Lectures on Vector Calculus

Kyle Broder – March 11th 2022

"*If only I could understand the meaning of* $d^2 = 0$."

– Henri Cartan

Introduction

Vector calculus is an extremely beautiful subject and is a subject that is essential to the modern development of calculus. To place the subject in the appropriate context, let us remind ourselves that long ago, we learned the calculus of functions of a single real variable. The theory is segregated into two primary branches: the *differential calculus* and the *integral calculus.*

The first branch gives us a set of tools for understanding how a function changes, and the rate at which this occurs; the main gadget to come out of this theory is the *derivative d*. The second branch treats infinite sums and gives us a reasonable theory for computing area and volume; the main apparatus which enables this is given by the *integral* $\int$.

The crowning achievement of the subject – the crescendo – is the *fundamental theorem of calculus*, a bridge between these two branches. In its most familiar incarnation, the fundamental theorem of calculus reads

$$\int_a^b f'(x)dx = f(b) - f(a).$$

The importance of this theorem cannot be understated.

After the elation that one experiences from learning this result of an almost divine and supernatural nature, we learn the corresponding higher-dimensional theory; namely, the calculus of several real variables $f : \mathbb{R}^n \to \mathbb{R}$. The subject is (more or less) identical to the one-variable story we learned earlier, with the additional hassle that comes from the book-keeping of indices.

The first glimpse of novel theory appears in vector calculus, the calculus of maps $f : \mathbb{R}^n \to \mathbb{R}^m$. The primary beauty of the subject is not in the presence of new concepts, such as *curl, divergence, vector fields*, etc. The beauty of the subject stems from the insights one attains concerning the previous (one and multi-variable) theories. We will see the fundamental theorem of calculus in its true, glorious generality:

$$\int_{\partial\Omega} \omega \;=\; \int_\Omega d\omega,$$

which, in this guise, is unanimously referred to as *Stokes' theorem.*

In contrast with the expedient belief that one typically walks away with after their first two courses in the subject, Stokes' theorem elucidates the deeper relationship between the differential calculus and integral calculus. It is not that derivatives and integrals are *opposite* to one another (whatever *opposite* means, we will use the more precise word *dual* from now on), it is that *the derivative is dual to the region of integration, and the duality is given by the integral.*

This duality, and Stokes' theorem, lead to the notions of cohomology (specifically, de Rham cohomology) – a theory for studying spaces employing the vector fields (or more generally, differential forms) which reside on them.

In a first course in calculus, we study functions, learn the relevant differentiation theory, this is followed by Riemann's theory of integration, and they are merged via the fundamental theorem of calculus. The present treatment of vector calculus will proceed in much the same way:

In the first chapter, we will introduce vector fields and differential forms – the main objects of study. We will discuss gradient fields, a particular class of vector fields that can be expressed in terms of functions, the association between vector fields and 1–forms, the exterior derivative, and the wedge product.

The differentiation theory is taken up in Chapter 2. In contrast with the one-variable and multi-variable calculus, there are two notions of derivatives of a vector field: curl and divergence. These are, in fact, both instances of one notion of derivative – the exterior derivative, which we meet in Chapter 1. To see that both curl and divergence are incarnations of the exterior derivative, we will discuss the Hodge $\star$–operator.

The ability to represent a vector field in terms of a function is dependent on the properties of the domain of a vector field. This topic is treated in Chapter 3.

The integration theory is exhibitted in chapters 3 and 4. Here, we introduce *line integrals* and *surface integrals* . These are then related via the various incarnations of Stokes' theorem (i.e., the fundamental theorem of calculus). There are three incarnations – *Green's theorem* , *Stokes' theorem* , and the *divergence theorem*.

Before getting into the details of the subject, let us remark that vector calculus is not the final point of this calculus theory. The ideas developed here, namely, the first glimpses of cohomology are extended (and treated more thoroughly) in the subjects of differential geometry, algebraic geometry, and algebraic topology (to name just a couple).

Further, the subject of one-variable complex analysis is arguably the (second) most beautiful of all the calculus theories, giving a theory of functions $f : \mathbb{C} \to \mathbb{C}$. The most beautiful of the calculus theories is the function theory of several complex variables $f : \mathbb{C}^n \to \mathbb{C}$ which is, unfortunately, not as well-known in comparison with the other theories mentioned.

Structure and purpose of the notes

The lecture notes are intended to treat the subject thoroughly, i.e., all statements are given proofs (or a proof is referenced for the reader's convenience). Each section has exercises whose completion is highly encouraged. There are many complete examples, and this is one of the key aspects of the notes. Answers (not solutions) will be given (eventually) at the back of the notes. An index is also given at the back of the book for ease of recalling definitions; theorems, definitions, and remarks are also hyperlinked if they are referenced later.

Let me now address the main question that is likely at the forefront of the reader's mind: Why write another vector calculus book? The answer is twofold: The first is that, the vector calculus literature is, in my mind, divided into two classes.

There is the computation-focused treatment, primarily catering to engineering students, which avoids any systematic development of the theory and presents the results in a very physical manner. This is the direction taken, for instance, in J. Stewarts' *Calculus* [**19**], and does a very good job at treating the subject from this lens. At this point, the reader is likely to interject with, *well, engineering students are the primary audience.* There is a pedagogical drawback to this approach, however, that not only afflicts the pure mathematics students, but also impacts the target audience; namely, the engineering students. The approach taken in Stewarts' book side-steps the systematic theory of differential forms that has been developed over the last century, in favor of the less intimidating ad-hoc approach involving the grad vector, the cross product, and so on. The price one pays for this is that, despite each individual lecture being rather straightforward and elementary, by the end of the teaching semester, students are left with a vast number of disparate and unrelated collection of facts.

On the other side, the high-brow approach of doing everything via forms, the enlightened approach taken, for instance, most notably in H. M. Schey's *Div, Grad, Curl, and all that*, the audience is evidently those who have seen the computation-focused approach and wish to achieve enlightenment through the theory of differential forms. This is not a criticism of Schey's book – the book has the sub-heading: *an informal text on vector calculus.* Harold M. Edwards – *Advanced Calculus – A Differential Forms Approach* [**4**] is a more appropriate illustration of this second class. Here, the theory is put first, but the intended audience is strange, at best. The amount of mathematical maturity required to digest [**4**] is likely only held by those students who have taken much more advanced courses than vector calculus.

The aim of the present book is to give an appropriate middle ground. The intention is to develop the theory properly, such that students walk away from the course with a coherent picture of the subject. Further, the material is presented in a digestable way. The first chapter may be labeled naive, but there is never a moment in the text where a false statement is given. Further, students who wish for a more detailed account of the developments are encouraged to see the appendix.

Errors/Typos/Misprints. Typos, errors, and misprints are likely to be ubiquitous throughout the notes. If any issues are found, please do not hesitate to inform me of them at `kyle.broder@anu.edu.au`.

Contents

Notation

- $\mathbb{N}$ – the natural numbers.
- $\mathbb{N}_0$ – the natural numbers including zero.
- $\mathbb{Z}$ – the integers.
- $\mathbb{R}$ – the real numbers.
- $\mathbb{C}$ – the complex numbers.
- Ω – a region in $\mathbb{R}^n$.
- $\partial\Omega$ – the boundary of a region Ω.
- $\mathcal{C}$ – a curve.
- $\mathcal{S}$ – a surface.
- $\mathcal{V}$ – a solid surface.
- $\mathcal{C}^0(\Omega)$ – the space of continuous functions $f : \Omega \to \mathbb{R}$.
- $\mathcal{C}^1(\Omega)$ – the space of continuously differentiable functions $f : \Omega \to \mathbb{R}$.
- $\mathcal{C}^k(\Omega)$ – the space of k–times continuously differentiable functions $f : \Omega \to \mathbb{R}$.
- $\mathcal{C}^\infty(\Omega)$ – the space of smooth functions $f : \Omega \to \mathbb{R}$.
- $\Lambda^0(\Omega)$ – the space of 0–forms on Ω.
- $\Lambda^1(\Omega)$ – the space of 1–forms on Ω.
- $\Lambda^2(\Omega)$ – the space of 2–forms on Ω.
- $\Lambda^3(\Omega)$ – the space of 3–forms on Ω.
- $\omega_{\mathbf{F}}$ – the 1–form associated to $\mathbf{F}$.
- $\mathrm{Vect}(\Omega)$ – the space of vector fields on Ω.
- $\mathbf{i}$ – the vector $(1, 0)$ or $(1, 0, 0)$.
- $\mathbf{j}$ – the vector $(0, 1)$ or $(0, 1, 0)$.
- $\mathbf{k}$ – the vector $(0, 0, 1)$.
- $\mathbf{0}$ – the zero vector.
- $\times$ – the cross product.
- $\cdot$ – the dot product.
- $\wedge$ – the wedge product.
- f_x, $\partial_x f$, $\frac{\partial f}{\partial x}$ – the partial derivative of f with respect to x.
- f_y, $\partial_y f$, $\frac{\partial f}{\partial y}$ – the partial derivative of f with respect to y.
- f_z, $\partial_z f$, $\frac{\partial f}{\partial z}$ – the partial derivative of f with respect to z.

- f_{xy} – the partial derivative of f_x with respect to y.
- f_{yx} – the partial derivative of f_y with respect to x.
- f_x – the partial derivative of f with respect to x.
- ∇ – the grad vector.
- ∇f – the gradient of f.
- d – the exterior derivative.
- det – the determinant.
- $\star$ – the Hodge $\star$–operator.
- curl($\mathbf{F}$) – the curl of $\mathbf{F}$.
- div($\mathbf{F}$) – the divergence of $\mathbf{F}$.

CHAPTER 1

Fields and Forms

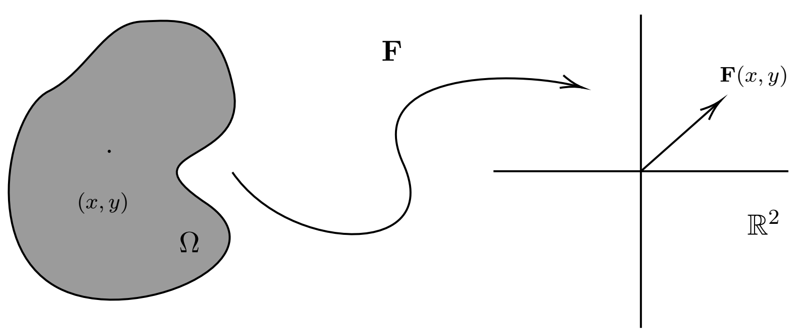
"The art of doing mathematics consists in finding that special case which contains all the germs of generality."

– David Hilbert

In the calculus of one variable, the main objects of study are functions $f : \mathbb{R} \to \mathbb{R}$. In the calculus of more than one variable, the main objects of study are functions $f : \mathbb{R}^n \to \mathbb{R}$. In vector calculus, the main objects of study are vector fields, i.e., maps $\mathbb{R}^n \to \mathbb{R}^n$.

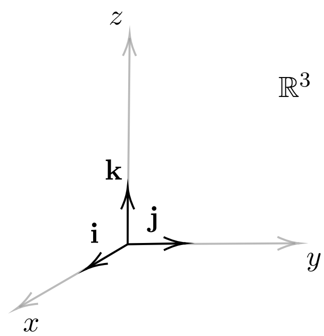
1.1. Definitions, First Examples, and Remarks on Regularity

Definition 1.1.1. (Vector field). Let Ω be a region (a subset) in $\mathbb{R}^2$. A *vector field* on Ω is a map $\mathbf{F} : \Omega \to \mathbb{R}^2$.



Remark 1.1.2. In other words, if we let (x, y) denote the coordinates on $\mathbb{R}^2$, then a vector field assigns to each point $(x, y) \in \Omega$, a vector $\mathbf{F}(x, y) \in \mathbb{R}^2$. We will write $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ for the standard basis on $\mathbb{R}^2$. With respect to this basis, one can express the data of a vector field in terms of two functions $P : \Omega \to \mathbb{R}$ and $Q : \Omega \to \mathbb{R}$:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}.$$



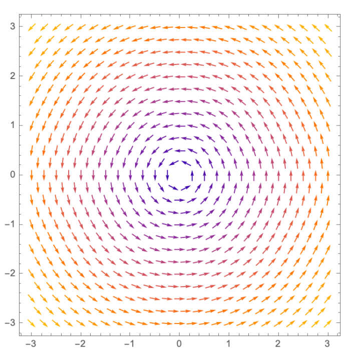
Of course, we can extend this to higher dimensions without much difficulty: If $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$, then a vector field on $\Omega \subseteq \mathbb{R}^3$ is specified by three functions $P, Q, R : \Omega \to \mathbb{R}$, where

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

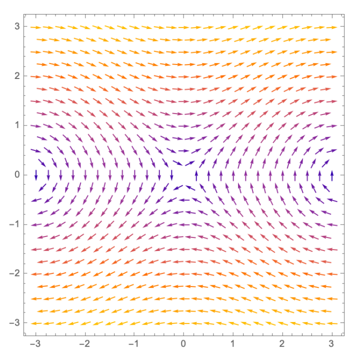
We refer to these functions P, Q, R as *component functions* for the vector field $\mathbf{F}$.

Example 1.1.3. The vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathrm{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ sends the point $(2, 1)$ to the vector $\mathbf{F}(2, 1) = -\mathbf{i} + 2\mathbf{j}$.

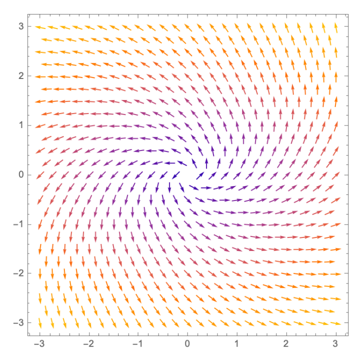
Examples of Vector Fields in $\mathbb{R}^2$. Here are some examples of vector fields on $\mathbb{R}^2$:



$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$.



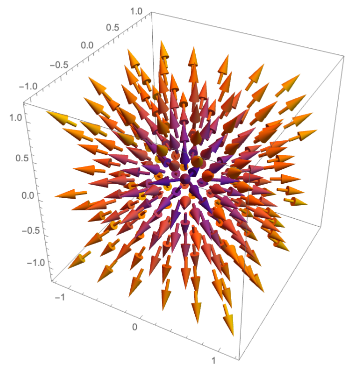
$\mathbf{F}(x, y) = y\mathbf{i} + \sin(x)\mathbf{j}$.



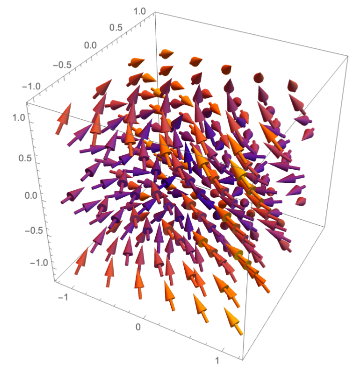
$\mathbf{F}(x, y) =$
$(2x - 3y)\mathbf{i} + (2x + 3y)\mathbf{j}$.

Remarks on dimension. We can also define vector fields on regions in $\mathbb{R}^3$, but these are more difficult to visualize. One can define vector fields on regions in $\mathbb{R}^n$, for any $n \in \mathbb{N}$. The difficulty in visualizing such vector fields increases with n, however. This is the reason for primarily sticking to the cases $n = 2$ and $n = 3$.

Examples of Vector fields in $\mathbb{R}^3$. Here are some examples of vector fields on $\mathbb{R}^3$:

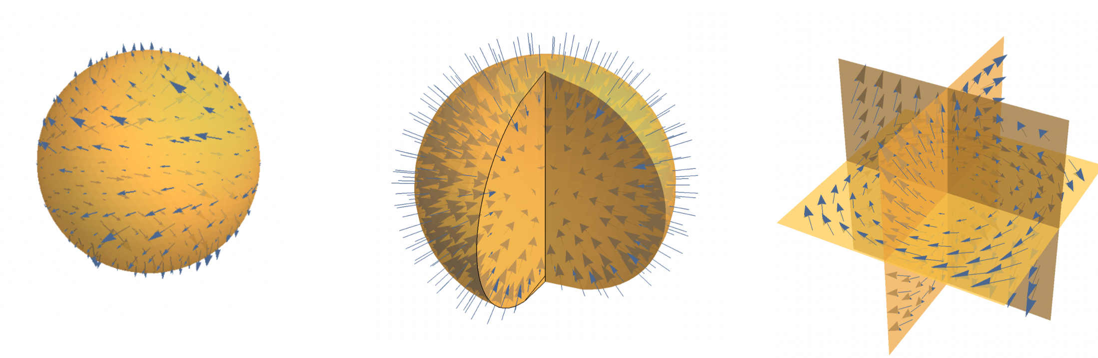


$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.



$\mathbf{F}(x, y, z) = (y - x)\mathbf{i} + -y\mathbf{j} + z^2\mathbf{k}$.

Visualization of Vector Fields. Oftentimes, we think of vector fields as lying over their domain, as indicated below:



Maxwell's equations. Vector calculus plays an essential role in Maxwell's theory of electromagnetism. The field generated in the presence of a magnetic material, or material supporting an electric charge, is an example of a vector field. If we let $\mathbf{B}, \mathbf{E}$ denote the magnetic and electric field respectively. Maxwell's equations, in a vacuum, state the following:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= \mathbf{0} \end{aligned} \qquad \begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= 0. \end{aligned}$$

We will understand the meaning of these equations using the theory developed in the next chapter.

Properties of Vector Fields. Many of the properties that hold for vectors extend to vector fields:

(i) (Addition). If $\mathbf{F}(x,y) = P\mathbf{i} + Q\mathbf{j}$ and $\mathbf{G}(x,y) = M\mathbf{i} + N\mathbf{j}$, then

$$\mathbf{F}(x,y) + \mathbf{G}(x,y) \ = \ (P+M)\mathbf{i} + (Q+N)\mathbf{j}.$$

(ii) (Scaling). If $\mathbf{F}(x,y) = P\mathbf{i} + Q\mathbf{j}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$f\mathbf{F}(x,y) = fP\mathbf{i} + fQ\mathbf{j}.$$

To develop an integration theory for vector fields, we will need to introduce the other main objects of the subjects:

Definition 1.1.4. (1–form). Let $\Omega \subseteq \mathbb{R}^2$ be a region. A 1*–form* on Ω is an expression of the following type

$$\omega \ = \ P(x,y)dx + Q(x,y)dy.$$

The set of 1–forms on a region $\Omega \subseteq \mathbb{R}^2$ is denoted by $\Lambda^1(\Omega)$.

Remark 1.1.5. If $\mathbf{F} : \Omega \to \mathbb{R}^2$ is a vector field, given by $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, then we have a 1–form

$$\omega_{\mathbf{F}} = P(x,y)dx + Q(x,y)dy.$$

We refer to $\omega_{\mathbf{F}}$ as the 1*–form associated to* $\mathbf{F}$.

Example 1.1.6. Let $\mathbf{F}(x,y) = 2x\mathbf{i} - 3y^2\mathbf{j}$. The associated 1–form is

$$\omega_{\mathbf{F}} = 2xdx - 3y^2dy.$$

Remark 1.1.7. It may appear at first glance that 1–forms are objects to keep the mathematicians happy, physicists only care about vector fields. This is certainly not the case – forms are essential for giving a coordinate-free description of Maxwell's equations, and this is necessary for the study of Maxwell's equations on curved spacetimes.
The notation we now use for 1–forms originated in Maxwell's 1855 paper and was then later used in the physics text written by Charles Delaunay.

Remark 1.1.8. Of course, given a 1–form $\omega = P(x,y)dx + Q(x,y)dy$, there is an *associated vector* field $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$.

In particular, we have an identification:

$$\begin{tikzcd} \mathrm{Vect}(\Omega) & \Lambda^1(\Omega) \\ \boxed{\begin{array}{c} \mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j} \\ \text{Vector fields on } \Omega \end{array}} \arrow[r, bend left] & \boxed{\begin{array}{c} \omega_{\mathbf{F}} = P(x,y)dx + Q(x,y)dy \\ 1\text{–forms on } \Omega \end{array}} \arrow[l, bend left] \end{tikzcd}$$

Example 1.1.9. The vector field associated to the 1–form $\omega = (3 - \sin(xy))dx + (x+1)dy$ is

$$\mathbf{F}(x,y) = (3 - \sin(xy))\mathbf{i} + (x+1)\mathbf{j}.$$

Properties of 1–forms. The properties of vector fields also transfer over to 1–forms:

(i) (Addition). If $\omega = Pdx + Qdy$ and $\eta = Adx + Bdy$, then

$$\omega + \eta \ = \ (P + A)dx + (Q + B)dy.$$

(ii) (Scaling). If $\omega = Pdx + Qdy$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$f\omega \ = fPdx + fQdy.$$

Example 1.1.10. Let $\omega = 2x^2dx + (3x - e^{-y})dy$ and $\eta = 9y\cos(y)dx + xdy$. Compute $\omega + \eta$.

Solution. From *Properties of 1–forms*, we have

$$\begin{aligned} \omega + \eta &= 2x^2dx + (3x - e^{-y})dy + 9y\cos(y)dx + xdy \\ &= (2x^2 + 9y\cos(y))dx + (4x - e^{-y})dy. \end{aligned}$$

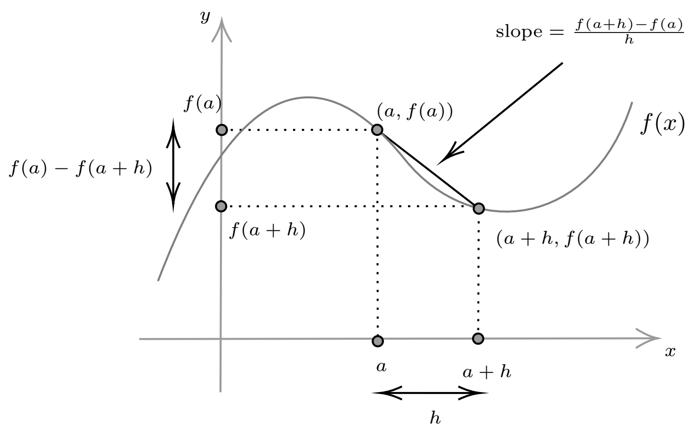
□

Reminder: Regularity of functions. In any theory of calculus, we need to make sense of derivatives, and in particular, when derivatives can be taken. Derivatives of vector fields will be taken up in the next chapter, but for the moment, we need to recall some of the regularity theory from the calculus we are already familiar with.

Reminder: Differentiability. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *differentiable* at a point $a \in \mathbb{R}$ if the limit

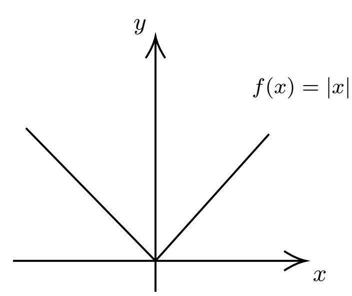
$$f'(a) \ := \ \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists and is finite. We refer to $f'(a)$ as the *derivative of f at the point a*.



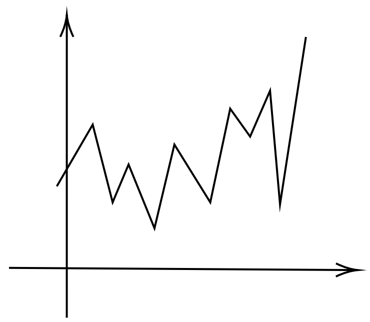
Definition 1.1.11. (Continuously differentiable). If f is differentiable at every point $a \in \mathbb{R}$ (or more generally, its domain, say, Ω), with continuous derivative at every point of its domain, then f is said to be *continuously differentiable*, and we write $f \in \mathcal{C}^1(\Omega)$. The class of functions for which the derivative $f' \in \mathcal{C}^1(\Omega)$ is continuously differentiable, is denoted by $\mathcal{C}^2(\Omega)$. Iterating this definition gives the set of functions of class $\mathcal{C}^k(\Omega)$, where $k \in \mathbb{N}_0$. The set of continuous functions is denoted by $\mathcal{C}^0(\Omega)$.

Example 1.1.12. The modulus function $f(x) = |x|$ is continuous for all $x \in \mathbb{R}$, but not differentiable at $x = 0$.

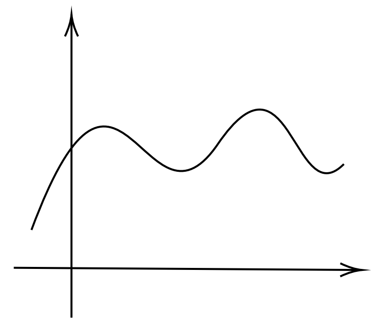


continuous but not differentiable

Definition 1.1.13. A function f is said to be $\mathcal{C}^\infty(\Omega)$, or *smooth*, if f is $\mathcal{C}^k(\Omega)$ for all $k \in \mathbb{N}_0$.



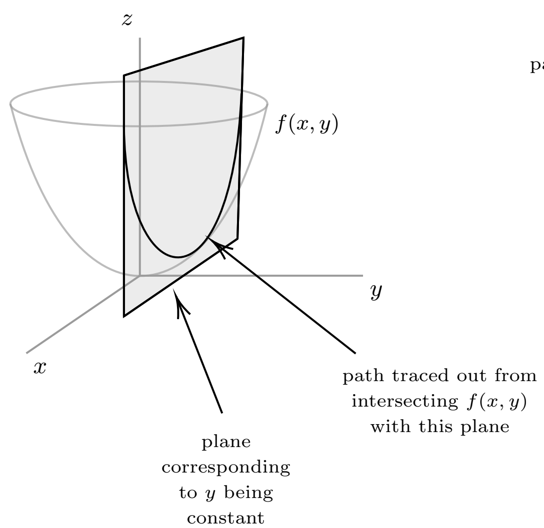
not smooth

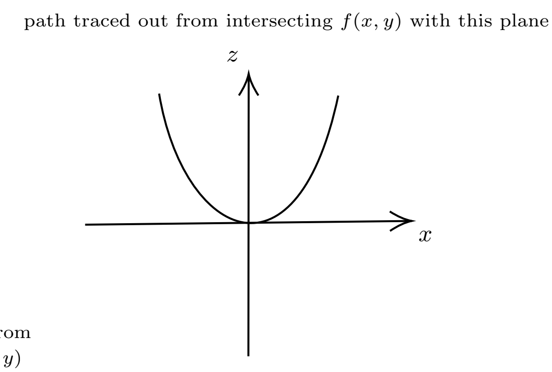


smooth

Reminder 1.1.14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function on $\mathbb{R}^2$, with (x, y) denoting the coordinates on $\mathbb{R}^2$. Recall that the *partial derivatives of f* are defined by

$$f_x \;:=\; \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}, \qquad f_y \;:=\; \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}.$$





Notational remark 1.1.15. We will interchange between the notation $\frac{\partial f}{\partial x}$, f_x, and $\partial_x f$ without acknowledgment, and without apology, from here on.

The following notations are equivalent:

$$\partial_x f \;=\; f_x \;=\; \frac{\partial f}{\partial x}$$

Example 1.1.16. If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = 3x + 5y^2$, then $f_x = 3$ and $f_y = 10y$.

Reminder 1.1.17. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function on $\mathbb{R}^2$ with (x, y) denoting the coordinates on $\mathbb{R}^2$. Let f_x and f_y denote the x and y partial derivatives of f, respectively. The *pure second-order partial derivatives* are defined by

$$f_{xx} := \lim_{h\to 0} \frac{f_x(x+h, y) - f(x, y)}{h}, \qquad f_{yy} := \lim_{h\to 0} \frac{f_y(x, y+h) - f_y(x, y)}{h}.$$

The *mixed second-order partial derivatives* are defined by

$$f_{xy} := \lim_{h\to 0} \frac{f_x(x, y+h) - f_x(x, y)}{h}, \qquad f_{yx} := \lim_{h\to 0} \frac{f_y(x+h, y) - f_y(x, y)}{h}.$$

Iterating this procedure defines partial derivatives of any order $k \in \mathbb{N}$.

Example 1.1.18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) := x^2 - 2xy^3 + 5y^2.$$

The first-order partial derivatives are $f_x = 2x - 2y^3$ and $f_y = -6xy^2 + 10y$. The pure second-order partial derivatives are $f_{xx} = 2$ and $f_{yy} = -12xy + 10$. The mixed second-order partial derivatives are $f_{xy} = -6y^2$ and $f_{yx} = -6y^2$.

Definition 1.1.19. (Smooth function). A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *smooth* if its partial derivatives to any order are continuous.

Remark 1.1.20. Unless stated otherwise, the functions we will consider here are smooth; this permits us to focus on the geometric concepts without regularity concerns. In particular, if f is smooth, then partial derivatives commute.

Unless otherwise stated, all functions are assumed smooth

Definition 1.1.21. (Smooth vector field). A vector field $\mathbf{F} : \Omega \to \mathbb{R}^n$ is said to be $\mathcal{C}^k$ (for $k \in \mathbb{N}_0$) if the component functions are k–times continuously differentiable. We refer to $\mathcal{C}^0$–vector fields as *continuous vector fields* and $\mathcal{C}^\infty$–vector fields as *smooth vector fields.*

Convention 1.1.22. Unless otherwise stated, a *vector field* is understood to be a *smooth vector field.*

Unless otherwise stated, all vector fields are assumed to be smooth.

Remark 1.1.23. Many of the results (if not all?) that we will treat in this course require vector fields to only be of class $\mathcal{C}^1$ or $\mathcal{C}^2$. To avoid these regularity considerations (which will take us too far afield), we simply assume that all vector fields (unless otherwise stated) are $\mathcal{C}^\infty$.

Clairaut's theorem. Let $\Omega \subseteq \mathbb{R}^2$ be a region in $\mathbb{R}^2$. Assume $f : \Omega \to \mathbb{R}$ has continuous second-order partial derivatives at a point $p \in \Omega$. Then

$$f_{xy}(p) \quad = \quad f_{yx}(p).$$

Proof. Let $f : \Omega \to \mathbb{R}$ have continuous second-order partial derivatives at a point $p \in \Omega$. Let $\mathcal{U} \ni p$ be a small neighbourhood of p contained in Ω. By changing coordinates if necessary, we can assume $\mathcal{U}$ is a small rectangle $[a, b] \times [c, d] \subset \Omega$. Observe that

$$\begin{aligned} \int_c^d \int_a^b f_{yx}(x,y)dxdy &= \int_c^d f_y(b,y) - f_y(a,y)dy \\ &= (f(b,d) - f(b,c)) - (f(a,d) - f(a,c)) \\ &= f(b,d) - f(b,c) - f(a,d) + f(a,c). \end{aligned}$$

Similarly,

$$\begin{aligned} \int_a^b \int_c^d f_{xy}(x,y)dydx &= \int_a^b f_x(x,d) - f_x(x,c)dx \\ &= (f(b,d) - f(a,d)) - (f(b,c) - f(a,c)) \\ &= f(b,d) - f(a,d) - f(b,c) + f(a,c). \end{aligned}$$

As a consequence,

$$\int_c^d \int_a^b f_{yx}(x,y)dxdy \quad = \quad \int_a^b \int_c^d f_{xy}(x,y)dydx.$$

Since the second-order partial derivatives vanish, the order of integration can be changed, and thus $f_{yx} = f_{xy}$ on the rectangle $[a, b] \times [c, d]$, proving the claim. $\square$

Remark 1.1.24. *Clairaut's theorem* states that, for functions with continuous second-order partial derivatives, the *second-order partial derivatives commute*: $f_{xy} = f_{yx}$. Further, the above proof of *Clairaut's theorem* hinges upon the fundamental theorem of calculus.

Definition 1.1.25. (Gradient). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. The *gradient of* f is the vector field

$$\nabla f \quad = \quad f_x\mathbf{i} + f_y\mathbf{j}.$$

Example 1.1.26. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x, y) := x^2y + 3\sin(y).$$

Compute the gradient of f.

Solution. The partial derivatives of f are given by

$$f_x = 2xy, \qquad f_y = x^2 + 3\cos(y).$$

Hence, the gradient of f is given by

$$\nabla f = 2xy\mathbf{i} + (x^2 + 3\cos(y))\mathbf{j}.$$

□

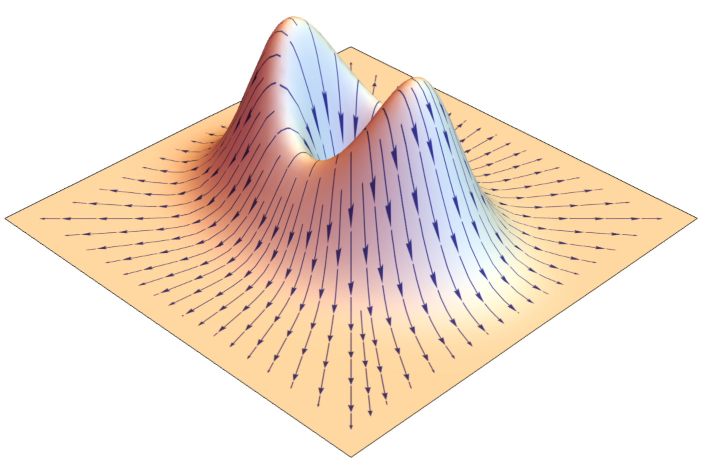
Remark 1.1.27. Observe that if $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, then ∇f is merely the familiar derivative $f'(x)$. In the one-dimensional case, the notation $f'(x) = f_x\mathbf{i}$ is, of course, unnecessary.

Remark 1.1.28. The gradient of f yields the vector field of steepest ascent on the graph of f. Indeed, let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Then for each point $p = (x, y) \in \mathbb{R}^2$, evaluating ∇f at p yields a vector $(\nabla f)(p)$ (the gradient of f at p). Given a vector $v \in \mathbb{R}^2$, the dot product with a unit vector v gives

$$(\nabla f)(p) \cdot v = |(\nabla f)(p)| \cos(\vartheta),$$

where ϑ is the angle between $(\nabla f)(p)$ and v. Since $\cos(\vartheta)$ is maximized at $\vartheta = 2k\pi$ $(k \in \mathbb{Z})$ with value 1, it follows that

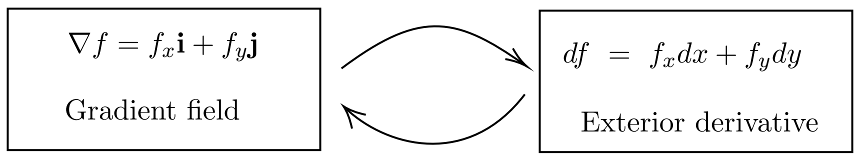
$$\max_v (\nabla f)(p) \cdot v = (\nabla f)(p) \cdot (\nabla f)(p) = |(\nabla f)(p)|^2.$$



The 1–form associated with the gradient vector field will play a central role:

Definition 1.1.29. (Exterior derivative). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. The *exterior derivative of* f is defined

$$df = f_x dx + f_y dy.$$



The exterior derivative is the 1–form associated to the gradient.

Example 1.1.30. Let Ω be region in $\mathbb{R}^2$, and let $f : \Omega \to \mathbb{R}$ be given by

$$f(x, y) = (x^2 + y^2) - \sin(y).$$

Compute the exterior derivative and gradient.

SOLUTION. The exterior derivative of f is

$$df = 2xdx + (2y - \cos(y))dy.$$

The gradient $\mathbf{F} = \nabla f$ is then just the vector field associated to df, i.e.,

$$\mathbf{F} = \nabla f = 2x\mathbf{i} + (2y - \cos(y))\mathbf{j}.$$

□

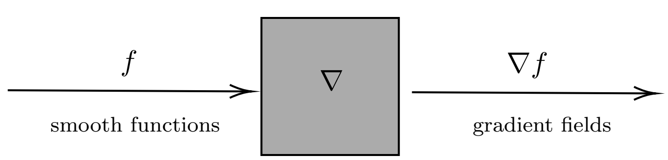
Remark 1.1.31. We have, in fact, already seen the exterior derivative when we studied implicit differentiation. For instance, if $x^2 + y^2 = 1$, then $2xdx + 2ydy = 0$, and therefore,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

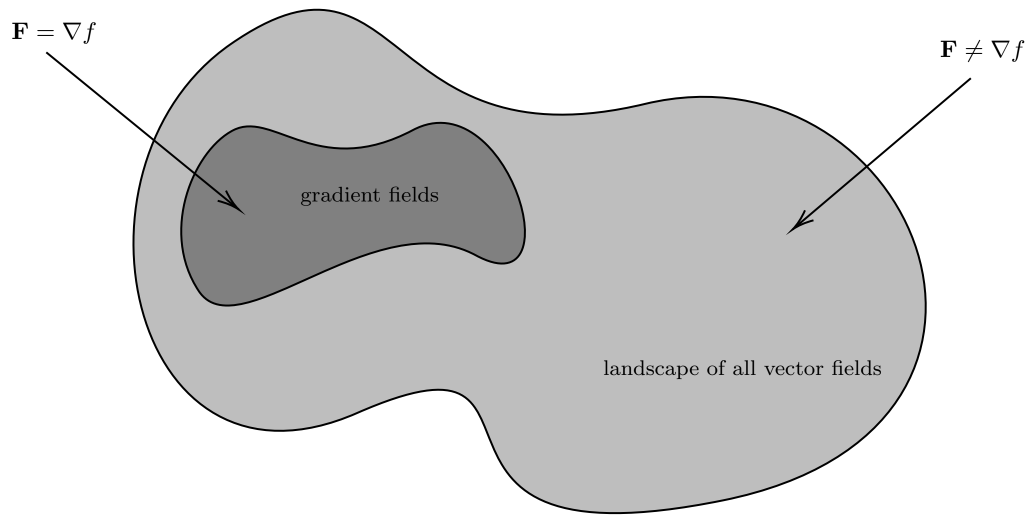
Definition 1.1.32. (Gradient field). A vector field $\mathbf{F} : \Omega \to \mathbb{R}^2$ is said to be a *gradient vector field* if there is a function $f : \Omega \to \mathbb{R}$ such that

$$\mathbf{F} = \nabla f.$$

Remark 1.1.33. We have seen that, if we have a smooth function $f : \Omega \to \mathbb{R}$, then we can produce a gradient field by computing ∇f. In this respect, the operator ∇ is viewed as a machine that inputs functions and outputs gradient fields:



If we stumble upon a vector field $\mathbf{F}$ in the wild, however, then it may *not* be a gradient field, in general:



Example 1.1.34. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x, y) := (y-2)\mathbf{i} + (3x - 2y)\mathbf{j}.$$

Show that $\mathbf{F}$ is not a gradient field.

SOLUTION. If $\mathbf{F}$ is a gradient field, then we can find a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f = f_x\mathbf{i} + f_y\mathbf{j}$. This implies that $f_x = (y-2)$ and $f_y = (3x-2y)$. Observe that

$$f_x \;=\; y-2 \quad\Longrightarrow\quad f(x,y) = xy - 2x + g(y),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a function independent of x. Computing the y partial derivative of the above result gives:

$$\frac{\partial f}{\partial y} \;=\; \frac{\partial}{\partial y}(xy - 2x + g(y)) \;=\; x + g'(y).$$

Comparing this with $f_y = (3x - 2y)$, we see that

$$\begin{aligned} (3x-2y) \;=\; x + g'(y) \quad &\Longrightarrow \quad g'(y) \;=\; 2x - 2y \\ &\Longrightarrow \quad g(y) \;=\; 2xy - y^2, \end{aligned}$$

but this implies that g depends on x, which is not true. Hence, $\mathbf{F}$ cannot be a gradient field. $\square$

Theorem 1.1.35. Let $\mathbf{F} : \Omega \to \mathbb{R}^2$ be a smooth vector field on a domain $\Omega \subseteq \mathbb{R}^2$ defined by $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$. If $\mathbf{F}$ is a gradient field, then the component functions must satisfy

$$P_y = Q_x.$$

Proof. Since $\mathbf{F}$ is a gradient field, there is a smooth function $f : \Omega \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. Since $\mathbf{F} = \nabla f$, we have

$$\mathbf{F} = f_x\mathbf{i} + f_y\mathbf{j},$$

and therefore, $P = f_x$ and $Q = f_y$. By *Clairaut's theorem*, the second-order partial derivatives of a smooth function commute; hence, it follows that

$$P_y = f_{yx} = f_{xy} = Q_x.$$

$\square$

Example 1.1.36. Show that the vector field

$$\mathbf{F}(x,y) = xy^2\mathbf{i} + x^2\mathbf{j}$$

is not a gradient field.

Solution. The component functions are $P(x,y) = xy^2$ and $Q(x,y) = x^2$. Hence, $P_y = 2xy$ and $Q_x = 2x$. Since $P_y \neq Q_x$, the vector field $\mathbf{F}$ is not a gradient field. $\square$

Remark 1.1.37. Note that *Theorem 1.1.35* does *not* assert that if $P_y = Q_x$, then $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ is a gradient field.

$$\boxed{\mathbf{F} = \nabla f \implies P_y = Q_x}$$

$$\boxed{P_y = Q_x \not\Longrightarrow \mathbf{F} = \nabla f}$$

More precisely, if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field on Ω with $Q_x = P_y$ everywhere, then $\mathbf{F}$ is *locally* a gradient field. That is, in some neighborhood of each point $(x,y) \in \Omega$, where $P_y = Q_x$, there is a smooth function f defined (only on) this small neighborhood such that $\mathbf{F} = \nabla f$ there. The standard example to show that $P_y = Q_x$ is not sufficient (in general) to have a global potential $f : \Omega \to \mathbb{R}$ is the vector field

$$\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

We will see this example time and time again throughout the course.

Example 1.1.38. Determine whether the vector field $\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ is a gradient vector field.

SOLUTION. If $\mathbf{F}$ is a gradient vector field, we can find a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. Hence,

$$\begin{aligned} \mathbf{F} = \nabla f \quad &\Longrightarrow \quad -y\mathbf{i} + x\mathbf{j} \ = \ f_x\mathbf{i} + f_y\mathbf{j} \\ &\Longrightarrow \quad f_x \ = \ -y, \qquad f_y \ = \ x. \end{aligned}$$

Integrating $f_x = -y$ with respect to x gives $f(x, y) = -xy + g(y)$, where g is a function depending only on y. Differentiating $f(x, y) = -xy + g(y)$ with respect to y gives

$$f_y \quad = \quad -x + g'(y).$$

Since we already know that $f_y = x$, it follows that

$$g'(y) = 2x.$$

In particular, g depends on x, which contradicts the definition of g. So $\mathbf{F}$ is not a gradient field. □

Theorem 1.1.39. Let $\mathbf{F} : \Omega \to \mathbb{R}^3$ be a smooth vector field on a domain $\Omega \subseteq \mathbb{R}^3$ defined by

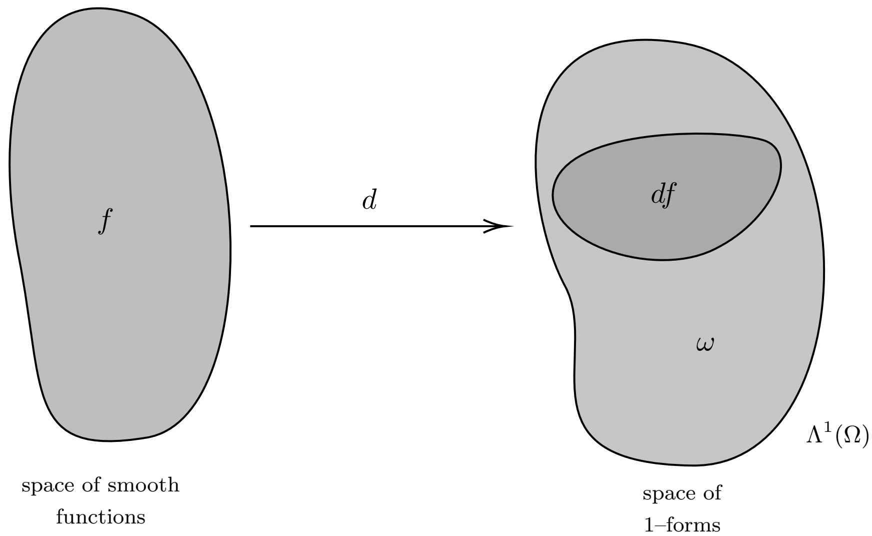
$$\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}.$$

If $\mathbf{F}$ is a gradient field, then the component functions must satisfy

$$P_y \ = \ Q_x, \qquad Q_z \ = \ R_y, \qquad P_z = R_x.$$

PROOF. The same as the proof of *Theorem 1.1.35.* □

Remark 1.1.40. The question of whether a vector $\mathbf{F}$ is a gradient field is equivalent to asking whether there is a smooth function f such that the associated 1–form $\omega_{\mathbf{F}} = df$. In this case, one says that the 1–form $\omega_{\mathbf{F}}$ is *exact*.



Remark 1.1.41. Much of our time in this course will be spent trying to understand exactly when a vector field $\mathbf{F}$ is a gradient field; or equivalently, when a 1–form is exact. This seemingly pedestrian endeavor leads to a surprisingly rich theory and is of central importance to mathematics.

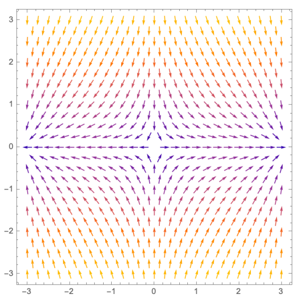
Remarks on Clairaut's theorem. It is curious that *Clairaut's theorem* bears the name of Clairaut, who, with company like Euler [5], Lagrange [11], and Cauchy [3], gave only a false (or incomplete) proof (see, e.g., [12]). The first counterexamples to the failure of mixed partial derivatives commuting were given by Lindelöf [12]. The first complete proof, however, was not given until 1873, by Schwarz [16].

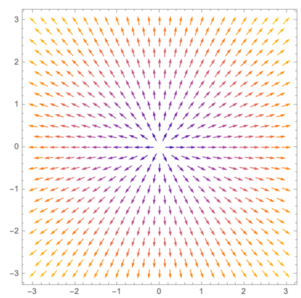
Exercises

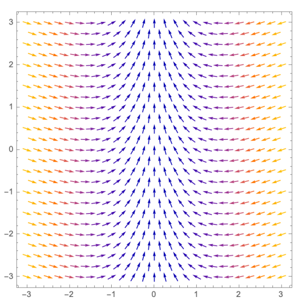
1. Sketch the vector fields

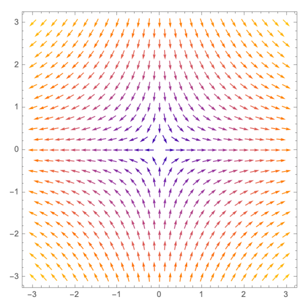
(i) $\mathbf{F}(x, y) = -y\mathbf{i} + 2x\mathbf{j}$.
(ii) $\mathbf{F}(x, y) = \mathbf{i} - \mathbf{j}$.
(iii) $\mathbf{F}(x, y) = (x^2 + \sin(y))\mathbf{i} - y\mathbf{j}$.
(iv) $\mathbf{F}(x, y) = \frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$.

2. Consider the vector fields (i) $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$, (ii) $\mathbf{F}(x, y) = \sin(x)\mathbf{i} - y\mathbf{j}$, (iii) $\mathbf{F}(x, y) = x\mathbf{i} - y\mathbf{j}$, and (iv) $\mathbf{F}(x, y) = -x\mathbf{i} + \cos(x)\mathbf{j}$. Match these vector fields to the vector fields sketched below:

(a) 

(c) 

(b) 

(d) 

3. Sketch the vector fields

(i) $\mathbf{F}(x, y) = (1 - \sin(x))\mathbf{i} + xe^{-y}\mathbf{j}$
(ii) $\mathbf{F}(x, y) = x^2\mathbf{i} + \log_e(y)\mathbf{j}$
(iii) $\mathbf{F}(x, y) = x\sqrt{1 - y^2}\mathbf{i} + 2y^3\mathbf{j}$
(iv) $\mathbf{F}(x, y) = \sin(y)e^{-x}\mathbf{i} + \cos(x)e^{-y}\mathbf{j}$

4. Let $\mathbf{F}, \mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ be smooth vector fields defined by $\mathbf{F}(x, y) = 2x\sin(y)\mathbf{i} - xy^2\mathbf{j}$ and $\mathbf{G}(x, y) = (1 + e^{-y})\mathbf{i} - \mathbf{j}$.

(i) Sketch the vector field $\mathbf{F}$.
(ii) Sketch the vector field $\mathbf{G}$.
(iii) Compute $\mathbf{F} + \mathbf{G}$.
(iv) Sketch the vector field $\mathbf{F} + \mathbf{G}$.

5. Determine the 1–forms associated to the vector fields:

(i) $\mathbf{F}(x,y) = (1-x^2)\mathbf{i} + 6y^3\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = e^{-x}\mathbf{i} + e^y \sin(x^2)\mathbf{j}$.
(iii) $\mathbf{F}(x,y) = (3x-y^2)\mathbf{i} + (y+1)^2\mathbf{j}$.
(iv) $\mathbf{F}(x,y,z) = (e^{1-zx} + e^{x+y})\mathbf{i} + (x^3+z^2)\mathbf{j} - x\mathbf{k}$.

6. Determine the vector fields associated to the 1–forms:

(i) $\omega = xy^3dx - 3yzdy + 2z^3dz$.
(ii) $\omega = xzdx + 2x^3dy + ydz$.
(iii) $\omega = 2z^2dx + (y-2z^2)dy + (z+4xy)dz$.
(iv) $\omega = (1+e^{-xz})dx + 2x\cos(z-y)dy + (x^2+7y)dz$.
(v) $\omega = (2x+z^2)dx + (xy - \sin(xz)dy - \cos(z)dz$

7. Consider the 1–forms $\omega = 2\sqrt{x^2+1}dx + (15y^2+1)dy$ and $\eta = (2-\tan(x))dx + e^x dy$. Determine

(i) $\omega + \eta$.
(ii) $3\omega - 4\eta$.
(iii) $x^2\omega - 3\sin(y)\eta$.

8. Consider the functions $f(x,y,z) = \cos(x) + \sin(x-z)$ and $g(x,y,z) = z\tan^{-1}(y/x)$.

(i) Compute $\omega = df$.
(ii) Compute $\eta = dg$.
(iii) Compute $3\sin(y)dy + xdx + (1-y)\omega - (2-\log_e(z))\eta$.

9. Consider the functions $f(x,y,z) = \sqrt{1+z^2}$ and $g(x,y,z) = e^{z^2} + \log_e(x-y)$.

(i) Compute $\omega = df$.
(ii) Compute $\eta = dg$.
(iii) Compute $4x^3dx + x^2\omega - 3y^2\eta$.

10. Let $\varphi = 2x^2dx + (x+y)dy$ and $\psi = -xdx + (x-3y)dy$. Compute

(i) $2\varphi + \psi$.
(ii) $\varphi - x^2\psi$.

11. Let $\alpha = x^3dx + yzdy - (x^2+y^2+z^2)dz$ and $\beta = y^2zdx - xzdy + (2x+1)dz$. Find

(i) $3\alpha - 4\beta$.
(ii) $x\alpha + y\beta$.

12. Compute the exterior derivative of the following functions:

(i) $f(x,y) = xy + x^3 e^{-y}$.
(ii) $f(x,y,z) = 1 - xz + \sin(xy+z)$.
(iii) $f(x,y,z) = \log_e(x) - \log_e(y) + \log_e(z)$.
(iv) $f(x,y,z) = e^{-z^2}\sqrt{x+y}$.

13. Determine whether the vector fields $\mathbf{F}$ given below are gradient fields:

(i) $\mathbf{F}(x,y) = xe^y\mathbf{i} + ye^x\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = (6x+5y)\mathbf{i} + (5x+4y)\mathbf{j}$.
(iii) $\mathbf{F}(x,y) = (1+2xy+\ln(x))\mathbf{i} + y^2\mathbf{j}$.
(iv) $\mathbf{F}(x,y,z) = (x^2-4zy)\mathbf{i} - 4x\mathbf{j}$.
(v) $\mathbf{F}(x,y,z) = \sqrt{z+x^2}\mathbf{i} + (x+2\sqrt{z})\mathbf{k}$.
(vi) $\mathbf{F}(x,y,z) = x\cos(y)\mathbf{i} + zy\cos(x)\mathbf{k}$.
(vii) $\mathbf{F}(x,y,z) = (y-x)\mathbf{i} + 2x^3\mathbf{j} + z^3\mathbf{k}$.
(viii) $\mathbf{F}(x,y,z) = xy^3\mathbf{i} + 4yz\mathbf{j} + 2x^3\mathbf{k}$.
(ix) $\mathbf{F}(x,y,z) = xz\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$.
(x) $\mathbf{F}(x,y,z) = 2x^2\mathbf{i} + (3y+z^2)\mathbf{j} + zy\mathbf{k}$.

14. Determine whether the following vector fields are gradient fields. If they are, find their potentials.

(i) $\mathbf{F}(x,y,z) := (x-y)\mathbf{i} - 2(y+x)\mathbf{j} + 3(x+z)\mathbf{k}$.
(ii) $\mathbf{F}(x,y,z) := \mathbf{i} + \mathbf{j} - \mathbf{k}$.
(iii) $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$.
(iv) $\mathbf{F}(x,y,z) = (1+e^{-z}x^2)\mathbf{i} + x\sin(z-y)\mathbf{j} + (x^3+9z)\mathbf{k}$.
(v) $\mathbf{F}(x,y,z) = (x^2+z^2)\mathbf{i} + (y-\sin(xy)\mathbf{j} - \cos(z)\mathbf{k}$.
(vi) $\mathbf{F}(x,y) = (4x-y^3)\mathbf{i} + x^2\mathbf{j}$.
(vii) $\mathbf{F}(x,y) = 2\sqrt{x^2+1}\mathbf{i} + (15y^2+1)\mathbf{j}$.
(viii) $\mathbf{F}(x,y,z) = (2-\tan(x))\mathbf{i} + e^x\mathbf{j}$.
(ix) $\mathbf{F}(x,y,z) = (e^z+e^x)\mathbf{i} + (y^3+x^2)\mathbf{j} - x\mathbf{k}$.
(x) $\mathbf{F}(x,y,z) = (\cos(x)+\sin(x-z))\mathbf{i} + (y^3+2)\mathbf{j} + (1+x)\mathbf{k}$.
(xi) $\mathbf{F}(x,y,z) = (\sin(z)+\log_e(x-1))\mathbf{i} + xy\mathbf{j} - z^2\mathbf{k}$.
(xii) $\mathbf{F}(x,y,z) = e^{z^2}\mathbf{i} + \sqrt{1+z^2}\mathbf{j} + \cos(x)\mathbf{k}$.

15. Determine whether

$$\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

is a gradient field.

16. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be smooth functions. Let

$$\mathbf{F}(x, y, z) := 2f(x)f'(x)\mathbf{i} + g'(y)h(z)\mathbf{j} + g(y)h'(z)\mathbf{k}.$$

Determine whether $\mathbf{F}$ is a gradient field.

17. Let

$$\mathbf{F}(x, y, z) := e^{z^2}\mathbf{i} + 2Byz^3\mathbf{j} + (Axze^{z^2} + 3By^2z^2)\mathbf{k}.$$

(i) Determine the values of the constants A and B such that $\mathbf{F}$ is a gradient field on $\mathbb{R}^3$.
(ii) Determine the potential function.

18. Recall that the dot product of two vectors $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$ is given by

$$u \cdot v = u_1v_1 + u_2v_2.$$

Let $\mathbf{F}(x, y) := (1 - x)\mathbf{i} + \mathbf{j}$ and $\mathbf{G}(x, y) = \mathbf{i} - (1 + y)\mathbf{j}$ be two smooth vector fields on $\mathbb{R}^2$.

(i) Compute the dot product $\mathbf{F}(x, y) \cdot \mathbf{G}(x, y)$, where $(x, y) \in \mathbb{R}^2$ is fixed.
(ii) Two vectors $u, v \in \mathbb{R}^2$ are said to be *orthogonal* if $u \cdot v = 0$. Determine the points $(x, y) \in \mathbb{R}^2$ (if any) where $\mathbf{F}(x, y)$ is orthogonal to $\mathbf{G}(x, y)$.

19. Let $\mathbf{F}(x, y) = 2x\cos(y)\mathbf{i} - 3ye^{-x}\mathbf{j}$ and $\mathbf{G}(x, y) = (1 + x)e^{1-y}\mathbf{i} + 2\mathbf{j}$.

(i) Compute the dot product $\mathbf{F}(x, y) \cdot \mathbf{G}(x, y)$, where $(x, y) \in \mathbb{R}^2$ is fixed.
(ii) Is the dot product of two vector fields a vector field? Explain.

20. A vector field $\mathbf{F}$ is said to be a *unit vector field* if $\|\mathbf{F}(x, y)\| = 1$ for all $(x, y) \in \mathbb{R}^2$. Determine which of the following (if any) are unit vector fields:

(i) $\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}$.
(ii) $\mathbf{F}(x, y) = \frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$.
(iii) $\mathbf{F}(x, y) = \sin(x)\mathbf{i} + \cos(y)\mathbf{j}$.

21. Building from the previous exercise, if the component functions of a vector field have unit length, i.e., if $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ with $\|P(x, y)\| = 1$ and $\|Q(x, y)\| = 1$ for all $(x, y) \in \mathbb{R}^2$) is $\mathbf{F}$ a unit vector field? Justify your answer.

22. Sketch the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{F}(x, y) \quad = \quad x\frac{x^2 - y^2}{x^2 + y^2}\mathbf{i} + y\frac{x^2 - y^2}{x^2 + y^2}. \tag{1.1.1}$$

23. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) := \frac{xy(x^2-y^2)}{x^2+y^2},$$

for $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, and $f(0,0) := 0$. Show that the second-order partial derivatives of f at the origin *do not* commute.

24. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{xy}{x^2+y^2}$$

for $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, and $f(0,0) = 0$. Show that, although this function is not continuous at the origin, it has first-order partial derivatives which are everywhere defined.

25. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) := \frac{x^2-y^2}{x^2+y^2},$$

for $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, and $f(0,0) = 0$.

(i) Show that $\lim_{x\to 0} \lim_{y\to 0} f(x,y)$ exists, and find its value.
(ii) Show that $\lim_{y\to 0} \lim_{x\to 0} f(x,y)$ exists, and find its value.
(iii) Do the values obtained in (i) and (ii) coincide?

26. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) \quad := \quad \begin{cases} x^2 \sin(1/x) + y^2 \sin(1/y), & xy \neq 0, \\ x^2 \sin(1/x), & x \neq 0,\ y = 0, \\ y^2 \sin(1/y), & x = 0,\ y \neq 0, \\ 0, & x = y = 0. \end{cases}$$

Show that f is differentiable but not continuously differentiable.

27. Do the second-order partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) := xy\frac{x^2-y^2}{x^2+y^2},$$

for $(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}$, and $f(0,0) = 0$, commute?

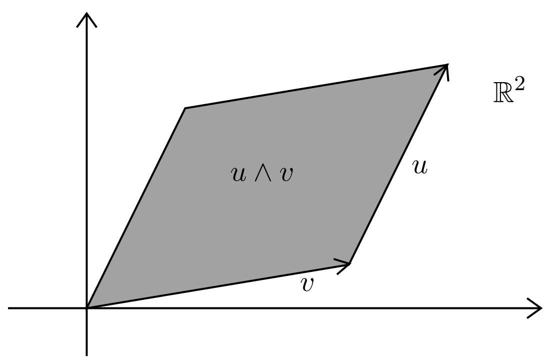
1.2. Forms and the Exterior Algebra

We have seen that 1–forms are given by expressions of the form

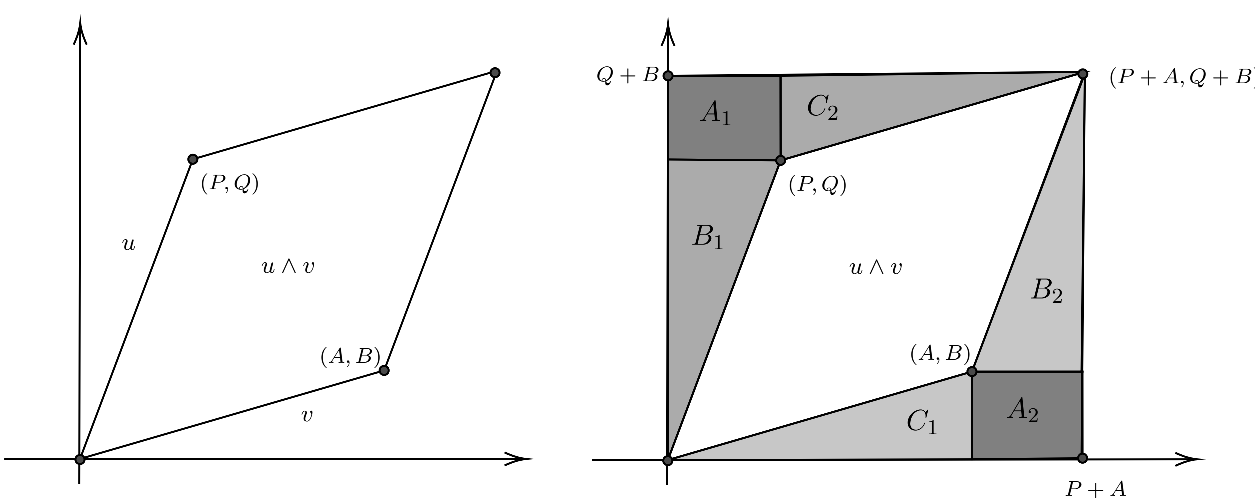
$$\omega = Pdx + Qdy,$$

where P, Q are smooth functions on $\mathbb{R}^2$. On $\mathbb{R}^3$, 1–forms are given by $\omega = Pdx + Qdy + Rdz$. As the notation indicates, 1–forms are important for an integration theory, where vector fields are not so appropriate. Forms have the additional advantage of coming with a multiplication, which we denote by $\wedge$, and call the *wedge product*.

The Wedge product. Let u, v be two 1–forms on $\mathbb{R}^2$. We want to introduce an operation on forms such that $u \wedge v$ gives the (signed) area of the parallelogram formed from u and v:



Area of parallelogram. To compute the area of this parallologram given by the forms $u = Pdx + Qdy$ and $v = Adx + Bdy$, we inscribe the parallologram in the rectangle as follows:



The area of the surrounding rectange is $|R| = (P+A)(Q+B)$. Moreover, we know that $|A_1| = |A_2|$, and $|B_1| = |B_2|$, and $|C_1| = |C_2|$. Hence, the area of the parallelogram $u \wedge v$ is

$$|u \wedge v| = |R| - 2|A_1| - 2|B_1| - 2|C_1|.$$

Since $|A_1| = PB$, $|B_1| = \frac{1}{2}PQ$ and $|C_1| = \frac{1}{2}BA$, it follows that

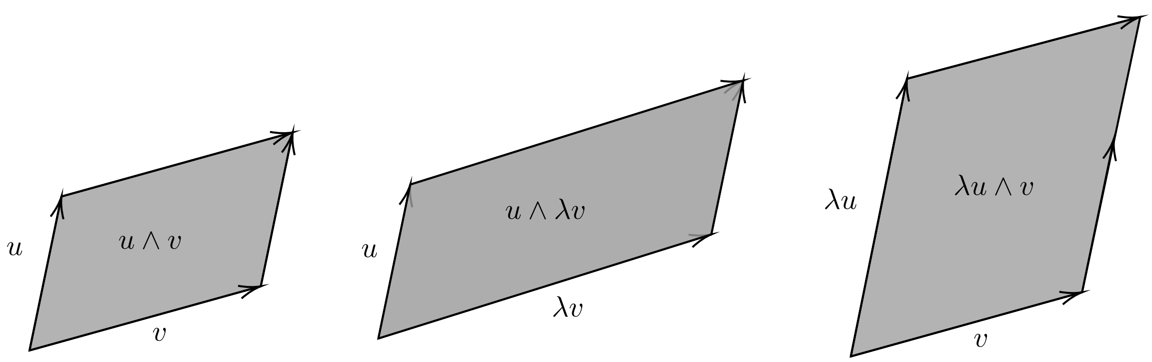
$$\begin{aligned}\text{Area of parallelogram} &= (P+A)(Q+B) - 2PB - 2\left(\frac{1}{2}PQ\right) - 2\left(\frac{1}{2}BA\right) \\ &= PQ + PB + QA + AB - 2PB - PQ - BA \\ &= PB - QA.\end{aligned}$$

Properties of parallelograms.

1. Scaling. If $u, v \in \mathbb{R}^2$ are vectors, and $\lambda \in \mathbb{R}$ is a scalar, then

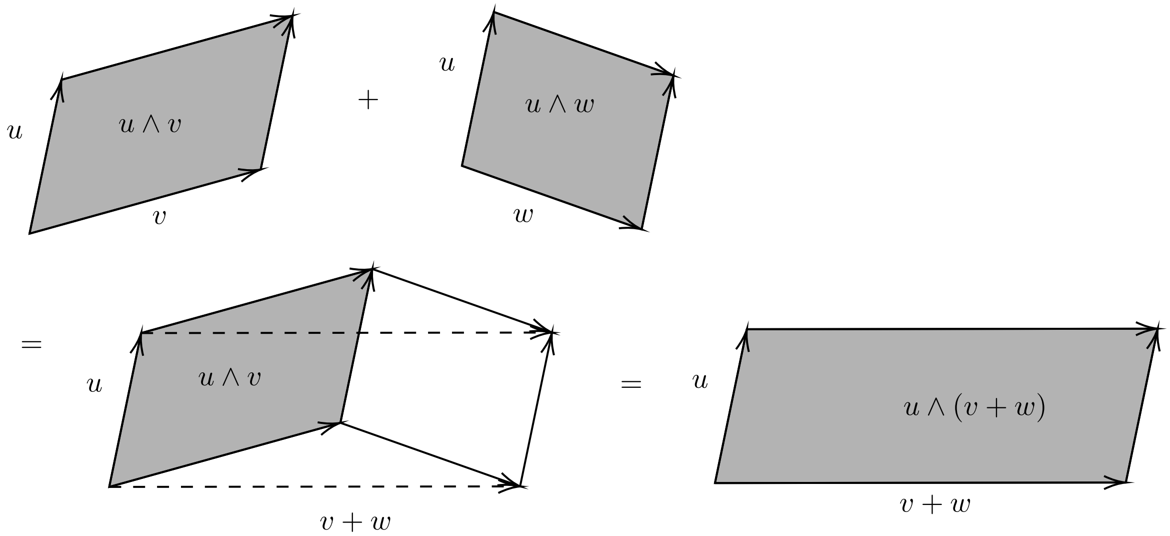
$$(\lambda u) \wedge v = u \wedge \lambda v = \lambda(u \wedge v).$$

That is, scaling a vector u by a constant λ scales the parallelogram by λ:



2. Distributive. If $u, v, w \in \mathbb{R}^2$ are vectors, then

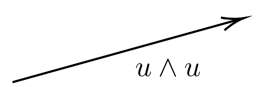
$$u \wedge (v + w) = u \wedge v + u \wedge w.$$



3. Nilpotence. If $u \in \mathbb{R}^2$ is a vector, then

$$u \wedge u = 0.$$

Visually, the nilpotent property describes the fact that the area of a line is zero:



Area of a line $= 0$

4. Anti-symmetry. For all vectors $u, v \in \mathbb{R}^2$, we have

$$u \wedge v = -v \wedge u.$$

This follows from the nilpotent and distributive properties. Indeed,

$$\begin{aligned} 0 \;=\; (u+v) \wedge (u+v) &= u \wedge u + u \wedge v + v \wedge u + v \wedge v \\ &= u \wedge v + v \wedge u. \end{aligned}$$

Summary of Properties of the Wedge Product.

(i) (Scaling). For all $u, v \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we have $(\lambda u) \wedge v = u \wedge (\lambda v) = \lambda(u \wedge v)$.
(ii) (Distributive). For all $u, v, w \in \mathbb{R}^2$, we have $u \wedge (v + w) = u \wedge v + u \wedge w$.
(iii) (Nilpotent). For all $u \in \mathbb{R}^2$, we have $u \wedge u = 0$.
(iv) (Anti-symmetry). For all $u, v \in \mathbb{R}^2$, we have $u \wedge v = -v \wedge u$.

Example 1.2.1. Let $u = Adx + Bdy$ and $v = Cdx + Ddy$ be two 1–forms on $\mathbb{R}^2$. Then

$$\begin{aligned} u \wedge v &= (Adx + Bdy) \wedge (Cdx + Ddy) \\ &= Adx \wedge Cdx + Adx \wedge Ddy + Bdx \wedge Cdx + Bdy \wedge Ddy \\ &= AC(dx \wedge dx) + AD(dx \wedge dy) + BC(dy \wedge dx) + BD(dy \wedge dy) \\ &= AD(dx \wedge dy) + BC(dy \wedge dx) \\ &= (AD - BC)dx \wedge dy. \end{aligned}$$

This recovers the formula for the determinant of a 2×2 matrix

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - BC.$$

Definition 1.2.2. A 2–form on a region $\Omega \subseteq \mathbb{R}^2$ is an expression of the form

$$\omega = f(x, y)dx \wedge dy.$$

A 2–form on a region $\Omega \subseteq \mathbb{R}^3$ is an expression of the form

$$\omega = Pdx \wedge dy + Qdy \wedge dz + Rdx \wedge dz,$$

where P, Q, R are functions of (x, y, z).
We denote by $\Lambda^2(\Omega)$ the set of 2–forms on Ω.

Remark 1.2.3. We sometimes refer to 2–forms on regions of $\mathbb{R}^2$ as *area forms* or *volume forms*.

Example 1.2.4. The expressions

$$\omega = 2xdx \wedge dy \qquad \text{and} \qquad \omega = \sin(xy)dx \wedge dy + x^2dy \wedge dz$$

are 2–forms on $\mathbb{R}^3$.

Remark 1.2.5. The expression $\eta = dy \wedge dx$ is a 2–form, by the anti-symmetric property of the wedge product, we can also write $\eta = -dx \wedge dy$.

Properties of 2–forms. The properties of 2–forms are the same as those of 1–forms:

(i) (Addition). If $\omega = Pdx\wedge dy+Qdx\wedge dz+Rdy\wedge dz$ and $\eta = Adx\wedge dy+dx\wedge dz+Cdy\wedge dz$, then

$$\omega + \eta = (P + A)dx \wedge dy + (Q + B)dx \wedge dz + (R + C)dy \wedge dz,$$

where P, Q, R, A, B, C are functions of (x, y, z).

(ii) (Scaling). If $\omega = Pdx \wedge dy + Qdx \wedge dz + Rdy \wedge dz$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is a function, then

$$f\omega = fPdx \wedge dy + fQdx \wedge dz + fRdy \wedge dz.$$

Example 1.2.6. Let $\omega = (x^2+y)dx\wedge dy+(1-z)dx\wedge dz+(z+y)dy\wedge dz$ and $\eta = dx\wedge dy-dy\wedge dz$. Compute

$$2\omega + 3\eta.$$

SOLUTION. Using the properties of 2–forms, we see that

$$\begin{aligned} 2\omega &= 2(x^2+y)dx\wedge dy + 2(1-z)dx\wedge dz + 2(z+y)dy\wedge dz, \\ 3\eta &= 3dx\wedge dy - 3dy\wedge dz. \end{aligned}$$

Hence,

$$2\omega + 3\eta = (2x^2+2y+3)dx\wedge dy + 2(1-z)dx\wedge dz + (2z+2y-3)dy\wedge dz.$$

□

Example 1.2.7. Let $\omega = xdx - ydy$ and $\eta = zdx + xdz$. Compute $\omega\wedge\eta$.

SOLUTION. We observe that

$$\begin{aligned} \omega\wedge\eta &= (xdx - ydy)\wedge(zdx + xdz) \\ &= xzdx\wedge dx + x^2dx\wedge dz - yzdy\wedge dx - xydy\wedge dz \\ &= x^2dx\wedge dz - yzdy\wedge dx - xydy\wedge dz \\ &= x^2dx\wedge dz + yzdx\wedge dy - xydy\wedge dz. \end{aligned}$$

□

Definition 1.2.8. A 3–form on a region $\Omega\subseteq\mathbb{R}^3$ is an expression of the form

$$\eta = f(x,y,z)dx\wedge dy\wedge dz,$$

where $f:\Omega\to\mathbb{R}$ is a smooth function.
The set of 3–forms on a region $\Omega\subseteq\mathbb{R}^3$ is denoted by $\Lambda^3(\Omega)$.

Properties of 3–forms. The properties of 3–forms are the same as those of 1–forms and 2–forms:

(i) (Addition). If $\omega = f(x,y,z)dx\wedge dy\wedge dz$ and $\eta = g(x,y,z)dx\wedge dy\wedge dz$, then

$$\omega + \eta = (f+g)dx\wedge dy\wedge dz.$$

(ii) (Scaling). If $\omega = fdx\wedge dy\wedge dz$ and $\lambda\in\mathbb{R}$ is a scalar, then

$$\lambda\omega = \lambda fdx\wedge dy\wedge dz.$$

Example 1.2.9. An example of a 3–form is given by

$$\omega = (2x^2y - xe^{-z})dx\wedge dy\wedge dz$$

Example 1.2.10. If Ω is a region in $\mathbb{R}^2$, then any 3–form on Ω is identically zero. This follows from the nilpotence property of the wedge product

$$dx \wedge dy \wedge dx = -dx \wedge dx \wedge dy = 0,$$

since $dx \wedge dx = 0$.

Remark 1.2.11. Similarly, if Ω is a region in $\mathbb{R}^3$, then any 4–form is identically zero.

Example 1.2.12. Let $\omega = 2xdx + 5e^{-xy}dy$ and $\eta = 4x^4 \sin(x+y)dx \wedge dz$. Compute $\omega \wedge \eta$.

SOLUTION.

$$\begin{aligned}
\omega \wedge \eta &= (2xdx + 5e^{-xy}dy) \wedge (4x^4 \sin(x+y)dx \wedge dz) \\
&= (2xdx) \wedge (4x^4 \sin(x+y)dx \wedge dz) + (5e^{-xy}dy) \wedge (4x^4 \sin(x+y)dx \wedge dz) \\
&= (2x)(4x^4 \sin(x+y))(dx \wedge dx \wedge dz) + (5e^{-xy})(4x^4 \sin(x+y))(dy \wedge dx \wedge dz).
\end{aligned}$$

Since $dx \wedge dx = 0$, we have $dx \wedge dx \wedge dz = 0$, so

$$\begin{aligned}
\omega \wedge \eta &= (5e^{-xy})(4x^4 \sin(x+y))(dy \wedge dx \wedge dz) \\
&= 20x^4 e^{-xy} \sin(x+y) dy \wedge dx \wedge dz \\
&= -20x^4 e^{-xy} \sin(x+y) dx \wedge dy \wedge dz.
\end{aligned}$$

□

Remark 1.2.13. The above example illustrates that the wedge product of a 1–form and a 2–form is a 3–form. More generally, the wedge product of a k–form with an ℓ–form is a $(k+\ell)$–form.

$$(k\text{–form}) \wedge (\ell\text{–form}) = (k+\ell)\text{–form})$$

$$\Lambda^k \wedge \Lambda^\ell = \Lambda^{k+\ell}$$

Exterior derivative for forms. We have seen that the exterior derivative of a function f gives a 1–form $df = f_x dx + f_y dy + f_z dz$. Similarly, the exterior derivative of a 1–form produces a 2–form:

Example 1.2.14. Let $\omega = (x^2 + y)dx - 3zdy + 2xdz$. Compute $d\omega$.

Solution. We have

$$\begin{aligned} d\omega &= (2xdx + dy) \wedge dx - 3dz \wedge dy + 2dx \wedge dz \\ &= 2xdx \wedge dx + dy \wedge dx - 3dz \wedge dy + 2dx \wedge dz \\ &= dy \wedge dx - 3dz \wedge dy + 2dx \wedge dz \\ &= 2dx \wedge dz - dx \wedge dy + 3dy \wedge dz. \end{aligned}$$

□

Example 1.2.15. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = (x^2 - z^3)\mathbf{i} + (\sin(y) + \cos(z))\mathbf{j} - z^3\mathbf{k}.$$

Compute the exterior derivative of the 1–form associated to $\mathbf{F}$.

Solution. The 1–form associated to $\mathbf{F}$ is

$$\omega_{\mathbf{F}} = (x^2 - z^3)dx + (\sin(y) + \cos(z))dy - z^3dz.$$

Then

$$\begin{aligned} d\omega_{\mathbf{F}} &= (2xdx - 3z^2dz) \wedge dx + (\cos(y)dy - \sin(z)dz) \wedge dy - 3z^2dz \wedge dz \\ &= 2xdx \wedge dx - 3z^2dz \wedge dx + \cos(y)dy \wedge dy - \sin(z)dz \wedge dy - 3z^2dz \wedge dz \\ &= -3z^2dz \wedge dx - \sin(z)dz \wedge dy, \end{aligned}$$

where the last equation follows from the nilpotence property. We can further simplify to

$$d\omega_{\mathbf{F}} = 3z^2dx \wedge dz + \sin(z)dy \wedge dz.$$

□

Remark 1.2.16. In general, the exterior derivative of a k–form is a $(k+1)$–form:

$$\begin{array}{ccccccc} \text{functions} & \xrightarrow{d} & \text{1–forms} & \xrightarrow{d} & \text{2–forms} & \xrightarrow{d} & \text{3–forms} \\ \Lambda^0(\Omega) & & \Lambda^1(\Omega) & & \Lambda^2(\Omega) & & \Lambda^3(\Omega) \end{array}$$

This motivates us to refer to functions as 0–*forms*:

Functions are 0–forms

$$\mathcal{C}^\infty(\Omega) = \Lambda^0(\Omega)$$

Example 1.2.17. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by $f = 2x^3 + \sin(y)$. The exterior derivative of f is

$$df = 6x^2 dx + \cos(y) dy.$$

The exterior derivative of df is then

$$\begin{aligned} d(df) &= d(6x^2 dx + \cos(y) dy) \\ &= 12x dx \wedge dx - \sin(y) dy \wedge dy \\ &= 0, \end{aligned}$$

since $dx \wedge dx = 0$ and $dy \wedge dy = 0$.

Theorem 1.2.18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with exterior derivative df. Then

$$d(df) = 0.$$

Proof. Let $f : \Omega \to \mathbb{R}$ be a smooth function on some open region in $\mathbb{R}^2$. The exterior derivative is then

$$df = f_x dx + f_y dy.$$

The exterior derivative of the 1–form df is then

$$\begin{aligned} d(df) &= d(f_x dx + f_y dy) \\ &= d(f_x) dx + d(f_y) dy \\ &= (f_{xx} dx + f_{xy} dy) \wedge dx + (f_{yx} dx + f_{yy} dy) \wedge dy \\ &= f_{xx} dx \wedge dx + f_{xy} dy \wedge dx + f_{yx} dx \wedge dy + f_{yy} dy \wedge dy \\ &= f_{xy} dy \wedge dx + f_{yx} dx \wedge dy \\ &= -f_{xy} dx \wedge dy + f_{yx} dx \wedge dy \\ &= (f_{yx} - f_{xy}) dx \wedge dy \ = \ 0, \end{aligned}$$

where the last equality follows from *Clairaut's theorem.* □

Remark 1.2.19. The above theorem holds for functions defined on $\mathbb{R}^n$ for any $n \in \mathbb{N}$.

Remark 1.2.20. Recall that a 1–form ω is said to be *exact* if $\omega = df$ for some smooth function f. The above theorem states that the exterior derivative of an exact 1–form is identically zero.

Summary 1.2.21.

$$
\begin{tikzcd}
\text{Functions} \arrow[d, leftrightarrow] & \text{Vector Fields} \arrow[d, leftrightarrow] & & \\
\Lambda^0\left(\mathbb{R}^3\right) \arrow[r, "d"] & \Lambda^1\left(\mathbb{R}^3\right) \arrow[r, "d"] & \Lambda^2\left(\mathbb{R}^3\right) \arrow[r, "d"] & \Lambda^3\left(\mathbb{R}^3\right) \\
\text{0–forms} & \text{1–forms} & \text{2–forms} & \text{3–forms}
\end{tikzcd}
$$

Functions are the same as 0–forms; vector fields are the same as 1–forms; the exterior derivative d increases the type of a form by $+1$.

Wedge product summary.

$$\Lambda^k\left(\mathbb{R}^3\right) \times \Lambda^\ell\left(\mathbb{R}^3\right) \xrightarrow{\wedge} \Lambda^{k+\ell}\left(\mathbb{R}^3\right)$$

The wedge product of a k–form and an ℓ–form is a form of type $(k+\ell)$.

Two key properties of the Wedge Product.

$$\boxed{\begin{array}{c} dx \wedge dx = 0 \\ \text{Nilpotence} \end{array}} \qquad \boxed{\begin{array}{c} dx \wedge dy = -dy \wedge dx \\ \text{Anti-symmetry} \end{array}}$$

Further directions. The fact that $d^2 = 0$ (i.e., the exterior derivative is nilpotent) leads to the theory of *de Rham cohomology*.

Exercises

1. Let $u = dx + 3dy$ and $v = 2dx + 5dy$ be two vectors 1–forms on $\mathbb{R}^2$.

(i) Compute $u \wedge v$.

(ii) Compute $\det \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

(iii) Compute $v \wedge u$.

(iv) Compute $\det \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$.

2. Let $u = 2xdx - zdy + 2ydz$ and $v = dx + 2xydy + (3-z)dz$ be two 1–forms on $\mathbb{R}^3$.

(i) Compute $u \wedge v$.

(ii) Compute

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{pmatrix}.$$

(iii) Let $\mathbf{F}_u = 2x\mathbf{i} - z\mathbf{j} + 2y\mathbf{k}$ and $\mathbf{F}_v = \mathbf{i} + 2xy\mathbf{j} + (3-z)\mathbf{k}$ be the vector fields associated to u, v. Compute $u \times v$, where $\times$ denotes the cross product.

(iv) What is the relation between the answers obtained from (i)–(iii)?

3. Determine the area of the parallelograms formed from the vectors

(i) $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $v = 3\mathbf{i} + 4\mathbf{j}$.

(ii) $\mathbf{u} = 3x\mathbf{i} + 2y\mathbf{j}$ and $v = (1-x)\mathbf{i} + 2y^2\mathbf{j}$.

4. Determine the types of the following forms:

(i) $dx \wedge dy - x^3 dz \wedge dx$.

(ii) $\tan^{-1}(\sin(z))dx - \cos(x^2)dy + 4\sqrt{1-x^2}dz$.

(iii) $x^2 + y^2$.

5. Let $\gamma = (x^2 - y^2)dx \wedge dy$ and $\eta = (x-y)dx \wedge dy$. Compute

(i) $y\gamma + x^2\eta$.

(ii) $-\gamma + (x+y)\eta$.

6. Compute $d\omega$, where

(i) $\omega = (x^2 + y^2)dx - ydy$.

(ii) $\omega = \sin(x)dz$.

(iii) $\omega = xdx \wedge dy + ydy \wedge dz$.

(iv) $\omega = yzdx + zy^2dy + xydz$.

(v) $\omega = xdx \wedge dy \wedge dz$.

7. Compute $d\varphi$, where

(i) $\varphi = (x^2 + y^3 z)dx + (y^2 - 2xz)dy + (x^4 + y^3 - z^2)dz$.
(ii) $\varphi = (x^2 + y^3 + z^4)dy \wedge dz + x^2 y^3 z^4 dz \wedge dx$.

8. Let $\varphi = x^2 dx - z^2 dy$ and $\psi = ydx - xdz$. Compute

(i) $d\varphi$.
(ii) $d\psi$.
(iii) $\varphi \wedge \psi$.

9. Let $\omega = dx + 2ydy$ and $\eta = dz$. Compute

(i) $\omega \wedge \eta$.
(ii) $\eta \wedge \omega$.
(iii) $4\omega \wedge 3\eta$.
(iv) $d\omega$.
(v) $d\eta$.
(vi) $\omega \wedge d\eta$.
(vii) $\eta \wedge d\omega$.
(viii) $d\omega \wedge d\eta$.

10. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $f = x^2 - 3y + \cos(z)$.

(i) Compute df.
(ii) Compute $d(df)$.
(iii) Did one need to compute $d(df)$ explicitly to know the result?

11. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = (\sin(x) - \cos(y))\mathbf{i} + 2xz\mathbf{j} - \tan(z^2)\mathbf{k}.$$

(i) Determine the 1–form $\omega_{\mathbf{F}}$ associated to $\mathbf{F}$.
(ii) Compute $d\omega_{\mathbf{F}}$.

12. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field given by

$$\mathbf{F}(x, y, z) = (x^3 + 2xe^y)\mathbf{i} + 2x^4 e^{1-y}\mathbf{j}.$$

(i) Determine the 1–form $\omega_{\mathbf{F}}$ associated to $\mathbf{F}$.
(ii) Compute $d\omega_{\mathbf{F}}$.

13. State, with justification, whether the following are true (T) or false (F):

(i) The wedge of two 1–forms is a 1–form.
(ii) The wedge of a 1–form and 2–form is 3–form.
(iii) The wedge of a p–form and q–form, for $p, q \in \mathbb{N}$, is a $(p + q)$–form.
(iv) If ω is a k–form, then $d\omega$ is a k–form.
(v) If ω is a p–form, then $d\omega$ is a $(p + 1)$–form.
(vi) If f is a smooth function, then df is a 2–form.
(vii) The wedge of two 2–form on $\mathbb{R}^3$ is always zero.

14. Extend the proof of *Theorem 1.2.18* to show that $(d \circ d)(f) = 0$ for any smooth function $f : \Omega \to \mathbb{R}$, where Ω is a domain in $\mathbb{R}^3$.

15. Extend the proof of *Theorem 1.2.18* to show that $(d \circ d)(f) = 0$ for any smooth function $f : \Omega \to \mathbb{R}$, where Ω is a domain in $\mathbb{R}^n$.

16. Set $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$; these are called *polar coordinates*. Convert the following 1–forms to polar coordinates:

(i) dx.
(ii) dy.
(iii) xdy.
(iv) $xdy - ydx$.
(v)
$$\frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy.$$
(vi)
$$\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx.$$

17. State, with justification, whether the following are true (T) or false (F):

(i) Functions are the same thing as 0–forms.
(ii) Every 1–form can be expessed as df for some smooth function f.
(iii) An area form is a 1–form on $\mathbb{R}^3$.
(iv) $xdx - ydy$ is a 2–form.

18. Let $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$. Show that
$$dx \wedge dy = rdrd\vartheta.$$

19. Let $\omega = 4x^3dx + \sin(z)dy + (1-y^2)dz$ and $\eta = (1-x)dy + (y-z)dz$.

(i) Compute $\omega \wedge \eta$.
(ii) Compute $d\omega$.
(iii) Compute $d\eta$.
(iv) Compute $\omega \wedge d\eta$.
(v) Compute $d\omega \wedge \eta$.
(vi) Compute $d\omega \wedge d\eta$.

20. State, with justification, whether the following are true or false:

(i) The wedge product of two 2–forms on $\mathbb{R}^3$ is zero.
(ii) The wedge product of two exact 1–forms is exact.
(iii) The wedge product of a function and a 2–form is a 2–form.
(iv) The wedge product of a 1–form and a 2–form on $\mathbb{R}^2$ is zero.
(v) If α is a k–form, and β is an ℓ–form, then $d(\alpha \wedge \beta)$ is a $(k+\ell)$–form.

21. Compute the exterior derivative of the following forms on $\mathbb{R}^3$:

(i) $\omega = (x+y^2)dx \wedge dy + \sin(z)dy \wedge dz + e^{-x}dx \wedge dz$.
(ii) $\omega = zdx \wedge dy + e^{-x}\tan(y)dy \wedge dz + (x+\sqrt{y^2+1})dx \wedge dz$.
(iii) $\omega = e^{x+y}dx \wedge dy + (1-e^z)dy \wedge dz + 2xdx \wedge dz$.
(iv) $\omega = \log_e(y)dx \wedge dy + \log_e(z)dy \wedge dz + \log_e(x)dx \wedge dz$.

22. Let $\omega = -ydx + xdy$ be a 1–form.

(i) Determine the function A such that $d\omega = Adx \wedge dy$.
(ii) Determine the vector field $\mathbf{F}_\omega$ associated to ω.
(iii) Compute the determinant

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{pmatrix}.$$

(iv) If $\star d\omega = Adz$, how does $\star d\omega$ relate to the answer obtained in (iii).

CHAPTER 2

Differentiation Theory

"It's true we pure mathematicians are connected to a different world. But it is a very real world nevertheless."

– Isadore Singer

In the previous chapter, we discussed the main objects of vector calculus: *vector fields* and *forms*. The exterior derivative was used to compute derivatives of forms. In the present chapter, we begin the differentiation theory for vector fields. In contrast to the familiar calculus of functions $f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R}^n \to \mathbb{R}$, there are two notions of derivative for a vector field: the *curl* and *divergence*:

$$\mathrm{curl}(\mathbf{F}) = \nabla \times \mathbf{F}, \qquad \mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}.$$

A unified perspective of divergence and curl is made possible with the exterior derivative and the Hodge $\star$–operator; that is, we will show that

$$\mathrm{curl}(\mathbf{F}) \ \sim \ \star d\omega_{\mathbf{F}}, \qquad \mathrm{div}(\mathbf{F}) \ \sim \ \star d \star \omega_{\mathbf{F}},$$

where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$ and $\sim$ means the vector field associated to.

2.1. The Curl of a Vector Field

Definition 2.1.1. Let Ω be a region on $\mathbb{R}^2$ or $\mathbb{R}^3$, on which, we have a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. The *curl* of $\mathbf{F}$ is the vector field defined by

$$\mathrm{curl}(\mathbf{F}) \ := \ \nabla \times \mathbf{F} \ = \ \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix}$$

where $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ is viewed as a vector (field) in $\mathbb{R}^3$, and $\times$ denotes the cross product.

Remark 2.1.2. It is common to refer to

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

as the *grad vector*.

Example 2.1.3. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$. Compute the curl of $\mathbf{F}$.

Solution. The curl is given by

$$\begin{aligned}\nabla \times \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xy & yz & zx \end{pmatrix} \\ &= \mathbf{i}\left(\partial_y(zx) - \partial_z(yz)\right) - \mathbf{j}\left(\partial_x(zx) - \partial_z(xy)\right) + \mathbf{k}\left(\partial_x(yz) - \partial_y(xy)\right) \\ &= -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}.\end{aligned}$$

□

Example 2.1.4. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Compute the curl of $\mathbf{F}$.

Solution. The curl is given by

$$\begin{aligned}\operatorname{curl}(\mathbf{F}) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{pmatrix} \\ &= \mathbf{i}(\partial_y(z) - \partial_z(y)) - \mathbf{j}(\partial_x(z) - \partial_z(x)) + \mathbf{k}(\partial_x(y) - \partial_y(x)) \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}.\end{aligned}$$

□

Remark 2.1.5. If we wish to compute the curl of a vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, given, say, by $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then we identify it with a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$, and compute the curl as before.

Example 2.1.6. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x, y) = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}.$$

Compute the curl of $\mathbf{F}$.

SOLUTION. The curl is given by

$$\begin{aligned}
\operatorname{curl}(\mathbf{F}) &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2-y^2 & 2xy & 0 \end{pmatrix} \\
&= (\partial_y(0) - \partial_z(2xy))\mathbf{i} - (\partial_x(0) - \partial_z(x^2-y^2))\mathbf{j} \\
&\qquad\qquad + (\partial_x(2xy) - \partial_y(x^2-y^2))\mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 4y\mathbf{k}.
\end{aligned}$$

□

Remark 2.1.7. If $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ is a vector field on $\mathbb{R}^2$, given by $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then

$$\operatorname{curl}(\mathbf{F}) = 0\mathbf{i} + 0\mathbf{j} + (Q_x - P_y)\mathbf{k}.$$

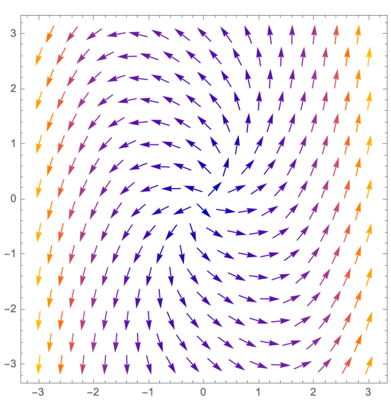
In particular, for vector fields on $\mathbb{R}^2$, we often consider the function

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} \;=\; Q_x - P_y.$$

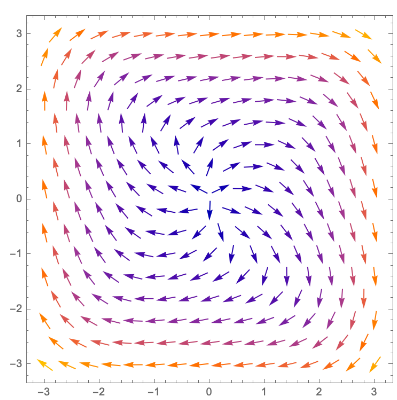
Remark 2.1.8. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field which models fluid flow. If you were to drop a twig into this fluid (keeping its center fixed), then $\operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} > 0$ means that the twig would rotate anti-clockwise, while $\operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} < 0$ means that the twig would rotate clockwise.

$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} > 0$ $\qquad\qquad$ $\operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} < 0$

Example 2.1.9. The following vector fields illustrative positive and negative curl, respectively.



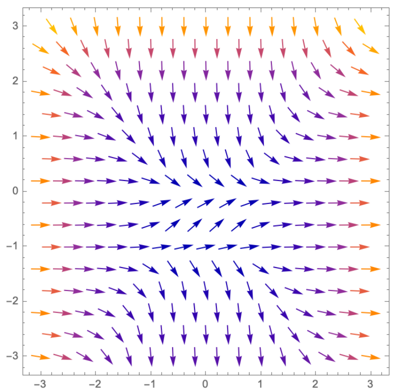
$\mathbf{F}(x,y) = (2x-y)\mathbf{i} + (y+x^3)\mathbf{j}$ has positive curl.



$\mathbf{F}(x,y) = (2x+y^3)\mathbf{i} + (y-x^3)\mathbf{j}$ has negative curl.

Definition 2.1.10. A vector field $\mathbf{F}$ is said to be *irrotational* if $\text{curl}(\mathbf{F}) = \mathbf{0}$.

Example 2.1.11. The following illustrates an irrotational vector field on $\mathbb{R}^2$:



$$\mathbf{F}(x, y) = (2x^4 + \cos(x))\mathbf{i} + (y^3 - 9ye^{y-1})\mathbf{j} \text{ is irrotational.}$$

Theorem 2.1.12. Gradient fields $\mathbf{F} = \nabla f$ are irrotational:

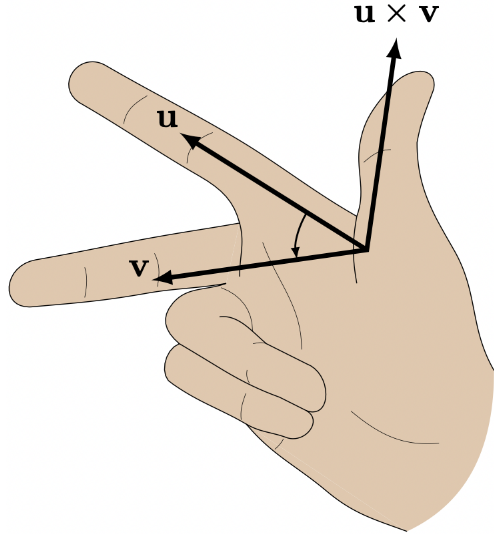
$$\text{curl}(\nabla f) = \mathbf{0}.$$

Proof. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Then $\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$ and

$$\begin{aligned} \text{curl}(\mathbf{F}) = \nabla \times \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{pmatrix} \\ &= \mathbf{i}(\partial_y f_z - \partial_z f_y) - \mathbf{j}(\partial_x f_z - \partial_z f_x) + \mathbf{k}(\partial_x f_y - \partial_y f_x) \\ &= \mathbf{i}(f_{yz} - f_{zy}) - \mathbf{j}(f_{xz} - f_{zx}) + \mathbf{k}(f_{xy} - f_{yx}) \\ &= \mathbf{0}, \end{aligned}$$

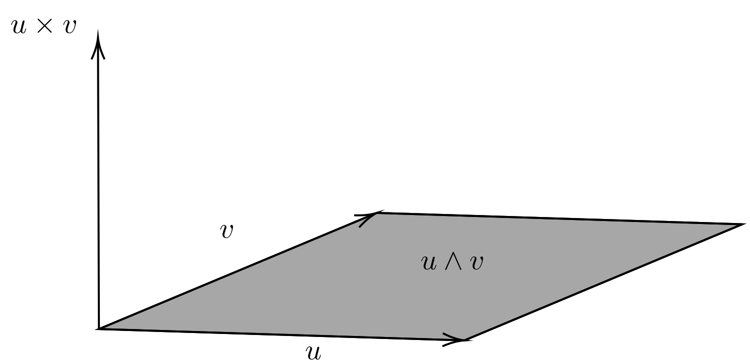
since the second-order partial derivatives of a smooth function commute by *Clairaut's theorem* . $\square$

Meaning of the cross product. Let $u, v \in \mathbb{R}^3$ be two vectors. The cross product $\times$ produces a vector $u \times v \in \mathbb{R}^3$ which is orthogonal to both u, v.



The right-hand rule.

Relationship to the wedge product. The magnitude of the cross product $|u \times v|$ is equal to the area of the parallelogram $u \wedge v$:



Remark 2.1.13. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. Then the associated 1–form is

$$\omega_{\mathbf{F}} = Pdx + Qdy + Rdz.$$

The exterior derivative of $\omega_{\mathbf{F}}$ is

$$\begin{aligned} d\omega_{\mathbf{F}} &= (P_x dx + P_y dy + P_z dz) \wedge dx + (Q_x dx + Q_y dy + Q_z dz) \wedge dy \\ &\qquad + (R_x dx + R_y dy + R_z dz) \wedge dz \\ &= P_y dy \wedge dx + P_z dz \wedge dx + Q_x dx \wedge dy + Q_z dz \wedge dy + R_x dx \wedge dz + R_y dy \wedge dz \\ &= (Q_x - P_y) dx \wedge dy + (R_y - Q_z) dy \wedge dz + (R_x - P_z) dx \wedge dz. \end{aligned}$$

Compare this with the curl of $\mathbf{F}$:

$$\text{curl}(\mathbf{F}) \quad = \quad (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}.$$

There is some similarity between $d\omega_{\mathbf{F}}$ and curl($\mathbf{F}$). But $d\omega_{\mathbf{F}}$ is a 2–form, and curl($\mathbf{F}$) is a vector field. Hence, we cannot make sense of $d\omega_{\mathbf{F}}$ being associated to a vector field.
Later, we will introduce the Hodge $\star$–operator which has the property that it maps 2–forms on $\mathbb{R}^3$ to 1–forms on $\mathbb{R}^3$, i.e., $\star : \Lambda^2(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3)$. Moreover, it satisfies

$$\star(dx \wedge dy) = dz, \qquad \star(dy \wedge dz) = dx, \qquad \star(dx \wedge dz) = -dy.$$

In particular,

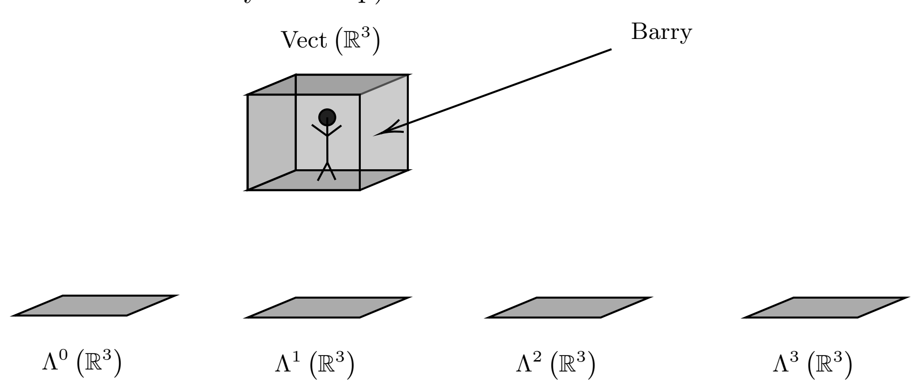
$$\star d\omega_{\mathbf{F}} \quad = \quad (Q_x - P_y)dz + (R_y - Q_z)dx + (P_z - R_x)dy,$$

and we see that $\star d\omega_{\mathbf{F}}$ is the 1–form associated to curl($\mathbf{F}$):

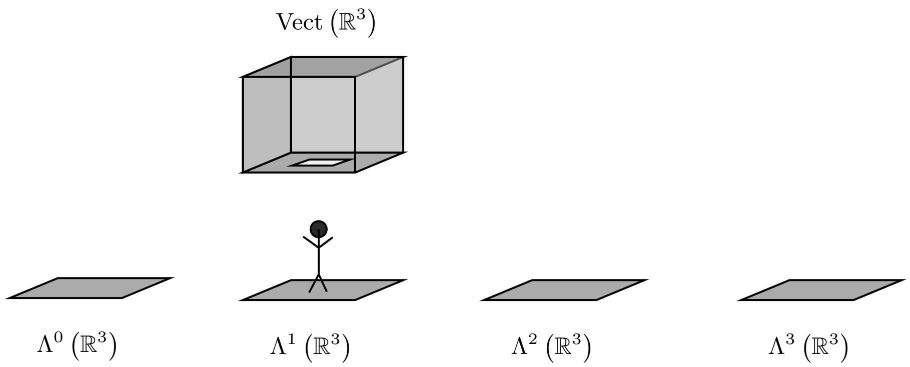
$\boxed{\star d\omega_{\mathbf{F}} \text{ is the 1–form associated to curl}(\mathbf{F})}$

Barry's world. The Hodge $\star$–operator and the exterior derivative d are often quite confusing concepts when they are first introduced. Nevertheless, one can think of the problem of finding expressions for the divergence and curl in terms of d and $\star$ as the following game:

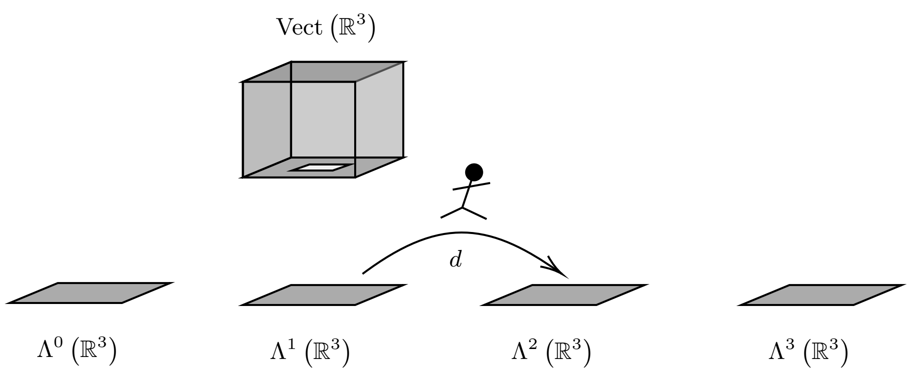
(i) To start off this, your player (Barry) is in the vector field position, which lies above the platform of 1–forms. List the forms of type $0, 1, 2, 3$ as follows (viewing them as platforms on which Barry can step):



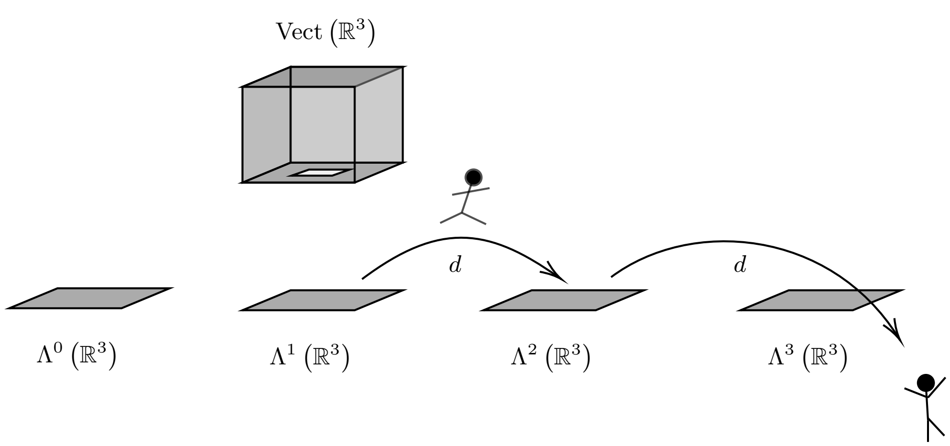
(ii) When we decide to pass to forms, we initiate the game by opening the trap door, releasing Barry onto the platform below – the platform of 1–forms:



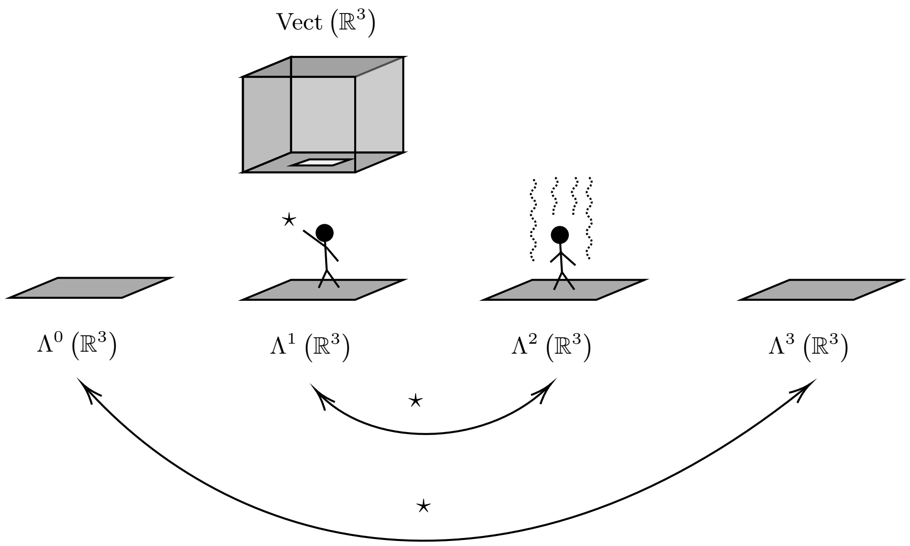
(iii) Barry is able to move to the next platform using the exterior derivative:



But, if he uses the exterior derivative twice in a row, then he falls:

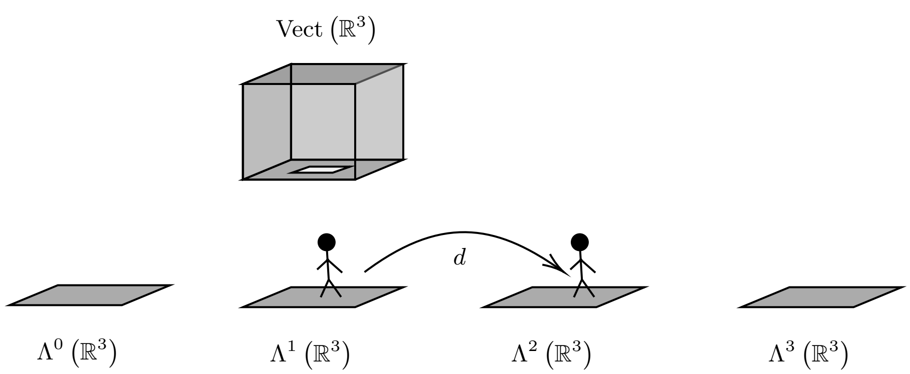


(iv) The Hodge $\star$–operator allows Barry to teleport between plats 0 and 3, and between platforms 1 and 2:

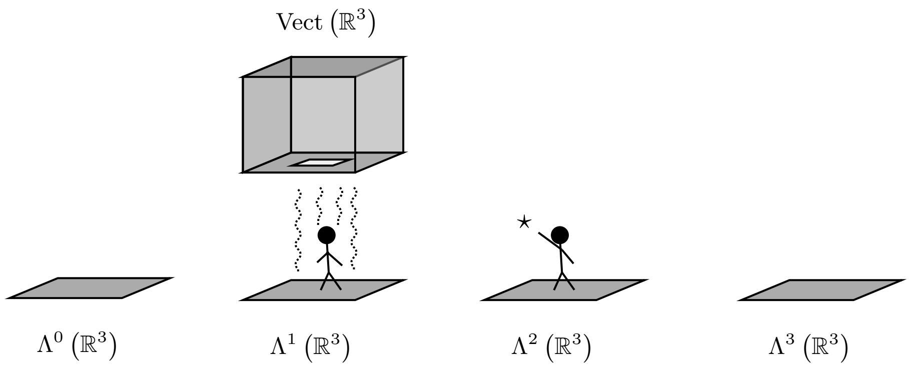


Barry explains: The curl of a vector field. The curl is a vector field, so can reach it by getting to the 1–forms platform. Here is how Barry does it:

(i) Apply the exterior derivative to get onto platform $\Lambda^2(\mathbb{R}^3)$:



(ii) We can't apply the exterior derivative, otherwise Barry falls. So we need to telelport Barry using the Hodge $\star$–operator, taking him back to platform $\Lambda^1(\mathbb{R}^3)$:



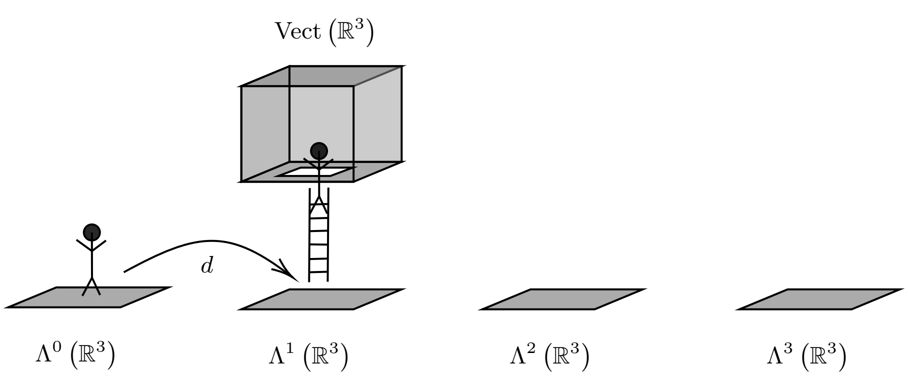
This gives lands Barry on $\Lambda^1(\mathbb{R}^3)$, which we can associate to a vector field. In other words:

$$\boxed{\star d\omega_{\mathbf{F}} \text{ is the 1–form associated to curl}(\mathbf{F})}$$

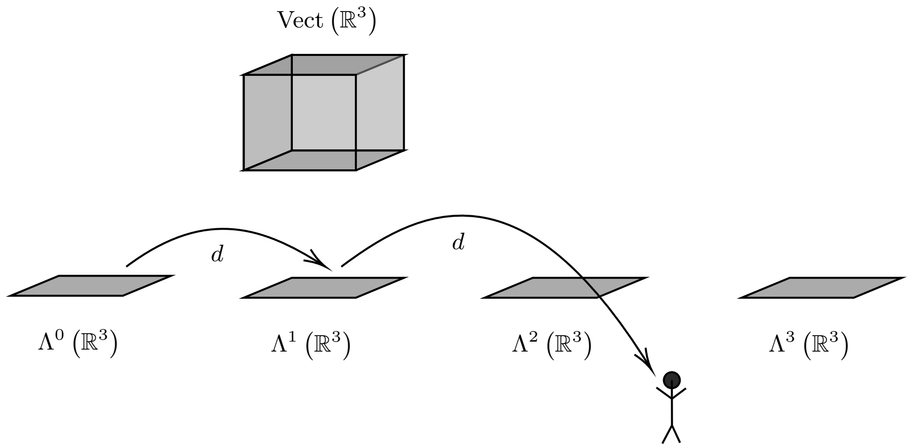
Barry explains: Curl of a gradient field is zero. Recall that we have seen the curl of a gradient field $\mathbf{F} = \nabla f$ vanishes, i.e.,

$$\operatorname{curl}(\nabla f) \quad = \quad \mathbf{0}.$$

We can understand this result without doing any calculation, merely following the results of Barry's world. The key point is that since $\mathbf{F}$ is a gradient field, the associated 1–form $\omega_{\mathbf{F}}$ comes from applying the exterior derivative df of a function, i.e., Barry doesn't start in the vector field box, he started on the $\Lambda^0(\mathbb{R}^3)$–platform, and then used the exterior derivative to get to the $\Lambda^1(\mathbb{R}^3)$–platform:



Hence, when we compute the curl, we first apply the exterior derivative, but since Barry has already done that, he falls:



$\mathrm{Vect}\left(\mathbb{R}^3\right)$
d
d
$\Lambda^0\left(\mathbb{R}^3\right)$
$\Lambda^1\left(\mathbb{R}^3\right)$
$\Lambda^2\left(\mathbb{R}^3\right)$
$\Lambda^3\left(\mathbb{R}^3\right)$

Exercises

1. Compute the curl of the following vector fields:

(i) $\mathbf{F}(x,y) = (2x - 3y^2)\mathbf{i} + x\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = 3\sqrt{x-1}\mathbf{i} + (2y^2 + x)\mathbf{j}$.
(iii) $\mathbf{F}(x,y,z) = (1 - \sin(x))\mathbf{i} + \tan(x)\mathbf{j}$.
(iv) $\mathbf{F}(x,y,z) = (e^{-z} + e^{-x})\mathbf{i} + (y^2 - x^3)\mathbf{j} + 2xz\mathbf{k}$.
(v) $\mathbf{F}(x,y,z) = (\sin(x) + \cos(x-y))\mathbf{i} + (z^2+1)\mathbf{j} + (1-x)\mathbf{k}$.
(vi) $\mathbf{F}(x,y,z) = (\cos(z^2) + \log_e(x^2+1))\mathbf{i} + (xyz)\mathbf{j} + (1 - xzy)\mathbf{k}$.
(vii) $\mathbf{F}(x,y,z) = (e^{-x+z^2})\mathbf{i} + \sqrt{1+z^3}\mathbf{j} + 2\sin(z)\mathbf{k}$.

2. Determine whether the vector fields $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined below are given by the curl of some other vector field:

(i) $\mathbf{F}(x,y,z) = zy^2\mathbf{i} + 2xz^2\mathbf{j} - 3z^3\mathbf{k}$.
(ii) $\mathbf{F}(x,y,z) = yz\mathbf{i} + xyz\mathbf{j} + xy\mathbf{k}$.

3. Let $\omega := \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$ be a constant vector. If (x,y,z) denote the coordinates on $\mathbb{R}^3$, let $\mathbf{v} := \omega \times \mathbf{r}$, where $\mathbf{r} := x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that

$$\omega = \frac{1}{2}\mathrm{curl}(\mathbf{v}).$$

4. Show that the vector field

$$\mathbf{F}(x,y,z) := 2x^3\mathbf{i} + (y^2+1)\mathbf{j} + (z^3 - 9)\mathbf{k},$$

is irrotational.

5. Show that the vector field

$$\mathbf{F}(x,y,z) = (y^2 + z^2)\mathbf{i} + (x^2 - z^2)\mathbf{j} + (x^9 + y^9)\mathbf{k}$$

is incompressible.

6. Let $\mathbf{F}(x,y,z) = x^2\mathbf{i} + (y - z^2)\mathbf{j} + xzy\mathbf{k}$ and $\mathbf{G}(x,y,z) = (1 - z^2)\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$.

(i) Compute curl($\mathbf{F}$).
(ii) Compute curl($\mathbf{G}$).
(iii) Determine the vector field $\mathbf{H} = \mathbf{F} + \mathbf{G}$.
(iv) Compute curl($\mathbf{H}$).
(v) How does the answer the answer in (iv) compare to the sum of the answers from (i) and (ii)?

7.

(i) Show by direct computation that

$$\operatorname{curl}(\mathbf{F}+\mathbf{G}) = \operatorname{curl}(\mathbf{F}) + \operatorname{curl}(\mathbf{G}).$$

(ii) Show that, for $\mathbf{F}$ a vector field, and f a smooth function, we have

$$\operatorname{curl}(f\mathbf{F}) = f\operatorname{curl}(\mathbf{F}) + (\nabla f) \times \mathbf{F}.$$

8. Prove that, for a smooth function f, we have

$$\operatorname{curl}(f\nabla f) = \mathbf{0}.$$

9. Let $\mathbf{F}(x,y,z) = (y - \sin(xy)\mathbf{i} + (x^2+z^2)\mathbf{j} - \tan(z)\mathbf{k}$ and $\mathbf{G}(x,y,z) = (5-2z^3)\mathbf{i} + x^2y\mathbf{j} + 5\mathbf{k}$.

(i) Compute $\operatorname{curl}(\mathbf{F})$.
(ii) Compute $\operatorname{curl}(\mathbf{G})$.
(iii) Determine the vector field $\mathbf{F} \times \mathbf{G}$.
(iv) Compute $\operatorname{curl}(\mathbf{F} \times \mathbf{G})$.

10. Compute the exterior derivative of the following 1–forms:

(i) $\omega = (x^2 - 3y)dx - 4xdy$.
(ii) $\omega = \sqrt{1-x^2}dx + (x + 2\sqrt{y})dy$.
(iii) $\omega = x\cos(y)dx + y\cos(x)dx$.
(iv) $\omega = (y-x)dx + 2x^3dy + z^3dz$.

11. For a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, explain why $\operatorname{curl}(\mathbf{F}) > 0$ meaningless. How can one make sense of $\operatorname{curl}(\mathbf{F}) > 0$ if $\mathbf{F}$ is a vector field on $\mathbb{R}^2$?

12. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Does $\operatorname{curl}(f)$ make sense? If $\mathbf{F}$ is a vector field, does $\nabla\mathbf{F}$ make sense?

13. Determine which of the following vector fields are irrotational:

(i) $\mathbf{F}(x,y) = (\sin(x)+2)\mathbf{i} + (1-\cos(y))\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = \sqrt{1+x^2}\mathbf{i} + \tan^{-1}(y^2+1)\mathbf{j}$.
(iii) $\mathbf{F}(x,y,z) = (2x\cos(x^2+y^2+z^3)+3x^2)\mathbf{i} + 2y\cos(x^2+y^2+z^3)\mathbf{j} + 3z^2\cos(x^2+y^2+z^3)\mathbf{k}$.
(iv) $\mathbf{F}(x,y) = (y\cos(1-xy)\csc(1-xy)+y)\mathbf{i} + (x\cos(1-xy)\csc(1-xy)+x)\mathbf{j}$.

14. Determine, by computing the curl, which of the following vector fields are gradient fields:

(i) $\mathbf{F}(x,y) = (2-e^{x+y})\mathbf{i} + 3\sqrt{1+x^2}\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = -2x(y^2-1)\mathbf{i} - 2x^2y\mathbf{j}$.
(iii) $\mathbf{F}(x,y,z) = (1-\sin(z))\mathbf{i} + 2y\mathbf{j} + z^3\mathbf{k}$.

16. Compute the exterior derivative of the 1–forms associated to the following vector fields:

(i) $\mathbf{F}(x, y) = -y\sin(x)\mathbf{i} + xe^{-y^2}\mathbf{j}$.
(ii) $\mathbf{F}(x, y) = \sqrt{x^2 + 3y^2}\mathbf{i} + \cos(x^2)\mathbf{j}$.

17. Let $\mathbf{F}$ be a gradient field. Compute the exterior derivative of the 1–form ω associated to $\mathbf{F}$.

18. Determine which of the following statements are true or false:

(i) Every gradient field is irrotational.
(ii) Every irrotational field is a gradient field.
(iii) Every irrotational field on $\mathbb{R}^2$ is a gradient field.
(iv) The exterior derivative of the 1–form associated to a gradient field is always zero.

19. Determine which of the following statements are true or false:

(i) The exterior derivative of a 1–form is a 2–form.
(ii) ∇f is the vector field associated to the 1–form df.
(iii) $\mathrm{curl}(\mathbf{F})$ is the vector field associated to the 2–form $d\omega$, where ω is the 1–form associated to $\mathbf{F}$.

20. A *complex lamellar vector field* is a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{F} \cdot (\nabla \times \mathbf{F}) \quad = \quad 0.$$

(i) Show that an irrotational vector field is a complex lamellar vector field.

21. A *Beltrami vector field* is a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{F} \times (\nabla \times \mathbf{F}) = \mathbf{0}.$$

(i) Determine whether an irrotational vector field is a Beltrami vector field.
(ii) Let

$$\mathbf{F}(x, y, z) := -\frac{z}{\sqrt{1+z^2}}\mathbf{i} + \frac{1}{\sqrt{1+z^2}}\mathbf{j}.$$

Compute the curl of $\mathbf{F}$.
(iii) Show that the vector field $\mathbf{F}$ defined in part (ii) is a Beltrami vector field.

22. Building on the previous exercise, show that if $\mathbf{F}$ is a Beltrami vector field, then $\mathbf{F}$ is parallel to $\mathrm{curl}(\mathbf{F})$.

23. Compute the exterior derivative $d\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to the following vector fields:

(i) $\mathbf{F}(x, y) = -xy\mathbf{i} + 2xe^{-y}\mathbf{j}$.
(ii) $\mathbf{F}(x, y) = \log_e(x)\mathbf{i} - x^2 \log_e(y)\mathbf{j}$.
(iii) $\mathbf{F}(x, y) = (x + \sin(xy))\mathbf{i} - yx\mathbf{j}$.
(iv) $\mathbf{F}(x, y) = \frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$.

24. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field

$$\mathbf{F}(x, y) = (x^3 - y^2)\mathbf{i} + (x^2 - \sin(y^3))\mathbf{j}.$$

(i) Compute $\mathrm{curl}(\mathbf{F})$.
(ii) Write down the associated 1–form $\omega_{\mathbf{F}}$.
(iii) Compute $d\omega_{\mathbf{F}}$.
(iv) How does the result of (i) compare with that of (iii)?

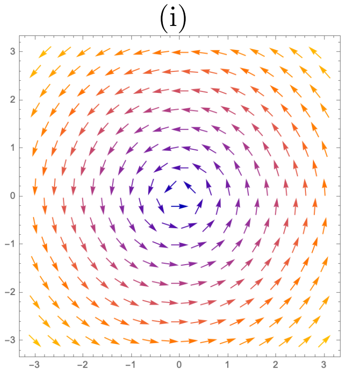
25. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field

$$\mathbf{F}(x, y) = (x \log_e(y) - \cos(y))\mathbf{i} + (\sin(x) - log_e(x + y))\mathbf{j}.$$

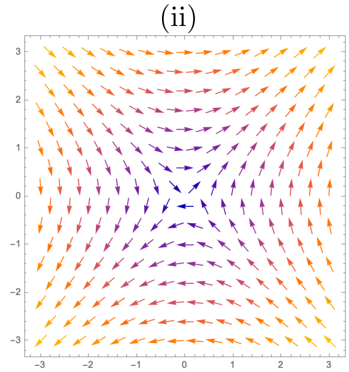
(i) Compute $\mathrm{curl}(\mathbf{F})$.
(ii) Write down the associated 1–form $\omega_{\mathbf{F}}$.
(iii) Compute $d\omega_{\mathbf{F}}$.
(iv) How does the result of (i) compare with that of (iii)?

26. Consider the following vector fields $\mathbf{F}$. State, with justification, whether $\mathbf{F}$ is a gradient field.

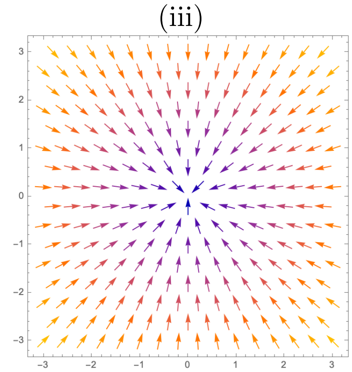
(i)



(ii)



(iii)



2.2. The Divergence of a Vector Field

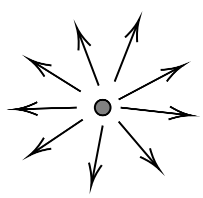
The curl of a vector field $\mathbf{F}$ was defined to be the cross product of ∇ with $\mathbf{F}$. This provides one candidate notion of derivative for a vector field. There is another notion of derivative, namely, the divergence.

Definition 2.2.1. (Divergence). Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. The *divergence* of $\mathbf{F}$ is defined

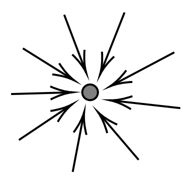
$$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}.$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ are smooth functions, then

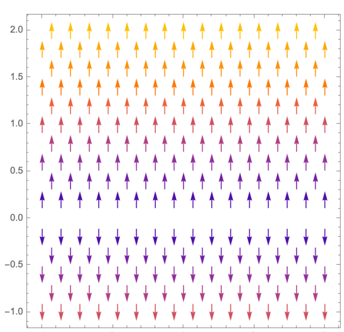
$$\mathrm{div}(\mathbf{F}) \quad = \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$



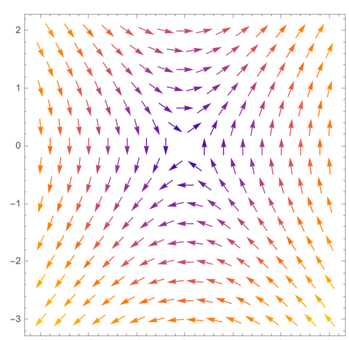
$\mathrm{div}(\mathbf{F}) > 0$



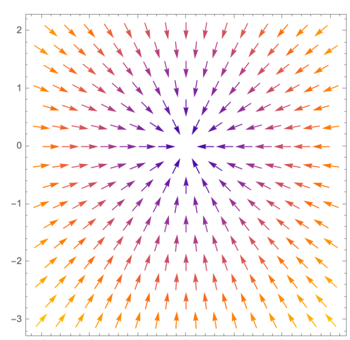
$\mathrm{div}(\mathbf{F}) < 0$



$\mathrm{div}(\mathbf{F}) > 0$



$\mathrm{div}(\mathbf{F}) = 0$



$\mathrm{div}(\mathbf{F}) < 0$

Example 2.2.2. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) = -xy\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$. Compute the divergence of $\mathbf{F}$.

Solution. We have

$$\mathrm{div}(\mathbf{F}) \quad = \quad \partial_x(-xy) + \partial_y(2z) + \partial_z(y) \quad = \quad -y.$$

$\square$

Example 2.2.3. Let ω be the 1–form on $\mathbb{R}^3$ given by

$$\omega = (x+z)dx + \sin(xy)dy.$$

Compute the divergence of the vector field $\mathbf{F}$ associated to ω.

Solution. The vector field $\mathbf{F}$ associated to ω is

$$\mathbf{F}(x,y) = (x+z)\mathbf{i} + \sin(xy)\mathbf{j} + 0\mathbf{k}.$$

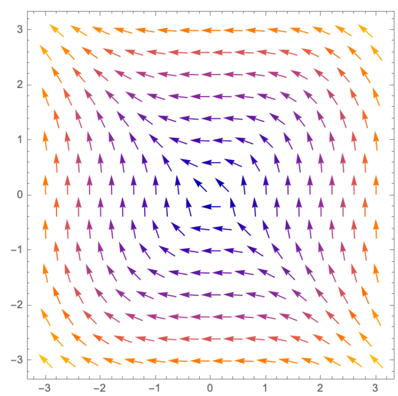
The divergence of $\mathbf{F}$ is then

$$\operatorname{div}(\mathbf{F}) \quad = \quad 1 + x\cos(y).$$

□

Definition 2.2.4. A vector field $\mathbf{F}$ is said to be *incompressible* if $\operatorname{div}(\mathbf{F}) = 0$.

Example 2.2.5. The following vector field is incompressible:



$$\mathbf{F}(x,y) = -y^2\mathbf{i} + x^2\mathbf{j}.$$

Example 2.2.6. Show that the vector field $\mathbf{F} = -z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$ is incompressible.

Solution. The divergence is

$$\operatorname{div}(\mathbf{F}) = \partial_x(-z) + \partial_y(x) + \partial_z(y^2) = 0.$$

□

Theorem 2.2.7. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then

$$\mathrm{div}(\mathrm{curl}(\mathbf{F})) = 0.$$

PROOF. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a smooth vector field on $\mathbb{R}^3$. Then

$$\begin{aligned}\mathrm{curl}(\mathbf{F}) &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix} \\ &= \mathbf{i}\,(R_y - Q_z) - \mathbf{j}(R_x - P_z) + \mathbf{k}(Q_x - P_y).\end{aligned}$$

Hence,

$$\begin{aligned}\mathrm{div}(\mathrm{curl}(\mathbf{F})) &= \partial_x(R_y - Q_z) - \partial_y(R_x - P_z) + \partial_z(Q_x - P_y) \\ &= R_{xy} - Q_{xz} - R_{yx} + P_{yz} + Q_{zx} - P_{zy} \\ &= (R_{xy} - R_{yx}) + (Q_{zx} - Q_{xz}) + (P_{yz} - P_{zy}).\end{aligned}$$

By *Clairaut's theorem*, the second-order partial derivatives of a smooth function commute, this proves the result. □

Definition 2.2.8. A vector field $\mathbf{F}$ is said to be *solenoidal* if there is a vector field $\mathbf{G}$ such that $\mathbf{F} = \mathrm{curl}(\mathbf{G})$.

Remark 2.2.9. *Theorem 2.2.7* asserts that every solenoidal vector field is incompressible.

Example 2.2.10. Show that the vector field $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ cannot be written as the curl of a vector field $\mathbf{G}$.

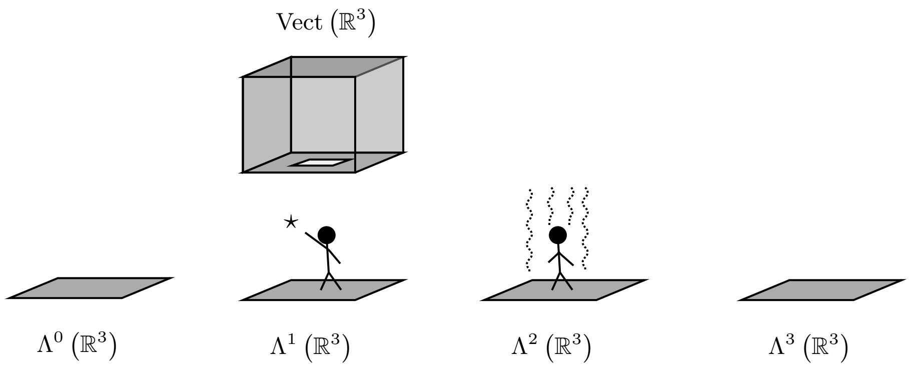
SOLUTION. If $\mathbf{F} = \mathrm{curl}(\mathbf{G})$, then $\mathrm{div}(\mathbf{F}) = \mathrm{div}(\mathrm{curl}(\mathbf{G}))$, which must be zero. Computing the divergence of $\mathbf{F}$, we have

$$\begin{aligned}\mathrm{div}(\mathbf{F}) &= \partial_x(xz) + \partial_y(xyz) + \partial_z(-y^2) \\ &= z + xz,\end{aligned}$$

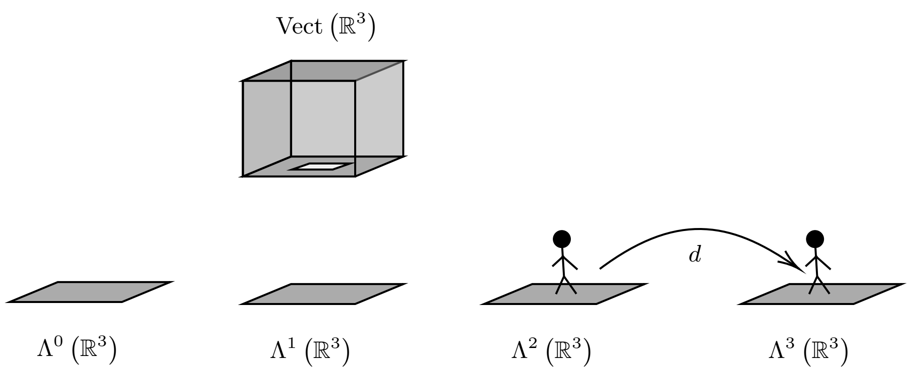
which is not zero. □

Barry explains: The Divergence. The divergence of a vector field is a scalar, so we need to land on platform $\Lambda^0(\mathbb{R}^3)$.

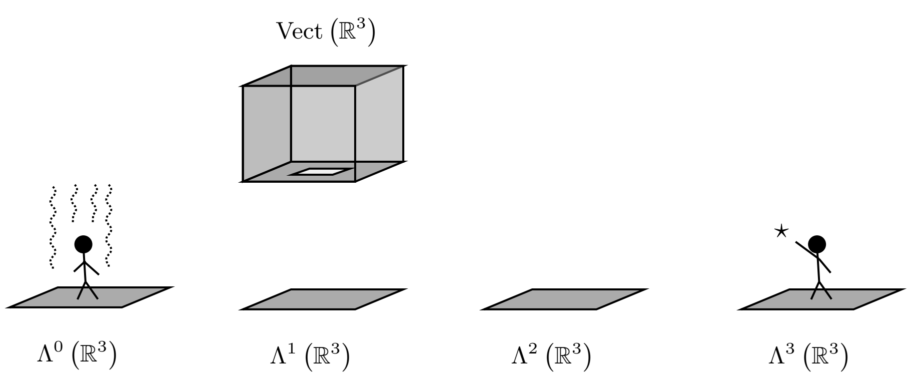
(i) If we apply the exterior derivative, then all we have left is the Hodge $\star$–operator, which traps Barry between the $\Lambda^1(\mathbb{R}^3)$–platform and the $\Lambda^2(\mathbb{R}^3)$–platform. So we have to first teleport using the Hodge $\star$–operator:



(ii) Applying the Hodge $\star$–operator again would take us back to where we started, so we need to use the exterior derivative, landing us on platform $\Lambda^3(\mathbb{R}^3)$:



(iii) Now we can teleport using the Hodge $\star$–operator to land us on the desired $\Lambda^0(\mathbb{R}^3)$ (or functions) platform.



Hence, we see that

$$\boxed{\operatorname{div}(\mathbf{F}) \; = \star d \star \omega_{\mathbf{F}}}$$

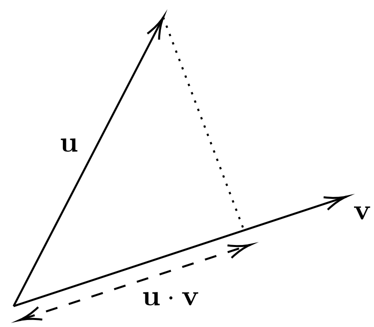
Reminder: The dot product. Recall that for two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, the *dot product* is defined

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example 2.2.11. Let $u = (1, -1, 2)$ and $v = (-3, 2, 6)$. Then

$$u \cdot v = 1(-3) + (-1)(2) + 2(6) = -3 - 2 + 12 = 7.$$

Remark: Meaning of the Dot Product. Suppose **u** and **v** are two vectors in $\mathbb{R}^n$. For the purposes of clarity, we take $n = 2$. The dot product $\mathbf{u} \cdot \mathbf{v}$ is the magnitude of the shadow cast over **v** by the vector **u**:



In other words, the dot product $\mathbf{u} \cdot \mathbf{v}$ measures the extent to which **u** resides over **v**.
From this, one achieves the following important property:

Properties of the dot product. Let **u** and **v** be two vectors in $\mathbb{R}^n$.

(i) If $\mathbf{u} \cdot \mathbf{v} = 0$, then **u** and **v** are orthogonal.
(ii) If $\mathbf{u} \cdot \mathbf{v} = 1$, then **u** and **v** are parallel.

Further directions: Holomorphic Functions. A rich source of examples of incompressible and irrotational vector fields come from complex analysis.

Definition. A function $f : \mathbb{C} \to \mathbb{C}$ is said to be holomorphic at $z_0 \in \mathbb{C}$ if

$$\lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists and is finite.

Example. Show that the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = z^2$$

is holomorphic.

SOLUTION. We compute

$$\begin{aligned} \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} &= \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{z_0^2 + 2z_0 \Delta z + (\Delta z)^2 - z_0^2}{\Delta z} \\ &= \lim_{\Delta z \to 0} 2z_0 + \Delta z \;=\; 2z_0. \end{aligned}$$

□

Theorem. A function $f : \mathbb{C} \to \mathbb{C}$, $f(z) = u(x, y) + \sqrt{-1}v(x, y)$ is holomorphic at $z_0 \in \mathbb{C}$ if and only if the *Cauchy–Riemann equations*

$$u_x = v_y, \qquad v_x = -u_y$$

hold at z_0.

PROOF. □

Example. Show that the function $f : \mathbb{C} \to \mathbb{C}$ defined by

$$f(z) = \overline{z}$$

is not holomorphic.

Proposition. The Cauchy–Riemann equations are equivalent to

$$\frac{\partial f}{\partial \bar{z}} \;:=\; \frac{1}{2}\left(\frac{\partial f}{\partial x} + \sqrt{-1}\frac{\partial f}{\partial y}\right) \;=\; 0.$$

Theorem. There is a one-to-one correspondence between holomorphic functions $f : \mathbb{C} \to \mathbb{C}$ and irrotational and incompressible vector fields $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ given by declaring $P = f$ and $Q = \sqrt{-1}f$.

Exercises

1. Compute the dot products of the following pairs of vectors:

(i) $u = \mathbf{i} - \mathbf{j}$ and $v = \mathbf{i} + 2\mathbf{j}$.

(ii) $u = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $v = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $f(x, y, z) := 2x^3y^2z^4$.

(i) Compute $\nabla \cdot \nabla f$.

(ii) Show that $(\nabla^2 - \nabla \cdot \nabla)f = 0$, where $\nabla^2 := \partial_x^2 + \partial_y^2 + \partial_z^2$ is the *Laplace operator*.

3. Determine the constant $\alpha \in \mathbb{R}$ such that the vector field

$$\mathbf{F}(x, y, z) := (x + 3y)\mathbf{i} + (y - 2z)\mathbf{j} + (x + \alpha z)\mathbf{k}$$

in incompressible, i.e., $\mathrm{div}(\mathbf{F}) = 0$.

4. Show that

(i) (additivity) $\mathrm{div}(\mathbf{F} + \mathbf{G}) = \mathrm{div}(\mathbf{F}) + \mathrm{div}(\mathbf{G})$.

(ii) (Leibniz rule) $\mathrm{div}(f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f\mathrm{div}(\mathbf{F})$.

5. Let $\mathbf{F}(x, y) = 2x^2\mathbf{i} + \sin(y)\mathbf{j} + zy\mathbf{k}$.

(i) Compute $\mathrm{div}(\mathbf{F})$.

(ii) Write the 1–form $\omega_{\mathbf{F}}$ associated to $\mathbf{F}$.

(iii) Compute the exterior derivative of $\alpha = 2x^2 dy \wedge dz - \sin(y)dx \wedge dz + zydx \wedge dy$.

(iv) How does the answer of (i) compare with that of (iii)?

6. Let $\mathbf{F}(x, y) = (1 - e^{-y})\mathbf{i} + (x^2 + y^2)\mathbf{j} + z^2\mathbf{k}$.

(i) Compute $\mathrm{div}(\mathbf{F})$.

(ii) Write the 1–form $\omega_{\mathbf{F}}$ associated to $\mathbf{F}$.

(iii) Compute the exterior derivative of $\alpha = (1-e^{-y})dy\wedge dz-(x^2+y^2)dx\wedge dz+z^2dx\wedge dy$.

(iv) How does the answer of (i) compare with that of (iii)?

7. Let $\mathbf{F}(x, y, z) = (1 - x^2)\mathbf{i} + (1 + y^2)\mathbf{j} + z\mathbf{k}$.

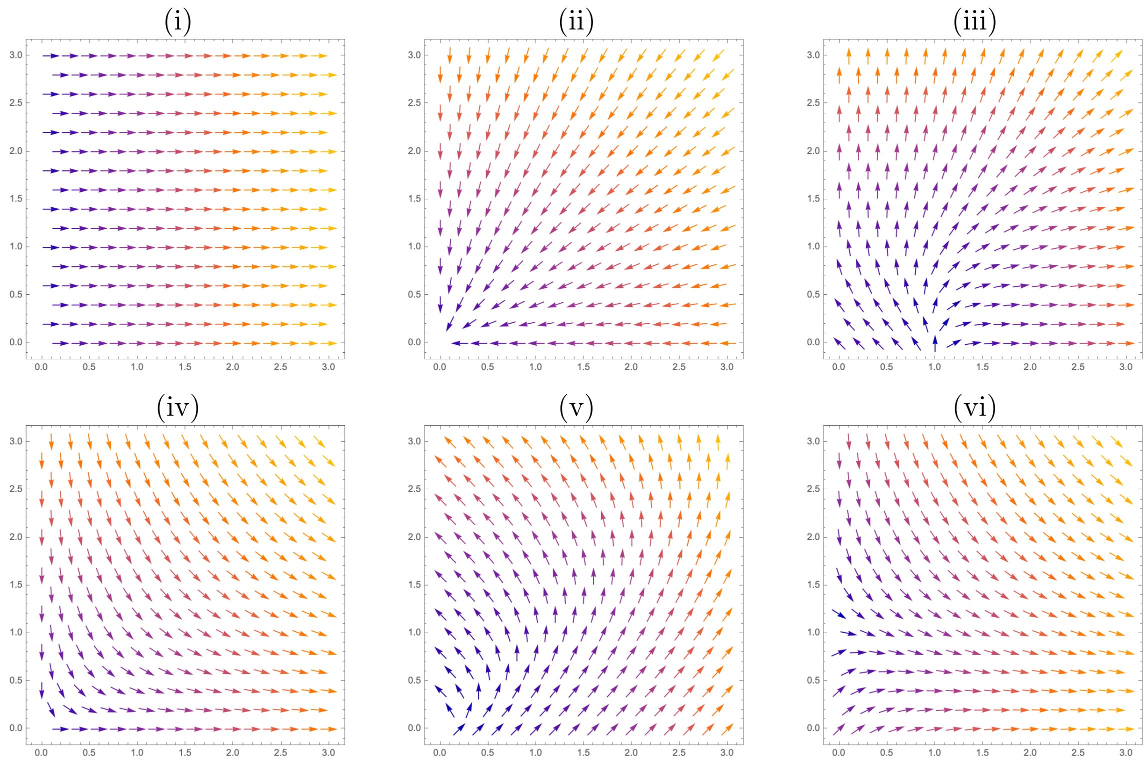
(i) Compute $\mathrm{curl}(\mathbf{F})$.

(ii) Compute $\mathrm{div}(\mathbf{F})$.

(iii) Compute $\mathrm{div}(\mathrm{curl}(\mathbf{F}))$.

(iv) Does one need to do the calculation explicitly in (iii) in order to know the result? Explain.

8. Determine whether the following vector fields have positive, negative, or zero divergence:



9. Show that the divergence is a linear operator, i.e., for any vector fields $\mathbf{F}, \mathbf{G}$, and any real numbers α, β, show that

$$\mathrm{div}(\alpha\mathbf{F} + \beta\mathbf{G}) = \alpha\mathrm{div}(\mathbf{F}) + \beta\mathrm{div}(\mathbf{G}).$$

10. Compute the divergence and curl of the vector field

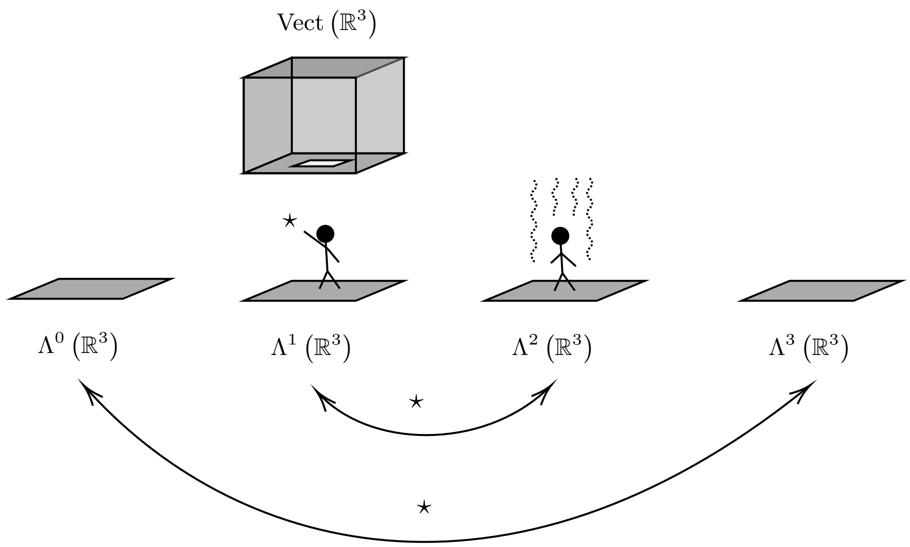
$$\mathbf{F}(x, y, z) = x^z\mathbf{i} + 2x\sin(y)\mathbf{j} + 2z\cos(y)\mathbf{k}.$$

11. Show that there is no vector field $\mathbf{F}$ such that

$$\mathrm{curl}(\mathbf{F}) = 2x\mathbf{i} + 3yz\mathbf{j} - xz^2\mathbf{k}.$$

2.3. The Hodge–$\star$ operator

In the previous sections, we have seen that the Hodge $\star$–operator (which has yet to be defined) has the property that it sends k-forms on $\mathbb{R}^3$ to $(3-k)$–forms on $\mathbb{R}^3$. In the Barry's world pictorial representation, this was represented by



The purpose of the present section is give to precise formulas for the Hodge $\star$–operator and, of course, give a definition.

Definition 2.3.1. (Hodge $\star$–operator). The Hodge $\star$–operator is the linear map

$$\star : \Lambda^k(\mathbb{R}^3) \to \Lambda^{3-k}(\mathbb{R}^3)$$

which sends a k–form $\alpha \in \Lambda^k(\mathbb{R}^3)$ to the $(3-k)$–form $\star\alpha$ which satisfies

$$\alpha \wedge (\star\alpha) \;=\; dx \wedge dy \wedge dz.$$

Example 2.3.2. Consider the 1–form $dx \in \Lambda^1(\mathbb{R}^3)$. Then $\star dx \in \Lambda^2(\mathbb{R}^3)$ is the 2–form such that

$$dx \wedge (\star dx) = dx \wedge dy \wedge dz.$$

In particular, we see that

$$\star dx = dy \wedge dz.$$

Example 2.3.3. Consider the 1–form $dy \in \Lambda^1(\mathbb{R}^3)$. Then $\star dy \in \Lambda^2(\mathbb{R}^3)$ is the 2–form such that

$$dy \wedge (\star dy) = dx \wedge dy \wedge dz.$$

In this case, we see that

$$dx \wedge dy \wedge dz \;=\; -dy \wedge dx \wedge dz \implies \star dy = -dx \wedge dz.$$

The remaining properties of the Hodge $\star$–operator are indicated in the following definition:

Definition 2.3.4. (Hodge $\star$–operator). The Hodge $\star$–operator is the linear map $\star : \Lambda^k(\mathbb{R}^3) \to \Lambda^{3-k}(\mathbb{R}^3)$ which maps a k–form on $\mathbb{R}^3$ to a form of type $(3-k)$, i.e., the Hodge $\star$–operator maps

(†) functions (i.e., 0–forms) to 3–forms:

$$\star f := f dx \wedge dy \wedge dz.$$

(†) 1–forms to 2–forms

$$\star(Pdx + Qdy + Rdz) = Pdy \wedge dz - Qdx \wedge dz + Qdx \wedge dy.$$

(†) 2–forms to 1–forms

$$\begin{aligned} \star(f dx \wedge dy) &= f dz \\ \star(f dy \wedge dz) &= f dx \\ \star(f dx \wedge dz) &= -f dy. \end{aligned}$$

(†) 3–forms to functions (i.e., 0–forms):

$$\star(f dx \wedge dy \wedge dz) = f.$$

Hodge $\star$–operator diagram. A more concise version of the Barry's world pictorial representation is given here:

$$\begin{tikzcd}
& \mathrm{Vect}\left(\mathbb{R}^3\right) \arrow[d, <->] & & \\
\Lambda^0\left(\mathbb{R}^3\right) \arrow[r, "d"] \arrow[rrr, <->, bend right, "\star"'] & \Lambda^1\left(\mathbb{R}^3\right) \arrow[r, "d"] \arrow[r, <->, bend right=60, "\star"'] & \Lambda^2\left(\mathbb{R}^3\right) \arrow[r, "d"] & \Lambda^3\left(\mathbb{R}^3\right)
\end{tikzcd}$$

Example 2.3.5. Compute $\star(2x^3dx + xydy + \sin(xy)dz)$.

Solution. We have

$$\begin{aligned}
\star(2x^3dx + xydy + \sin(xy)dz) &= 2x^3(\star dx) + xy(\star dy) + \sin(xy)(\star dz) \\
&= 2x^3dy \wedge dz - xydx \wedge dz + \sin(xy)dx \wedge dy.
\end{aligned}$$

□

Example 2.3.6. Compute

$$\star(xydx \wedge dy - z\cos(y)dx \wedge dz + e^{-z}dy \wedge dz).$$

Solution. We know that $\star(dx \wedge dy) = dz$, and $\star(dx \wedge dz) = -dy$, and $\star(dy \wedge dz) = dx$. Hence,

$$\star(xydx \wedge dy - z\cos(y)dx \wedge dz + e^{-z}dy \wedge dz) = xydz + z\cos(y)dy + e^{-z}dx$$

□

Example 2.3.7. Determine the vector field associated to

$$\star(2x^3dx \wedge dy + \log_e(x+y)dx \wedge dz).$$

Solution. We know that $\star(dx \wedge dy) = dz$ and $\star(dx \wedge dz) = -dy$. Hence,

$$\star(2x^3dx \wedge dy + \log_e(x+y)dx \wedge dz) = 2x^3dz - \log_e(x+y)dy.$$

Since this is a 1–form, we can associate to it a vector field

$$\mathbf{F} = 0\mathbf{i} - \log_e(x+y)\mathbf{j} + 2x^3\mathbf{k}.$$

□

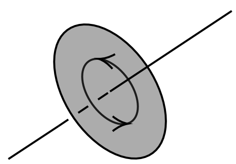
Divergence and Curl as derivatives. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field on $\mathbb{R}^3$. Let us write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ for smooth functions $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$. We have seen that the curl of $\mathbf{F}$ is the *vector field*

$$\operatorname{curl}(\mathbf{F}) \quad = \quad \nabla \times \mathbf{F}.$$

The divergence, on the other hand, is the (scalar-valued) *function*

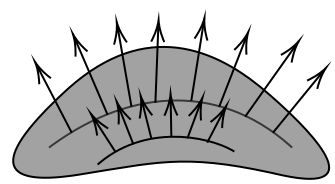
$$\operatorname{div}(\mathbf{F}) \quad = \quad \nabla \cdot \mathbf{F}.$$

I claim that both the curl and the divergence are notions of *derivatives* of the vector field $\mathbf{F}$.



curl measures the
variation
of a vector field
in two dimensions

$\operatorname{curl}(\mathbf{F}) = \star d\omega_{\mathbf{F}}$



divergence measures the
variation
of a vector field
in three dimensions

$\operatorname{div}(\mathbf{F}) = \star d \star \omega_{\mathbf{F}}$

We know that vector fields can be identified with 1–forms (and vice versa), so let $\omega_{\mathbf{F}}$ be the 1–form associated to $\mathbf{F}$. Applying the exterior derivative to $\omega_{\mathbf{F}}$ produces the 2–form $d\omega_{\mathbf{F}}$. Since 2–forms are not associated to vector fields (only 1–forms are), we would like an operator which allows us to change the type of the form, while maintaining the information it contains. This is achieved via the Hodge $\star$–operator:

Theorem 2.3.8. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with associated 1–form ω. The curl of $\mathbf{F}$ is the vector field associated to the 1–form $\star d\omega$.

Proof. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field given by $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. The associated 1–form is $\omega = Pdx + Qdy + Rdz$. Hence,

$$\begin{aligned}
d\omega &= d(P)dx + d(Q)dy + d(R)dz \\
&= (P_x dx + P_y dy + P_z dz) \wedge dx + (Q_x dx + Q_y dy + Q_z dz) \wedge dy \\
&\qquad\qquad + (R_x dx + R_y dy + R_z dz) \wedge dz \\
&= P_y dy \wedge dx + P_z dz \wedge dx + Q_x dx \wedge dy + Q_z dz \wedge dy + R_x dx \wedge dz + R_y dy \wedge dz \\
&= (Q_x - P_y) dx \wedge dy + (R_x - P_z) dx \wedge dz + (R_y - Q_z) dy \wedge dz.
\end{aligned}$$

The Hodge $\star$–operator applied to $d\omega$ is then

$$\begin{aligned}\star d\omega &= (Q_x - P_y)(\star dx \wedge dy) + (R_x - P_z)(\star dx \wedge dz) + (R_y - Q_z)(\star dy \wedge dz)\\ &= (Q_x - P_y)dz - (R_x - P_z)dy + (R_y - Q_z)dx.\end{aligned}$$

In particular, $\star d\omega$ is the 1–form associated to curl($\mathbf{F}$). □

Let us now give an alternative proof of the fact that we saw previously; namely, that div(curl($\mathbf{F}$)) = 0, using our knowledge of the Hodge $\star$–operator:

Theorem. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then

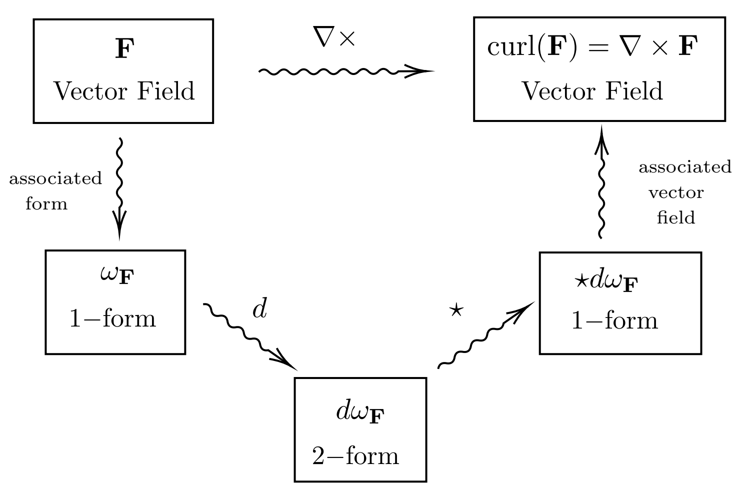
$$\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0.$$

PROOF. Let $\omega_{\mathbf{F}}$ be the 1–form associated to $\mathbf{F}$. Then

$$\begin{aligned}\operatorname{div}(\operatorname{curl}(\mathbf{F})) &= \star d \star (\star d\omega_{\mathbf{F}})\\ &= \star d \star^2 d\omega_{\mathbf{F}}\\ &= \star d(d\omega_{\mathbf{F}}) \;=\; 0,\end{aligned}$$

where the second equality makes use of $\star^2 = \mathrm{id}$, and the third equality makes use $d^2 = 0$. □

Summary of the above proof. A summary of the above proof is given by the diagram:



Essential properties of the Hodge $\star$–operator.

$$\begin{aligned}\star dx &= dy \wedge dz & \qquad \star(dx \wedge dy) &= dz\\ \star dy &= -dx \wedge dz & \qquad \star(dy \wedge dz) &= dx\\ \star dz &= dx \wedge dy & \qquad \star(dx \wedge dz) &= -dy.\end{aligned}$$

$$\begin{aligned}\star(1) &= dx \wedge dy \wedge dz \\ 1 &= \star(dx \wedge dy \wedge dz).\end{aligned}$$

Summary of the derivatives in terms of forms. If $\mathbf{F}$ is a vector field with associated 1–form $\omega_{\mathbf{F}}$, then the two notions of derivative are summarized here:

notion of derivative	vector field language	form language	type
curl	$\nabla \times \mathbf{F}$	$\star d\omega_{\mathbf{F}}$	vector field
divergence	$\nabla \cdot \mathbf{F}$	$\star d \star \omega_{\mathbf{F}}$	function

Aside: The Laplace–Beltrami Operator. One of the most important uses of the Hodge $\star$–operator is in its use to define the *codifferential* $d^* : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$.

Definition. Let Ω be a domain in $\mathbb{R}^n$. The *codifferential* $d^* : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ is defined by the formula

$$d^* \ := \ (-1)^{n(k-1)+1} \star d\star,$$

where d is the exterior derivative $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$, and $\star : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ is the Hodge $\star$–operator.

Definition. The *Laplace–Beltrami operator* $\Delta_d : \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ is defined by

$$\Delta_d := d^*d + dd^*.$$

Definition. A k–form $\omega \in \Lambda^k(\Omega)$ is said to be *harmonic* if

$$\Delta_d\omega = 0.$$

That is, the kernel of $\Delta_d : \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ defines the space of harmonic k–forms.

Exercises

1. Compute the Hodge $\star$–operator of the following forms on $\mathbb{R}^3$:

(i) $dx + dz$.
(ii) $xdy + \sin^3(y)dx$.
(iii) 4.
(iv) $7x^9 dx \wedge dy \wedge dz$.
(v) $x^3 dx + (x-y)dy + (x-z)dz$.
(vi) $zdx \wedge dy - xdy \wedge dz + ydx \wedge dz$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function given by

$$f(x, y, z) = xe^{-y} + z\cos(y).$$

(i) Compute df.
(ii) Compute $\star df$.

3. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = (2x^3 - y^2)\mathbf{i} + z^4 \sin(x)\mathbf{j} + 5z\mathbf{k}.$$

(i) Compute $d\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $\star d\omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{curl}(\mathbf{F})$ using part (ii).

4. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = y\cos(x)\mathbf{i} + e^{-z}\mathbf{j} + 2\sin(y)\mathbf{k}.$$

(i) Compute $d\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $\star d\omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{curl}(\mathbf{F})$ using part (ii).

5. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = e^{y-x}\mathbf{i} + (1 - xz)\mathbf{j} + 2e^z\mathbf{k}.$$

(i) Compute $d\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $\star d\omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{curl}(\mathbf{F})$ using part (ii).

6. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = \sin(xy)\mathbf{i} + \cos(xz)\mathbf{j} + \sin(yz)\mathbf{k}.$$

(i) Compute $\star\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $d \star \omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{div}(\mathbf{F})$ using part (ii).

7. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = 2z^9\mathbf{i} + 4xy^4\mathbf{j} + 10\mathbf{k}.$$

(i) Compute $\star\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $d \star \omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{div}(\mathbf{F})$ using part (ii).

8. Determine (with justification) whether the following are true or false:

(i) The curl of a vector field is the vector field associated to the 1–form $\star d \star \omega_{\mathbf{F}}$.
(ii) The curl of a vector field is the vector field associated to the 1–form $\star d\omega_{\mathbf{F}}$.
(iii) The Hodge $\star$–operator satisfies $\star^2 = 1$.
(iv) The divergence of a vector field is given by $\star d \star \omega_{\mathbf{F}}$.
(v) For any smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, we have $d(\star f) = 0$.

9. Determine (with justification) whether the following are true or false:

(i) If f is a smooth function, then f is a 0–form.
(ii) If $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function, then df is a 2–form.
(iii) If $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, then $\star f$ is a 3–form.
(iv) If ω is a 2–form on $\mathbb{R}^3$, then $\star\omega$ can be associated to a vector field.
(v) If α is a k–form, then $\star(\star\alpha)$ is a k–form.
(vi) If α is a k–form, and β is a $(k+1)$–form, then $\alpha \wedge \beta$ is a $(k+2)$–form.

10. Determine (with justification) whether the following are true or false:

(i) If ω is a 2–form, then $\star\omega$ is a 1–form.
(ii) If η is a 1–form, then $\star\eta$ is a 1–form.
(iii) If ω is a 1–form, then $\star d \star d\omega$ is a 1–form.
(iv) If ω is a 2–form, then $\star d \star d \star \omega$ is a 1–form.

11. Determine which of the following are scalars (**S**), vector fields (**VF**), 1–forms (**1**), 2–forms (**2**), 3–forms (**3**):

(i) $\mathrm{curl}(\mathbf{F}) \cdot \mathbf{k}$.
(ii) $\mathrm{curl}(\mathbf{F})$.
(iii) $\mathrm{div}(\mathbf{F})$.

(iv) df.

(v) $d\omega$, where ω is a 1–form.

(vi) $x^2 dx \wedge dy \wedge dz$.

(vii) $dx \wedge dx$.

(viii) $\omega_{\mathbf{F}}$.

(ix) $\star d\omega_{\mathbf{F}}$.

(x) $\star d \star \omega_{\mathbf{F}}$.

CHAPTER 3

Integration Theory – Curves

"Everything useful in mathematics has been devised for a purpose. Even if you don't know it, the guy who did it first, he knew what he was doing. Banach didn't just develop Banach spaces for the sake of it. He wanted to put many spaces under one heading. Without knowing the examples, the whole thing is pointless."

– Sir. Michael Atiyah

The integration theory of vector calculus is distinct from the familiar integration theory in that we are integrating of curves $\mathcal{C}$, surfaces $\mathcal{S}$, and solid regions $\mathcal{V}$. The fundamental theorem of calculus (FTC) tells us that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

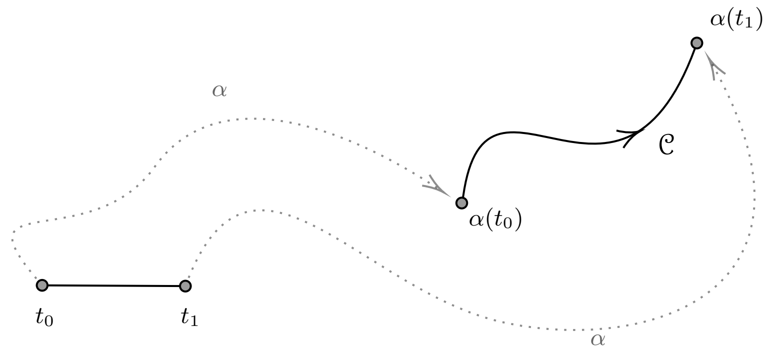
The purpose of the present chapter is to discuss the extension (or better, different incarnation) of the FTC vector fields (or really, 1–forms) on curves. For curves $\mathcal{C}$, the FTC will be the *fundamental theorem of line integrals*

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(b) - f(a).$$

This will then allow us to explore the notion of *path independence*, one of the fundamental properties of gradient fields.

3.1. Line Integrals

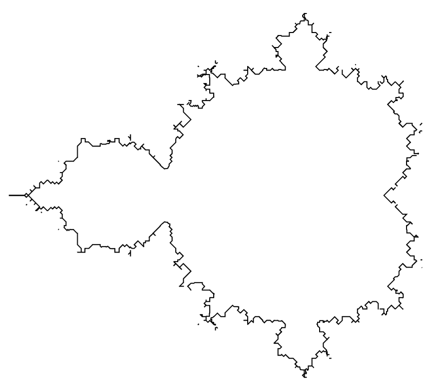
Definition 3.1.1. A *curve* $\mathcal{C} \subset \mathbb{R}^n$ is (locally) the image of a continuous map (which we call a *parametrization* for $\mathcal{C}$) $\alpha : [t_0, t_1] \to \mathbb{R}^n$, with $t_0, t_1 \in \mathbb{R} \cup \{-\infty, \infty\}$. A curve is said to be *smooth* if it affords a smooth parametrization. .



A smooth parametrization α of a curve $\mathcal{C}$.

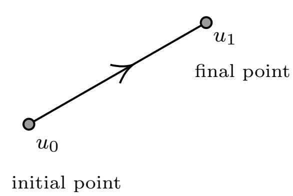
Example 3.1.2. The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t, |t|)$ is continuous, but is not smooth.

Example 3.1.3. The Mandelbrot set is a highly non-smooth curve:



A highly non-smooth curve.

Formula for the parametrization of a line. The formula for parametrizing a line between two points u_0 and u_1 (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is $\alpha(t) = (1-t)u_0 + tu_1$, where $0 \le t \le 1$.



$$\alpha(t) = (1-t)u_0 + tu_1$$

Example 3.1.4. Find a parametrization for the line passing between the points $(0, 1)$ and $(2, 3)$.

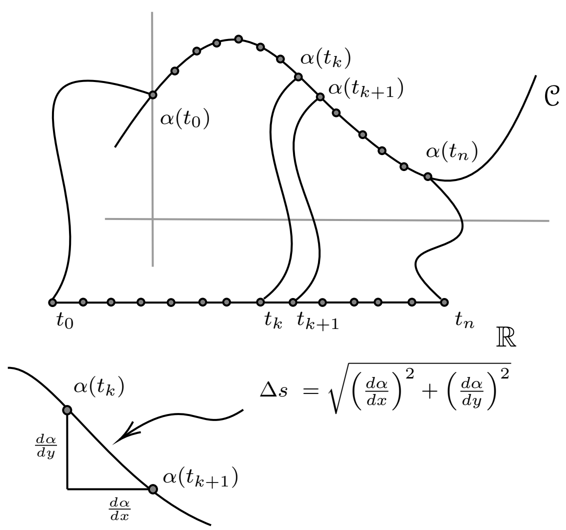
SOLUTION. The above formula tells us that $\alpha : [0,1] \to \mathbb{R}^2$ given by

$$\begin{aligned}\alpha(t) &= (1-t)(0,1) + t(2,3) \\ &= (0, 1-t) + (2t, 3t) \\ &= 2t\mathbf{i} + (1-t+3t)\mathbf{j} \;=\; 2t\mathbf{i} + (1+2t)\mathbf{j}\end{aligned}$$

yields the desired parametrization. □

Definition 3.1.5. Let $\mathcal{C}$ be a curve in $\mathbb{R}^3$. Let $\alpha(t) : [t_0, t_1] \to \mathbb{R}^3$ be a parametrization for $\mathcal{C}$. The *arc length* of $\mathcal{C}$ is given by

$$\int_{t_0}^{t_1} |\dot{\alpha}(t)| dt \;=\; \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$



Example 3.1.6. Let $\mathcal{C}$ be the curve given by the straight line connecting the points $(1, -2, 4)$ and $(0, 5, -1)$. Find a parametrization of $\mathcal{C}$ and compute its arc length.

SOLUTION. We want a map $\alpha : [0,1] \to \mathbb{R}^3$ which is linear, and satisfies $\alpha(0) = (1, -2, 4)$ and $\alpha(1) = (0, 5, -1)$. The desired path is given by

$$\alpha(t) = (1-t) \cdot (1, -2, 4) + t \cdot (0, 5, -1) = (1-t)\mathbf{i} + (-2+7t)\mathbf{j} + (4-5t)\mathbf{k}.$$

To compute its arc length, we note that

$$\alpha'(t) \;=\; -\mathbf{i} + 7\mathbf{j} - 5\mathbf{k},$$

and therefore,

$$|\alpha'(t)| \;=\; \sqrt{(-1)^2 + (7)^2 + (-5)^2} \;=\; \sqrt{75}.$$

Hence, the arc length of $\mathcal{C}$ is given by

$$\int_0^1 \sqrt{75}dt \quad = \quad \sqrt{75}.$$

$\square$

Example 3.1.7. Let $\mathcal{C}$ be the curve given by the unit circle in $\mathbb{R}^2$, centered at the origin. Find a parametrization of $\mathcal{C}$ and compute its arc length.

Solution. A parametrization of the unit circle is given by $\alpha : [0, 2\pi] \to \mathbb{R}^2$, where $\alpha(t) = (\cos(t), \sin(t))$. Here, $\dot{\alpha}(t) = (-\sin(t), \cos(t))$, and

$$|\dot{\alpha}(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = \sqrt{\sin^2(t) + \cos^2(t)} = 1.$$

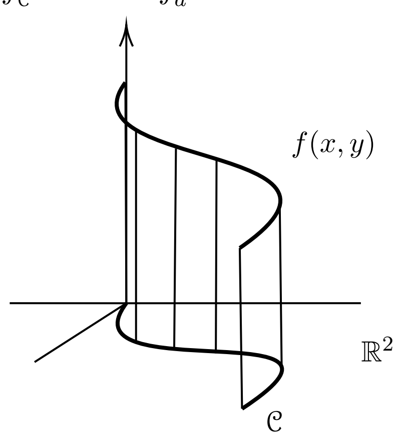
Hence, the arc length of $\mathcal{C}$ is given by

$$\int_0^{2\pi} dt \quad = \quad 2\pi.$$

$\square$

Definition 3.1.8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and let $\mathcal{C}$ be a smooth curve. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a smooth function parametrizing $\mathcal{C}$. The *line integral of f along* $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} f ds \quad = \quad \int_a^b f(\alpha(t))|\alpha'(t)|dt.$$



Evaluating line integrals of functions. Let f be a smooth function and $\mathcal{C}$ a curve. The steps to evaluating a line integral are as follows:

(1) Parametrize the curve $\mathcal{C}$ by a smooth function $\alpha : [t_0, t_1] \to \mathbb{R}^n$.
(2) Compute the derivative $\alpha'(t)$ of the parametrization.
(3) Compute the norm $|\alpha'(t)|$.
(4) Express the function f in terms of the parametrization $\alpha(t)$.

(5) Evaluate the integral $\int_{t_0}^{t_1} f(\alpha(t)) \cdot |\alpha'(t)| dt$.

Example 3.1.9. Evaluate the line integral

$$\int_{\mathcal{C}} f ds,$$

where $f(x,y) = x$ and $\mathcal{C}$ is the arc of the parabola $y = x^2$ from $(1,1)$ to $(2,4)$.

SOLUTION. Parametrize the curve $\mathcal{C}$ by $\alpha : [1,2] \to \mathbb{R}^2$, given by $\alpha(t) = t\mathbf{i} + t^2\mathbf{j}$. Then $\alpha'(t) = \mathbf{i} + 2t\mathbf{j}$ and hence,

$$|\alpha'(t)| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}.$$

Now,

$$\int_{\mathcal{C}} f ds = \int_1^2 t\sqrt{1 + 4t^2} dt = \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right).$$

□

Example 3.1.10. Find the mass of the wire in the shape of an arc given by $x^2 + y^2 = 1$ for $x \geq 0$ and $y \geq 0$, where the density function is $\rho(x,y) = x^2 + y$.

SOLUTION. The mass of the wire is given by the line integral of the density function. Hence, we need to evaluate

$$\int_{\mathcal{C}} \rho ds,$$

where $\mathcal{C}$ is the curve describing the wire.
Parametrize the wire using the function $\alpha : [0, \pi/2] \to \mathbb{R}^2$ given by $\alpha(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$. Then $\alpha'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$ and hence,

$$|\alpha'(t)| = \sqrt{\sin^2(t) + \cos^2(t)} = 1.$$

With respect to this parametrization,

$$\rho = \cos^2(t) + \sin(t),$$

and therefore, the mass of the wire is

$$\begin{aligned}
\int_{\mathcal{C}} \rho ds &= \int_0^{\pi/2} (\cos^2(t) + \sin(t)) dt \\
&= \int_0^{\pi/2} \left(\frac{1}{2}(\cos(2t) + 1) + \sin(t)\right) dt \\
&= \left[\frac{1}{4}\sin(2t) + \frac{1}{2}t - \cos(t)\right]_0^{\pi/2} \\
&= \frac{\pi}{4} + 1.
\end{aligned}$$

□

It becomes simpler to evaluate line integrals of 1–forms:

Example 3.1.11. Let $\mathcal{C}$ be the curve given by $\alpha : [0,1] \to \mathbb{R}^2$ be the curve parametrized by $\alpha(t) = t^2\mathbf{i} + t^3\mathbf{j}$. Evaluate the line integral

$$\int_{\mathcal{C}} \omega,$$

where $\omega = 2xdx + xydy$.

Solution. The parametrization $\alpha(t) = t^2\mathbf{i} + t^3\mathbf{j}$ informs us that $x = t^2$ and $y = t^3$. Hence, $dx = 2tdt$ and $dy = 3t^2dy$. The 1–form ω then reads

$$\begin{aligned} \omega \;=\; 2xdx + xydy \;&=\; 2(t^2)(2tdt) + (t^2)(t^3)(3t^2dt) \\ &=\; 4t^3dt + 3t^7dt \;=\; (4t^3 + 3t^7)dt. \end{aligned}$$

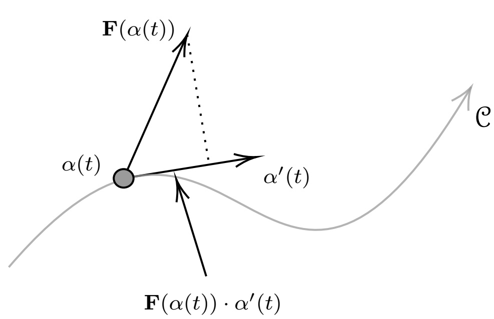
Integrating gives

$$\int_{\mathcal{C}} \omega \;=\; \int_0^1 (4t^3 + 3t^7)dt \;=\; \frac{11}{8}.$$

□

Definition 3.1.12. Let $\mathbf{F}$ be a smooth vector field on a smooth curve $\mathcal{C}$. Let $\alpha : [a,b] \to \mathbb{R}^n$ be a smooth function parametrizing $\mathcal{C}$. The *line integral of $\boldsymbol{F}$ along* $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_a^b \mathbf{F}(\alpha(t)) \cdot \alpha'(t)dt.$$



$\mathbf{F}(\alpha(t)) \cdot \alpha'(t)$ expresses the *amount of $\boldsymbol{F}$ which lies over* $\mathcal{C}$

Evaluating line integrals of vector fields. Let $\mathbf{F}$ be a vector field and $\mathcal{C}$ a curve. The steps to evaluating a line integral are as follows:

(1) Parametrize the curve $\mathcal{C}$ by a smooth function $\alpha : [t_0, t_1] \to \mathbb{R}^n$.
(2) Compute the derivative $\alpha'(t)$ of the parametrization.
(3) Express the vector field $\mathbf{F}$ in terms of the parametrization $\alpha(t)$.
(4) Compute the dot product $\mathbf{F}(\alpha(t)) \cdot \alpha'(t)$, producing a function of t.
(5) Evaluate the integral $\int_{t_0}^{t_1} \mathbf{F}(\alpha(t)) \cdot \alpha'(t) dt$.

Example 3.1.13. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) := z\mathbf{i} + x\mathbf{j} + y\mathbf{k}.$$

Let $\mathcal{C}$ be the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^3$, where $\alpha(t) := (t, t^2, 3)$. Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Solution. The velocity of the parametrization is $\alpha'(t) = (1, 2t, 0)$. We also note that $x(t) = t$, $y(t) = t^2$ and $z(t) = 3$. Hence,

$$\mathbf{F}(x, y, z) = 3\mathbf{i} + t\mathbf{j} + t^2\mathbf{k} = (3, t, t^2),$$

and subsequently,

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 (3, t, t^2) \cdot (1, 2t, 0) dt \\ &= \int_0^1 \left[3 + 2t^2 + 0\right] dt \\ &= \int_0^1 (3 + 2t^2) dt = \left[3t + \frac{2}{3}t^3\right]_0^1 = 3 + \frac{2}{3} = \frac{11}{3}. \end{aligned}$$

□

Example 3.1.14. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) := y\mathbf{i} - x\mathbf{j} + \mathbf{k}.$$

Let $\mathcal{C}$ be the curve traced out by $\alpha(t) = (\cos(t), -\sin(t), t/2\pi)$, where $0 \leq t \leq 2\pi$. Compute the work done by $\mathbf{F}$ along $\mathcal{C}$.

SOLUTION. The work done by $\mathbf{F}$ along $\mathcal{C}$ is simply $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. Hence,

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin(t), -\cos(t), 1) \cdot (-\sin(t), -\cos(t), 1/2\pi) dt \\ &= \int_0^{2\pi} \left(\sin^2(t) + \cos^2(t) + \frac{1}{2\pi} \right) dt \\ &= \int_0^{2\pi} \left(1 + \frac{1}{2\pi} \right) dt \\ &= 2\pi \left(1 + \frac{1}{2\pi} \right). \end{aligned}$$

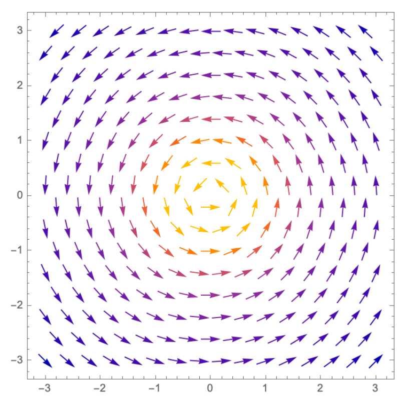
□

Example 3.1.15. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

Let $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Compute the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$



The vector field $\mathbf{F}(x, y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$.

SOLUTION. The curve $\mathcal{C}$ can be parametrized by the curve $\alpha : [0, 2\pi] \to \mathbb{R}^2$, $\alpha(t) = (\cos(t), \sin(t))$. Of course,

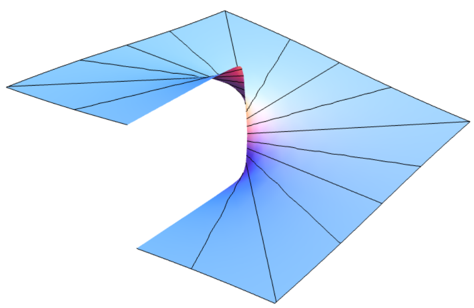
$$\alpha'(t) = (-\sin(t), \cos(t)),$$

and therefore

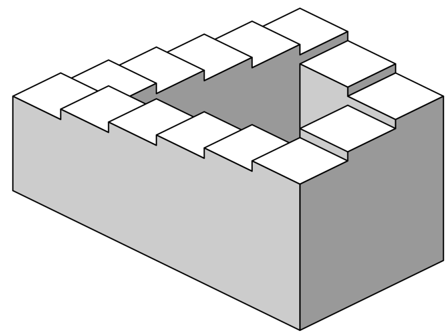
$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) dt \\ &= \int_0^{2\pi} [\sin^2(t) + \cos^2(t)] dt = \int_0^{2\pi} dt = 2\pi. \end{aligned}$$

□

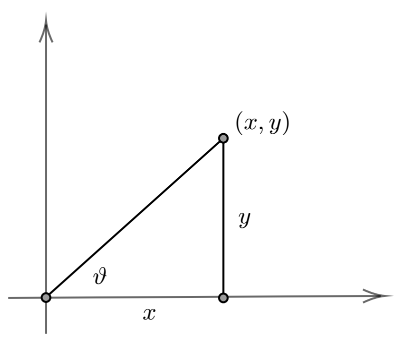
Example 3.1.16.

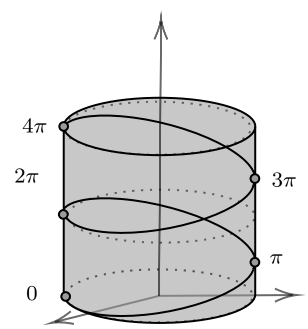


The function $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$.



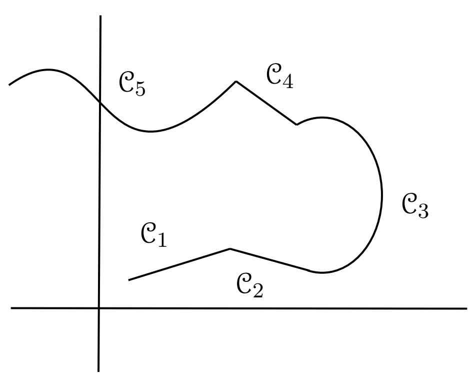
M. C. Escher's staircase gives a pictorial representation of $\tan^{-1}\left(\frac{y}{x}\right)$.





The function $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ can be interpretted as the *angle function*, which is understood to be a function *over* the plane.

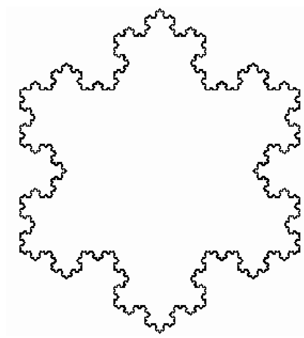
Definition 3.1.17. A curve $\mathcal{C}$ is said to be *piecewise smooth* if there is a finite number of smooth curves $\mathcal{C}_\alpha$, where $1 \leq \alpha \leq m$, such that $\mathcal{C} = \bigcup_{\alpha=1}^m \mathcal{C}_\alpha$ and the initial point of $\mathcal{C}_{\alpha+1}$ is the endpoint of $\mathcal{C}_\alpha$.



Definition 3.1.18. Let $\mathcal{C} = \bigcup_{\alpha=1}^{m} \mathcal{C}_\alpha$ be a piecewise smooth curve in $\Omega \subseteq \mathbb{R}^n$. Then, for a vector field $\mathbf{F} : \Omega \to \mathbb{R}^n$, we define

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \sum_{\alpha=1}^{m} \int_{\mathcal{C}_\alpha} \mathbf{F} \cdot d\mathbf{r}.$$

Example 3.1.19. The Koch curve is not piecewise smooth



Key Properties of Line integrals of Functions. Let $\mathcal{C}$ be a smooth curve in $\Omega \subset \mathbb{R}^n$. Some important properties of line integrals include:

(i) (Linearity). If $f, g : \Omega \to \mathbb{R}$ are smooth functions and $\lambda, \mu \in \mathbb{R}$ are constants, then

$$\int_{\mathcal{C}} (\lambda f + \mu g) ds \;=\; \lambda \int_{\mathcal{C}} f ds + \mu \int_{\mathcal{C}} g ds.$$

(ii) (Independence of reparametrization). The value of

$$\int_{\mathcal{C}} f ds$$

is independent of the choice of parametrization $\alpha : [0,1] \to \Omega$ for $\mathcal{C}$.

Indeed, let $\beta : [0,1] \to [0,1]$ be a smooth function such that $\beta(0) = 0$ and $\beta(1) = 1$. Let $\alpha : [0,1] \to \Omega$ be a smooth parametrization, and $\widetilde{\alpha} : [0,1] \to \Omega$ a reparametrization defined by $\widetilde{\alpha}(t) := \alpha(\beta(t))$. Computing with respect to the reparametrization $\widetilde{\alpha}$, we have

$$\begin{aligned}
\int_{\mathcal{C}} f ds &= \int_0^1 f(\widetilde{\alpha}(t)) |\widetilde{\alpha}'(t)| dt \\
&= \int_0^1 f(\alpha(\beta(t)) |\beta'(t) \alpha'(\beta(t))| dt \\
&= \int_0^1 f(\alpha(\tau)) |\alpha'(\tau)| d\tau \\
&= \int_{\mathcal{C}} f ds,
\end{aligned}$$

where we changed variables $\tau := \beta(t)$ in the second-last equality.

(iii) (Orientation-dependence). Let $\widehat{\alpha} : [0,1] \to \Omega$ be the parametrization of $\mathcal{C}$ given by $\widehat{\alpha}(t) := \alpha(1-t)$, i.e., traversing the curve in reverse. If $-\mathcal{C}$ is the curve parametrized by $\widehat{\alpha}$, then

$$\int_{-\mathcal{C}} f ds \quad = \quad -\int_{\mathcal{C}} f ds.$$

(iv) (Additive on paths). If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves such that the endpoint of $\mathcal{C}_1$ is the initial point of $\mathcal{C}_2$, then

$$\int_{\mathcal{C}_1+\mathcal{C}_2} f ds \quad = \quad \int_{\mathcal{C}_1} f ds + \int_{\mathcal{C}_2} f ds.$$

Remark 3.1.20. The reader should compare the above properties with the properties familiar to Riemann integration:

(i) (Linearity). If $f, g : [a,b] \to \mathbb{R}$ are smooth functions, then

$$\int_a^b (\lambda f(x) + \mu g(x))dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx$$

for all $\lambda, \mu \in \mathbb{R}$.

(ii) (Independence of reparametrization). The value of

$$\int_a^b f(x)dx$$

(iii) (Orientation-dependence).

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

(iv) (Additive on paths).

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

Exercises

1. Find a parametrization for the following curves:

(i) The line connecting $(0, 3)$ to $(-1, 4)$.
(ii) The line connecting $(-1, 4, 3)$ to $(6, 2, -3)$.
(iii) The circle centered at the origin of radius 4.
(iv) The sphere centered at the point $(0, 3, 1)$ of radius 9.

2. Evaluate the following line integrals:

(i) $\int_{\mathcal{C}} xy ds$, where $\mathcal{C}$ is the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^2$, $\alpha(t) = (t^2 - 1)\mathbf{i} + 6t\mathbf{j}$.
(ii) $\int_{\mathcal{C}} y/(x^2+1) ds$, where $\mathcal{C}$ is the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^2$, $\alpha(t) = t\mathbf{i} + t\mathbf{j}$.
(iii) $\int_{\mathcal{C}} xe^y ds$, where $\mathcal{C}$ is the line segment which connects $(0, 1)$ to $(-3, 5)$.
(iv) Let $\mathcal{C}$ consist of the line segments connecting $(0, 0)$ to $(-1, 0)$ and $(-1, 0)$ to $(3, 1)$. Compute $\int_{\mathcal{C}} x\sqrt{y + 1} dx + (1 - x)y dy$.

3. Evaluate the following line integrals $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where

(i) $\mathbf{F}(x, y) = (x - 3y^2)\mathbf{i} + y\mathbf{j}$ and $\mathcal{C}$ is the unit circle in $\mathbb{R}^2$ centered at the origin.
(ii) $\mathbf{F}(x, y) = (1 - \sin(x))\mathbf{i} + y\mathbf{j}$ and $\mathcal{C}$ is the curve parametrized by $\alpha(t) = t^2\mathbf{i} + (1 - t^2)\mathbf{j}$, for $0 \leq t \leq 1$.
(iii) $\mathbf{F}(x, y) = \sin(x)\mathbf{i} - \sin(y)\mathbf{j}$ and $\mathcal{C}$ is the curve parametrized by $\alpha(t) = t^3\mathbf{i} + t^2\mathbf{j}$ for $0 \leq t \leq 2$.

4. Let $f = 2xyz^2$ and $\mathbf{F}(x, y, z) = xy\mathbf{i} - z\mathbf{j} + x^2\mathbf{k}$. Let $\mathcal{C}$ be parametrized by $\alpha : [0, 1] \to \mathbb{R}^3$, $\alpha(t) = (t^2, 2t, t^3)$.

(i) Evaluate $\int_{\mathcal{C}} f d\mathbf{r}$.
(ii) Evaluate $\int_{\mathcal{C}} \mathbf{F} d\mathbf{r}$.

5. A piece of wire is bent into the shape of a semi-circle $x^2 + y^2 = 9$, where $y \geq 0$. Suppose the density of the wire is constant, equal to ρ.

(i) Find the mass of the wire.
(ii) Find the center of mass of the wire.

6. Find the work done by the force field $\mathbf{F}(x, y) = x\mathbf{i} + (2 - y)\mathbf{j}$ as an object moves along the curve parametrized by $\alpha : [0, 2\pi] \to \mathbb{R}^2$, $\alpha(t) = (t - \sin(t))\mathbf{i} + (1 - \cos(t))\mathbf{j}$.

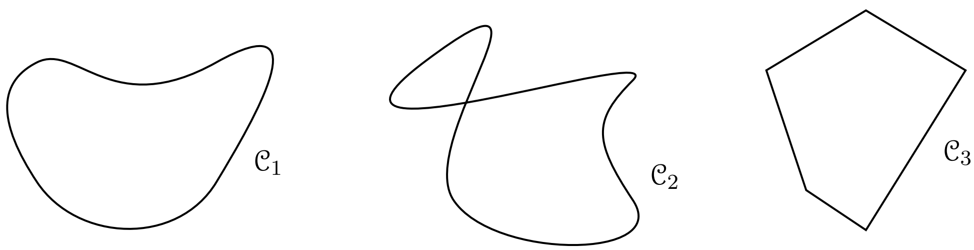
7. Find the work done by the force field $\mathbf{F}(x, y, z) = yx\mathbf{i} + zy\mathbf{j} + zx\mathbf{k}$ as an object moves along the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^2$, $\alpha(t) = t^2\mathbf{i} + t\mathbf{j} + t^3\mathbf{k}$.

8. Let $\mathbf{F}(x, y) = 3xy\mathbf{i} - y^2\mathbf{j}$. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ if $\mathcal{C}$ is the curve given by $y = 2x^2$, from the origin to $(1, 2)$.

9. Find the work done in moving a particle once around the circle $\mathcal{C}$ of radius 3 centered at the origin if

$$\mathbf{F}(x, y, z) = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 2y + 4z)\mathbf{k}.$$

10. Determine which of the following curves are piecewise smooth:



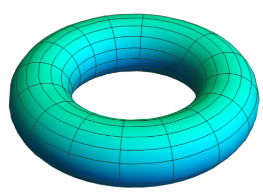
11. Let $f : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}$ be the function $f(x, y) = \tan^{-1}(y/x)$.

(i) Compute ∇f.

(ii) If $\mathbf{F} = \nabla f$ and $\mathcal{C}$ is the unit circle centered at the origin, compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

12. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by $\mathbf{F}(x, y, z) = (x, y, z)$. Let $\mathbb{T}^2$ denote the torus, parametrized by $x(s, t) := (r + \cos(t))\cos(s)$, $y(s, t) := (r + \cos(t))\sin(s)$, $z(s, t) := \sin(t)$, for $0 \le s, t \le 2\pi$. Evaluate

$$\int_{\mathbb{T}^2} \mathbf{F} \cdot d\mathbf{r}.$$



13. Let $\mathcal{C}$ be the curve parametrized by $x(t) = \cos(t)$, $y(t) = \sin(t)$, and $z(t) = \sin(t)$, for $0 \le t \le 2\pi$.

(i) Sketch $\mathcal{C}$.

(ii) Evaluate $\int_{\mathcal{C}} 2xe^{2y} dx + (2x^2 e^{2y} + 2y\cot(z))dy - y^2 \csc^2(z) dz$.

3.2. Path Dependence of Line Integrals

The fundamental theorem of calculus states that

$$\int_a^b f'(x)dx = f(b) - f(a),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. We now recognize the 1–form $f'(x)dx$ as the exterior derivative $df = f'(x)dx$. Hence, the fundamental theorem of calculus can be written as

$$\int_a^b df = f(b) - f(a).$$

In particular, the integral of the exterior derivative depends only on the endpoints. This extends to line integrals:

Theorem 3.2.1. (Fundamental theorem of line integrals). Let $\mathcal{C}$ be a smooth curve described by a smooth map $\alpha : [a, b] \to \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} \quad = \quad f(\alpha(b)) - f(\alpha(a)).$$

Proof. It suffices to prove the theorem for $n = 2$, since the case for general $n \in \mathbb{N}$ is the same. To this end, let $\alpha : [a, b] \to \mathbb{R}^2$ be a smooth parametrization for $\mathcal{C}$. Then

$$\begin{aligned}
\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f \cdot \alpha'(t)dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} \right) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} \right) dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} \right) dt \\
&= \int_a^b \frac{d}{dt} f(\alpha(t))dt \;=\; f(\alpha(b)) - f(\alpha(a)),
\end{aligned}$$

where the last equality follows from the fundamental theorem of calculus (in one variable). □

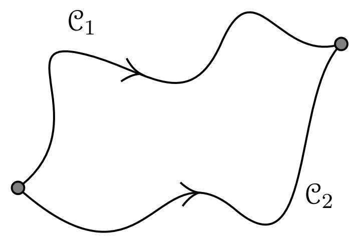
Definition 3.2.2. Let $\mathbf{F}$ be a smooth vector field on (some domain in) $\mathbb{R}^n$. We say that the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

is *independent of path* if, for all smooth paths $\mathcal{C}_1, \mathcal{C}_2$ which share their endpoints, we have

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} \quad = \quad \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}.$$

Figure 3.2.3. Two smooth paths which share their endpoints:



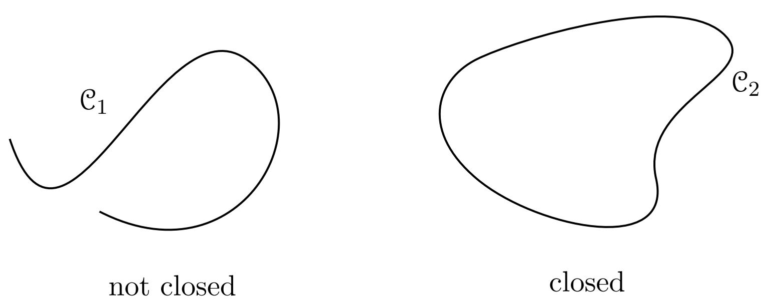
Immediate from the fundamental theorem of line integrals is the following:

Corollary 3.2.4. If $\mathbf{F}$ is a gradient field, then $\mathbf{F}$ has the path independence property.

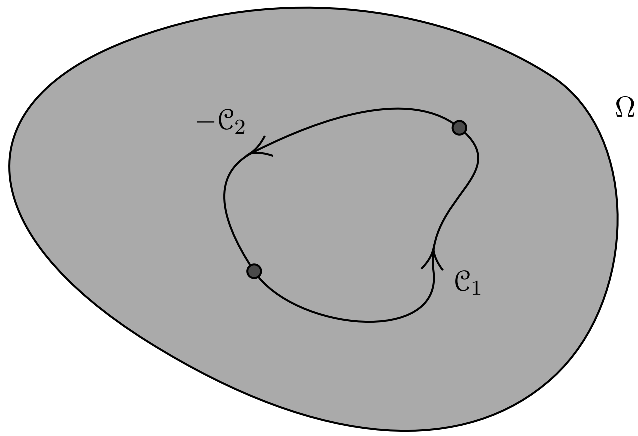
$$\underset{\text{Gradient field}}{\mathbf{F} = \nabla f} \longrightarrow \underset{\text{path independent}}{\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}}$$

The following shows that if $\mathbf{F} : \Omega \to \mathbb{R}^2$ is a vector field which fails to be *smooth* in Ω, then $\mathbf{F} = df$ does not imply that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is *not* independent of path.

Definition 3.2.5. A curve $\mathcal{C}$ in $\mathbb{R}^n$ is said to be *closed* if the endpoints coincide. That is, if $\mathcal{C}$ is described by a curve $\alpha : [a, b] \to \mathbb{R}^n$, then the curve is closed if $\alpha(a) = \alpha(b)$.



Theorem 3.2.6. Let $\Omega \subseteq \mathbb{R}^n$ be a domain (i.e., a path-connected open subset of $\mathbb{R}^n$). Let $\mathbf{F} : \Omega \to \mathbb{R}^n$ be a smooth vector field on Ω. The line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path in Ω if and only if $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths $\mathcal{C}$ in Ω.



A closed loop can be decomposed into two paths

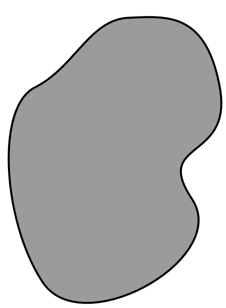
Example 3.2.7. Let $\mathbf{F}(x,y) : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$ be the vector field

$$\mathbf{F}(x,y) := -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

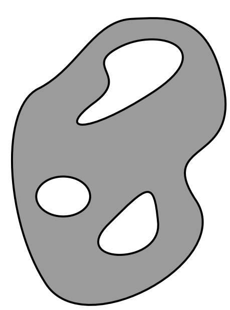
Then $\mathbf{F} = \nabla f$, where $f = \tan^{-1}(y/x)$.

In the previous sections, we have seen that gradient fields satisfy the path-independence property. Further, this is equivalent to $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed loops $\mathcal{C}$. It is natural to ask whether a vector field satisfying the path-independence property is necessarily a gradient field. On $\mathbb{R}^3$ this is true (see *Theorem 3.2.15* below), but in general it depends on the presence of holes in the domain of the vector field. A region in $\mathbb{R}^n$ which has no holes, is called a simply connection domain:

Definition 3.2.8. Let $\Omega \subseteq \mathbb{R}^2$ be a region in the plane. Denote by $\partial\Omega$ the boundary of Ω. If $\partial\Omega$ is path connected, we say that Ω is *simply connected.*

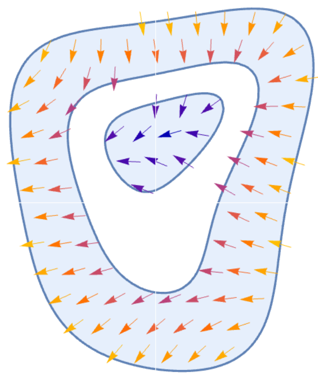


simply connected



Not simply connected

Example 3.2.9.



A vector field on a non-simply connected region.

Example 3.2.10. Euclidean space $\mathbb{R}^n$ is simply connected.

Example 3.2.11. Let

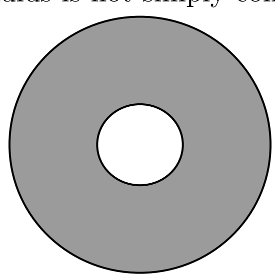
$$\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

denote the (closed) unit disk in the plane. The boundary $\partial\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected. Hence, Ω is simply connected.

Example 3.2.12. Let $\Omega := \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$ denote the (closed) annulus. The boundary is

$$\partial\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2\}.$$

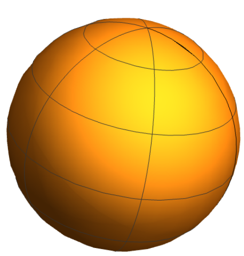
Since $\partial\Omega$ is not connected, the annulus is not simply connected.

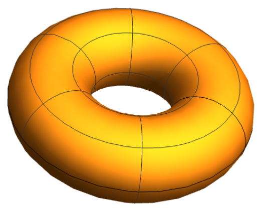


Example 3.2.13.

The sphere is simply connected

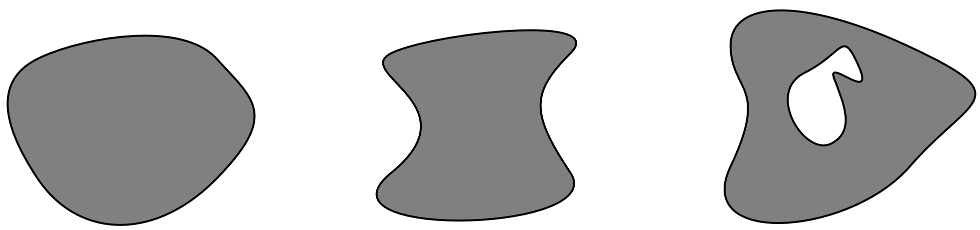
The torus is not simply connected





The following result is clear

Proposition 3.2.14. A convex set in $\mathbb{R}^n$ is simply connected.

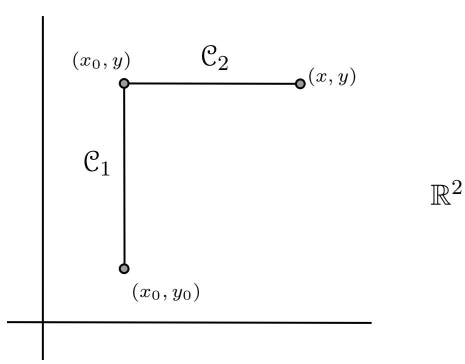


Theorem 3.2.15. Let $\mathbf{F}$ be a vector field which is smooth on $\mathbb{R}^3$. If $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path in $\mathbb{R}^3$, then $\mathbf{F}$ is a gradient vector field.

PROOF. We need to produce a function f such that $\mathbf{F} = \nabla f$. Fix a point $(x_0, y_0) \in \mathbb{R}^2$, I claim that

$$f(x,y) := \int_{(x_0,y_0)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$$

is the desired potential. To see this, observe that since $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is path independent, we can choose the linear paths which connect (x_0, y_0) to (x_0, y), followed by the linear path which connects (x_0, y) to (x, y):



Hence,

$$f(x,y) \;=\; \int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r}.$$

The curve $\mathcal{C}_1$ does not vary in x, while $\mathcal{C}_2$ does not vary in y. Therefore,

$$f_x \;=\; \frac{\partial}{\partial x}\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} + \frac{\partial}{\partial x}\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} \;=\; \frac{\partial}{\partial x}\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r}, \tag{3.2.1}$$

and

$$f_y \;=\; \frac{\partial}{\partial y}\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} + \frac{\partial}{\partial x}\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} \;=\; \frac{\partial}{\partial y}\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r}. \tag{3.2.2}$$

Let us write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, and write $\omega_{\mathbf{F}} = Pdx + Qdy$ for the associated 1–form. We may write

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{\mathcal{C}} Pdx + Qdy.$$

Since $\mathcal{C}_1$ does not vary in x, along $\mathcal{C}_1$ we have $dx = 0$. Therefore,

$$\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{\mathcal{C}_1} Pdx + Qdy \;=\; \int_{\mathcal{C}_1} Qdy,$$

and from (3.2.2), we have:

$$f_y \;=\; \frac{\partial}{\partial y}\int_{\mathcal{C}_1} \mathbf{F}\cdot d\mathbf{r} \;=\; \frac{\partial}{\partial y}\int_{\mathcal{C}_1} Qdy \;=\; \frac{\partial}{\partial y}\int_{(x_0,y_0)}^{(x_0,y)} Q(x,t)dt.$$

By the (one-variable) fundamental theorem of calculus, we have $f_y = Q$. Similarly, since $\mathcal{C}_2$ does not vary in y, along $\mathcal{C}_2$ we have $dy = 0$. Therefore,

$$\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} \;=\; \int_{\mathcal{C}_2} Pdx + Qdy \;=\; \int_{\mathcal{C}_2} Pdx,$$

and from (3.2.1) we have

$$f_x \;=\; \frac{\partial}{\partial x}\int_{\mathcal{C}_2} \mathbf{F}\cdot d\mathbf{r} \;=\; \frac{\partial}{\partial x}\int_{\mathcal{C}_2} Pdx \;=\; \frac{\partial}{\partial x}\int_{(x_0,y)}^{(x,y)} P(t,y)dt.$$

By the (one variable) fundamental theorem of calculus, we have $f_x = P$. Hence,

$$\mathbf{F} \;=\; P\mathbf{i} + Q\mathbf{j} \;=\; f_x\mathbf{i} + f_y\mathbf{j} \;=\; \nabla f.$$

□

Recall that in *Theorem 2.1.12*, we saw that all gradient fields are irrotational:

$$\underset{\text{gradient field}}{\mathbf{F} = \nabla f} \quad\longrightarrow\quad \underset{\text{irrotational}}{\mathrm{curl}(\mathbf{F}) = \mathbf{0}}$$

On simply connected regions, the converse is true:

Theorem 3.2.16. Let $\mathbf{F} : \Omega \to \mathbb{R}^3$ be a smooth vector field on a simply connected region $\Omega \subseteq \mathbb{R}^3$. If $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$, then $\mathbf{F}$ is a gradient vector field.

We will give the proof in *Theorem 4.4.5*.

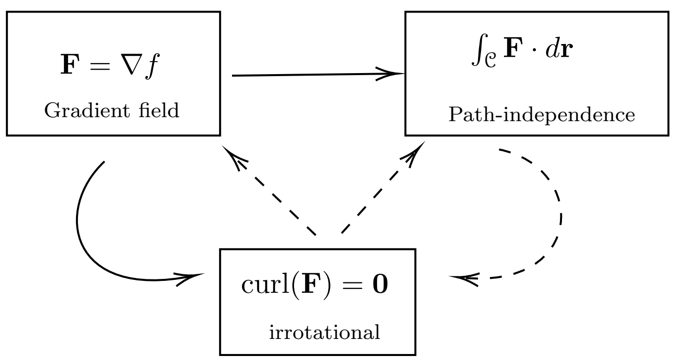
Theorem 3.2.17. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Suppose $\operatorname{div}(\mathbf{F}) = 0$. Then $\mathbf{F} = \operatorname{curl}(\mathbf{G})$ for some $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$.

Remark 3.2.18. The above theorem is not true, in general, not even for simply connected regions. For instance, consider the vector field $\mathbf{F} : \mathbb{S}^3 \to \mathbb{R}^3$ defined by

$$\mathbf{F}(x, y, z) := \frac{x}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}.$$

This vector field is incompressible but is not solenoidal, i.e., $\mathbf{F}$ is not the curl of a vector field $\mathbf{G}$.

Summary 3.2.19. In terms of the vector field $\mathbf{F}$ alone, we have the following diagram:



solid lines = always true; dashed lines = true on simply connected domains

Exercises

1. Evaluate the line integrals $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ by first determining whether $\mathbf{F}$ is a gradient field, and if so, using the fundamental theorem of line integrals. If the vector field is not a gradient field, evaluate the line integral directly.

(i) $\mathcal{C}$ is the unit circle in $\mathbb{R}^2$ centered at the origin, and

$$\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}.$$

(ii) $\mathcal{C}$ is the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^2$, $\alpha(t) = \sqrt{t}\mathbf{i} + (1 + t^3)\mathbf{j}$ and

$$\mathbf{F}(x, y) = x^2 y\mathbf{i} - yx^4\mathbf{j}.$$

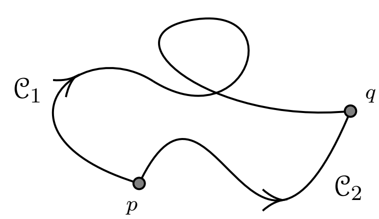
(iii) $\mathcal{C}$ is the line segment from $(0, 1, -3)$ to $(2, -4, 1)$ and

$$\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}.$$

(iv) $\mathcal{C}$ is the curve parametrized by $\alpha : [0, 2\pi] \to \mathbb{R}^3$, $\alpha(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$ and

$$\mathbf{F}(x, y, z) = (2xz + \sin(y))\mathbf{i} + x\cos(y)\mathbf{j} + x^2\mathbf{k}.$$

2. Let $p, q \in \mathbb{R}^2$ be two points. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two paths from p to q as indicated in the following diagram below:



Suppose $\mathbf{F}$ is a gradient field on $\mathbb{R}^2$, and $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 10$. Evaluate $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$.

3. Determine (with justification) whether the following regions in $\mathbb{R}^2$ are simply connected:

(i) $\mathbb{R}^2$.
(ii) $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
(iii) $A(0, 1) := \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$.

4. Determine (with justification) whether the following statements are true or false:

(i) The union of simply connected regions is simply connected.
(ii) The intersection of simply connected regions is simply connected.
(iii) If X is a simply connection region in $\mathbb{R}^2$, then $X \backslash \{p\}$ is simply connected, where $p \in X$ is some point.

5. Let $\mathbf{F}$ be a smooth vector field on a region $\Omega \subseteq \mathbb{R}^n$. Suppose, moreover, that $\mathbf{F}$ is irrotational, i.e., $\text{curl}(\mathbf{F}) = \mathbf{0}$.

(i) If $\Omega = \mathbb{R}^2$, is $\mathbf{F} = \nabla f$ for some function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$? Justify your answer.
(ii) If Ω is a convex set in $\mathbb{R}^3$, is $\mathbf{F} = \nabla f$ for some smooth function $f : \Omega \to \mathbb{R}$? Justify your answer.
(iii) If Ω is a simply connected domain, is $\mathbf{F}$ a gradient field?

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f'(x) = 0$ for all $x \in \mathbb{R}$.

(i) Show that $f(x) = c$ for some $c \in \mathbb{R}$.
(ii) Suppose $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is such that $f'(x) = 0$ for all $x \in \mathbb{R}\backslash\{0\}$. Is f constant?

7. Determine (with justification) whether the following statements are true or false:

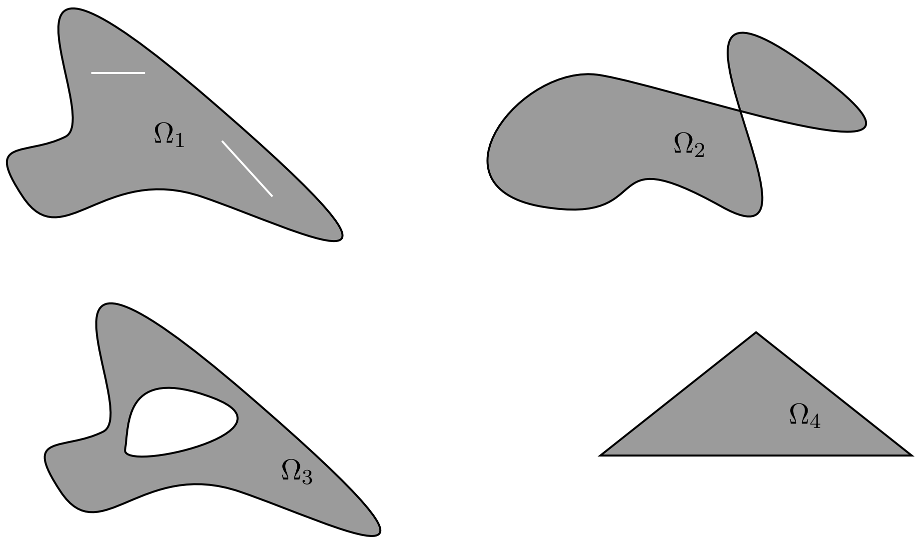
(i) Every irrotational vector field (i.e., a vector field $\mathbf{F}$ such that $\text{curl}(\mathbf{F}) = \mathbf{0}$) is a gradient field.
(ii) An irrotational vector field on a convex set is a gradient field.
(iii) The line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path if $\mathbf{F}$ is an irrotational vector field on the annulus $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$.
(iv) The line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}$ is irrotational.

8. Is every vector field on a simply connected domain in $\mathbb{R}^2$, the curl of another vector field?

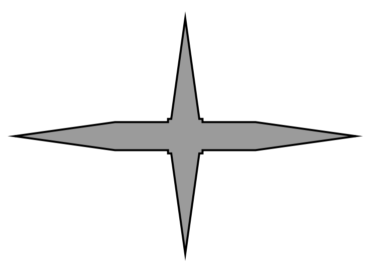
9. State (with justification) whether the following statements are true or false:

(i) The intersection of simply connected sets is simply connected.
(ii) The union of convex sets is convex.

10. Determine which of the following regions are (i) simply connected, or (ii) convex:



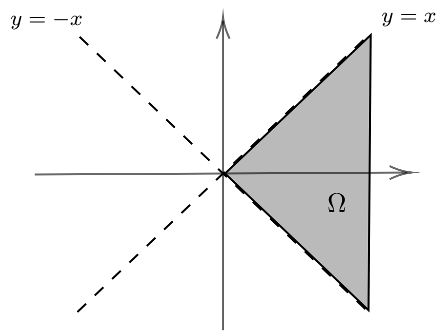
11. Is the region given below simply connected?



12. Consider the vector field

$$\mathbf{F}(x,y) = -\frac{y}{x^2\sqrt{1-\frac{y^2}{x^2}}}\mathbf{i} + \frac{1}{x\sqrt{1-\frac{y^2}{x^2}}}\mathbf{j}.$$

(i) Show that $\mathrm{curl}(\mathbf{F}) = \mathbf{0}$.
(ii) Show that $\mathbf{F} = \nabla f$ for some f. State the function f explicitly.
(iii) Is $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ path independent for all curves $\mathcal{C} \subset \Omega$, where Ω is the region described below:



13. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x, y) = (x + y)\mathbf{i} + y\mathbf{j}.$$

(i) Let $\mathcal{C}_1$ be the line connecting the points $(0, 0)$ and $(1, 1)$. Determine a parametrization for $\mathcal{C}_1$.

(ii) Compute $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$ using the parametrization for $\mathcal{C}_1$ given by part (i).

(iii) Let $\mathcal{C}_2$ be the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^2$, $\alpha(t) = (t, t^2)$. Compute $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$.

(iv) Does $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ satisfy the path independence property?

(v) Without doing any computation, is $\mathbf{F}$ a gradient field?

(vi) Is $\mathbf{F}$ an irrotational vector field?

CHAPTER 4

Integration Theory – Surfaces and Beyond

"Good mathematics is proving things that should be true. Great mathematics is proving things that shouldn't."

– Misha Gromov

The previous chapter extended the fundamental theorem of calculus (FTC) to line integrals. In this chapter, we will extend this to regions in $\mathbb{R}^2$, yielding *Green's theorem*. The two-dimensional version of line-integrals, namely, *surface integrals*, are then presented, and the corresponding version of the FTC is given. In this guise, it is referred to as *Stokes' theorem*. Finally, for solid surfaces in $\mathbb{R}^3$, the FTC is arises under the name of the *divergence theorem*. For the reader's convenience, we will begin by recalling some known-results concerning the computation of double and triple integrals.

4.1. Multiple integrals

We assume the reader has some familiarity with multiple integrals, we will recall the results here for their convenience.

Iterated integrals. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. The iterated integral is defined

$$\int_a^b \int_c^d f(x, y) dy dx \;=\; \int_a^b \left(\int_c^d f(x, y) dy \right) dx. \tag{4.1.1}$$

The first (or inner) integral on the right-hand side of (4.1.1) is evaluated by fixing x and viewing $f(x, y)$ as a function of the variable y alone. This is analogous to how partial derivatives are computed. The result of this (definite) integration is a function $F(x)$ which is independent of y. The resulting iterated integral is obtained by evaluating this integral of a one-variable function $\int_a^b F(x)dx$ in the familiar manner:

Example 4.1.1. Evaluate the iterated integral

$$\int_0^1 \int_2^5 xy^2 dy dx.$$

Solution. From (4.1.1), we have

$$\begin{aligned}\int_0^1 \int_2^5 xy^2 dy dx &= \int_0^1 \left(\int_2^5 xy^2 dy\right) dx \\ &= \int_0^1 \left[\frac{1}{3}xy^3\right]_2^5 dx \\ &= \int_0^1 \left[\frac{1}{3}x(5^3 - 2^3)\right] dx \\ &= \int_0^1 \frac{117}{3} x dx = \left[\frac{117}{6}x^2\right]_0^1 = \frac{117}{6}.\end{aligned}$$

□

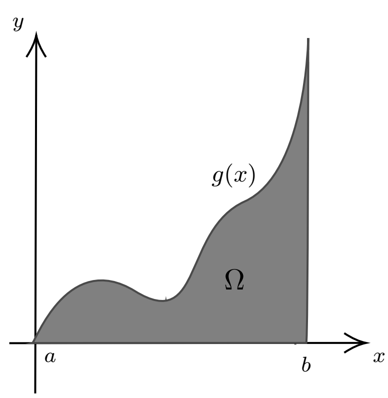
One can observe that in these cases, the order of integration does not matter, more formally:

Fubini's Theorem. If $f : [a, b] \times [c, d] \to \mathbb{R}$ is a continuous function, then

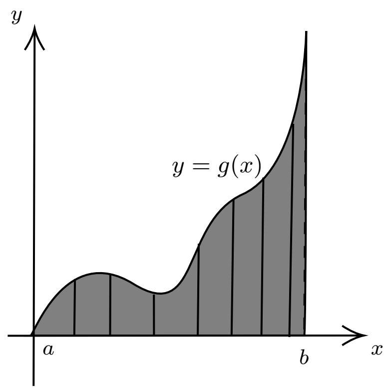
$$\int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy.$$

Remark 4.1.2. Fubini's theorem applies to functions which are continuous on *rectangles* and where the integration is taken over *rectangles*. The situation is more delicate when considering integrals over more general regions:

Integration over general regions. Consider the following region in $\mathbb{R}^2$:

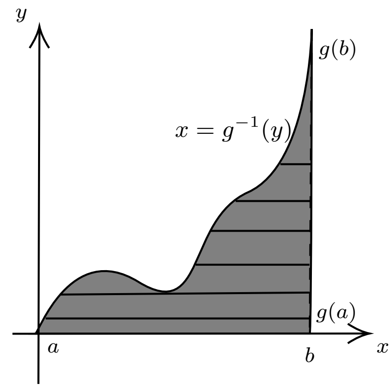


If we want to evaluate the integral of a function $f : \Omega \to \mathbb{R}$, then the order of integration will matter:



integrating with respect to y then x

$a \leq x \leq b \qquad 0 \leq y \leq g(x)$



integrating with respect to x then y

$g^{-1}(y) \leq x \leq b \quad g(a) \leq y \leq g(b)$

Example 4.1.3. Let Ω be the region bounded between the parabola $y = x^2 + 1$, the coordinate axes, and the line $x = 1$. Evaluate

$$\iint_\Omega ye^{-x} dA.$$

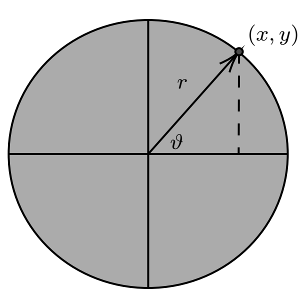
SOLUTION. The region is most simply described by $0 \leq x \leq 1$ and $0 \leq y \leq x^2 + 1$. Hence

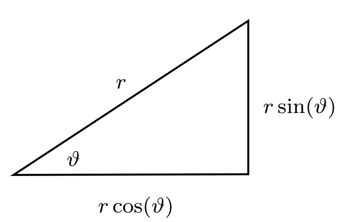
$$\begin{aligned} \iint_\Omega ye^{-x} dA &= \int_0^1 \int_0^{x^2+1} ye^{-x} dy dx \\ &= \int_0^1 \frac{1}{2}(x^2+1)^2 e^{-x} dx \;=\; \frac{29}{2} - \frac{38}{e}. \end{aligned}$$

□

Polar coordinates: For circular regions of integration (such as a circle), it is often convenient to pass to *polar coordinates*:

$$x = r\cos(\vartheta) \qquad y = r\sin(\vartheta)$$





Proposition 4.1.4. In polar coordinates $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$, the area form is

$$\boxed{dA = r\ dr\ d\vartheta}$$

PROOF. If $x = r\cos(\vartheta)$, then $dx = \cos(\vartheta)dr - r\sin(\vartheta)d\vartheta$. If $y = r\sin(\vartheta)$, then $dy = \sin(\vartheta)dr + r\cos(\vartheta)d\vartheta$. Then

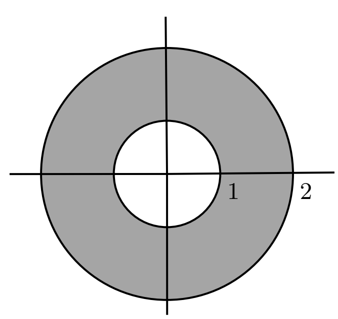
$$\begin{aligned} dx \wedge dy &= (\cos(\vartheta)dr - r\sin(\vartheta)d\vartheta) \wedge (\sin(\vartheta)dr + r\cos(\vartheta)d\vartheta) \\ &= \cos(\vartheta)\sin(\vartheta)dr \wedge dr + r\cos^2(\vartheta)dr \wedge d\vartheta \\ &\qquad\qquad -r\sin^2(\vartheta)d\vartheta \wedge dr - r^2\sin(\vartheta)\cos(\vartheta)d\vartheta \wedge d\vartheta \\ &= r\cos^2(\vartheta)dr \wedge d\vartheta - r\sin^2(\vartheta)d\vartheta \wedge dr \\ &= r(\cos^2(\vartheta) + \sin^2(\vartheta))dr \wedge d\vartheta \\ &= rdr \wedge d\vartheta. \end{aligned}$$

□

Example 4.1.5. Compute the double integral

$$\iint_\Omega (x^2 + y^2)dA,$$

where Ω is the annulus



$$\Omega = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}.$$

SOLUTION. Because the region Ω is circular, we can use polar coordinates: $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$. The associated volume form is $dxdy = rdrd\vartheta$. Hence, using the fact that

$x^2 + y^2 = r^2$,

$$\begin{aligned}
\iint_\Omega (x^2+y^2)dA &= \int_0^{2\pi}\int_1^2 r^2 \cdot r dr d\vartheta \\
&= \int_0^{2\pi}\int_1^2 r^3 dr d\vartheta \\
&= 2\pi \int_1^2 r^3 dr \\
&= 2\pi \left[\frac{1}{4}r^4\right]_1^2 = \frac{15\pi}{2}.
\end{aligned}$$

□

Example 4.1.6. Let Ω be the region bounded by the semi-circle $x = \sqrt{9-y^2}$ and the y–axis. Compute

$$\iint_\Omega e^{-x^2-y^2} dA.$$

SOLUTION. The region Ω is circular, so can be described via polar coordinates $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$. The region Ω is described in polar coordinates by $-\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}$ and $0 \le r \le 3$. Now,

$$\begin{aligned}
\iint_\Omega e^{-x^2-y^2} dA &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^3 e^{-r^2} r dr d\vartheta \\
&= \pi \int_0^3 r e^{-r^2} dr.
\end{aligned}$$

Let $u = -r^2$, then $du = -2rdr$ and

$$\begin{aligned}
\iint_\Omega e^{-x^2-y^2} dA &= \pi \int_0^3 r e^{-r^2} dr \\
&= \pi \int_0^{-9} r e^u \frac{du}{-2r} \\
&= -\frac{\pi}{2}\int_0^{-9} e^u du \\
&= \frac{\pi}{2}\int_{-9}^0 e^u du \\
&= \frac{\pi}{2}(1-e^{-9}).
\end{aligned}$$

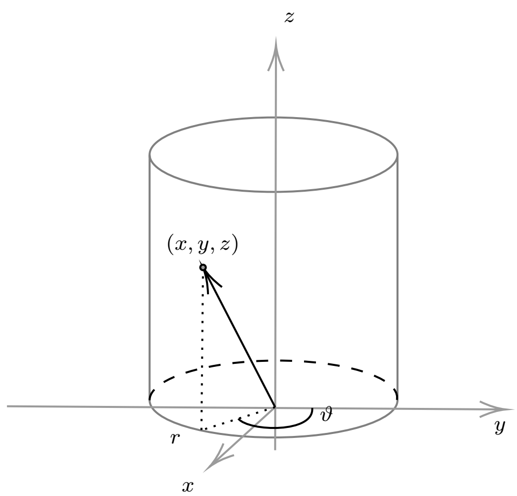
□

Cylindrical coordinates: In cylindrical coordinates

$$x = r\cos(\vartheta), \qquad y = r\sin(\vartheta), \qquad z = z.$$

The associated volume form is

$$\boxed{dV = r\ dr\ d\vartheta\ dz}$$



Example 4.1.7. Evaluate the integral

$$\iiint_\Omega x(x^2+y^2)dV,$$

where Ω is the region inside the cylinder $x^2+y^2=9$ and between the planes $z=0$ and $z=3$.

Solution. Passing to cylindrical coordinates, we have $x = r\cos(\vartheta)$, $y = r\sin(\vartheta)$, and $z = z$, for $r \in [0,3]$, $\vartheta \in [0,2\pi]$, and $z \in [0,3]$. Hence,

$$\begin{aligned}
\iiint_\Omega x(x^2+y^2)dV &= \int_0^3 \int_0^{2\pi} \int_0^3 r\cos(\vartheta)(r^2) r dr d\vartheta dz \\
&= 3\int_0^{2\pi}\int_0^3 r^4\cos(\vartheta) dr d\vartheta \\
&= 3\int_0^{2\pi}\cos(\vartheta)d\vartheta \int_0^3 r^4 dr \\
&= 3\left[\sin(\vartheta)\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^3 \\
&= \frac{3}{4}(\sin(2\pi)-\sin(0))(3^4-0) \ = \ 0.
\end{aligned}$$

$\square$

Spherical coordinates. In spherical coordinates

$$x \;=\; r\sin(\vartheta)\cos(\phi), \qquad y \;=\; r\sin(\vartheta)\sin(\phi), \qquad z \;=\; r\cos(\vartheta).$$

The associated *spherical volume form* (i.e., the volume form in spherical coordinates) is

$$\boxed{\quad dV = r^2\sin(\vartheta)\ dr\ d\vartheta\ d\phi \quad}$$

PROOF. We have

$$\begin{aligned}
dx &= \sin(\vartheta)\cos(\phi)dr + r\cos(\vartheta)\cos(\phi)d\vartheta - r\sin(\vartheta)\sin(\phi)d\phi\\
dy &= \sin(\vartheta)\sin(\phi)dr + r\cos(\vartheta)\sin(\phi)d\vartheta + r\sin(\vartheta)\cos(\phi)d\phi\\
dz &= \cos(\vartheta)dr - r\sin(\vartheta)d\vartheta.
\end{aligned}$$

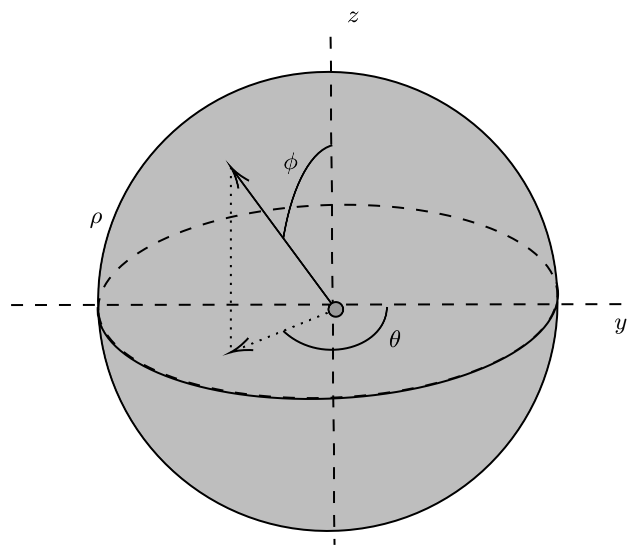
First compute

$$\begin{aligned}
dy\wedge dz &= (\sin(\vartheta)\sin(\phi)dr + r\cos(\vartheta)\sin(\phi)d\vartheta + r\sin(\vartheta)\cos(\phi)d\phi)\wedge(\cos(\vartheta)dr - r\sin(\vartheta)d\vartheta)\\
&= -r\sin^2(\vartheta)\sin(\phi)dr\wedge d\vartheta + r\cos^2(\vartheta)\sin(\phi)d\vartheta)\wedge dr\\
&\quad + r\sin(\vartheta)\cos(\vartheta)\cos(\phi)d\phi\wedge dr - r^2\sin^2(\vartheta)\cos(\phi)d\phi\wedge d\vartheta\\
&= -r\sin(\phi)dr\wedge d\vartheta - r\sin(\vartheta)\cos(\vartheta)\cos(\phi)dr\wedge d\phi + r^2\sin^2(\vartheta)\cos(\phi)d\vartheta\wedge d\phi.
\end{aligned}$$

Finally, compute

$$\begin{aligned}
dx\wedge dy\wedge dz &= dx\wedge\big(-r\sin(\phi)dr\wedge d\vartheta - r\sin(\vartheta)\cos(\vartheta)\cos(\phi)dr\wedge d\phi + r^2\sin^2(\vartheta)\cos(\phi)d\vartheta\wedge d\phi\big)\\
&= (\sin(\vartheta)\cos(\phi)dr + r\cos(\vartheta)\cos(\phi)d\vartheta - r\sin(\vartheta)\sin(\phi)d\phi)\\
&\quad \wedge\big(-r\sin(\phi)dr\wedge d\vartheta - r\sin(\vartheta)\cos(\vartheta)\cos(\phi)dr\wedge d\phi + r^2\sin^2(\vartheta)\cos(\phi)d\vartheta\wedge d\phi\big)\\
&= r^2\sin^3(\vartheta)\cos^2(\phi)dr\wedge d\vartheta\wedge d\phi - r^2\sin(\vartheta)\cos^2(\vartheta)\cos^2(\phi)d\vartheta\wedge dr\wedge d\phi\\
&\quad + r^2\sin(\vartheta)\sin^2(\phi)d\phi\wedge dr\wedge d\vartheta\\
&= r^2\left(\sin^3(\vartheta)\cos^2(\phi) + \sin(\vartheta)\cos^2(\vartheta)\cos^2(\phi) + \sin(\vartheta)\sin^2(\phi)\right)dr\wedge d\vartheta\wedge d\phi\\
&= r^2\sin(\vartheta)\left(\sin^2(\vartheta)\cos^2(\phi) + \cos^2(\vartheta)\cos^2(\phi) + \sin^2(\phi)\right)dr\wedge d\vartheta\wedge d\phi\\
&= r^2\sin(\vartheta)(\cos^2(\phi) + \sin^2(\phi))dr\wedge d\vartheta\wedge d\phi\\
&= r^2\sin(\vartheta)dr\wedge d\vartheta\wedge d\phi.
\end{aligned}$$

$\square$



Example 4.1.8. Let $\mathcal{V}$ be the hemi-sphere given by $0 \leq z \leq \sqrt{1-x^2-y^2}$. Compute

$$\iiint_{\mathcal{V}} (x^2+y^2)dV.$$

Solution. In spherical coordinates, $x = r\sin(\vartheta)\cos(\phi)$, $y = r\sin(\vartheta)\sin(\phi)$, $z = r\cos(\vartheta)$. Then

$$x^2+y^2 \;=\; r^2\sin^2(\vartheta)\cos^2(\phi) + r^2\sin^2(\vartheta)\sin^2(\phi) \;=\; r^2\sin^2(\vartheta)$$

and $dV = r^2\sin(\vartheta)drd\vartheta d\phi$. Hence,

$$\begin{aligned}
\iiint_{\mathcal{V}} (x^2+y^2)dV &= \int_0^{\pi}\int_0^{2\pi}\int_0^1 (r^2\sin^2(\vartheta))r^2\sin(\vartheta)drd\vartheta d\phi \\
&= \left(\int_0^{\pi} d\phi\right)\left(\int_0^{2\pi}\sin^3(\vartheta)d\vartheta\right)\left(\int_0^1 r^4dr\right) \\
&= \pi\left(\int_0^{2\pi}\sin^3(\vartheta)d\vartheta\right)\left(\int_0^1 r^4dr\right) \\
&= \pi\cdot 0\cdot\frac{1}{5} \;=\; 0.
\end{aligned}$$

□

Exercises

1. Evaluate the following iterated integrals

(i) $\int_0^1 \int_0^2 (x^2 - y^2) dx dy$.

(ii) $\int_0^1 \int_0^\pi (\sqrt{x} + \sin(y)) dy dx$.

(iii) $\int_0^1 \int_0^v \sqrt{1 - v^2} du dv$.

2. Evaluate

$$\int_\Omega x^2 e^{x^2+y^2} dx dy,$$

where $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4,\ y \geq 0\}$.

3. Let Ω be the region bounded by $y = x^2$, $x = 2$ and $y = 0$.

(i) Evaluate $\int_\Omega xy dA$ by first integrating with respect to x.
(ii) Evaluate $\int_\Omega xy dA$ by first integrating with respect to y.

4. Evaluate

$$\int_0^1 \int_x^{2x} e^{x+y} dy dx.$$

5. Use spherical coordinates to compute the integrals

(i)

$$\iiint_\Omega 4(x^2 + y^2 + z^2)^2 dV,$$

where Ω is the ball $x^2 + y^2 + z^2 \leq 4$.

(ii)

$$\iiint_\Omega z e^{(x^2+y^2+z^2)} dV,$$

where Ω is the region bounded between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$.

6. Evaluate the double integrals $\iint_\Omega f(x, y) dA$, where

(i) $f(x, y) = x^3 y^6$ and $\Omega := \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 3,\ -x \leq y \leq x\}$.
(ii) $f(x, y) = \frac{5y}{x^2+4}$ and $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{x}\}$.
(iii) $f(x, y) = 1 - e^{-y^2}$ and $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1, 0 \leq x \leq y\}$.
(iv) $f(x, y) = 4x\sqrt{y^2 - x^2}$ and $\Omega = \{(x, y) \in \mathbb{R}^2 : 0 \leq y \leq 1,\ 0 \leq x \leq y\}$.

7. Evaluate the double integrals $\iint_\Omega f(x,y)dA$, where

(i) $f(x,y) = x\sin(y)$ and Ω is the region bounded by $y=0$, $y=x^2$, and $x=1$.
(ii) $f(x,y) = \frac{1}{3}(x+y)$ and Ω is the region bounded by $y=\sqrt{x}$ and $y=x^2$.
(iii) $f(x,y) = y^2 - x$ and Ω is the region bounded by $x=y^2$ and $x=3-2y^2$.
(iv) $f(x,y) = ye^x$ and Ω is the triangular region with vertices $(0,0)$, $(1,1)$, and $(5,3)$.

8. Sketch the region of integration and change the order of integration for the following double integrals:

(i)
$$\int_0^2 \int_0^{1-x} (x^2 - y)dydx.$$
(ii)
$$\int_1^2 \int_0^{\ln(x)} e^{-\ln(x^3)}dydx.$$
(iii)
$$\int_0^{\pi/9} \int_0^{\sin(x)} e^{\cos(x^2)-9}dydx.$$
(iv)
$$\int_0^1 \int_{\tan^{-1}(x)}^{\pi/3} (x-y)dydx.$$

9. Evaluate the following double integrals by reversing the order:

(i)
$$\int_0^1 \int_{2x}^2 e^{x^2}dxdy.$$
(ii)
$$\int_0^4 \int_{\sqrt{y}}^1 \sqrt{x^3+1}dxdy.$$
(iii)
$$\int_0^3 \int_y^9 y\cos(x^2)dxdy.$$
(iv)
$$\int_0^1 \int_x^1 x^3\sin(y^3)dydx.$$

10. Evaluate
$$\iint_\Omega (x^2\tan(x) + y^3 + 4)dA,$$
where $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\}$.

11. Use polar coordinates to find the area of the following regions:

(i) two loops of the rose $r = \cos(3\vartheta)$.

(ii) the region enclosed by the lemniscate $r^2 = 4\cos(2\vartheta)$.

(iii) the region enclosed by the cardioid $r = 1 - \sin(\vartheta)$.

(iv) the region bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

12. Evaluate the following integrals using cylindrical coordinates

(i)
$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz dz dx dy.$$

(ii)
$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2+x^2+y^2} (x^2+y^2)^{\frac{3}{2}} dz dy dx.$$

13. For the region $\Omega := \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 4\}$, compute
$$\iint_\Omega \sqrt{1 + x^2 + y^2} dx dy.$$

14. The sphere $x^2 + y^2 + z^2 = 25$ has a hole bored through it by the cylinder $x^2 + y^2 = 4$. Find the volume of that part of the sphere that is removed.

15. Sketch the region of integration and evaluate
$$\int_0^1 \int_x^{2x} e^{x+2y} dy dx.$$

4.2. Green's theorem

The fundamental theorem of line integrals informs us that if $\mathbf{F} : \Omega \to \mathbb{R}^2$ is a gradient field, i.e., $\mathbf{F} = \nabla f$, where $f : \Omega \to \mathbb{R}$ is a smooth function, then

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(\alpha(1)) - f(\alpha(0)),$$

for any smooth parametrization $\alpha : [0,1] \to \mathbb{R}^2$ of $\mathcal{C}$. We have seen that it is more natural to express this in terms of forms:

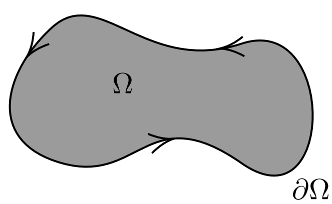
$$\int_{\mathcal{C}} df = f(\alpha(1)) - f(\alpha(0)).$$

In particular, we have

$$\int_{\mathcal{C}} d(0\text{–form}) \quad = \quad \text{values of 0–form on boundary of } \mathcal{C} \quad = \quad \int_{\partial\mathcal{C}} (0\text{–form}).$$

Now, if we want to integrate the 2–form $d(1\text{–form})$, then we integrate over a region $\Omega \subseteq \mathbb{R}^2$ in the plane. In particular, we would expect that

$$\iint_{\Omega} d(1\text{–form}) \quad = \quad \text{values of 1–form on boundary of } \Omega \quad = \quad \int_{\partial\Omega} (1\text{–form}).$$



This is exactly Green's theorem:

Theorem 4.2.1. (Green's theorem). Let $\mathcal{C}$ be a positively oriented, piecewise smooth, simple closed curve in the plane. Let $\Omega \subset \mathbb{R}^2$ be the region bounded by $\mathcal{C}$. If $\mathbf{F} : \Omega \to \mathbb{R}^2$ is a smooth vector field, then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \quad = \quad \iint_{\Omega} \operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} dA.$$

Proof. Let $\omega_{\mathbf{F}} = Pdx + Qdy$ be the 1–form associated to $\mathbf{F}$. Then from *Theorem 2.3.8*, we know that $\operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} dA = d\omega_{\mathbf{F}}$. Hence, if

$$\int_{\partial\Omega} \omega_{\mathbf{F}} \quad = \quad \iint_{\Omega} d\omega_{\mathbf{F}}, \tag{4.2.1}$$

then

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \ = \ \int_{\mathcal{C}} \omega_{\mathbf{F}} \ = \ \int_{\partial\Omega} \omega_{\mathbf{F}} \ = \ \iint_{\Omega} d\omega_{\mathbf{F}} \ = \ \iint_{\Omega} \operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} dA.$$

It remains to (4.2.1), we will consider only the case when Ω is the rectangle

$$\Omega = \{(x, y) \in \mathbb{R}^2 : a \le x \le b,\ c \le y \le d\}$$

for the moment. The general case is obtained by gluing together the result we get for rectangles, which is more technical, and will be treated later.
If $\omega_{\mathbf{F}} = Pdx + Qdy$, then $d\omega_{\mathbf{F}} = P_y dy \wedge dx + Q_x dx \wedge dy = (Q_x - P_y) dx \wedge dy$. Hence,

$$\iint_\Omega d\omega_{\mathbf{F}} \;=\; \int_c^d \int_a^b (Q_x - P_y) dx \wedge dy.$$

First, compute

$$\int_c^d \int_a^b Q_x(x, y) dx dy \;=\; \int_c^d [Q(x, y)]_a^b \, dy \;=\; \int_c^d (Q(b, y) - Q(a, y)) dy,$$

and similarly,

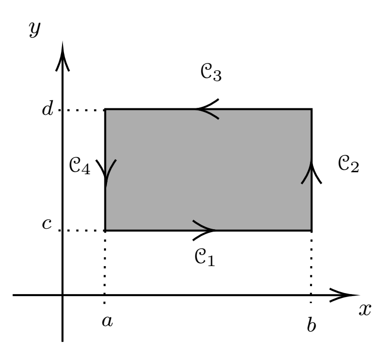
$$\int_c^d \int_a^b P_y(x, y) dx dy \;=\; \int_a^b \int_c^d P_y(x, y) dy dx \;=\; \int_a^b (P(x, d) - P(x, c)) dx.$$

This shows that

$$\iint_\Omega d\omega_{\mathbf{F}} \;=\; \int_c^d (Q(b, y) - Q(a, y)) dy - \int_a^b (P(x, d) - P(x, c)) dx. \tag{4.2.2}$$

On the other hand, the boundary of Ω is given by the curves

$$\begin{aligned}
\mathcal{C}_1 &:= \{\alpha_1(t) = t\mathbf{i} + c\mathbf{j} \ : \ a \le t \le b\},\\
\mathcal{C}_2 &:= \{\alpha_2(t) = b\mathbf{i} + t\mathbf{j} \ : \ c \le t \le d\},\\
-\mathcal{C}_3 &:= \{\alpha_3(t) = t\mathbf{i} + d\mathbf{j} \ : \ a \le t \le b\},\\
-\mathcal{C}_4 &:= \{\alpha_4(t) = a\mathbf{i} + t\mathbf{j} \ : \ c \le t \le d\},
\end{aligned}$$



Hence, we see that

$$\int_{\partial\Omega} \omega \;=\; \int_{\mathcal{C}_1} \omega + \int_{\mathcal{C}_2} \omega + \int_{-\mathcal{C}_3} \omega + \int_{-\mathcal{C}_4} \omega.$$

For $\mathcal{C}_1$, the parametrization is given by $\alpha_1 : [a,b] \to \mathbb{R}^2$, $\alpha(t) = t\mathbf{i} + c\mathbf{j}$. Therefore,

$$\int_{\mathcal{C}_1} \omega \;=\; \int_{\mathcal{C}_1} P(x,y)dx + Q(x,y)dy \;=\; \int_a^b P(t,c)dt,$$

since $dy = 0$ along $\mathcal{C}_1$. Similarly,

$$\begin{aligned}
\int_{\mathcal{C}_2} \omega &= \int_c^d Q(b,t)dt, \\
\int_{\mathcal{C}_3} \omega &= \int_a^b P(t,d)dt, \\
\int_{\mathcal{C}_4} \omega &= \int_c^d Q(a,t)dt.
\end{aligned}$$

Combining these results gives

$$\begin{aligned}
\int_{\partial\Omega} \omega &= \int_a^b P(t,c)dt + \int_c^d Q(b,t)dt - \int_a^b P(t,d)dt - \int_c^d Q(a,t)dt \\
&= \int_a^b (P(t,c) - P(t,d))dt + \int_c^d (Q(b,t) - Q(a,t))dt.
\end{aligned}$$

Replacing t with x in the first integral, and t with y in the second integral, we get

$$\int_{\partial\Omega} \omega \;=\; \int_a^b (P(x,c) - P(x,d))dx + \int_c^d (Q(b,y) - Q(a,y))dy,$$

which is the expression we obtained for $\iint_\Omega d\omega_{\mathbf{F}}$ in (4.2.2). □

Example 4.2.2. Let $\mathcal{C} \subset \mathbb{R}^2$ be the circle $\{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 9\}$. Compute

$$\int_{\mathcal{C}} (3y - e^{\sin(x)})dx + (7x + \sqrt{y^4 + 1})dy.$$

Solution. Let $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$ be the region with boundary $\partial\Omega = \mathcal{C}$. By Green's theorem,

$$\begin{aligned}
&\int_{\mathcal{C}} (3y - e^{\sin(x)})dx + (7x + \sqrt{y^4 + 1})dy \\
=\;& \int_\Omega \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4+1}) - \frac{\partial}{\partial y}(3y - e^{\sin(x)})\right] dxdy \\
=\;& \int_\Omega (7-3)\, dxdy \\
=\;& 4\int_\Omega dxdy \\
=\;& 4\text{area}(\Omega) \;=\; 4(9\pi) \;=\; 36\pi.
\end{aligned}$$

□

Example 4.2.3. Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution. Parametrize the ellipse by $x = a\cos(\theta)$ and $y = b\sin(\theta)$, for $0 \le \theta \le 2\pi$. Green's theorem applied to $1 = Q_x - P_y$ gives

$$\begin{aligned} \text{Area} = \int_D 1 dx \wedge dy &= \frac{1}{2}\int_{\mathcal{C}} xdy - ydx \\ &= \frac{1}{2}\int_0^{2\pi} a\cos(\theta)(b\cos(\theta))d\theta - (b\sin(\theta))(-a\sin(\theta))d\theta \\ &= \frac{ab}{2}\int_0^{2\pi} d\theta \\ &= \pi ab. \end{aligned}$$

□

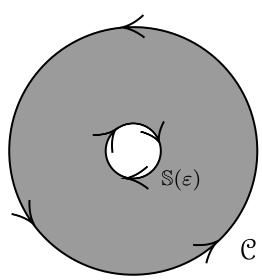
Example 4.2.4. Let $\mathbf{F} : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

Let $\mathcal{C} \subset \mathbb{R}^2 \backslash \{0\}$ be any closed path which circles the origin. Show that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \quad = \quad 2\pi.$$

Solution. Let $\varepsilon > 0$. Let $\mathbb{S}(\varepsilon) := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = \varepsilon^2\}$. Consider the region formed from removing $\mathbb{S}(\varepsilon)$ from the region bounded by $\mathcal{C}$. The positively-oriented boundary is given by $\mathcal{C} \cup (-\mathbb{S}(\varepsilon))$.



A parametrization of $\mathbb{S}(\varepsilon)$ is given by $x = \varepsilon\cos(\theta)$, $y = \varepsilon\sin(\theta)$. Green's theorem tells us that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} + \int_{-\mathbb{S}(\varepsilon)} \mathbf{F} \cdot d\mathbf{r} \quad = \quad \int_{D(\varepsilon)} (Q_x - P_y) dx \wedge dy,$$

where $P = \frac{-y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$. Compute

$$P_y \;=\; \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad Q_x \;=\; \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Hence,

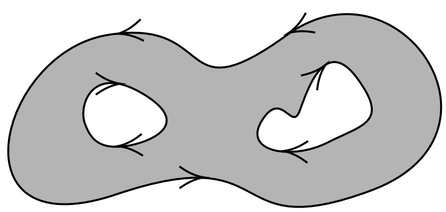
$$\int_{D(\varepsilon)} (Q_x - P_y)dx \wedge dy \;=\; \int_{D(\varepsilon)} \left(\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} \right) dx \wedge dy \;=\; 0,$$

and so

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; -\int_{-\mathbb{S}(\varepsilon)} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_{\mathbb{S}(\varepsilon)} \mathbf{F} \cdot d\mathbf{r}.$$

□

Orientation of Boundary. When attempting to determine the orientation of the boundary curves, the standard heuristic is the following: If you image walking along the boundary curves, the orientation is positive if the interior region is always to the left. For instance, if the boundary of Ω has multiple components, then a positive orientation is given by:



Aside for experts. The validity of Green's theorem under various relaxations of the regularity can be found in [8] and the references therein.

Exercises

1. Use Green's theorem to compute the line integrals $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where

(i) $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ and $\mathcal{C}$ is the unit circle.
(ii) $\mathbf{F}(x, y) = x\cos(y)\mathbf{i} + y\sin(x)\mathbf{j}$ and $\mathcal{C}$ is the boundary of the region between the circles $\mathbb{S}^1(1)$ and $\mathbb{S}^1(2)$.
(iii) $\mathbf{F}(x, y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ and $\mathcal{C}$ is the boundary of the square with vertices $(-1, -1)$, $(-1, 1)$, $(1, 1)$, and $(1, -1)$.
(iv) $\mathbf{F}(x, y) = (x\sin(y^2) - y^2)\mathbf{i} + (x^2 y\cos(y^2) + 3x)\mathbf{j}$ and $\mathcal{C}$ is the trapezoid with vertices $(0, -2)$, $(1, -1)$, $(1, 1)$, and $(0, 2)$.

2. Use Green's theorem to evaluate

$$\int_{\mathcal{C}} -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

where

(i) $\mathcal{C}$ is the arc of the parabola $y = \frac{1}{4}x^2 + 1$ from $(-2, 2)$ to $(2, 2)$.
(ii) $\mathcal{C}$ is the arc of the parabola $y = x^2 - 2$ from $(-2, 2)$ to $(2, 2)$.
(iii) What do the results of part (i) and (ii) indicate about the path independence or path dependence of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$?

2. Find the area of the region bounded by the curve parametrized by $\alpha(t) = \sin^3(t)\mathbf{i} + \cos^3(t)\mathbf{j}$, where $0 \le t \le 2\pi$.

3. Let $\mathbf{F} : \mathbb{R}^2 \backslash \{(1, 0)\} \to \mathbb{R}^2$ be the vector field

$$\mathbf{F}(x, y) := -\frac{y}{(x-1)^2 + y^2}\mathbf{i} + \frac{x-1}{(x-1)^2 + y^2}\mathbf{j}.$$

Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathcal{C}$ is the ellipse $\frac{x^2}{25} + \frac{y^2}{36} = 1$.

4. Let $\mathcal{C}$ be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$ oriented anti-clockwise. Use Green's theorem to evaluate

$$\int_{\mathcal{C}} \sqrt{9 + 3x^3}dx + 4xydy.$$

5. Use Green's theorem to evaluate

$$\int_{\mathcal{C}} x^2 ydx + yx^3 dy,$$

where $\mathcal{C}$ is the circle $(x-1)^2 + (y+1)^2 = 9$ oriented clockwise.

6. Let $\mathbf{F} : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x,y) := \frac{y^3}{(x^2+y^2)^2}\mathbf{i} - \frac{xy^2}{(x^2+y^2)^2}\mathbf{j}.$$

(i) Let $\mathcal{C}$ be the unit circle. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.
(ii) Evaluate $\int_{\mathcal{C}_0} \mathbf{F} \cdot d\mathbf{r}$, where $\mathcal{C}_0$ is the ellipse $\frac{x^2}{9} + \frac{y^2}{36} = 1$.

7. Find a smooth, simple, closed, counterclockwise oriented curve $\mathcal{C} \subset \mathbb{R}^2$ which maximizes the line integral

$$\int_{\mathcal{C}} (y^3 - y)dx - 2x^3 dy.$$

Is this maximum unique?

8. Evaluate the following line integrals using Green's theorem:

(i) $\int_{\mathcal{C}} (x^2 + y)dx - y^2 dy$, where $\mathcal{C}$ is the rectangle with vertices $(0,0)$, $(0,-2)$, $(2,2)$, and $(-1,1)$.
(ii) $\int_{\mathcal{C}} (x-y)dx + (x+y^4)dy$, where $\mathcal{C}$ is the triangle with vertices $(0,0)$, $(0,2)$, and $(1,1)$.
(iii) $\int_{\mathcal{C}} e^{-xy}dx + xdy$, where $\mathcal{C}$ is the circle centered at the origin with radius 4.
(iv) $\int_{\mathcal{C}} \sin(y)dx - \cos(x)dy$, where $\mathcal{C}$ is the parallelogram formed from the vectors $\mathbf{u} = (1,1)$ and $\mathbf{v} = (-1,2)$.
(v) $\int_{\mathcal{C}} (y^2 - \tan^{-1}(x))dx + (4x + \cos(y))dy$, where $\mathcal{C}$ is the boundary of the region enclosed by the parabola and the line $y = 3$.

9. Evaluate the line integrals $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ using Green's theorem, where

(i) $\mathbf{F}(x,y) = y^6\mathbf{i} + xy^5\mathbf{j}$ and $\mathcal{C}$ is the ellipse $4x^2 + y^2 = 1$.
(ii) $\mathbf{F}(x,y) = -x(x+y)\mathbf{i} + x^2y\mathbf{j}$ and $\mathcal{C}$ is the curve given by the line connecting $(0,0)$ to $(1,0)$, then the line connecting $(1,0)$ to $(0,1)$, then back to the origin.

10. Use Green's theorem to determine the area of the following regions:

(i) The region bounded by the hypocycloid $\alpha : [0, 2\pi] \to \mathbb{R}^2$,

$$\alpha(t) = \cos^3(t)\mathbf{i} + \sin^3(t)\mathbf{j}.$$

(ii) The region bounded by the curve parametrized by $\alpha : [0, 2\pi] \to \mathbb{R}$,

$$\alpha(t) = \cos(t)\mathbf{i} + \sin^3(t)\mathbf{j}.$$

(iii) The region bounded by the ellipse given by $2x^2 + 3y^2 = 4y$.
(iv) The region bounded by the ellipse given by $9x^2 + 4y^2 = 3x$.

11. Green's theorem requires the component functions of the vector field $\mathbf{F}$ to have continuous first-order partial derivatives. Let $\Omega = [0,1] \times [0,1]$ denote the unit square in $\mathbb{R}^2$. Let $\mathbf{F}(x,y) = 0\mathbf{i} + Q(x,y)\mathbf{j}$, where

$$Q(x,y) := \begin{cases} x^2 y \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(i) Show that the partial derivative Q_x exists.
(ii) Show that Q_x is not continuous on $\{(x,y) : x = 0,\ 0 < y \leq 1\}$.
(iii) Verify that Green's theorem holds for $\mathbf{F} : \Omega \to \mathbb{R}^2$.

12. Produce an example of a vector field for which Green's theorem does not hold.

13. Verify the statement of Green's theorem for

$$\int_{\mathcal{C}} xydx + x^2ydy,$$

where $\mathcal{C}$ is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$.

14. Use Green's theorem to evaluate $\int_{\mathcal{C}} \omega$, where

$$\omega = (1 + \tan x)dx + (x^2 + e^{-y})dy,$$

where $\mathcal{C}$ is the positively-oriented boundary of the region enclosed by the curves $y = \sqrt{x}$, $x = 1$ and $y = 0$.

15. Let $\mathcal{C}$ be the boundary of the square given by $0 \leq x \leq \frac{\pi}{3}$ and $0 \leq y \leq \frac{\pi}{3}$. Compute

$$\int_{\mathcal{C}} (\cos(x)\cos(y) + 3^{x^2})dx + (\sin(x)\sin(y) + \sqrt{y^4+1})dy.$$

16. Let $\mathcal{C}$ be the boundary of the region enclosed by the ellipse $x^2 + 4y^2 = 9$, lying above the line $x + 2y = 3$. Compute

$$\int_{\mathcal{C}} -2x^2y^2dx + 4x^3ydy.$$

17. Let Ω be the triangular region bounded by the curves $y = 1$, $x = 2$, and $y = x$. Let $\alpha = -ye^x dx + e^x dy$. Compute

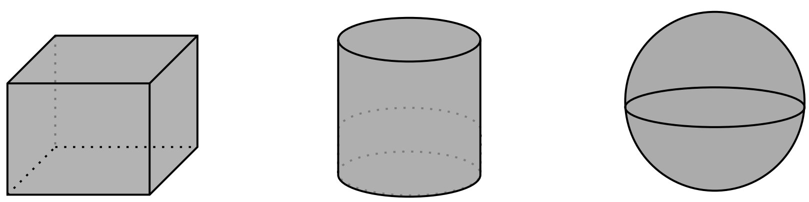
(i) the exterior derivative $d\alpha$.
(ii) the double integral $\iint_{\Omega} d\alpha$.
(iii) the line integral $\int_{\partial\Omega} \alpha$.

4.3. Surface integrals

Green's theorem applies to regions Ω in the plane. There is no reason to restrict to regions in $\mathbb{R}^2$, however. There is a more general Green's theorem which holds for vector fields defined on surfaces on $\mathbb{R}^2$. This version of the theorem is commonly referred to as the *Kelvin–Stokes theorem* and requires us to develop the machinery of surface integrals – two-dimensional versions of line integrals.

Reminder 4.3.1. Recall that a curve $\mathcal{C}$ is said to be *smooth* if there is a smooth parametrization $\alpha : I \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$ is an interval. In a similar manner, we have:

Definition 4.3.2. A surface $\mathcal{S}$ is said to be *smooth* if there is a smooth parametrization $\alpha : I \times J \to \mathbb{R}^n$, for intervals $I, J \subset \mathbb{R}$. If $\mathcal{S}$ can be expressed as the (finite) disjoint union of smooth surfaces, then $\mathcal{S}$ is said to be a *piecewise smooth surface*.



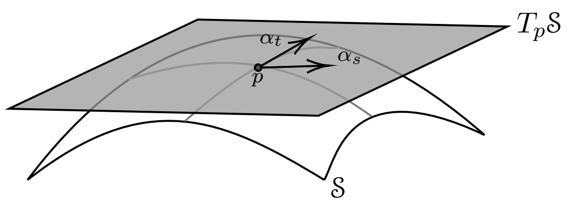
piecewise smooth piecewise smooth smooth

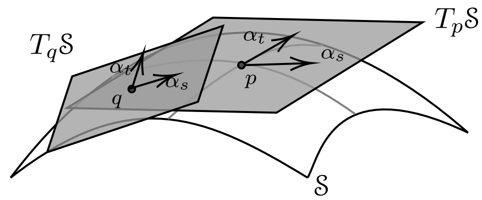
Remark 4.3.3. Throughout, we will use $\mathcal{S}$ to denote a surface in $\mathbb{R}^3$. Moreover, unless otherwise stated, all surfaces are assumed to be piecewise smooth.

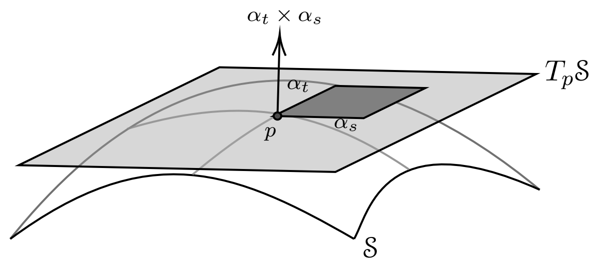
Reminder: Line Integrals. In defining the line integral of a (scalar) function f and subsequently, a vector field $\mathbf{F}$, we first obtained an expression for the arc length. The arc length of the curve $\mathcal{C}$ was given by adding up (i.e., integrating) the norms of the tangent vectors of a smooth parametrization $\alpha : [a, b] \to \mathbb{R}^n$:

$$\text{arc length}(\mathcal{C}) \ := \ \int_{\mathcal{C}} ds \ := \ \int_a^b |\alpha'(t)| dt.$$

In a similar manner, the surface area of a surface $\mathcal{S}$ will be given by adding up (i.e., integrating) the area of tangent planes formed by the two coordinate partial derivatives of a smooth parametrization $\alpha : [a, b] \times [c, d] \to \mathbb{R}^n$ (viewed as tangent vectors):







With $\alpha : [a,b] \times [c,d] \to \mathbb{R}^3$, $(s,t) \mapsto \alpha(s,t)$ denoting the parametrization, we write α_s and α_t for the tangent vectors. The area of the parallelogram $\alpha_s \wedge \alpha_t$ is given by $|\alpha_s \times \alpha_t|$.

Surface area formula. Hence, the surface area of a surface $\mathcal{S}$ parametrized by $\alpha : [a,b] \times [c,d] \to \mathbb{R}^3$ is given by

$$\text{surface area}(\mathcal{S}) \quad := \quad \int_{\mathcal{S}} d\mathbf{S} \;\; := \;\; \int_a^b \int_c^d \alpha_s \wedge \alpha_t.$$

Example 4.3.4. Let $\mathbb{S}^2$ be the unit sphere in $\mathbb{R}^3$:

$$\mathbb{S}^2 := \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

Compute the surface area of $\mathbb{S}^2$.

Solution. A parametrization for $\mathbb{S}^2$ is given by the spherical coordinate representation

$$\alpha : [0,2\pi] \times [0,\pi] \to \mathbb{R}^3, \qquad \alpha(\vartheta,\phi) = \cos(\vartheta)\sin(\phi)\mathbf{i} + \sin(\vartheta)\sin(\phi)\mathbf{j} + \cos(\phi)\mathbf{k}.$$

From *Spherical coordinates*, we know that

$$\alpha_\vartheta \wedge \alpha_\phi = \sin(\phi) d\vartheta \wedge d\phi.$$

Hence,

$$\begin{aligned} \text{surface area}(\mathbb{S}^2) &= \int_0^\pi \int_0^{2\pi} \sin(\phi) d\vartheta \wedge d\phi \\ &= 2\pi \left[-\cos(\phi)\right]_0^\pi \;=\; 4\pi. \end{aligned}$$

□

Example 4.3.5. Compute the surface integral

$$\iint_{\mathcal{S}} y^2 dS,$$

where $\mathcal{S} = \{x^2 + y^2 + z^2 = 1\}$.

Solution. Again, a parametrization for $\mathcal{S}$ is given by the spherical coordinate representation

$$\alpha : [0, 2\pi] \times [0, \pi] \to \mathbb{R}^3, \qquad \alpha(\vartheta, \phi) = \cos(\vartheta)\sin(\phi)\mathbf{i} + \sin(\vartheta)\sin(\phi)\mathbf{j} + \cos(\phi)\mathbf{k}.$$

From *Spherical coordinates*, we know that

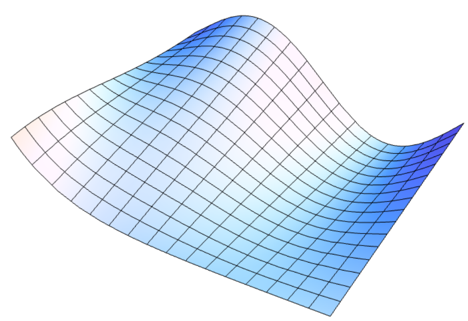
$$\alpha_\vartheta \wedge \alpha_\phi = \sin(\phi) d\vartheta \wedge d\phi.$$

Hence,

$$\begin{aligned} \iint_{\mathcal{S}} y^2 dS &= \int_0^\pi \int_0^{2\pi} (\sin(\vartheta)\sin(\phi))^2 \sin(\phi) d\vartheta d\phi \\ &= \int_0^\pi \sin^3(\phi) d\phi \int_0^{2\pi} \sin^2(\vartheta) d\vartheta = \frac{4\pi}{3}. \end{aligned}$$

□

Surface integrals of graphs of functions. Suppose our surface $\mathcal{S} \subset \mathbb{R}^3$ is given by the graph of a (smooth) function $f : \mathbb{R}^2 \to \mathbb{R}$.



The graph of a smooth function.

The previous ideas can easily be applied to give a formula for the surface integral over the graph of f. Indeed, if $\mathcal{S}$ is the graph of f, then a parametrization is given by

$$\alpha : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \alpha(s, t) = s\mathbf{i} + t\mathbf{j} + f(s, t)\mathbf{k}.$$

Hence, the coordinate partial derivatives of α are $\alpha_s = \mathbf{i} + 0\mathbf{j} + f_s\mathbf{k}$ (where $f_s = \frac{\partial f}{\partial s}$) and $\alpha_t = 0\mathbf{i} + \mathbf{j} + f_t\mathbf{k}$ (where $f_t = \frac{\partial f}{\partial t}$). The cross product is computed:

$$\alpha_s \times \alpha_t = \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_s \\ 0 & 1 & f_t \end{pmatrix} = -f_s\mathbf{i} - f_t\mathbf{j} + \mathbf{k}.$$

The norm is then

$$|\alpha_s \times \alpha_t| = \sqrt{1 + (f_s)^2 + (f_t)^2},$$

and we have:

Formula for the surface integral over the graph of a function. If $\mathcal{S}$ is a surface given by the graph of a smooth function $f : \Omega(\subseteq \mathbb{R}^2) \to \mathbb{R}^3$, then

$$\int_{\mathcal{S}} d\mathbf{S} = \iint_{\Omega} \sqrt{1 + (f_s)^2 + (f_t)^2} dsdt.$$

If $g : \mathcal{S} \to \mathbb{R}$ is a smooth function on the surface $\mathcal{S}$ (given by the graph of f), then

$$\int_{\mathcal{S}} gd\mathbf{S} = \iint_{\Omega} g(x(s,t), y(s,t), z(s,t))\sqrt{1 + (f_s)^2 + (f_t)^2} dsdt.$$

Example 4.3.6. Evaluate $\int_{\mathcal{S}} ydS$, where $\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 : 0 \le x \le 1, 0 \le y \le 2, z = x + y^2\}$.

Solution. Since $g(x,y) = x + y^2$, we have $\partial_x g = 1$ and $\partial_y g = 2y$. Apply the formula for the surface integral when the surface is given by $z = g(x,y)$ to get:

$$\begin{aligned} \int_{\partial\Omega} ydS &= \int_{\Omega} y\sqrt{1 + (\partial_x g)^2 + (\partial_y g)^2} dA \\ &= \int_0^1 \int_0^2 y\sqrt{1 + 1^2 + (2y)^2} dxdy \\ &= \int_0^1 dx \int_0^2 y\sqrt{2 + 4y^2} dy \\ &= \int_0^2 y\sqrt{2 + 4y^2} dy \\ &= \frac{13\sqrt{2}}{3}. \end{aligned}$$

□

Formula for the surface integral of a vector field over the graph of a function. If $\mathbf{F} : \mathcal{S} \to \mathbb{R}^3$ is a smooth vector field on the surface $\mathcal{S}$ (given by the graph of f), then

$$\int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\Omega} \mathbf{F} \cdot (\alpha_s \times \alpha_t) ds dt.$$

If $\mathbf{F} = P(s,t)\mathbf{i} + Q(s,t)\mathbf{j} + R(s,t)\mathbf{k}$, then since $\alpha_s \times \alpha_t = -f_s\mathbf{i} - f_t\mathbf{j} + \mathbf{k}$, we have

$$\mathbf{F} \cdot (\alpha_s \times \alpha_t) = -P(s,t)f_s - Q(s,t)f_t + R(s,t).$$

In other words,

$$\int_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \iint_{\Omega} (R(s,t) - P(s,t)f_s - Q(s,t)f_t) ds dt.$$

Example 4.3.7. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$\mathbf{F}(x,y,z) = -y\mathbf{i} + x\mathbf{j} + 3z\mathbf{k}.$$

If $\mathcal{S}$ is the upper-hemisphere of radius 1, compute

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

Solution. We have $\alpha(\vartheta, \phi) = \sin(\phi)\cos(\vartheta)\mathbf{i} + \sin(\phi)\sin(\vartheta)\mathbf{j} + \cos(\phi)\mathbf{k}$, where $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \vartheta \le 2\pi$. We know that

$$\alpha_\phi \times \alpha_\vartheta = \sin^2(\phi)\cos(\vartheta)\mathbf{i} + \sin^2(\phi)\sin(\vartheta)\mathbf{j} + \sin(\phi)\cos(\phi)\mathbf{k},$$

and moreover,

$$\mathbf{F}(\vartheta, \phi) = -\sin(\phi)\sin(\vartheta)\mathbf{i} + \sin(\phi)\cos(\vartheta)\mathbf{j} + 3\sin(\phi)\cos(\phi)\mathbf{k}.$$

Hence,

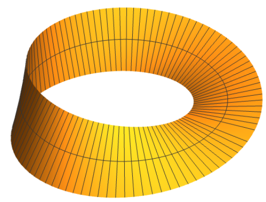
$$\begin{aligned}
\mathbf{F}(\vartheta,\phi) \cdot (\alpha_\phi \times \alpha_\vartheta) &= (-\sin(\phi)\sin(\vartheta)\mathbf{i} + \sin(\phi)\cos(\vartheta)\mathbf{j} + 3\sin(\phi)\cos(\phi)\mathbf{k}) \\
&\quad \cdot \big(\sin^2(\phi)\cos(\vartheta)\mathbf{i} + \sin^2(\phi)\sin(\vartheta)\mathbf{j} + \sin(\phi)\cos(\phi)\mathbf{k}\big) \\
&= -\sin^3(\phi)\sin(\vartheta)\cos(\vartheta) + \sin^3(\phi)\cos(\vartheta)\sin(\vartheta) + 3\sin^2(\phi)\cos^2(\phi) \\
&= 3\sin^2(\phi)\cos^2(\phi).
\end{aligned}$$

Computing the surface integral now gives

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 3\sin^2(\phi)\cos^2(\phi) d\vartheta d\phi \\
&= 6\pi \int_0^{\frac{\pi}{2}} \sin^2(\phi)\cos^2(\phi) d\phi \\
&= \frac{3\pi^2}{8}.
\end{aligned}$$

□

Orientation. The Möbius strip is the prototypical example of a non-orientable surface:



The Möbius strip.

Historical remarks. As early as 1760 Lagrange had given an explicit expression for the element of surface dS in the process of calculating surface areas. It was not until 1811, however, in the second edition of his Mécanique analytique, that Lagrange introduced the general notion of a surfae integral.

Exercises

1. Compute the area of the following regions:

(i) The helicoid surface parametrized by the function $f : [0,1] \times [0,2\pi] \to \mathbb{R}$, defined by $f(x,y) := x\cos(y)\mathbf{i} + x\sin(y)\mathbf{j} + y\mathbf{k}$.

(ii) The part of the paraboloid $y = z^2 + x^2$ which lies in the sphere $x^2 + y^2 + z^2 = 9$.

(iii) The hyperbolic surface $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$.

(iv) The surface given by rotating $z = e^{-y}$, for $0 \le y \le 1$, around the y–axis.

(v) The part of the sphere $x^2 + y^2 + z^2 = 1$ which lies between $z = -\frac{1}{2}$ and $z = \frac{1}{2}$.

2. Compute the following surface integrals $\int_{\mathcal{S}} f dS$, where

(i) $f(x,y,z) = x+y+z$ and $\mathcal{S}$ is the hemi-sphere $\mathcal{S} = \{(x,y,z) : x^2+y^2+z^2 = 1, z \ge 0\}$.

(ii) $f(x,y,z) = xy+yz$ and $\mathcal{S}$ is the triangular region with vertices $(0,1,0)$, $(1,0,2)$ and $(0,0,3)$.

(iii) $f(x,y,z) = z + \sqrt{x+y}$ and $\mathcal{S} = \{(x,y,z) : x^2 + y^2 = 9,\ -1 \le z \le 1\}$.

3. Find the center of mass of the hemisphere $x^2+(y-1)^2+(z+1)^2 = 4$ assuming the density is constant.

4. Find the mass of the cone $z = 2\sqrt{x^2+y^2}$, where $0 \le z \le 6$, if $\rho(x,y,z) := 1 - z$ is the density function.

5. Let $\rho = 10$ denote the density of a fluid. Let $\mathbf{v} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$ denote the velocity of the fluid. Determine the rate of flow upward through the paraboloid

$$z = 9 - \frac{1}{4}(x^2 + y^2)$$

if $x^2 + y^2 \le 49$.

6. Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit sphere. Evaluate the surface integral

$$\int_{\mathbb{S}^2} (x^2 + y + z) dA.$$

7. Evaluate

$$\int_{\Omega} (x^3 - 3xy^2) dx dy,$$

where

$$\Omega = \{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 \le 9,\ (x-1)^2 + y^2 \ge 1\}.$$

8. Let $\mathcal{S}$ be the part of the surface $x^2 + y^2 + z = 2$ that lies above the square $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$.

(i) Compute $\int_{\mathcal{S}} \frac{x^2+y^2}{\sqrt{1+x^2+y^2}} dS$.

(ii) Compute the flux of $\mathbf{F} = -x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ upward through $\mathcal{S}$.

9. Let $\mathcal{S}$ be the part of the surface $f(x, y) = xy(x + 1)$ that lies over the square $(x, y) \in [0, 3] \times [0, 3]$.

(i) Compute $\int_{\mathcal{S}} \frac{x^2(y+1)}{\sqrt{1+x^2+y^2}} dS$.

(ii) Compute the flux of $\mathbf{F} = x^2\mathbf{i} + (y - x)\mathbf{j} + (z + 3)\mathbf{k}$ upward through $\mathcal{S}$.

10. Find the area of the part of the surface $z = \sqrt{y^3 + 1}$ that lies above the square $(x, y) \in [-1, 0] \times [1, 0]$.

11. Let $\mathcal{S}$ be the surface given by the equation

$$x^2 + y^2 = \cos^2(z),$$

lying between the planes $z = 0$ and $z = \frac{\pi}{2}$. Evaluate

$$\int_{\mathcal{S}} \sqrt{1 + \sin^2(z)} dS.$$

4.4. Stokes' theorem

The fundamental theorem of calculus

$$\int_a^b f'(x)dx \;=\; f(b) - f(a)$$

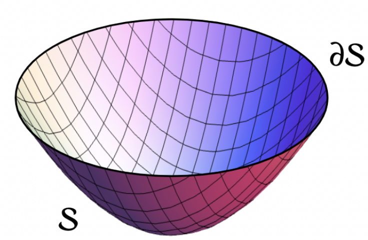
relates the integral of $f'(x)$ over $[a,b] \subseteq \mathbb{R}$ to the behaviour of the $f(x)$ on the boundary $\{a,b\}$. The fundamental theorem of line integrals

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = \nabla f(b) - \nabla f(a)$$

relates the integral of a gradient field $\mathbf{F} = \nabla f$ over $\mathcal{C}$ to the behaviour of its potential f on the boundary of $\mathcal{C}$. Green's theorem

$$\iint_\Omega \mathrm{curl}(\mathbf{F}) \cdot \mathbf{k} dA \;=\; \int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{r}$$

relates the curl of $\mathbf{F}$, or equivalently $d\omega_{\mathbf{F}}$, over $\Omega \subset \mathbb{R}^2$, to the behaviour of $\mathbf{F}$, or equivalently $\omega_{\mathbf{F}}$, over the boundary of Ω. Stokes' theorem relates the surface integral of $\mathrm{curl}(\mathbf{F})$, or equivalently $d\omega_{\mathbf{F}}$, over a surface $\mathcal{S}$, to the behaviour of $\mathbf{F}$, or equivalently $\omega_{\mathbf{F}}$, over the boundary of $\mathcal{S}$:



Theorem 4.4.1. (Stokes' theorem). Let $\mathcal{S}$ be an oriented piecewise-smooth surface, with smooth boundary $\partial\mathcal{S}$. Let $\mathbf{F} : \mathcal{S} \to \mathbb{R}^3$ be a smooth vector field on $\mathcal{S}$. Then

$$\int_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

Proof. Let $\omega_{\mathbf{F}} = Pdx + Qdy + Rdz$ be the 1–form associated to $\mathbf{F}$. Then from **??**, we know that $\mathrm{curl}(\mathbf{F})dV = d\omega_{\mathbf{F}}$. Hence, if

$$\int_{\partial\mathcal{S}} \omega_{\mathbf{F}} \;=\; \iint_{\mathcal{S}} d\omega_{\mathbf{F}}, \tag{4.4.1}$$

then

$$\int_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_{\partial\mathcal{S}} \omega_{\mathbf{F}} \;=\; \iint_{\mathcal{S}} d\omega_{\mathbf{F}} \;=\; \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F})dV$$

It remains to (4.4.1), which will be treated later. $\square$

Remark 4.4.2. (Green's theorem). Observe that if $\mathcal{S}$ is a region $\Omega \subset \mathbb{R}^2$, and $\mathbf{F} : \Omega \to \mathbb{R}^2$, then $\mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S} = \mathrm{curl}(\mathbf{F}) \cdot \mathbf{k} dA$, and we recover Green's theorem

$$\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{r} = \iint_{\Omega} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{k} dA.$$

Example 4.4.3. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) := 2z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$. Let $\mathbb{S}^2_+ := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \geq 0\}$. Verify Stokes' theorem in this case.

Solution. We want to verify that the result obtained from computing the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ coincides with the result obtained from evaluating the surface integral $\int_{\mathbb{S}^2_+} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S}$. Let us first compute the line integral: The boundary of $\mathbb{S}^2_+$ is given by the curve $\mathcal{C} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ z = 0\}$. This is parametrized by the curve $\alpha : [0, 2\pi] \to \mathbb{R}^3$ defined by $\alpha(t) := (\cos(t), \sin(t), 0)$. Hence,

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (0, \cos(t), \sin(t)) \cdot (-\sin(t), \cos(t), 0) dt \\ &= \int_0^{2\pi} \cos^2(t) dt \ = \ \pi. \end{aligned}$$

Let us now compute the surface integral: The curl of $\mathbf{F}$ is $\mathrm{curl}(\mathbf{F}) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. In spherical coordinates, $\alpha : [0, 2\pi] \times [0, \pi/2] \to \mathbb{R}^2$, where $\alpha(\vartheta, \phi) = \sin(\vartheta)\cos(\phi)\mathbf{i} + \sin(\vartheta)\sin(\phi)\mathbf{j} + \cos(\vartheta)\mathbf{k}$. Hence,

$$\alpha_\phi \times \alpha_\vartheta = \sin^2(\phi)\cos(\vartheta)\mathbf{i} + \sin^2(\phi)\sin(\vartheta)\mathbf{j} + \sin(\phi)\cos(\phi)\mathbf{k},$$

and we compute

$$\begin{aligned} \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) d\mathbf{S} &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \left(\sin^2(\phi)(\cos(\vartheta) + 2\sin(\vartheta)) + \sin(\phi)\cos(\phi) \right) d\phi d\vartheta \\ &= \int_0^{2\pi} \left(\frac{1}{4}(2 + \pi\cos(\vartheta) + 2\pi\sin(\vartheta)) \right) d\vartheta \ = \ \pi. \end{aligned}$$

$\square$

Example 4.4.4. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) := -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, and let $\mathcal{C}$ be the curve given by the intersection of $y + z = 2$ and $x^2 + y^2 = 1$. Orient $\mathcal{C}$ to be counterclockwise when viewed from above. Evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

using Stokes' theorem.

Solution. The curl of $\mathbf{F}$ is calculated to be

$$\mathrm{curl}(\mathbf{F}) \ = \ (1 + 2y)\mathbf{k}.$$

There are many surfaces with boundary $\mathcal{C}$, but the most convenient choice is the elliptical region S given in the plane $y+z=2$ that is bounded by $\mathcal{C}$. If we orient S upward, then $\mathcal{C}$ has the induced positive orientation. The projection D of S on the (x,y)–plane is the disk $x^2+y^2 \leq 1$. Hence, with $z = g(x,y) = 2-y$, we have

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} &= \iint_S \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} \\
&= \int_D (1+2y)dA \\
&= \int_0^{2\pi}\int_0^1 (1+2r\sin\vartheta)r\,dr\,d\vartheta \\
&= \int_0^{2\pi}\left[\frac{1}{2}r^2+\frac{2}{3}r^3\sin\vartheta\right]_0^1 d\vartheta \;=\; \pi.
\end{aligned}$$

□

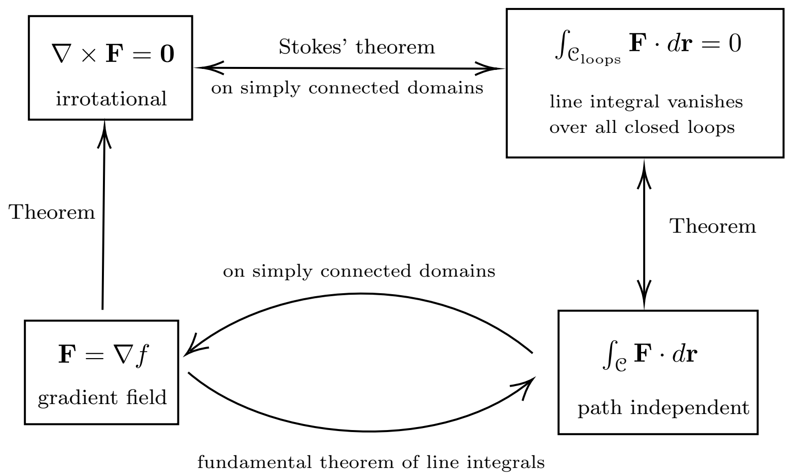
Theorem 4.4.5. Let $\mathbf{F}$ be an irrotational vector field on a simply connected region Ω. Then, for any curve $\mathcal{C}$ in Ω, the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ is independent of path. Moreover, $\mathbf{F}$ is a gradient field.

PROOF. Suppose $\mathbf{F}$ is an irrotational vector field. Then $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$. By Stokes' theorem *Theorem 4.4.1*, for any closed loop $\mathcal{C}$, we have

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} \;=\; \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} \;=\; 0.$$

In particular, the line integral over $\mathbf{F}$ over all closed loops $\mathcal{C}$ is zero. By *Theorem 3.2.6*, it follows that the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ is independent of path. By *Theorem 3.2.15*, it follows that $\mathbf{F}$ is a gradient field. □

Irrotational Vector Fields are Gradient Fields on Simply Connected Regions.


$\nabla \times \mathbf{F} = \mathbf{0}$
irrotational
Stokes' theorem
on simply connected domains
$\int_{\mathcal{C}_{\text{loops}}} \mathbf{F} \cdot d\mathbf{r} = 0$
line integral vanishes
over all closed loops
Theorem
Theorem
on simply connected domains
$\mathbf{F} = \nabla f$
gradient field
$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$
path independent
fundamental theorem of line integrals

Exercises

1. Use Stokes' theorem to evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where

(i) $\mathbf{F}(x,y,z) = (x-y)\mathbf{i} + (y-z)\mathbf{j} + z^2\mathbf{k}$ and $\mathcal{C}$ is the triangle with vertices $(1,0,0)$, $(0,1,3)$ and $(0,2,-1)$.

(ii) $\mathbf{F}(x,y,z) = e^{-x}\mathbf{i} - e^{-y}\mathbf{j} + e^{-z}\mathbf{k}$ and $\mathcal{C}$ is the boundary of the paraboloid $z = 2 - x^2 - y^2$ in the first orthant.

2. Let $\mathbf{F}(x,y,z) := xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$ and let $\mathbb{H} := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1, z \geq 0\}$. Evaluate the surface integral

$$\int_{\mathbb{H}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

3. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. For any sphere $\mathbb{S}^2(r) \subset \mathbb{R}^3$, compute the surface integral

$$\int_{\mathbb{S}^2} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

4. Let $\Omega \subseteq \mathbb{R}^3$ be a surface with boundary $\partial\Omega$. Let $u, v : \mathbb{R}^3 \to \mathbb{R}$ be two smooth functions. Show that

$$\int_{\partial\Omega} (u\nabla v) \cdot d\mathbf{r} = \int_{\Omega} (\nabla u \times \nabla v) \cdot d\mathbf{S}.$$

5. Let $\mathcal{C}$ be a smooth curve in $\mathbb{R}^3$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Show that

$$\int_{\mathcal{C}} (f\nabla f) \cdot d\mathbf{r} \quad = \quad 0.$$

6. Let $\mathcal{C}$ be a smooth curve in $\mathbb{R}^3$. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be two smooth functions. Show that

$$\int_{\mathcal{C}} (f\nabla g + f\nabla f) \cdot d\mathbf{r} = 0.$$

7. Let $\mathcal{C}$ be the curve given by the intersection of the plane whose normal vector is $\mathbf{n} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, with the cylinder $x^2 + y^2 = 4$. Orient this curve anti-clockwise, when viewed from above.

(i) Determine the equation of the plane.

(ii) Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ using Stokes' theorem.

8. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$\mathbf{F}(x, y, z) := (x^3 - axz^2)\mathbf{i} + (x^2y + bz)\mathbf{j} + cy^2z\mathbf{k}.$$

Let $\mathcal{C}$ be the curve given by the intersection of the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with the plane $x + y + z = 0$. Determine the values of $a, b, c \in \mathbb{R}$ such that

$$\int_S \mathbf{F} \cdot d\mathbf{S}$$

is independent of the surface S whose boundary is $\partial S = \mathcal{C}$.

9. Determine (with justification) whether the following statements are true or false:

(i) Gradient fields are irrotational on simply connected domains.

(ii) Irrotational vector fields are gradient fields on convex domains.

(iii) If $\mathbf{F}$ is an irrotational vector field, then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for all curves $\mathcal{C}$.

10. Use Stokes' theorem to evaluate

$$\int_{\mathcal{C}} y dx + z dy + x dz,$$

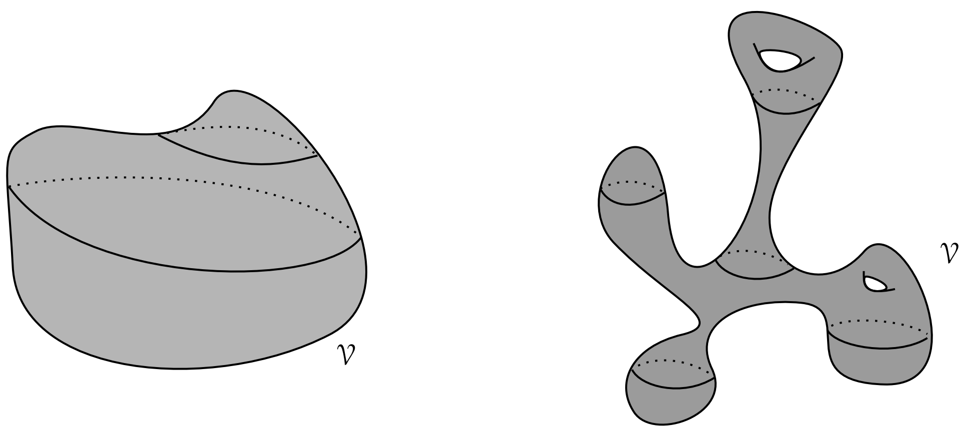
where $\mathcal{C}$ is the intersection of $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$.

4.5. The Divergence Theorem

Green's theorem related the double integral of the curl of a two-dimension vector field $\mathbf{F}$, i.e., $\iint_\Omega \operatorname{curl}(\mathbf{F})dA$, to the line integral of $\mathbf{F}$ over the curve which is the boundary of Ω, i.e., $\int_{\partial\Omega} \mathbf{F}\cdot d\mathbf{r}$. The Stokes' theorem from the previous section extends this (more or less) without change: $\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = \int_{\partial\mathcal{S}} \mathbf{F}\cdot d\mathbf{r}$. Of course, Green's theorem relates the line integral to a double integral, while Stokes' theorem relates the surface integral to a line integral, but this change is minor – Green's theorem is merely a specific instance of Stokes' theorem.

We saw that the curl of a vector field is a particular notion of derivative for a vector field. Indeed, if $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$, then the 1–form associated to $\operatorname{curl}(\mathbf{F})$ just $\star d\omega_{\mathbf{F}}$. Stokes' theorem then merely translates to

$$\iint_{\mathcal{S}} d\omega_{\mathbf{F}} = \int_{\partial\mathcal{S}} \omega_{\mathbf{F}}.$$



We have seen another notion of derivative for a vector field, however, namely, the divergence $\operatorname{div}(\mathbf{F})$. In terms of forms, the divergence is merely the vector field associated to the 1–form $\star d \star \omega_{\mathbf{F}}$. In the same way that we obtained Stokes' theorem (and hence, Green's theorem), we obtain the divergence theorem:

Theorem 4.5.1. (Divergence theorem). Let $\mathcal{V} \subseteq \mathbb{R}^3$ be a region in $\mathbb{R}^3$ with boundary $\partial\mathcal{V}$. Let $\mathbf{F} : \mathcal{V} \to \mathbb{R}^3$ be a smooth vector field on Ω. Then

$$\int_{\partial\mathcal{V}} \mathbf{F}\cdot d\mathbf{S} \;=\; \int_{\mathcal{V}} \operatorname{div}(\mathbf{F})dV.$$

PROOF. Let $\omega_{\mathbf{F}} = Pdx + Qdy + Rdz$ be the 1–form associated to $\mathbf{F}$. Then from **??**, we know that $\mathrm{div}(\mathbf{F})dV = d\star\omega_{\mathbf{F}}$. Hence, if

$$\int_{\partial\mathcal{V}} \omega_{\mathbf{F}} \;=\; \iint_{\mathcal{V}} d\omega_{\mathbf{F}}, \tag{4.5.1}$$

then

$$\int_{\partial\mathcal{V}} \mathbf{F}\cdot d\mathbf{S} \;=\; \int_{\partial\mathcal{V}} \omega_{\mathbf{F}} \;=\; \iiint_{\mathcal{V}} d\omega_{\mathbf{F}} \;=\; \iiint_{\mathcal{V}} \mathrm{div}(\mathbf{F})dV$$

It remains to (4.5.1), which will be treated later. □

Example 4.5.2. Compute the flux of the vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, where $\mathbf{F}(x,y,z) := z\mathbf{i} + y\mathbf{j} + x\mathbf{k}$, over the unit sphere $\mathbb{S}^2 := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$.

SOLUTION. The divergence of $\mathbf{F}$ is

$$\mathrm{div}(\mathbf{F}) \;=\; \partial_x(z) + \partial_y(y) + \partial_z(x) \;=\; 1.$$

The sphere $\mathbb{S}^2$ is the boundary of the ball $\mathbb{B}^2 := \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 1\}$. Hence,

$$\begin{aligned} \int_{\mathbb{S}^2} \mathbf{F}\cdot d\mathbf{S} \;=\; \int_{\mathbb{B}^2} \mathrm{div}(\mathbf{F})dV \;&=\; \int_{\mathbb{B}^2} 1dV \\ &=\; \mathrm{vol}(\mathbb{B}^2) \;=\; \frac{4\pi}{3}. \end{aligned}$$

□

Example 4.5.3. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x,y,z) = (4x + y^{102} - z^{23}, y^2 + \sin(xe^{z^{10}}), z+1).$$

Suppose $\Omega := \{(x,y,z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq y \leq 3,\ 0 \leq z \leq 2\}$. Compute

$$\int_{\partial\Omega} \mathbf{F}\cdot d\mathbf{S}.$$

SOLUTION. Compute the divergence:

$$\mathrm{div}(\mathbf{F}) \;=\; 5 + 2y.$$

By the divergence theorem,

$$\begin{aligned} \int_{\partial\Omega} \mathbf{F}\cdot d\mathbf{S} \;&=\; \int_{\Omega} \mathrm{div}(\mathbf{F})dV \\ &=\; \int_0^1\int_0^3\int_0^2 (5+2y)dxdydz \\ &=\; 2\int_0^3 (5+2y)dy \\ &=\; 2\left[5y + y^2\right]_0^3 \\ &=\; 2(15+9) \;=\; 48. \end{aligned}$$

$\square$

Gauss' law. Let $\mathbf{E}$ be an electric field with vacuum permittivity ε_0. Denote by ρ the volume charge density. Gauss' law states that the surface integral of $\mathbf{E}$ over a closed surface $\mathcal{S}$ is given by

$$\iint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0},$$

where Q is the charge. The divergence theorem offers a different expression for Gauss' law. Indeed, let $\mathcal{V} \subseteq \mathbb{R}^3$ be the region such that $\partial\mathcal{V} = \mathcal{S}$. By the divergence theorem,

$$\iiint_{\mathcal{V}} \operatorname{div}(\mathbf{E}) dV \quad = \quad \iint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{S} \quad = \quad \frac{Q}{\varepsilon_0}.$$

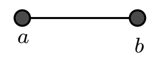
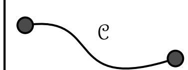
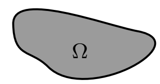
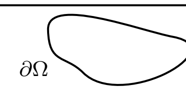
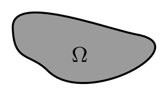
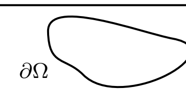
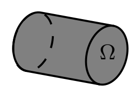
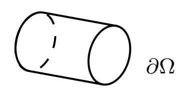
The charge Q is given by $Q = \iiint_{\mathcal{V}} \rho dV$, and therefore,

$$\iiint_{\mathcal{V}} \operatorname{div}(\mathbf{E}) dV = \iiint_{\mathcal{V}} \frac{\rho}{\varepsilon_0} dV.$$

Since this holds for all regions $\mathcal{V}$ with boundary $\partial\mathcal{S}$, the integrands must coincide, and therefore,

$$\operatorname{div}(\mathbf{E}) = \frac{\rho}{\varepsilon_0}.$$

Summary of Main Theorems.

Theorem	Statement	Notion of derivative	Region of integration	Boundary
Fundamental theorem of calculus	$\int_a^b f'(x)dx = f(b) - f(a)$	$f'(x)$		
Fundamental theorem of line integrals	$\int_{\mathcal{C}} \nabla f dr = f(b) - f(a)$	∇f		
Green's theorem	$\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{r} = \int_{\Omega} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{k} dA$	$\mathrm{curl}(\mathbf{F}) \cdot \mathbf{k}$		
Stokes' theorem	$\int_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) dA$	$\mathrm{curl}(\mathbf{F})$		
Divergence theorem	$\int_{\partial\mathcal{V}} \mathbf{F} \cdot d\mathbf{S} = \int_{\mathcal{V}} \mathrm{div}(\mathbf{F}) dV$	$\mathrm{div}(\mathbf{F})$		

Exercises

1. Let $\mathbf{F} : \Omega \to \mathbb{R}^3$ be a vector field which is always normal to $\partial\Omega$. Show that

$$\int_{\Omega} \mathrm{curl}(\mathbf{F}) dV = 0.$$

2. Suppose $\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r} = -\frac{1}{c}\frac{\partial}{\partial t}\int_{\Omega} \mathbf{H} \cdot d\mathbf{S}$, where Ω is any surface bounded by the closed curve $\mathcal{C}$. Show that

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}.$$

3. Let $\mathbf{F} : \mathbb{S}^2 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) = (x - 3y)\mathbf{i} + (4 + 9z^3)\mathbf{j} + (y^2 - 10z)\mathbf{k}.$$

(i) Compute $\mathrm{div}(\mathbf{F})$.
(ii) Compute $\int_{\mathbb{S}^2} \mathbf{F} \cdot d\mathbf{S}$.

4. Let Σ be some closed surface in $\mathbb{R}^3$. Let ρ denote the density of some fluid in Σ. The mass of the fluid in Σ is then given by

$$\int_{\Sigma} \rho dV,$$

while the total mass of the fluid flowing out of Σ in unit time is given by

$$\int_{\partial\Sigma} \rho\mathbf{v} \cdot d\mathbf{S}.$$

Show that

$$\frac{\partial\rho}{\partial t} + \mathrm{div}(\rho\mathbf{v}) = 0.$$

5. Let Σ be some closed surface in $\mathbb{R}^3$. Let p denote the pressure of a fluid inside Σ. The total force acting on the volume is given by

$$-\int_{\partial\Sigma} p d\mathbf{S}.$$

(i) Determine the function φ such that $-\int_{\partial\Sigma} p d\mathbf{S} = -\int_{\Sigma} \varphi dV$.
(ii) Use Newton's second law of motion to show that

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p,$$

where $\mathbf{v}$ is the velocity of the fluid, and ρ is the density.
(iii) Show that

$$\frac{\partial\mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla\mathbf{v} = -\frac{1}{\rho}\nabla p.$$

(iv) Determine the corresponding equation when the fluid experiences a graviational force $\mathbf{g}$.

6. Let $\mathcal{V}$ be a region in $\mathbb{R}^3$ for which the divergence theorem holds. Show that the volume of $\mathcal{V}$ is given by

$$\operatorname{vol}(\mathcal{V}) = \frac{1}{3}\iint_{\partial\mathcal{V}} x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

7. Let $\mathbf{F}(x, y, z) = 2x\mathbf{i} + 3xy\mathbf{j} - yz^2\mathbf{k}$ and let $\mathcal{S}$ be the surface bounded by the planes $x = 0$, $x + y = 2$, $y = 0$, $z = 0$, and $z = 2$. Compute the surface integral

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

8. Find the flux of $\mathbf{F}(x, y, z) = xz^2\mathbf{i} + (x^2y - z^3)\mathbf{j} + (2xy + y^2z)\mathbf{k}$ outwards, across the entire surface of the hemisphere bounded by $z = \sqrt{r^2 - x^2 - y^2}$ and $z = 0$.

9. Compute

$$\int_{\mathcal{C}} x^2y^3 dx + dy + z dz,$$

where $\mathcal{C}$ is the curve given by traversing along the circle $x^2 + y^2 = r^2$ anti-clockwise.

10. Use the divergence theorem to evaluate the following surface integrals

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where

(i) $\mathbf{F}(x, y, z) = 3y^2z^3\mathbf{i} + 9x^2yz^2\mathbf{j} - 4xy^2\mathbf{k}$ and $\mathcal{S}$ is the surface of the region bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$.

(ii) $\mathbf{F}(x, y, z) = -xz\mathbf{i} - yz\mathbf{j} + z^2\mathbf{k}$ and $\mathcal{S}$ is the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$.

(iii) $\mathbf{F}(x, y, z) = 3xy\mathbf{i} + y^2\mathbf{j} - x^2y^4\mathbf{k}$ and $\mathcal{S}$ is the surface of the tetrahedron with vertices $(0, 1, 0)$, $(0, 0, 0)$, $(1, 0, 0)$, and $(0, 0, 1)$.

(iv) $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and $\mathcal{S}$ is the unit sphere centered at the origin in $\mathbb{R}^3$.

11. Let $\mathbf{F}(x, y, z) = y^2\mathbf{i} + xz\mathbf{j} - (x + y)\mathbf{k}$.

(i) Find the vector field $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl}(\mathbf{G})$.

(ii) Hence, evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem, where $\mathcal{S}$ is any closed surface.

12. Let $\mathcal{S}$ be the surface of the solid that lies above the xy–plane and below the surface $z = 2 - x^4 - y^4$, where $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. Evaluate

$$\iint_{\mathcal{S}} e^y \tan(z) dx + y\sqrt{3 - y^2} dy + x \sin(y) dz.$$

13. Let $\mathcal{S}$ be the surface of the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z - y + 3 = 0$. Evaluate

$$\iint_{\mathcal{S}} ye^{z^2} dx + y^2 dy + e^{xy} dz.$$

14. Let $\mathbf{F}$ be a vector field such that $\mathbf{F} \neq \text{curl}(\mathbf{G})$ for any vector field $\mathbf{G}$.

(i) What can be concluded about the divergence of $\mathbf{F}$?
(ii) Hence, what can be concluded about the surface integrals $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$?

15. Let $\mathcal{S}$ be the surface $z = 1 - x^2 - y^2$ for $x^2 + y^2 \leq 1$, and $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$.

(i) Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ directly.
(ii) Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem.

16. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field such that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \pi(r^3 + 2r^4),$$

for every $r > 0$, where $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = r^2$. Determine the value of $\nabla \cdot \mathbf{F}$ at the origin.

17. State which of the following theorem: (**G**) Green's theorem, (**S**) Stokes' theorem, (**D**) Divergence theorem, would be most appropriate to address the following problems:

(i) Computing a line integral by computing a surface integral.
(ii) Compute a double integral by computing a line integral.
(iii) Compute a surface integral by computing a triple integral.
(iv) Showing that a line integral of an irrotational vector field on a simply connected domain is independent of path.

CHAPTER 5

The High Road to Vector Calculus

"If there is one thing in mathematics that fascinates me more than anything else (and doubtless always has), it is neither 'number' nor 'size,' but always form."

– Alexander Grothendieck

The results we have presented – namely, the Clairaut theorem, Green's theorem, Stokes' theorem, and the divergence theorem, have been known since the nineteenth century. The manner in which it has been presented, however, is relatively more modern, and is really due to Henri Poincaré. The present chapter is intended to exhibit the theory which has been lying in the background.

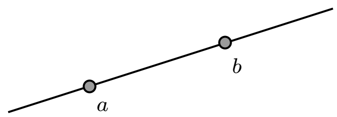
We will not exhibit the more intimidating generality of Stokes' theorem on orientable manifolds, but see how the machinery we have already developed can be used in surprising and unexpected ways.

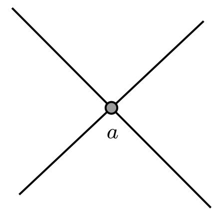
Throughout the course, I have hopefully, with some success, indoctrinated the reader into believing the supernatural and miraculous nature of the fundamental theorem of calculus. We have seen four new variants of this theorem: The fundamental theorem of line integrals, Green's theorem, Stokes' theorem, and the divergence theorem. In all cases, the fundamental theorem has related the integral of a derivative $\int_M d\omega$ to the integral over the boundary $\int_{\partial M} \omega$. This is no small observation – It indicates a *duality* between forms and their exterior derivative, and regions of integration and their boundaries.

5.1. Duality

Duality is one of the central notions in modern mathematics, and physics, so let us give some details concerning its meaning. Following [2], duality is not a theorem, but a *principle*; fundamentally, duality gives two different points of view when looking at the same thing (two sides of a coin).

Example. The most elementary example of a duality is given by points and lines in $\mathbb{R}^2$. Given two points, $a, b \in \mathbb{R}^2$, there is a unique line which passes through them. On the other hand, two lines $\mathcal{L}_1, \mathcal{L}_2 \subseteq \mathbb{R}^2$ meet at a unique point $a \in \mathbb{R}^2$ so long as they are not parallel.





This asymmetry in the duality, namely that one has to impose the restriction that lines are not paralell to intersect at a unique point, can be bypassed by adding a point *at* ∞. This leads one to *projective geometry.* In this case, the duality between lines and points is perfectly symmetric.

Examples. The fundamental example coming from physics is Maxwell's discovery of the duality between electricity and magnetism; specifically, the electric field $\mathbf{E}$ is dual to the magnetic field $\mathbf{B}$.
We have, in fact, been using a duality since the beginning: the duality between vector fields and 1–forms.
Another duality that we have seen quite often is the Hodge $\star$–operator, which exhibits a duality between k–forms and $(3-k)$–forms on $\mathbb{R}^3$. More generally, it exhibits a duality between k–forms and $(n-k)$–forms on $\mathbb{R}^n$.

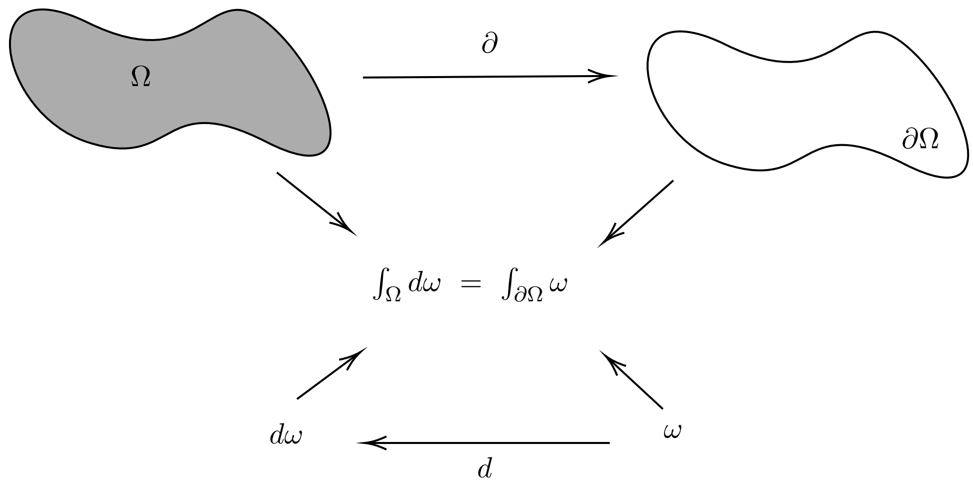
Remark. We have remarked already that a duality expresses some relationship between two classes objects, allowing us to view them as two sides of a single coin. The method of passing between these two sides (or of flipping the coin, so to speak) is referred to as the *dual pairing.*

Example. For the Hodge duality between k–forms and $(3-k)$–forms, the dual pairing is given by the Hodge $\star$–operator.
For the duality between vector fields and 1–forms, the duality is given by the dot product.

We now want to understand how the fundamental theorem of calculus is an example of a duality. The two sides of the duality are given by

(i) regions of integration Ω,
(ii) differential forms ω.

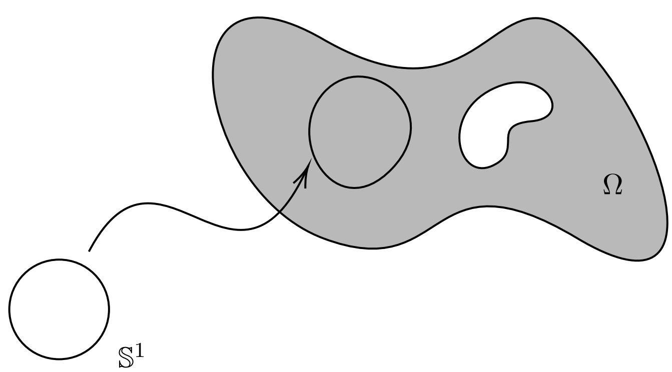
Note that in both cases, we have a derivative operator: For (i), the derivative operator is denoted by ∂, and maps a region of integration Ω to its boundary $\partial\Omega$; for (ii), the derivative operator is denoted by d, and is the exterior derivative mapping ω to $d\omega$. In this instance, the dual pairing is given by integration.



5.2. Homology

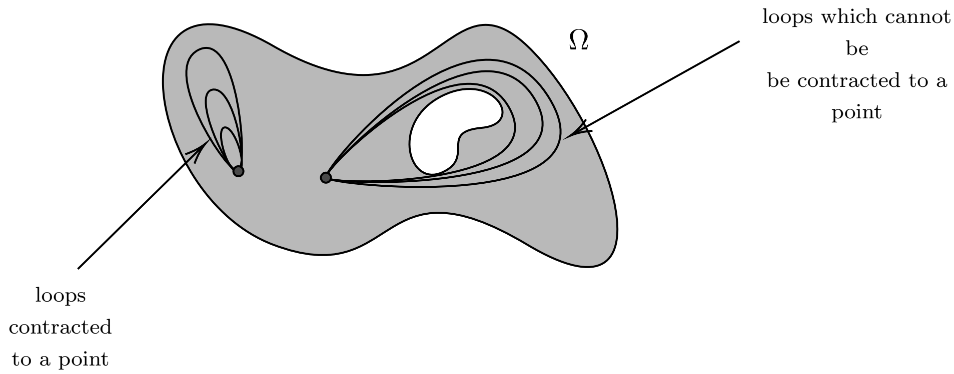
Recall that we have seen that the exterior derivative d is nilpotent, in the sense that $d^2 = 0$. [1] The same is true for the operator ∂. Indeed, if Ω is a region of integration, the boundary is $\partial\Omega$, but the boundary of $\partial\Omega$ is then empty. This allows one to define homology and cohomology. Homology is a tool for studying the properties of a space Σ by looking at maps *into* Ω.

For instance, recall that a domain $\Omega \subset \mathbb{R}^2$ is simply connected if it (more or less) has no holes. We can test for simply connectedness by looking at loops in Ω.



If every loop can be contracted to a point, then Ω must be simply connected:

[1]This computation essentially reduces to Clairaut's theorem. As we showed in the proof of *Clairaut's theorem*, however, *Clairaut's theorem* follows from the fundamental theorem of calculus.



The notion of simply connectedness is understood by means of homotopy theory (a branch of algebraic topology).

Definition. Let $\Omega \subset \mathbb{R}^n$ be a domain. The fundamental group $\pi_1(\Omega)$ of Ω is the space of all (homotopy classes) of loops in Ω.

Theorem. A domain $\Omega \subset \mathbb{R}^n$ is simply connected if and only if $\pi_1(\Omega) = 0$.

5.3. Cohomology

On the other hand, recall that every gradient field $\mathbf{F} = \nabla f$ is irrotational, i.e.,

$$\operatorname{curl}(\nabla f) = \mathbf{0}.$$

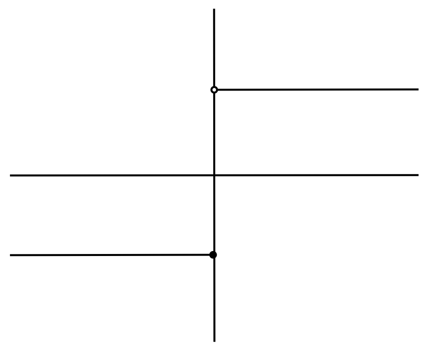
But, an irrotational vector field will fail to be a gradient field, in general, unless the domain of the vector field is simply connected. From the manner in which the subject was developed, we viewed this as a statement about vector fields. We can nevertheless flip the theorem on its head, however, and view it as a statement about the domain of the vector field. This leads to the notion of cohomology:

Definition. Let Ω be a domain in $\mathbb{R}^n$. We write $\mathcal{Z}^k(\Omega)$ for the space of closed k–forms on Ω, i.e., k–forms ω such that $d\omega = 0$. We write $\mathcal{B}^k(\Omega)$ for the subspace (of $\mathcal{Z}^k(\Omega)$) of exact k–forms, i.e., k–forms ω such that $\omega = d\eta$ for some $(k-1)$–form $\eta \in \Lambda^{k-1}(\Omega)$.

Definition. The *de Rham cohomology groups* $H^k_{\mathrm{DR}}(\Omega, \mathbb{R})$ we defined to be the quotient vector spaces

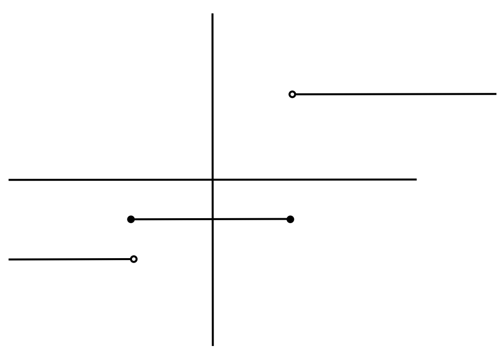
$$H^k_{\mathrm{DR}}(\Omega, \mathbb{R}) \ := \ \mathcal{Z}^k(\Omega)/\mathcal{B}^k(\Omega).$$

Example. The de Rham cohomology group $H^0_{\mathrm{DR}}(\Omega, \mathbb{R})$ is not so hard to understand. Indeed, consider a function $f : \mathbb{R} \to \mathbb{R}$ such that $f'(x) = 0$ for all $x \in \mathbb{R}$. Then, from our calculus of a single real variable, we know that f must be constant. Suppose now that $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ satisfies $f'(x) = 0$ for all $x \in \mathbb{R}\backslash\{0\}$; is f constant? The answer is no, as illustrated by the following example:



The heaviside function satisfies $f'(x) = 0$ for all $x \in \mathbb{R}\backslash\{0\}$, but is not constant.

Similarly, if $f : \mathbb{R}\backslash\{-1, 1\} \to \mathbb{R}$ satisfies $f'(x) = 0$ for all $x \in \mathbb{R}\backslash\{-1, 1\}$, then f is permitted to assume 3 distinct values:



Hence, we see that $H^0_{\mathrm{DR}}(\Omega, \mathbb{R})$ measures the number of connected components of Ω.

Example. For $k = 1$, the group $H^1_{\mathrm{DR}}(\Omega, \mathbb{R})$ measures the failure of every closed 1–form being an exact 1–form, i.e., when a 1–form ω can be written as $\omega = df$ for some function $f : \Omega \to \mathbb{R}$. In terms of vector fields, since $\mathrm{curl}(\mathbf{F})$ is the vector field associated to $\star d\omega_{\mathbf{F}}$, this is equivalent to asking whether an irrotational vector field is a gradient field. In **??**, we saw this was the case if Ω was simply connected. If Ω is a domain in $\mathbb{R}^2$, we can flip this theorem on its head, and view **??** not as a statement about vector fields, but a statement about the properties of the domain Ω:

Theorem. Let $\Omega \subseteq \mathbb{R}^2$ be a region in $\mathbb{R}^2$. If every irrotational vector field on Ω is a gradient field, then Ω is simply connected.

PROOF. If every irrotational vector field on Ω is a gradient field, then every 1–form $\omega \in \Lambda^1(\Omega)$ is exact, i.e., $\omega = df$, for some smooth function $f : \Omega \to \mathbb{R}$. By [20, p. 142], the fundamental group $\pi_1(\Omega)$ is free, and by Hurewicz's theorem, $H^1_{\mathrm{DR}}(\Omega, \mathbb{R})$ is the free abelian group with the same number of generators as $\pi_1(\Omega)$. Hence, if $H^1_{\mathrm{DR}}(\Omega, \mathbb{R}) = 0$, then $\pi_1(\Omega) = 0$, and Ω is simply connected. □

Remark. If Ω is an open set in $\mathbb{R}^3$, it is no longer true that, if every irrotational vector field on Ω is a gradient field, then Ω is simply connected. By de Rham's theorem, we know that $H^1_{\mathrm{DR}}(\Omega, \mathbb{R}) = 0$ if and only if $H_1(\Omega, \mathbb{Z}) = 0$, where $H_1(\Omega, \mathbb{Z})$ denotes the first singular homology group of Ω. Of course, the vanishing of $H_1(\Omega, \mathbb{Z}) = 0$ does not imply the vanishing of $\pi_1(\Omega)$, the most famous example being the exterior of the Alexander Horned sphere:

Example. One of the core examples throughout the course was the vector field

$$\mathbf{F}(x, y) \ := \ -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

This provides an example of a vector field whose line integral around any closed loop containing the origin is non-zero (c.f., **??**). This is despite the fact that

$$\mathbf{F} \ = \ \nabla \tan^{-1}\left(\frac{y}{x}\right).$$

The key point here is that the potential (in this case, $\tan^{-1}(y/x)$) is not smooth at the origin. If one would like gradient fields to have all the nice properties (e.g., path independence), then the potential is required to be smooth (or at, least $\mathcal{C}^1$). The 1–form $\omega_{\mathbf{F}}$ is closed, but not exact, and generates $H^1_{\mathrm{DR}}(\mathbb{R}^2 \backslash \{0\}, \mathbb{R})$.

Example. For $k = 2$, the group $H^2_{\mathrm{DR}}(\Omega, \mathbb{R})$ measures the failure of every closed 2–form being exact. Since $\mathrm{div}(\mathbf{F}) = \star d \star \omega_{\mathbf{F}}$, and $\star$ is linear, the second de Rham cohomology group measures the extent to which every incompressible vector field (i.e., $\mathrm{div}(\mathbf{F}) = 0$) is the curl of some vector field ($\mathbf{F} = \mathrm{curl}(\mathbf{G})$). The sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ is an example of a simply connected space (hence, $H^1_{\mathrm{DR}}(\mathbb{S}^2, \mathbb{R}) = 0$), but $H^2_{\mathrm{DR}}(\mathbb{S}^2, \mathbb{R}) \neq 0$.

5.4. Hodge Theory

Recall that the Hodge $\star$–operator is a linear map sending k–forms on $\mathbb{R}^n$ to $(n-k)$–forms on $\mathbb{R}^n$, i.e.,

$$\star : \Lambda^k(\mathbb{R}^n) \to \Lambda^{n-k}(\mathbb{R}^n).$$

One of the most important uses of the Hodge $\star$–operator is that it permits us to define the Hodge Laplacian, and subsequently, harmonic forms – these give canonical representatives of cohomology classes and can be used to prove substantial theorems.

Before defining the Hodge Laplacian, we need to make the following preliminary definition:

Definition. Let d denote the exterior derivative. The *codifferential* $\delta : \Lambda^k(\mathbb{R}^n) \to \Lambda^{k-1}(\mathbb{R}^n)$ is defined by

$$\delta \ := \ (-1)^{\cdots} \star d \star .$$

Appendix – Foundational Theory of Differential Forms

The main components of the text do not give a very rigorous definition of the wedge product and exactly what differential forms are. These details are collected in the present appendix.

A.1. Linear Algebra

To begin speaking about the formal definition of a differential form, we need to remind ourselves of some notions from linear algebra:

Definition. A (real) vector space is a set V together with two operations:

(i) (vector addition). If $u, v \in V$, then $u + v \in V$.
(ii) (scalar multiplication). If $v \in V$ and $\lambda \in \mathbb{R}$, then $\lambda v \in V$.

Definition. Let V and W be two vector spaces. A map $f : V \to W$ is said to be *linear* if

(i) (additivity). $f(u + v) = f(u) + f(v)$, for all $u, v \in V$.
(ii) (homogeneity). $f(\lambda v) = \lambda f(v)$, for all $\lambda \in \mathbb{R}$, $v \in V$.

Definition. A linear map $f : V \to W$ between vector spaces is said to be an *isomorphism* if f is one-to-one and onto.

Definition. Let V be a vector space. Let $V^k := V \times \cdots \times V$ (k–times) be the k–fold cartesian product. A map $f : V^k \to \mathbb{R}$ is said to be *multi-linear* if it is linear in each of its arguments.

Remark. When $k = 2$, a multilinear map is said to be *bilinear*.

Definition. Let V be a finite-dimensional real vector space. We define a *covector* on V to be a linear map $\omega : V \to \mathbb{R}$. The space of all covectors forms a vector space – the *dual space* to V – and is denoted by V^*.

Proposition. Let V be a finite-dimensional vector space. Let $\{v_1, ..., v_n\}$ be any basis for V. Then the covectors $\varepsilon^1, ..., \varepsilon^n$ defined by

$$\varepsilon^k(v_j) \;=\; \begin{cases} 1, & j = k, \\ 0, & \text{otherwise.} \end{cases}$$

form a basis for V^* – the *dual basis* to $\{v_1, ..., v_n\}$.

Definition. Let V be a finite-dimensional (real) vector space. A *covariant k–tensor* (respectively, a *contravariant ℓ–tensor*) on V is a multi-linear map $f : V^k \to \mathbb{R}$ (respectively, a multi-linear map $f : (V^*)^\ell \to \mathbb{R}$). A tensor *of type* (k, ℓ) is a multi-linear map

$$f : (V^*)^\ell \times V^k \to \mathbb{R}.$$

The space of all covariant k–tensors is denoted by $T^k(V)$, the space of contravariant ℓ–tensors on V is denoted by $T_\ell(V)$, and the space of all (k, ℓ)–tensors is denoted by $T^k_\ell(V)$.

Remark. The spaces $T^k(V)$ and $T_\ell(V)$ form real vector spaces with respect to the obvious operations.

Examples.

(i) A covariant 1–tensor is a covector $\omega : V \to \mathbb{R}$. Hence, $T^1(V)$ coincides with the dual vector space V^*.
(ii) A covariant 2–tensor is a real-valued bilinear form $B : V \times V \to \mathbb{R}$. The dot product on $\mathbb{R}^n$ is an example of a covariant 2–tensor.
(iii) If we view the determinant of an $n \times n$ matrix as a multi-linear map on the columns of the matrix (viewed as vectors) then the determinant is a covariant n–tensor on $\mathbb{R}^n$.

Proposition. Let V be a real vector space of dimension n. Let $\{v_1, ..., v_n\}$ be a basis for V with dual basis $\{\varepsilon^1, ..., \varepsilon^n\}$. The covariant k–tensors of the form

$$\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},$$

for $1 \leq i_1, ..., i_k \leq n$ is a basis for $T^k(V)$. Hence, $\dim T^k(V) = n^k$.

Proposition. Let V and W be finite-dimensional real vector spaces. If $A : V \times W \to Z$ is a bilinear map into any vector space Z, there is a unique linear map $\widehat{A} : V \otimes W \to X$ such that the diagram

$$\begin{tikzcd}
V \times W \arrow[r, "A"] \arrow[d, "\pi"'] & Z \\
V \otimes W \arrow[ur, "\hat{A}"'] &
\end{tikzcd}$$

Definition. Let V be a finite-dimensional vector space. The *tensor algebra* of V is defined

$$T(V) \ := \ \bigoplus_{k=0}^{\infty} T^k(V).$$

The multiplication on $T(V)$ is defined via the tensor product:

$$T^k(V) \otimes T^\ell(V) \longrightarrow T^{k+\ell}(V),$$

extended linearly to all of $T(V)$.

Definition. Let $T(V)$ denote the tensor algebra of a finite-dimensional vector space V. The quotient of $T(V)$ by the two-sided ideal generated by $v \otimes v$, for $v \in V$, defines the *exterior algebra.*

Definition. Let $\pi : T(V) \to \Lambda(V)$ denote the quotient map. We define the *wedge product* of two elements $\alpha, \beta \in \Lambda(V)$ by

$$\alpha \wedge \beta \ := \ \pi(A \otimes B),$$

where $\pi(A) = \alpha$ and $\pi(B) = \beta$.

Remark. The reader may easily verify that the wedge product is well-defined, independent of the choice of representatives.

Remark. The exterior algebra affords the grading

$$\Lambda(V) \ = \ \bigoplus_{k=0}^{\infty} \Lambda^k(V),$$

where $\Lambda^k(V)$ denotes the *kth exterior algebra.* The kth exterior algebra forms a subspace $\Lambda^k(V) \subseteq \Lambda(V)$ spanned by the wedge of k elements of V.

Remark. If $\{v_1, ..., v_n\}$ is a basis for V, then a basis for $\Lambda^k(V)$ is given by

$$\{e_{i_1} \wedge \cdots \wedge e_{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

In particular, $\dim \Lambda^k(V) = \binom{n}{k}$.

A.2. Topological Spaces

The standard reference for this section is [**?**].

Definition. A *topological space* is a pair (X, τ), where X is a non-empty set and τ is a family of subsets which satisfy:

(i) $\emptyset \in \tau$ and $X \in \tau$.

(ii) τ is closed under arbitrary unions in the sense that if $\mathcal{U}_\alpha \in \tau$, then

$$\bigcup_\alpha \mathcal{U}_\alpha \in \tau.$$

(iii) τ is closed under finite intersections in the sense that if $\mathcal{U}_\alpha \in \tau$, then

$$\bigcap_\alpha \mathcal{U}_\alpha \in \tau.$$

The family of subsets τ is called a *toplogy*, and the elements of a topology are called *open sets*. The complement of an open set is called a *closed set*.

Definition. Let (X, τ) be a topological space. We say that (X, τ) is a *Hausdorff space* if for every pair of distinct points $x, y \in X$, there exists $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, $x \notin \mathcal{V}$, and $y \notin \mathcal{U}$.

Remark. In a Hausdorff topological space, a sequence of points can have at most one limit.

Definition. Let (X, τ) and (Y, σ) be two topological spaces. A map $f : X \to Y$ is said to be *continuous* if for every $V \in \sigma$, we have $f^{-1}(V) \in \tau$.

In other words, a map is continuous if the preimage of an open set is open.

Definition. A continuous map $f : (X, \tau) \to (Y, \sigma)$ is a *homeomorphism* if f is invertible with $f^{-1} : (Y, \sigma) \to (X, \tau)$ continuous.

A.3. Smooth Manifolds

Let M be a connected Hausdorff topological space with a countable base of open sets. For any arbitrary indexing set A, we assume M admits a covering $\mathcal{U} := (\mathcal{U}_\alpha)_{\alpha \in A}$ by connected open sets $\mathcal{U}_\alpha \subset M$ which are homeomorphic to balls $\mathbb{B}_\alpha \subset \mathbb{R}^n$. The pair $(\mathcal{U}_\alpha, \varphi_\alpha)$, where $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{B}_\alpha$ is a homeomorphism, is called a *chart*, and the set of charts $\mathscr{A} := \{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is said to be the *atlas* of the covering $\mathcal{U}$.

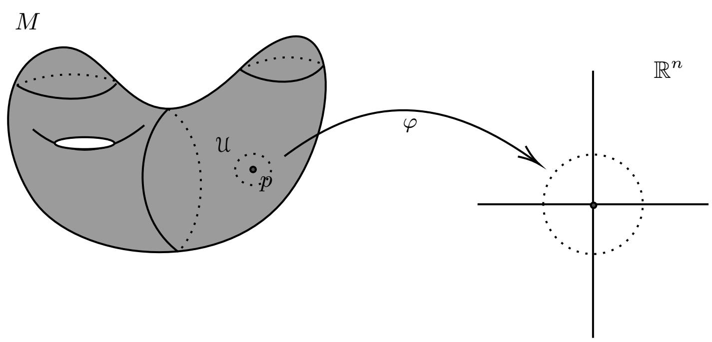
Remark. The charts permit one to locally identify a neighborhood of a point in M with a neighborhood of the origin in some Euclidean space $\mathbb{R}^n$. In particular, if $(x_1, ..., x_n)$ denote the coordinates on $\mathbb{R}^n$, these coordinates can be pulled back via the homeomorphism $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{B}_\alpha \subset \mathbb{R}^n$ to furnish *local coordinates* on M, and hence, $\mathcal{U}_\alpha$ is sometimes called a *coordinate chart*.
On any overlap of $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, the composition

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_{\alpha\beta}) \longrightarrow \varphi_\beta(\mathcal{U}_{\alpha\beta})$$

defines a homeomorphism between open subsets of $\mathbb{R}^n$, which we call *transition maps*. These transition maps allow one to make sense of the *regularity* of M. Namely, if the transition maps are of class $\mathcal{C}^k$, for some $k \in \mathbb{N}$, we say that the atlas $\mathscr{A}$ is a $\mathcal{C}^k$*–atlas*. If $\mathscr{A}$ is a $\mathcal{C}^k$–atlas for all $k \in \mathbb{N}$, we say $\mathscr{A}$ is a $\mathcal{C}^\infty$*–atlas* or a *smooth atlas*.

Remark. To remove the dependence on the choice, let us declare two $\mathcal{C}^k$–atlases $\mathscr{A}$ and $\mathscr{B}$ to be *equivalent* if their union is a $\mathcal{C}^k$–atlas. This defines an equivalence relation on the $\mathcal{C}^k$–atlases of M and ensures that the transition maps from the charts of one atlas to the charts of the other atlas have the same regularity as the regularity of the constituent transition maps for each atlas separately.

Definition. A $\mathcal{C}^k$*–manifold* is a connected Hausdorff topological space M endowed with an equivalence class of $\mathcal{C}^k$–atlases. The dimension of the balls to which the domains of the covering $\mathcal{U}$ are homeomorphic is called the (real) *dimension* of M, and is denoted $\dim_{\mathbb{R}} M$.



Definition. Let M be a manifold of class $\mathcal{C}^k$. A function $f : M \to \mathbb{R}$ is said to be of class $\mathcal{C}^\ell$, for some $\ell \leq k$, if the composite map $f \circ \varphi_\alpha^{-1}$ is of class $\mathcal{C}^\ell$ on the open set $\varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$. Similarly, if M and N are two $\mathcal{C}^k$ manifolds, a map $f : M \to N$ is said to be of class $\mathcal{C}^\ell$, for some $\ell \leq k$, if the composite map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is of class $\mathcal{C}^\ell$ on the open set $\varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$. These notions are clearly well-defined.

The tangent space. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. To any point $x \in \Omega$, we can assign a *tangent space* to Ω at x, which we denote by $T_x\Omega$. We identify $T_x\Omega = \{(x, v) : x \in \Omega,\ v \in \mathbb{R}^n\}$, and define a *tangent vector* to Ω at x to be an element of $T_x\Omega$. The tangent space $T_x\Omega$ is a (real) vector space of dimension n. This coincides with our familiar understanding of tangent vectors to functions, which is seen as follows:
Let $v_x \in T_x\Omega$ be a tangent vector. Then we can define a map $D_v|_x : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$ which is defined to be the directional derivative of f at x in the direction of v:

$$D_v|_x f \quad := \quad D_v f(x) \ = \ \frac{d}{dt}\bigg|_{t=0} f(x + tv).$$

This operation is linear and satisfies the Leibniz rule:

$$D_v|_x(fg) = f(x) D_v|_x g + g(x) D_v|_x f.$$

If we write $v_p = \sum_{k=1}^{n} v^k e_k|_p$ in terms of the standard basis of $\mathbb{R}^n$ (restricted to Ω), then, by the chain rule, $D_v|_x f$ can be written as

$$D_v|_p f \;=\; v^k \frac{\partial f}{\partial x_k}(p).$$

Definition. Let $p \in M$. A linear map $X : \mathcal{C}^\infty(M) \to \mathbb{R}$ is said to be a *derivation at* p if it satisfies the Leibniz rule:

$$X(fg) \;=\; f(p)X(g) + g(p)X(f).$$

We let $\mathrm{Der}_p(M)$ denote the set of all derivations of $\mathcal{C}^\infty(M)$ at p.

Remark. It is an elementary exercise to show that $\mathrm{Der}_p(M)$ forms a (real) vector space under the operations $(X+Y)(f) = X(f) + Y(f)$, and $(\lambda X)(f) = \lambda(X(f))$ for all $\lambda \in \mathbb{R}$, $f \in \mathcal{C}^\infty(M)$, and $X, Y \in \mathrm{Der}_p(M)$.

Theorem. For any $p \in M$, the map $v_p \mapsto D_v|_p$ is an isomorphism from T_pM onto $\mathrm{Der}_p(M)$.

Corollary. For any $p \in M$, the derivations

$$\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p,$$

defined by

$$\left.\frac{\partial}{\partial x_k}\right|_p f \;=\; \frac{\partial f}{\partial x_k}(p),$$

form a basis for T_pM.

Definition. Let M be a smooth manifold with T_pM the tangent space to M at $p \in M$. The *cotangent space* at p, denoted by T_p^*M, is the dual space to T_pM:

$$T_p^*M \;:=\; (T_pM)^*.$$

Example. Let $\partial_{x_1}|_p, ..., \partial_{x_n}|_p$ denote the coordinate partial derivatives at a point $p \in \Omega$, which we can view as either tangent vectors or as derivations. These provide a basis for the tangent space $T_x\Omega$, and the corresponding dual basis is defined $dx_1, ..., dx_n$.

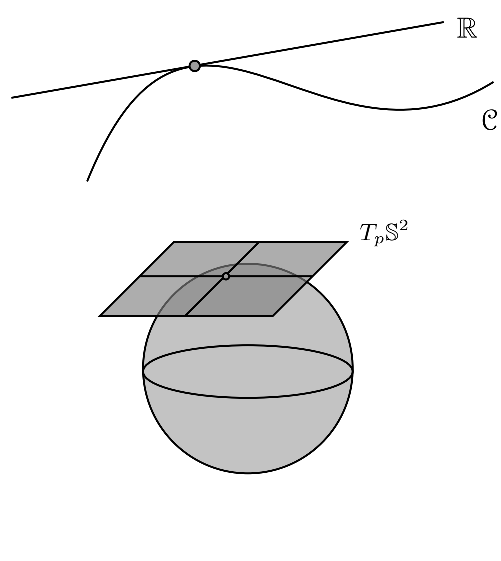
A.4. Vector Bundles

This linear algebraic picture works if we keep the point $p \in \Omega$ fixed. But, of course, this is very restrictive – this would demand that all functions are constants we do not allow p to vary. If we allow p to vary, however, then we need to permit the tangent spaces $T_p\Omega$, and the cotangent spaces $T_p^*\Omega$ to vary.
This leads to the notion of a vector bundle. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. For each $p \in \Omega$, let V_p be a (real) vector space of dimension k. Set $V = \coprod_{p\in\Omega} V_p$ and let $\pi : V \to \Omega$ be the map which projects V_p to p.

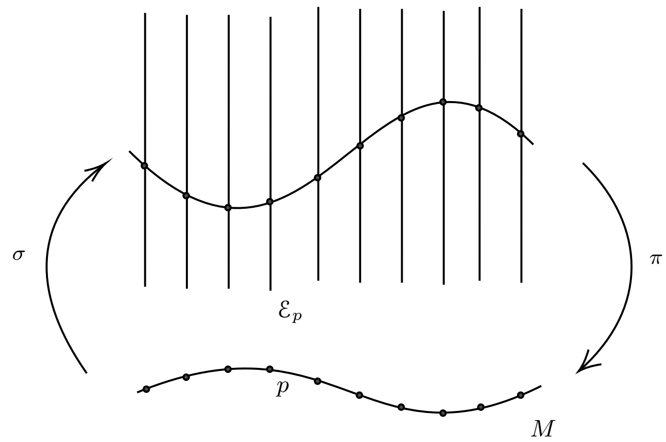
Definition. Let $\pi : \mathcal{E} \to M$ be a smooth map of smooth manifolds. We say that π is a *rank k vector bundle* if the fibers $\mathcal{E}_p := \pi^{-1}(p)$ are vector spaces of rank k and for any point $p \in M$, there is an open neighbourhood $\mathcal{U} \subset X$ of p such that $\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{R}^k$.

Example. Let $TM = \coprod_{p\in M} T_pM$. Then $\pi : TM \to M$ denotes the *tangent bundle*. Let $T^*M := \coprod_{p\in M} T_p^*M$. Then $\pi : T^*M \to M$ is called the *cotangent bundle*.

Example. The tangent to a curve gives an example of the tangent bundle:



Definition. Let $\pi : \mathcal{E} \to \Omega$ be a vector bundle. A section of $\mathcal{E}$ is a smooth map $\sigma : \Omega \to \mathcal{E}$ such that $\pi \circ \sigma = \mathrm{id}$.



Example.

(i) If $\mathcal{E} \to \Omega$ is the trivial bundle with $\mathcal{E}_p = \mathbb{R}$, then a section $\sigma : \Omega \to \mathcal{E}$ is just a function $\sigma : \Omega \to \mathbb{R}$.

(ii) A vector field is a smooth section of the tangent bundle.

(iii) A 1–form is a smooth section of the cotangent bundle.

(iv) A k–form is a smooth section of the kth exterior power of the cotangent bundle bundle.

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