}{2}, \\ 0, & \text{otherwise}, \end{cases}$$

defines a probability density distribution.

Proof. We require that $\int_{-\infty}^{\infty} p(x)dx = 1$. Hence, we observe that

$$\begin{aligned} \int_{-\infty}^{\infty} p(x)dx = 1 \quad &\Longrightarrow \quad \int_{-\infty}^{\infty} (1 - kx^2)dx = 1 \\ &\Longrightarrow \quad \int_0^{\frac{1}{2}} (1 - kx^2)dx = 1 \\ &\Longrightarrow \quad x - \frac{k}{3}x^3 \Big|_0^{\frac{1}{2}} = 1 \\ &\Longrightarrow \quad \frac{1}{2} - \frac{k}{24} = 1 \\ &\Longrightarrow \quad \frac{k}{24} = -\frac{1}{2} \\ &\Longrightarrow \quad k = -12. \end{aligned}$$

□

Example 7.6.2. Suppose that X has probability density function given by

$$p(x) := \begin{cases} \frac{1}{5\pi} x \cos\left(x - \frac{\pi}{2}\right), & 2\pi \leq x \leq 3\pi, \\ 0, & \text{otherwise}. \end{cases}$$

Evaluate $\mathbb{P}\left(X < \frac{5\pi}{2}\right)$.

Proof. We simply observe that by integrating by parts, we have

$$\begin{aligned}
\int_{2\pi}^{\frac{5\pi}{2}} \frac{1}{5\pi} x \cos\left(x - \frac{\pi}{2}\right) dx &= \frac{1}{5\pi} \int_{2\pi}^{\frac{5\pi}{2}} x \cos\left(x - \frac{\pi}{2}\right) dx \\
&= \frac{1}{5\pi} \left[x \sin\left(x - \frac{\pi}{2}\right)\right]_{2\pi}^{\frac{5\pi}{2}} - \frac{1}{5\pi} \int_{2\pi}^{\frac{5\pi}{2}} \sin\left(x - \frac{\pi}{2}\right) dx \\
&= \frac{1}{5\pi} \left[x \sin\left(x - \frac{\pi}{2}\right)\right]_{2\pi}^{\frac{5\pi}{2}} + \frac{1}{5\pi} \left[\cos\left(x - \frac{\pi}{2}\right)\right]_{2\pi}^{\frac{5\pi}{2}} \\
&= \frac{1}{5\pi} \left[x \sin\left(x - \frac{\pi}{2}\right) + \cos\left(x - \frac{\pi}{2}\right)\right]_{2\pi}^{\frac{5\pi}{2}} \\
&= \frac{1}{5\pi} \left[\frac{5\pi}{2} \sin(2\pi) + \cos(2\pi) - 2\pi \sin\left(\frac{3\pi}{2}\right) - \cos\left(\frac{3\pi}{2}\right)\right] \\
&= \frac{1}{5\pi} [0 + 1 + 2\pi - 0] \\
&= \frac{1 + 2\pi}{5\pi}.
\end{aligned}$$

□

The last object of study in this section, will be the corresponding formulae for the mean, variance, standard deviation, and also how to determine the mode of a continuous random variable. All proofs are ommitted since they are obvious.

Theorem 7.6.3. The mean of a continuous random variable X is given by

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x p(x) dx,$$

where $p(x)$ denotes the probability density function for X.

Theorem 7.6.4. The variance of a continuous random variable X is given by

$$\mathbb{V}\text{ar}(X) = \int_{-\infty}^{\infty} (x - \mu)^2 p(x) dx,$$

where $p(x)$ denotes the probability density function for X and $\mu = \mathbb{E}(X)$, as given in Theorem 7.5.3.

Moreover, the standard deviation is simply given by $\sigma(X) = \sqrt{\mathbb{V}\mathrm{ar}(X)}$.

Example 7.6.5. Let X be a continuous random variable with probability density function given by

$$p(x) := \begin{cases} x, & 0 \leq x \leq 1, \\ 2 - x, & 1 \leq x \leq 2, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the variance and standard deviation of X.

Proof. The mean is given by

$$\begin{aligned} \mu = \mathbb{E}(X) &= \int_{-\infty}^{\infty} p(x)dx \\ &= \int_0^1 x \cdot x dx + \int_1^2 x \cdot (2 - x) dx \\ &= \int_0^1 x^2 dx + \int_1^2 2x - x^2 dx \\ &= \frac{1}{3}x^3 \Big|_0^1 + \left[x^2 - \frac{1}{3}x^3\right]_1^2 \\ &= \frac{1}{3} + \left[4 - \frac{1}{3} \cdot 8 - 1 + \frac{1}{3}\right] \\ &= \frac{1}{3} + 3 - \frac{8}{3} + \frac{1}{3} \\ &= 1. \end{aligned}$$

Therefore, the variance is given by

$$\begin{aligned}
\mathbb{V}\text{ar}(X) &= \int_{-\infty}^{\infty} (x-\mu)^2 p(x)dx \\
&= \int_{-\infty}^{\infty} (x-1)^2 p(x)dx \\
&= \int_0^1 (x-1)^2 x dx + \int_1^2 (x-1)^2(2-x)dx \\
&= \int_0^1 (x^2-2x+1)xdx + \int_1^2 (x^2-2x+1)(2-x)dx \\
&= \int_0^1 (x^3-2x^2+x)dx + \int_1^2 (2x^2-4x+2-x^3+2x^2-x)dx \\
&= \int_0^1 (x^2-2x^2+x)dx + \int_1^2 2-5x+4x^2-x^3)dx \\
&= \frac{1}{3}x^3 - \frac{2}{3}x^3 + \frac{1}{2}x^2\bigg|_0^1 + \left[2x - \frac{5}{2}x^2 + \frac{4}{3}x^3 - \frac{1}{4}x^4\right]_1^2 \\
&= \frac{1}{3} - \frac{2}{3} + \frac{1}{2} + \left[4 - \frac{5}{2}(4) + \frac{4}{3}(8) - \frac{1}{4}(16) - 2 + \frac{5}{2} - \frac{4}{3} + \frac{1}{4}\right] \\
&= \frac{1}{3} - \frac{2}{3} + \frac{1}{2} + 4 - 10 + \frac{32}{3} - 4 - 2 + \frac{5}{2} - \frac{4}{3} + \frac{1}{4} \\
&= \frac{1}{6}.
\end{aligned}$$

Therefore, we also see that the standard deviation is given by

$$\sigma(X) = \frac{1}{\sqrt{6}}.$$

□

Example 7.6.5. Let X be a continuous random variable with probability density function given by

$$p(x) := \begin{cases} \frac{3}{5}(2-x^2), & 0 \leq x \leq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Determine the mode of $p(x)$.

Proof. The mode is the highest point of $p(x)$. Therefore, we look at the turning point of p as well as the endpoints of p to determine the maximum of $p(x)$ on the interval $0 \leq x \leq 1$. The turning point is given by

$$p'(x) = 0 \quad \implies \quad -\frac{6}{5}x = 0 \implies x = 0.$$

Therefore, the turning point occurs at $\left(0, \frac{6}{5}\right)$. Inserting $x = 1$ for the end-point, we see that $p(1) = \frac{3}{5}(2-1) = \frac{3}{5} < \frac{6}{5}$, so the mode is $x = 0$. □

Exercises

Q1. Determine the value of $k \in \mathbb{R}$ such that

$$p(x) := \begin{cases} 1 - 4kx, & 0 \leq x \leq 2, \\ 0, & \text{otherwise}, \end{cases}$$

defines a probability distribution.

Q2. Determine the value of $k \in \mathbb{R}$ such that

$$p(x) := \begin{cases} \cos\left(\frac{k\pi}{3}x\right), & 0 \leq x \leq \pi, \\ 0, & \text{otherwise}, \end{cases}$$

defines a probability distribution.

Q3. Determine the value of $k \in \mathbb{R}$ such that

$$q(x) := \begin{cases} x^2(1 - kx), & 0 \leq x \leq 4, \\ 0, & \text{otherwise}, \end{cases}$$

defines a probability distribution.

Q4. Determine the value of $k \in \mathbb{R}$ such that

$$p(x) := \begin{cases} kxe^{-x^2}, & 0 \leq x \leq 10, \\ 0, & \text{otherwise}, \end{cases}$$

defines a probability distribution.

Q5. Let X be a continuous random variable with probability density function given by

$$f(x) = \begin{cases} 1 + 2x - ax^2, & 0 \leq x \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

a. Determine the value of $a \in \mathbb{R}$.
b. Calculate the mean of X.
c. Calculate the variance of X.
d. Calculate the standard deviation of X.
e. Calculate the mode of X.

Q6. Let X be a continuous random variable with probability density function given by

$$g(x) := \begin{cases} 1 - a\sqrt{x}, & 0 \leq x \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

a. Determine the value of $a \in \mathbb{R}$.
b. Calculate the mean of X.
c. Calculate the variance of X.
d. Calculate the standard deviation of X.
e. Calculate the mode of X.

Q7. Let X be a continuous random variable with probability density function given by

$$f(x) := \begin{cases} \frac{3}{5} - kxe^{-x}, & 0 \leq x \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

a. Determine the value of $k \in \mathbb{R}$.
b. Calculate $\mathbb{P}\left(X < \frac{1}{2}\right)$.
c. Calculate $\mathbb{P}\left(X < \frac{1}{2} | X < \frac{3}{4}\right)$.

Q8. Let X be a continuous random variable with probability density function given by

$$p(x) := \begin{cases} ke^{-x}\cos(x), & 0 \leq x \leq 2\pi, \\ 0, & \text{otherwise}. \end{cases}$$

a. Determine the value of $k \in \mathbb{R}$.
b. Calculate $\mathbb{P}(X < \pi)$.
c. Calculate $\mathbb{P}\left(\pi < x < \frac{3\pi}{2}\right)$.
d. Calculate $\mathbb{P}\left(X < \pi | X < \frac{5\pi}{4}\right)$.

Q9. Let X be a continuous random variable with probability density function given by

$$p(x) = \begin{cases} k + \frac{2}{k} x e^{-x^2}, & 0 \leq x \leq 3, \\ 0, & \text{otherwise.} \end{cases}$$

a. Determine the value of $k \in \mathbb{R}$.

b. Caclulate the mode of $p(x)$.

Q10. Let X be a continuous random variable with probability density function given by

$$p(x) := \begin{cases} k e^{-kx+2}, & x \geq 0, \\ 0, & \text{otherwise.} \end{cases}$$

a. Determine the value of $k \in \mathbb{R}$.

b. Determine the mode of $p(x)$.

1.7 Analysis Exercises

Q1. Consider a deck of cards consisting of 6 suits (spades, hearts, diamonds, clubs, swords and shields), 13 ranks (1,2,3,4,5,6,7,8,9,10,A, K,Q).

a. Determine the number of ways in which we may draw a full house.

b. Determine the number of ways in which we may draw a flush. We permit straight flushes in the definition of a flush.

c. Determine the number of ways in which we draw a straight. We order a suit in the following manner: $1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9 < 10 < A < K < Q$, where A denotes the ace, Q the Queen and K the King.

d. A standard hand consists of 5 cards and a straight flush consists of 5 cards from the same suit in a row. An example of a straight flush is the hand $(2, 3, 4, 5, 6)$, while $(1, 3, 4, 5, 6)$ is not a straight flush.

Q2. Determine the number of ways in which 5 women and 6 men can be seated around a table if

a. the women all sit together.

b. the women do not sit together.

Q3. Determine the value of $k \in \mathbb{R}$ such that

$$p(x) := \begin{cases} \log_e |x|, & 1 \leq x \leq k, \\ 0, & \text{otherwise} \end{cases}$$

defines a probability distribution.

Q4. Suppose that X is binomially distributed with mean 4 and standard deviation of $\frac{1}{2}$.

a. $\mathbb{P}(X = 3|X < 4)$.

b. $\mathbb{P}(X = 5|X > 4)$.

Q5. Suppose that A, B are two events such that $\mathbb{P}(A \cap B) = \frac{2}{5}$ and $\mathbb{P}(A \cap B') = \frac{3}{7}$.

a. Determine $\mathbb{P}(A)$.

b. Determine $\mathbb{P}(B)$.

c. Determine $\mathbb{P}(A|B)$.

d. Determine $\mathbb{P}(B'|A)$.

e. Determine $\mathbb{P}(A'|B')$.

Q6. Suppose the volume of a cone of height h and base radius $r = \frac{1}{2}$ was used to define a probability density function, where $\mathbb{P}(X = k)$ is given by the volume of the cone at height k.

a. Determine the maximal height $h_{\max}$ of the cone.

b. Evaluate $\mathbb{P}\left(X < \frac{1}{2}h_{\max}|X > \frac{1}{3}h_{\max}\right)$.

Q7. Let $f(x) = x^2$ and $g(x) = kx$. Determine the value $k \in \mathbb{R}$ such that $h(x) := f(x) - g(x)$ defines a probability density function on $[0, k]$.