Chapter 1

Vectors.

1.1 Elementary Definitions and Operations

Let us begin with some definitions.

Definition 8.1.1. A vector space V is set such that

† for all $v, w \in V$, $v + w \in V$. That is to say, V is closed under addition.

† for all $v \in V$ and $\lambda \in \mathbb{R}$, $\lambda v \in V$. That is to say, V is closed under scalar multiplication.

A vector is an element of a vector space.

The most notable example of a vector space for our purposes here will be $\mathbb{R}^2$. Vectors in $\mathbb{R}^2$ may simply be considered as arrows.

Let $v = \langle v_1, v_2 \rangle$ and $w = \langle w_1, w_2 \rangle$ be two vectors in $\mathbb{R}^2$. To add vectors, we add component-wise, that is,

$$v + w = \langle v_1, v_2 \rangle + \langle w_1, w_2 \rangle = \langle v_1 + w_1, v_2 + w_2 \rangle.$$

If $\lambda \in \mathbb{R}$ and $v = \langle v_1, v_2 \rangle$, to multiply λ and v, we again do it component-wise,

$$\lambda \cdot \langle v_1, v_2 \rangle = \langle \lambda v_1, \lambda v_2 \rangle.$$

Example 8.1.2. Let $v = \langle 1, 2 \rangle$ and $w = \langle -1, 1 \rangle$. Evaluate $v + w$ and $2v - w$.

Proof. We simply observe that

$$\begin{aligned} v + w &= \langle 1, 2\rangle + \langle -1, 1\rangle \\ &= \langle 1-1, 2+1\rangle \\ &= \langle 0, 3\rangle. \end{aligned}$$

Similarly, we have

$$\begin{aligned} 2v - w &= 2\langle 1, 2\rangle - \langle -1, 1\rangle \\ &= \langle 2, 4\rangle - \langle -1, 1\rangle \\ &= \langle 2+1, 4-1\rangle \\ &= \langle 3, 3\rangle. \end{aligned}$$

□

Common notation that is used for vectors in $\mathbb{R}^2$, $v = \langle v_1, v_2\rangle$ is

$$v = v_1\mathbf{i} + v_2\mathbf{j}.$$

Similarly, in $\mathbb{R}^3$, for vectors $v = \langle v_1, v_2, v_3\rangle$, we have

$$v = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Example 8.1.3. Consider the vectors $u = 2\mathbf{i}+3\mathbf{j}+\mathbf{k}$ and $v = 3\mathbf{i}+\mathbf{j}+5\mathbf{k}$. Compute the following

(a) $u + v$.

Proof. We simply observe that

$$\begin{aligned} u + v &= 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} + 3\mathbf{i} + \mathbf{j} + 5\mathbf{k} \\ &= (2+3)\mathbf{i} + (3+1)\mathbf{j} + (1+5)\mathbf{k} \\ &= 5\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}. \end{aligned}$$

□

(b) $u - v$.

Proof. Similarly, we have

$$\begin{aligned} u - v &= 2\mathbf{i} + 3\mathbf{j} + \mathbf{k} - (3\mathbf{i} + \mathbf{j} + 5\mathbf{k}) \\ &= (2-3)\mathbf{i} + (3-1)\mathbf{j} + (1-5)\mathbf{k} \\ &= -\mathbf{i} - 2\mathbf{j} - 4\mathbf{k}. \end{aligned}$$

$\square$

(c) $3 \cdot u$.

Proof. We also have

$$3u = 3(2\mathbf{i} + 3\mathbf{j} + \mathbf{k}) = 6\mathbf{i} + 9\mathbf{j} + 3\mathbf{k}.$$

$\square$

Exercises

Q1. Let $\mathscr{C}(\mathbb{R})$ denote the space of continuous functions $f : \mathbb{R} \to \mathbb{R}$. Show that $\mathscr{C}(\mathbb{R})$ is a vector space. What are the vectors in $\mathscr{C}(\mathbb{R})$.

Q2. Let $\mathbb{Z}$ denote the integers. Determine whether $\mathbb{Z}$ is a vector space.

Q3. Let $\mathbb{Q}$ denote the rational numbers. Determine whether $\mathbb{Q}$ is a vector space.

Q4. Let $\mathbb{C}$ denote the complex numbers. Determine whether $\mathbb{C}$ is a vector space.

Q5. Let $\mathscr{C}^1(\mathbb{R})$ denote the space of continuously differentiable functions $f : \mathbb{R} \to \mathbb{R}$. Deteremine whether $\mathscr{C}^1(\mathbb{R})$ is a vector space.

Q6. Does every vector space V contain the zero vector $\mathbf{0}$? That is, a vector $w \in V$ such that for all $v \in V$, $v + w = w + v = v$.

Q7. Is addition of vectors commutative, that is, if $v, w \in V$, does $v + w = w + v$?

Q8. Let $v = \langle -2, 0 \rangle$, $w = \langle 1, 4 \rangle$ and $\lambda = 3$. Determine the following vectors.

a. $v + w$.

b. $v - w$.

c. λv.

d. λw.

e. $\lambda v + w$.

f. $w - \lambda v$.

Q9. Let $v = \langle 3, -4 \rangle$, $w = \langle 6, 2 \rangle$ and $\lambda = -7$. Determine the following vectors.

a. $v + w$.

b. $v - w$.

c. λv.

d. λw.

e. $\lambda v + w$.

f. $w - \lambda v$.

Q10. Let $v = \mathbf{i} + \frac{1}{2}\mathbf{j} - 4\mathbf{k}$ and $w = -2\mathbf{i} + 6\mathbf{j} + 1\mathbf{k}$. Determine the following vectors.

a. $v + w$.

b. $v - 3w$.

c. $2v + w$.

d. $4w$.

e. $\frac{1}{2}v$.

f. $4w - \frac{1}{2}v$.

1.2 Normed and Inner Product Spaces

We have so far defined a vector space to be a set with a well defined notion of addition and an action of multiplication of multiplication by scalars. We are yet however, to discuss a notion of distance or magnitude. That is, we proceed to answer the questions of 'how far away are two vectors?' and 'how big are these vectors?'.

Definition 8.2.1. A metric on a vector space V is a map $d : V \times V \to [0, \infty]$ which satisfies the following conditions.

† (Uniqueness). $d(u, v) = 0 \iff u = v$, for all $u, v \in V$.

† (Positivity). $d(u, v) \geq 0$, for all $u, v \in V$.

† (Symmetry). $d(u, v) = d(v, u)$, for all $u, v \in V$.

† (Triangle Inequality). $d(u, w) \leq d(u, v) + d(v, w)$, for all $u, v, w \in V$.

A vector space equipped with a metric is called a metric space. Examples of metric spaces include $\mathbb{R}^2$ with the metric

$$d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2},$$

where $u = \langle u_1, u_2 \rangle$ and $v = \langle v_1, v_2 \rangle$. Another example of a metric space is $\mathscr{C}([0, 1])$ equipped with the metric

$$d(f, g) = \max_{x \in [0,1]} |f(x) - g(x)| .$$

Our main examples of a metric space however are $\mathbb{R}^2$ equipped with the Euclidean metric

$$d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2}, \quad u = \langle u_1, u_2 \rangle, v = \langle v_1, v_2 \rangle,$$

and $\mathbb{R}^3$ equipped with the Euclidean metric

$$d(u, v) = \sqrt{(u_1 - v_1)^2 + (u_2 - v_2)^2 + (u_3 - v_3)^2},\ u = \langle u_1, u_2, u_3 \rangle, v = \langle v_1, v_2, v_3 \rangle.$$

Example 8.2.2. Let $v = \langle 2, 3 - 1 \rangle$ and $w = \langle -3, 4, 2 \rangle$. Determine the distance between v and w.

Proof. We simply observe that

$$\begin{aligned} d(v,w) &= \sqrt{(v_1-w_1)^2+(v_2-w_2)^2+(v_3-w_3)^2} \\ &= \sqrt{(2-(-3))^2+(3-4)^2+((-1)-2)^2} \\ &= \sqrt{5^2+(-1)^2+(-3)^2} \\ &= \sqrt{25+1+9} \\ &= \sqrt{35}. \end{aligned}$$

□

A notion of size in mathematics is formalised by what is referred to as a norm.

Definition 8.2.3. A norm is a function $\|\cdot\| : V \times V \to [0,\infty)$ which satisfies the following conditions.

† (Uniqueness). $\|v\| = 0 \iff v = 0$, for all $v \in V$.

† (Positivity). $\|v\| \geq 0$, for all $v \in V$.

† (Homogeneity). $\|\lambda v\| = |\lambda| \cdot \|v\|$, for all $\lambda \in \mathbb{R}, v \in V$.

† (Triangle Inequality). $\|u+v\| \leq \|u\| + \|v\|$, for all $u, v \in V$.

A vector space equipped with a norm is a normed space. Notice that if $\|\cdot\|$ is a norm, then $\|v-w\|$ naturally defines a metric. We therefore see that every normed space is a metric space. The main examples of normed spaces that we will consider here is $\mathbb{R}^2$ equipped with the Euclidean norm

$$\|v\| = \sqrt{v_1^2+v_2^2}$$

and $\mathbb{R}^3$ equipped with the Euclidean norm

$$\|v\| = \sqrt{v_1^2+v_2^2+v_3^2}.$$

Example 8.2.4. Let $v = \langle 1, 5, -3 \rangle$. Determine the norm of v.

Proof. It is easy to see that

$$\|v\| = \sqrt{1^2+5^2+(-3)^2} = \sqrt{1+25+9} = \sqrt{35}.$$

□

The last natural notion that we would like to endow a vector space with is that of an angle between two vectors. This is given by a so-called inner product.

Definition 8.2.5. An inner product is map $(\cdot,\cdot) : V \times V \to \mathbb{R}$ with the following properties.

† $(v, w) = 0 \iff v = w$, for all $v, w \in V$.

† $(\lambda v, w) = \lambda(v, w) = (v, \lambda w)$, for all $\lambda \in \mathbb{R}, v, w \in V$.

† $(v, w) = (w, v)$, for all $v, w \in V$.

† $(u + v, w) = (u, w) + (v, w) = (v, u + w)$, for all $u, v, w \in V$.

A vector space equipped with an inner product is referred to as an inner product space. The main examples of inner product spaces that we will consider here is $\mathbb{R}^2$ equipped with the inner product

$$(v, w) = v_1w_1 + v_2w_2,\ v = \langle v_1, v_2\rangle, w = \langle w_1, w_2\rangle,$$

and $\mathbb{R}^3$ equipped with the inner product

$$(v, w) = v_1w_1 + v_2w_2 + v_3w_3,\ v = \langle v_1, v_2, v_3\rangle, w = \langle w_1, w_2, w_3\rangle.$$

This inner product is often referred to as the dot product.

We say that two vectors $v, w \in V$ are parallel if $(v, w) = 1$, and say that two vectors $v, w \in V$ are orthogonal, or perpendicular, if $(v, w) = 0$.

Example 8.2.6. Let $v = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ and $w = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$ be two vectors in $\mathbb{R}^3$. Determine whether these vectors are orthogonal.

Proof. Recall that two vectors in $\mathbb{R}^3$ are orthogonal if $(v, w) = 0$. We therefore observe that

$$\begin{aligned}(v, w) &= 2 \cdot 1 + (-1) \cdot 2 + 3 \cdot (-1) \\ &= 2 - 2 - 3 = -3.\end{aligned}$$

The vectors v and w are therefore not orthogonal with respect to the dot product. □

Example 8.2.7. Let $r(t) = r\cos t\mathbf{i} + r\sin t\mathbf{j}$ denote the position of a small rock on the end of a string of length r that is being swung in a circular motion. Show that the velocity of the rock is orthogonal to the position of the rock at any time t.

Proof. Using some elementary calculus, we see that the velocity is given by

$$v(t) = -r\sin t\mathbf{i} + r\cos t\mathbf{j}.$$

Then, by calculating the dot product of $r(t)$ and $v(t)$, we see that

$$\begin{aligned}(r(t), v(t)) &= [r\cos t]\cdot[-r\sin t] + [r\sin t]\cdot[r\cos t]\\ &= -r^2\cos t\sin t + r^2\sin t\cos t = 0.\end{aligned}$$

□

We also have the rather useful characterisation of the dot product on $\mathbb{R}^2$.

Theorem 8.2.8. Let $v, w \in \mathbb{R}$ be two vectors. Then the dot product of v and w is given by

$$v \cdot w = |v|\,|w|\cos\vartheta.$$

Example 8.2.9. Determine the angle between the vectors $v = \mathbf{i}-3\mathbf{j}+2\mathbf{k}$ and $w = -2\mathbf{i}+\mathbf{j}+4\mathbf{k}$.

Proof. We first observe that $|v| = \sqrt{1^2+(-3)^2+2^2} = \sqrt{1+9+4} = \sqrt{14}$, and $|w| = \sqrt{(-2)^2+1^2+4^2} = \sqrt{4+1+16} = \sqrt{21}$. Moreover, $v\cdot w = -2-3+8 = 3$, so we see that

$$\cos\vartheta = \frac{v\cdot w}{|v|\,|w|} = \frac{3}{\sqrt{14}\sqrt{21}} \implies \vartheta = \cos^{-1}\left(\frac{3}{\sqrt{14}\sqrt{21}}\right).$$

□

Exercises

Q1. Calculate the magnitude of the following vectors.

a. $v = 2\mathbf{i} - \mathbf{j} - \frac{1}{2}\mathbf{k}$.

b. $v = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

c. $v = -\frac{2}{3}\mathbf{i} + \frac{4}{3}\mathbf{j} + 2\mathbf{k}$.

d. $v = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.

e. $v = 2\mathbf{i} - 7\mathbf{j} + \frac{3}{4}\mathbf{k}$.

f. $v = 3\mathbf{j} + 2\mathbf{k}$.

Q2. Determine the distance between the vectors $v, w \in \mathbb{R}^2$, where

a. $v = 3\mathbf{i} - 2\mathbf{j}$ and $w = -\mathbf{i} + 5\mathbf{j}$.
b. $v = \frac{1}{3}\mathbf{i} - 5\mathbf{j}$ and $w = 2\mathbf{i} + 4\mathbf{j}$.
c. $v = 4\mathbf{i} - 5\mathbf{j}$ and $w = \frac{1}{3}\mathbf{j}$.
d. $v = 7\mathbf{i} + \frac{3}{2}\mathbf{j}$ and $w = 9\mathbf{i} - \mathbf{j}$.

Q3. Determine the angle between the following vectors.

a. $v = 3\mathbf{i} - 2\mathbf{j}$ and $w = -\mathbf{i} + 5\mathbf{j}$.
b. $v = \frac{1}{3}\mathbf{i} - 5\mathbf{j}$ and $w = 2\mathbf{i} + 4\mathbf{j}$.
c. $v = 4\mathbf{i} - 5\mathbf{j}$ and $w = \frac{1}{3}\mathbf{j}$.
d. $v = 7\mathbf{i} + \frac{3}{2}\mathbf{j}$ and $w = 9\mathbf{i} - \mathbf{j}$.

Q4. Let $v = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. Provide an example of a vector $w \in \mathbb{R}^3$ which is orthogonal to v.

Q5. Let $v = -3\mathbf{i} + 5\mathbf{j} + \mathbf{k}$. Provide an example of a vector $w \in \mathbb{R}^3$ which is parallel to w.

Q6. Show that the magnitude of the vector $v = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$ is constant, independent of $t \in \mathbb{R}$.

Q7. Let $r(t)$ denote the position of a particle at time $t > 0$. If

$$r(t) := 3t^2\mathbf{i} + 2\ln(10t + 4)\mathbf{j} + \frac{1}{t^2 + 1}\mathbf{k},$$

determine the distance from the origin at $t = 1$.

Q8. Let $\mathcal{R}[0,1]$ denote the space of Riemann integrable functions $f : [0,1] \to \mathbb{R}$. Show that

$$\|f\| := \int_0^1 |f(x)|\, dx,$$

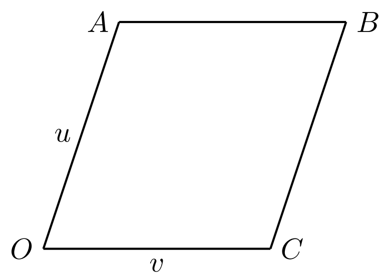
determines a norm on $\mathcal{R}[0,1]$ and determine the norm of the vectors $f(x) = x$ and $f(x) = \frac{1}{3}x^2$.

1.3 Geometry - Vector Proofs

In this section we apply our new found understanding of vectors to classical Euclidean geometry.

Example 8.3.1. Prove that the diagonals of a rhombus are perpendicular.

Proof. Let $OABC$ denote the rhombus. Moreover, let v be the vector OC and u be the vector OA, as seen below.



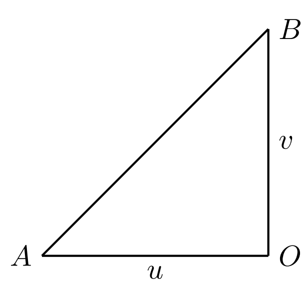
The diagonals of the rhombus are given by $OB = v+u$ and $CA = -v+u$. To show that the diagonals are perpendicular, we need to show that $OB \cdot CA = 0$, where $\cdot$ denotes the dot product. Hence we see that

$$\begin{aligned} OB \cdot CA &= (v+u)\cdot(u-v) \\ &= v\cdot u - v\cdot v + u\cdot u - u\cdot v \\ &= |u|^2 - |v|^2 . \end{aligned}$$

The result then follows from the fact that $|u| = |v|$, given that all sides of a rhombus are of equal length. □

Example 8.3.2. Prove Pythagoras' theorem.

Proof. Let OAB denote the triangle with $OA = u$ and $OB = u$, as seen below.



Hence we see that $AB = v - u$. The magnitude of AB is given by

$$|AB|^2 = (v-u)\cdot(v-u) = |v|^2 + |u|^2 - 2v\cdot u.$$

We know that $u \cdot v = 0$ however, since u and v are perpendicular. So

$$|AB|^2 = |v|^2 + |u|^2 = |OA|^2 + |OB|^2,$$

which yields the desired result. □

In a vector space V, two vectors that are linearly dependent, in some sense, represent two vectors which describe the same amount of information.

Definition 8.3.4. Let V be a vector space. We say that the vectors $\{v_1, ..., v_n\} \subseteq V$ are linearly independent if we cannot find scalars $\alpha_1, ..., a_n \in \mathbb{R}$ such that

$$\alpha_1 v_1 + \cdots + \alpha_n v_n = 0.$$

Example 8.3.5. Determine the value(s) of $k \in \mathbb{R}$ such that $v = \langle k, 12, 2\rangle$ and $u = \langle 2, -3, k\rangle$ are linearly dependent.

Proof. We simply observe that if u and v are linearly dependent, there is some real number $\lambda \in \mathbb{R}$ such that $u = \lambda v$. That is, $\langle 2, -3, k\rangle = \lambda\langle k, 12, 2\rangle$. This tells us that $2 = \lambda k$, $-3 = 12\lambda$ and $k = 2\lambda$. Hence we see that

$$12\lambda - 3 \implies \lambda = -\frac{1}{4},$$

and

$$k = 2\lambda \implies k = 2 \cdot -\frac{1}{4} \implies -\frac{1}{2}.$$

□

Exercises

Q1. Prove that the diagonals of a parallelogram bisect each other.

Q2. Prove the sine rule for any triangle.

Q3. Prove the cosine rule for any triangle.

Q4. Prove that the sum of the squares of the lengths of the diagonals of any parallelogram is equal to the sum of the squares of the lengths of the sides.

Q5. Prove that if the diagonals of a parallelogram are of equal length then the parallelogram is a rectangle.

Q6. Prove that if the midpoints of the sides of a square are joined then another square is formed.

Q7. Show that the angle subtended by a diameter of a circle is a right angle.

Q8. Let $u = 2\mathbf{i}+3\mathbf{k}$ and $v = \mathbf{i}+\frac{3}{2}\mathbf{j}$. Determine whether u and v are linearly independent.

Q9. Let AB and CD denote diameters of a circle. Show that $ABCD$ is a rectangle.

Q10. Let $ABCD$ denote a square with circle of radius $r > 0$ inscribed in it. Denote by p an abitrary point on the circle.

a. Prove that $AP \cdot AP = 3r^2 - 2OP \cdot OA$.

b. Hence, or otherwise, determine $AP^2+BP^2+CP^2+DP^2$ in terms of r.

Q11. Let $u = \langle 3, 3, -6\rangle$, $v = \langle 1, -7, 6\rangle$ and $w = \langle -2, -5, 2\rangle$. Determine the values of $a, b \in \mathbb{R}$ such that $z := u + a \cdot v + b \cdot t$ is parallel to the vector $e_1 = \langle 1, 0, 0\rangle$.

Q12. Let $u = 2\mathbf{i} + 3\mathbf{j} - 4\lambda\mathbf{k}$, for some $\lambda \in \mathbb{R}$, and $v = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k}$. Determine the value(s) of $\lambda \in \mathbb{R}$ such that u and v are linearly dependent.

Q13. Determine whether the vectors $u = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$ and $v = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ are linearly independent.

Q14. Write the vector $v = 2\mathbf{i} - 4\mathbf{j}$ in terms of the vectors $e_1 = \mathbf{i} + 0\mathbf{j}$ and $e_2 = 0\mathbf{i} + \mathbf{j}$.

Q15. Determine whether the polynomials $p(x) = 1-x^2$ and $q(x) = 2+x-x^2$ are linearly independent.

Q16. Determine the value(s) of $\lambda \in \mathbb{R}$ such that $p(x) = 1 - \lambda x + x^2$ and $q(x) = 2\lambda + x^2$ are linearly dependent.

Q17. Take ABC to be a right-angled triangle with the right angle occuring at B. If we suppose that $AC = 2\mathbf{i} + 4\mathbf{j}$ and AB is parallel to $\mathbf{i} + \mathbf{j}$, determine the vector AB.

Q18. Prove that perpendicular vectors are linearly independent.

Q19. Prove that the medians of a triangle are concurrent.

1.4 Analysis Exercises

Q1. Let $\mathcal{PQR}$ be a right-angled triangle with the right angle occurring at $\mathcal{Q}$. Take $\mathcal{PR} = 2\mathbf{i} + 4\mathbf{j}$ and $\mathcal{PQ}$ to be parallel to the vector $\mathbf{i} + \mathbf{j}$.

a. Sketch the vectors on an appropriate cartesian plane $\mathbb{R}^2$.

b. Determine the vector $\mathcal{PQ}$.

c. Determine the vector $\mathcal{QR}$.

Q2. Let $u = 2\mathbf{i} + 2\mathbf{j} + m\mathbf{k}$ and $v = 2\mathbf{i} + m\mathbf{j} + 2\mathbf{k}$. Determine the value of $m \in \mathbb{R}$ such that the vectors are linearly dependent.

Q3. (Dr. Lloyd Gunatilake). Consider an isosceles triangle with sides AB and BC. Let $\overrightarrow{\mathrm{AB}} = \mathbf{b}$ and $\overrightarrow{\mathrm{AC}} = \mathbf{a}$ and $\overrightarrow{\mathrm{AP}} = m\overrightarrow{\mathrm{AC}}$, where $m \in \mathbb{R}$ is constant. The angle $\angle$APB is a right angle.

a. By considering the vectors $\overrightarrow{\mathrm{PA}}$ and $\overrightarrow{\mathrm{PB}}$ show that $\mathbf{a} \cdot \mathbf{b} = m\,|\mathbf{a}|^2$.

b. If $\alpha = \angle BCP$, show that

$$\cos\alpha = \frac{\mathbf{a}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\cdot|\mathbf{b}|}.$$

c. If $\vartheta = \angle BAP$, determine a similar expression for $\cos\vartheta$.

d. Hence, show that AP = PC.

Q4. Let $v_1 = 3\mathbf{i} + \mathbf{j}$ and $v_2 = -\frac{2}{5}\mathbf{i} + \frac{6}{5}\mathbf{j}$. Show that $v_1 \cdot v_2$ are perpendicular.

Q5. Using the inner product defined by

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)dx,$$

show that $f(x) = \cos(\pi x)$ is perpendicular to $g(x) = \cos(3\pi x)$.

Q6. Let $v_1 = -\frac{7\sqrt{3}}{3}\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ and $v_2 = \mathbf{i} + \sqrt{3}\mathbf{j} + 2\sqrt{3}\mathbf{k}$.

a. Determine $|v_1|$.

b. Determine $|v_2|$.

c. Determine the value of $m \in \mathbb{R}$ such that the vector

$$u = -\frac{7\sqrt{3}}{3}\mathbf{i} + \mathbf{j} + m\mathbf{k}$$

which is parallel to v_2.

d. Determine the value of $n \in \mathbb{R}$ such that

$$w = n\mathbf{i} + \mathbf{j} - 2\mathbf{j}$$

forms an angle of $\frac{2\pi}{3}$ with v_2.

Q7. Let $u = 7\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, $v = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $w = 5\mathbf{i} + \mathbf{j} + \mathbf{k}$.

a. Show that $2u = 3v + w$.

b. Hence, or otherwise, solve the matrix equation

$$\begin{pmatrix} 7 & 3 \\ 2 & 1 \\ 5 & 3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \\ 1 \end{pmatrix}.$$

Q8. Let $v_1 = 2\mathbf{j} + 3\mathbf{k}$, $v_2 = -8\mathbf{k}$ and $v_3 = -1\mathbf{i} + 3\mathbf{j} + \mathbf{k}$.

a. Show that the vectors are linearly independent.

b. Are the vectors v_1 and v_2 orthogonal?

Q9. Show that any set of vectors $\{v_1, v_2, v_3\}$ in $\mathbb{R}^2$ are linearly dependent.

Q10. Show that every vector space contains the zero vector.

Q11. Determine whether the vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; v_2 = \begin{pmatrix} -2 \\ 5 \\ -3 \\ 0 \end{pmatrix}, \; v_3 = \begin{pmatrix} 5 \\ 6 \\ 1 \\ 5 \end{pmatrix}, \; v_4 = \begin{pmatrix} 1 \\ 1 \\ 4 \\ 6 \end{pmatrix},$$

are linearly independent.