Calculus Practice Exam 1 Solutions

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Question 1. Recall that a function f is said to be differentiabe at $x \in \mathbb{R}$ if the limit

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

exists and is finite. Determine whether the following functions are differentiable and calculate their derivatives.

a. $f(x) = 4x - 5$.

Proof. We simply observe that

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{4(x + \Delta x) - 5 - (4x - 5)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4x + 4\Delta x - 5 - 4x + 5}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4\Delta x}{\Delta x} \\
&= 4 \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x} \\
&= 4.
\end{aligned}$$

$\square$

b. $f(x) = 5 - x^2$.

Proof. We simply observe that

$$\begin{aligned}
\lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \lim_{\Delta x \to 0} \frac{5 - (x + \Delta x)^2 - (5 - x^2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{5 - (x^2 + 2x \cdot \Delta x + \Delta x^2) - 5 + x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{5 - x^2 - 2x \cdot \Delta x - \Delta x^2 - 5 + x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-2x \cdot \Delta x - \Delta x^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} -2x - \Delta x \\
&= -2x.
\end{aligned}$$

$\square$

c. $f(x) = 1 + 2x + 3x^3$.

Proof. We simply observe that

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{1+2(x+\Delta x)+3(x+\Delta x)^3-(1+2x+3x^3)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{1+2x+2\Delta x+3(x^3+3\cdot\Delta x^2\cdot x+3\Delta x\cdot x^2+\Delta x^3)-1-2x-3x^3}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{2\Delta x+9\Delta x^2\cdot x+3\Delta x\cdot x^2+\Delta x^3}{\Delta x}\\
&= \lim_{\Delta x\to 0} 2+9\Delta x+3x^2+\Delta x^2\\
&= 3x^2+2.
\end{aligned}$$

□

d. $f(x) = x^4$.

Proof. We simply observe that

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^4-x^4}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{x^4+4x^3\cdot\Delta x+6x^2\Delta x^2+4x\cdot\Delta x^3+\Delta x^4-x^4}{\Delta x}\\
&= \lim_{\Delta x\to 0} 4x^3+6x^2\Delta x+4x\cdot\Delta x^2+\Delta x^3\\
&= 4x^3.
\end{aligned}$$

□

e. $f(x) = 2|x|+1$.

Proof. Recall that we define

$$|x| = \begin{cases} x, & x\geq 0,\\ -x, & x<0.\end{cases}$$

Therefore,

$$f(x) = 2|x|+1 \quad = \begin{cases} 2x+1, & x\geq 0,\\ -2x+1, & x<0.\end{cases}$$

It is clear that f is not differentiable at zero. Differentiating away from zero, to the right of zero however, we have

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{2(x+\Delta x)+1-(2x+1)}{\Delta x}\\
&= \lim_{\Delta x\to 0}\frac{2x+2\Delta x+1-2x-1}{\Delta x}\\
&= 2.
\end{aligned}$$

Similarly, to the left of zero, we have

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{-2(x+\Delta x)+1-(-2x+1)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{-2x-2\Delta x+1+2x-1}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{-2\Delta x}{\Delta x}\\ &= -2.\end{aligned}$$

We therefore see that

$$f'(x) = \begin{cases} 2, & x>0,\\ -2, & x<0.\end{cases}$$

□

f. $f(x) = x^4 + x^2 + 1$.

Proof. We simply observe that

$$\begin{aligned}\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{(x+\Delta x)^4+(x+\Delta x)^2+1-(x^4+x^2+1)}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{x^4+4x^3\Delta x+6x^2\Delta x^2+4x\Delta x^3+\Delta x^4}{\Delta x}\\ &\quad+\lim_{\Delta x\to 0}\frac{x^2+2x\Delta x+\Delta x^2+1}{\Delta x}-\lim_{\Delta x\to 0}\frac{x^4+x^2+1}{\Delta x}\\ &= \lim_{\Delta x\to 0}\frac{4x^3\Delta x+6x^2\Delta x^2+4x\Delta x^3+\Delta x^4+2x\Delta x+\Delta x^2}{\Delta x}\\ &= \lim_{\Delta x\to 0} 4x^3+6x^2\Delta x+4x\Delta x^2+\Delta x^3+2x+\Delta x\\ &= 4x^3+2x.\end{aligned}$$

□

Question 2. Determine where the following functions are not differentiable. [Hint: Recall that for a function to be differentiable, the left derivative must equal the right derivative.]

a. $f(x) = |x-3| + 1$.

Proof. It is clear that f is not differentiable at $x = 3$. □

b. $f(x) = 2|x+1| + 3$.

Proof. It is clear that f is not differentiable at $x = -1$. □

c. $f(x) = \frac{1}{3}|2-4x| + \frac{2}{5}$.

Proof. It is clear that f is not differentiable at $x = \frac{1}{2}$. □

d. $f(x) = |x-1|^2 + 1$.

Proof. It is clear that f is not differentiable at $x = 1$. □

Question 3. In this exercise, we assume that all polynomials are differentiable on the entire real line $\mathbb{R}$. One may differentiate the following functions by using the formula given in Question 1, or we may simply use the rule

$$f(x) = x^n \implies f'(x) = nx^{n-1}, \quad n \in \mathbb{N},$$

which was proved in the notes. Differentiate the following functions

a. $f(x) = x^2 - 5x + 6$.

Proof. It is easy to see that $f'(x) = 2x - 5$. □

b. $f(x) = 2x + 1$.

Proof. It is easy to see that $f'(x) = 2$. □

c. $f(x) = 4x - 3x^2 + 1$.

Proof. It is easy to see that $f'(x) = 4 - 6x$. □

d. $f(x) = 2x^3 + 5x^5 + 6x^7$.

Proof. It is easy to see that $f'(x) = 6x^2 + 25x^4 + 42x^6$. □

e. $f(x) = \frac{1}{3}x^3 + 4x$.

Proof. It is easy to see that $f'(x) = x^2 + 4$. □

Question 4. The formula

$$f(x) = x^n \implies f'(x) = nx^{n-1}$$

that was given in the previous exercise may be extended to fractional powers. That is,

$$f(x) = x^{\frac{1}{2}} \implies f'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}.$$

Use this extension of the formula to differentiate the following functions.

a. $f(x) = \sqrt[3]{x}$.

Proof. It is easy to see that

$$f'(x) = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3x^{\frac{2}{3}}}.$$

□

b. $f(x) = \frac{1}{2}\sqrt{x} + 1$.

Proof. It is easy to see that

$$f'(x) = \frac{1}{4\sqrt{x}}.$$

□

c. $f(x) = 3x - 5\sqrt{x} + 4x^3$.

Proof. It is easy to see that $f'(x) = 3 - \frac{5}{2\sqrt{x}} + 12x^2$. □

d. $f(x) = \frac{2}{x}$.

Proof. It is easy to see that

$$f'(x) = -\frac{2}{x^2}.$$

$\square$

e. $f(x) = \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x}$.

Proof. It is easy to see that

$$f'(x) = -\frac{3}{x^4} - \frac{2}{x^3} - \frac{1}{x^2}.$$

$\square$

f. $f(x) = \frac{1}{3\sqrt{x}}$.

Proof. It is easy to see that

$$f'(x) = -\frac{1}{6x^{\frac{3}{2}}}.$$

$\square$

Question 5. Consider the function

$$f(x) := \sum_{k=1}^{n} 4x^{\frac{4}{5}k-\frac{1}{3}}.$$

a. Determine whether $f(x)$ is differentiable.

Proof. For each $k \in [1, n]$, the function $f_k(x) := 4x^{\frac{4}{5}k-\frac{1}{3}}$ is differentiable. Since the sum of differentiable functions is differentiable, it follows that $f(x)$ is differentiable. $\square$

b. Evaluate $f'(x)$ where $f(x)$ is differentiable.

Proof. It is easy to see that

$$f'(x) = \frac{16}{5} \sum_{k=1}^{n} x^{\frac{4}{5}k-\frac{4}{3}}.$$

$\square$

Question 6. Consider the function

$$f(x) := \frac{1}{x^2 - 5x + 6}.$$

a. Determine where $f(x)$ is continuous.

Proof. The function $f(x)$ is continuous at all points where $x^2 - 5x + 6 \neq 0$. It is clear that $x^2 - 5x + 6 = 0$ exactly when $x = 3$ or $x = 2$. The function f is therefore continuous for all $x \in \mathbb{R} \backslash \{2, 3\}$. $\square$

b. Determine the values of A and B such that

$$f(x) = \frac{A}{x-3} + \frac{B}{x-2}.$$

Proof. We simply observe that

$$\begin{aligned} \frac{1}{x^2 - 5x + 6} &= \frac{A}{x-3} + \frac{B}{x-2} \\ \therefore\ 1 &= A(x-2) + B(x-3). \end{aligned}$$

Inserting $x = 2$, we see that $B = -1$ and. Inserting $x = 3$, we see that $A = 1$. It so follows that

$$f(x) = \frac{1}{x-3} - \frac{1}{x-2}.$$

$\square$

c. Determine where $f(x)$ is differentiable.

Proof. Being the composition of differentiable functions on $\mathbb{R} \backslash \{2, 3\}$, it follows that $f(x)$ is differentiable on $\mathbb{R} \backslash \{2, 3\}$. $\square$

d. Evaluate $f'(x)$.

Proof. Beyond what we have currently done, but the answer is

$$f'(x) = -\frac{1}{(x-3)^2} + \frac{1}{(x-2)^2}.$$

$\square$