Exponentials and Logarithms

Kyle Broder – ANU – MSI – 2017

Introduction to the Exponential Function

We introduce the exponential function $f(x) = e^x$ to be a continuous function that grows faster than any polynomial function. Recall that a polynomial is a function $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = a_n x^n + \cdots + a_1 x + a_0$, where the $a_k \in \mathbb{R}$, for $0 \leq k \leq n$.

Some basic properties of the exponential function are:

1. $e^x \cdot e^y = e^{x+y}$ (products are taken to sums),
2. $e^x / e^y = e^{x-y}$ (equivalent to 1),
3. $(e^x)^y = e^{xy}$ (powers are taken to products),
4. $e^0 = 1$.

In order to solve exponential equations, we need to introduce the logarithm function.

The logarithm $f(x) = \log_e(x)$ is the inverse function of e^x. That is, if we compose $\log_e(x)$ with e^x, we get the identity function $f(x) = x$.

Some basic properties of the logarithm function are

1. $\log_e(x) + \log_e(y) = \log_e(xy)$ (sums are taken to products),
2. $\log_e(x) - \log_e(y) = \log_e(x/y)$ (equivalent to 1),
3. $\log_e(x^y) = y \log_e(x)$ (powers are taken to products)
4. $\log_e(1) = 0$.

Example 1. Prove that $\log_e(e) = 1$.

Proof. Let $\log_e(e) = k$, for some $k \in \mathbb{R}$. Then $e^k = e$. It is then clear that $k = 1$ and therefore $\log_e(e) = 1$. $\square$

Example 2. Solve the equation $e^{2x+1} = 5$ for $x \in \mathbb{R}$.

Proof.

$$\begin{aligned} e^{2x+1} = 5 \quad &\implies \quad \log_e(e^{2x+1}) = \log_e(5) \\ &\implies \quad (2x+1)\log_e(e) = \log_e(5) \\ &\implies \quad 2x + 1 = \log_e(5) \\ &\implies \quad 2x = \log_e(5) - 1 \\ &\implies \quad x = \frac{1}{2}\left(\log_e(5) - 1\right). \end{aligned}$$

$\square$

Question 1. Solve the following exponential equations.

a. $e^{3x-5} = 10$.
b. $e^{2x-7} = 1$.
c. $2e^{x} = \frac{1}{3}$.
d. $e^{x} \cdot e^{2x+4} = 2$.
e. $e^{x-6} = \frac{3}{e^{x+2}}$.

Example 3. Solve the logarithmic equation $\log_e(x) + \log_e(3x+1) = 2$.

Proof. We observe that

$$\begin{aligned} \log_e(x) + \log_e(3x+1) = 2 \quad &\implies \quad \log_e(x \cdot (3x+1)) = 2 \\ &\implies \quad x(3x+1) = e^2 \\ &\implies \quad 3x^2 + x - e^2 = 0 \\ &\implies \quad x = \frac{-1 \pm \sqrt{1^2 - 4(3)(-e^2)}}{3(2)} \\ &\implies \quad x = \frac{-1 \pm \sqrt{1 + 12e^2}}{6}, \\ &\implies \quad x = \frac{-1 + \sqrt{1 + 12e^2}}{6}, \end{aligned}$$

where the last implication is realised since we may only take positive values for x in the domain of the logarithm. $\square$

Question 2. Solve the following logarithmic equations for $x \in \mathbb{R}$.

a. $\log_e(x) + \log_e(3) - \log_e(4) = \log_e(x+1)$.
b. $\log_e(2x) + \log_e(4) = 2\log_e(2)$.
c. $\log_e(7-x) + 4\log_e(1) = 2\log_e(x)$.

Question 3. Suppose that $\mathscr{S}$ represents the amount (in milligrams per cubic meter) of sarin gas in a particular bag. The amount of sarin gas that leaks out of the bag into a train station after t hours is given by an equation of the form

$$\mathscr{S} = -2 + 2e^{kt}.$$

a. Determine the initial amount of sarin gas in the train.
b. Suppose that in one hour the amount of sarin in the train reaches a lethal concentration of 35 mg per cubic meter. Determine the value of k.
c. Determine the amount of sarin in the train after 2 hours.

Question 4.

a. Solve $\log_3(6-x) - \log_3(4-x) = 2$ for x, where $x < 4$.
b. Solve $3e^{x} = 5 + 8e^{-x}$ for $x \in \mathbb{R}$.

Question 5. Solve the following equation

$$\log_e(x) - \lambda = \log_e(2\sqrt{x}),$$

in terms of λ, where $x > 0$ and $\lambda \in \mathbb{R}$.

Question 6. Suppose that $f : (-\infty, 2) \to \mathbb{R}$ is defined by

$$f(x) = \log_e(2 - x)$$

and suppose that $g : [-2, \infty) \to \mathbb{R}$ is defined by $g(x) = \sqrt{x+2}$. What is the largest domain that we could define the following functions on?

a. $f + g$.
b. $f - g$.
c. $f \cdot g$.
d. $f \circ g$.
e. $g \circ f$.

Example 4. Show that

$$\log_e(x) = \frac{1}{\log_x(e)}.$$

Proof. Let $\log_e(x) = k$. Then we observe that

$$\begin{aligned}
\log_e(x) = k \quad &\implies \quad e^k = x \\
&\implies \quad \log_x(e^k) = \log_x(x) \\
&\implies \quad k\log_x(e) = \log_x(x) \\
&\implies \quad k\log_x(e) = 1 \\
&\therefore \quad k = \frac{1}{\log_x(e)}.
\end{aligned}$$

□

Question 7. Determine the value(s) of $k \in \mathbb{R}$ such that the following equation has only one solution for $x \in \mathbb{R}$,

$$\log_e(x) - 3k\log_x(e) + 2 = 0.$$

Question 8. Determine the value(s) of $k \in \mathbb{R}$ such that the following equation has no solutions for $x \in \mathbb{R}$,

$$\log_e(x) - \log_x(e) + k = 3.$$

Question 9. Let $\mu > 0$ be a fixed real number. Solve the equation

$$e^{\mu x} = 3 + \frac{1}{e^{\mu x}}.$$

Question 10. Plot the following functions on their maximal domains.

a. $f(x) = e^{x-3}$.
b. $f(x) = 2e^{3x+1} - 4$.
c. $f(x) = \frac{1}{2}e^{\frac{1}{2}x+5}$.
d. $f(x) = \frac{2}{e^{x+10}} - 3$.
e. $f(x) = 2\log_e(x) + 1$.
f. $f(x) = 3\log_e(5x+2) - \frac{4}{5}$.
g. $f(x) = 4\log_e(2-x) - 4$.
h. $f(x) = 2\log_e(x) + 3e^x$.
i. $f(x) = e^{2|x|}$.
j. $f(x) = |e^x|$.
k. $f(x) = \log_e|x+1| - 2$.
l. $f(x) = -\log_e\left|x - \frac{1}{2}\right| + 13$.
m. $f(x) = \log_e(1/x)$.
n. $f(x) = \log_e\left(\frac{1}{|x|}\right)$.

Question 11. Describe the transformations (in order) that are required to map the function f to the function g, where

a. $f(x) = e^x$, $g(x) = -e^{2x+1}$.
b. $f(x) = e^x$, $g(x) = 4e^{x+5} - 15$.
c. $f(x) = \log_e(x)$, $g(x) = 2\log_e(x-3) + 1$.
d. $f(x) = \log_e(x)$, $g(x) = \log_e(6-2x) - 5$.