Chapter 1

Differentiation Theory.

Let us begin by defining what "Calculus" is. *Calculus*, in Latin, literally means *small pebble used for counting on an abacus.* In essence, calculus studies change in a continuous manner. Calculus is broken up into two branches, differential calculus and integral calculus. Differential calculus, which is the canonical starting point, is the study of very small, infinitesimal, change. Integral calculus is a way for us to convert finite sums to infinite sums and allow us to evaluate areas that would have previously been other too daunting or impossible.

Calculus was invented in the century in the 17th century independently by Isaac Newton and Gottfried Leibniz.

1.1 Continuity and Limits.

In this section we will build on the theory that we developed in the elementary function theory chapter.

Let us recall that a function f should be thought of like a box. A box which you are allowed to put only certain objects into (these are the elements in the domain). Once they are placed in the box, you can shake the box (this is the application of the function, for example multiplication by two), and open it to see what happened to the objects you put in (this will give you the elements in the range).

We now begin to discuss a cornerstone of calculus, the notion of continuity. An imprecise definition of continuity is often that a function is continuous if it can be drawn without lifting the pen off paper. One should bear in mind that this does **not** define continuity of a function, rather it is a model[1]. This definition of "no pen off paper" is useless mathematically, and fails to accurately describe the essence continuity.

To begin talking about the continuity of a function $f : \mathbb{R} \to \mathbb{R}$, we need to introduce the notion of a limit. We say that a function $f(x)$ has a limit point at $f(x_0)$ if $f(x) \to f(x_0)$ as $x \to x_0$. The standard notation for this is

$$\lim_{x \to x_0} f(x) = f(x_0).$$

Limits have the following properties, of which we will not prove here.

Theorem 5.1.1. Suppose that $\lim_{x \to x_0} f(x) = \ell_1$ and $\lim_{x \to x_0} g(x) = \ell_2$, then

(i) $\lim_{x \to x_0} (f + g)(x) = \ell_1 + \ell_2$.

(ii) $\lim_{x \to x_0} (f - g)(x) = \ell_1 - \ell_2$.

(iii) $\lim_{x \to x_0} (f \cdot g)(x) = \ell_1 \cdot \ell_2$.

(iv) $\lim_{x \to x_0} \frac{f}{g}(x) = \frac{\ell_1}{\ell_2}$, if $\ell_2 \neq 0$.

Proof. We postpone the proofs to section 5.2. □

We now outline below some examples of how to compute limits.

Example 5.1.2. Calculate $\lim_{x \to 3} x^2 - 9$.

Proof. To compute this limit we simply need insert the value of $x = 3$ into the function $x^2 - 9$. Therefore,

$$\lim_{x \to 3} x^2 - 9 = (3)^2 - 9 = 0.$$

□

Example 5.1.3. Calculate the limit $\lim_{x \to 2} \frac{x^2 - 4}{x - 2}$.

[1] And a rather ineffective one at that, see Example 4.5.3 and the Intermediate Value Theorem (Theorem 4.7.1) of *Introduction to Analysis*.

Proof. Again using a similar process as before, we simply insert the value of $x = 2$ into the function $\frac{x^2-4}{x-2}$. Therefore,

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \frac{(2)^2-4}{2-2} = \frac{0}{0}.$$

This is an example of an indeterminant form.[2] So we have to manipulate the equation first, prior to inserting the value of x. Observe that we can write

$$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x\to 2} x+2 = 4.$$

$\square$

Example 5.1.4. Calculate the limit $\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)$.

Proof. We simply observe that

$$\begin{aligned}
\lim_{x\to\infty}\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right) &= \lim_{x\to\infty} \frac{\left(\sqrt{x^2+x}-\sqrt{x^2-x}\right)\left(\sqrt{x^2+x}+\sqrt{x^2-x}\right)}{\sqrt{x^2+x}+\sqrt{x^2-x}} \\
&= \lim_{x\to\infty} \frac{2x}{\sqrt{x^2+x}+\sqrt{x^2-x}} \\
&= \lim_{x\to\infty} \frac{2}{\sqrt{1+\frac{1}{x}}+\sqrt{1-\frac{1}{x}}} \\
&= 1.
\end{aligned}$$

$\square$

By now the question has perhaps been raised as to when does a limit exist? We say that a limit exists as $x \to x_0$ if $\lim_{x\to x_0} f(x) = f(x_0)$ from both directions. That is, as x approaches x_0 from the left, $f(x)$ should approach $f(x_0)$, and if x approaches x_0 from the right, $f(x)$ should again approach $f(x_0)$. By letting x_0^- and x_0^+ denote the left and right of x_0 respectively, we can express that a limit exists if and only if

$$\lim_{x\to x_0^-} f(x) = \lim_{x\to x_0^+} f(x).$$

Example 5.1.5. If we consider the function $f(x) = |x|$,

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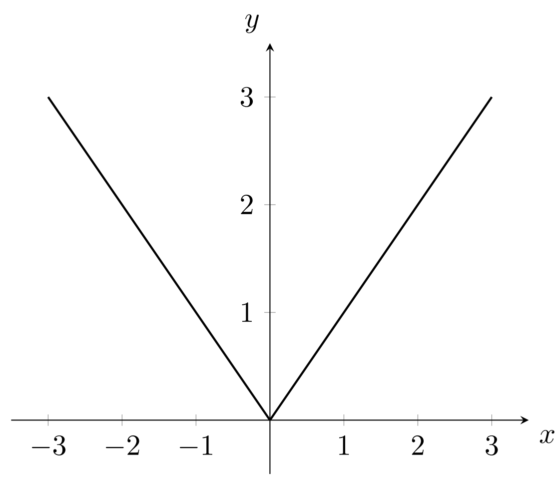
[2]An indeterminant form is a mathematical expression that is not definitively or precisely determined. Indeterminant form look like one of the following : $\frac{0}{0}, 0\cdot\infty, \infty - \infty, 0^0, \infty^0, 1^\infty$.



then

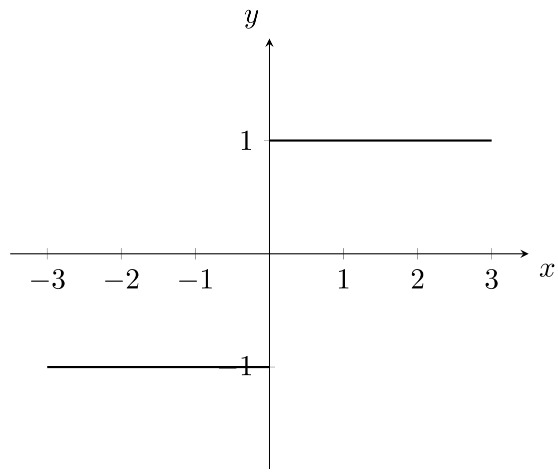
$$\lim_{x\to 0^-} |x| = 0, \text{ and } \lim_{x\to 0^+} |x| = 0 \text{ also.}$$

Therefore $\lim_{x\to x_0} |x|$ exists and is equal[3] to zero.

Example 5.1.6. If we consider the function

$$g(x) = \begin{cases} 1, & x > 0, \\ 0, & x = 0, \\ -1, & x < 0. \end{cases}$$

[3]It is worth remarking that when we say that a limit is equal to something, the word 'equal' is quite distorting. The limit says nothing about what happens to $f(x)$ when $x = x_0$, just what happens when x gets arbitrarily close to x_0. So it would be better to write $\lim_{x\to 0} |x|$ approaches 0 on either side of x_0. This is rather cumbersome however, so we simply say that $\lim_{x\to 0} |x| = 0$.



Then as we approach $x = 0$ from the left, $g(x)$ approaches -1, and $\lim_{x\to 0^-} g(x) = -1$. As we approach $x = 0$ from the right however, $g(x)$ approaches 1, and $\lim_{x\to 0^+} g(x) = 1$. Therefore, since

$$\lim_{x\to 0^-} g(x) = -1 \neq 1 = \lim_{x\to 0^+} g(x),$$

the limit does not exist at $x = 0$.

Definition 5.1.7 We say that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x_0 \in \mathbb{R}$ if

$$\lim_{x\to x_0} f(x) = f(x_0).$$

We introduce a more useful definition in the next section.

Exercises.

Q1. Caclulate the following limits.

a. $\lim_{x\to 3} x^2 + 1$.

b. $\lim_{x\to -1} (x - 6)^3$.

c. $\lim_{x\to \pi} \sin(x)$.

d. $\lim_{x\to \frac{1}{2}} |x|$.

e. $\lim_{x\to 2} \frac{x^2-5x+6}{x-2}$.

f. $\lim_{x\to 1} \frac{x^2+2x+1}{x-1}$.

Q2. Let

$$f(x) := \begin{cases} 1, & x \geq 0, \\ -1, & x < 0. \end{cases}$$

Evaluate $\lim_{x\to 0} f(x)$.

Q3. Let

$$f(x) = \begin{cases} g(x), & x \in \mathbb{R}\backslash\{0\}, \\ 1, & \text{otherwise}, \end{cases}$$

where $g(x) = 0$. Evaluate $\lim_{x\to 0} f(x)$.

Q4. Evaluate

$$\lim_{x\to 0} x \sin\left(\frac{1}{x}\right).$$

Q5. Evaluate

$$\lim_{x\to 0} x \cos\left(\frac{1}{x}\right).$$

Q6. Evaluate

$$\lim_{x\to\infty} \frac{x+1}{x^2-1}.$$

Q7. Evaluate

$$\lim_{x\to\infty} \frac{e^{-x}+1}{e^{-x}-1}.$$

Q8. Determine whether the function

$$f(x) := \begin{cases} -x, & x < 0, \\ 2, & x = 0, \\ x, & x > 0 \end{cases}$$

is continuous for all $x \in \mathbb{R}$.

Q9. Determine whether the function

$$f(x) := \begin{cases} x^2, & x > 0, \\ -x, & \text{otherwise} \end{cases}$$

is continuous for all $x \in \mathbb{R}$.

Q10. Determine whether the function

$$f(x) := \begin{cases} \sin(x), & x \in (-\pi, 0] \cup [\pi, 2\pi), \\ \cos(x), & x \in (0, \pi), \\ 0, & \text{otherwise}, \end{cases}$$

is continuous for all $x \in \mathbb{R}$.

Q11. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is continuous on some set $\Omega_1 \subseteq \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is continuous on some set $\Omega_2 \subseteq \mathbb{R}$. Determine the set on which

a. $f + g$ is continuous.

b. $f \cdot g$ is continuous.

c. f/g is continuous.

Q12. Suppose that f is continuous on the set $[-5, 4)$ and $3 - f(x) > 0$ for all $x \in (-1, 4)$. Determine the set on which

$$g(x) := \sqrt{f(x) - 3}$$

is continuous.

1.2 Formal Continuity Theory

Now that we have an understanding of some elementary notions of continuity and limits, and have some examples to keep in mind, we look at a more formal treatment of such ideas.

Definition 5.2.1. A function $f(x)$ approaches a value $f(x_0)$ as x approaches x_0, allowing us to write

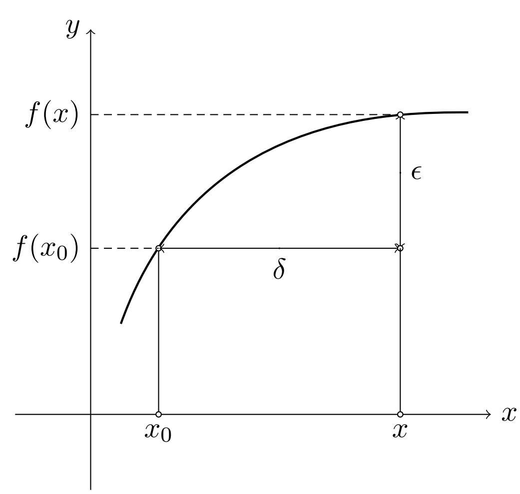
$$\lim_{x \to x_0} f(x) = f(x_0),$$

if for all $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon.$$

This may sometimes be equivalently written : $\forall \epsilon > 0 \, \exists \delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

The ϵ term gives us a way of formally expressing how far away $f(x)$ is from $f(x_0)$, and similarly, the δ terms gives us a way of formally expressing how far away x is from x_0. The above definition essentially says that if $f(x)$ is within some distance of $f(x_0)$, then I can find a corresponding distance between x and x_0. Moreover, as $\epsilon \to 0$, i.e. gets smaller and smaller, $\delta \to 0$ and gets smaller and smaller[4]. The associated picture is given by



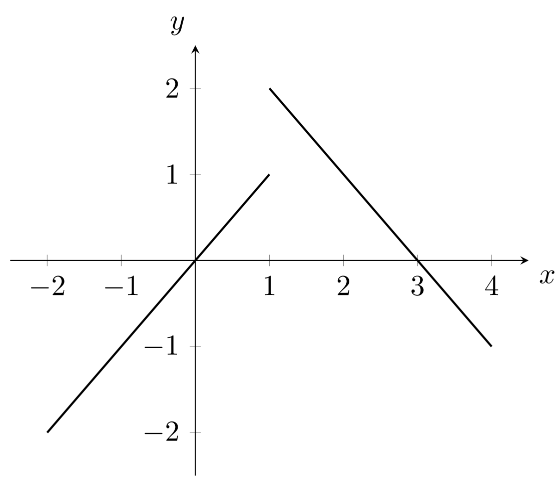
[4]Notice that again, in the definitions, ϵ and δ are greater than zero. One should recall that a limit gives us no information about what actually happens when $x = x_0$, only when x gets very close to x_0. This is formalised by defining $\epsilon > 0$ and $\delta > 0$.

The definition is perhaps best illustrated in example of a function f that does not have a limit at some x_0.

Example 5.2.2. Prove that $\lim_{x\to 1} f(x)$ does not exists, where f is defined by

$$f(x) = \begin{cases} x, & x < 1, \\ 3 - x, & x \geq 1. \end{cases}$$

The associated picture is



Proof. Suppose that $\lim_{x\to 1} f(x)$ exists. Then for all $\epsilon > 0$ there exists a $\delta > 0$ such that

$$|x - 1| < \delta \implies |f(x) - f(1)| < \epsilon.$$

Since this must hold for every $\epsilon > 0$, take $\epsilon = \frac{1}{2}$. Moreover, consider that if $|x - 1| < \delta$, then

$$1 - \delta < x < 1 + \delta.$$

Observe that $x_1 = 1 - \frac{\delta}{2}$ and $x_2 = 1 + \frac{\delta}{2}$ both satisfy this condition. So

$$|f(x_1) - f(x)| < \epsilon = \frac{1}{2}, \quad \text{and} \quad |f(x_2) - f(x)| < \epsilon = \frac{1}{2}.$$

We also note that

$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1 - (3 - x_2)| \\ &= \left|1 - \frac{\delta}{2} - 3 + 1 + \frac{\delta}{2}\right| \\ &= 1. \end{aligned}$$

Therefore, using the triangle inequality, we see that

$$\begin{aligned} |f(x_1) - f(x_2)| &= |f(x_1) - f(1) + f(1) - f(x_2)| \\ &= |(f(x_1) - f(1)) + (f(1) - f(x_2))| \\ &\leq |f(x_1) - f(1)| + |f(1) - f(x_2)| \\ &< \frac{1}{2} + \frac{1}{2} = 1. \end{aligned}$$

This subsequently shows that

$$1 = |f(x_1) - f(x_2)| < 1,$$

and we have our desired contradiction. $\square$

Let us now proceed with some elementary examples of proving limits using this definition.

Example 5.2.3. Prove that $\lim_{x\to 5} 7 = 7$.

Proof. For any $\epsilon > 0$ we want to find a $\delta > 0$ such that

$$|x - 5| < \delta \implies |f(x) - 7| = |7 - 7| = |0| < \epsilon.$$

Since $\epsilon > 0$ always, the inequality is trivially true, and is subsequently true for every choice of $\delta > 0$. $\square$

Example 5.2.4. Prove that $\lim_{x\to -\frac{3}{2}} 1 - 4x = 7$.

Proof. For any $\epsilon > 0$ we want to find a $\delta > 0$ such that

$$\left|x + \frac{3}{2}\right| < \delta \implies |(1 - 4x) - 7| < \epsilon.$$

The general approach for these types of proofs, is to manipulate the $|f(x) - f(x_0)|$ expression such that it looks like the $|x - x_0|$ expression. Using this idea, observe that

$$|(1 - 4x) - 7| = |-6 - 4x| = \left|(-4)\left(\frac{3}{2} + x\right)\right| = |-4|\cdot\left|\frac{3}{2} + x\right| = 4\left|x + \frac{3}{2}\right| < \epsilon.$$

We know that $\left|x + \frac{3}{2}\right| < \delta$ however, so

$$4\left|x + \frac{3}{2}\right| < 4\delta = \epsilon.$$

Taking $\delta = \dfrac{\epsilon}{4}$ completes the proof. $\square$

Example 5.2.5. Prove that $\lim_{x\to 3} x^2 - 9 = 0$.

Proof. For any $\epsilon > 0$ we want to find a $\delta > 0$ such that

$$|x-3| < \delta \implies \left|(x^2-9) - 0\right| < \epsilon.$$

Proceeding with the typical approach, we observe that

$$\left|(x^2-9)-0\right| = \left|x^2-9\right| = |x-3|\cdot|x+3| < \epsilon.$$

Now, arbitrarily assume $\delta \leq 1$[5]. Then since $|x-3| < \delta \leq 1$, we have

$$\begin{aligned} -1 < \ & x-3 & < 1 \\ 2 < \ & x & < 4 \\ 5 < \ & x+3 & < 7. \end{aligned}$$

Therefore, $|x+3| < 7$, and we obtain

$$|x-3|\cdot|x+3| < |x-3|\cdot(7) < 7\delta = \epsilon.$$

Taking $\delta = \min\{1, \epsilon/7\}$ completes the proof. ☐

Example 5.2.6. Prove that

$$\lim_{x\to -6} \frac{x+4}{2-x} = \frac{-1}{4}.$$

Proof. For any $\epsilon > 0$ we want to find a $\delta > 0$ such that

$$|x+6| < \delta \implies \left|\left(\frac{x+4}{2-x}\right) + \frac{1}{4}\right| < \epsilon.$$

We proceed in the usual manner, realising that

$$\left|\frac{x+4}{2-x} + \frac{1}{4}\right| = \left|\frac{4(x+4)+(2-x)}{4(2-x)}\right| = \left|\frac{3x+18}{4(2-x)}\right| = \left|\frac{3(x+6)}{4(2-x)}\right| = \left|\frac{3}{2}\right|\cdot\frac{|x+6|}{|2-x|} = \frac{3}{4}\frac{|x+6|}{|2-x|} < \epsilon.$$

[5]This is a valid assumption given that once we find a δ that works, all smaller values of δ work also.

Now, arbitrarily assume $\delta \leq 1$. Then since $|x+6| < \delta \leq 1$, we have

$$\begin{array}{rcl} -1 < & x+6 & < 1 \\ -7 < & x & < -5 \\ 7 < & |2-x| & < 9 \\ \dfrac{1}{9} < & \dfrac{1}{|2-x|} & < \dfrac{1}{7}. \end{array}$$

Therefore,

$$\frac{3}{4}\frac{|x+6|}{|2-x|} < \frac{3}{4} \cdot \frac{1}{7} \cdot |x+6| < \frac{3}{28}\delta = \epsilon.$$

Taking $\delta = \min\{1, 28\epsilon/3\}$ completes the proof. □

Moving away from examples, we prove Theorem 5.1.1, upon which we restate the theorem for the purposes of clarity.

Theorem 5.2.7. Suppose that $\lim_{x\to x_0} f(x) = \ell_1$ and $\lim_{x\to x_0} g(x) = \ell_2$, then

(i) $\lim_{x\to x_0}(f+g)(x) = \ell_1 + \ell_2$.

(ii) $\lim_{x\to x_0}(f-g)(x) = \ell_1 - \ell_2$.

(iii) $\lim_{x\to x_0}(f\cdot g)(x) = \ell_1 \cdot \ell_2$.

(iv) $\lim_{x\to x_0} \frac{f}{g}(x) = \frac{\ell_1}{\ell_2}$, if $\ell_2 \neq 0$.

Proof. (i) Since $\lim_{x\to x_0} f(x) = \ell_1$, the for any $\epsilon > 0$ there exists a $\delta_1 > 0$ such that

$$|x - x_0| < \delta_1 \implies |f(x) - \ell_1| < \epsilon.$$

Similarly, there exists a $\delta_2 > 0$ such that

$$|x - x_0| < \delta_2 \implies |g(x) - \ell_2| < \epsilon.$$

Suppose that $|x - x_0| < \delta = \min(\delta_1, \delta_2)$. Then

$$\begin{aligned} |(f+g)(x) - (\ell_1+\ell_2)| &= |(f(x)-\ell_1) + (g(x)-\ell_2)| \\ &\leq |f(x)-\ell_1| + |g(x)-\ell_2| < 2\epsilon. \end{aligned}$$

(ii) The argument here is analogous to the one seen in (i). The reader is left to explicitly write this proof out if she so desires.

(iii) Again, suppose that $|x - x_0| < \delta = \min(\delta_1, \delta_2)$. Then

$$\begin{aligned}
|(f \cdot g)(x) - \ell_1 \cdot \ell_2| &= |f(x)g(x) - \ell_1\ell_2| \\
&= |f(x)g(x) - \ell_2 \cdot f(x) + \ell_2 \cdot f(x) - \ell_1 \cdot \ell_2|^{6} \\
&= |f(x)(g(x) - \ell_2) + \ell_2(f(x) - \ell_1)| \\
&\leq |f(x)| \cdot |g(x) - \ell_2| + |\ell_2| \cdot |f(x) - \ell_1| \\
&\leq (|f(x)| + |\ell_2|) \cdot \epsilon \\
&\leq (|f(x) - \ell_1| + |\ell_1| + |\ell_1| + |\ell_2|) \cdot \epsilon \\
&\leq (\epsilon + |\ell_1| + |\ell_2|) \cdot \epsilon \\
&\leq (1 + |\ell_1| + |\ell_2|) \cdot \epsilon.
\end{aligned}$$

Taking $\epsilon < 1$ and x such that $|x - x_0| < \delta = \min(\delta_1, \delta_2)$ completes the proof.

(iv) Note that if $\ell_2 \neq 0$, then there exists a $\delta_3 > 0$ such that

$$|g(x) - \ell_2| < \frac{|\ell_2|}{2} \implies |g(x)| > \frac{|\ell_2|}{2} \text{ if } |x - x_0| < \delta_3.$$

To see this, suppose that $\epsilon = |\ell_2|/2$ and $|x - x_0| < \min(\delta_1, \delta_2, \delta_3)$. Then

$$\begin{aligned}
\left|\left(\frac{f}{g}\right)(x) - \frac{\ell_1}{\ell_2}\right| &= \left|\frac{f(x)}{g(x)} - \frac{\ell_1}{\ell_2}\right| \\
&= \frac{|\ell_2 f(x) - \ell_1 g(x)|}{|g(x)\ell_2|} \\
&\leq \frac{2}{|\ell_2|^2} |\ell_2 f(x) - \ell_1 g(x)| \\
&= \frac{2}{|\ell_2|^2} |\ell_2(f(x) - \ell_1) + \ell_1(\ell_2 - g(x))| \\
&\leq \frac{2}{|\ell_2|^2} (|\ell_2| \cdot |f(x) - \ell_1| + |\ell_1| \cdot |\ell_2 - g(x)|) \\
&\leq \frac{2}{|\ell_2|^2} (|\ell_1| + |\ell_2|) \cdot \epsilon.
\end{aligned}$$

This completes the proof.

□

Just as we proved with sequences of real numbers, a similar result holds for real valued functions.

Theorem 5.2.8. Suppose $\lim_{x\to x_0} f(x)$ exists, then this limit is unique. That is, if

$$\lim_{x\to x_0} f(x) = \ell_1 \text{ and } \lim_{x\to x_0} f(x) = \ell_2,$$

then $\ell_1 = \ell_2$.

Proof. Suppose that $\lim_{x\to x_0} f(x) = \ell_1$ and $\lim_{x\to x_0} f(x) = \ell_2$. Then for all $\epsilon > 0$, there exist δ_1 and δ_2, both greater than zero such that

$$|x - x_0| < \delta_1 \implies |f(x) - \ell_1| < \epsilon,$$

and

$$|x - x_0| < \delta_2 \implies |f(x) - \ell_2| < \epsilon.$$

Suppose that $\delta = \min(\delta_1, \delta_2)$, then

$$\begin{aligned} |\ell_1 - \ell_2| &= |\ell_1 - f(x) + f(x) - \ell_2|^7 \\ &\leq |\ell_1 - f(x)| + |f(x) - \ell_2| < 2\epsilon. \end{aligned}$$

Therefore,

$$|\ell_1 - \ell_2| < 2\epsilon \to 0, \implies \ell_1 = \ell_2.$$

□

If we wish to discuss limits as x tends to ∞, then we need to make some small modifications to the definition we have used previously.

Definition 5.2.9. We say that $f(x)$ approaches the limit ℓ as x approaches ∞, and express this as

$$\lim_{x\to\infty} f(x) = \ell,$$

if f is defined on some interval (α, ∞) and, for all $\epsilon > 0$ there exists a number β such that

$$x > \beta \implies |f(x) - \ell| < \epsilon.$$

Example 5.2.10. Prove that $\lim_{x\to\infty} f(x) = 1$, where

$$f(x) = 1 - \frac{2}{x^2}.$$

Proof. For any $\epsilon > 0$ we want to find some β such that

$$x > \beta \implies |f(x) - 1| = \left|1 - \frac{2}{x^2} - 1\right| < \epsilon.$$

Observe that

$$\left|1 - \frac{2}{x^2} - 1\right| = \left|\frac{1}{x^2}\right| = \frac{1}{x^2} < \epsilon, \text{ if } x > \frac{1}{\sqrt{\epsilon}}.$$

□

Example 5.2.11. Show that $\lim_{x\to\infty} \cos x$ does not exist.

Proof. Let $(x_n) = 2\pi n$ and $(y_n) = 2\pi n + \pi/2$ be two sequences. Each of these sequences tend to ∞ as $n \to \infty$. Therefore,

$$\lim_{n\to\infty} \cos(x_n) = 1 \neq 0 = \lim_{n\to\infty} \cos(y_n),$$

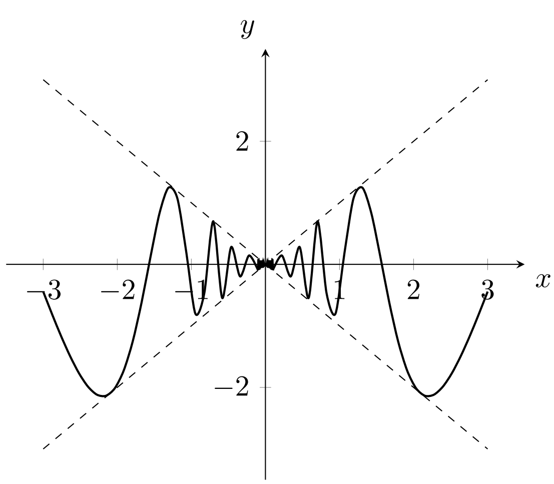
and therefore

$$\lim_{x\to\infty} \cos x$$

does not exist. □

We turn our attention now to the limits of a functions of a different nature. Consider the function

$$f(x) = x \sin\left(\frac{1}{x}\right),$$



and suppose we want to determine

$$\lim_{x\to 0} f(x).$$

One may wish to calculate it directly, but as $x \to 0, \frac{1}{x} \to \infty$ and moreover, $\sin x$ does not converge as $x \to \infty$. So the problem is not that simple. What we can do however, is realised that $\left|\sin \frac{1}{x}\right| \leq 1$ for all $x \in \mathbb{R}$. Subsequently, $\left|x \sin \frac{1}{x}\right| = |x| \cdot \left|\sin \frac{1}{x}\right| \leq |x|$, and we can easily determine $\lim_{x\to 0} |x|$. We therefore have that

$$-x \leq x \sin \frac{1}{x} \leq x,$$

and so

$$\lim_{x\to 0} -x \leq \lim_{x\to 0} x \sin \frac{1}{x} \leq \lim_{x\to 0} x.$$

With the fact that both x and $-x$ tend to zero as $x \to 0$, we see that

$$0 \leq \lim_{x\to 0} x \sin \frac{1}{x} \leq 0,$$

which allows us to conclude that

$$\lim_{x\to 0} x \sin \frac{1}{x} = 0.$$

What we did here was squeeze our function between two other functions, of which had the same limit. This is encapsulated in what is aptly named the Squeeze theorem.

Theorem 5.2.12. (The Squeeze Theorem) - Suppose that $[a, b] \subset \mathbb{R}$ with $a < x_0 < b$, and let f, g and h be functions defined on $[a, b]\backslash\{x_0\}$ such that for all $x \in [a, b]\backslash\{x_0\}$,

$$f(x) \leq h(x) \leq g(x).$$

Further suppose that

$$\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \ell,$$

then

$$\lim_{x\to x_0} h(x) = \ell.$$

Proof. Since $\lim_{x\to x_0} f(x) = \lim_{x\to x_0} g(x) = \ell$, that means that for any $\epsilon > 0$ we can find a $\delta_1, \delta_2 > 0$ such that

$$|x - x_0| < \delta_1 \implies |f(x) - \ell| < \epsilon,$$

and

$$|x - x_0| < \delta_2 \implies |g(x) - \ell| < \epsilon.$$

Therefore, by taking $\delta = \min(\delta_1, \delta_2)$ and observing that since

$$f(x) \leq h(x) \leq g(x),$$

we also have

$$f(x) - \ell \leq h(x) - \ell \leq g(x) - \ell,$$

hence, if $|x - x_0| < \delta$, then

$$-\epsilon < f(x) - \ell \leq h(x) - \ell \leq g(x) - \ell < \epsilon.$$

More concisely, this says that for $\epsilon > 0$,

$$|x - x_0| < \delta \implies |h(x) - \ell| < \epsilon.$$

□

Theorem 5.2.13. Suppose that f and g are continuous functions. Then

(i) $f + g$ is continuous.

(ii) $f - g$ is continuous.

(iii) $f \cdot g$ is continuous.

(iv) f/g is continuous.

Proof. The arguments for (i) - (iii) are analogous to those seen in the proof of Theorem 3.4.2. □

A very important theorem states the compositition of continuous functions is also continuous.

Theorem 5.2.14. Suppose that $f : X \to \mathbb{R}$ and $g : Y \to \mathbb{R}$ are functions such that $f(X) \subset Y$. Suppose further that f is continuous at $x_0 \in X$ and g is continuous at $f(x_0) \in B$, then the composition $g \circ f : X \to \mathbb{R}$ is continuous at x_0 also.

Proof. Suppose that $\epsilon > 0$ and note that since g is continuous at $f(x_0)$, then there exists a $\delta_1 > 0$ such that

$$|y - f(x_0)| < \delta_1 \implies |g(y) - g(f(x_0))| < \epsilon.$$

Moreover, f is continuous at x_0, so there exists a $\delta_2 > 0$ such that

$$|x - x_0| < \delta_2 \implies |f(x) - f(x_0)| < \epsilon.$$

Taking $\delta = \min(\delta_1, \delta_2)$ yields that

$$|x - x_0| < \delta \implies |g(f(x)) - g(f(x_0))| < \epsilon.$$

□

Exercises

Q1. Evaluate the following limits and prove your result.

a. $\lim_{x\to 4} x + 5$.

b. $\lim_{x\to 1} x^2 + 2$.

c. $\lim_{x\to 0} x - 3x^4$.

d. $\lim_{x\to -3} 7 - 5x^2$.

Q2. Evaluate the following limit, and prove your result.

$$\lim_{x\to 2} \frac{4-x}{1+3x}.$$

Q3. Evaluate the following limit, and prove your result

$$\lim_{x\to -3} \frac{3+5x}{2x-3}.$$

Q4. Evaluate the following limit, and prove your result

$$\lim_{x\to 1} \frac{1+x}{2-x}.$$

Q5. Show that the following functions are continuous at $x = 3$.

a. $f(x) = 2x + 3$.

b. $f(x) = x^2 - 5x + 6$.

c. $f(x) = x^2 - 4$.

d. $f(x) = \frac{1}{x-3}$.

e. $f(x) = |x + 2|$.

f. $f(x) = \frac{2+5x}{1+6x}$.

Q6. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \sqrt{1+x^2}.$$

Show that f is continuous for all $x \in \mathbb{R}$.

Q7. Let $f : (0,\infty) \to \mathbb{R}$ be the function defined by

$$f(x) := \frac{1}{x}.$$

Show that f is continuous for all $x \in (0,\infty)$.

Q8. Let $f : (0,\infty) \to \mathbb{R}$ be the function defined by $f(x) := \log_e(x)$. Show that f is continuous for all $x \in (0,\infty)$.

Q9. Show that the function $f(x) := \sqrt{x}$ is continuous for all $x \in \mathbb{R}_{\geq 0}$.

Q10. Show that the function

$$f(x) := \frac{1}{\sqrt{2e^x + \sin x}}$$

is continuous for all $x \in \mathbb{R}$.

Q11. Show that the function

$$f(x) = \sin^3(x) + \frac{1}{\sqrt{x^2+9}} - \tan^{-1}(x)$$

is continuous for all $x \in \mathbb{R}$.

Q12. Determine the domain on which

$$f(x) := \sin^{-1}(x) + \cos^{-1}(x)$$

is continuous and prove your result.

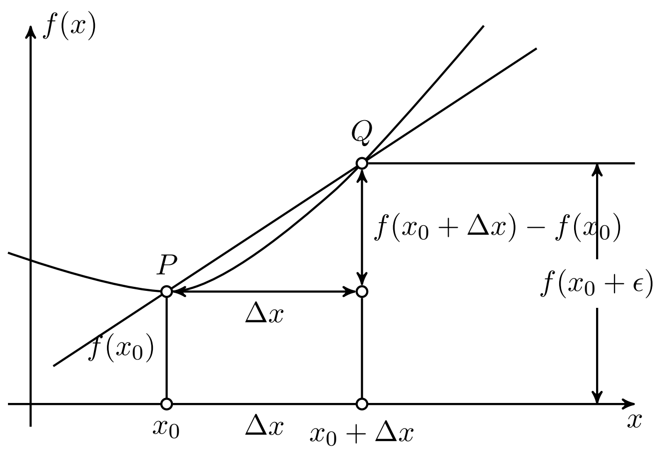
1.3 Differentiation Theory – First Principles.

Definition 5.3.1. A function f is said to be differentiable at x if the limit

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is defined. Moreover, the derivative of f at x is equal to the value of this limit.

The associated picture is given by



Definition 5.3.2. We say that a function f is continuously differentiable, or $f \in \mathscr{C}^1$, if the derivative of f, namely f' is continuous.

Theorem 5.3.3. For any positive integers $n \in \mathbb{Z}$, the function $f(x) = x^n$ is differentiable for all $x \in \mathbb{R}$ with derivative $f'(x) = nx^{n-1}$.

Proof. We simply observe that

$$\begin{aligned}
\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} &= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left((x+\Delta x)^n - x^n\right)\\
&= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(\sum_{k=0}^{n}\binom{n}{k}x^k(\Delta x)^{n-k} - x^n\right)\\
&= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\left(x^n + \sum_{k=0}^{n-1}\binom{n}{k}x^k(\Delta x)^{n-k} - x^n\right)\\
&= \lim_{\Delta x\to 0}\frac{1}{\Delta x}\sum_{k=0}^{n-1}\binom{n}{k}x^k(\Delta x)^{n-k}\\
&= \lim_{\Delta x\to 0}\sum_{k=0}^{n-1}\binom{n}{k}x^k(\Delta x)^{n-k-1}\\
&= \lim_{\Delta x\to 0}\sum_{k=0}^{n-2}\binom{n}{k}x^k(\Delta x)^{n-k-1} + \lim_{\Delta x\to 0}\binom{n}{n-1}x^{n-1}\\
&= \lim_{\Delta x\to 0}\binom{n}{n-1}x^{n-1}\\
&= nx^{n-1}.
\end{aligned}$$

This proves the result. □

⚠**Remark 5.3.4**. The above theorem does not prove that if $f(x) = x^n$ then $f'(x) = nx^{n-1}$ for $n \in \mathbb{Q}$. While this result is true, we need the chain rule to prove it for $\mathbb{Q}$-valued indices.

Example 5.3.5. Show that the function $f(x) = x^2 - 5x + 6$ is differentiable for all $x \in \mathbb{R}$.

Proof. We simply observe that

$$
\begin{aligned}
& \lim_{\Delta x \to 0} \frac{f(x+\Delta) - f(x)}{\Delta x} \\
= & \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - 5(x+\Delta x) + 6 - (x^2 - 5x + 6)}{\Delta x} \\
= & \lim_{\Delta x \to 0} \frac{x^2 + 2x \cdot \Delta x + (\Delta x)^2 - 5x - 5\Delta x + 6 - x^2 + 5x - 6}{\Delta x} \\
= & \lim_{\Delta x \to 0} \frac{2x \cdot \Delta x + (\Delta x)^2 - 5\Delta x}{\Delta x} \\
= & \lim_{\Delta x \to 0} 2x + \Delta x - 5 \\
= & \ 2x - 5.
\end{aligned}
$$

□

We now prove the chain rule, which will be of great importance in both the development of the differentiation theory and computations.

Theorem 5.3.6. (Chain Rule). Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at x_0 and $g : \mathbb{R} \to \mathbb{R}$ be differentiable at $f(x_0)$. Then the composition $g(f(x))$ is differentiable at x_0 with

$$\frac{d}{dx} g(f(x)) = f'(x) g'(f(x)).$$

Proof. Suppose f and g satisfy the hypotheses of the theorem. Then we simply observe that

$$
\begin{aligned}
[f(g(x))]' &= \lim_{\Delta x \to 0} \frac{f(g(x+\Delta x)) - f(g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f[g(x) + (g(x+\Delta x) - g(x))] - f(g(x))}{\Delta x}.
\end{aligned}
$$

To simplify notation, let

$$\overline{g}_\Delta = g(x+\Delta x) - g(x),$$

and bear in mind that $\overline{g}_\Delta \to 0$ as $h \to 0$. Substituting this into our expression yields

$$\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{f(g(x)+\overline{g}_\Delta) - f(g(x))}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(g(x)+\overline{g}_\Delta) - f(g(x))}{\overline{g}_\Delta} \cdot \frac{\overline{g}_\Delta}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(g(x)+\overline{g}_\Delta) - f(g(x))}{\overline{g}_\Delta} \cdot \lim_{\Delta x \to 0} \frac{\overline{g}_\Delta}{\Delta x} \\
&= \lim_{\overline{g}_\Delta \to 0} \frac{f(g(x)+\overline{g}_\Delta) - f(g(x))}{\overline{g}_\Delta} \cdot \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(g(x)) \cdot g'(x).
\end{aligned}$$

□

With the chain rule now readily available, we may now prove the stronger version of Theorem 5.3.3.

Theorem 5.3.7. For any real number $n \in \mathbb{Q}$, the function $f(x) = x^n$ is differentiable with derivative $f'(x) = nx^{n-1}$.

Proof. Let $f(x) = x^n$ with $n \in \mathbb{Q}$. Since $n \in \mathbb{Q}$, choose $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ such that $n = \frac{p}{q}$ and set $g(x) := x^q$. Since $q \in \mathbb{N}$, Theorem 5.3.3 tells us that $g'(x) = qx^{q-1}$. Moreover, we see that the composition

$$h(x) := g(f(x)) = [x^{\frac{p}{q}}]^q = x^p.$$

Similarly, since $p \in \mathbb{Z}$, Theorem 5.3.3 tells us that $h'(x) = px^{p-1}$. Applying the chain rule to $h(x)$, we see that

$$h'(x) = f'(x)g'(f(x)).$$

Coalescing our work so far, we see that

$$\begin{aligned}
px^{p-1} = h'(x) &= f'(x)g'(f(x)) \\
&= f'(x) \cdot q[f(x)]^{q-1} \\
&= f'(x) \cdot q[x^{\frac{p}{q}}]^{q-1} \\
&= f'(x) \cdot qx^{\frac{p(q-1)}{q}}.
\end{aligned}$$

Hence we see that

$$\begin{aligned}
f'(x) &= px^{p-1} \cdot \frac{1}{qx^{\frac{p(q-1)}{q}}} \\
&= \frac{p}{q} x^{p-1} \cdot x^{\frac{p}{q}(1-q)} \\
&= \frac{p}{q} x^{p-1+\frac{p}{q}(1-q)} \\
&= \frac{p}{q} x^{p-1+\frac{p}{q}-p} \\
&= \frac{p}{q} x^{\frac{p}{q}-1} \\
&= nx^{n-1},
\end{aligned}$$

as required. □

Theorem 5.3.8. Suppose that f and g are differentiable at $x \in \mathbb{R}$. Then the function $f + g$ is differentiable at x, with

$$\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x).$$

Proof. Suppose that $f(x)$ and $g(x)$ are differentiable at $x \in \mathbb{R}$. We consider that

$$\begin{aligned}
\frac{d}{dx}[f(x) + g(x)] &= \lim_{\Delta x \to 0} \frac{(f+g)(x+\Delta x) - (f+g)(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) + g(x+\Delta x) - f(x) - g(x)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} + \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} \\
&= f'(x) + g'(x).
\end{aligned}$$

This proves the result. □

Example 5.3.9. Determine the domain on which the function

$$f(x) = x^4 + 4\sqrt{x}$$

is differentiable and calculate the derivative at all such points.

Proof. The function x^4 is differentiable everywhere and the function $\sqrt{x}$ is differentiable for all $x > 0$. We therefore see that the function $f(x) = x^4 + 4\sqrt{x}$ is differentiable for all $x > 0$. We then simply observe that

$$f'(x) = 4x^3 + \frac{2}{\sqrt{x}}.$$

□

Example 5.3.10. Let $f(x) = 3(1 - 4x)^7$. Evaluate $f'(x)$.

Proof. Using the chain rule, we simply observe that

$$\begin{aligned} f'(x) &= 3 \cdot (-4) \cdot 7 \cdot (1 - 4x)^6 \\ &= -84(1 - 4x)^6. \end{aligned}$$

□

Example 5.3.11. Determine the equation of the tangent line to the curve $f(x) = x^2$ at the point $x = 1$.

Proof. The slope of the tangent line at $x = 1$ is given by $f'(1)$. The derivative of f is given by $f'(x) = 2x$ and so we see that $f'(1) = 2$. The equation of the tangent line is therefore given by $y = 2x + c$, for some $c \in \mathbb{R}$. To determine the value of c, we use the fact that $(1, 1)$ lies on the tangent line. Thus, $1 = 2(1) + c$ and $c = -1$. So the equation of the tangent line is given by $y = 2x - 1$. □

Exercises.

Q1. Show that the following functions are differentiable and calculate their derivatives.

a. $f(x) = 2x + 1$.

b. $f(x) = 3x - 5$.

c. $f(x) = 9 - x$.

d. $f(x) = x^2$.

e. $f(x) = 5 + 2x - x^2$.

f. $f(x) = x^2 - 5x + 6$.

Q2. Show that the function

$$f(x) = |x|$$

is not differentiable at $x = 0$.

Q3. Determine where the following functions are differentiable.

a. $f(x) = \sqrt{x}$.

b. $f(x) = \frac{1}{x}$.

c. $f(x) = \sqrt{x^2 - 3x}$.

d. $f(x) = |x - 3|$.

e. $f(x) = \frac{1}{(x-9)^2}$.

f. $f(x) = \frac{2x-3}{x+1}$.

g. $f(x) = \frac{4x}{3-x}$.

h. $f(x) = \frac{1}{\sqrt{x^2-9}}$.

Q4. Let f be the function defined by

$$f(x) := \frac{3x-1}{4x+5}.$$

a. Determine the domain on which f is differentiable.

b. Show that f is differentiable on this domain and calculate the derivative of f for all points on this domain.

Q5. Let f be the function defined by

$$f(x) = \frac{1}{x^2+1}.$$

a. Determine the domain on which f is differentiable.

b. Show that f is differentiable on this domain and calculate the derivative of f for all points on this domain.

Q6. Consider the function $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{3-x}.$$

Show that f is differentiable on $\mathbb{R}\backslash\{3\}$ and compute $f'(x)$.

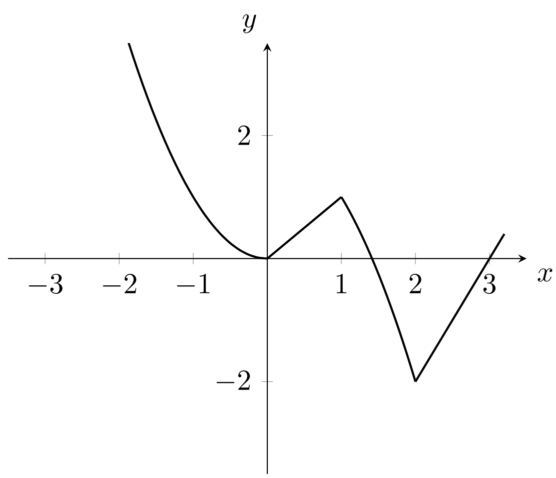
Q7. Consider the function $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ defined by

$$f(x) = \frac{5+x}{4-2x}.$$

Show that f is differentiable on $\mathbb{R}\backslash\{2\}$ and compute $f'(x)$.

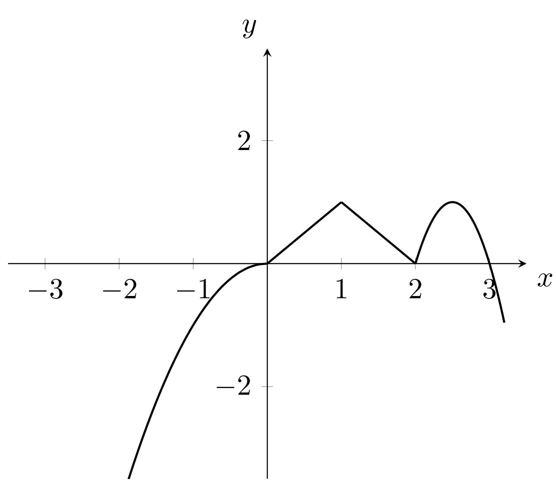
Q8. If $f : \mathbb{R} \to \mathbb{R}$ is continuous for all $x \in \mathbb{R}$. Is it necessarily true that f is differentiable for all $x \in \mathbb{R}$? If not, provide a counterexample.

Q9. Consider the function f whose graph is given by



Determine all points $x \in \mathbb{R}$ where f is not differentiable.

Q10. Consider the function g whose graph is given by



Determine all points $x \in \mathbb{R}$ where g is not differentiable.

Q11. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is differentiable, then f is continuous.

Q12. Evaluate the derivatives of the following functions.

a. $f(x) = 3x^2 + 5x + 1.$
b. $f(x) = 4 - 10x^{13} + 7x.$
c. $f(x) = 1 + x.$
d. $f(x) = 2x + 4x^2.$
e. $f(x) = \frac{1}{4}(x-3)^2.$
f. $f(x) = (6-x)^2 + 4.$

Q13. Evaluate the derivatives of the following functions.

a. $f(x) = x^2 - 5x + 6.$
b. $f(x) = 2x + 1.$
c. $f(x) = 4x - 3x^2 + 1.$
d. $f(x) = 2x^3 + 5x^5 + 6x^7.$
e. $f(x) = \frac{1}{3}x^3 + 4x.$
f. $f(x) = 4x^{10} + \frac{1}{\pi}x.$

Q14. Evaluate the derivatives of the following functions.

a. $f(x) = \sqrt[3]{x}.$
b. $f(x) = \frac{1}{2}\sqrt{x} + 1.$
c. $f(x) = 3x - 5\sqrt{x} + 4x^3.$
d. $f(x) = \frac{2}{x}.$
e. $f(x) = \frac{1}{x^3} + \frac{1}{x^2} + \frac{1}{x}.$
f. $f(x) = \frac{1}{3\sqrt{x}}.$

Q15. Evaluate the following derivatives.

a. $f(x) = 2\sqrt{x-3} + 1.$
b. $f(x) = 3\sqrt{x+5} + 2x.$
c. $f(x) = \sqrt{2x+3} + 4x^3.$
d. $f(x) = \frac{1}{2}\sqrt{3x-5} + 2x + 5.$
e. $f(x) = \sqrt{4x-6} + \sqrt{x} + 3.$
f. $f(x) = \sqrt{3-7x} + \sqrt{4x-3}.$

Q16. Evaluate the following derivatives.

a. $f(x) = \frac{3}{x-5} + 1.$
b. $f(x) = \frac{1}{x+4} + \frac{3}{x-5}.$
c. $f(x) = \frac{5}{x+1} + \frac{2}{x-1}.$
d. $f(x) = \frac{3}{5x+4} + \frac{3}{4-5x}.$
e. $f(x) = \frac{7}{2x+1} - \frac{3}{5x-1}.$

Q17. Consider the function

$$f(x) = \frac{1}{x^2 - 8x + 15}.$$

a. Determine the values of A and B such that

$$f(x) = \frac{A}{x-3} + \frac{B}{x-5}.$$

b. Hence, differentiate $f(x)$.

Q18. Evaluate the following derivatives.

a. $f(x) = \sqrt{4x^3 + 2x + 1}$.

b. $f(x) = 2\sqrt{3 - x - x^2 - x^3 - x^4}$.

c. $f(x) = 5\sqrt{2x + 5x^3 + 6x^7}$.

d. $h(x) = \sqrt{f(x) + 1} + g(x)$, where $f(x)$ and $g(x)$ are differentiable on the entire real line and $f(x) \geq -1$ for all $x \in \mathbb{R}$.

Q19. Let

$$f(x) = \sqrt{1 + \sqrt{3x + 1}}.$$

Determine $f'(x)$.

Q20. Evaluate the following derivatives.

a. $f(x) = 2(x - 3)^4$.

b. $f(x) = 4(x^2 - 5x + 6)^{13} + 2x + 5$.

c. $f(x) = (x^2 + 2x + 1)^6 - 3x - 13$.

Q21. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable on some set $\Omega_1 \subset \mathbb{R}$, and suppose that $g : \mathbb{R} \to \mathbb{R}$ on some set $\Omega_2 \subset \mathbb{R}$. Determine the domain on which $f + g$ is differentiable.

1.4 Differentiation of Transcendental Functions.

In this section we discuss the differentiation techniques used to evaluate the derivatives of the functions $f(x) = e^x, f(x) = \log_e(x), f(x) = \sin x, f(x) = \cos x$. A summary of the derivatives is given by

† $f(x) = e^{g(x)} \implies f'(x) = g'(x)e^{g(x)}$.

† $f(x) = \log_e(g(x)) \implies f'(x) = \frac{g'(x)}{g(x)}$.

† $f(x) = \sin(g(x)) \implies f'(x) = g'(x)\cos(g(x))$.

† $f(x) = \cos(g(x)) \implies f'(x) = -g'(x)\sin(g(x))$.

Theorem 5.4.1. The function $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = e^x$, is differentiable for all $x \in \mathbb{R}$ with $f'(x) = e^x$.

Proof. Let us first obtain a useful representation of $f(x) = e^x$ that will make our proof simpler. To this end, we observe that

$$\begin{aligned}
e^x &:= \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\frac{x^k}{n^k} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n(n-1)\cdots(n-k+1)}{n^k}\frac{x^k}{k!} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n(n-1)\cdots(n-k+1)}{\underbrace{n\cdot n\cdots n\cdot n}_{k \text{ times}}}\frac{x^k}{k!} \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\frac{n}{n}\left(1-\frac{1}{n}\right)\cdots\left(1-\frac{k-1}{n}\right)\frac{x^k}{k!} \\
&= \sum_{k=0}^{\infty}\frac{x^k}{k!}.
\end{aligned}$$

With this representation of $f(x) = e^x$, we see that

$$\begin{aligned} f'(x) = \frac{d}{dx}e^x &= \frac{d}{dx}\left(\sum_{k=0}^{\infty}\frac{x^k}{k!}\right) \\ &= \sum_{k=1}^{\infty}\frac{d}{dx}\frac{x^k}{k!} \\ &= \sum_{k=1}^{\infty}\frac{kx^{k-1}}{k!} \\ &= \sum_{k=1}^{\infty}\frac{x^{k-1}}{(k-1)!} \\ &= \sum_{j=0}^{\infty}\frac{x^j}{j!} = e^x, \end{aligned}$$

as required. □

Example 5.4.2. Evaluate the derivative of $f(x) = e^{x^2-5x+6}$.

Proof. We simply observe that

$$f'(x) = (2x-5)e^{x^2-5x+6}.$$

□

Theorem 5.4.3. The function $f : (0, \infty) \to \mathbb{R}$ defined by $f(x) = \log_e(x)$, is differentiable for all $x \in (0, \infty)$ with $f'(x) = \frac{1}{x}$.

Proof. Let $f(x) = \log_e(x)$ and $g(x) = e^x$. We then see that

$$h(x) := g(f(x)) = x.$$

Hence, by using the chain rule, we see that

$$h'(x) = f'(x)g'(f(x)) = 1.$$

We know that $g'(x) = e^x$, so $g'(f(x)) = e^{\log_e(x)} = x$. So it follows that

$$h'(x) = f'(x) \cdot x = 1 \implies f'(x) = \frac{1}{x},$$

as required. □

Example 5.4.4. Evaluate the derivative of $f(x) = \log_e(x^3 + 4x + 1)$.

Proof. It is easy to see that

$$f'(x) = \frac{3x^2 + 4}{x^3 + 4x + 1}.$$

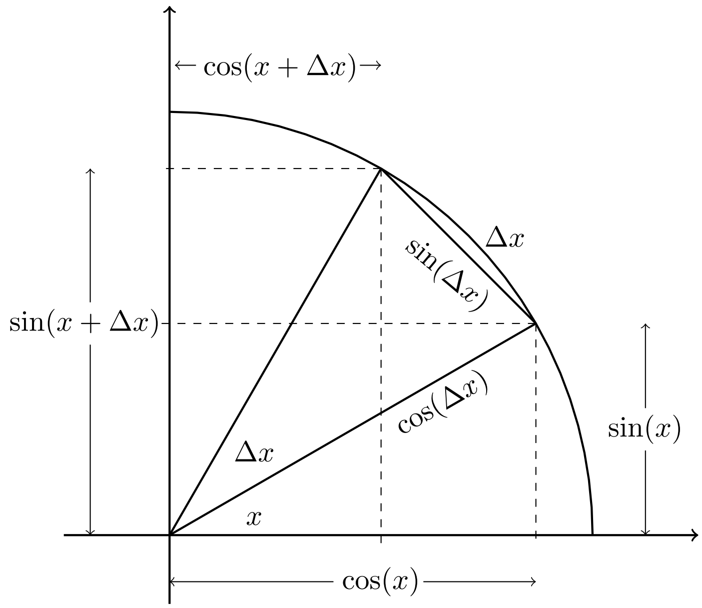
□

Theorem 5.4.5. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \sin x$ is differentiable for all $x \in \mathbb{R}$ with $f'(x) = \cos x$.

Proof. We simply observe that

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{\sin(x + \Delta x) - \sin(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sin(x)\cos(\Delta x) + \cos(x)\sin(\Delta x) - \sin(x)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sin(x)[\cos(\Delta x) - 1]}{\Delta x} + \lim_{\Delta x \to 0} \frac{\cos(x)\sin(\Delta x)}{\Delta x} \\ &= \sin(x) \lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} + \cos(x) \lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}. \end{aligned}$$

We therefore make the claim that $\lim_{\Delta x \to 0} \frac{\cos(\Delta x) - 1}{\Delta x} = 0$ and $\lim_{\Delta x \to 0} \frac{\sin(\Delta x)}{\Delta x}$. To investigate these limits, we consider the following picture



From the above diagram, we see that the desired limits fall out. Moreover, it can be argued

$$\cos(x) = \lim_{\Delta x \to 0} \frac{\sin(x+\Delta x) - \sin(x)}{\Delta x}$$

and

$$\sin(x) = \lim_{\Delta x \to 0} \frac{\cos(x) - \cos(x+\Delta x)}{\Delta x}$$

purely geometrically. □

Example 5.4.6. Evaluate the derivative of $f(x) = 3\sin(\pi x - \pi)$.

Proof. Without too much difficulty, we see that

$$f'(x) = 3\pi\cos(\pi x - x).$$

□

Exercises.

Q1. Using the chain rule, differentiate the following functions.

a. $f(x) = e^{3-x} + 1$.

b. $f(x) = e^{2x-1} + 4x + 12$.

c. $f(x) = e^{7x+2} + 3$.

d. $f(x) = \frac{1}{3}e^{4-x} + \frac{3}{7}$.

e. $f(x) = \sqrt{3}e^{4x+13} + 2$.

f. $f(x) = \frac{7}{5}e^{7-x}$.

Q2. Using the chain rule, differentiate the following functions.

a. $f(x) = e^{\sqrt{x+1}}$.

b. $f(x) = e^{x^2-5x+6}$.

c. $f(x) = 3e^{4x^2+\sqrt{x-3}+2} + 3x +$

5.

d. $f(x) = \frac{4}{3}e^{2(x-5)^3+1} + 4$.

e. $f(x) = \frac{\sqrt{2}}{5}e^{-x} + 4x^3$.

Q3. Consider the function

$$f(x) = \frac{1}{\sqrt{3}\exp\left(\sqrt{5x + x^2 - 1}\right)}.$$

Evaluate $f'(x)$.

Q4. Using the chain rule, differentiate the following functions.

a. $f(x) = \log_e(x-3) + 1$.

b. $f(x) = \frac{1}{3}\log_e(4-2x) + 4$.

c. $f(x) = \frac{3}{5}\log_e(x+7) + 2x + 3$.

d. $f(x) = \sqrt{3}\log_e(4x+5) + x^2$.

e. $f(x) = \log_e(x) + 1$.

Q5. Consider the function

$$f(x) = \frac{2}{3\log_e(x)}.$$

Evaluate $f'(x)$.

Q6. Let $|x|$ denote the absolute value function and consider the function

$$f(x) := \log_e |x|.$$

Evaluate $f'(x)$ and state the exact domain on which f is differentiable.

Q7. Using the chain rule, differentiate the following functions.

a. $f(x) = \sin(x + \pi) - 3$.
b. $f(x) = 2\cos(x - \pi/2) + 1$.
c. $f(x) = 3\sin(2(x + \pi)) + x$.
d. $f(x) = \frac{1}{4}\cos(-x) + \frac{1}{x^2}$.
e. $f(x) = \sin^2(x) + \cos^2(x)$.
f. $f(x) = \cos^3(x) + \sin^2(x - \pi)$.
g. $f(x) = 2\sin^3(x) + 5x - 3$.
h. $f(x) = 4\cos^7(x) + 2x$.

Q8. Consider the function

$$f(x) := \sec(x) := \frac{1}{\cos x}.$$

a. Determine the maximal domain on which $f(x)$ is defined.
b. Evaluate $f'(x)$.
c. Let $g(x)$ be the function given by applying the following transformations to $f(x)$:
   1. Dilate by factor 3 from the x-axis.
   2. Reflect about the y-axis.
   3. Translate by 2 units in the positive x-direction.

   Write the equation for $g(x)$.
d. Evaluate $g'(x)$.

Q9. Consider the function

$$f(x) := \csc(x) := \frac{1}{\sin x}.$$

a. Determine the maximal domain of $f(x)$.
b. Evaluate $f'(x)$.

Q10. Consider the functions

$$\sinh(x) := \frac{e^x - e^{-x}}{2} \text{ and } \cosh(x) := \frac{e^x + e^{-x}}{2}.$$

a. Show that the derivative of $\sinh(x)$ is $\cosh(x)$.
b. Determine the derivative of $\cosh(x)$.

Q11. Using the chain rule, differentiate the following functions.

a. $f(x) = \sinh(2x + 4) + 4x + 6$.
b. $f(x) = 3\cosh(4x - 3) + 4x^2$.

Q12. Consider the function

$$f(x) = \sinh(\sqrt{x}) = \frac{e^{\sqrt{x}} - e^{-\sqrt{x}}}{2}.$$

Evaluate $f'(x)$.

Q13. Consider the function

$$f(x) = \sin(x)\cos(x).$$

Evaluate $f'(x)$.

Q14. Let $f(x) = \sin x + \cos x$ and let $g(x) = [f(x)]^2 - \sin 2x$. Show that $g'(x) = 0$ for all $x \in \mathbb{R}$.

Q15. Let $f(x) = \sqrt{\log_e(x-1) - 3}$. Determine the maximal domain on which f is differentiable and evaluate $f'(x)$.

Q16. Let $f(x) = \log_e(x \sin x)$. Determine the maximal domain on which f is differentiable and evaluate $f'(x)$.

Q17. Let $f : (0, \pi) \to \mathbb{R}$ be the function defined by $f(x) := \cot x$. Show that $f'(x) = -\csc^2(x)$, where $\csc x := \frac{1}{\sin x}$.

Q18. Show that for all $0 < p < 1$,

$$(\cos\vartheta)^p \leq \cos(p\vartheta), \qquad 0 < \vartheta < \frac{\pi}{2}.$$

1.5 Differentiation of Products and Quotients.

Suppose we have a function $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ that both differentiable. We are yet to discuss a method of evaluating the derivative of $f(x) \cdot g(x)$ and $\frac{f(x)}{g(x)}$. Conveniently, we have formulas for both.

Theorem 5.5.1. (Product Rule). If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$, the derivative of $f \cdot g$ is given by

$$\frac{d}{dx}[f(x) \cdot g(x)] = f(x) \cdot g'(x) + f'(x) \cdot g(x).$$

Proof. We simply observe that

$$\begin{aligned}
&\frac{d}{dx}[f(x) \cdot g(x)] \\
= \; & \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x} \\
= \; & \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) + f(x+\Delta x)g(x) - f(x+\Delta x)g(x) - f(x)g(x)}{\Delta x} \\
= \; & \lim_{\Delta x \to 0} \frac{f(x+\Delta x)[g(x+\Delta x) - g(x)] + [f(x+\Delta x) - f(x)]g(x)}{\Delta x} \\
= \; & \lim_{\Delta x \to 0} f(x+\Delta x) \lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} + g(x) \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \\
= \; & f(x) \cdot g'(x) + g(x) \cdot f'(x),
\end{aligned}$$

as required. □

Example 5.5.2. Evaluate the derivative of $f(x) = e^x \sin x$.

Proof. The derivative is simply given by

$$f'(x) = e^x \cos x + e^x \sin x = e^x(\sin x + \cos x).$$

□

Theorem 5.5.3. (Quotient Rule). If $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is differentiable at $x \in \mathbb{R}$, and $g(x) \neq 0$, then the derivative of f/g is given by

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x) \cdot f'(x) - f(x) \cdot g'(x)}{[g(x)]^2}.$$

Proof. Apply the product rule to $f(x) \cdot [g(x)]^{-1}$. □

Example 5.5.4. Evaluate the derivative of $f(x) = \tan x$.

Proof. We begin by observing that

$$f(x) = \tan x = \frac{\sin x}{\cos x}.$$

Then it is immediate that

$$\begin{aligned} f'(x) &= \frac{\cos x \cdot \cos x + \sin x \cdot \sin x}{\cos^2 x} \\ &= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\ &= \frac{1}{\cos^2 x} = \sec^2 x. \end{aligned}$$

□

Exercise

Q1. Using the product rule, differentiate the following functions.

a. $f(x) = xe^x$.
b. $f(x) = (x-1)e^{3x}$.
c. $f(x) = x^2 e^{4-6x}$.
d. $f(x) = \sqrt{x}e^{\sqrt{x}}$.
e. $f(x) = 4xe^{\cos x}$.
f. $f(x) = e^{7-x}\sin x$.

Q2. Using the product rule, differentiate the following functions.

a. $f(x) = x \sin x$.
b. $f(x) = x \cos x$.
c. $f(x) = 3x \cos(2x)$.
d. $f(x) = (x^2 - 5x + 6)\sin(x)$.
e. $f(x) = 4x^3 \cos(x^4)$.
f. $f(x) = \sqrt{x}\sin(\sqrt{x-1})$.

Q3. Using the product rule, differentiate the following functions.

a. $f(x) = e^x \log_e(x)$.
b. $f(x) = e^x \sin x$.
c. $f(x) = \sqrt{x}\log_e(x)$.
d. $f(x) = 3x^{\frac{1}{3}}\log_e(x-5) + 1$.
e. $f(x) = \sin(x)\cos(3x)$.
f. $f(x) = \sin(x+\pi)\log_e(x+\pi)$.

Q4. Consider the function

$$f(x) := \cot(x).$$

Evaluate $f'(x)$.

Q5. Evaluate the derivatives of the following functions.

a.
$$f(x) = \frac{x-3}{x+2}.$$

b.
$$f(x) = \frac{3x+1}{4x+7}.$$

c.
$$f(x) = \frac{4x+6}{3-x}.$$

d.
$$f(x) = \frac{3-6x}{5-x}.$$

e.
$$f(x) = \frac{2}{x^2-5x+6}.$$

f.
$$f(x) = \frac{2x-6}{x^2+12x+1}.$$

g.
$$f(x) = \frac{x^2+2x+1}{x^2+4x+1}.$$

h.
$$f(x) = \frac{x+3}{x^2-9}.$$

i.
$$f(x) = \frac{4x^3+5x+1}{x^3+2x+1}.$$

Q6. Evaluate the derivatives of the following functions.

a.
$$f(x) = \frac{\sqrt{x}+\sqrt{-x}}{2x+3}.$$

b.
$$f(x) = \frac{\sin x + \cos x}{\tan x}.$$

c.
$$f(x) = \frac{3x+\cos(x^2)}{2x-4}.$$

Q7. Evaluate the derivatives of the following functions.

a. $f(x) = \sin 2x$.
b. $f(x) = \sec x$.
c. $f(x) = \cos^2 x$.
d. $f(x) = \csc(3x)$.
e. $f(x) = \sec x \tan x + \cos^2(x+\pi)$.
f. $f(x) = \tan^4 x + \sec^3 x + \frac{3}{5}\cot x$.

Q8. Let $f(x) = \log_e |x| + \cot(x)$. Determine the maximal domain on which f is differentiable.

Q9. Let $f(x) = 2\exp(-x^2) + \cos x$. Determine the maximal domain on which f is differentiable.

Q10. Let $f(x) = |x|$. Determine the maximal domain on which f is differentiable.

Q11. Evaluate the derivatives of the following functions.

a.
$$f(x) = (x^2 + 1)e^{-x}.$$

b.
$$f(x) = \log_e(x)\sqrt{x^2 + 5}.$$

c.
$$f(x) = \frac{1}{x^3 + x}.$$

d.
$$f(x) = e^{-\sqrt{x}} + x.$$

e.
$$f(x) = x^2 + x^3.$$

f.
$$f(x) = \frac{5x - 4}{x^2 - 5x + 6}.$$

Q12. Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable on some set $\Omega_1 \subseteq \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is differentiable on some set $\Omega_2 \subseteq \mathbb{R}$.

a. Determine the domain on which $f+g$ and $f-g$ are differentiable.
b. Determine the domain on which $f \cdot g$ is differentiable.
c. Determine the domain on which f/g is differentiable.
d. Determine the domain on which $\sqrt{f} \cdot g$ is differentiable.
e. Determine the domain on which $|f|$ is differentiable.
f. Let $f(x) = \exp(x)$ and $g(x) = \sqrt{x}$. On what domain is $h(x) = f(x) \cdot g(x)$ differentiable?
g. Let $f(x) = |x|$ and $g(x) = \log_e(-x)$. On which domain in $f(x) + g(x)$ differentiable?

1.6 Stationary and Inflection Points

In this section we recall and build upon some of the valuable information that can be gathered about a function f by looking at its derivative f'. We use the notation f'' to denote the second derivative of the function f. Note that we have assumed that both f' exists and that it is itself differentiable.

Definition 5.6.1. We say that a function f has a stationary point at x_0 if $f'(x_0) = 0$.

Stationary points take the form of local minima, local maxima and stationary points of inflection. In order to determine the nature of a stationary point, we use the second derivative of f, f''.

Example 5.6.2. Let $f(x) = x^2 - 5x + 6$. Determine the turning point of f.

Proof. The turning point is exactly the stationary point of f. We therefore consider that the derivative of f is given by $f'(x) = 2x-5$. Setting $f'(x) = 0$, we see that $x = \frac{5}{2}$. So the turning point occurs at $x = \frac{5}{2}$. To determine the corresponding value of $f(x)$, we simply observe that

$$f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^2 - 5\left(\frac{5}{2}\right) + 6 = \frac{25}{4} - \frac{25}{2} + 5 = -\frac{1}{4}.$$

□

Theorem 5.6.3. Suppose that f and f' are differentiabie at x_0. Then f has a

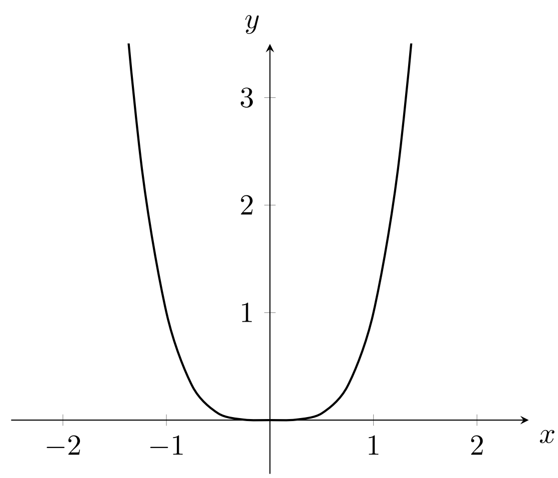
(i) local minimum at x_0 if $f'(x_0) = 0$ and $f''(x_0) > 0$,

(ii) local maximum at x_0 if $f'(x_0) = 0$ and $f''(x_0) < 0$,

(iii) a point of inflection if f'' changes sign about the point x_0.

Condition (iii) says that the function f has a stationary point of inflection if and only if $f''(x_0)$ and f'' changes sign about the point x_0. Many students are not aware of the condition that the second derivative f'' must change sign. It is often assumed that $f''(x_0) = 0$ is sufficient for there to be an inflection point at of f at x_0. We illustrate the falacy of this with some elementary examples.

Example 5.6.4. Consider the function $f(x) = x^4$.

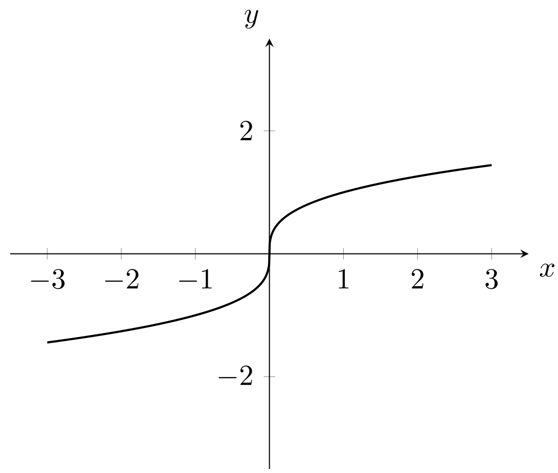


Proof. Let us compute the first and second derivatives. Clearly,

$$\begin{aligned} f'(x) &= 4x^3, \quad \text{and} \\ f''(x) &= 12x^2. \end{aligned}$$

Setting $f'(x) = 4x^3 = 0$, we see that f has a stationary point at $x = 0$. Moreover, setting $f''(x) = 12x^2 = 0$, we see that f is assumed to have a stationary point of inflection at $x = 0$. Looking at the graph of $f(x) = x^4$ however, it is clear that there is a local minimum at $x = 0$. Indeed if we look at the function $f''(x) = 12x^2$, $f'' \geq 0$ for all $x \in \mathbb{R}$. In particular, there is no change of sign. □

Example 5.6.5. Consider the function $g(x) = x^{1/3}$.



Proof. Just as we did in the previous example, let us compute the first and second derivatives. We see that

$$\begin{aligned} f'(x) &= \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}, \quad \text{and} \\ f''(x) &= \frac{-2}{9}x^{-5/3} = \frac{-2}{9x^{5/3}}. \end{aligned}$$

Setting $f'(x) = \frac{1}{3x^{2/3}} = 0$, we see that f' is not defined at $x = 0$. Moreover, setting $f''(x) = \frac{-2}{9x^{5/3}} = 0$, we see that f'' is not defined at $x = 0$ either. It is clear from the graph however that there is a point of inflection[8] at $x = 0$. Indeed if we look at the graph of the second derivative, about $x = 0$, it changes sign. Hence we have established that $f''(x) = 0$ is not sufficient nor necessary for f to have a point of inflection at x. □

Exercises

Q1. Determine the stationary points of the following functions.

a. $f(x) = x^2 - 5x + 6$.

b. $f(x) = x^2 + 2x + 1$.

c. $f(x) = x^3 + 2x + 1$.

d. $f(x) = x^4 + 1$.

e. $f(x) = x^3 - 2x^2 + x + 1$.

f. $f(x) = 4x^3 - 12x$.

g. $f(x) = 3x^3 - x^2 + 4x - 5$.

h. $f(x) = 2x^3 + 5x + \sqrt{3}$.

Q2. Determine the stationary points of the following functions.

[8] non-stationary

a. $f(x) = \sin x$.
b. $f(x) = \cos x$.
c. $f(x) = 4\sin(3x - \pi)$.
d. $f(x) = \frac{2}{5}\cos\left(\frac{1}{2} - x\right)$.

Q3. Determine the stationary points of the following functions.

a. $f(x) = x \sin x$.
b. $f(x) = x^2 \cos(-x)$.
c. $f(x) = e^x \sin x$.
d. $f(x) = e^{-x}\cos(x^2)$.

Q4. a. Let $f(x) := \tan x$. Determine where f is differentiable.

b. Evaluate $f'(x)$ at all points that f was determined to be differentiable in part (a).

c. Hence, or otherwise, determine the stationary points of $g(x) := \tan^2 x$.

d. Using the second derivative test, or otherwise, determine the nature of the stationary points that were determined in part (c).

Q5. Let $\psi : [0, 4\pi] \to \mathbb{R}$ be the function defined by $\psi : x \mapsto x - \sin x$. Determine whether ψ has any points of inflection. Are any of these points stationary?

Q6. Show that for all $a, b, c \in \mathbb{R}$, the function $f(x) := ax^2 + bx + c$ has no points of inflection.

Q7. The radius $r > 0$ and height $h > 0$ of a solid circular cylinder $\mathscr{C}$ vary in such a way that the volume of the cylinder is always 250π.

a. Show that the total surface area $\mathscr{A}$ of the cylinder is given by

$$\mathscr{A} = 2\pi r^2 + \frac{500\pi}{r}.$$

b. What is the minimum surface area of $\mathscr{C}$?

Q8. Let $f(x) = x^2 - 1$ and $g(x) = e^{-x}$.

a. Determine the domain on which f and g are differentiable.

b. Determine the domain on $f(g(x))$ is differentiable.

c. Does the domain on which $g(f(x))$ is differentiable coincide with the domain found in part (b)?

d. Determine the stationary points of $f(g(x))$.

Q9. Describe the graph of $f(x)$ if $f'(x) = 0$ when $x = 3$ and $x = -2$, $f''(3) = 4$ and $f''(-2) = -5$.

Q10. Suppose that $f(x)$ satisfies $f'(3) = 0$, and $f'(x) > 0$ for all $x \in (3, 5)$. Determine which of the following is true:

a. $0 > f''(3) > f''(5)$.
b. $0 < f''(3) < f''(5)$.
c. $f''(3) < 0 < f''(5)$.
d. $f''(5) < 0 < f''(3)$.

Q11. Suppose that $f(x) = x^3 + bx^2 + cx + d$, where $b, c, d \in \mathbb{R}$. Determine the maximum number of turning points of f.

Q12. Let $f(x) := a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, be a polynomial of degree n with coefficients $a_i \in \mathbb{R}$ for all $0 \le i \le n$. Determine the maximum number of stationary points of f.

1.7 Applications of Calculus

We are now going to consider an application of the calculus techniques that we have learned up until this point. This is best illustrated through examples.

Example 5.7.1. Suppose that 1000 cm^2 of cardboard is available to make a box with square base and open top. Determine the length of the base which maximises the volume.

Proof. Let x be the length of the base. The volume of the box is then given by $V(x) = x^2 h$, where h is the height. The outside surface area is given by $A(x) = 4xh + x^2$, which we know is equal to 1000. It therefore follows that

$$4xh + x^2 = 1000 \quad \Longrightarrow \quad h = \frac{1000 - x^2}{4x}.$$

We may therefore write

$$V(x) = x^2 \left(\frac{1000 - x^2}{4x} \right) = 250x - \frac{x^3}{4}.$$

Since we wish to optimise the area, we determine the local max of $V(x)$. To this end, we observe that

$$\begin{aligned} \frac{dV}{dx} = 250 - \frac{3}{4}x^2 = 0 \quad &\Longrightarrow \quad x^2 = \frac{1000}{3} \\ &\Longrightarrow \quad x = \sqrt{\frac{1000}{3}}. \end{aligned}$$

To show that this does in fact maximise the volume, we look at the second derivative of $V(x)$, which is given by

$$\frac{d^2V}{dx^2} = -\frac{3}{2}x.$$

If we take $x = \sqrt{\frac{1000}{3}}$, it follows that $\frac{d^2V}{dx^2} < 0$ and so we have a local maximum. $\square$

Exercises

Q1. Find two numbers $x, y \in \mathbb{R}$ whose difference is 10 and whose product is a minimum.

Q2. Find two numbers $x, y \in \mathbb{R}$ whose product is 12 and whose sum is a minimum.

Q3. Find the dimensions of a rectangle which has area 10 m^2 and whose perimeter is as small as possible.

Q4. A cylinder is inscribed in a circle of radius $r = 4$. Determine the largest volume of the cylinder.

Q5. A cylinder is inscribed in a cone of base radius $r = 4$ and height $h = 12$. Determine the largest volume of the cylinder.

Q6. Let $f(x) = \frac{1}{x}$, for $x > 0$. Determine the point on the curve $f(x)$ which is closest to the origin.

Q7. The route that James takes when driving to school may be modelled by the hyperbolic equation $xy = 6$, where x denotes the horizontal position and y denotes the vertical position from a bird's-eye view. As James approaches the Mill park shopping centre, which occurs at the point $(3, 2)$ on the hyperbola, his vertical position is decreasing at a rate of 5 metres per second. At what rate is James' horizontal position changing when James reaches the Mill Park shopping centre?

Q8. Consider a water tank which has the shape of an inverted circular cone with base radius 1 metre and a height of 2 metres. Suppose that water is pumped into the tank at a rate of 0.5 m^3 per minute. Determine the rate at which the water level rises.

Q9. A cylindrical tank has a radius of 10 m and is initially filled with water. If the water is released at a rate of 4 m^3 per minute, determine how fast the depth of the water is changing when the tank is half empty.

1.8 Differentiation of Inverse Circular Functions

Listed below are the rules for differentiating the inverse circular functions. Observe that there is a change in the derivative depending on the value of $x \in \mathbb{R}$.

† $f(x) = \sin^{-1}(g(x)) \implies f'(x) = \frac{g'(x)}{\sqrt{1-(g(x))^2}}, \quad -1 < g(x) < 1.$

† $f(x) = \cos^{-1}(g(x)) \implies f'(x) = \frac{-g'(x)}{\sqrt{1-(g(x))^2}}, \quad -1 < g(x) < 1.$

† $f(x) = \tan^{-1}(g(x)) \implies f'(x) = \frac{g'(x)}{1+(g(x))^2}, \quad g(x) \in \mathbb{R}.$

Example 5.8.1. Differentiate each of the following functions, stating the domain.

a. $y = \sin^{-1}(\frac{x+2}{3})$.

Proof. We simply observe that

$$\begin{aligned}
\frac{d}{dx}\left(\sin^{-1}\left(\frac{x+2}{3}\right)\right) &= \left(\frac{x+2}{3}\right)' \cdot \frac{1}{\sqrt{1-(\frac{x+2}{3})^2}} \\
&= \frac{1}{3} \cdot \frac{1}{\sqrt{1-\frac{(x+2)^2}{9}}} \\
&= \frac{1}{3}\frac{3}{\sqrt{9-(x+2)^2}} \\
&= \frac{1}{\sqrt{9-(x+2)^2}}
\end{aligned}$$

□

b. $y = \tan^{-1}(\frac{x}{x-5})$

$$\begin{aligned}
\frac{d}{dx}\left(\tan^{-1}(\frac{x}{x-5})\right) &= \left(\frac{x}{x-5}\right)' \cdot \frac{1}{1+(\frac{x}{x-5})^2} \\
&= \left(\frac{x-5-x}{(x-5)^2}\right) \cdot \frac{1}{1+(\frac{x}{x-5})^2} \\
&= \left(\frac{-5}{(x-5)^2}\right) \cdot \frac{(x-5)^2}{(x-5)^2+x^2} \\
&= \frac{-5}{x^2+(x^2-10x+25)} \\
&= \frac{-5}{2x^2-10x+25}
\end{aligned}$$

Example 5.8.2. Differentiate the function $y = \sin^{-1}\left(\frac{2}{x}\right)$.

Proof. It is easy to see that

$$\begin{aligned}
\frac{d}{dx}\left(\sin^{-1}\left(\frac{2}{x}\right)\right) &= \left(\frac{2}{x}\right)' \cdot \frac{1}{\sqrt{1-(\frac{2}{x})^2}} \\
&= \frac{-2}{x^2} \cdot \frac{1}{\sqrt{1-(\frac{2}{x})^2}} \\
&= \frac{-2x}{x^2\sqrt{x^2-4}} \\
&= \frac{-2}{x\sqrt{x^2-4}}
\end{aligned}$$

□

Example 5.8.3. Find the equation of the tangent line when $x = \frac{1}{6}$ for the function $f(x) = 2\sin^{-1}(3x)$.

Proof. The derivative of $f(x)$ is given by

$$\begin{aligned}
f'(x) &= 2 \cdot 3 \cdot \frac{1}{\sqrt{1-(3x)^2}} \\
&= \frac{6}{\sqrt{1-9x^2}}.
\end{aligned}$$

Evaluating the derivative at $x = \frac{1}{6}$, we have

$$f'\left(\frac{1}{6}\right) = 6 \cdot \frac{1}{\sqrt{1 - \frac{9}{36}}} = 6 \cdot \frac{1}{5/6} = \frac{36}{5}.$$

So the equation of the tangent line to $f(x)$ at $x = \frac{1}{6}$ has the form $y = \frac{36}{6}x + c$ for some $c \in \mathbb{R}$. To determine the value of c, we use the fact that $(1/6, f(1/6))$ lies on the curve $f(x)$. Note that

$$f\left(\frac{1}{6}\right) = 2\sin^{-1}\left(\frac{1}{2}\right) = 2 \cdot \frac{\pi}{6} = \frac{\pi}{3}.$$

Plugging this into our equation for the tangent line, we have

$$\begin{aligned} \frac{\pi}{3} &= \frac{36}{5}\left(\frac{1}{6}\right) + c \\ \therefore\ c &= \frac{\pi}{3} - \frac{6}{5} = \frac{1}{15}\left(5\pi - 18\right). \end{aligned}$$

Hence, the equation of the tangent line is given by

$$y = \frac{36}{5}x + \frac{1}{15}(5\pi - 18).$$

□

Exercises

Q1 Differentiate the following functions, with respect to x.

(a) $y = 2\sin^{-1}(x)$	(b) $y = 3\cos^{-1}(2x)$	(c) $y = \sin^{-1}(\frac{x}{2})$
(d) $y = \cos^{-1}(\frac{x}{6})$	(e) $y = \tan^{-1}(\frac{x}{3})$	(f) $y = 2\cos^{-1}(\frac{x}{3})$
(g) $y = 4\tan^{-1}(\frac{2x}{3})$	(h) $y = 2\cos^{-1}(\frac{2x}{5})$	(i) $y = \frac{3}{2}\sin^{-1}(\frac{x}{4})$
(j) $y = \frac{2}{5}\cos^{-1}(3x)$	(k) $y = \frac{1}{2}\tan^{-1}(2x)$	(l) $y = \frac{4}{5}\tan^{-1}(3x)$

Q2 Differentiate the following functions, with respect to x.

(a) $y = \sin^{-1}(\frac{x+1}{2})$ (b) $y = \cos^{-1}(\frac{x+2}{4})$ (c) $y = \tan^{-1}(\frac{2x-3}{5})$

(d) $y = \cos^{-1}(\frac{x}{3} + \frac{1}{5})$ (e) $y = \tan^{-1}(\frac{x+7}{3})$ (f) $y = \sin^{-1}(\frac{2x+\pi}{3})$

Q3 Differentiate the function

$$f(x) = \sin^{-1}(x) \cdot \cos^{-1}(x).$$

Q4 Differentiate the function

$$f(x) = \tan^{-1}(\cos^{-1}(x)).$$

Q5 Differentiate the function

$$f(x) = x \cdot \sin^{-1}(\sqrt{x}).$$

Q6 Differentiate the function

$$g(x) = \left|\tan^{-1}(x)\right|.$$

Q7 Differentiate the function

$$h(x) = \sin^{-1}\left(\frac{f(x)}{\sqrt{g(x)}}\right),$$

where f and g are differentiable everywhere and $g(x) > 0$ for all $x \in \mathbb{R}$.

Q8 Differentiate the function

$$f(x) = \frac{\tan^{-1}(\sqrt{x})}{\sin^{-1}(\sqrt{f(x)})},$$

where $f(x) > 0$ and differentiable for all $x \in \mathbb{R}$.

1.9 Implicit Differentiation

Implicit differentiation is a method of differentiating a function when y cannot be represented as an explicit function of x. For example $xy + \frac{x}{y^2+x} = \frac{2}{x}$, $\sqrt{x^2+y^2} = \frac{2xy}{x^2+y^2}$ are examples of implicit functions.

If we wish to find the derivative of y with respect to x, the derivative is trivially $\frac{dy}{dx}$. What is the derivative of y^2 with respect to x? Using the method of implicit differentiation:

$$y^2 \implies 2y \cdot \frac{dy}{dx} = 2\frac{d(y^2)}{dx}.$$

Example 5.9.1. Find $\frac{dy}{dx}$ if

$$y^2 - x^3 + \frac{6}{y} = 2x - y.$$

Proof. We simply observe that

$$\begin{aligned}
y^2 - x^3 + \frac{6}{y} &= 2x - y \\
2y \cdot \frac{dy}{dx} - 3x^2 - \frac{6}{y^2} \cdot \frac{dy}{dx} &= 2 - 1 \cdot \frac{dy}{dx} \\
2y \cdot \frac{dy}{dx} - \frac{6}{y^2} \cdot \frac{dy}{dx} + \frac{dy}{dx} &= 2 + 3x^2 \\
\frac{dy}{dx}\left(2y + 1 - \frac{6}{y^2}\right) &= 2 + 3x^2 \\
\frac{dy}{dx} &= \frac{2 + 3x^2}{\left(2y + 1 - \frac{6}{y^2}\right)} \\
&= \frac{2 + 3x^2y^2}{2y^3 + y^2 - 6}
\end{aligned}$$

□

Example 5.9.2. Find the value(s) for x such that $\frac{dy}{dx} = -1$ if

$$\frac{(x-1)^2}{4} + \frac{(y+3)^2}{16} = 1.$$

Proof. We simply observe that

$$\begin{aligned}
\frac{(x-1)^2}{4}+\frac{(y+3)^2}{16} &= 1\\
\frac{(x-1)}{2}+\frac{(y+3)}{8}\cdot\frac{dy}{dx} &= 0\\
\frac{(y+3)}{8}\cdot\frac{dy}{dx} &= \frac{1-x}{2}\\
\frac{dy}{dx} &= \frac{8(1-x)}{2(y+3)}\\
&= \frac{4(1-x)}{y+3}\\
\therefore \frac{4(1-x)}{y+3} &= -1\\
4-4x &= -3-y\\
y &= -7+4x.
\end{aligned}$$

We now substitute the above equation into the original equation $\frac{(x-1)^2}{4}+\frac{(y+3)^2}{16}=1$. This yields

$$\begin{aligned}
\frac{(x-1)^2}{4}+\frac{(4x-3)^2}{16} &= 1\\
4(x-1)^2+(4x-3)^2 &= 16\\
4(x^2-2x+1)+(16x^2-24x+9) &= 16\\
4x^2-8x+4+16x^2-24x+9 &= 16\\
20x^2-32x-3 &= 0.
\end{aligned}$$

Hence we see that

$$x=\frac{8\pm\sqrt{79}}{10}.$$

Note that the last assertion follows easily from an application of the quadratic formula. □

Exercises

Q1. Determine $\frac{dy}{dx}$ for each of the following expressions

a. $x^2y = 2x - 3$ b. $\frac{x}{y} - x = 2y$ c. $\frac{x+y}{x-y} = \frac{1}{y}$

d. $2\sin^{-1}(y) = x$ e. $\cos(x) = \sin(y)$ f. $\tan(y) = \sec^2(y)$

Q2. Find $\frac{dy}{dx}$ if

$$\log_e(x^2 + y) = \frac{3}{\tan^{-1}(x)}.$$

Q3. Find $\frac{dy}{dx}$ if

$$e^{\frac{x}{y}} + e^{x^2+y^2} = 1.$$

Q4. Evaluate $\frac{dr}{d\vartheta}$ if $r = \cos\vartheta + 3\sin^{-1}(\vartheta)$.

Q5. Determine $\frac{dy}{dx}$ for the expression

$$\frac{f(x)}{f(y)} + 2f(x) = 2y,$$

where f is differentiable and nonzero for all $x \in \mathbb{R}$.

Q6. Determine $\frac{dy}{dx}$ for the expression

$$\frac{f(x+y)}{x+y} = 2x - \frac{3}{y},$$

where f is differentiable for all $x \in \mathbb{R}$.

Q7. Find the value(s) of x such that $\frac{dy}{dx} = 1$ for

$$\frac{(x+1)^2}{16} + \frac{y^2}{4} = 1.$$

Q8. Find the value(s) of x such that $\frac{dy}{dx} = \frac{-1}{2}$ for

$$\frac{x^2}{4} - \frac{y^2}{25} = 1.$$

Q9. Determine the slope of the tangent line to the curve $y = x\sqrt{x+y}$ at the point $(1, 0)$.

1.10 Review Exercises

Q1. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x^2 e^{3-x}.$$

Determine the stationary points of f and classify their nature.

Q2. Let $f : \mathbb{R} \backslash \left\{-\frac{3}{4}\right\}$ be the function defined by

$$f(x) = \frac{2-x}{4x+3}.$$

Show that f is differentiable on $\mathbb{R} \backslash \left\{-\frac{3}{4}\right\}$ and evaluate $f'(x)$.

Q3. Let f be the function defined by

$$f(x) = \tan^{-1}(\sqrt{x+4}) + \sec(x).$$

Evaluate $f'(x)$.

Q4. A cubular block of ice melts at a rate of 2 cm^3 per second. Determine the rate at which the length of the cube is decreasing.

Q5. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = e^{2-x}(x^2 + 4x + 6).$$

Determine whether f has an inflection point at $x = 0$.

Q6. A right circular cylinder is inscribed in a sphere of radius $r > 0$. Find the largest possible volume of the cylinder.

Q7. Suppose that

$$xy + \frac{1}{\sqrt{x+1}} = y^2 + \frac{1}{x-y}.$$

Evaluate $\frac{dy}{dx}$.

Q8. Prove that

$$\tan x - x > 0$$

for all $0 < x < \frac{\pi}{2}$.

[Hint: First show that $f(x) := \tan x - x$ is an increasing function for all $0 < x < \frac{\pi}{2}$.]

Q9. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = \frac{x}{x+1}.$$

Determine the stationary points of f and classify their nature.

Q10. Evaluate the limit

$$\lim_{x \to 0} x \sin\left(\frac{1}{x}\right).$$

Q11. A ladder of length 3m rests against a vertical wall. If the bottom of the ladder slides away at a speed of 1m per second, determine the rate at which the angle between the top of the ladder and the wall is changing when the angle is at $\vartheta = \frac{\pi}{6}$.

Q12. Let f be a function which is continuously differentiable. Evaluate the derivative of

$$g(x) := \frac{1 + \sqrt{f(x)+1}}{x},$$

where $f(x) + 1 \geq 0$ for all $x \in \mathbb{R}$.

Q13. Define the curve γ be

$$\sqrt{x} + \sqrt{y} = \sqrt{\lambda},$$

for some $\lambda \geq 0$. Show that the axes intercepts of any tangent line to the curve γ is equal to λ.

Q14. Evaluate the limit

$$\lim_{x \to 1} \frac{\sin(x-1)}{x^2 + x - 2}.$$

Q15. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x\,|x|\,.$$

Determine whether f has a point of inflection at $x = 0$.

Q16. Find the point on the hyperbola $y^2 = 9 + x^2$ that is closes to the points $(3, 0)$.

Q17. Evaluate the limit

$$\lim_{x \to \infty} \frac{\cos^2 x}{x^2}.$$

Q18. Find all points on the curve

$$x^2y^2 + xy = 3$$

where the slope of the tangent line is 1.

Q19. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = x \cos x.$$

Determine the equation of the tangent line at $x = \pi$.

Q20. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by

$$f(x) = x \sin x.$$

Determine the stationary points of f.

Q21. Evaluate the following limit

$$\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}.$$

Q22. Consider the function $f : \mathbb{R} \backslash \{3\} \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{3 - x}.$$

Show that f is differentiable on $\mathbb{R} \backslash \{3\}$ and compute $f'(x)$.

Q23. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{4}{\sqrt{x + \pi}} + \sqrt{x}.$$

Explain why f is differentiable on $(0, \infty)$ and compute $f'(x)$.

Q24. Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

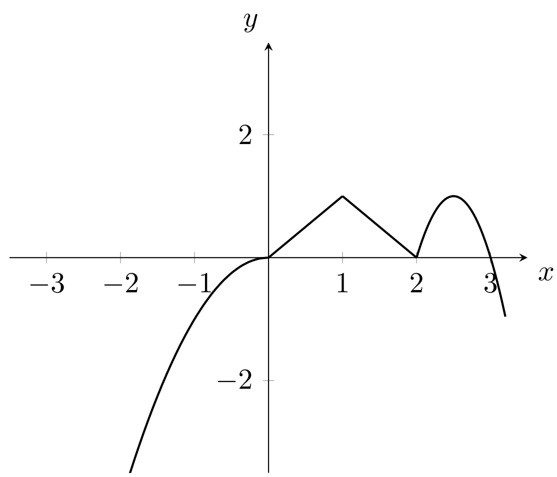
This function forms the prototypical example of a function that is continuous but *not* differentiable.

a. [10 marks]. Explain exactly what is meant by a function not being differentiable.

b. [5 marks]. Determine the point(s) where f is not differentiable.

c. [5 marks]. Compute the derivative of f at the points where f is differentiable.

Q25. Consider the function f whose graph is given below.



Determine the domain of $f'(x)$ and sketch $f'(x)$ on this domain. [Hint: If you struggle with the sketch, just describe where the derivative is positive, negative, zero, undefined, etc.]

Q26. Evaluate the following limit

$$\lim_{x \to -1} \frac{x^2 + 2x + 1}{x + 1}.$$

Q27. Consider the function $f : \mathbb{R}\backslash\{3\} \to \mathbb{R}$ defined by

$$f(x) = \frac{5 + x}{4 - 2x}.$$

Show that f is differentiable on $\mathbb{R}\backslash\{2\}$ and compute $f'(x)$.

Q28. Consider the function $f : [0, \infty) \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{\sqrt{x - 1}} + \frac{3}{(x - 6)^{\frac{3}{7}}}.$$

Explain why f is differentiable on $(1, \infty)$ and compute $f'(x)$.

Q29. Determine where the following functions are not differentiable.

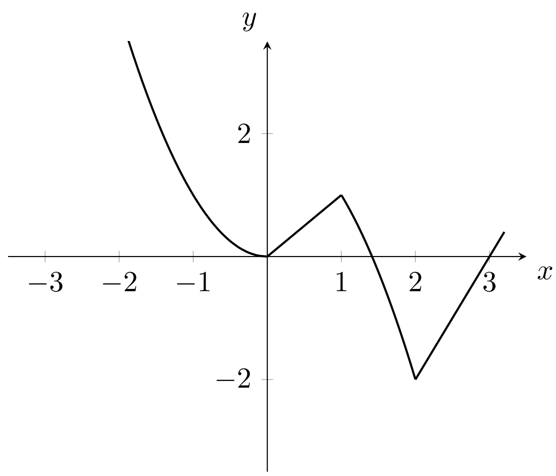
a. [5 marks]. $f(x) = 2\,|x-3| + 1$.

b. [5 marks]. $f(x) = 4\,|2x-5| - 7$.

c. [5 marks]. $f(x) = \frac{3}{2}\,|7-x|$.

d. [5 marks]. $f(x) = 1/\,|x|$.

Q30. Consider the function f whose graph is given below.



Determine the domain of $f'(x)$ and sketch $f'(x)$ on this domain.