Elementary Function Theory – Lecture 1

In this first lecture we introduce the theory behind functions. Particular emphasis will be placed on polynomial functions of degree one and degree two.

1. Foundations

Exercise 1. Define what is meant by a *function*. Provide an example of a map that is a function and a map that is not a function. Define the domain, range, and codomain of a function.

Exercise 2. Define what is meant by a *polynomial*.

Exercise 3. Determine which of the following functions are polynomials.

(a) $f(x) = 2x + 3$.

(b) $f(x) = x^2 - 5x + 102$.

(c) $f(x) = \sqrt{x}$.

(d) $f(x) = \frac{1}{x}$.

(e) $f(x) = |x|$.

(f) $f(x) = \sqrt{x} - \sqrt{x} + x^2$.

Exercise 4. Solve the following equations:

(a) $2x + 7 = -3$.

(b) $\frac{3-x}{12} = x$.

Exercise 5. Let $f(x) := ax + b$ and $g(x) = cx + d$, where $a, b, c, d \in \mathbb{R}$. Determine the conditions on $a, b, c, d \in \mathbb{R}$ such that f and g

(a) have a single point of intersection.

(b) have no point of intersection.

(c) have an infinite number of points of intersection.

Exercise 6. Let $f(x) = 2x - 3$ and let $g(x) = 5 - x$. Determine the point of intersection, if it exists.

Exercise 7. Determine the value of $k \in \mathbb{R}$ such that $f(x) = kx+2$ never intersects $g(x) = 4x-1$.

Exercise 8. Determine the values of $x \in \mathbb{R}$ such that

$$2x + 3 > 4x - 6.$$

Exercise 9. Determine the angle made between the x-axis and the line

$$y = 1 + \sqrt{3}x.$$

Exercise 10. Determine the angle made between the x-axis and the line

$$y = 2\sqrt{6} - 2x.$$

Exercise 11. Consider the system of linear equations

$$\begin{cases} \lambda x + 8y & = 0, \\ 4x + (\lambda + 2)y & = 4. \end{cases}$$

Determine the value of $\lambda \in \mathbb{R}$ such that the system has an infinite number of solutions.

2. Quadratic Equations

Exercise 1. Expand

(a) $(x-3)(x-1)$.

(b) $(2x-3)(x+1)$.

(c) $(x-3)(x+3)$.

Exercise 2. Factorise the expression

(a) x^2+2x+1.

(b) x^2-9.

(c) x^2+6x+9.

(d) x^2+1.

Exercise 3. Let $f(x) := x^2 + bx + c$, where $b, c \in \mathbb{R}$. Show that the roots of f, i.e., the solutions of $f(x) = 0$ are determined by the

$$x = \frac{1}{2}\left(-b \pm \sqrt{b^2 - 4c}\right).$$

Exercise 4. Determine the roots of the following quadratics:

(a) $5 - x + x^2$.

(b) $3x^2 - 8x + 1$.

(c) $2x^2 - 5x + \frac{1}{3}$.

Exercise 5. Define the discriminant Δ and explain its importance. Is Δ a function?

Exercise 6. Calculate the discriminant of the following quadratics. Hence, determine the number of roots of each quadratic (over $\mathbb{R}$).

(a) $x^2 + 4x - 1$.

(b) $x^2 - 5x + 6$.

(c) $1 - x^2$.

Exercise 7. Suppose that $f : \mathbb{R} \longrightarrow \mathbb{R}$ is a quadratic with two roots, determine the sign of the discriminant.

Exercise 8. Complete the square for the following quadratic equations.

(a) $f(x) = x^2 + 2x + 1$.

(b) $f(x) = 3x^2 - 8x + 1$.

(c) $f(x) = 5 - x + x^2$.

Exercise 9. Determine the value(s) of $k \in \mathbb{R}$ such that $f(x) = x^2 - kx$ intersections $g(x) = 2x - 3$ exactly once.

Exercise 10. Determine the value(s) of $k \in \mathbb{R}$ such that $f(x) = 4x - k$ intersects $g(x) = kx^2 - 1$ two times.

Exercise 11. The volume of water in a tank, V m^3, over a 10 month period is given by the function

$$V(t) = 2t^2 - 16t + 40,$$

where t is the number of months, $0 \leq t \leq 10$.

(a) Determine the initial volume of water in the tank.

(b) Determine the maximum volume of water in the tank.

(c) Determine the minimum volume of water in the tank.

Exercise 12. A section of a roller–coaster at an amusement park follows a parabolic path. The function

$$h(t) = t^2 - 12t + 48,$$

where $0 \leq t \leq 11$, models the height above the ground of the front of one of the carriages, where t is the time in seconds and h is the height in meters.

(a) Find the lowest point of this section of the ride.

(b) Find the time taken for the carriage to reach the lowest point.

(c) Find the highest point above the ground.

(d) Find the domain and range of the function.

(e) Sketch the function.

Exercise 13. Sketch the graph of

$$f(x) = \frac{1}{x^2 + 3x - 10},$$

stating the domain and range.

Exercise 14. Sketch the graph of

$$f(x) = \frac{1}{x^2 + 9x + 18},$$

stating the domain and range.