MATH1023 (WORKING) NOTES – MULTIVARIABLE CALCULUS AND MODELLING

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Abstract. Notes for MATH1023, taught at the University of Sydney, Semester 2, 2020. For further details, see the books by Cullen–Zill [1] and J. Stewart [2]. Please inform me of any issues, typos, or errors found in the present manuscript – contact email: `kyle.broder@anu.edu.au`

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§1.1. Differential Equations – Separable ODEs

Definition 1.1.1. Recall that an ordinary differential equation (ODE)

$$\frac{dy}{dx} = f(x,y)$$

is *separable* if we can write $f(x,y) = g(x) \cdot h(y)$, for some functions g and h.

Example 1.1.2.

(i) The ODE

$$\frac{dy}{dx} = x^2(y+1)$$

is separable.

(ii) The ODE

$$\frac{dy}{dx} = xy + 1$$

is not separable.

Proof. Suppose this ODE is separable. Then we can find functions $g(x)$ and $h(y)$ such that $xy + 1 = g(x)h(y)$.If $x = 0$, this implies that $1 = g(0)h(y) \implies h(y) = \frac{1}{g(0)}$. Similarly, if $y = 0$, we have $1 = g(x)h(0) \implies g(x) = \frac{1}{h(0)}$. Inserting this into the original equation, we see that $xy + 1 = \frac{1}{g(0)h(0)}$, but this is not possible since the left-hand side is variable in (x, y), while the right-hand side is constant. □

§1.2. Models

So far, we have considered the following models:

1.2.1. Linear growth: Fix a constant $k \in \mathbb{R}$. A function $N = N(t)$ is said to grow *linearly* if

$$\frac{dN}{dt} = k.$$

The solutions of this equation are given by $N(t) = kt + c$, where $c = N(0)$ is the initial condition.

1.2.2. Exponential growth: Fix some $k \in \mathbb{R}$. A function $N = N(t)$ is said to grow *exponentially* if

$$\frac{dN}{dt} = kN.$$

The solutions of this equation are given by $N(t) = Ae^{kt}$, where $A = N(0)$ is the initial condition.

1.2.3. Logistic growth: Fix constants k and α. A function $N = N(t)$ is said to grow *logistically* if

$$\frac{dN}{dt} = kN(\alpha - N).$$

To compute the solution of this equation, we use the knowledge of separable ODEs to compute:

$$\begin{aligned}
\frac{dN}{dt} = kN(\alpha - N) \quad &\implies \quad \frac{dN}{N(\alpha - N)} = kdt \\
&\implies \quad \int \frac{1}{N(\alpha - N)} dN = kt + c \\
&\implies \quad \int \frac{1}{\alpha N} dN - \int \frac{1}{\alpha(N - \alpha)} dN = kt + c \\
&\implies \quad \frac{1}{\alpha} \ln|N| - \frac{1}{\alpha} \ln|N - \alpha| = kt + c \\
&\implies \quad \ln\left|\frac{N}{N - \alpha}\right| = \alpha kt + c\alpha \\
&\implies \quad \frac{N}{N - \alpha} = Ae^{\alpha kt + c\alpha} \\
&\implies \quad N(1 - Ae^{\alpha kt + c\alpha}) = -\alpha Ae^{\alpha kt + c\alpha} \\
&\implies \quad N = -\frac{\alpha Ae^{\alpha kt + c\alpha}}{1 - Ae^{\alpha kt + c\alpha}} \\
&\implies \quad N = -\frac{\alpha Be^{\alpha kt}}{1 - Be^{\alpha kt}} \\
&\implies \quad N = -\frac{\alpha}{Ce^{-\alpha kt} - 1}
\end{aligned}$$

1.2.4. Remark. There is a very nice video from 3Blue1Brown: https://www.youtube.com/watch?v=Kas0tIxDvrg&ab_channel=3Blue1Brown

§1.3. Linear ODEs

Recall that an ODE is said to be *linear* if the highest power of y occurring in ODE is 1.

Example 1.3.1.

(i) The ODE
$$\frac{dy}{dx} + xy = x^3$$
is linear.

(ii) The ODE
$$\frac{d^2y}{dx^2} + x^9\sqrt{x+1}y = 1 - \sqrt{\sin(x)}$$
is linear.

(iii) The ODE
$$\frac{dy}{dx} + xy^2 = y$$
is not linear.

Remark 1.3.2. A better definition of linear ODE is that the set of solutions to the equation form a vector space.

Recall from linear algebra that a vector space is a set V, whose elements were called *vectors*, such that

(i) (vector addition). $u, v \in V \implies u + v \in V$.
(ii) (scalar multiplication). $v \in V, \lambda \in \mathbb{R} \implies \lambda v \in V$.

Independent solutions of a linear ODE should therefore be understood in the sense that the solutions are linearly independent vectors (in the vector space of solutions to the ODE).

Definition 1.3.3. The *order* of an ODE is the highest number of derivatives occurring in the equation.

Example 1.3.4.

(i) The ODE
$$\frac{dy}{dx} + xy = x^3$$
is of first order.

(ii) The ODE
$$\frac{d^2y}{dx^2} + x^9\sqrt{x+1}y = 1 - \sqrt{\sin(x)}$$
is of second order.

(iii) The ODE
$$\frac{dy}{dx} + xy^2 = y$$
is of first order.

1.3.5. Method of solving first-order linear ODEs: Let
$$p(x)y' + p(x)y = q(x)$$
be a (homogeneous) first-order linear ODE. The *integrating factor* for this equation is given by the function
$$\mathfrak{I}(x) \ := \ \exp\left(\int p(x)dx\right).$$

§1.4. Second-Order ODEs

These are discussed at length here: https://www.youtube.com/watch?v=5slwtitULSE&t=0s&ab_channel=KyleBroder

The *characteristic equation* of a second-order ODE is the quadratic polynomial one gets when you make the substitution

$$y = e^{\lambda x}$$

. In more detail, let

$$y'' + py' + qy = 0$$

be a second-order linear homogeneous ODE with constant coefficients, i.e., p and q are real numbers.
Inserting $y = e^{\lambda x}$, we see that the equation becomes

$$\lambda^2 e^{\lambda x} + p\lambda e^{\lambda x} + qe^{\lambda x} = 0.$$

For all $x \in \mathbb{R}$, the exponential function $e^{\lambda x}$ is positive, and the above equation reduces to the aforementioned characteristic polynomial:

$$\lambda^2 + p\lambda + q = 0.$$

It is clear that the values of λ are given by the quadratic formula:

$$\lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}.$$

Three cases are distinguished by the (sign of the) discriminant $\Delta = p^2 - 4q$:

(i) If $\Delta > 0$, the characteristic equation has two real roots.
(ii) If $\Delta = 0$, the characteristic equation has one real root with multiplicity 2.[1]
(iii) If $\Delta < 0$, the characteristic equation has two complex conjugate roots.

Question 1.4.1. Why are there only two roots to the characteristic equation? Why is it the case that if there are two complex roots, the complex roots are necessarily conjugate?

[1]This means that not only does the function vanish at the given point, but its derivative does too.

Example 1.4.2. Solve the ODE

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$$

.

Solution. In the usual way, let $y = e^{\lambda x}$ so that

$$\frac{dy}{dx} = \lambda e^{\lambda x}$$

and

$$\frac{d^2y}{dx^2} = \lambda^2 e^{\lambda} x.$$

Inserting this into the given equation, we get

$$\lambda^2 + 2\lambda + 1 = 0.$$

We can write

$$\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 \implies \lambda = -1$$

and $y = e^{-x}$.
The second independent solution is given by $y = xe^{-x}$. □

Remark 1.4.3. To justify the above assertion, we need to factor in the observation that $\lambda = -1$ has multiplicity =2. In particular, not only does the characteristic polynomial

$$p(\lambda) := \lambda^2 + 2\lambda + 1 = 0$$

at

$$\lambda = -1$$

, but the derivative of the characteristic polynomial

$$p'(\lambda) = 2\lambda + 2$$

also vanishes at $\lambda = -1$. To this end, perturb the solution e^{-x} by a small factor, which we will denote by $\varepsilon > 0$. That is, we consider a small perturbation to the left: $e^{\varepsilon - x}$, and a small perturbation to the right: $e^{-(\varepsilon + x)}$. The derivative of the characteristic equation, $p'(\lambda)$, affects these perturbed solutions since the perturbation is determined by the "directions" of the solution. The characteristic equation itself does not give you this information concerning "directions".

We now take the average of these two perturbations, or the average of these two "directions":

$$\frac{e^{\varepsilon-x}-e^{-(\varepsilon+x)}}{2\varepsilon}.$$

To get an understanding of how to evaluate this term, we recall that the exponential function admits the following Taylor series expansion:

$$e^x=\sum_{k=0}^{\infty}\frac{x^k}{k!}.$$

In particular, this implies that

$$e^{\varepsilon-x}=\sum_{k=0}^{\infty}\frac{(\varepsilon-x)^k}{k!}$$

and

$$e^{-\varepsilon-x}=\sum_{k=0}^{\infty}\frac{(-\varepsilon-x)^k}{k!}=\sum_{k=0}^{\infty}(-1)^k\frac{(x+\varepsilon)}{k!}.$$

We will not discuss any issues of convergence here, other than mentioning that the convergence is uniform for both series, and the terms of the series can be subtracted term-by-term: So we get that the above ratio is given by

$$\begin{aligned}\mathcal{R} &= \frac{1}{2\varepsilon}\left[\sum_{k=0}^{\infty}\frac{(\varepsilon-x)^k}{k!}-\sum_{k=0}^{\infty}(-1)^k\frac{(\varepsilon+x)^k}{k!}\right]\\ &= \frac{1}{2\varepsilon}\left[1+\varepsilon x-(1-\varepsilon x)\right]\\ &\quad+\frac{1}{2\varepsilon}\left[\sum_{k=2}^{\infty}\frac{(\varepsilon-x)^k}{k!}-\sum_{k=2}^{\infty}(-1)^k\frac{(\varepsilon+x)^k}{k!}\right]\\ &= xe^{-x}\end{aligned}$$

as $\varepsilon\to 0$.

§2.1. Review of Calculus in one-variable

Definition 2.1.1. We say that a function $f:\mathbb{R}\to\mathbb{R}$ is *continuous* at $x_0\in\mathbb{R}$ if

$$\lim_{x\to x_0}f(x)=f(x_0).$$

Remark 2.1.2. The squeeze theorem: Consider the function

$$f(x) = x \sin\left(\frac{1}{x}\right),$$

What we did here was to squeeze our function between two other functions, of which had the same limit. This is encapsulated in what is aptly named the Squeeze theorem.

Theorem 2.1.3. (The Squeeze Theorem) - Suppose that $[a, b] \subset \mathbb{R}$ with $a < x_0 < b$, and let f, g and h be functions defined on $[a, b] \backslash \{x_0\}$ such that for all $x \in [a, b] \backslash \{x_0\}$,

$$f(x) \leq h(x) \leq g(x).$$

Further suppose that

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = \ell,$$

then

$$\lim_{x \to x_0} h(x) = \ell.$$

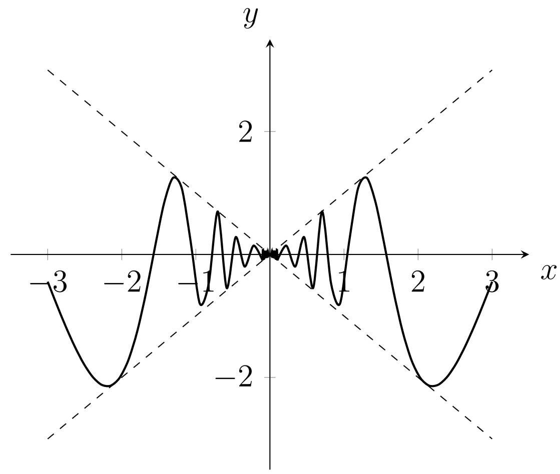
Theorem 2.1.4. Suppose that f and g are continuous functions. Then

(i) $f + g$ is continuous.
(ii) $f - g$ is continuous.
(iii) $f \cdot g$ is continuous.
(iv) f/g is continuous at all points where $g \neq 0$.

Definition 2.1.5. Let D be a subset of the real line $\mathbb{R}$. Suppose $f : D \to \mathbb{R}$ is continuous. We say that f admits a *continuous extension* to $\widetilde{D} \supset D$ if there exists a continuous function $\widetilde{f} : \widetilde{D} \to \mathbb{R}$ such that $\widetilde{f}(x) = f(x)$ for all $x \in D$.

Example 2.1.6. The function $f : \mathbb{R} \backslash \{0\} \to \mathbb{R}$ defined by $f(x) = x \sin\left(\frac{1}{x}\right)$ admits a continuous extension $\widetilde{f} : \mathbb{R} \to \mathbb{R}$ which can be given explicitly:

$$\widetilde{f}(x) = \begin{cases} f(x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$



This follows from the squeeze theorem: since $|\sin(x)| \leq 1$, it follows that $\left|\sin\left(\frac{1}{x}\right)\right| \leq 1$ and subsequent $\left|x \sin\left(\frac{1}{x}\right)\right| \leq x$.

Example 2.1.7. The function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by $f(x) = \frac{1}{x}$ does not admit a continuous extension to $\mathbb{R}$.

Example 2.1.8. A dumb example is the following: the function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ defined by $f(x) = \frac{x}{x}$ admits a continuous extension to $\mathbb{R}$.

Definition 2.1.9. A function f is said to be *differentiable at* x if the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists (and is finite). Moreover, the *derivative* of f at x is equal to the value of this limit.

Example 2.1.10. Polynomials are differentiable on all of $\mathbb{R}$.

Example 2.1.11. The function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = |x|$ is continuous on $\mathbb{R}$ and differentiable on $\mathbb{R}\backslash\{0\}$.

Question 2.1.12. Is the inverse of a differentiable function differentiable?

§2.2. Continuity in $\mathbb{R}^2$

Definition 2.2.1. We say that a function $f : \mathbb{R}^2 \to \mathbb{R}$ is *continuous* at the point $(x, y) \in \mathbb{R}^n$ if

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0).$$

Example 2.2.2. The function $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ defined by

$$f(x,y) = \frac{x^2 - y^2}{x^2 + y^2}$$

is not continuous at the origin $(0,0)$.[2]

Solution. Indeed, consider the path that approaches $(0,0)$ along the x-axis, namely, when $y = 0$. This yields

$$\lim_{(x,0)\to(0,0)} \frac{x^2}{x^2} \quad = \quad 1, \qquad \forall x \neq 0.$$

Similarly, consider the path that approaches $(0,0)$ along the y-axis, namely, when $x = 0$. This yields

$$\lim_{(0,y)\to(0,0)} \frac{-y^2}{y^2} \quad = \quad -1, \qquad \forall y \neq 0.$$

Therefore, since two different approaches did not yield the same limiting value, we conclude that f is not continuous at $(0,0)$. $\square$

Definition 2.2.3. Let D be a subset of $\mathbb{R}^2$. Suppose $f : D \to \mathbb{R}$ is continuous. We say that f admits a *continuous extension* to $\widetilde{D} \supset D$ if there exists a continuous function $\widetilde{f} : \widetilde{D} \to \mathbb{R}$ such that $\widetilde{f}(x) = f(x)$ for all $x \in D$.

Example 2.2.4. Let $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ be the function defined by

$$f(x,y) = \frac{5x^3y^2}{x^3 + y^2}.$$

This function admits a continuous extension $\widetilde{f} : \mathbb{R}^2 \to \mathbb{R}$.

Proof. The continuous extension is defined by

$$\widetilde{f}(x) = \begin{cases} f(x), & x \in \mathbb{R}^2 \backslash \{(0,0)\}, \\ 0, & (x,y) = (0,0). \end{cases}$$

It suffices to show that $\widetilde{f}$ is continuous, i.e., if $|(x,y)|^2 = x^2 + y^2 < \delta$ for some small $\delta > 0$, then $|f(x,y)|^2 < \varepsilon$ for some small ε. This is seen as follows:

$$\begin{aligned} \left| \frac{5x^3y}{x^3 + y^2} - 0 \right| = \left| \frac{5x^3y}{x^3 + y^2} \right| \quad &= \quad \frac{5\,|x^3| \cdot |y|}{|x^3 + y^2|} \\ &\leq \quad 5\,|y| \leq 5\sqrt{x^2 + y^2} < \epsilon. \end{aligned}$$

[2]More precisely, f does admit a continuous extension to all of $\mathbb{R}^2$.

Therefore, by taking $\delta = \epsilon/5$, we see that $\widetilde{f}$ is continuous at $(0,0)$. □

§2.3. Partial Derivatives

Definition 2.3.1. The *directional derivative* of a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ at the point (x, y) in the direction of a unit vector $v = \langle v_1, v_2 \rangle$ is defined to be the limit

$$\nabla_v f(x,y) \;:=\; \lim_{h \to 0} \frac{f(x + h \cdot v_1, y + h \cdot v_2) - f(x,y)}{h}.$$

Remark 2.3.2. It is clear how to generalize the above definition to smooth functions $f : \mathbb{R}^n \to \mathbb{R}$.

Example 2.3.3. Compute the directional derivative of $f(x,y,z) = x^2 + y^2 + 2xyz$ in direction of $v = \langle \frac{1}{\sqrt{3}}. \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$.

Proof. We proceed to compute the limit

$$
\begin{aligned}
\nabla_v f(x,y,z) &= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(f\left(x + \frac{\Delta x}{\sqrt{3}}, y + \frac{\Delta x}{\sqrt{3}}, z + \frac{\Delta x}{\sqrt{3}} \right) - f(x,y,z) \right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(\left(x + \frac{\Delta x}{\sqrt{3}} \right)^2 + \left(y + \frac{\Delta x}{\sqrt{3}} \right)^2 + 2\left(x + \frac{\Delta x}{\sqrt{3}} \right) \left(y + \frac{\Delta x}{\sqrt{3}} \right) \left(z + \frac{\Delta x}{\sqrt{3}} \right) \right. \\
&\quad \left. - x^2 - y^2 - 2xyz \right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(x^2 + \frac{2x\Delta x}{\sqrt{3}} + \frac{(\Delta x)^2}{3} + y^2 + \frac{2y\Delta x}{\sqrt{3}} + \frac{(\Delta x)^2}{3} \right. \\
&\quad \left. + 2\left(xy + x\frac{\Delta x}{\sqrt{3}} + y\frac{\Delta x}{\sqrt{3}} + \frac{(\Delta x)^2}{3} \right) \left(z + \frac{\Delta x}{\sqrt{3}} \right) - x^2 - y^2 - 2xyz \right) \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x} \left(x^2 + \frac{2x\Delta x}{\sqrt{3}} + \frac{(\Delta x)^2}{3} + y^2 + \frac{2y\Delta x}{\sqrt{3}} + \frac{(\Delta x)^2}{3} + 2xyz + 2xy\frac{\Delta x}{\sqrt{3}} \right. \\
&\quad \left. + 2xz\frac{\Delta x}{\sqrt{3}} + 2x\frac{(\Delta x)^2}{3} + 2yz\frac{\Delta x}{\sqrt{3}} + 2y\frac{(\Delta x)^2}{3} + 2z\frac{(\Delta x)^2}{3} + 2\frac{(\Delta x)^3}{3\sqrt{3}} - x^2 - y^2 - 2xyz \right) \\
&= \lim_{\Delta x \to 0} \frac{x^2}{\Delta x} + \frac{2x}{\sqrt{3}} + \frac{\Delta x}{3} + \frac{y^2}{\Delta x} + \frac{2y}{\sqrt{3}} + \frac{\Delta x}{3} + \frac{2xyz}{\Delta x} + \frac{2xy}{\sqrt{3}} + \frac{2xz}{\sqrt{3}} + \\
&\quad + \frac{2x\Delta x}{3} + \frac{2yz}{\sqrt{3}} + \frac{2y\Delta x}{3} + \frac{2z\Delta x}{3} + \frac{2z\Delta x}{3} + \frac{2(\Delta x)^2}{3\sqrt{3}} - \frac{x^2}{\Delta x} - \frac{y^2}{\Delta x} - \frac{2xyz}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x^2}{\Delta x} + \frac{2x}{\sqrt{3}} + \frac{y^2}{\Delta x} + \frac{2y}{\sqrt{3}} + \frac{2xyz}{\Delta x} + \frac{2xy}{\sqrt{3}} + \frac{2xz}{\sqrt{3}} + \frac{2yz}{\sqrt{3}} - \frac{x^2}{\Delta x} - \frac{y^2}{\Delta x} - \frac{2xyz}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x}{\sqrt{3}} + \frac{2y}{\sqrt{3}} + \frac{2xy}{\sqrt{3}} + \frac{2xz}{\sqrt{3}} + \frac{2yz}{\sqrt{3}} \\
&= \frac{2}{\sqrt{3}} \left(x + y + xy + xz + yz \right).
\end{aligned}
$$

□

The directional derivative in the direction of a basis vector $e_1 = (1,0)$, or $e_2 = (0,1)$, is called a *partial derivative*:

Definition 2.3.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. The *partial derivative of f with respect to the variable x* is defined to be the limit

$$\frac{\partial f}{\partial x} := \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}.$$

Similarly, the *partial derivative of f with respect to y* is the limit

$$\frac{\partial f}{\partial y} := \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}.$$

Example 2.3.5. Compute the partial derivatives of $f(x,y) = x^3 + 2xy^3 + \sqrt{xy}$ with respect to x and y.

Proof. To compute the partial derivative of f with respect to x, we assume that y is constant:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \frac{\partial}{\partial x}(x^3 + 2xy^3 + \sqrt{xy}) \\ &= 3x^2 + 2y^3 + \frac{y}{2\sqrt{xy}}. \end{aligned}$$

Similarly, to compute the partial derivative of f with respect to y, we assume that x is constant:

$$\begin{aligned} \frac{\partial f}{\partial y} &= \frac{\partial}{\partial y}(x^3 + 2xy^3 + \sqrt{xy}) \\ &= 6xy^2 + \frac{x}{2\sqrt{xy}}. \end{aligned}$$

□

Example 2.3.6. Compute the partial derivatives of $f(x,y) = x^3 + x^2y^3 - 3y^2$.

Proof. We observe that the first order partial derivatives are given by

$$\frac{\partial f}{\partial x} = 3x^2 + 2xy^2 \quad \text{and} \quad \frac{\partial f}{\partial y} = 3x^2y^2 - 6y.$$

We can differentiate $\frac{\partial f}{\partial x}$ with respect to x, this yields

$$\frac{\partial^2 f}{\partial x^2} = 6x + 2y^2.$$

We can differentiate $\frac{\partial f}{\partial y}$ with respect to x, this yields

$$\frac{\partial^2 f}{\partial y \partial x} = 6xy^2.$$

Similarly, we can differentiate $\frac{\partial f}{\partial x}$ with respect to y, this yields

$$\frac{\partial^2}{\partial x \partial y} = 4xy.$$

We can differentiate $\frac{\partial f}{\partial y}$ with respect to y, this yields

$$\frac{\partial^2 f}{\partial y^2} = 6x^2y - 6.$$

$\square$

An important thing to note from the above example is that

$$\frac{\partial^2 f}{\partial x \partial y} = 4xy \neq 6xy^2 = \frac{\partial^2 f}{\partial y \partial x}.$$

That is, the partial derivatives do not necessarily commute. We do have the following theorem however.

Theorem 2.3.7. (Clairaut's Theorem). Suppose that $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives (this is the case if f is smooth) the partial derivatives commute:

$$\partial^2_{xy} f = \partial^2_{yx} f.$$

Definition 2.3.8. The *gradient vector* is the vector $\nabla := (\partial_x, \partial_y)$, and the *gradient of* f is

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right).$$

Theorem 2.3.9. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ has continuous partial derivatives. Let $v = (v_1, v_2)$ be a unit vector. The directional derivative of f in the direction of v is given by

$$\nabla_v f(x, y) = \nabla f \cdot v,$$

where ∇f is the gradient of f and $\cdot$ is the dot product.

Example 2.3.10. (c.f., Example 2.3.3). Observe that

$$\nabla f = (2x + 2yz, 2y + 2xz, 2xy).$$

Therefore,

$$\begin{aligned}\nabla_v f = \nabla f \cdot v &= (2x + 2yz, 2y + 2xz, 2xy) \cdot \frac{1}{\sqrt{3}}(1, 1, 1) \\ &= \frac{1}{\sqrt{3}}(2x + 2yz + 2y + 2xz + 2xy) \\ &= \frac{2}{\sqrt{3}}(x + y + xy + xz + yz).\end{aligned}$$

Theorem 2.3.11. Suppose that f has continuous partial derivatives. The directional derivative $\nabla_v f$ is optimised when $\nabla_v f = \nabla f$. That is, the directional derivative $\nabla_v f$ is maximimised when v is parallel to the gradient and the maximum is given by $|\nabla f|$.

§2.4. Critical Points of Functions on $\mathbb{R}^2$

Recall that in one variable calculus the maximum of a smooth function $f : \mathbb{R} \to \mathbb{R}$ is attained either on the boundary of its domain or at a *critical point*, i.e., a point p at which $f'(p) = 0$.

Definition 2.4.1. A smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ is said to have a *critical point* at (x, y) if

$$\frac{\partial f}{\partial x}(x, y) = \frac{\partial f}{\partial y}(x, y) = 0.$$

Example 2.4.2. Determine the critical points, if they exist, of the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x, y) = 3xye^{-x^2-y^4}.$$

Proof. We simply compute

$$\begin{aligned}\frac{\partial f}{\partial x} &= 3ye^{-x^2-y^4} + 3xy(-2x)e^{-x^2-y^4} \\ &= 3ye^{-x^2-y^4}(1 - 2x^2); \\ \frac{\partial f}{\partial y} &= 3xe^{-x^2-y^4} + 2xy(-4y^3)e^{-x^2-y^4} \\ &= xe^{-x^2-y^4}(3 - 8y^4).\end{aligned}$$

The equation

$$\frac{\partial f}{\partial x} = 3ye^{-x^2-y^4}(1-2x^2) = 0$$

has the solutions $y=0$ and $x = \pm\frac{1}{2}$. The equation

$$\frac{\partial f}{\partial y} = xe^{-x^2-y^4}(3-8y^4) = 0$$

has the solutions $x=0$ and $y = \pm\sqrt[4]{\frac{3}{8}}$. We therefore see that the critical points occur at $(0,0)$ and $\left(\pm\frac{1}{2}, \pm\sqrt[4]{\frac{3}{8}}\right)$. □

Definition 2.4.3. For a function $f : \mathbb{R}^2 \to \mathbb{R}$, we define the *Jacobi matrix* of second derivatives (or the *Hessian*) to be the following matrix of partial derivatives

$$\mathcal{J}_f \ \ := \ \ \begin{pmatrix} \partial^2_{xx} f & \partial^2_{xy} f \\ \partial^2_{yx} f & \partial^2_{yy} f \end{pmatrix}.$$

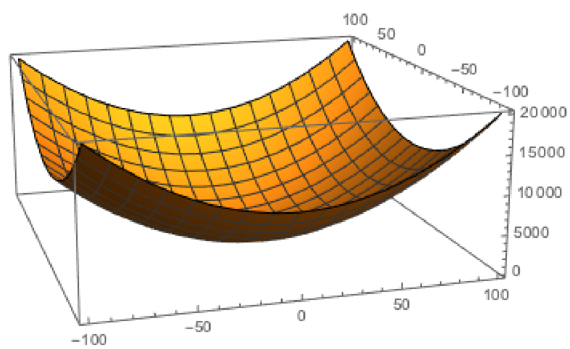
Observe that for f smooth,

$$\det(\mathcal{J}_f) \ \ = \ \ (\partial^2_{xx} f)(\partial^2_{yy} f) - |\partial^2_{xy} f|^2$$

Example 2.4.4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = x^2 + y^2.$$

This function is often referred to as a *paraboloid* (since it looks like a parabola in the (x,z) and (y,z) plane.



Observe that

$$\partial^2_{xx} f = 2, \qquad \partial^2_{xy} f = 0, \qquad \partial^2_{yy} f = 2.$$

From the graph of f, it is clear that f has a critical point at $(0,0)$ which is a local minimum. Analytically, this is seen from the fact that

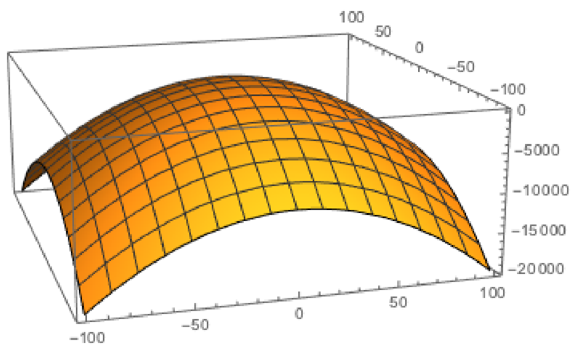
$$\partial_x f(0,0) = 2x|_{(0,0)} = 0, \qquad \partial_y f(0,0) = 2y|_{(0,0)} = 0.$$

Moreover, both $\partial^2_{xx} f$ and $\partial^2_{yy} f$ are positive.

Example 2.4.5. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = -x^2 - y^2.$$

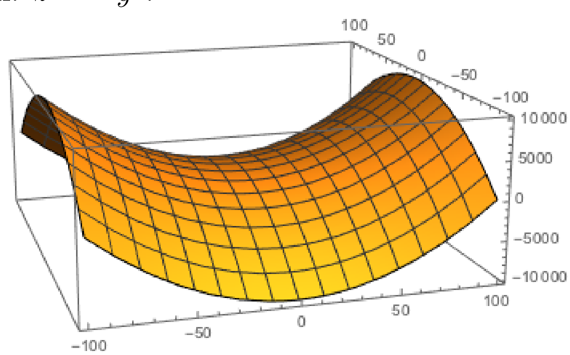
This is just the reflection about the z–axis of the paraboloid considered in the previous example. Hence, we expect there to be a local maximum at $(0,0)$. Note here that $\partial^2_{xx} f$ and $\partial^2_{yy} f$ are both negative.



Example 2.4.6. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = x^2 - y^2.$$

Here, the second-order partial derivatives are given by $\partial^2_{xx} f = 2$ and $\partial^2_{yy} f = -2$. In the (x,z)–plane, f looks like the standard parabola $z = x^2$, but in the (y,z)–plane, it looks like its relfection: $z = -y^2$.



We call this type of critical point a *saddle point*.

Theorem 2.4.7. (Second Derivative Test). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with a crtical point at (x,y). Then

(i) f has a local minimum at (x,y) if $\det \mathfrak{J}(x,y) > 0$ and $\partial^2_{xx} f > 0$.
(ii) f has a local maximum at (x,y) if $\det \mathfrak{J}(x,y) > 0$ and $\partial^2_{xx} f < 0$.

(iii) f has a saddle point at x^0 if $\det \mathfrak{J}(x^0) < 0$.

Remark 2.4.8. Observe that

$$\det(\mathfrak{J}_f) = (\partial_{xx}^2 f)(\partial_{yy}^2 f) - |\partial_{xy}^2 f|^2.$$

For simplicity, assume for the moment that $\partial_{xy}^2 f = 0$. Then

(i) $\det(\mathfrak{J}_f) > 0$ if both $\partial_{xx}^2 f$ and $\partial_{yy}^2 f$ are positive, or if both $\partial_{xx}^2 f$ and $\partial_{yy}^2 f$ are negative.

(ii) $\det(\mathfrak{J}_f) < 0$ if $\partial_{xx}^2 f$ is the opposite sign to $\partial_{yy}^2 f$.

Note that $\det(\mathfrak{J}_f) > 0$ implies that f has either a local maximum or local minimum. To determine which one it is, we look at the sign of $\partial_{xx}^2 f$ (we could equally well have looked at the sign of $\partial_{yy}^2 f$). If $\det(\mathfrak{J}_f) < 0$, then we have a saddle point.

Caution 2.4.9. . We warn the reader that the second derivative test is inconclusive in the case of $\det \mathfrak{J} = 0$.

Example 2.4.10. . Determine the nature of the critical points for the function

$$f(x,y) = 3xye^{-x^2-y^4}.$$

Proof. The critical points of f occur at $(0,0)$ and $\left(\pm\frac{1}{2}, \pm\sqrt[4]{\frac{3}{8}}\right)$. Let us now calculate the Jacobi matrix. To this end, we observe that

$$\mathfrak{J}(f) \quad = \quad e^{-x^2-y^4}\begin{pmatrix} 6xy(2x^2-3) & 3(2x^2-1)(4y^4-1) \\ 3(2x^2-1)(4y^4-1) & 12xy^3(4y^4-5) \end{pmatrix}.$$

At the critical point $(0,0)$, we evaluate the second derivative with respect to x, yielding

$$6xy(2x^2-3)e^{-x^2-y^4}|_{(0,0)} \quad = \quad 0.$$

The Jacobian is evaluated to be

$$\begin{aligned} \det \mathfrak{J}(f)(0,0) \quad &= \quad e^{-x^2-y^4}\det\begin{pmatrix} 6xy(2x^2-3) & 3(2x^2-1)(4y^4-1) \\ 3(2x^2-1)(4y^4-1) & 12xy^3(4y^4-5) \end{pmatrix}\Bigg|_{(0,0)} \\ &= \quad \det\begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix} = -9 < 0. \end{aligned}$$

Using the second derivative test, we see that f has a saddle point at $(0,0)$. We leave it to the reader to determine the nature of the other four stationary points. $\square$

§2.5. Selected Exercises and Problems

Q1 Determine whether the function

$$f(x,y) = \begin{cases} \frac{\sqrt{x^2-4y}}{x^2+y^2}, & (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}, \\ 0, & \text{otherwise} \end{cases}$$

is continuous at $(0,0)$.

Q2 Determine whether the function

$$f(x,y) = \begin{cases} \frac{1}{xy}(x^2+y^2)e^{-x^2-y^2}, & (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}, \\ 0, & \text{otherwise,} \end{cases}$$

is continuous at $(0,0)$.

Q3 Determine whether the function

$$f(x,y) = \begin{cases} \frac{1}{x^2+y} \sin(xy), & (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}, \\ 0, & \text{otherwise} \end{cases}$$

is continuous at $(0,0)$.

Q4 Show that the function

$$f(x,y) = \begin{cases} \frac{x^2-y^2}{x^2+y^2}, & (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}, \\ 0, & \text{otherwise} \end{cases}$$

is not continuous at $(0,0)$.

Q5 Show that the function

$$f(x,y) = x \tan^{-1}\left(\frac{y}{x}\right)$$

is continuous at $(0,0)$.

Q6 Determine whether the function

$$f(x,y) = \frac{x^2+y}{x^2+y}$$

is continuous at $(0,0)$.

Q7 Prove that the function

$$f(x,y) = \begin{cases} \frac{x^2y}{y^2+x^4}, & (x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}, \\ 0, & \text{otherwise} \end{cases}$$

is not continuous at $(0,0)$.

Q8 Evaluate the following limit
$$\lim_{(x,y)\to(0,0)} \frac{1}{(x^2+y^2)^2 - 2x^2y^2} e^{-\frac{1}{(x+y)^2-2xy}}.$$

Q9 Evaluate the following limit
$$\lim_{(x,y)\to(0,0)} \frac{1}{\sqrt{x^2+y^2}} \log_e(|y| + e^{|x|}).$$

Q10 Evaluate the following limit
$$\lim_{(x,y)\to(0,0)} \cos\left(\frac{y}{\sqrt{x^2+y^2}}\right).$$

Q11 Evaluate the following limit
$$\lim_{(x,y)\to(0,0)} \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right) e^{-x^2-y^2}.$$

Q12 Evaluate the following limit
$$\lim_{(x,y)\to(0,0)} \sin\left(\sqrt{x^2+y^2}\right) \cos\left(\frac{1}{\sqrt{x^2+y^2}}\right).$$

Q13 Evaluate the following limit
$$\lim_{(x,y)\to(0,0)} \frac{e^{-x^2} - y}{x^2+y^2}.$$

Q14 Evaluate f_x, f_y, f_{xx}, f_{yy}, f_{xy} and f_{yx} for the following functions.

(a) $f(x,y) = x^3 - 3y^2$.
(b) $f(x,y) = e^{-x^2+y^2}$.
(c) $f(x,y) = x^2 e^{-xy^2}$.
(d) $f(x,y) = \sqrt{x^2-y^2} + \sqrt{x+e^y}$.
(e) $f(x,y) = \sqrt{x} + e^{\frac{x}{y+x^2}}$.
(f) $f(x,y) = \log_e |xy|$.

Q15 Verify Clairaut's theorem for all functions considered in Question 14.

Q16 We say that a function $u : \mathbb{R}^2 \to \mathbb{R}$ is *harmonic* if
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

Prove that the following functions are harmonic.

(a) $f(x,y) = e^x \sin y$.
(b) $f(x,y) = e^x \cos y$.
(c) $f(x,y) = \ln(x^2+y^2)$.
(d) $f(x,y) = x^2 + y^2 - 2z^2$.

Q17 Determine whether the function

$$f(x,y) = \log_e(e^x + e^y)$$

is harmonic.

Q18 Evaluate the directional derivatives of all functions in Question 14 with respect to the following vectors.

(a) $v = \langle 1, 1\rangle$.
(b) $v = \langle 1, 2\rangle$.
(c) $v = \langle \frac{1}{\sqrt{3}}, \frac{2}{\sqrt{5}}\rangle$.
(d) $v = \langle 2, 1\rangle$.
(e) $v = \langle 0, 1\rangle$.
(f) $v = \langle 1, 0\rangle$.

Q19 Determine the vector v such that $\nabla_v f = 1$, where $f = e^{-x^2 y}$.

Q20 Determine the vector v such that $\nabla_v f = 1$, where $f(x,y) = xe^{-x^2+y^2}$.

Q21 Determine the vector v such that $\nabla_v f = 0$, where $f(x,y) = \sqrt{x} - \log_e(|y| + x^3)$.

Q22 Prove the following properties of the gradient vector ∇.

(a) $\nabla(\alpha f + \beta g) = \alpha\nabla f + \beta\nabla g, \quad \forall \alpha, \beta \in \mathbb{R}$.
(b) $\nabla(f \cdot g) = \nabla f \cdot g + f \cdot \nabla g$.
(c) $\nabla\left(f/g\right) = \frac{g\cdot\nabla f - f\cdot\nabla g}{g^2}$.

Q23 Determine the critical points, and their nature, of the following functions.

(a) $f(x,y) = x^2 + y^2$.
(b) $f(x,y) = x^3 - 4y + x$.
(c) $f(x,y) = xe^{-x^2+y^2}$.
(d) $f(x,y) = \sqrt{x^2 + y}$.
(e) $f(x,y) = |xy|$.

References

[1] Cullen, M. R., Zill, D., *Differential Equations with boundary-value problems*, Seventh Edition, 2009 Brooks/Cole, Cengage Learning

[2] Stewart, J., *Calculus: Early transcendentals*, Fifth Edition, Thomson Brooks Cole, Belmont CA, 2003.