Integration Techniques

Kyle Broder – ANU – MSI – 2017

ELEMENTARY EXAMPLES

A list of standard integrals is included below:

† $x^n \implies \int x^n dx = \frac{x^{n+1}}{n+1} + C$
† $(ax+b)^n, n \neq 1, \implies \int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$
† $e^{ax+b} \implies \int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$
† $\frac{1}{ax+b} \implies \int \frac{1}{ax+b} dx = \frac{1}{a} \log_e |x| + C$

Question 1. Integrate the following functions with respect to x.

a. $f(x) = x^4 + 3x^2 + 5x + 1$.
b. $f(x) = 4x^3 - 6x + \frac{1}{2}$.
c. $f(x) = 5x^6 - 7x + 1$.
d. $f(x) = 6x^5 + x^4 + x + 2$.
e. $f(x) = 1$.
f. $f(x) = 6x^{13} + 7$.

Question 2. Integrate the following functions with respect to x.

a. $f(x) = \sqrt{5x+1}$.
b. $f(x) = 4\sqrt{3x-4} + 1$.
c. $f(x) = 6x - 3\sqrt{2x-8} + 8$.
d. $f(x) = \frac{1}{3}x^3 - 3\sqrt{\frac{3}{5} - x}$.

Question 3. Integrate the following functions with respect to x.

a. $f(x) = (x-3)^2 + (x-4)^3 + (x-6)^3$.
b. $f(x) = (x-1)^2 + (x-1)^3 + (x-3)^6$.
c. $f(x) = \frac{3}{5}(x-6)^5 + \sqrt{2}x + 1$.

Question 4. Integrate the following functions with respect to x.

a. $f(x) = \frac{2}{(x-3)^3} + 1$.
b. $f(x) = \frac{5}{(x+1)^7} - 5$.
c. $f(x) = \frac{3}{6(x-1)^8} + \frac{4}{(x+2)^2} + \frac{9}{(x+7)^5} + 2x + \sqrt{4x+1}$.

Question 5. Integrate the following functions with respect to x.

a. $f(x) = 2e^{x+3} + 4$.
b. $f(x) = 4e^{6-4x} - 2$.
c. $f(x) = 8e^{3x+1} - 76$.
d. $f(x) = \frac{1}{e^{4x+1}} - 4x + \sqrt{4x+9} - (3x+6)^3 + 10$.

Question 6. Evaluate the following expression.

$$\int \sum_{k=1}^{3} \frac{3}{(4x-k)^3} dx.$$

Question 7. Integrate the following expressions with respect to x.

a. $f(x) = \frac{4}{x+3}$.
b. $f(x) = \frac{2}{4x-7} + \sqrt{5x+9}$.
c. $f(x) = \frac{3x+1}{4x+5}$.

Question 8. Evaluate the following definite integral

$$\int_0^1 e^{4x+3} + \sqrt{4x-1} + \frac{1}{x+4} - \frac{3}{(x-3)^7} + 8dx$$

Question 9. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} \frac{1}{2}x^2 - 4x + 1, & 0 < x < 1, \\ x + 3, & 1 < x < 10, \\ 0, & \text{otherwise}. \end{cases}$$

Evaluate the integral

$$\int_{-\infty}^{\infty} f(x)dx.$$

Question 10. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

Evaluate the integral

$$\int_{-3}^{3} f(x)dx.$$

Question 11. Evaluate the integral

$$\int_{-1}^{1} |x-4| \, dx.$$

Question 12. Evaluate the integral

$$\int_{-1}^{1} \frac{1}{|x-2|} dx.$$

Question 13. Evaluate the integral

$$\int_{-3}^{3} e^{|x+1|} dx.$$

Integration of Trigonometric Functions.

Integration by Substitution.

Example 1. Evaluate the following expressions

(a) $\int \frac{2x-5}{\sqrt{x^2-5x+1}}dx$

$$
\begin{aligned}
\text{let } u &= x^2-5x+1 \\
\frac{du}{dx} &= 2x-5 \\
\therefore dx &= \frac{du}{2x-5} \\
\therefore \int \frac{2x-5}{\sqrt{x^2-5x+1}}dx &= \int \frac{2x-5}{\sqrt{u}} \cdot \frac{du}{2x-5} \\
&= \int \frac{1}{\sqrt{u}}du \\
&= \int u^{\frac{-1}{2}}du \\
&= \frac{u^{\frac{1}{2}}}{\frac{1}{2}} + C \\
&= 2\sqrt{u} + C \\
&= 2\sqrt{x^2-5x+1} + C
\end{aligned}
$$

(b) $\int \frac{x^2+3}{(\frac{1}{3}x^3+3x-8)^5}dx$

$$\text{let } u = \frac{1}{3}x^3+3x-8$$

$$
\begin{aligned}
\frac{du}{dx} &= x^2+3 \\
\therefore dx &= \frac{du}{x^2+3} \\
\therefore \int \frac{x^2+3}{(\frac{1}{3}x^3+3x-8)^5}dx &= \int \frac{x^2+3}{u^5} \cdot \frac{du}{x^2+3} \\
&= \int \frac{1}{u^5}du \\
&= \int u^{-5}du \\
&= \frac{u^{-4}}{-4} + C \\
&= \frac{-1}{4} \cdot \frac{1}{(\frac{1}{3}x^3+3x-8)^4} + C
\end{aligned}
$$

(c) $\int \frac{\log_e(3x)}{2x} dx$

$$\begin{aligned}
\text{let } u &= \log_e(3x) \\
\frac{du}{dx} &= \frac{1}{x} \\
\therefore dx &= du \cdot x \\
\therefore \int \frac{\log_e(3x)}{2x} dx &= \int \frac{u}{2x} \cdot du \cdot x \\
&= \int \frac{u}{2} du \\
&= \frac{u^2}{4} + C \\
&= \frac{1}{4} \cdot (\log_e(3x))^2 + C
\end{aligned}$$

Example 2. Use appropriate substitutions to evaluate the following expressions

(a) $\int \frac{2x-5}{\sqrt{x+6}} dx$

$$\begin{aligned}
\text{let } u &= x + 6 \\
\therefore x &= u - 6 \\
\frac{du}{dx} &= 1 \\
\therefore dx &= du \\
\therefore \int \frac{2x-5}{\sqrt{x+6}} dx &= \int \frac{2(u-6)-5}{\sqrt{u}} du \\
&= \int \frac{2u}{\sqrt{u}} du - \int \frac{17}{\sqrt{u}} du \\
&= \int 2u^{\frac{1}{2}} du - \int 17u^{\frac{-1}{2}} du \\
&= \frac{2u^{\frac{3}{2}}}{\frac{3}{2}} - \frac{17u^{\frac{1}{2}}}{\frac{1}{2}} + C \\
&= \frac{4u^{\frac{3}{2}}}{3} - 34\sqrt{u} + C \\
&= \frac{4}{3}(x+6)^{\frac{3}{2}} - 34\sqrt{x+6} + C
\end{aligned}$$

(b) $\int 6x^2\sqrt{x+9}\ dx$

$$\begin{aligned}
\text{let } u &= x+9\\
x &= u-9\\
\frac{du}{dx} &= 1\\
\therefore du &= dx\\
\therefore \int 6x^2\sqrt{x+9}\ dx &= \int 6(u-9)^2\sqrt{u}du\\
&= 6\int (u^2-18u+81)\sqrt{u}du\\
&= 6\int (u^{\frac{5}{2}}-18u^{\frac{3}{2}}+81u^{\frac{1}{2}})du\\
&= 6\left(\frac{2}{7}u^{\frac{7}{2}}-45u^{\frac{5}{2}}+\frac{162}{3}u^{\frac{3}{2}}\right)+C\\
&= \frac{12}{7}(x+9)^{\frac{7}{2}}-270(x+9)^{\frac{5}{2}}+324(x+9)^{\frac{3}{2}}+C.
\end{aligned}$$

Example 3. Determine the function $f(x)$ if $f'(x)=\frac{x}{\sqrt{x-1}}$ and $f(1)=0$.

We begin by integrating $f'(x)$.

$$\begin{aligned}
\text{let } u &= x-1\\
\therefore x &= u+1\\
\frac{du}{dx} &= 1\\
\therefore du &= dx\\
\therefore \int \frac{x}{\sqrt{x-1}}dx &= \int \frac{u+1}{\sqrt{u}}du\\
&= \int u^{\frac{1}{2}}+u^{\frac{-1}{2}}\ du\\
&= \frac{2}{3}u^{\frac{3}{2}}+2u^{\frac{1}{2}}+C\\
&= \frac{2}{3}(x-1)^{\frac{3}{2}}+2\sqrt{x-1}+C\\
\therefore\ \text{since } f(1) &= 0\\
0 &= \frac{2}{3}(0)^{\frac{3}{2}}+2\sqrt{0}+C\\
\therefore C &= 0\\
\therefore f(x) &= \frac{2}{3}(x-1)^{\frac{3}{2}}+2\sqrt{x-1}
\end{aligned}$$

Question 14. Evaluate the following.

(a)

$$\int \frac{2x}{x^2-5}dx.$$

(b)

$$\int \frac{x}{\sqrt{3-x^2}} dx.$$

(c)

$$\int \frac{2x+6}{(x^2+6x-5)^5} dx.$$

Question 15. Evaluate the following.

(a)

$$\int x^5 \cdot e^{x^6} dx.$$

(b)

$$\int \frac{2\log_e(x)}{x} dx.$$

(c)

$$\int \frac{-x^3}{e^{x^4}} dx.$$

Question 16. Evaluate the following.

a.

$$\int \sin(x)\cos^2(x)dx.$$

b.

$$\int 3\tan(x)\sec^2(x)dx.$$

c.

$$\int 4\cos(x)e^{\sin(x)}dx.$$

d.

$$\int 2\tan(x)dx.$$

e.

$$\int \cot(x)dx.$$

f.

$$\int \frac{\sin(x)}{1-\cos(x)} dx.$$

g.

$$\int 2\sin(x)\sqrt{1+\cos(x)}dx.$$

h.

$$\int \frac{3x^2\sin(x^2)}{x} dx.$$

i.

$$\int \frac{\log_e(2x+1)}{2x+1} dx.$$