Chapter 1

Mechanics.

This chapter serves as an elementary introduction to the mathematics behind classical mechanics. Loosely speaking, classical mechanics is the physics behind the motion of macro-sized objects such as cars, tennis balls, objects that we deal with all the time. The beauty of this chapter is that it serves as the primary application of theory of vectors we saw in Chapter 8 and the Calculus covered in Chapter 5 and 6.

1.1 Displacement, Velocity and Acceleration

If we fix an origin, then the distance (with positive and negative signs recorded) from the origin is referred to as the displacement. The rate of change of the displacement with respect to time is the velocity. The rate of change of the velocity with respect to time is the acceleration. Hence, we see that

$$\boxed{\text{Displacement}} \xrightarrow{\text{Differentiate}} \boxed{\text{Velocity}} \xrightarrow{\text{Differentiate}} \boxed{\text{Acceleration}}.$$

Or equivalently,

$$\boxed{\text{Displacement}} \xleftarrow{\text{Integrate}} \boxed{\text{Velocity}} \xleftarrow{\text{Integrate}} \boxed{\text{Acceleration}}.$$

Example 9.1.1. Suppose that the position of an object is given by $x(t) = t^2 - 5t + 1$. Determine the equations for the velocity and the acceleration.

Proof. We simply observe that the velocity is given by

$$\frac{d}{dt}x(t) = x'(t) = 2t - 5.$$

Moreover, the acceleration is given by

$$\frac{d}{dt}x'(t) = x''(t) = 2.$$

□

Example 9.1.2. The acceleration of an object is given by

$$a(t) = 12 - \frac{1}{t}.$$

Moreover, the initial position and velocity at both zero. Determine the expressions for the displacement and velocity.

Proof. The velocity is given by

$$\begin{aligned} v(t) &= \int_0^t a(s)ds \\ &= \int_0^t 12 - s^2 ds \\ &= 12s - \frac{1}{3}s^3 \Big|_0^t \\ &= 12t - \frac{1}{3}t^3 - \left(12(0) - \frac{1}{3}(0)^3\right) \\ &= 12t - \frac{1}{3}t^3. \end{aligned}$$

The displacement is given by

$$\begin{aligned} x(t) &= \int_0^t v(s)ds \\ &= \int_0^t 12s - \frac{1}{3}s^3 ds \\ &= \frac{12}{2}s^2 - \frac{1}{12}s^4 \Big|_0^t \\ &= 6t^2 - \frac{1}{12}t^4. \end{aligned}$$

□

Example 9.1.3. Suppose that the position of a particle is given by $\mathbf{r}(t) = 3\mathbf{i} + 4t^2\mathbf{j}$.

a. Determine the magnitude of $\mathbf{r}(t)$.

Proof. Using Pythagoras, we simply observe that

$$\begin{aligned} |\mathbf{r}(t)| &= \sqrt{3^2 + (4t^2)^2} \\ &= \sqrt{9 + 16t^4}. \end{aligned}$$

□

b. Determine the velocity $\mathbf{v}(t)$.

Proof. We simply differentiate to see that

$$\begin{aligned} \mathbf{v}(t) &= \frac{d}{dt}\mathbf{r}(t) \\ &= 0\mathbf{i} + 8t\mathbf{j} \\ &= 8t\mathbf{j}. \end{aligned}$$

□

Exercises

Q1. Determine the velocity and acceleration of an object whose displacement is given by the following expressions.

a. $x(t) = 1 + 2t$.
b. $x(t) = t^2 - 3t + 1$.
c. $x(t) = \cos(t) + 1$.
d. $x(t) = 2t + e^{-t}$.
e. $x(t) = \frac{1}{t^2+1}$.
f. $x(t) = 1 - 4t + e^{2t-t^3}$.

Q2. Determine when the velocity is greatest for the objects whose position is given by the following expressions.

a. $x(t) = 1 + t^2$.
b. $x(t) = 1 - 2t^2 + 4t$.
c. $x(t) = e^{-t}\cos(t)$.
d. $x(t) = 1 + \sin(2t)$.
e. $x(t) = e^{2t}\cos(t) + e^{2t}\sin(t)$.
f. $x(t) = 3e^t + 2e^{-t}$.

Q3. Determine the velocities of the objects whose accelerations are given by the following expressions with initial condition $v(0) = 1$.

a. $a(t) = 1 + 2t$.

b. $a(t) = 1 - e^{-4t}$.

c. $a(t) = 4 + \cos(t^2)$.

d. $a(t) = \frac{1}{t+1} - 2t$.

e. $a(t) = t^3 + e^{-4t}$.

f. $a(t) = \frac{1}{\sqrt{1+t^2}} + t$.

Q4. Determine the displacements of the objects whose velocities are given by the following expressions with initial condition $x(0) = 0$.

a. $v(t) = 2t + e^{-4t}$.

b. $v(t) = \frac{1}{t^2-1} + \tan(t)$.

c. $v(t) = |t - 1|$.

d. $v(t) = 4t - t^3$.

e. $v(t) = 2t^3 + t\log_e(t + 1)$.

f. $v(t) = \frac{t}{t^2+1}$.

Q5. Suppose the velocity of an object is given by

$$v(t) = \frac{(t-2)^2}{\sqrt{2-t}}.$$

Determine the equation for the displacement if $x(0) = 4$.

Q6. Suppose the velocity of an object is given by

$$v(t) = \frac{e^{3t}}{e^t - 1}.$$

Determine the equation for the displacement if $x(0) = 1$.

Q7. Suppose the velocity of an object is given by

$$v(t) = \cos^2(t)\sin^3(t).$$

Determine the equation for the displacement if $x(0) = 0$.

Q8. Suppose the velocity of an object is given by

$$v(t) = \frac{1}{2}\tan^2\left(\frac{t}{2}\right)\sec^4\left(\frac{t}{2}\right).$$

Determine the equation for the displacement if $x(0) = 0$.

Q9. Suppose the velocity of an object is given by

$$v(t) = \frac{1}{3}\cos^6(t)\sin^3(t).$$

Determine the equation for the displacement if $x(0) = \frac{1}{2}$.

Q10. Suppose the velocity of an object is given by

$$v(t) = \frac{t^2 + 5t + 1}{(t^2 + 1)(2 - t)}.$$

Determine the equation for the displacement if $x(0) = 2$.

Q11. Determine the velocities of the objects whose position is given by the following expressions.

a. $\mathbf{r}(t) = t^2\mathbf{i} - 3t\mathbf{j}$.

b. $\mathbf{r}(t) = \mathbf{i} - \frac{1}{t^2+1}\mathbf{j}$.

c. $\mathbf{r}(t) = 2\cos(3t)\mathbf{i} + 3\sin(3t)\mathbf{j}$.

d. $\mathbf{r}(t) = 3\tan(t)\mathbf{i} + 2t\mathbf{j}$.

e. $\mathbf{r}(t) = 4t\log_e(1 - t)\mathbf{i} + 4t^3\mathbf{j}$.

f. $\mathbf{r}(t) = (t^2 - 1)\mathbf{i} + 8t\mathbf{j}$.

Q12. Determine the positions of the objects whose velocity is given by the following expressions with initial condition $\mathbf{r}(0) = 0\mathbf{i} + 0\mathbf{j}$.

a. $\mathbf{v}(t) = (1-2t)\mathbf{i} + \log_e(1-t)\mathbf{j}$.

b. $\mathbf{v}(t) = \left(\frac{1}{t^2+1}\right)\mathbf{i} + t\mathbf{j}$.

c. $\mathbf{v}(t) = \cos(2t)\mathbf{i} - 3\tan(t)\mathbf{j}$.

d. $\mathbf{v}(t) = \frac{1}{t-4}\mathbf{i} + 7t^4\mathbf{j}$.

e. $\mathbf{v}(t) = \frac{4t+1}{3-t}\mathbf{i} + (5t + 4)\mathbf{j}$.

f. $\mathbf{v}(t) = t\sqrt{t^2 + 1}\mathbf{i} + (2 - t)\mathbf{j}$.

Q13. Let $\mathbf{r}(t) = 2t\mathbf{i} + 5t\mathbf{j} + 8t^3\mathbf{k}$ be the position of a particle. Determine the angle between the position and the velocity.

1.2 Motion Under Constant Acceleration

In this section we look at solving problems where the acceleleration is independent of time. Indeed, suppose that $\frac{dv}{dt} = a$, then we see that

$$\begin{aligned} \frac{dv}{dt} = a \quad &\Longrightarrow \quad dv = adt \\ &\Longrightarrow \quad \int dv = \int adt \\ &\Longrightarrow \quad v = v_0 + at, \end{aligned}$$

where v_0 is the initial velocity. Similarly, suppose that $\frac{dx}{dt} = v$, then we see that

$$\begin{aligned} \frac{dx}{dt} = v \quad &\Longrightarrow \quad \frac{dx}{dt} = v_0 + at \\ &\Longrightarrow \quad dx = (v_0 + at)dt \\ &\Longrightarrow \quad \int dx = \int (v_0 + at)dt \\ &\Longrightarrow \quad x = v_0 t + \frac{1}{2}at^2. \end{aligned}$$

An obvious formula that requires no calculus to derive is given by

$$x = \frac{v + v_0}{2}t,$$

or just as easily,

$$t = \frac{v - v_0}{a}.$$

With these in mind, we see that

$$\begin{aligned} x = v_0 t + \frac{1}{2}at^2 \quad &\Longrightarrow \quad x = v_0\left(\frac{v - v_0}{a}\right) + \frac{1}{2}a\left(\frac{v - v_0}{a}\right)^2 \\ &\Longrightarrow \quad x = \frac{v_0}{a}(v - v_0) + \frac{1}{2a}(v - v_0)^2 \\ &\Longrightarrow \quad 2ax = 2v_0(v - v_0) + (v - v_0)^2 \\ &\Longrightarrow \quad 2ax = 2v_0 v - 2v_0^2 + (v^2 - 2vv_0 + v_0^2) \\ &\Longrightarrow \quad 2ax = 2v_0 v - 2v_0^2 + v_0^2 + v^2 - 2vv_0 \\ &\Longrightarrow \quad 2ax = v^2 - v_0^2 \\ &\Longrightarrow \quad v^2 = v_0^2 + 2ax. \end{aligned}$$

Example 9.2.1. Suppose that Jono has a car which is able to brake with a constant deceleration of 4.3 m/s^2. Determine how long it will take the car to stop if it is initially travelling at 27 m/s and how far it travels in this time.

Proof. The initial velocity is $v_0 = 27$, the acceleration is $a = -4.3$ and the final velocity is $v = 0$. Using the formula $v = v_0 + at$, we see that

$$t = \frac{v - v_0}{a} = \frac{-27}{-4.3} = \frac{27}{4.3} \approx 6.3.$$

Using the formula $x = v_0 t + \frac{1}{2}at^2$, we see that

$$\begin{aligned} x &= 27 \cdot 6.3 + \frac{1}{2} \cdot -4.3 \cdot 6.3^2 \\ &\approx 84.8. \end{aligned}$$

□

A very important example of situations that involve constant acceleration are free fall problems since gravity has a constant acceleration of $g = 9.8$ m/s^2.

Exercises

Q1. Solbee is driving her car at a constant speed of 80 $km\ h^{-1}$ in a school zone when she passes a stationary police car. The policeman sets off in pursuit, accelerating uniformly for 10 seconds until he reaches a speed of 100 $km\ h^{-1}$. He then maintains this speed until he catches Solbee. How long does it take for the Policeman to reach Solbee and put her away?

Q2. Ishan is investigating the effect of gravity on objects of different masses. He holds a tomato and a bowling ball in his hands.

a. Assuming that air resistance is negligible, which object will reach the ground first if Ishan drops both objects at the same time?

b. Suppose Ishan drops the tomato from a height of 2 meters above the ground. What will the velocity of the tomato be when it hits the ground?

Q3. Determine the velocity required to launch a ball vertically from the ground to a height of 40 metres.

Q4. An object falls from a height of 2000m above the ground. How fast is the object moving when it hits the ground?

Q5. Khursh is driving his car at 84 km/h when he realises that there is a barrier on the road 32 metres away to prevent people from passing. At this point he slams on the brakes of his car. Unfortunately, it was too late and his car hit the barrier 3 seconds later.

a. Assuming his deceleration is uniform, what is the magnitude of the car's deceleration.

b. Determine the speed of the car upon impact.

c. If the mass of the car is given to be 1400 kg and the force that is felt on the car upon impact is given by $F = ma$, where m is the mass and a is the acceleration, what is the force that Khursh feels upon hitting the barrier?

d. What is meant by the fact that F is negative in part c?

Q6. A bowling ball hits the ground with a velocity of 37 ms^{-1}.

a. Determine the height from which it fell.

b. Determine the length of time that it was falling.

Q7. Layla throws a spanner vertically upwards with an initial velocity of 14 ms^{-1} at an initial height of 1.2 m.

a. Determine the maximum height that the spanner reaches.

b. Determine the time taken for the spanner to hit the ground.

c. Determine the velocity of the spanner upon impact.

Q8. David likes to throw his eraser up in the air during his exams since he can never answer any of the questions. If David throws his eraser with an initial velocity of 8 ms^{-1} from a height of 0.6 m, determine the time taken for the eraser to reach the ground.

Q9. Barnsley is new pilot and is flying horizontally at a height of 50m with velocity 900 kmh^{-1}. The ground is initially flat, but after 6 minutes in the air, the ground slopes up at an angle of $\vartheta = 7.2°$. If Barnsley does not change the direction of the plane, how long will it take before Barnsley hiits the ground?

Q10. The Queensland government has decided to change all speed limits in rural areas from 100 kmh^{-1} to 130 kmh^{-1}. If Ling's commute to work is typically 40 minutes and occurs entirely in rural areas, how much time does he save from this government decision?

Q11. If a rock reaches a height of h after being launched with a velocity of v, what velocity is required for the rock to reach a height of $2h$?

Q12. If a rock reaches a height of h, it takes time t to reach the ground. At what height must the rock reach for it to take time $2t$ to reach the ground?

Q13. Dhiviya uses a slingshot to launch stones vertically up in the air. During the course of the trajectory, the stone has a velocity v at height h_0 and a velocity of $\frac{1}{3}v$ at a height $h_1 = h_0 + 4$.

a. Determine the value of v.

b. Determine the maximum height of the stone.

c. Determine the time taken for the stones to reach the ground.

d. At what velocity are the stones travelling when they hit the ground?

Q14. Two balls are released from rest into free fall one second apart.

a. How long after the first ball is in free fall does it take for the two balls to be 5 m apart?

b. How many metres do the balls have to fall in order to be 5m apart?

Q15. An object moves with a velocity given by $\mathbf{v}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j}$. Is this object moving at a constant acceleration? Justify your answer.

1.3 Projectile Motion

A body which moves in $\mathbb{R}^2$ with some initial velocity v_0, acted on only by the constant acceleration of gravity g is referred to as a projectile. The motion of a projectile, is suitably called projectile motion. Given that the vectors $e_1 := \mathbf{i} + 0\mathbf{j}$ and $e_2 := 0\mathbf{i} + \mathbf{j}$ are linearly independent, it follows that the vertical motion and horizontal motion of a projectile are independent.

The beauty is that even though the motion is two-dimensional, we may solve problems by treating them as two one-dimensional problems.

Let us therefore begin by looking at the horizontal component of the motion. To this end, we first note that the action of gravity on an object is vertical, so the horizontal component of the velocity of an object remains constant throughout the motion. If v_0 denotes the initial velocity of the projectile, which is launched at an angle ϑ, then it is clear that the horizontal velocity is given by

$$v_x = v_0 \cdot \cos\vartheta.$$

The vertical component of the velocity, unlike the horizontal component, is acted on by gravity, and so the vertical velocity is not in general constant. The vertical velocity is given by

$$v_y = v_0 \cdot \sin\vartheta - gt,$$

where g denotes gravity and in particular, the initial vertical velocity is given by

$$v_y - v_0 \cdot \sin\vartheta.$$

Example 9.3.1. A ball is launched from the top of cliff of height 15m with a velocity of 5 ms^{-1} angled at 30° above the horizontal.

a. Determine the highest point that the ball reaches above the ground.

Proof. For our purposes of solving for the height, we are concerned only with the vertical component of the velocity. The initial vertical velocity is given by

$$u_x = v_0 \cdot \sin 30^\circ = 5 \cdot \sin 30^\circ = \frac{5}{2}.$$

The vertical velocity at the hightest point is 0, so $v_x = 0$; and the acceleration is constant at -9.8. We therefore see that

$$\begin{aligned} v_x^2 = u_x^2 + 2ah \quad &\implies \quad 0^2 = \left(\frac{5}{2}\right)^2 - 2gh \\ &\implies \quad h = \frac{1}{2g} \cdot \frac{25}{4} \\ &\implies \quad h \approx 0.32 \text{ m} \\ &\implies \quad h \approx 15.32 \text{ m above the ground.} \end{aligned}$$

$\square$

b. Determine the time taken for the ball to reach the ground.

Proof. If we start from the highest point of the motion, the initial vertical velocity is $u_x = 0$. Moreover, the acceleration remains -9.8 and the displacement is given by $h = 15.96$. We therefore see that

$$\begin{aligned} h = u_x t + \frac{1}{2}at^2 \quad &\implies \quad 2h = 2u_x t + at^2 \\ &\implies \quad 2h = at^2 \\ &\implies \quad t = \sqrt{\frac{2h}{a}} \\ &\implies \quad t = \sqrt{\frac{2h}{g}} \\ &\implies \quad t = \sqrt{\frac{2 \cdot 15.32}{9.8}} \approx 1.768 \text{ s.} \end{aligned}$$

Moreover, the time to get to the highest point is given by observing that $v_x = 0$, $u_x = \frac{5\sqrt{3}}{2}$, $a = -9.8$ and therefore

$$\begin{aligned} v_x = u_x + at \quad &\implies \quad at = -u_x \\ &\implies \quad t = -\frac{u_x}{a} \\ &\implies \quad t = \frac{5}{2 \cdot 9.8} \\ &\implies \quad t \approx 0.255. \end{aligned}$$

Therefore the time taken to reach the ground is given by

$$t \approx 1.768 + 0.255 = 2.023 \text{ s.}$$

$\square$

c. Determine the velocity of the ball when it hits the ground.

Proof. Starting at the highest point of the motion, we have an initial vertical velocity of $u_x = 0$. We know that $h = 15.96$ and the acceleration is given by $a = -9.8$. Hence we see that

$$v_x = u_x + at \quad \Longrightarrow \quad v_x = 0 - 9.8 \cdot 15.32 \approx 150.136 \text{ ms}^{-1}.$$

□

d. Determine the horizontal displacement of the ball when it reaches the ground.

Proof. Since the horizontal velocity is constant, it is immediate that the horizontal displacement R is given by

$$\begin{aligned} R = u_h t &= v_0 \cdot \cos\vartheta \cdot t \\ &\approx \frac{5\sqrt{3}}{2} \cdot 2.023 \approx 8.76 \text{ m}. \end{aligned}$$

□

Exercises

Q1. Flint fires an arrow from his bow 25 m above the ground at a speed of 185 ms^{-1}.

a. How long does ittake for the projectile to hit the ground?

b. How far, horizontally, does the projectile travel?

Q2. A model plane is thrown in the air from a height of 10m above the ground. The initial speed of the launch is 10 ms^{-1} at an angle of $\vartheta = 30°$.

a. Determine the maximum height that the model plane reaches.

b. Determine the the time it takes for the model plane to reach the highest point in its trajectory.

c. Determine the time it takes to reach the ground.

d. Determine the horizontal distance the model plane travels.

Q3. Jesse likes to ride his motorcylce and do tricks. The greatest height that he has been able to reach when taking off from a ramp angled at 9° to the horizontal is 62 m.

a. Determine the velocity that Jesse is travelling at when he takes off.

b. Jesse tries really hard to make it to a height of 72 m, what velocity is required when he leaves the ramp to get to a height of 72 m?

Q4. Skinny D is playing cricket with his friends and hits a cricket ball with a velocity of 23 ms^{-1} and an angle of 40°. He is trying to hit the cricket ball so that it lands on his neighbour's roof which is a horizontal distance of 14m away and a vertical distance 3m away.

a. Determine whether Skinny D hits the ball onto the roof.

b. If not, determine the velocity required to launch the ball on the roof if the angle remains at 40°.

Q5. Salmon are launching off a waterfall of height 3m at an angle of 20° from the horizontal. A bear is readily waiting for the salmon at a height of 2.7m to catch the salmon. If the salmon are initially swimming with a velocity of 10 ms^{-1}, determine the following.

a. The velocity of the salmon when the bear catches them.

b. The horizontal distance the bear must place itself in order to catch the salmon.

c. The velocity of the salmon upon landing in the water at the bottom of the waterfall.

d. The velocity of the bear if it is to stumble and fall into the water.

e. Which object would fall faster, the salmon or the bear?

Q6. Determine the angle ϑ which will maximise the horizontal range of a projectile which is launched with an initial velocity v_0 at an angle of ϑ.

Q7. A projectile is launched with an initial velocity v_0 at an angle ϑ and attains a maximum height h. If v_0 is 4 times its velocity at the maximum height, determine the value of ϑ.

Q8. Jocko has just moved to the unit states with his friend Rocko. Jocko buys a rifle from the local firearms store which launches bullets at a velocity of 260 ms^{-1}. After having a few beers one night Jocko and Rocko thought it would be a good idea to go out shooting tin cans. As Rocko begins to set up the cans, Jocko has accidentally fired the rifle at an angle of $20°$ below the horizontal and hit Rocko's foot.

a. Determine the horizontal distance between Rocko's foot and the rifle.

b. If it takes Rocko 0.2 seconds to realise what happened, how long after the rifle had been shot does it take Rocko to react to the pain?

Q9. Flying possums are able to launch themselves between trees at a velocity of 1 ms^{-1} and can launch themselves at a maximum angle of $60°$ relative to the horizontal. What is the furtherest distance that any two trees can be from each other such that a flying possum may able to land on the other tree if it leaves the tree initially at a height of 4m above the ground?

Q10. Blinky is a Koala that tends to have a few too many Eucalyptus leaves during the day, Him and his girlfriend Nutsy have been together for roughly three years now, but Nutsy has had enough. When Blinky is sleeping, Nutsy kicks him out of a tree at an angle of $10°$ below the horizontal at a height of 3m above the ground, the launches Blinky with an initial velocity of 2 ms^{-1}.

a. How long does it take Blinky to reach the ground?

b. How fast is Blinky moving when he hits the ground?

c. How far away from his home (the exact point in the tree) is Blinky when he lands on the ground?

Q11. Upon hearing of Blinky and Nutsy separating, Mayor Pelican flies over to see how Blinky is doing. Mayor Pelican arranges another tree for Blinky to stay in, which is 15m horizontally away from his previous home with Nutsy. Nutsy is not pleased by the fact that she can still see Blinky every day. In order to drive him out of the area, Nutsy builds a slingshot with the help of Danny Dingo which fires pine cones at speeds of up to 10 ms^{-1}. If Blinky's new home is 15m away horizontally and 0.5m below vertically from his old home, is it possible for Nutsy to hit Blinky with her pine cones?

Q12. Suppose a ball is launched from the ground at a velocity of 15 ms^{-1} at an angle of $30°$. After travelling for one second, the ball lands in a chamber that reverses the direction of gravity. That is, gravity now acts upwards as opposed to downwards. If the chamber has a height of 10m from the ground, determine how long it takes for the ball to hit the roof of the chamber.

Q13. A sharpener is horizontally thrown off a table of height 1.3m with an initial velocity v_0. The speed at which the ball hits the ground is $v = 3v_0$.

a. Determine the value of v_0.

b. Determne the of the velocity when the sharpener hits the floor.

Q14. Michael always shoots hoops at an angle of 48° from the horizontal from a height of 1.9 m above the ground and 2 m horizontally from the hoop. The hoop has a height of 2.9 m. Determine the initial velocity that Michael must throw the ball with to make the shot.

Q15. Michael also has a trick shot. If he stands in the same position as in the previous question, he is able to throw the ball towards the ground and rebound it such that it still lands in the hoop. The point on the ground which he rebounds the ball from is 0.6 m away from him. If the angle at which he throws the ball is 60° below the horizontal, what initial velocity does Michael throw the ball in order to make the trick shot?

1.4 Newton's Laws of Motion

Newton's laws of motion describe the ways in which objects that we encounter on a daily basis behave upon applying a force. Before proceeding with this classical theory however, we begin with some necessary definitions.

Definition 9.4.1. We define the momentum of an object with velocity v and mass m to be the quantity

$$p(t) = m \cdot v(t), \qquad m \in \mathbb{R}.$$

With this definition in mind, we have the following definition.

Definition 9.4.2. We define the force on an object with moment p at time t to be the quantity

$$F(t) := \frac{dp}{dt}.$$

That is, the force on an objective is the rate of change of the object's momentum with respect to time. We invite the reader to verify that this definition agrees with their physical intuition. Force is measured in kg m s^{-2} often referred to as Newtons and denoted by N. To provide some intuition for this unit 1 N is approximately 100 grams, which is roughly the weight of an apple.

Note that as a consequence of the above definitions, we see that

$$F(t) = \frac{dp}{dt} = \frac{d}{dt}p(t) \quad = \quad \frac{d}{dt}mv(t) = m\frac{dv}{dt} = m \cdot a(t).$$

This is actually the statement of Newton's second law.

Particular forces that we care about are:

† Weight - This force is due to gravity g. If an object has m, the weight of an object is given by $W = mg$.

† The normal forice - This is the reaction force induced by the weight force acting on a surface. It is a consequence of Newton's third law which we will dicuss shortly. The normal force is the induced force that exists perpendicular to the surface.

† Frictional Forces - These are retarding forces that act against the motion of a body. Common examples of frictional forces are air resistance and the friction on a surface.

Definition 9.4.3. We define the resultant force, or net force, Σ on an object to be the sum of all forces acting on the object. Indeed, this notion is worth considering given that force is a vector.

We now state Newton's Laws of Motion.

Newton's First Law. An object that is at rest will remain at rest unless acted on by an external. A body moving at a constant velocity in a straight line will remain at a constant velocity in a straight line unless acted on by an external force. This is phenomenon is referred to as *inertia*.

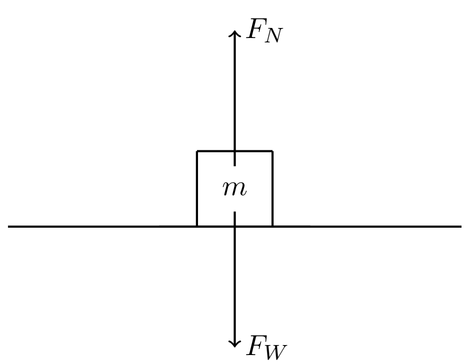
Newton's Second Law. The force on an object is proportional to its acceleration,

$$F = ma.$$

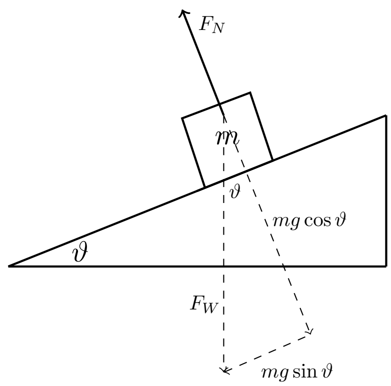
Newton's Third Law. An object A acting on an object B with force F_{AB} will experience of a force of equal magnitude, but opposite in direction.

Let us now discuss the normal force in a little more depth now that we have an understanding of Newton's Laws of Motion.

If we have an object of mass m on a surface, then the object has a weight $F_W = mg$ which acts on this surface. Newton's third law then states that there is a force in reaction to this weight force which acts on the object, as shown in the diagram with $F_N = -F_W$,



It is this reaction force that is referred to as the normal force. The above diagram which labels all forces acting on an object is often referred to as a free body diagram. If our surface if angled, the situation is a little more interesting. Indeed, if we angle our surface by some angle ϑ, then we see that



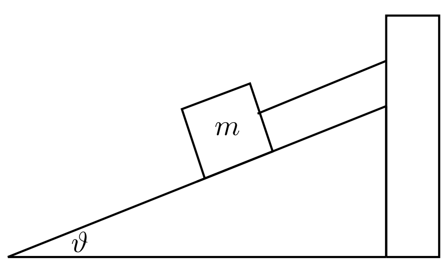
From the diagram we see that the weight force is decomposed into the sum

$$F_W = mg\cos\vartheta + mg\sin\vartheta =: F_N + F_R,$$

where F_N denotes the normal force and F_R denotes the frictional force. That is,

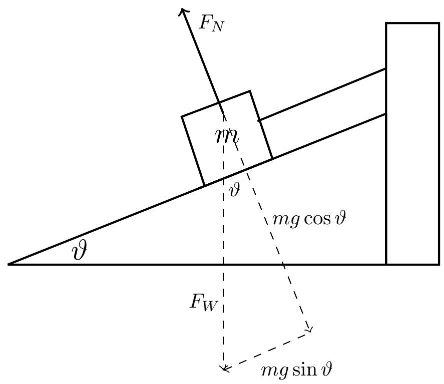
$$F_N = mg\cos\vartheta, \quad F_R = mg\sin\vartheta.$$

Example 9.4.1. Consider a block of mass $m = 2$ kg on a ramp which is angled $\vartheta = 35°$ from the horizontal, held stationary by a massless rope as seen in the diagram below.



a. Determine the tension in the rope.

Proof. Let us first draw the associated free body diagram



We therefore see that the tension in the rope is given by

$$T = mg\cos\vartheta = 2 \cdot 9.8 \cdot \cos(35^\circ) \approx 16.06 \text{ N}.$$

□

b. Determine the normal force acting on the block.

Proof. From the free body diagram, we see immediately that

$$F_N = mg\sin\vartheta = 2 \cdot 9.8 \cdot \sin(35^\circ) \approx 11.24 \text{ N}.$$

□

c. Determine the magnitude of the acceleration of the block if the rope is cut.

Proof. The net force on the block is $F = mg\sin\vartheta$, therefore we see that

$$a = \frac{1}{m}F = g\sin\vartheta \approx 5.62.$$

□

Exercises

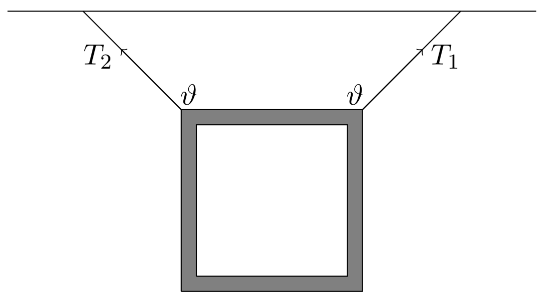
Q1. Thuva does not enjoy studying much, instead he likes to put all of his textbooks in a crate, attach a string to the crate, and drag it across the floor of the tutorial room. The crate has a mass of 60kg. Thuva pulls the crate with 120 Newtons of force at an angle of $\frac{\pi}{4}$ from the horizontal. Calculate the horizontal and veritcal components of the force.

Q2. Jack has not been going to the gym recently, and subsequently, him and his bike have a combined mass of 80kg. It would usually take him only 5.0 seconds to accelerate to 8.0 ms^{-1}. Now however, it takes him 10 seconds.

a. Determine Jack's acceleration during this time.

b. Calculate how weak the force is from Jack's legs in order to produce the acceleration determined in part (a).

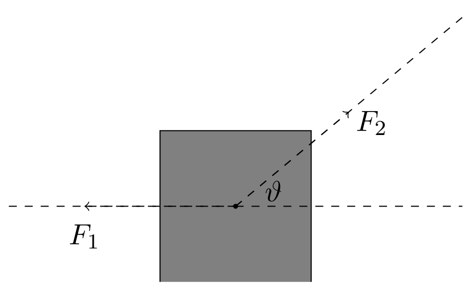
Q3. A picture frame of mass 2 kg is suspended as shown in the diagram below. The tension in string on the right is given to be $T_1 = 15$ N.



a. Determine the tension in the second rope, T_2.

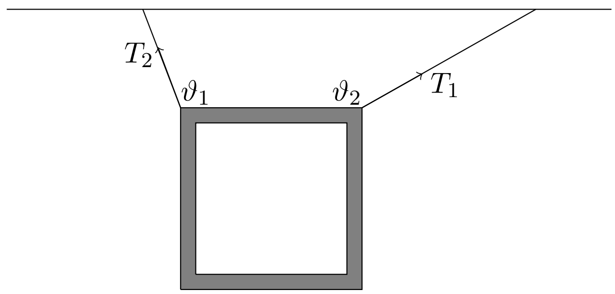
b. Determine the value of ϑ such that picture frame remains upright.

Q4. Consider two forces F_1 and F_2 pulling on a crate. F_1 has a magnitude of 120 N, while the second force F_2 has a magnitude of 150 N. Determine the value of ϑ such that the crate moves at a constant velocity.



Q5. A picture frame of mass 3 kg is suspended as shown in the diagram below. The values of $\vartheta_1 = 110°$ and $\vartheta_2 = 165°$. Suppose the value of

T_1 is 8 N. Determine the value of T_2.



Q6. Jimmy does not have a girlfriend, and therefore spends his Friday nights riding up and down an elevator. Jimmy has a mass of 75 kg.

a. If the elevator is stationary, what is the reaction force that Jimmy experiences?

b. Suppose the elevator now accelerates down at a rate of 2 ms^{-2}. What is the reaction force that Jimmy experiences?

c. Does the reaction force change if the elevator accelerates up at a rate of 2 ms^{-2}?

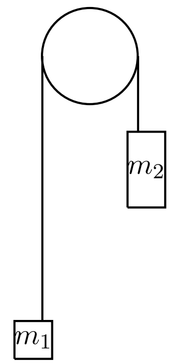
Q7. A load of mass 200 kg is raised by a cable. Determine the tension in the cable if

a. The load is lifted at a constant speed of 1.5 ms^{-1}.

b. The load is lifted with an acceleration of 0.5 ms^{-2}.

Q8. Jamie has bought a new motorcycle helmet that he claims will prevent him from any serious injury in a car accident. The helmet is able to withstand a force of 8000 N. Eager to test out his new helmet, Jamie finds a long strip of road with a solid brick wall at the end of it. The combined mass of Jamie and his car is known to 850 kg and he accelerates at a rate of 12.5 ms^{-2}. Determine Jamie's fate.

Q9. Two blocks are connected by a smooth rope as shown in the figure below. The value of m_1 is 2 kg, and the value of m_2 is 4kg.

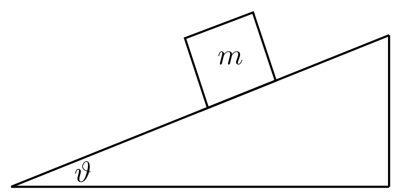


a. Determine the magnitude of the acceleration for m_1.

b. Determine the magnitude of the acceleration for m_2.

c. What is the tension in the rope?

Q10. A block of mass $m = 3.0$ kg sits on a frictionless plane angled at $\vartheta = 30°$ from the horizontal as shown in the figure below.

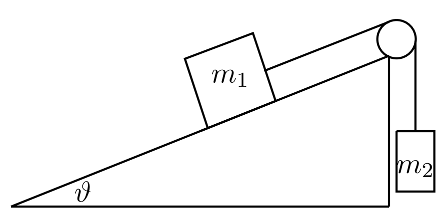


a. Label all forces acting on the block.

b. Determine the normal force acting on the block.

c. Determine the acceleration of the block.

Q11. A block of mass $m_1 = 3.50$ kg sits on a frictionless plane angled at $\vartheta = 25°$, which is connected by a rope to a second block of mass $m_2 = 2.40$ kg as shown in the figure below.



a. Determine the magnitude of the acceleration in each block.

b. Determine the direction of the acceleration of m_2.

c. Determine the tension in the rope.

Q12. A train of mass 30×10^3 kg is moving west at a velocity of 12 m/s and collides with three trucks, each of which have a mass of 10×10^3 kg and are initially stationary. If after the collision the train and trucks move together, what is the velocity of this combined locomotive after the collision?

Q13. Consider the situation as described in the previous question. During the collision, let the magnitude of the average force exerted on the trucks be denoted by $\mathcal{F}_1$ and the magnitude of the average force exerted by the trucks on the train be denoted by $\mathcal{F}_2$. Determine the relation between $\mathcal{F}_1$ and $\mathcal{F}_2$. Justify your answer.

1.5 Collisions

We now look at an application of Newton's laws of motion to collision problems. We invite the reader to recall that momenum was defined to be the product of the mass of an object and the velocity of that object; that is,

$$p = mv.$$

Force was subsequently defined to be the first derivative of momentum with respect to time t. A fundamental phenomenon that occurs in our world is that momenum is conserved.

Conservation of Momentum. If a system is not acted on by an external force, the momenum of the system is constant.

If we consider the simple situation of two objects of mass m_1, m_2 and velocities u_1, u_2 respectively, colliding, the law of conservation of momentum demands that

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2,$$

where v_1, v_2 denote the velocities of the objects after the collision.

Example 9.5.1. Suppose we have two cars of mass 1000 kg and 800 kg, each moving at a velocity of 80 km h^{-1} and 40 km h^{-1} in the opposing direction, respectively. If the cars stick together after the collision, what is the velocity of the two cars after the collision?

Proof. We observe that $m_1 = 1000, m_2 = 800, u_1 = 80$ and $u_2 = -40$. The law of conversation of momentum then tells us that

$$m_1u_1 + m_2u_2 = (m_1 + m_2)v,$$

for some v to be determined. We therefore see that

$$\begin{aligned} v &= \frac{m_1u_1 + m_2u_2}{m_1 + m_2} \\ &= \frac{1000 \cdot 80 + 800 \cdot (-40)}{1000 + 800} \\ &= \frac{80000 - 32000}{1800} \\ &= \frac{800 - 320}{18} \\ &\approx 26.67. \end{aligned}$$

□

Exercises

Q1. Paul has bought fireworks and releases one with an initial velocity of 30 ms^{-1} at an angle of 55° to the horizontal. At the highest point in its trajectory, the fireworks explode into two fragments with the same mass. The speed of the first fragment is immediately zero and falls vertically down. Determine how far the second fragment lands.

Q2. A 3000kg semi-trailor is travelling east at 84 km h^{-1} and is accelerating at 44 km h^{-1}. Determine the change in momentum of the truck.

Q3. What does it mean if the total momenum of a system has changed? Justify your answer.

Q4. Billy is a big man weighing 132 kg. He doesn't like where the furniture is in his house and decides to push it as far away from himself as he can. He pushes a chair which weighs 62kg such that it has a velocity of 0.5 ms^{-1}. What velocity did Billy carry as a consequence of moving the chair?

Q5. A train is travelling at 240 kmh^{-1} and collides with a stationary cow of mass 180 kg. If the train has a mass of 4 tonnes and the two objects do not stick together, what is the velocity of the cow after the collision?

Q6. A box of mass 4kg lies stationary on the edge of a table which is 1.5 m above the ground. A second box of mass 2kg collides with the 4kg box with a velocity of 5 ms^{-1}. Determine the time taken for the first box to reach the ground if the second box is immediately stationary after the collision.

Q7. Whiskey Jim gets into quite a few bar fights on a Friday night. Given that he is 194cm tall and weighs 165kg, physical confrontation is no big deal for him. One night he gets into a fight with Big Wheeler Ted, who weighs in at 146kg. If Whiskey Jim charges with a velocity of 3 ms^{-1} while Big Wheeler Ted is initially startionary, determine the velocity of both men if Whiskey Jim grabs hold of Big Wheeler Ted.

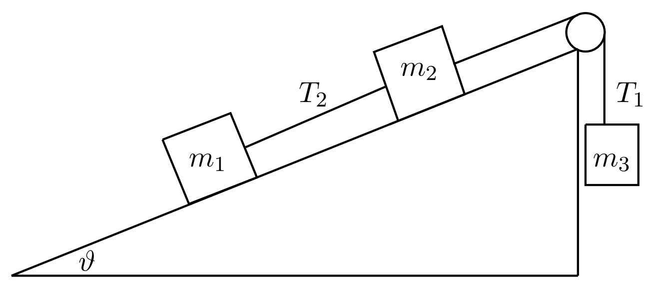
Q8. With an acceleration of 4 ms^{-1}, Kazuya can produce a punch that has a force of 320 N. If Kazuya hits paul with this punch at this velocity and Paul is initially startionary, with mass 92 kg, what is his velocity after the punch? We assume Kazuya remains stationary after the punch.

Q9. Phil and Spud have bought special suits that allow them to launch and catch a bowling bowl with their chest. The bowling ball they are using has a mass of 12kg and Phil launches the bowling ball out of his chest with an initial velocity of 5 ms^{-1} at an angle of 40°. Assuming that Phil and Spud are the same height, how much does Spud weigh in relation to Phil's weight?

Q10. A tennis ball of mass 0.1 kg moving at 10 ms^{-1} is struck by a tennis racket of mass 0.2 kg moving at 3 ms^{-1}. Assuming the racket is stationary after the hit, what is the velocity of the ball after the collision.

1.6 Analysis Exercises

Q1. The diagram below shows three blocks of mass $m_1 = 2\text{kg}, m_2 = 3\text{kg}$ and $m_3 = 4\text{kg}$ on a smooth plane which has an angle of inclination of ϑ relative to the horizontal. The masses are held together by a rope of negligible mass.



a. Write down an equation for the motion of m_3.

b. Write down an equation for the motion of m_1.

c. Write down an equation for the motion of m_2.

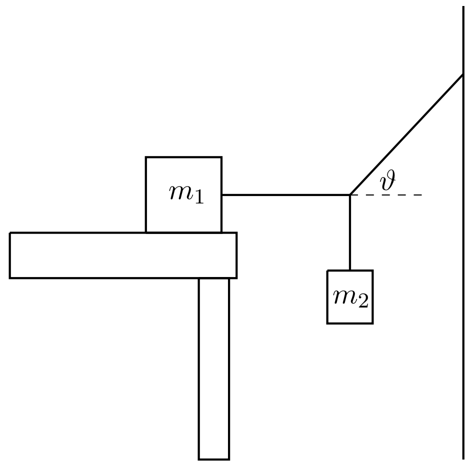
d. Find the value of ϑ such that the system is in equillibrium.

Q2. Chucky the monkey climbs up a rope of negligible mass which runs over a tree branch of negligible friction and is attached to a rock on the ground of mass 14 kg. Suppose Chucky has a mass of 9kg.

a. Determine the minimal acceleration that Chucky must attain in order to lift the rock off the ground.

b. Suppose Chucky succeeds in lifting the rock from the ground and decides to rest for a moment. Determine the tension in the rope.

Q3. The diagram below shows a block of mass $m_1 = 10\text{kg}$ resting on a table and another block of mass m_2 which is suspended from a cable of negligible mass as shown in the diagram. The cable is attached to the block of mass m_1 and the wall.



a. Suppose $m_2 = 3$kg. Determine the value of ϑ such that the system is stationary.

b. Suppose $\vartheta = \frac{\pi}{6}$. Determine the value of m_2 such that the system is stationary.

Q4. Donald hits a ball at an angle of ϑ from the horizontal at a speed of v_0 ms^{-1}.

a. Determine an expression for the maximal height of the ball.

b. Determine an expression for the velocity of the ball when it reaches its maximal height.

c. Suppose the ball hits a wall which is R metres from the Don's position. Determine an expression for the height of the ball when the ball hits the wall.

Q5. (Uniform Circular Motion). Suppose a particle moves around a circle of radius r with its position given by

$$x(t) = r \cos t\mathbf{i} + r \sin t\mathbf{j}.$$

a. Determine an expression for the velocity $v(t)$.

b. Show that the velocity of the particle is always perpendicular to its position.

c. Determine an expression for the acceleration $a(t)$.

d. Determine the direction of the force on the particle.

e. Show that the magnitude of $a(t)$ is constant.

Q6. (Dr. Lloyd Gunatilake). Two particles are moving in 3D-space. Both particles have non-intersecting line paths. The position vector of particle A is given by

$$\mathbf{r}(t) = \mathbf{i} - 2\mathbf{j} + t(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}),$$

where the vector $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ is parallel to the path of A. The position of particle B is given by

$$\mathbf{s}(t) = \mathbf{j} + t(3\mathbf{i} + \mathbf{j} - 5\mathbf{k}),$$

where the vector $3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$ is parallel to the path of B. Take all distances to be in metres and t is time in seconds.

a. The vector $\mathbf{a} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, where $x > 0$, is a unit vector perpendicular to both $2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and $3\mathbf{i} + \mathbf{j} - 5\mathbf{k}$. Find the values of x, y, and z.

b. Find the coordinates of the positions of both particles at $t = 0$.

c. Hence, find the minimum distance between the path of partical A and the path of particle B.

Q7. (Line Integrals). Let $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1,\ y \geq 0\}$. In this exercise, we look at how to evaluate an integral of a function $f(x, y)$ *along* a the curve which traces the boundary of Ω.

a. Show that if $x = \cos t$ and $y = \cos t$, then $x^2 + y^2 = 1$.

b. Show that if $-1 \leq x \leq 1$ and $0 \leq y \leq 1$, then $0 \leq t \leq \pi$.

c. The formula for evaluate the area under a the curve $f(x, y)$ along the curve γ whose boundary is given by $x = x(t)$ and $y = y(t)$, is

$$\int_\gamma f(x(t), y(t)) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

With $x = \cos t$ and $y = \sin t$, evaluate

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

d. If $f(x, y) = 4 + x^2 y$, show that

$$f(t) = 4 + \cos^2 t \sin t.$$

e. Hence, show that

$$\int_\gamma f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt=\int_0^\pi 4+\cos^2 t\sin t dt.$$

f. Hence, determine the value of A, where

$$A=\int_\gamma f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt.$$

Q8. (Line Integral, Continued).

If $\Omega=\{(x,y)\in\mathbb{R}^2 : x^2+y^2=16, -4\le x\le 0\}$ and $f(x,y)=xy^4$, use the procedure outlined in the previous exercise to evaluate

$$\int_\gamma f(x(t),y(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}dt.$$