Lectures on Vector Calculus

Kyle Broder – April 4th 2022

"*If only I could understand the meaning of* $d^2 = 0$."

– Henri Cartan

Introduction

Vector calculus is an extremely beautiful subject and is a subject that is essential to the modern development of calculus. To place the subject in the appropriate context, let us remind ourselves that long ago, we learned the calculus of functions of a single real variable. The theory is segregated into two primary branches: the *differential calculus* and the *integral calculus.*

The first branch gives us a set of tools for understanding how a function changes, and the rate at which this occurs; the main gadget to come out of this theory is the *derivative* d. The second branch treats infinite sums and gives us a reasonable theory for computing area and volume; the main apparatus which enables this is given by the *integral* $\int$.

The crowning achievement of the subject – the crescendo – is the *fundamental theorem of calculus*, a bridge between these two branches. In its most familiar incarnation, the fundamental theorem of calculus reads

$$\int_a^b f'(x)dx = f(b) - f(a).$$

The importance of this theorem cannot be understated.

After the elation that one experiences from learning this result of an almost divine and supernatural nature, we learn the corresponding higher-dimensional theory; namely, the calculus of several real variables $f : \mathbb{R}^n \to \mathbb{R}$. The subject is (more or less) identical to the one-variable story we learned earlier, with the additional hassle that comes from the book-keeping of indices.

The first glimpse of novel theory appears in vector calculus, the calculus of maps $f : \mathbb{R}^n \to \mathbb{R}^m$. The primary beauty of the subject is not in the presence of new concepts, such as *curl, divergence, vector fields*, etc. The beauty of the subject stems from the insights one attains concerning the previous (one and multi-variable) theories. We will see the fundamental theorem of calculus in its true, glorious generality:

$$\int_{\partial\Omega} \omega \ = \ \int_{\Omega} d\omega,$$

which, in this guise, is unanimously referred to as *Stokes' theorem.*

In contrast with the expedient belief that one typically walks away with after their first two courses in the subject, Stokes' theorem elucidates the deeper relationship between the differential calculus and integral calculus. It is not that derivatives and integrals are *opposite* to one another (whatever *opposite* means, we will use the more precise word *dual* from now on), it is that *the derivative is dual to the region of integration, and the duality is given by the integral.*

This duality, and Stokes' theorem, lead to the notions of cohomology (specifically, de Rham cohomology) – a theory for studying spaces employing the vector fields (or more generally, differential forms) which reside on them.

In a first course in calculus, we study functions, learn the relevant differentiation theory, this is followed by Riemann's theory of integration, and they are merged via the fundamental theorem of calculus. The present treatment of vector calculus will proceed in much the same way:

In the first chapter, we will introduce vector fields and differential forms – the main objects of study. We will discuss gradient fields, a particular class of vector fields that can be expressed in terms of functions, the association between vector fields and 1–forms, the exterior derivative, and the wedge product.

The differentiation theory is taken up in Chapter 2. In contrast with the one-variable and multi-variable calculus, there are two notions of derivatives of a vector field: curl and divergence. These are, in fact, both instances of one notion of derivative – the exterior derivative, which we meet in Chapter 1. To see that both curl and divergence are incarnations of the exterior derivative, we will discuss the Hodge $\star$–operator.

The ability to represent a vector field in terms of a function is dependent on the properties of the domain of a vector field. This topic is treated in Chapter 3.

The integration theory is exhibitted in chapters 3 and 4. Here, we introduce *line integrals* and *surface integrals* . These are then related via the various incarnations of Stokes' theorem (i.e., the fundamental theorem of calculus). There are three incarnations – *Green's theorem* , *Stokes' theorem* , and the *divergence theorem.*

Before getting into the details of the subject, let us remark that vector calculus is not the final point of this calculus theory. The ideas developed here, namely, the first glimpses of cohomology are extended (and treated more thoroughly) in the subjects of differential geometry, algebraic geometry, and algebraic topology (to name just a couple).

Further, the subject of one-variable complex analysis is arguably the (second) most beautiful of all the calculus theories, giving a theory of functions $f : \mathbb{C} \to \mathbb{C}$. The most beautiful of the calculus theories is the function theory of several complex variables $f : \mathbb{C}^n \to \mathbb{C}$ which is, unfortunately, not as well-known in comparison with the other theories mentioned.

Structure and purpose of the notes

The lecture notes are intended to treat the subject thoroughly, i.e., all statements are given proofs (or a proof is referenced for the reader's convenience). Each section has exercises whose completion is highly encouraged. There are many complete examples, and this is one of the key aspects of the notes. Answers (not solutions) will be given (eventually) at the back of the notes. An index is also given at the back of the book for ease of recalling definitions; theorems, definitions, and remarks are also hyperlinked if they are referenced later.

Let me now address the main question that is likely at the forefront of the reader's mind: Why write another vector calculus book? The answer is twofold: The first is that, the vector calculus literature is, in my mind, divided into two classes.

There is the computation-focused treatment, primarily catering to engineering students, which avoids any systematic development of the theory and presents the results in a very physical manner. This is the direction taken, for instance, in J. Stewarts' *Calculus* [**25**], and does a very good job at treating the subject from this lens. At this point, the reader is likely to interject with, *well, engineering students are the primary audience.* There is a pedagogical drawback to this approach, however, that not only afflicts the pure mathematics students, but also impacts the target audience; namely, the engineering students. The approach taken in Stewarts' book side-steps the systematic theory of differential forms that has been developed over the last century, in favor of the less intimidating ad-hoc approach involving the grad vector, the cross product, and so on. The price one pays for this is that, despite each individual lecture being rather straightforward and elementary, by the end of the teaching semester, students are left with a vast number of disparate and unrelated collection of facts.

On the other side, the high-brow approach of doing everything via forms, the enlightened approach taken, for instance, most notably in H. M. Schey's *Div, Grad, Curl, and all that*, the audience is evidently those who have seen the computation-focused approach and wish to achieve enlightenment through the theory of differential forms. This is not a criticism of Schey's book – the book has the sub-heading: *an informal text on vector calculus.* Harold M. Edwards – *Advanced Calculus – A Differential Forms Approach* [**6**] is a more appropriate illustration of this second class. Here, the theory is put first, but the intended audience is strange, at best. The amount of mathematical maturity required to digest [**6**] is likely only held by those students who have taken much more advanced courses than vector calculus.

The aim of the present book is to give an appropriate middle ground. The intention is to develop the theory properly, such that students walk away from the course with a coherent picture of the subject. Further, the material is presented in a digestable way. The first chapter may be labeled naive, but there is never a moment in the text where a false statement is given. Further, students who wish for a more detailed account of the developments are encouraged to see the appendix.

Errors/Typos/Misprints. Typos, errors, and misprints are likely to be ubiquitous throughout the notes. If any issues are found, please do not hesitate to inform me of them at `kyle.broder@anu.edu.au`.

Contents

Notation

- $\mathbb{N}$ – the natural numbers.
- $\mathbb{N}_0$ – the natural numbers including zero.
- $\mathbb{Z}$ – the integers.
- $\mathbb{R}$ – the real numbers.
- $\mathbb{C}$ – the complex numbers.
- Ω – a region in $\mathbb{R}^n$.
- $\partial\Omega$ – the boundary of a region Ω.
- $\mathcal{C}$ – a curve.
- $\mathcal{S}$ – a surface.
- $\mathcal{V}$ – a solid surface.
- $\mathcal{C}^0(\Omega)$ – the space of continuous functions $f : \Omega \to \mathbb{R}$.
- $\mathcal{C}^1(\Omega)$ – the space of continuously differentiable functions $f : \Omega \to \mathbb{R}$.
- $\mathcal{C}^k(\Omega)$ – the space of k–times continuously differentiable functions $f : \Omega \to \mathbb{R}$.
- $\mathcal{C}^\infty(\Omega)$ – the space of smooth functions $f : \Omega \to \mathbb{R}$.
- $\Lambda^0(\Omega)$ – the space of 0–forms on Ω.
- $\Lambda^1(\Omega)$ – the space of 1–forms on Ω.
- $\Lambda^2(\Omega)$ – the space of 2–forms on Ω.
- $\Lambda^3(\Omega)$ – the space of 3–forms on Ω.
- $\omega_{\mathbf{F}}$ – the 1–form associated to $\mathbf{F}$.
- $\mathrm{Vect}(\Omega)$ – the space of vector fields on Ω.
- $\mathbf{i}$ – the vector $(1, 0)$ or $(1, 0, 0)$.
- $\mathbf{j}$ – the vector $(0, 1)$ or $(0, 1, 0)$.
- $\mathbf{k}$ – the vector $(0, 0, 1)$.
- $\mathbf{0}$ – the zero vector.
- $\times$ – the cross product.
- $\cdot$ – the dot product.
- $\wedge$ – the wedge product.
- f_x, $\partial_x f$, $\frac{\partial f}{\partial x}$ – the partial derivative of f with respect to x.
- f_y, $\partial_y f$, $\frac{\partial f}{\partial y}$ – the partial derivative of f with respect to y.
- f_z, $\partial_z f$, $\frac{\partial f}{\partial z}$ – the partial derivative of f with respect to z.

- f_{xy} – the partial derivative of f_x with respect to y.
- f_{yx} – the partial derivative of f_y with respect to x.
- f_x – the partial derivative of f with respect to x.
- ∇ – the grad vector.
- ∇f – the gradient of f.
- d – the exterior derivative.
- det – the determinant.
- $\star$ – the Hodge $\star$–operator.
- curl($\mathbf{F}$) – the curl of $\mathbf{F}$.
- div($\mathbf{F}$) – the divergence of $\mathbf{F}$.

CHAPTER 1

Vector Fields and Differential Forms

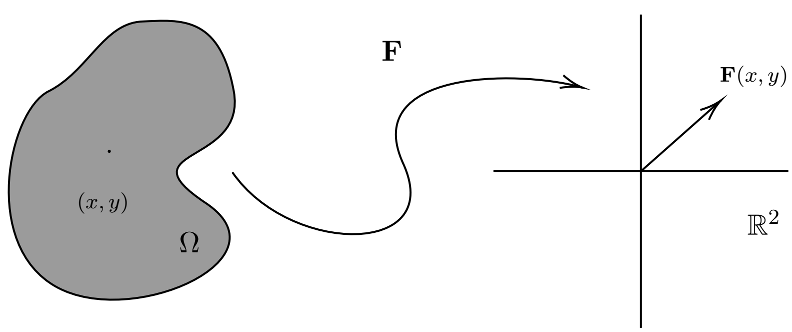
"The art of doing mathematics consists in finding that special case which contains all the germs of generality."

– David Hilbert

In the calculus of one variable, the main objects of study are functions $f : \mathbb{R} \to \mathbb{R}$. In the calculus of more than one variable, the main objects of study are functions $f : \mathbb{R}^n \to \mathbb{R}$. In vector calculus, the main objects of study are vector fields, i.e., maps $\mathbb{R}^n \to \mathbb{R}^n$.

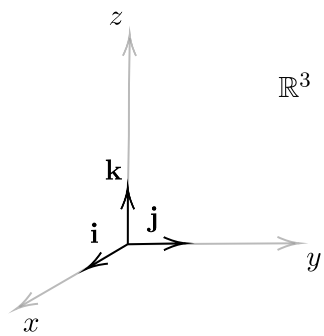
1.1. Definitions, First Examples, and Remarks on Regularity

Definition 1.1.1. (Vector field). Let Ω be a region (a subset) in $\mathbb{R}^2$. A *vector field* on Ω is a map $\mathbf{F} : \Omega \to \mathbb{R}^2$.



Remark 1.1.2. In other words, if we let (x, y) denote the coordinates on $\mathbb{R}^2$, then a vector field assigns to each point $(x, y) \in \Omega$, a vector $\mathbf{F}(x, y) \in \mathbb{R}^2$. We will write $\mathbf{i} = (1, 0)$ and $\mathbf{j} = (0, 1)$ for the standard basis on $\mathbb{R}^2$. With respect to this basis, one can express the data of a vector field in terms of two functions $P : \Omega \to \mathbb{R}$ and $Q : \Omega \to \mathbb{R}$:

$$\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}.$$



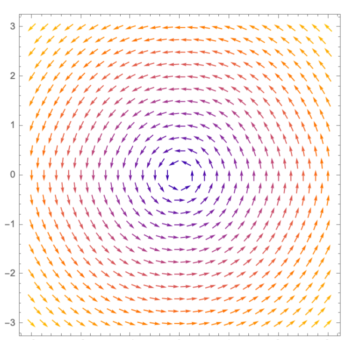
Of course, we can extend this to higher dimensions without much difficulty: If $\mathbf{i} = (1, 0, 0)$, $\mathbf{j} = (0, 1, 0)$, and $\mathbf{k} = (0, 0, 1)$, then a vector field on $\Omega \subseteq \mathbb{R}^3$ is specified by three functions $P, Q, R : \Omega \to \mathbb{R}$, where

$$\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}.$$

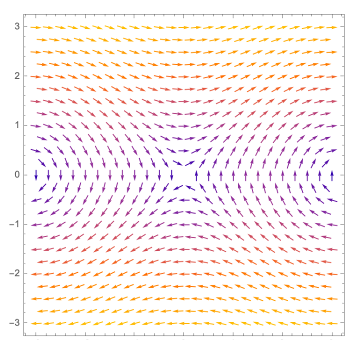
We refer to these functions P, Q, R as *component functions* for the vector field $\mathbf{F}$.

Example 1.1.3. The vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\mathrm{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$ sends the point $(2, 1)$ to the vector $\mathbf{F}(2, 1) = -\mathbf{i} + 2\mathbf{j}$.

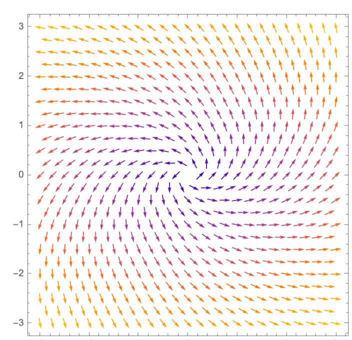
Examples of Vector Fields in $\mathbb{R}^2$. Here are some examples of vector fields on $\mathbb{R}^2$:



$\mathbf{F}(x, y) = -y\mathbf{i} + x\mathbf{j}$.



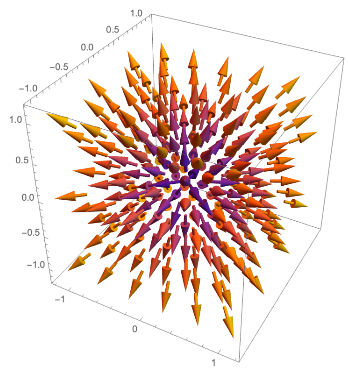
$\mathbf{F}(x, y) = y\mathbf{i} + \sin(x)\mathbf{j}$.



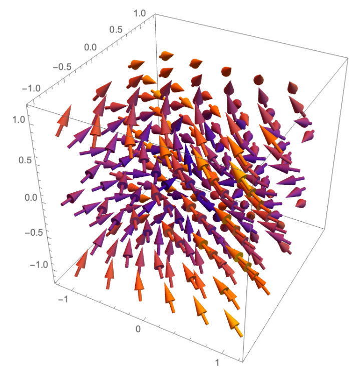
$\mathbf{F}(x, y) =$
$(2x - 3y)\mathbf{i} + (2x + 3y)\mathbf{j}$.

Remarks on dimension. We can also define vector fields on regions in $\mathbb{R}^3$, but these are more difficult to visualize. One can define vector fields on regions in $\mathbb{R}^n$, for any $n \in \mathbb{N}$. The difficulty in visualizing such vector fields increases with n, however. This is the reason for primarily sticking to the cases $n = 2$ and $n = 3$.

Examples of Vector fields in $\mathbb{R}^3$. Here are some examples of vector fields on $\mathbb{R}^3$:

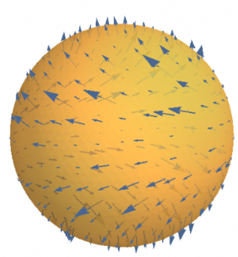


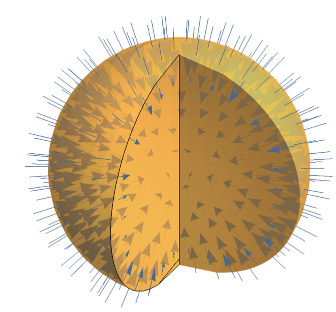
$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

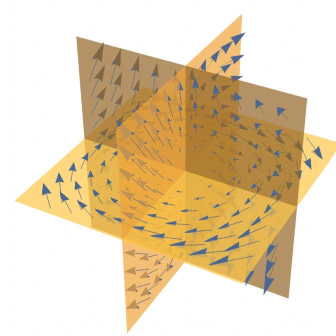


$\mathbf{F}(x, y, z) = (y - x)\mathbf{i} + -y\mathbf{j} + z^2\mathbf{k}$.

Visualization of Vector Fields. Oftentimes, we think of vector fields as lying over their domain, as indicated below:







Maxwell's equations. Vector calculus plays an essential role in Maxwell's theory of electromagnetism. The field generated in the presence of a magnetic material, or material supporting an electric charge, is an example of a vector field. If we let $\mathbf{B}, \mathbf{E}$ denote the magnetic and electric field respectively. Maxwell's equations, in a vacuum, state the following:

$$\begin{aligned} \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= \mathbf{0} \end{aligned} \qquad\qquad \begin{aligned} \nabla \cdot \mathbf{E} &= 0 \\ \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= 0. \end{aligned}$$

We will understand the meaning of these equations using the theory developed in the next chapter.

Properties of Vector Fields. Many of the properties that hold for vectors extend to vector fields:

(i) (Addition). If $\mathbf{F}(x,y) = P\mathbf{i} + Q\mathbf{j}$ and $\mathbf{G}(x,y) = M\mathbf{i} + N\mathbf{j}$, then

$$\mathbf{F}(x,y) + \mathbf{G}(x,y) \;=\; (P+M)\mathbf{i} + (Q+N)\mathbf{j}.$$

(ii) (Scaling). If $\mathbf{F}(x,y) = P\mathbf{i} + Q\mathbf{j}$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$f\mathbf{F}(x,y) = fP\mathbf{i} + fQ\mathbf{j}.$$

To develop an integration theory for vector fields, we will need to introduce the other main objects of the subjects:

Definition 1.1.4. (1–form). Let $\Omega \subseteq \mathbb{R}^2$ be a region. A 1*–form* on Ω is an expression of the following type

$$\omega \;=\; P(x,y)dx + Q(x,y)dy.$$

The set of 1–forms on a region $\Omega \subseteq \mathbb{R}^2$ is denoted by $\Lambda^1(\Omega)$.

Remark 1.1.5. If $\mathbf{F} : \Omega \to \mathbb{R}^2$ is a vector field, given by $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$, then we have a 1–form

$$\omega_{\mathbf{F}} = P(x,y)dx + Q(x,y)dy.$$

We refer to $\omega_{\mathbf{F}}$ as the 1*–form associated to* $\mathbf{F}$.

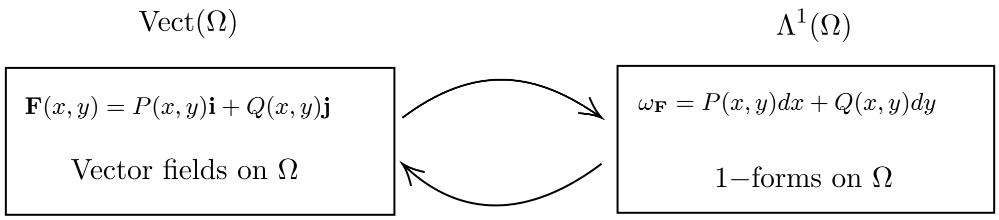
Example 1.1.6. Let $\mathbf{F}(x,y) = 2x\mathbf{i} - 3y^2\mathbf{j}$. The associated 1–form is

$$\omega_{\mathbf{F}} = 2xdx - 3y^2dy.$$

Remark 1.1.7. It may appear at first glance that 1–forms are objects to keep the mathematicians happy, physicists only care about vector fields. This is certainly not the case – forms are essential for giving a coordinate-free description of Maxwell's equations, and this is necessary for the study of Maxwell's equations on curved spacetimes.
The notation we now use for 1–forms originated in Maxwell's 1855 paper and was then later used in the physics text written by Charles Delaunay.

Remark 1.1.8. Of course, given a 1–form $\omega = P(x,y)dx + Q(x,y)dy$, there is an *associated vector* field $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$.

In particular, we have an identification:



Example 1.1.9. The vector field associated to the 1–form $\omega = (3 - \sin(xy))dx + (x+1)dy$ is

$$\mathbf{F}(x, y) = (3 - \sin(xy))\mathbf{i} + (x+1)\mathbf{j}.$$

Properties of 1–forms. The properties of vector fields also transfer over to 1–forms:

(i) (Addition). If $\omega = Pdx + Qdy$ and $\eta = Adx + Bdy$, then

$$\omega + \eta \;=\; (P + A)dx + (Q + B)dy.$$

(ii) (Scaling). If $\omega = Pdx + Qdy$ and $f : \mathbb{R}^2 \to \mathbb{R}$ is a function, then

$$f\omega \;= fPdx + fQdy.$$

Example 1.1.10. Let $\omega = 2x^2dx + (3x - e^{-y})dy$ and $\eta = 9y\cos(y)dx + xdy$. Compute $\omega + \eta$.

Solution. From *Properties of* 1*–forms*, we have

$$\begin{aligned} \omega + \eta &= 2x^2dx + (3x - e^{-y})dy + 9y\cos(y)dx + xdy \\ &= (2x^2 + 9y\cos(y))dx + (4x - e^{-y})dy. \end{aligned}$$

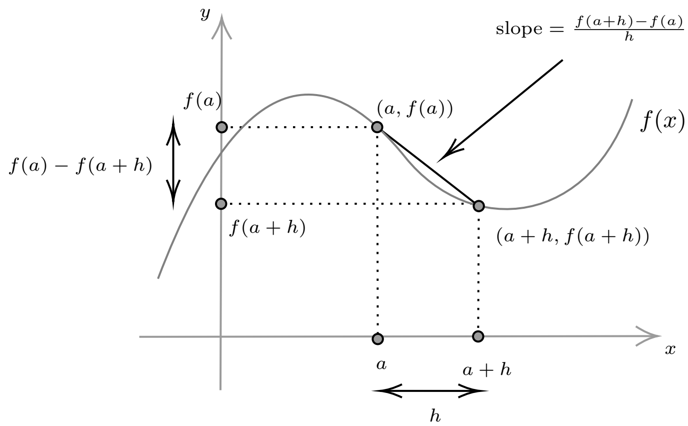
□

Reminder: Regularity of functions. In any theory of calculus, we need to make sense of derivatives, and in particular, when derivatives can be taken. Derivatives of vector fields will be taken up in the next chapter, but for the moment, we need to recall some of the regularity theory from the calculus we are already familiar with.

Reminder: Differentiability. Recall that a function $f : \mathbb{R} \to \mathbb{R}$ is said to be *differentiable* at a point $a \in \mathbb{R}$ if the limit

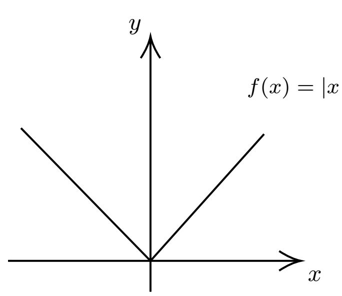
$$f'(a) \;:=\; \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

exists and is finite. We refer to $f'(a)$ as the *derivative of f at the point a*.



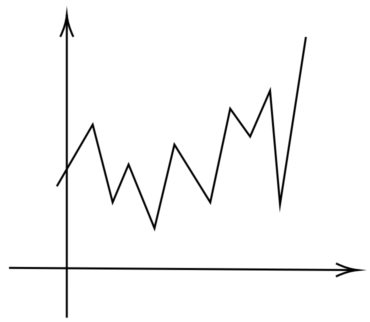
Definition 1.1.11. (Continuously differentiable). If f is differentiable at every point $a \in \mathbb{R}$ (or more generally, its domain, say, Ω), with continuous derivative at every point of its domain, then f is said to be *continuously differentiable*, and we write $f \in \mathcal{C}^1(\Omega)$. The class of functions for which the derivative $f' \in \mathcal{C}^1(\Omega)$ is continuously differentiable, is denoted by $\mathcal{C}^2(\Omega)$. Iterating this definition gives the set of functions of class $\mathcal{C}^k(\Omega)$, where $k \in \mathbb{N}_0$. The set of continuous functions is denoted by $\mathcal{C}^0(\Omega)$.

Example 1.1.12. The modulus function $f(x) = |x|$ is continuous for all $x \in \mathbb{R}$, but not differentiable at $x = 0$.

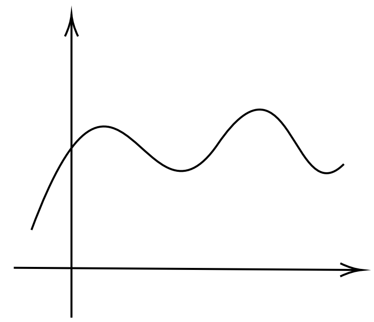


continuous but not differentiable

Definition 1.1.13. A function f is said to be $\mathcal{C}^\infty(\Omega)$, or *smooth*, if f is $\mathcal{C}^k(\Omega)$ for all $k \in \mathbb{N}_0$.



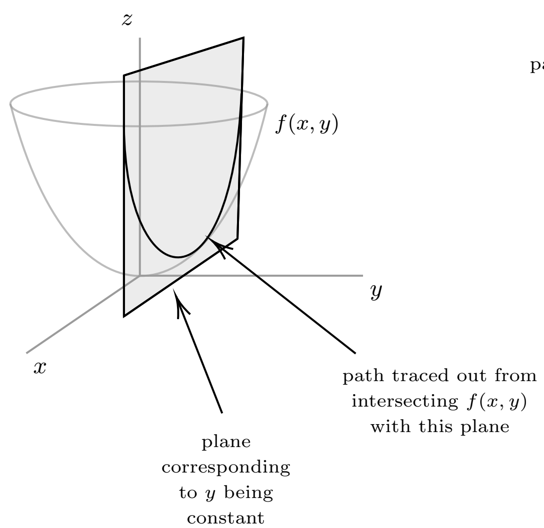
not smooth

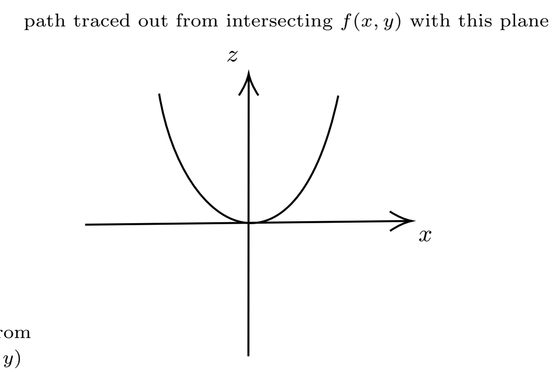


smooth

Reminder 1.1.14. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function on $\mathbb{R}^2$, with (x, y) denoting the coordinates on $\mathbb{R}^2$. Recall that the *partial derivatives of* f are defined by

$$f_x \;:=\; \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}, \qquad f_y \;:=\; \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h}.$$





Notational remark 1.1.15. We will interchange between the notation $\frac{\partial f}{\partial x}$, f_x, and $\partial_x f$ without acknowledgment, and without apology, from here on.

The following notations are equivalent:

$$\partial_x f \;=\; f_x \;=\; \frac{\partial f}{\partial x}$$

Example 1.1.16. If $f : \mathbb{R}^2 \to \mathbb{R}$ is given by $f(x, y) = 3x + 5y^2$, then $f_x = 3$ and $f_y = 10y$.

Reminder 1.1.17. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function on $\mathbb{R}^2$ with (x, y) denoting the coordinates on $\mathbb{R}^2$. Let f_x and f_y denote the x and y partial derivatives of f, respectively. The *pure second-order partial derivatives* are defined by

$$f_{xx} := \lim_{h \to 0} \frac{f_x(x+h, y) - f(x, y)}{h}, \qquad f_{yy} := \lim_{h \to 0} \frac{f_y(x, y+h) - f_y(x, y)}{h}.$$

The *mixed second-order partial derivatives* are defined by

$$f_{xy} := \lim_{h \to 0} \frac{f_x(x, y+h) - f_x(x, y)}{h}, \qquad f_{yx} := \lim_{h \to 0} \frac{f_y(x+h, y) - f_y(x, y)}{h}.$$

Iterating this procedure defines partial derivatives of any order $k \in \mathbb{N}$.

Example 1.1.18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x, y) := x^2 - 2xy^3 + 5y^2.$$

The first-order partial derivatives are $f_x = 2x - 2y^3$ and $f_y = -6xy^2 + 10y$. The pure second-order partial derivatives are $f_{xx} = 2$ and $f_{yy} = -12xy + 10$. The mixed second-order partial derivatives are $f_{xy} = -6y^2$ and $f_{yx} = -6y^2$.

Definition 1.1.19. (Smooth function). A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be *smooth* if its partial derivatives to any order are continuous.

Remark 1.1.20. Unless stated otherwise, the functions we will consider here are smooth; this permits us to focus on the geometric concepts without regularity concerns. In particular, if f is smooth, then partial derivatives commute.

! Unless otherwise stated, all functions are assumed smooth

Definition 1.1.21. (Smooth vector field). A vector field $\mathbf{F} : \Omega \to \mathbb{R}^n$ is said to be $\mathcal{C}^k$ (for $k \in \mathbb{N}_0$) if the component functions are k–times continuously differentiable. We refer to $\mathcal{C}^0$–vector fields as *continuous vector fields* and $\mathcal{C}^\infty$–vector fields as *smooth vector fields*.

Convention 1.1.22. Unless otherwise stated, a *vector field* is understood to be a *smooth vector field*.

! Unless otherwise stated, all vector fields are assumed to be smooth.

Remark 1.1.23. Many of the results (if not all?) that we will treat in this course require vector fields to only be of class $\mathcal{C}^1$ or $\mathcal{C}^2$. To avoid these regularity considerations (which will take us too far afield), we simply assume that all vector fields (unless otherwise stated) are $\mathcal{C}^\infty$.

Clairaut's theorem. Let $\Omega \subseteq \mathbb{R}^2$ be a region in $\mathbb{R}^2$. Assume $f : \Omega \to \mathbb{R}$ has continuous second-order partial derivatives at a point $p \in \Omega$. Then

$$f_{xy}(p) \quad = \quad f_{yx}(p).$$

PROOF. Let $f : \Omega \to \mathbb{R}$ have continuous second-order partial derivatives at a point $p \in \Omega$. Let $\mathcal{U} \ni p$ be a small neighbourhood of p contained in Ω. By changing coordinates if necessary, we can assume $\mathcal{U}$ is a small rectangle $[a,b] \times [c,d] \subset \Omega$. Observe that

$$\begin{aligned} \int_c^d \int_a^b f_{yx}(x,y)dxdy &= \int_c^d f_y(b,y) - f_y(a,y)dy \\ &= (f(b,d) - f(b,c)) - (f(a,d) - f(a,c)) \\ &= f(b,d) - f(b,c) - f(a,d) + f(a,c). \end{aligned}$$

Similarly,

$$\begin{aligned} \int_a^b \int_c^d f_{xy}(x,y)dydx &= \int_a^b f_x(x,d) - f_x(x,c)dx \\ &= (f(b,d) - f(a,d)) - (f(b,c) - f(a,c)) \\ &= f(b,d) - f(a,d) - f(b,c) + f(a,c). \end{aligned}$$

As a consequence,

$$\int_c^d \int_a^b f_{yx}(x,y)dxdy \quad = \quad \int_a^b \int_c^d f_{xy}(x,y)dydx.$$

Since the second-order partial derivatives vanish, the order of integration can be changed, and thus $f_{yx} = f_{xy}$ on the rectangle $[a,b] \times [c,d]$, proving the claim. □

Remark 1.1.24. *Clairaut's theorem* states that, for functions with continuous second-order partial derivatives, the *second-order partial derivatives commute*: $f_{xy} = f_{yx}$. Further, the above proof of *Clairaut's theorem* hinges upon the fundamental theorem of calculus.

Definition 1.1.25. (Gradient). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. The *gradient of* f is the vector field

$$\nabla f \quad = \quad f_x \mathbf{i} + f_y \mathbf{j}.$$

Example 1.1.26. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) := x^2 y + 3\sin(y).$$

Compute the gradient of f.

SOLUTION. The partial derivatives of f are given by

$$f_x = 2xy, \qquad f_y = x^2 + 3\cos(y).$$

Hence, the gradient of f is given by

$$\nabla f = 2xy\mathbf{i} + (x^2 + 3\cos(y))\mathbf{j}.$$

□

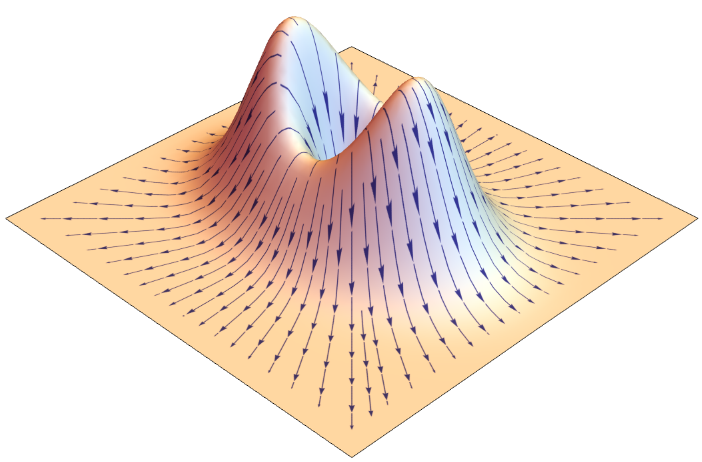
Remark 1.1.27. Observe that if $f : \mathbb{R} \to \mathbb{R}$ is a smooth function, then ∇f is merely the familiar derivative $f'(x)$. In the one-dimensional case, the notation $f'(x) = f_x\mathbf{i}$ is, of course, unnecessary.

Remark 1.1.28. The gradient of f yields the vector field of steepest ascent on the graph of f. Indeed, let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Then for each point $p = (x, y) \in \mathbb{R}^2$, evaluating ∇f at p yields a vector $(\nabla f)(p)$ (the gradient of f at p). Given a vector $v \in \mathbb{R}^2$, the dot product with a unit vector v gives

$$(\nabla f)(p) \cdot v = |(\nabla f)(p)| \cos(\vartheta),$$

where ϑ is the angle between $(\nabla f)(p)$ and v. Since $\cos(\vartheta)$ is maximized at $\vartheta = 2k\pi$ $(k \in \mathbb{Z})$ with value 1, it follows that

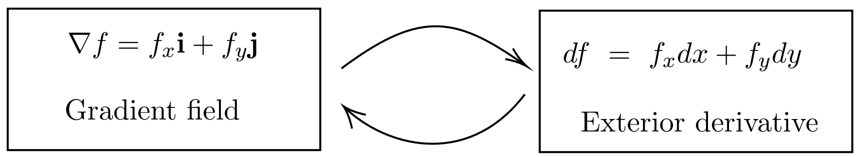
$$\max_v (\nabla f)(p) \cdot v = (\nabla f)(p) \cdot (\nabla f)(p) = |(\nabla f)(p)|^2.$$



The 1–form associated with the gradient vector field will play a central role:

Definition 1.1.29. (Exterior derivative). Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. The *exterior derivative of* f is defined

$$df = f_x dx + f_y dy.$$



The exterior derivative is the 1–form associated to the gradient.

Example 1.1.30. Let Ω be region in $\mathbb{R}^2$, and let $f : \Omega \to \mathbb{R}$ be given by

$$f(x, y) = (x^2 + y^2) - \sin(y).$$

Compute the exterior derivative and gradient.

Solution. The exterior derivative of f is

$$df \;=\; 2xdx + (2y - \cos(y))dy.$$

The gradient $\mathbf{F} = \nabla f$ is then just the vector field associated to df, i.e.,

$$\mathbf{F} \;=\; \nabla f \;=\; 2x\mathbf{i} + (2y - \cos(y))\mathbf{j}.$$

□

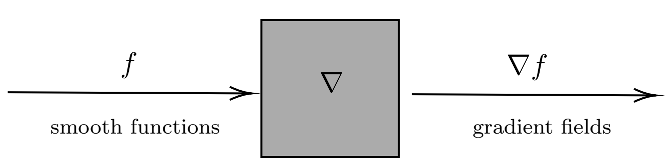
Remark 1.1.31. We have, in fact, already seen the exterior derivative when we studied implicit differentiation. For instance, if $x^2 + y^2 = 1$, then $2xdx + 2ydy = 0$, and therefore,

$$\frac{dy}{dx} = -\frac{x}{y}.$$

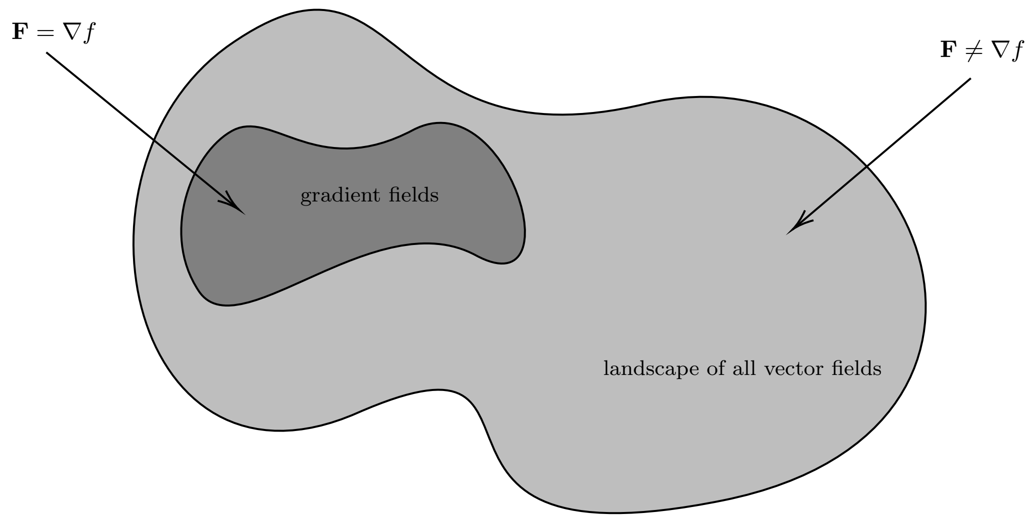
Definition 1.1.32. (Gradient field). A vector field $\mathbf{F} : \Omega \to \mathbb{R}^2$ is said to be a *gradient vector field* if there is a function $f : \Omega \to \mathbb{R}$ such that

$$\mathbf{F} \;=\; \nabla f.$$

Remark 1.1.33. We have seen that, if we have a smooth function $f : \Omega \to \mathbb{R}$, then we can produce a gradient field by computing ∇f. In this respect, the operator ∇ is viewed as a machine that inputs functions and outputs gradient fields:



If we stumble upon a vector field $\mathbf{F}$ in the wild, however, then it may *not* be a gradient field, in general:



Example 1.1.34. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x, y) := (y - 2)\mathbf{i} + (3x - 2y)\mathbf{j}.$$

Show that $\mathbf{F}$ is not a gradient field.

SOLUTION. If $\mathbf{F}$ is a gradient field, then we can find a smooth function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathbf{F} = \nabla f = f_x\mathbf{i} + f_y\mathbf{j}$. This implies that $f_x = (y - 2)$ and $f_y = (3x - 2y)$. Observe that

$$f_x \;=\; y - 2 \quad\Longrightarrow\quad f(x, y) = xy - 2x + g(y),$$

where $g : \mathbb{R} \to \mathbb{R}$ is a function independent of x. Computing the y partial derivative of the above result gives:

$$\frac{\partial f}{\partial y} \;=\; \frac{\partial}{\partial y}(xy - 2x + g(y)) \;=\; x + g'(y).$$

Comparing this with $f_y = (3x - 2y)$, we see that

$$\begin{aligned} (3x - 2y) \;=\; x + g'(y) \quad &\Longrightarrow \quad g'(y) \;=\; 2x - 2y \\ &\Longrightarrow \quad g(y) \;=\; 2xy - y^2, \end{aligned}$$

but this implies that g depends on x, which is not true. Hence, $\mathbf{F}$ cannot be a gradient field. $\square$

Theorem 1.1.35. Let $\mathbf{F} : \Omega \to \mathbb{R}^2$ be a smooth vector field on a domain $\Omega \subseteq \mathbb{R}^2$ defined by $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$. If $\mathbf{F}$ is a gradient field, then the component functions must satisfy

$$P_y = Q_x.$$

PROOF. Since $\mathbf{F}$ is a gradient field, there is a smooth function $f : \Omega \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. Since $\mathbf{F} = \nabla f$, we have

$$\mathbf{F} = f_x\mathbf{i} + f_y\mathbf{j},$$

and therefore, $P = f_x$ and $Q = f_y$. By *Clairaut's theorem*, the second-order partial derivatives of a smooth function commute; hence, it follows that

$$P_y = f_{yx} = f_{xy} = Q_x.$$

$\square$

Example 1.1.36. Show that the vector field

$$\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2\mathbf{j}$$

is not a gradient field.

SOLUTION. The component functions are $P(x, y) = xy^2$ and $Q(x, y) = x^2$. Hence, $P_y = 2xy$ and $Q_x = 2x$. Since $P_y \neq Q_x$, the vector field $\mathbf{F}$ is not a gradient field. $\square$

Remark 1.1.37. Note that *Theorem 1.1.35* does *not* assert that if $P_y = Q_x$, then $\mathbf{F}(x, y) = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a gradient field.

$$\boxed{\mathbf{F} = \nabla f \implies P_y = Q_x}$$

$$\boxed{P_y = Q_x \not\Longrightarrow \mathbf{F} = \nabla f}$$

More precisely, if $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$ is a vector field on Ω with $Q_x = P_y$ everywhere, then $\mathbf{F}$ is *locally* a gradient field. That is, in some neighborhood of each point $(x, y) \in \Omega$, where $P_y = Q_x$, there is a smooth function f defined (only on) this small neighborhood such that $\mathbf{F} = \nabla f$ there. The standard example to show that $P_y = Q_x$ is not sufficient (in general) to have a global potential $f : \Omega \to \mathbb{R}$ is the vector field

$$\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

We will see this example time and time again throughout the course.

Example 1.1.38. Determine whether the vector field $\mathbf{F}(x,y) = -y\mathbf{i}+x\mathbf{j}$ is a gradient vector field.

SOLUTION. If $\mathbf{F}$ is a gradient vector field, we can find a function $f : \mathbb{R} \to \mathbb{R}$ such that $\mathbf{F} = \nabla f$. Hence,

$$\begin{aligned} \mathbf{F} = \nabla f \quad &\Longrightarrow \quad -y\mathbf{i} + x\mathbf{j} \;=\; f_x\mathbf{i} + f_y\mathbf{j} \\ &\Longrightarrow \quad f_x \;=\; -y, \qquad f_y \;=\; x. \end{aligned}$$

Integrating $f_x = -y$ with respect to x gives $f(x,y) = -xy + g(y)$, where g is a function depending only on y. Differentiating $f(x,y) = -xy + g(y)$ with respect to y gives

$$f_y \;=\; -x + g'(y).$$

Since we already know that $f_y = x$, it follows that

$$g'(y) = 2x.$$

In particular, g depends on x, which contradicts the definition of g. So $\mathbf{F}$ is not a gradient field. $\square$

Conservation of Energy. Let m be the mass of a particle in $\mathbb{R}^3$. Let $\mathbf{v}$ denote the velocity of the particle. The kinetic energy of the particle is given by

$$E_{\text{kin}} \;=\; \frac{1}{2}m\|\mathbf{v}\|^2 \;=\; \frac{1}{2}m\mathbf{v}\cdot\mathbf{v}.$$

Newton's second law of motion states that

$$\mathbf{F} \;=\; m\frac{d\mathbf{v}}{dt}.$$

If $\mathbf{F}$ is a gradient field, then $\mathbf{F} = \nabla f$ for some $f \in \mathcal{C}^\infty(\mathbb{R}^3)$. This implies that

$$\nabla f \;=\; m\frac{d\mathbf{v}}{dt}.$$

Taking the dot product with $\mathbf{v}$, we rewrite the above equation as

$$\mathbf{v}\cdot\nabla f - m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt} \;=\; 0.$$

This is equivalent to

$$\frac{d}{dt}(E_{\text{kin}} - f) \;=\; 0,$$

which informs us that the total energy $E_{\text{kin}} - f$ is constant.

Remark 1.1.39. This is the reason for commonly referring to gradient fields as *conservative vector fields*. The reason for avoiding this terminology in the present book is that the high-esteemed label of a vector field being 'conservative' is likely to lead to confusion. Indeed, we have seen that the vector field

$$\mathbf{F} = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

satisfies $\mathbf{F} = \nabla f$, where $f = \tan^{-1}(y/x)$. But it is not the case that the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$ is path independent. This, of course, is due to the fact that the potential $f = \tan^{-1}(y/x)$ is not smooth at the origin.

Theorem 1.1.40. Let $\mathbf{F} : \Omega \to \mathbb{R}^3$ be a smooth vector field on a domain $\Omega \subseteq \mathbb{R}^3$ defined by

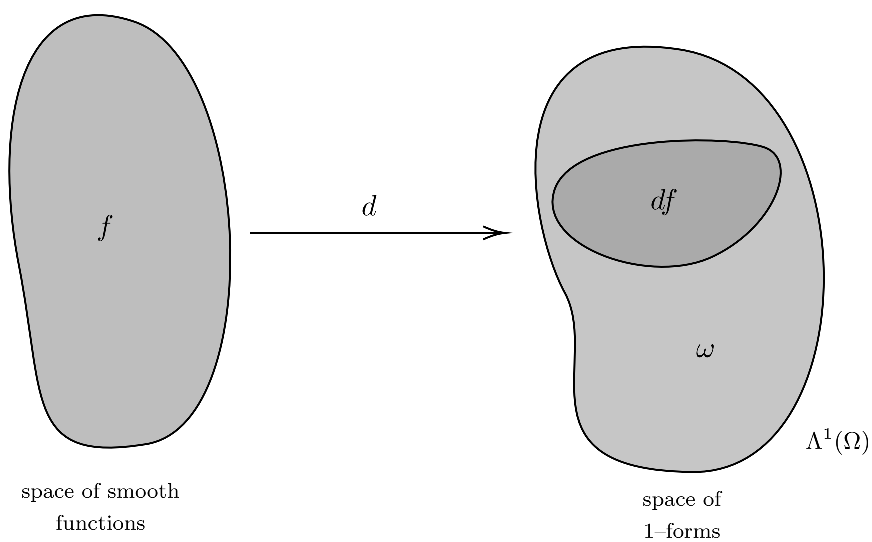
$$\mathbf{F}(x, y, z) = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}.$$

If $\mathbf{F}$ is a gradient field, then the component functions must satisfy

$$P_y = Q_x, \qquad Q_z = R_y, \qquad P_z = R_x.$$

PROOF. The same as the proof of *Theorem 1.1.35*. ☐

Remark 1.1.41. The question of whether a vector $\mathbf{F}$ is a gradient field is equivalent to asking whether there is a smooth function f such that the associated 1–form $\omega_{\mathbf{F}} = df$. In this case, one says that the 1–form $\omega_{\mathbf{F}}$ is *exact*.



Remark 1.1.42. Much of our time in this course will be spent trying to understand exactly when a vector field $\mathbf{F}$ is a gradient field; or equivalently, when a 1–form is exact. This seemingly pedestrian endeavor leads to a surprisingly rich theory and is of central importance to mathematics.

Remarks on Clairaut's theorem. It is curious that *Clairaut's theorem* bears the name of Clairaut, who, with company like Euler [7], Lagrange [16], and Cauchy [5], gave only a false (or incomplete) proof (see, e.g., [17]). The first counterexamples to the failure of mixed partial derivatives commuting were given by Lindelöf [17]. The first complete proof, however, was not given until 1873, by Schwarz [22].

Exercises

1. Sketch the vector fields

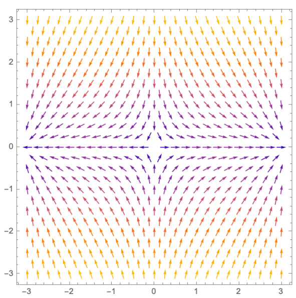
(i) $\mathbf{F}(x,y) = -y\mathbf{i} + 2x\mathbf{j}$.

(ii) $\mathbf{F}(x,y) = \mathbf{i} - \mathbf{j}$.

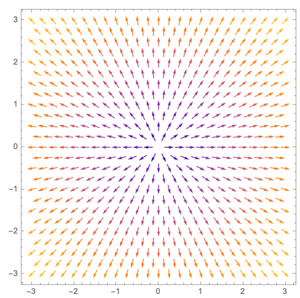
(iii) $\mathbf{F}(x,y) = (x^2 + \sin(y))\mathbf{i} - y\mathbf{j}$.

(iv) $\mathbf{F}(x,y) = \frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$.

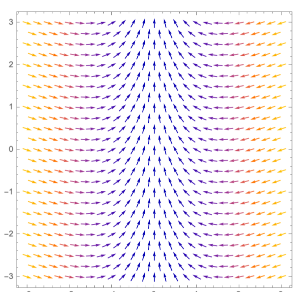
2. Consider the vector fields (i) $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$, (ii) $\mathbf{F}(x,y) = \sin(x)\mathbf{i} - y\mathbf{j}$, (iii) $\mathbf{F}(x,y) = x\mathbf{i} - y\mathbf{j}$, and (iv) $\mathbf{F}(x,y) = -x\mathbf{i} + \cos(x)\mathbf{j}$. Match these vector fields to the vector fields sketched below:



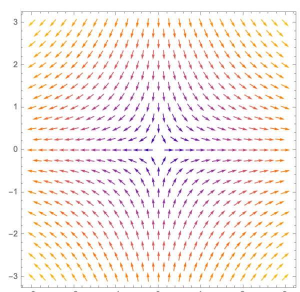
(a)



(c)



(b)



(d)

3. Sketch the vector fields

(i) $\mathbf{F}(x,y) = (1 - \sin(x))\mathbf{i} + xe^{-y}\mathbf{j}$

(ii) $\mathbf{F}(x,y) = x^2\mathbf{i} + \log_e(y)\mathbf{j}$

(iii) $\mathbf{F}(x,y) = x\sqrt{1-y^2}\mathbf{i} + 2y^3\mathbf{j}$

(iv) $\mathbf{F}(x,y) = \sin(y)e^{-x}\mathbf{i} + \cos(x)e^{-y}\mathbf{j}$

4. Let $\mathbf{F}, \mathbf{G} : \mathbb{R}^2 \to \mathbb{R}^2$ be smooth vector fields defined by $\mathbf{F}(x,y) = 2x\sin(y)\mathbf{i} - xy^2\mathbf{j}$ and $\mathbf{G}(x,y) = (1 + e^{-y})\mathbf{i} - \mathbf{j}$.

(i) Sketch the vector field $\mathbf{F}$.

(ii) Sketch the vector field $\mathbf{G}$.

(iii) Compute $\mathbf{F} + \mathbf{G}$.

(iv) Sketch the vector field $\mathbf{F} + \mathbf{G}$.

5. Determine the 1–forms associated to the vector fields:

(i) $\mathbf{F}(x,y) = (1-x^2)\mathbf{i} + 6y^3\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = e^{-x}\mathbf{i} + e^y \sin(x^2)\mathbf{j}$.
(iii) $\mathbf{F}(x,y) = (3x-y^2)\mathbf{i} + (y+1)^2\mathbf{j}$.
(iv) $\mathbf{F}(x,y,z) = (e^{1-zx} + e^{x+y})\mathbf{i} + (x^3+z^2)\mathbf{j} - x\mathbf{k}$.

6. Determine the vector fields associated to the 1–forms:

(i) $\omega = xy^3dx - 3yzdy + 2z^3dz$.
(ii) $\omega = xzdx + 2x^3dy + ydz$.
(iii) $\omega = 2z^2dx + (y-2z^2)dy + (z+4xy)dz$.
(iv) $\omega = (1+e^{-xz})dx + 2x\cos(z-y)dy + (x^2+7y)dz$.
(v) $\omega = (2x+z^2)dx + (xy - \sin(xz)dy - \cos(z)dz$

7. Consider the 1–forms $\omega = 2\sqrt{x^2+1}dx + (15y^2+1)dy$ and $\eta = (2-\tan(x))dx + e^x dy$. Determine

(i) $\omega + \eta$.
(ii) $3\omega - 4\eta$.
(iii) $x^2\omega - 3\sin(y)\eta$.

8. Consider the functions $f(x,y,z) = \cos(x) + \sin(x-z)$ and $g(x,y,z) = z\tan^{-1}(y/x)$.

(i) Compute $\omega = df$.
(ii) Compute $\eta = dg$.
(iii) Compute $3\sin(y)dy + xdx + (1-y)\omega - (2-\log_e(z))\eta$.

9. Consider the functions $f(x,y,z) = \sqrt{1+z^2}$ and $g(x,y,z) = e^{z^2} + \log_e(x-y)$.

(i) Compute $\omega = df$.
(ii) Compute $\eta = dg$.
(iii) Compute $4x^3dx + x^2\omega - 3y^2\eta$.

10. Let $\varphi = 2x^2dx + (x+y)dy$ and $\psi = -xdx + (x-3y)dy$. Compute

(i) $2\varphi + \psi$.
(ii) $\varphi - x^2\psi$.

11. Let $\alpha = x^3dx + yzdy - (x^2+y^2+z^2)dz$ and $\beta = y^2zdx - xzdy + (2x+1)dz$. Find

(i) $3\alpha - 4\beta$.
(ii) $x\alpha + y\beta$.

12. Compute the exterior derivative of the following functions:

(i) $f(x,y) = xy + x^3e^{-y}$.
(ii) $f(x,y,z) = 1 - xz + \sin(xy+z)$.
(iii) $f(x,y,z) = \log_e(x) - \log_e(y) + \log_e(z)$.
(iv) $f(x,y,z) = e^{-z^2}\sqrt{x+y}$.

13. Determine whether the vector fields $\mathbf{F}$ given below are gradient fields:

(i) $\mathbf{F}(x,y) = xe^y\mathbf{i} + ye^x\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = (6x+5y)\mathbf{i} + (5x+4y)\mathbf{j}$.
(iii) $\mathbf{F}(x,y) = (1+2xy+\ln(x))\mathbf{i} + y^2\mathbf{j}$.
(iv) $\mathbf{F}(x,y,z) = (x^2-4zy)\mathbf{i} - 4x\mathbf{j}$.
(v) $\mathbf{F}(x,y,z) = \sqrt{z+x^2}\mathbf{i} + (x+2\sqrt{z})\mathbf{k}$.
(vi) $\mathbf{F}(x,y,z) = x\cos(y)\mathbf{i} + zy\cos(x)\mathbf{k}$.
(vii) $\mathbf{F}(x,y,z) = (y-x)\mathbf{i} + 2x^3\mathbf{j} + z^3\mathbf{k}$.
(viii) $\mathbf{F}(x,y,z) = xy^3\mathbf{i} + 4yz\mathbf{j} + 2x^3\mathbf{k}$.
(ix) $\mathbf{F}(x,y,z) = xz\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$.
(x) $\mathbf{F}(x,y,z) = 2x^2\mathbf{i} + (3y+z^2)\mathbf{j} + zy\mathbf{k}$.

14. Determine whether the following vector fields are gradient fields. If they are, find their potentials.

(i) $\mathbf{F}(x,y,z) := (x-y)\mathbf{i} - 2(y+x)\mathbf{j} + 3(x+z)\mathbf{k}$.
(ii) $\mathbf{F}(x,y,z) := \mathbf{i} + \mathbf{j} - \mathbf{k}$.
(iii) $\mathbf{F}(x,y) = -y\mathbf{i} + x\mathbf{j}$.
(iv) $\mathbf{F}(x,y,z) = (1+e^{-z}x^2)\mathbf{i} + x\sin(z-y)\mathbf{j} + (x^3+9z)\mathbf{k}$.
(v) $\mathbf{F}(x,y,z) = (x^2+z^2)\mathbf{i} + (y-\sin(xy)\mathbf{j} - \cos(z)\mathbf{k}$.
(vi) $\mathbf{F}(x,y) = (4x-y^3)\mathbf{i} + x^2\mathbf{j}$.
(vii) $\mathbf{F}(x,y) = 2\sqrt{x^2+1}\mathbf{i} + (15y^2+1)\mathbf{j}$.
(viii) $\mathbf{F}(x,y,z) = (2-\tan(x))\mathbf{i} + e^x\mathbf{j}$.
(ix) $\mathbf{F}(x,y,z) = (e^z+e^x)\mathbf{i} + (y^3+x^2)\mathbf{j} - x\mathbf{k}$.
(x) $\mathbf{F}(x,y,z) = (\cos(x)+\sin(x-z))\mathbf{i} + (y^3+2)\mathbf{j} + (1+x)\mathbf{k}$.
(xi) $\mathbf{F}(x,y,z) = (\sin(z)+\log_e(x-1))\mathbf{i} + xy\mathbf{j} - z^2\mathbf{k}$.
(xii) $\mathbf{F}(x,y,z) = e^{z^2}\mathbf{i} + \sqrt{1+z^2}\mathbf{j} + \cos(x)\mathbf{k}$.

15. Determine whether

$$\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$$

is a gradient field.

16. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be smooth functions. Let

$$\mathbf{F}(x,y,z) := 2f(x)f'(x)\mathbf{i} + g'(y)h(z)\mathbf{j} + g(y)h'(z)\mathbf{k}.$$

Determine whether $\mathbf{F}$ is a gradient field.

17. Let

$$\mathbf{F}(x,y,z) := e^{z^2}\mathbf{i} + 2Byz^3\mathbf{j} + (Axze^{z^2} + 3By^2z^2)\mathbf{k}.$$

(i) Determine the values of the constants A and B such that $\mathbf{F}$ is a gradient field on $\mathbb{R}^3$.
(ii) Determine the potential function.

18. Recall that the dot product of two vectors $u = (u_1, u_2), v = (v_1, v_2) \in \mathbb{R}^2$ is given by

$$u \cdot v = u_1 v_1 + u_2 v_2.$$

Let $\mathbf{F}(x,y) := (1-x)\mathbf{i} + \mathbf{j}$ and $\mathbf{G}(x,y) = \mathbf{i} - (1+y)\mathbf{j}$ be two smooth vector fields on $\mathbb{R}^2$.

(i) Compute the dot product $\mathbf{F}(x,y) \cdot \mathbf{G}(x,y)$, where $(x,y) \in \mathbb{R}^2$ is fixed.
(ii) Two vectors $u, v \in \mathbb{R}^2$ are said to be *orthogonal* if $u \cdot v = 0$. Determine the points $(x,y) \in \mathbb{R}^2$ (if any) where $\mathbf{F}(x,y)$ is orthogonal to $\mathbf{G}(x,y)$.

19. Let $\mathbf{F}(x,y) = 2x\cos(y)\mathbf{i} - 3ye^{-x}\mathbf{j}$ and $\mathbf{G}(x,y) = (1+x)e^{1-y}\mathbf{i} + 2\mathbf{j}$.

(i) Compute the dot product $\mathbf{F}(x,y) \cdot \mathbf{G}(x,y)$, where $(x,y) \in \mathbb{R}^2$ is fixed.
(ii) Is the dot product of two vector fields a vector field? Explain.

20. A vector field $\mathbf{F}$ is said to be a *unit vector field* if $\|\mathbf{F}(x,y)\| = 1$ for all $(x,y) \in \mathbb{R}^2$. Determine which of the following (if any) are unit vector fields:

(i) $\mathbf{F}(x,y) = x\mathbf{i} + y\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = \frac{y}{\sqrt{x^2+y^2}}\mathbf{i} + \frac{x}{\sqrt{x^2+y^2}}\mathbf{j}$.
(iii) $\mathbf{F}(x,y) = \sin(x)\mathbf{i} + \cos(y)\mathbf{j}$.

21. Building from the previous exercise, if the component functions of a vector field have unit length, i.e., if $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ with $\|P(x,y)\| = 1$ and $\|Q(x,y)\| = 1$ for all $(x,y) \in \mathbb{R}^2$) is $\mathbf{F}$ a unit vector field? Justify your answer.

22. Sketch the vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$\mathbf{F}(x,y) \quad = \quad x\frac{x^2-y^2}{x^2+y^2}\mathbf{i} + y\frac{x^2-y^2}{x^2+y^2}. \tag{1.1.1}$$

23. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function defined by

$$f(x,y) := \frac{xy(x^2-y^2)}{x^2+y^2},$$

for $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, and $f(0,0) := 0$. Show that the second-order partial derivatives of f at the origin *do not* commute.

24. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) = \frac{xy}{x^2+y^2}$$

for $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, and $f(0,0) = 0$. Show that, although this function is not continuous at the origin, it has first-order partial derivatives which are everywhere defined.

25. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) := \frac{x^2-y^2}{x^2+y^2},$$

for $(x,y) \in \mathbb{R}^2 - \{(0,0)\}$, and $f(0,0) = 0$.

(i) Show that $\lim_{x\to 0} \lim_{y\to 0} f(x,y)$ exists, and find its value.
(ii) Show that $\lim_{y\to 0} \lim_{x\to 0} f(x,y)$ exists, and find its value.
(iii) Do the values obtained in (i) and (ii) coincide?

26. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) \;:=\; \begin{cases} x^2 \sin(1/x) + y^2 \sin(1/y), & xy \neq 0, \\ x^2 \sin(1/x), & x \neq 0,\ y = 0, \\ y^2 \sin(1/y), & x = 0,\ y \neq 0, \\ 0, & x = y = 0. \end{cases}$$

Show that f is differentiable but not continuously differentiable.

27. Do the second-order partial derivatives of $f : \mathbb{R}^2 \to \mathbb{R}$ defined by

$$f(x,y) := xy\frac{x^2-y^2}{x^2+y^2},$$

for $(x,y) \in \mathbb{R}^2 \backslash \{(0,0)\}$, and $f(0,0) = 0$, commute?

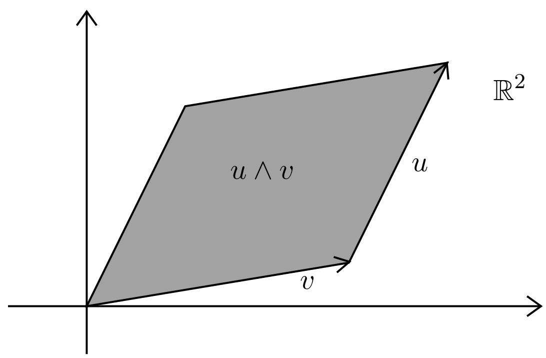
1.2. Forms and the Exterior Algebra

We have seen that 1–forms are given by expressions of the form

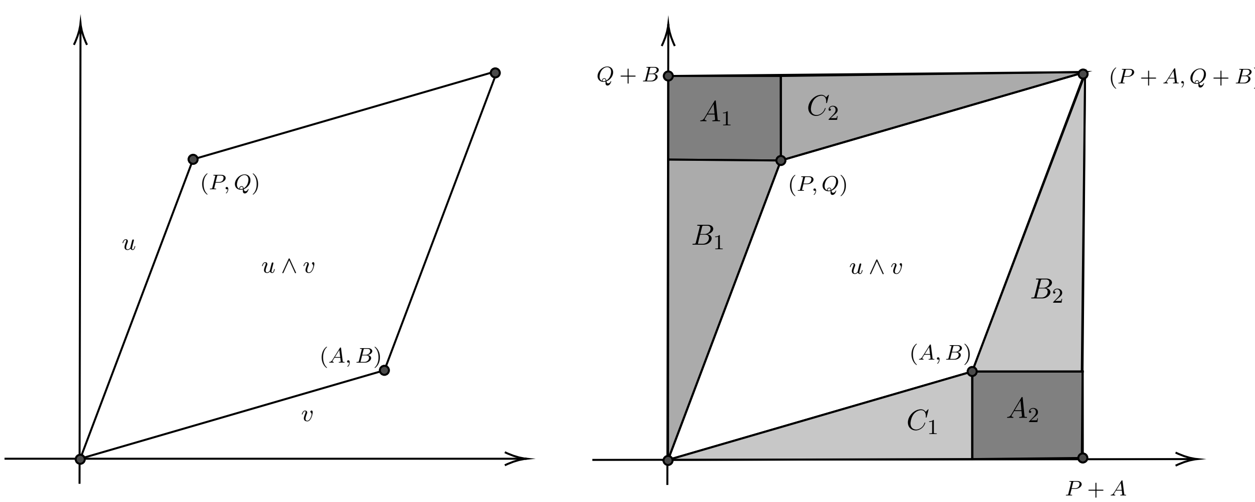
$$\omega = Pdx + Qdy,$$

where P, Q are smooth functions on $\mathbb{R}^2$. On $\mathbb{R}^3$, 1–forms are given by $\omega = Pdx + Qdy + Rdz$. As the notation indicates, 1–forms are important for an integration theory, where vector fields are not so appropriate. Forms have the additional advantage of coming with a multiplication, which we denote by $\wedge$, and call the *wedge product*.

The Wedge product. Let u, v be two 1–forms on $\mathbb{R}^2$. We want to introduce an operation on forms such that $u \wedge v$ gives the (signed) area of the parallelogram formed from u and v:



Area of parallelogram. To compute the area of this parallologram given by the forms $u = Pdx + Qdy$ and $v = Adx + Bdy$, we inscribe the parallologram in the rectangle as follows:



The area of the surrounding rectange is $|R| = (P + A)(Q + B)$. Moreover, we know that $|A_1| = |A_2|$, and $|B_1| = |B_2|$, and $|C_1| = |C_2|$. Hence, the area of the parallelogram $u \wedge v$ is

$$|u \wedge v| = |R| - 2|A_1| - 2|B_1| - 2|C_1|.$$

Since $|A_1| = PB$, $|B_1| = \frac{1}{2}PQ$ and $|C_1| = \frac{1}{2}BA$, it follows that

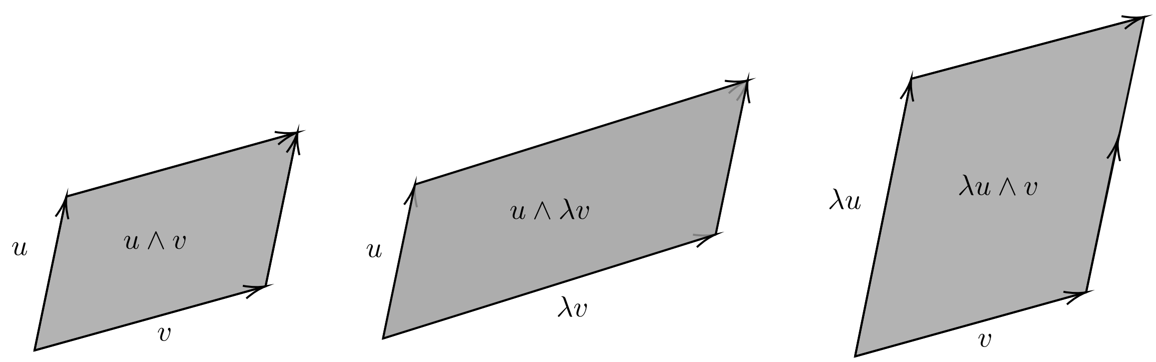
$$\begin{aligned}
\text{Area of parallelogram} &= (P + A)(Q + B) - 2PB - 2\left(\frac{1}{2}PQ\right) - 2\left(\frac{1}{2}BA\right) \\
&= PQ + PB + QA + AB - 2PB - PQ - BA \\
&= PB - QA.
\end{aligned}$$

Properties of parallelograms.

1. Scaling. If $u, v \in \mathbb{R}^2$ are vectors, and $\lambda \in \mathbb{R}$ is a scalar, then

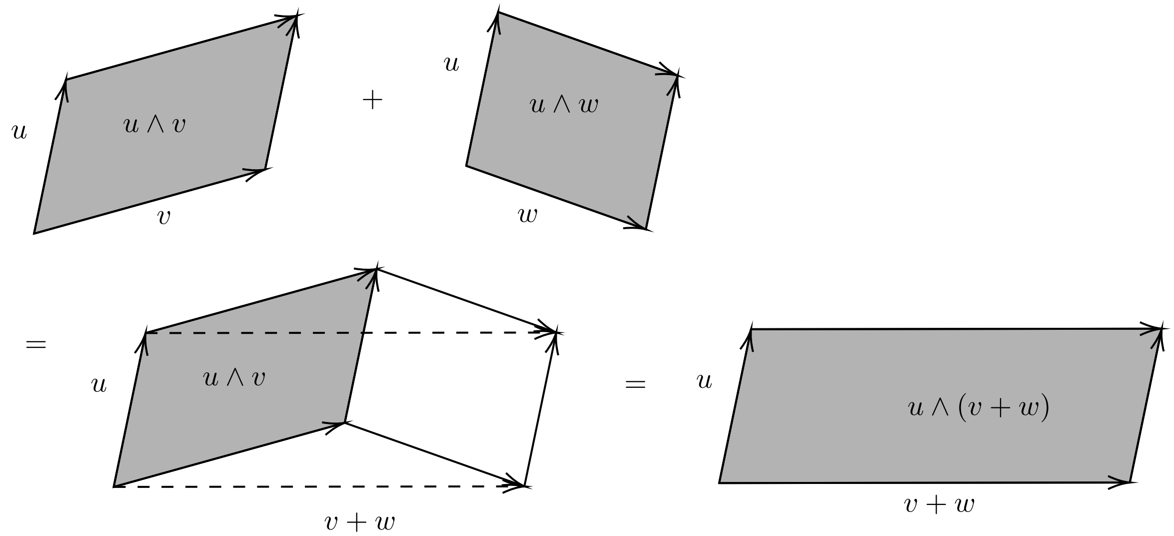
$$(\lambda u) \wedge v = u \wedge \lambda v = \lambda(u \wedge v).$$

That is, scaling a vector u by a constant λ scales the parallelogram by λ:



2. Distributive. If $u, v, w \in \mathbb{R}^2$ are vectors, then

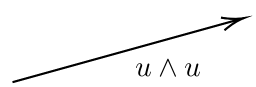
$$u \wedge (v + w) = u \wedge v + u \wedge w.$$



3. Nilpotence. If $u \in \mathbb{R}^2$ is a vector, then

$$u \wedge u = 0.$$

Visually, the nilpotent property describes the fact that the area of a line is zero:



Area of a line = 0

4. Anti-symmetry. For all vectors $u, v \in \mathbb{R}^2$, we have

$$u \wedge v = -v \wedge u.$$

This follows from the nilpotent and distributive properties. Indeed,

$$\begin{aligned} 0 \;=\; (u+v) \wedge (u+v) &= u \wedge u + u \wedge v + v \wedge u + v \wedge v \\ &= u \wedge v + v \wedge u. \end{aligned}$$

Summary of Properties of the Wedge Product.

(i) (Scaling). For all $u, v \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$, we have $(\lambda u) \wedge v = u \wedge (\lambda v) = \lambda(u \wedge v)$.
(ii) (Distributive). For all $u, v, w \in \mathbb{R}^2$, we have $u \wedge (v + w) = u \wedge v + u \wedge w$.
(iii) (Nilpotent). For all $u \in \mathbb{R}^2$, we have $u \wedge u = 0$.
(iv) (Anti-symmetry). For all $u, v \in \mathbb{R}^2$, we have $u \wedge v = -v \wedge u$.

Example 1.2.1. Let $u = Adx + Bdy$ and $v = Cdx + Ddy$ be two 1–forms on $\mathbb{R}^2$. Then

$$\begin{aligned} u \wedge v &= (Adx + Bdy) \wedge (Cdx + Ddy) \\ &= Adx \wedge Cdx + Adx \wedge Ddy + Bdx \wedge Cdx + Bdy \wedge Ddy \\ &= AC(dx \wedge dx) + AD(dx \wedge dy) + BC(dy \wedge dx) + BD(dy \wedge dy) \\ &= AD(dx \wedge dy) + BC(dy \wedge dx) \\ &= (AD - BC)dx \wedge dy. \end{aligned}$$

This recovers the formula for the determinant of a 2×2 matrix

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = AD - BC.$$

Definition 1.2.2. A 2–form on a region $\Omega \subseteq \mathbb{R}^2$ is an expression of the form

$$\omega = f(x, y)dx \wedge dy.$$

A 2–form on a region $\Omega \subseteq \mathbb{R}^3$ is an expression of the form

$$\omega = Pdx \wedge dy + Qdy \wedge dz + Rdx \wedge dz,$$

where P, Q, R are functions of (x, y, z).
We denote by $\Lambda^2(\Omega)$ the set of 2–forms on Ω.

Remark 1.2.3. We sometimes refer to 2–forms on regions of $\mathbb{R}^2$ as *area forms* or *volume forms*.

Example 1.2.4. The expressions

$$\omega = 2xdx \wedge dy \qquad \text{and} \qquad \omega = \sin(xy)dx \wedge dy + x^2dy \wedge dz$$

are 2–forms on $\mathbb{R}^3$.

Remark 1.2.5. The expression $\eta = dy \wedge dx$ is a 2–form, by the anti-symmetric property of the wedge product, we can also write $\eta = -dx \wedge dy$.

Properties of 2–forms. The properties of 2–forms are the same as those of 1–forms:

(i) (Addition). If $\omega = Pdx \wedge dy + Qdx \wedge dz + Rdy \wedge dz$ and $\eta = Adx \wedge dy + dx \wedge dz + Cdy \wedge dz$, then

$$\omega + \eta = (P + A)dx \wedge dy + (Q + B)dx \wedge dz + (R + C)dy \wedge dz,$$

where P, Q, R, A, B, C are functions of (x, y, z).

(ii) (Scaling). If $\omega = Pdx \wedge dy + Qdx \wedge dz + Rdy \wedge dz$ and $f : \mathbb{R}^3 \to \mathbb{R}$ is a function, then

$$f\omega = fPdx \wedge dy + fQdx \wedge dz + fRdy \wedge dz.$$

Example 1.2.6. Let $\omega = (x^2+y)dx\wedge dy+(1-z)dx\wedge dz+(z+y)dy\wedge dz$ and $\eta = dx\wedge dy-dy\wedge dz$. Compute

$$2\omega + 3\eta.$$

SOLUTION. Using the properties of 2–forms, we see that

$$\begin{aligned} 2\omega &= 2(x^2+y)dx\wedge dy + 2(1-z)dx\wedge dz + 2(z+y)dy\wedge dz, \\ 3\eta &= 3dx\wedge dy - 3dy\wedge dz. \end{aligned}$$

Hence,

$$2\omega + 3\eta = (2x^2+2y+3)dx\wedge dy + 2(1-z)dx\wedge dz + (2z+2y-3)dy\wedge dz.$$

□

Example 1.2.7. Let $\omega = xdx - ydy$ and $\eta = zdx + xdz$. Compute $\omega\wedge\eta$.

SOLUTION. We observe that

$$\begin{aligned} \omega\wedge\eta &= (xdx - ydy)\wedge(zdx + xdz) \\ &= xzdx\wedge dx + x^2dx\wedge dz - yzdy\wedge dx - xydy\wedge dz \\ &= x^2dx\wedge dz - yzdy\wedge dx - xydy\wedge dz \\ &= x^2dx\wedge dz + yzdx\wedge dy - xydy\wedge dz. \end{aligned}$$

□

Definition 1.2.8. A 3–form on a region $\Omega\subseteq\mathbb{R}^3$ is an expression of the form

$$\eta = f(x,y,z)dx\wedge dy\wedge dz,$$

where $f:\Omega\to\mathbb{R}$ is a smooth function.
The set of 3–forms on a region $\Omega\subseteq\mathbb{R}^3$ is denoted by $\Lambda^3(\Omega)$.

Properties of 3–forms. The properties of 3–forms are the same as those of 1–forms and 2–forms:

(i) (Addition). If $\omega = f(x,y,z)dx\wedge dy\wedge dz$ and $\eta = g(x,y,z)dx\wedge dy\wedge dz$, then

$$\omega + \eta = (f+g)dx\wedge dy\wedge dz.$$

(ii) (Scaling). If $\omega = fdx\wedge dy\wedge dz$ and $\lambda\in\mathbb{R}$ is a scalar, then

$$\lambda\omega = \lambda fdx\wedge dy\wedge dz.$$

Example 1.2.9. An example of a 3–form is given by

$$\omega = (2x^2y - xe^{-z})dx\wedge dy\wedge dz$$

Example 1.2.10. If Ω is a region in $\mathbb{R}^2$, then any 3–form on Ω is identically zero. This follows from the nilpotence property of the wedge product

$$dx \wedge dy \wedge dx = -dx \wedge dx \wedge dy = 0,$$

since $dx \wedge dx = 0$.

Remark 1.2.11. Similarly, if Ω is a region in $\mathbb{R}^3$, then any 4–form is identically zero.

Example 1.2.12. Let $\omega = 2xdx + 5e^{-xy}dy$ and $\eta = 4x^4 \sin(x+y)dx \wedge dz$. Compute $\omega \wedge \eta$.

Solution.

$$\begin{aligned} \omega \wedge \eta &= (2xdx + 5e^{-xy}dy) \wedge (4x^4 \sin(x+y)dx \wedge dz) \\ &= (2xdx) \wedge (4x^4 \sin(x+y)dx \wedge dz) + (5e^{-xy}dy) \wedge (4x^4 \sin(x+y)dx \wedge dz) \\ &= (2x)(4x^4 \sin(x+y))(dx \wedge dx \wedge dz) + (5e^{-xy})(4x^4 \sin(x+y))(dy \wedge dx \wedge dz). \end{aligned}$$

Since $dx \wedge dx = 0$, we have $dx \wedge dx \wedge dz = 0$, so

$$\begin{aligned} \omega \wedge \eta &= (5e^{-xy})(4x^4 \sin(x+y))(dy \wedge dx \wedge dz) \\ &= 20x^4 e^{-xy} \sin(x+y) dy \wedge dx \wedge dz \\ &= -20x^4 e^{-xy} \sin(x+y) dx \wedge dy \wedge dz. \end{aligned}$$

□

Remark 1.2.13. The above example illustrates that the wedge product of a 1–form and a 2–form is a 3–form. More generally, the wedge product of a k–form with an ℓ–form is a $(k+\ell)$–form.

$$(k\text{–form}) \wedge (\ell\text{–form}) \; = (k+\ell)\text{–form})$$

$$\Lambda^k \wedge \Lambda^\ell \; = \; \Lambda^{k+\ell}$$

Exterior derivative for forms. We have seen that the exterior derivative of a function f gives a 1–form $df = f_x dx + f_y dy + f_z dz$. Similarly, the exterior derivative of a 1–form produces a 2–form:

Example 1.2.14. Let $\omega = (x^2 + y)dx - 3zdy + 2xdz$. Compute $d\omega$.

Solution. We have

$$\begin{aligned} d\omega &= (2xdx + dy) \wedge dx - 3dz \wedge dy + 2dx \wedge dz \\ &= 2xdx \wedge dx + dy \wedge dx - 3dz \wedge dy + 2dx \wedge dz \\ &= dy \wedge dx - 3dz \wedge dy + 2dx \wedge dz \\ &= 2dx \wedge dz - dx \wedge dy + 3dy \wedge dz. \end{aligned}$$

□

Example 1.2.15. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = (x^2 - z^3)\mathbf{i} + (\sin(y) + \cos(z))\mathbf{j} - z^3\mathbf{k}.$$

Compute the exterior derivative of the 1–form associated to $\mathbf{F}$.

Solution. The 1–form associated to $\mathbf{F}$ is

$$\omega_{\mathbf{F}} = (x^2 - z^3)dx + (\sin(y) + \cos(z))dy - z^3dz.$$

Then

$$\begin{aligned} d\omega_{\mathbf{F}} &= (2xdx - 3z^2dz) \wedge dx + (\cos(y)dy - \sin(z)dz) \wedge dy - 3z^2dz \wedge dz \\ &= 2xdx \wedge dx - 3z^2dz \wedge dx + \cos(y)dy \wedge dy - \sin(z)dz \wedge dy - 3z^2dz \wedge dz \\ &= -3z^2dz \wedge dx - \sin(z)dz \wedge dy, \end{aligned}$$

where the last equation follows from the nilpotence property. We can further simplify to

$$d\omega_{\mathbf{F}} = 3z^2dx \wedge dz + \sin(z)dy \wedge dz.$$

□

Remark 1.2.16. In general, the exterior derivative of a k–form is a $(k+1)$–form:

$$\begin{array}{ccccccc} \text{functions} & \xrightarrow{d} & \text{1–forms} & \xrightarrow{d} & \text{2–forms} & \xrightarrow{d} & \text{3–forms} \\ \Lambda^0(\Omega) & & \Lambda^1(\Omega) & & \Lambda^2(\Omega) & & \Lambda^3(\Omega) \end{array}$$

This motivates us to refer to functions as 0–*forms*:

Functions are 0–forms

$$\mathcal{C}^\infty(\Omega) = \Lambda^0(\Omega)$$

Example 1.2.17. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by $f = 2x^3 + \sin(y)$. The exterior derivative of f is

$$df = 6x^2 dx + \cos(y) dy.$$

The exterior derivative of df is then

$$\begin{aligned} d(df) &= d(6x^2 dx + \cos(y) dy) \\ &= 12x dx \wedge dx - \sin(y) dy \wedge dy \\ &= 0, \end{aligned}$$

since $dx \wedge dx = 0$ and $dy \wedge dy = 0$.

Theorem 1.2.18. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function with exterior derivative df. Then

$$d(df) = 0.$$

PROOF. Let $f : \Omega \to \mathbb{R}$ be a smooth function on some open region in $\mathbb{R}^2$. The exterior derivative is then

$$df = f_x dx + f_y dy.$$

The exterior derivative of the 1–form df is then

$$\begin{aligned} d(df) &= d(f_x dx + f_y dy) \\ &= d(f_x) dx + d(f_y) dy \\ &= (f_{xx} dx + f_{xy} dy) \wedge dx + (f_{yx} dx + f_{yy} dy) \wedge dy \\ &= f_{xx} dx \wedge dx + f_{xy} dy \wedge dx + f_{yx} dx \wedge dy + f_{yy} dy \wedge dy \\ &= f_{xy} dy \wedge dx + f_{yx} dx \wedge dy \\ &= -f_{xy} dx \wedge dy + f_{yx} dx \wedge dy \\ &= (f_{yx} - f_{xy}) dx \wedge dy \ = \ 0, \end{aligned}$$

where the last equality follows from *Clairaut's theorem.* □

Remark 1.2.19. The exterior derivative is the unique linear map

$$d : \Lambda^\bullet(X) \longrightarrow \Lambda^\bullet(X)$$

such that

(i) $d : \Lambda^k(X) \longrightarrow \Lambda^{k+1}(X)$.
(ii) $d(f) = df$ (the ordinary differential) for $f \in \Lambda^0(X)$.
(iii) (Leibniz rule). If $\sigma \in \Lambda^k(X)$ and $\tau \in \Lambda^\bullet(X)$, then

$$d(\sigma \wedge \tau) \ = \ (d\sigma) \wedge \tau + (-1)^k \sigma \wedge d\tau.$$

(iv) (Nilpotence). $d^2 = 0$.

We will give a complete proof of this fact in the appendix.

Remark 1.2.20. Recall that a 1–form ω is said to be *exact* if $\omega = df$ for some smooth function f. The above theorem states that the exterior derivative of an exact 1–form is identically zero.

Summary 1.2.21.

$$
\begin{tikzcd}
\text{Functions} \arrow[d, leftrightarrow] & \text{Vector Fields} \arrow[d, leftrightarrow] & & \\
\Lambda^0\left(\mathbb{R}^3\right) \arrow[r, "d"] & \Lambda^1\left(\mathbb{R}^3\right) \arrow[r, "d"] & \Lambda^2\left(\mathbb{R}^3\right) \arrow[r, "d"] & \Lambda^3\left(\mathbb{R}^3\right) \\
\text{0–forms} & \text{1–forms} & \text{2–forms} & \text{3–forms}
\end{tikzcd}
$$

Functions are the same as 0–forms; vector fields are the same as 1–forms; the exterior derivative d increases the type of a form by $+1$.

Wedge product summary.

$$\Lambda^k\left(\mathbb{R}^3\right) \times \Lambda^\ell\left(\mathbb{R}^3\right) \xrightarrow{\wedge} \Lambda^{k+\ell}\left(\mathbb{R}^3\right)$$

The wedge product of a k–form and an ℓ–form is a form of type $(k+\ell)$.

Two key properties of the Wedge Product.

$$\boxed{\begin{array}{c} dx \wedge dx = 0 \\ \text{Nilpotence} \end{array}} \qquad \boxed{\begin{array}{c} dx \wedge dy = -dy \wedge dx \\ \text{Anti-symmetry} \end{array}}$$

Remarks. The treatment of the wedge product and differential forms we have given thus far has pedagogy as the primary focus. For students wanting greater levels of rigour, we encourage them to read Chapter 6.

Further reading. Additional references for the wedge product are given in the comprehensive differential geometry text by Lee [**?**]. Winitzki's linear algebra text [**32**] is devoted to the exterior algebra.

Exercises

1. Let $u = dx + 3dy$ and $v = 2dx + 5dy$ be two vectors 1–forms on $\mathbb{R}^2$.

(i) Compute $u \wedge v$.

(ii) Compute $\det \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$.

(iii) Compute $v \wedge u$.

(iv) Compute $\det \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$.

2. Let $u = 2xdx - zdy + 2ydz$ and $v = dx + 2xydy + (3 - z)dz$ be two 1–forms on $\mathbb{R}^3$.

(i) Compute $u \wedge v$.

(ii) Compute

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2x & -z & 2y \\ 1 & 2xy & 3 - z \end{pmatrix}.$$

(iii) Let $\mathbf{F}_u = 2x\mathbf{i} - z\mathbf{j} + 2y\mathbf{k}$ and $\mathbf{F}_v = \mathbf{i} + 2xy\mathbf{j} + (3 - z)\mathbf{k}$ be the vector fields associated to u, v. Compute $\mathbf{F}_u \times \mathbf{F}_v$, where $\times$ denotes the cross product.

(iv) What is the relation between the answers obtained from (i)–(iii)?

3. Determine the area of the parallelograms formed from the vectors

(i) $\mathbf{u} = 2\mathbf{i} - \mathbf{j}$ and $v = 3\mathbf{i} + 4\mathbf{j}$.

(ii) $\mathbf{u} = 3x\mathbf{i} + 2y\mathbf{j}$ and $v = (1 - x)\mathbf{i} + 2y^2\mathbf{j}$.

4. Determine the types of the following forms:

(i) $dx \wedge dy - x^3 dz \wedge dx$.

(ii) $\tan^{-1}(\sin(z))dx - \cos(x^2)dy + 4\sqrt{1 - x^2}dz$.

(iii) $x^2 + y^2$.

5. Let $\gamma = (x^2 - y^2)dx \wedge dy$ and $\eta = (x - y)dx \wedge dy$. Compute

(i) $y\gamma + x^2\eta$.

(ii) $-\gamma + (x + y)\eta$.

6. Compute $d\omega$, where

(i) $\omega = (x^2 + y^2)dx - ydy$.

(ii) $\omega = \sin(x)dz$.

(iii) $\omega = xdx \wedge dy + ydy \wedge dz$.

(iv) $\omega = yzdx + zy^2dy + xydz$.

(v) $\omega = xdx \wedge dy \wedge dz$.

7. Compute $d\varphi$, where

(i) $\varphi = (x^2 + y^3 z)dx + (y^2 - 2xz)dy + (x^4 + y^3 - z^2)dz$.
(ii) $\varphi = (x^2 + y^3 + z^4)dy \wedge dz + x^2 y^3 z^4 dz \wedge dx$.

8. Let $\varphi = x^2 dx - z^2 dy$ and $\psi = ydx - xdz$. Compute

(i) $d\varphi$.
(ii) $d\psi$.
(iii) $\varphi \wedge \psi$.

9. Let $\omega = dx + 2ydy$ and $\eta = dz$. Compute

(i) $\omega \wedge \eta$.
(ii) $\eta \wedge \omega$.
(iii) $4\omega \wedge 3\eta$.
(iv) $d\omega$.
(v) $d\eta$.
(vi) $\omega \wedge d\eta$.
(vii) $\eta \wedge d\omega$.
(viii) $d\omega \wedge d\eta$.

10. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $f = x^2 - 3y + \cos(z)$.

(i) Compute df.
(ii) Compute $d(df)$.
(iii) Did one need to compute $d(df)$ explicitly to know the result?

11. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = (\sin(x) - \cos(y))\mathbf{i} + 2xz\mathbf{j} - \tan(z^2)\mathbf{k}.$$

(i) Determine the 1–form $\omega_{\mathbf{F}}$ associated to $\mathbf{F}$.
(ii) Compute $d\omega_{\mathbf{F}}$.

12. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field given by

$$\mathbf{F}(x, y, z) = (x^3 + 2xe^y)\mathbf{i} + 2x^4 e^{1-y}\mathbf{j}.$$

(i) Determine the 1–form $\omega_{\mathbf{F}}$ associated to $\mathbf{F}$.
(ii) Compute $d\omega_{\mathbf{F}}$.

13. State, with justification, whether the following are true (T) or false (F):

(i) The wedge of two 1–forms is a 1–form.
(ii) The wedge of a 1–form and 2–form is 3–form.
(iii) The wedge of a p–form and q–form, for $p, q \in \mathbb{N}$, is a $(p + q)$–form.
(iv) If ω is a k–form, then $d\omega$ is a k–form.
(v) If ω is a p–form, then $d\omega$ is a $(p + 1)$–form.
(vi) If f is a smooth function, then df is a 2–form.
(vii) The wedge of two 2–form on $\mathbb{R}^3$ is always zero.

14. Extend the proof of *Theorem 1.2.18* to show that $(d \circ d)(f) = 0$ for any smooth function $f : \Omega \to \mathbb{R}$, where Ω is a domain in $\mathbb{R}^3$.

15. Extend the proof of *Theorem 1.2.18* to show that $(d \circ d)(f) = 0$ for any smooth function $f : \Omega \to \mathbb{R}$, where Ω is a domain in $\mathbb{R}^n$.

16. Set $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$; these are called *polar coordinates*. Convert the following 1–forms to polar coordinates:

(i) dx.
(ii) dy.
(iii) xdy.
(iv) $xdy - ydx$.
(v)
$$\frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy.$$
(vi)
$$\frac{x}{x^2+y^2}dy - \frac{y}{x^2+y^2}dx.$$

17. State, with justification, whether the following are true (T) or false (F):

(i) Functions are the same thing as 0–forms.
(ii) Every 1–form can be expessed as df for some smooth function f.
(iii) An area form is a 1–form on $\mathbb{R}^3$.
(iv) $xdx - ydy$ is a 2–form.

18. Let $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$. Show that

$$dx \wedge dy = rdrd\vartheta.$$

19. Let $\omega = 4x^3dx + \sin(z)dy + (1-y^2)dz$ and $\eta = (1-x)dy + (y-z)dz$.

(i) Compute $\omega \wedge \eta$.
(ii) Compute $d\omega$.
(iii) Compute $d\eta$.
(iv) Compute $\omega \wedge d\eta$.
(v) Compute $d\omega \wedge \eta$.
(vi) Compute $d\omega \wedge d\eta$.

20. State, with justification, whether the following are true or false:

(i) The wedge product of two 2–forms on $\mathbb{R}^3$ is zero.
(ii) The wedge product of two exact 1–forms is exact.
(iii) The wedge product of a function and a 2–form is a 2–form.
(iv) The wedge product of a 1–form and a 2–form on $\mathbb{R}^2$ is zero.
(v) If α is a k–form, and β is an ℓ–form, then $d(\alpha \wedge \beta)$ is a $(k+\ell)$–form.

21. Compute the exterior derivative of the following forms on $\mathbb{R}^3$:

(i) $\omega = (x+y^2)dx \wedge dy + \sin(z)dy \wedge dz + e^{-x}dx \wedge dz$.
(ii) $\omega = zdx \wedge dy + e^{-x}\tan(y)dy \wedge dz + (x+\sqrt{y^2+1})dx \wedge dz$.
(iii) $\omega = e^{x+y}dx \wedge dy + (1-e^z)dy \wedge dz + 2xdx \wedge dz$.
(iv) $\omega = \log_e(y)dx \wedge dy + \log_e(z)dy \wedge dz + \log_e(x)dx \wedge dz$.

22. Let $\omega = -ydx + xdy$ be a 1–form.

(i) Determine the function A such that $d\omega = Adx \wedge dy$.
(ii) Determine the vector field $\mathbf{F}_\omega$ associated to ω.
(iii) Compute the determinant

$$\det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ -y & x & 0 \end{pmatrix}.$$

(iv) If $\star d\omega = Adz$, how does $\star d\omega$ relate to the answer obtained in (iii).

$\star$**23.** Determine on which of the following open sets $\mathcal{U}$, does there exist a smooth function $f : \mathcal{U} \to \mathbb{R}$ such that

$$\omega \ := \ -\frac{ydx + xdy}{x^2+y^2} \ = \ df.$$

(i) The upper half-plane $\{(x,y) : y > 0\}$.
(ii) The union of the upper half-plane and the right half-plane.
(iii) The left half-plane.
(iv) The lower half-plane.
(v) The complement of the negative x–axis.
(vi) The annulus $\{(x,y) : 1 < x^2+y^2 < 2\}$.
(vii) The points of the form $(re^t\cos(t), re^t\sin(t))$, $0 < t < 4\pi$, $\frac{1}{2} < r < 2$.

24. Let

$$\omega \ := \ \frac{xdx + ydy}{(x^2+y^2)^2}.$$

Is ω exact on $\mathbb{R}^2 \backslash \{(0,0)\}$?

$\star$**25.** For any point $p = (x_0, y_0) \in \mathbb{R}^2$, define a 1–form ω_p on $\mathbb{R}^2 \backslash \{p\}$ by the formula

$$\omega_p \quad := \quad \frac{(y_0 - y)dx + (x - x_0)dy}{(x - x_0)^2 + (y - y_0)^2}.$$

(i) For any two points $p, q \in \mathbb{R}^2$, show that the 1–form

$$\omega \; := \; \omega_p - \omega_q$$

is exact on $\mathbb{R}^2 \backslash \mathcal{L}$, where $\mathcal{L}$ is the line segment connecting p to q.

(ii) Find a function f such that $df = \omega$.

CHAPTER 2

Differentiation Theory

"It's true we pure mathematicians are connected to a different world. But it is a very real world nevertheless."

– Isadore Singer

In the previous chapter, we discussed the main objects of vector calculus: *vector fields* and *forms*. The exterior derivative was used to compute derivatives of forms. In the present chapter, we begin the differentiation theory for vector fields. In contrast to the familiar calculus of functions $f : \mathbb{R} \to \mathbb{R}$ or $f : \mathbb{R}^n \to \mathbb{R}$, there are two notions of derivative for a vector field: the *curl* and *divergence*:

$$\text{curl}(\mathbf{F}) = \nabla \times \mathbf{F}, \qquad \text{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}.$$

A unified perspective of divergence and curl is made possible with the exterior derivative and the Hodge $\star$–operator; that is, we will show that

$$\text{curl}(\mathbf{F}) \ \sim \ \star d\omega_{\mathbf{F}}, \qquad \text{div}(\mathbf{F}) \ \sim \ \star d \star \omega_{\mathbf{F}},$$

where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$ and $\sim$ means the vector field associated to.

2.1. The Curl of a Vector Field

Definition 2.1.1. Let Ω be a region on $\mathbb{R}^2$ or $\mathbb{R}^3$, on which, we have a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. The *curl* of $\mathbf{F}$ is the vector field defined by

$$\text{curl}(\mathbf{F}) \ := \ \nabla \times \mathbf{F} \ = \ \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix}$$

where $\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$ is viewed as a vector (field) in $\mathbb{R}^3$, and $\times$ denotes the cross product.

Remark 2.1.2. It is common to refer to

$$\nabla = \frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}$$

as the *grad vector*.

Example 2.1.3. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) = xy\mathbf{i} + yz\mathbf{j} + zx\mathbf{k}$. Compute the curl of $\mathbf{F}$.

SOLUTION. The curl is given by

$$\begin{aligned} \nabla \times \mathbf{F} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ xy & yz & zx \end{pmatrix} \\ &= \mathbf{i}\,(\partial_y(zx) - \partial_z(yz)) - \mathbf{j}\,(\partial_x(zx) - \partial_z(xy)) + \mathbf{k}\,(\partial_x(yz) - \partial_y(xy)) \\ &= -y\mathbf{i} - z\mathbf{j} - x\mathbf{k}. \end{aligned}$$

$\square$

Example 2.1.4. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Compute the curl of $\mathbf{F}$.

SOLUTION. The curl is given by

$$\begin{aligned} \operatorname{curl}(\mathbf{F}) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x & y & z \end{pmatrix} \\ &= \mathbf{i}(\partial_y(z) - \partial_z(y)) - \mathbf{j}(\partial_x(z) - \partial_z(x)) + \mathbf{k}(\partial_x(y) - \partial_y(x)) \\ &= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k}. \end{aligned}$$

$\square$

Remark 2.1.5. If we wish to compute the curl of a vector field $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$, given, say, by $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then we identify it with a vector field $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + 0\mathbf{k}$, and compute the curl as before.

Example 2.1.6. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x, y) = (x^2 - y^2)\mathbf{i} + 2xy\mathbf{j}.$$

Compute the curl of $\mathbf{F}$.

SOLUTION. The curl is given by

$$
\begin{aligned}
\operatorname{curl}(\mathbf{F}) &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ x^2-y^2 & 2xy & 0 \end{pmatrix} \\
&= (\partial_y(0) - \partial_z(2xy))\mathbf{i} - (\partial_x(0) - \partial_z(x^2-y^2))\mathbf{j} \\
&\qquad\qquad\qquad\qquad + (\partial_x(2xy) - \partial_y(x^2-y^2))\mathbf{k} \\
&= 0\mathbf{i} + 0\mathbf{j} + 4y\mathbf{k}.
\end{aligned}
$$

□

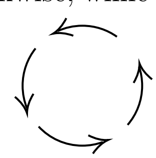
Remark 2.1.7. If $\mathbf{F}:\mathbb{R}^2 \to \mathbb{R}^2$ is a vector field on $\mathbb{R}^2$, given by $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then

$$\operatorname{curl}(\mathbf{F}) = 0\mathbf{i} + 0\mathbf{j} + (Q_x - P_y)\mathbf{k}.$$

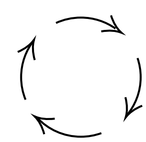
In particular, for vector fields on $\mathbb{R}^2$, we often consider the function

$$\operatorname{curl}(\mathbf{F}) \cdot \mathbf{k} \;=\; Q_x - P_y.$$

Remark 2.1.8. Let $\mathbf{F}:\mathbb{R}^2 \to \mathbb{R}^2$ be a vector field which models fluid flow. If you were to drop a twig into this fluid (keeping its center fixed), then $\operatorname{curl}(\mathbf{F})\cdot\mathbf{k} > 0$ means that the twig would rotate anti-clockwise, while $\operatorname{curl}(\mathbf{F})\cdot\mathbf{k} < 0$ means that the twig would rotate clockwise.

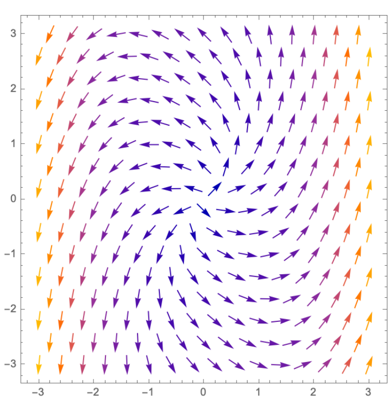


$\operatorname{curl}(\mathbf{F})\cdot\mathbf{k} > 0$

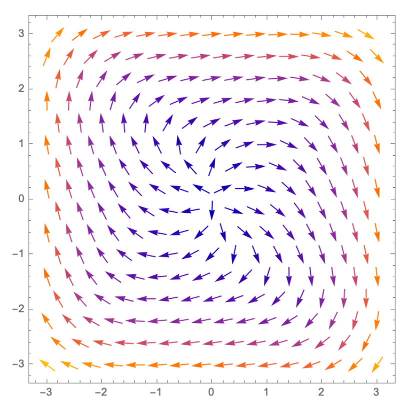


$\operatorname{curl}(\mathbf{F})\cdot\mathbf{k} < 0$

Example 2.1.9. The following vector fields illustrative positive and negative curl, respectively.



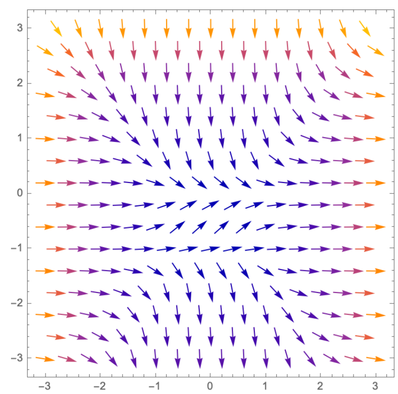
$\mathbf{F}(x,y) = (2x-y)\mathbf{i} + (y+x^3)\mathbf{j}$ has positive curl.



$\mathbf{F}(x,y) = (2x+y^3)\mathbf{i} + (y-x^3)\mathbf{j}$ has negative curl.

Definition 2.1.10. A vector field $\mathbf{F}$ is said to be *irrotational* if $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$.

Example 2.1.11. The following illustrates an irrotational vector field on $\mathbb{R}^2$:



$$\mathbf{F}(x, y) = (2x^4 + \cos(x))\mathbf{i} + (y^3 - 9ye^{y-1})\mathbf{j} \text{ is irrotational.}$$

Theorem 2.1.12. Gradient fields $\mathbf{F} = \nabla f$ are irrotational:

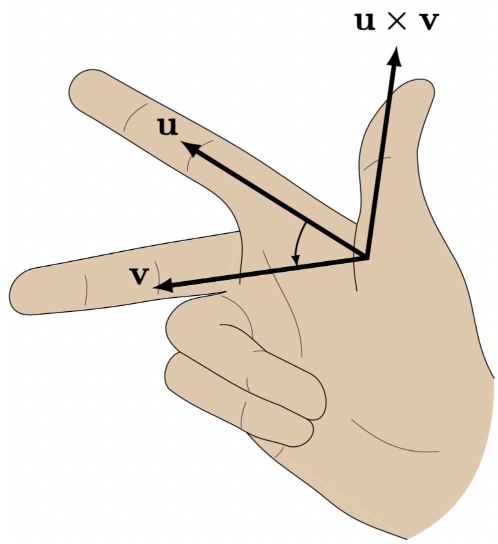
$$\operatorname{curl}(\nabla f) = \mathbf{0}.$$

Proof. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Then $\mathbf{F} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$ and

$$\begin{aligned}
\operatorname{curl}(\mathbf{F}) = \nabla \times \mathbf{F} &= \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ f_x & f_y & f_z \end{pmatrix} \\
&= \mathbf{i}(\partial_y f_z - \partial_z f_y) - \mathbf{j}(\partial_x f_z - \partial_z f_x) + \mathbf{k}(\partial_x f_y - \partial_y f_x) \\
&= \mathbf{i}(f_{yz} - f_{zy}) - \mathbf{j}(f_{xz} - f_{zx}) + \mathbf{k}(f_{xy} - f_{yx}) \\
&= \mathbf{0},
\end{aligned}$$

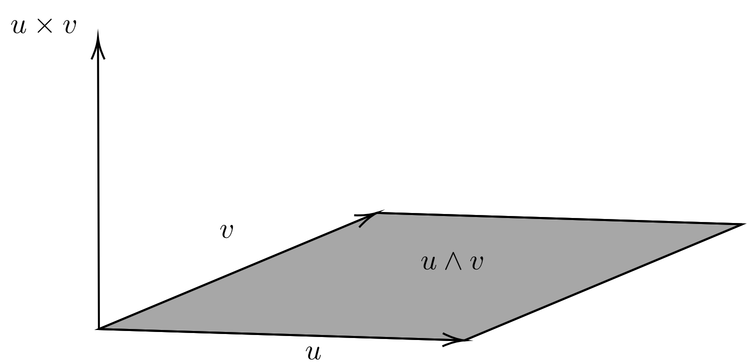
since the second-order partial derivatives of a smooth function commute by *Clairaut's theorem*
. ☐

Meaning of the cross product. Let $u, v \in \mathbb{R}^3$ be two vectors. The cross product $\times$ produces a vector $u \times v \in \mathbb{R}^3$ which is orthogonal to both u, v.



The right-hand rule.

Relationship to the wedge product. The magnitude of the cross product $|u \times v|$ is equal to the area of the parallelogram $u \wedge v$:



Remark 2.1.13. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F} = P\mathbf{i}+Q\mathbf{j}+R\mathbf{k}$. Then the associated 1–form is

$$\omega_{\mathbf{F}} = Pdx + Qdy + Rdz.$$

The exterior derivative of $\omega_{\mathbf{F}}$ is

$$\begin{aligned}
d\omega_{\mathbf{F}} &= (P_x dx + P_y dy + P_z dz) \wedge dx + (Q_x dx + Q_y dy + Q_z dz) \wedge dy \\
&\qquad +(R_x dx + R_y dy + R_z dz) \wedge dz \\
&= P_y dy \wedge dx + P_z dz \wedge dx + Q_x dx \wedge dy + Q_z dz \wedge dy + R_x dx \wedge dz + R_y dy \wedge dz \\
&= (Q_x - P_y) dx \wedge dy + (R_y - Q_z) dy \wedge dz + (R_x - P_z) dx \wedge dz.
\end{aligned}$$

Compare this with the curl of $\mathbf{F}$:

$$\operatorname{curl}(\mathbf{F}) \;=\; (R_y - Q_z)\,\mathbf{i} + (P_z - R_x)\mathbf{j} + (Q_x - P_y)\mathbf{k}.$$

There is some similarity between $d\omega_{\mathbf{F}}$ and curl($\mathbf{F}$). But $d\omega_{\mathbf{F}}$ is a 2–form, and curl($\mathbf{F}$) is a vector field. Hence, we cannot make sense of $d\omega_{\mathbf{F}}$ being associated to a vector field.
Later, we will introduce the Hodge $\star$–operator which has the property that it maps 2–forms on $\mathbb{R}^3$ to 1–forms on $\mathbb{R}^3$, i.e., $\star : \Lambda^2(\mathbb{R}^3) \to \Lambda^1(\mathbb{R}^3)$. Moreover, it satisfies

$$\star(dx \wedge dy) = dz, \qquad \star(dy \wedge dz) = dx, \qquad \star(dx \wedge dz) = -dy.$$

In particular,

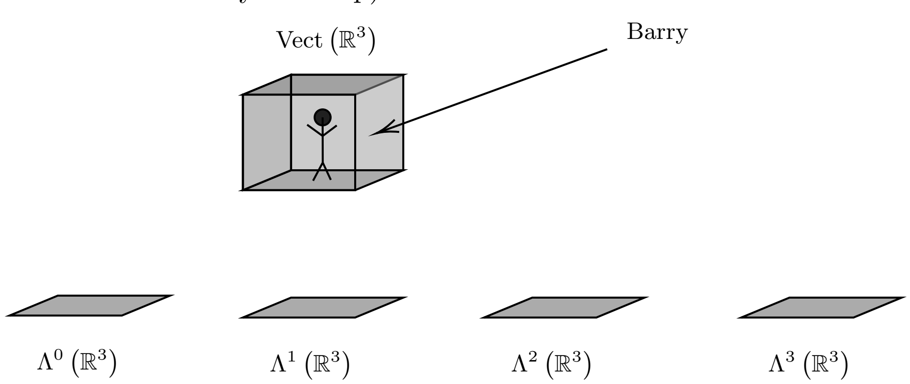
$$\star d\omega_{\mathbf{F}} \;=\; (Q_x - P_y)dz + (R_y - Q_z)dx + (P_z - R_x)dy,$$

and we see that $\star d\omega_{\mathbf{F}}$ is the 1–form associated to curl($\mathbf{F}$):

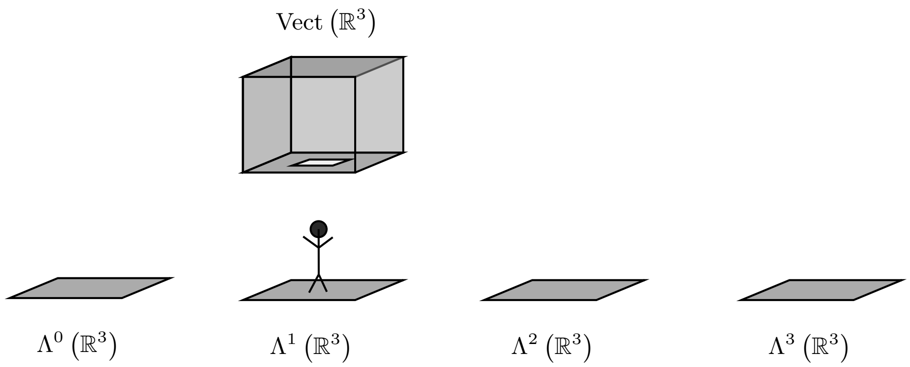
$$\boxed{\star d\omega_{\mathbf{F}} \text{ is the 1–form associated to } \operatorname{curl}(\mathbf{F})}$$

Barry's world. The Hodge $\star$–operator and the exterior derivative d are often quite confusing concepts when they are first introduced. Nevertheless, one can think of the problem of finding expressions for the divergence and curl in terms of d and $\star$ as the following game:

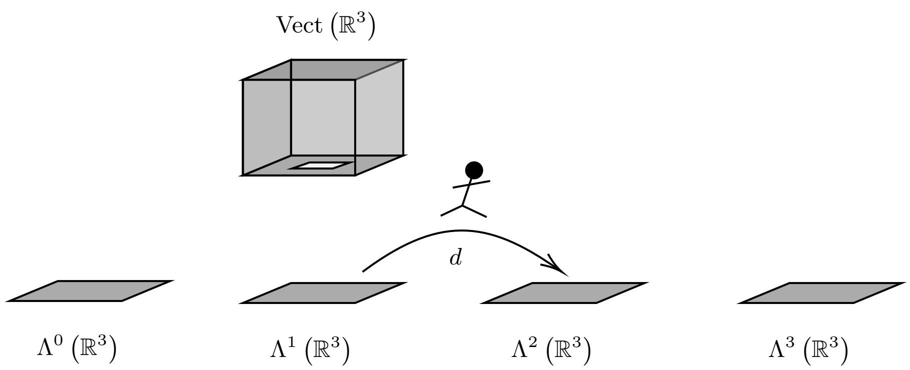
(i) To start off this, your player (Barry) is in the vector field position, which lies above the platform of 1–forms. List the forms of type $0, 1, 2, 3$ as follows (viewing them as platforms on which Barry can step):



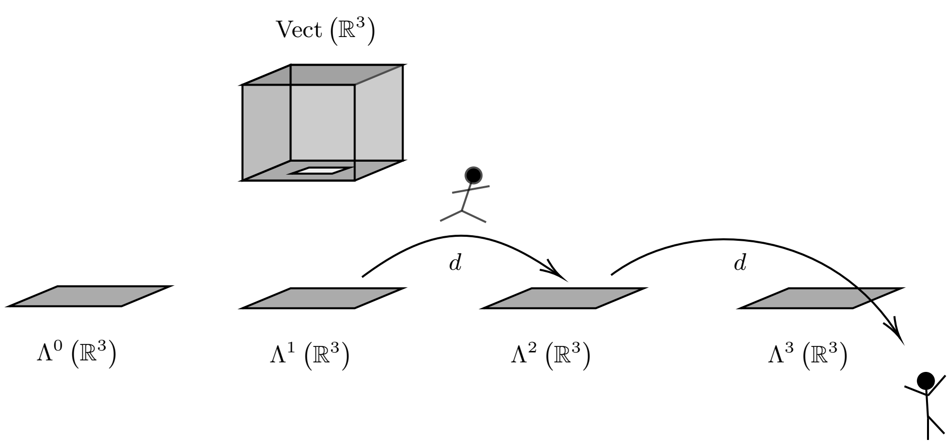
(ii) When we decide to pass to forms, we initiate the game by opening the trap door, releasing Barry onto the platform below – the platform of 1–forms:



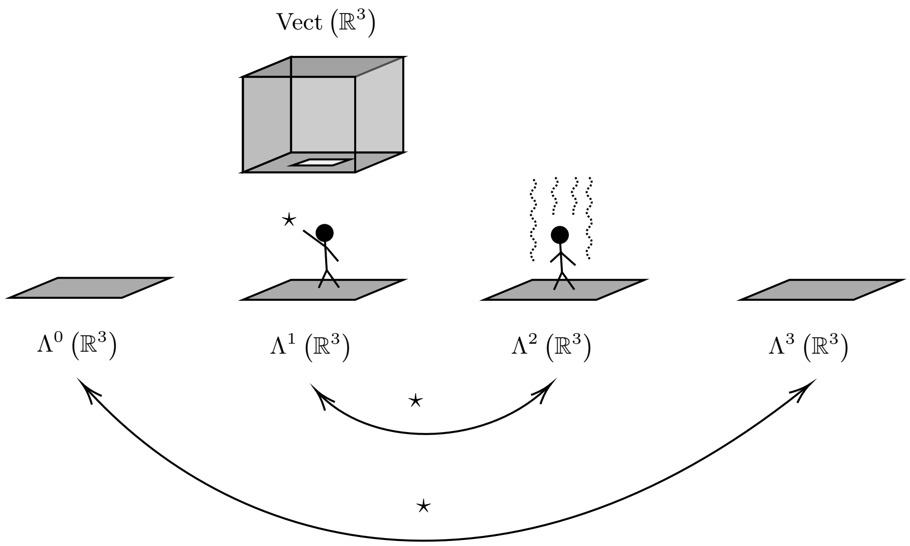
(iii) Barry is able to move to the next platform using the exterior derivative:



But, if he uses the exterior derivative twice in a row, then he falls:

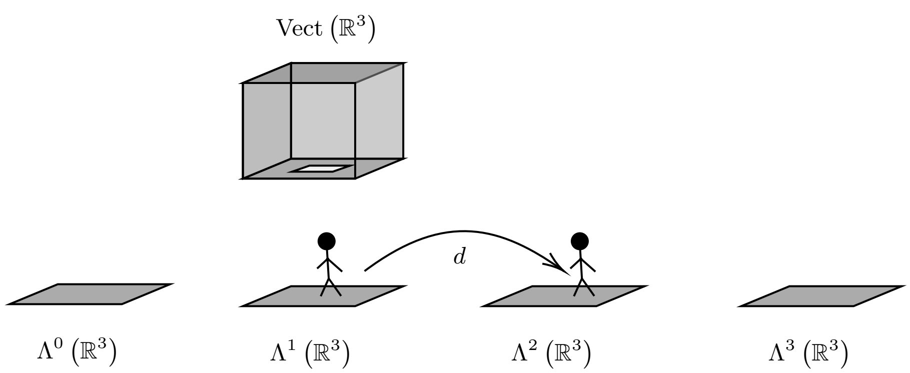


(iv) The Hodge $\star$–operator allows Barry to teleport between plats 0 and 3, and between platforms 1 and 2:

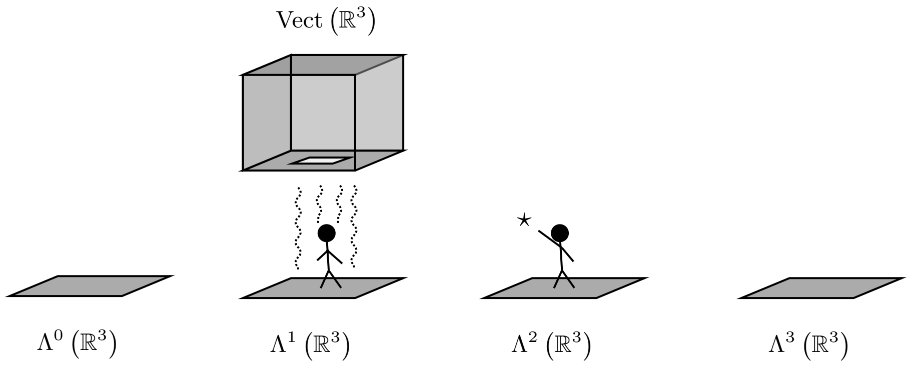


Barry explains: The curl of a vector field. The curl is a vector field, so can reach it by getting to the 1–forms platform. Here is how Barry does it:

(i) Apply the exterior derivative to get onto platform $\Lambda^2(\mathbb{R}^3)$:



(ii) We can't apply the exterior derivative, otherwise Barry falls. So we need to telelport Barry using the Hodge $\star$–operator, taking him back to platform $\Lambda^1(\mathbb{R}^3)$:



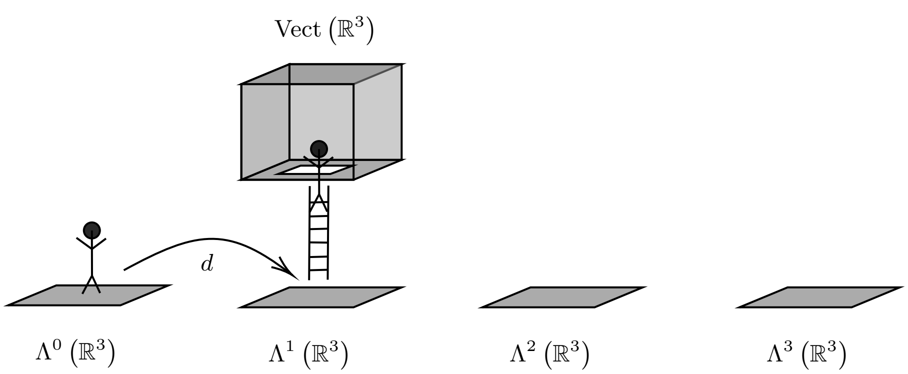
This gives lands Barry on $\Lambda^1(\mathbb{R}^3)$, which we can associate to a vector field. In other words:

$$\boxed{\star d\omega_{\mathbf{F}} \text{ is the 1–form associated to } \operatorname{curl}(\mathbf{F})}$$

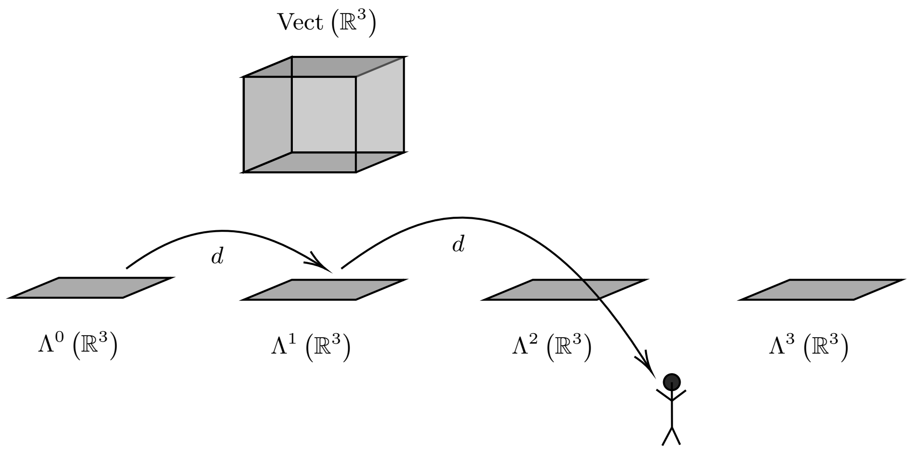
Barry explains: Curl of a gradient field is zero. Recall that we have seen the curl of a gradient field $\mathbf{F} = \nabla f$ vanishes, i.e.,

$$\operatorname{curl}(\nabla f) \quad = \quad \mathbf{0}.$$

We can understand this result without doing any calculation, merely following the results of Barry's world. The key point is that since $\mathbf{F}$ is a gradient field, the associated 1–form $\omega_{\mathbf{F}}$ comes from applying the exterior derivative *df* of a function, i.e., Barry doesn't start in the vector field box, he started on the $\Lambda^0(\mathbb{R}^3)$–platform, and then used the exterior derivative to get to the $\Lambda^1(\mathbb{R}^3)$–platform:



Hence, when we compute the curl, we first apply the exterior derivative, but since Barry has already done that, he falls:



Why the cross product exists in $\mathbb{R}^3$. The wedge product can be used to give some insight into why the cross product exists on $\mathbb{R}^3$ (and not in $\mathbb{R}^4$, for instance). Indeed, we can build the cross product from the wedge product.

Historical remark. The notion of curl was introduced by Helmholtz [**12**] (see also [**27**]) in the context of hydrodynamics.

Exercises

1. Compute the curl of the following vector fields:

(i) $\mathbf{F}(x, y) = (2x - 3y^2)\mathbf{i} + x\mathbf{j}$.
(ii) $\mathbf{F}(x, y) = 3\sqrt{x-1}\mathbf{i} + (2y^2 + x)\mathbf{j}$.
(iii) $\mathbf{F}(x, y, z) = (1 - \sin(x))\mathbf{i} + \tan(x)\mathbf{j}$.
(iv) $\mathbf{F}(x, y, z) = (e^{-z} + e^{-x})\mathbf{i} + (y^2 - x^3)\mathbf{j} + 2xz\mathbf{k}$.
(v) $\mathbf{F}(x, y, z) = (\sin(x) + \cos(x - y))\mathbf{i} + (z^2 + 1)\mathbf{j} + (1 - x)\mathbf{k}$.
(vi) $\mathbf{F}(x, y, z) = (\cos(z^2) + \log_e(x^2 + 1))\mathbf{i} + (xyz)\mathbf{j} + (1 - xzy)\mathbf{k}$.
(vii) $\mathbf{F}(x, y, z) = (e^{-x+z^2})\mathbf{i} + \sqrt{1 + z^3}\mathbf{j} + 2\sin(z)\mathbf{k}$.

2. Let $\omega := \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$ be a constant vector. If (x, y, z) denote the coordinates on $\mathbb{R}^3$, let $\mathbf{v} := \omega \times \mathbf{r}$, where $\mathbf{r} := x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Show that

$$\omega = \frac{1}{2}\mathrm{curl}(\mathbf{v}).$$

3. Show that the vector field

$$\mathbf{F}(x, y, z) := 2x^3\mathbf{i} + (y^2 + 1)\mathbf{j} + (z^3 - 9)\mathbf{k},$$

is irrotational.

4. Let $\mathbf{F}(x, y, z) = x^2\mathbf{i} + (y - z^2)\mathbf{j} + xzy\mathbf{k}$ and $\mathbf{G}(x, y, z) = (1 - z^2)\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$.

(i) Compute $\mathrm{curl}(\mathbf{F})$.
(ii) Compute $\mathrm{curl}(\mathbf{G})$.
(iii) Determine the vector field $\mathbf{H} = \mathbf{F} + \mathbf{G}$.
(iv) Compute $\mathrm{curl}(\mathbf{H})$.
(v) How does the answer the answer in (iv) compare to the sum of the answers from (i) and (ii)?

5.

(i) Show by direct computation that

$$\mathrm{curl}(\mathbf{F} + \mathbf{G}) = \mathrm{curl}(\mathbf{F}) + \mathrm{curl}(\mathbf{G}).$$

(ii) Show that, for $\mathbf{F}$ a vector field, and f a smooth function, we have

$$\mathrm{curl}(f\mathbf{F}) = f\mathrm{curl}(\mathbf{F}) + (\nabla f) \times \mathbf{F}.$$

6. Prove that, for a smooth function f, we have

$$\mathrm{curl}(f\nabla f) = \mathbf{0}.$$

7. Let $\mathbf{F}(x,y,z) = (y - \sin(xy)\mathbf{i} + (x^2+z^2)\mathbf{j} - \tan(z)\mathbf{k}$ and $\mathbf{G}(x,y,z) = (5-2z^3)\mathbf{i} + x^2y\mathbf{j} + 5\mathbf{k}$.

(i) Compute curl($\mathbf{F}$).
(ii) Compute curl($\mathbf{G}$).

8. Compute the exterior derivative of the following 1–forms:

(i) $\omega = (x^2 - 3y)dx - 4xdy$.
(ii) $\omega = \sqrt{1-x^2}dx + (x + 2\sqrt{y})dy$.
(iii) $\omega = x\cos(y)dx + y\cos(x)dx$.
(iv) $\omega = (y-x)dx + 2x^3dy + z^3dz$.

9. For a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$, explain why curl($\mathbf{F}$) > 0 meaningless. How can one make sense of curl($\mathbf{F}$) > 0 if $\mathbf{F}$ is a vector field on $\mathbb{R}^2$?

10. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. Does curl(f) make sense? If $\mathbf{F}$ is a vector field, does $\nabla\mathbf{F}$ make sense?

11. Determine which of the following vector fields are irrotational:

(i) $\mathbf{F}(x,y) = (\sin(x)+2)\mathbf{i} + (1-\cos(y))\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = \sqrt{1+x^2}\mathbf{i} + \tan^{-1}(y^2+1)\mathbf{j}$.
(iii) $\mathbf{F}(x,y,z) = (2x\cos(x^2+y^2+z^3)+3x^2)\mathbf{i} + 2y\cos(x^2+y^2+z^3)\mathbf{j} + 3z^2\cos(x^2+y^2+z^3)\mathbf{k}$.
(iv) $\mathbf{F}(x,y) = (y\cos(1-xy)\csc(1-xy)+y)\mathbf{i} + (x\cos(1-xy)\csc(1-xy)+x)\mathbf{j}$.

12. Using the curl, is the vector field

$$\mathbf{F}(x,y) = (2 - e^{x+y})\mathbf{i} + 3\sqrt{1+x^2}\mathbf{j}$$

a gradient field?

13. Compute the exterior derivative of the 1–forms associated to the following vector fields:

(i) $\mathbf{F}(x,y) = -y\sin(x)\mathbf{i} + xe^{-y^2}\mathbf{j}$.
(ii) $\mathbf{F}(x,y) = \sqrt{x^2+3y^2}\mathbf{i} + \cos(x^2)\mathbf{j}$.

14. Let $\mathbf{F}$ be a gradient field. Compute the exterior derivative of the 1–form ω associated to $\mathbf{F}$.

15. Determine which of the following statements are true or false:

(i) Every gradient field is irrotational.
(ii) Every irrotational field is a gradient field.
(iii) Every irrotational field on $\mathbb{R}^2$ is a gradient field.
(iv) The exterior derivative of the 1–form associated to a gradient field is always zero.

16. Determine which of the following statements are true or false:

(i) The exterior derivative of a 1–form is a 2–form.
(ii) ∇f is the vector field associated to the 1–form df.
(iii) $\mathrm{curl}(\mathbf{F})$ is the vector field associated to the 2–form $d\omega$, where ω is the 1–form associated to $\mathbf{F}$.

17. A *complex lamellar vector field* is a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{F} \cdot (\nabla \times \mathbf{F}) \quad = \quad 0.$$

(i) Show that an irrotational vector field is a complex lamellar vector field.

18. A *Beltrami vector field* is a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ such that

$$\mathbf{F} \times (\nabla \times \mathbf{F}) = \mathbf{0}.$$

(i) Determine whether an irrotational vector field is a Beltrami vector field.
(ii) Let

$$\mathbf{F}(x, y, z) := -\frac{z}{\sqrt{1+z^2}}\mathbf{i} + \frac{1}{\sqrt{1+z^2}}\mathbf{j}.$$

Compute the curl of $\mathbf{F}$.
(iii) Show that the vector field $\mathbf{F}$ defined in part (ii) is a Beltrami vector field.

19. Building on the previous exercise, show that if $\mathbf{F}$ is a Beltrami vector field, then $\mathbf{F}$ is parallel to $\mathrm{curl}(\mathbf{F})$.

20. Compute the exterior derivative $d\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to the following vector fields:

(i) $\mathbf{F}(x, y) = -xy\mathbf{i} + 2xe^{-y}\mathbf{j}$.
(ii) $\mathbf{F}(x, y) = \log_e(x)\mathbf{i} - x^2 \log_e(y)\mathbf{j}$.
(iii) $\mathbf{F}(x, y) = (x + \sin(xy))\mathbf{i} - yx\mathbf{j}$.

21. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field

$$\mathbf{F}(x, y) = (x^3 - y^2)\mathbf{i} + (x^2 - \sin(y^3))\mathbf{j}.$$

(i) Compute $\mathrm{curl}(\mathbf{F})$.
(ii) Write down the associated 1–form $\omega_{\mathbf{F}}$.
(iii) Compute $d\omega_{\mathbf{F}}$.
(iv) How does the result of (i) compare with that of (iii)?

22. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field

$$\mathbf{F}(x, y) = (x \log_e(y) - \cos(y))\mathbf{i} + (\sin(x) - \log_e(x + y))\mathbf{j}.$$

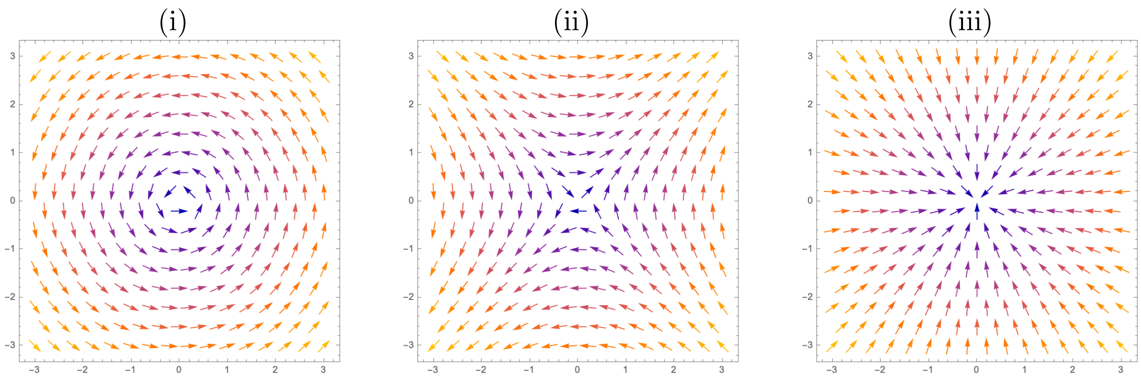
(i) Compute $\operatorname{curl}(\mathbf{F})$.

(ii) Write down the associated 1–form $\omega_{\mathbf{F}}$.

(iii) Compute $d\omega_{\mathbf{F}}$.

(iv) How does the result of (i) compare with that of (iii)?

23. Consider the following vector fields $\mathbf{F}$. State, with justification, whether $\mathbf{F}$ is a gradient field.



2.2. The Divergence of a Vector Field

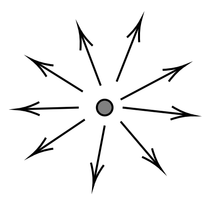
The curl of a vector field $\mathbf{F}$ was defined to be the cross product of ∇ with $\mathbf{F}$. This provides one candidate notion of derivative for a vector field. There is another notion of derivative, namely, the divergence.

Definition 2.2.1. (Divergence). Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. The *divergence* of $\mathbf{F}$ is defined

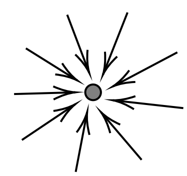
$$\mathrm{div}(\mathbf{F}) = \nabla \cdot \mathbf{F}.$$

If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$, where $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$ are smooth functions, then

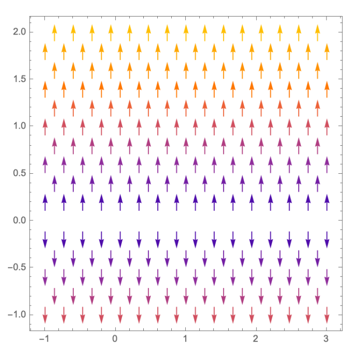
$$\mathrm{div}(\mathbf{F}) \quad = \quad \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}.$$



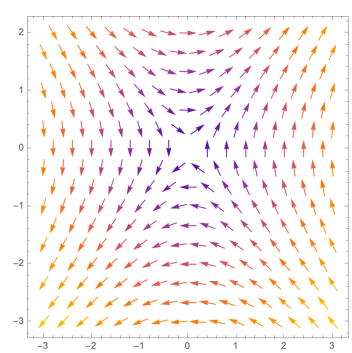
$\mathrm{div}(\mathbf{F}) > 0$



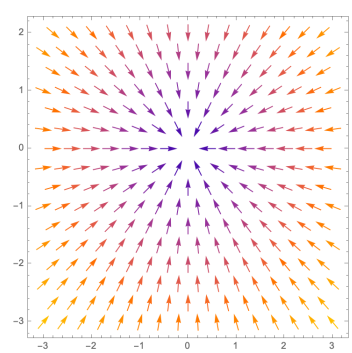
$\mathrm{div}(\mathbf{F}) < 0$



$\mathrm{div}(\mathbf{F}) > 0$



$\mathrm{div}(\mathbf{F}) = 0$



$\mathrm{div}(\mathbf{F}) < 0$

Example 2.2.2. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) = -xy\mathbf{i} + 2z\mathbf{j} + y\mathbf{k}$. Compute the divergence of $\mathbf{F}$.

Solution. We have

$$\mathrm{div}(\mathbf{F}) \quad = \quad \partial_x(-xy) + \partial_y(2z) + \partial_z(y) \quad = \quad -y.$$

$\square$

Example 2.2.3. Let ω be the 1–form on $\mathbb{R}^3$ given by

$$\omega = (x+z)dx + \sin(xy)dy.$$

Compute the divergence of the vector field $\mathbf{F}$ associated to ω.

Solution. The vector field $\mathbf{F}$ associated to ω is

$$\mathbf{F}(x,y) = (x+z)\mathbf{i} + \sin(xy)\mathbf{j} + 0\mathbf{k}.$$

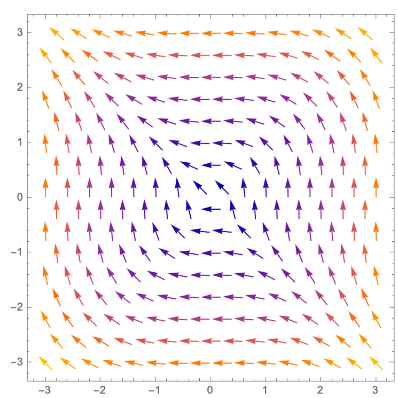
The divergence of $\mathbf{F}$ is then

$$\operatorname{div}(\mathbf{F}) = 1 + x\cos(xy).$$

□

Definition 2.2.4. A vector field $\mathbf{F}$ is said to be *incompressible* if $\operatorname{div}(\mathbf{F}) = 0$.

Example 2.2.5. The following vector field is incompressible:



$$\mathbf{F}(x,y) = -y^2\mathbf{i} + x^2\mathbf{j}.$$

Example 2.2.6. Show that the vector field $\mathbf{F} = -z\mathbf{i} + x\mathbf{j} + y^2\mathbf{k}$ is incompressible.

Solution. The divergence is

$$\operatorname{div}(\mathbf{F}) = \partial_x(-z) + \partial_y(x) + \partial_z(y^2) = 0.$$

□

Theorem 2.2.7. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then

$$\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0.$$

PROOF. Let $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ be a smooth vector field on $\mathbb{R}^3$. Then

$$\begin{aligned} \operatorname{curl}(\mathbf{F}) &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{pmatrix} \\ &= \mathbf{i}\,(R_y - Q_z) - \mathbf{j}(R_x - P_z) + \mathbf{k}(Q_x - P_y). \end{aligned}$$

Hence,

$$\begin{aligned} \operatorname{div}(\operatorname{curl}(\mathbf{F})) &= \partial_x(R_y - Q_z) - \partial_y(R_x - P_z) + \partial_z(Q_x - P_y) \\ &= R_{xy} - Q_{xz} - R_{yx} + P_{yz} + Q_{zx} - P_{zy} \\ &= (R_{xy} - R_{yx}) + (Q_{zx} - Q_{xz}) + (P_{yz} - P_{zy}). \end{aligned}$$

By *Clairaut's theorem*, the second-order partial derivatives of a smooth function commute, this proves the result. □

Definition 2.2.8. A vector field $\mathbf{F}$ is said to be *solenoidal* if there is a vector field $\mathbf{G}$ such that $\mathbf{F} = \operatorname{curl}(\mathbf{G})$.

Remark 2.2.9. *Theorem 2.2.7* asserts that every solenoidal vector field is incompressible.

Example 2.2.10. Show that the vector field $\mathbf{F}(x, y, z) = xz\mathbf{i} + xyz\mathbf{j} - y^2\mathbf{k}$ cannot be written as the curl of a vector field $\mathbf{G}$.

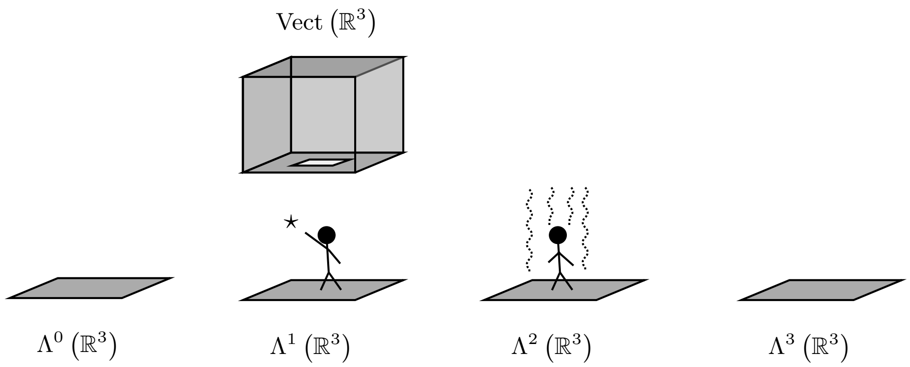
SOLUTION. If $\mathbf{F} = \operatorname{curl}(\mathbf{G})$, then $\operatorname{div}(\mathbf{F}) = \operatorname{div}(\operatorname{curl}(\mathbf{G}))$, which must be zero. Computing the divergence of $\mathbf{F}$, we have

$$\begin{aligned} \operatorname{div}(\mathbf{F}) &= \partial_x(xz) + \partial_y(xyz) + \partial_z(-y^2) \\ &= z + xz, \end{aligned}$$

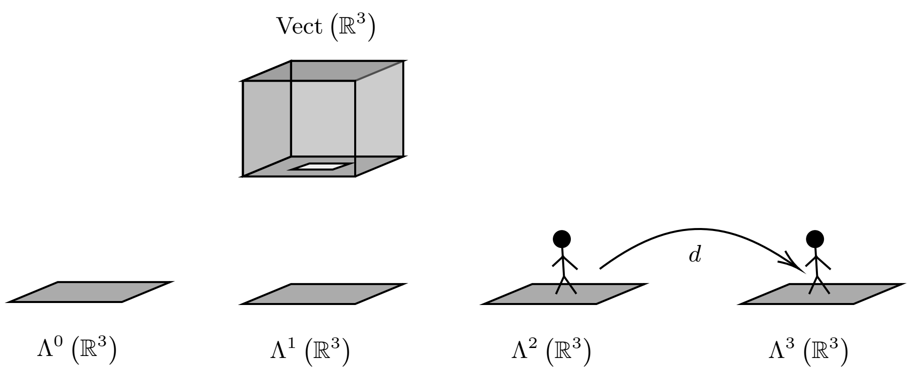
which is not zero. □

Barry explains: The Divergence. The divergence of a vector field is a scalar, so we need to land on platform $\Lambda^0(\mathbb{R}^3)$.

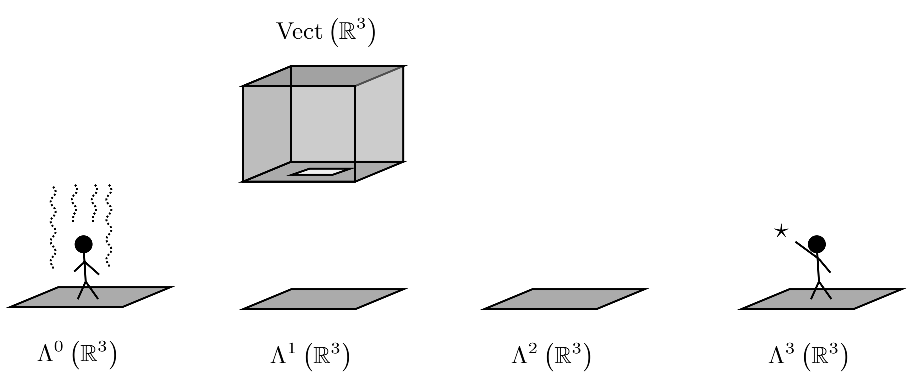
(i) If we apply the exterior derivative, then all we have left is the Hodge $\star$–operator, which traps Barry between the $\Lambda^1(\mathbb{R}^3)$–platform and the $\Lambda^2(\mathbb{R}^3)$–platform. So we have to first teleport using the Hodge $\star$–operator:



(ii) Applying the Hodge $\star$–operator again would take us back to where we started, so we need to use the exterior derivative, landing us on platform $\Lambda^3(\mathbb{R}^3)$:



(iii) Now we can teleport using the Hodge $\star$–operator to land us on the desired $\Lambda^0(\mathbb{R}^3)$ (or functions) platform.



Hence, we see that

$$\boxed{\operatorname{div}(\mathbf{F}) \;= \star d \star \omega_{\mathbf{F}}}$$

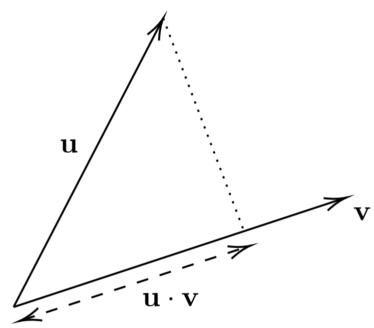
Reminder: The dot product. Recall that for two vectors $u = (u_1, u_2, u_3)$ and $v = (v_1, v_2, v_3)$, the *dot product* is defined

$$u \cdot v = u_1 v_1 + u_2 v_2 + u_3 v_3.$$

Example 2.2.11. Let $u = (1, -1, 2)$ and $v = (-3, 2, 6)$. Then

$$u \cdot v = 1(-3) + (-1)(2) + 2(6) = -3 - 2 + 12 = 7.$$

Remark: Meaning of the Dot Product. Suppose $\mathbf{u}$ and $\mathbf{v}$ are two vectors in $\mathbb{R}^n$. For the purposes of clarity, we take $n = 2$. The dot product $\mathbf{u} \cdot \mathbf{v}$ is the magnitude of the shadow cast over $\mathbf{v}$ by the vector $\mathbf{u}$:



In other words, the dot product $\mathbf{u} \cdot \mathbf{v}$ measures the extent to which $\mathbf{u}$ resides over $\mathbf{v}$.
From this, one achieves the following important property:

Properties of the dot product. Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^n$.

(i) If $\mathbf{u} \cdot \mathbf{v} = 0$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
(ii) If $\mathbf{u} \cdot \mathbf{v} = 1$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.

Exercises

1. Compute the dot products of the following pairs of vectors:

(i) $u = \mathbf{i} - \mathbf{j}$ and $v = \mathbf{i} + 2\mathbf{j}$.

(ii) $u = -3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$ and $v = 5\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function defined by $f(x, y, z) := 2x^3y^2z^4$.

(i) Compute $\nabla \cdot \nabla f$.

(ii) Show that $(\nabla^2 - \nabla \cdot \nabla)f = 0$, where $\nabla^2 := \partial_x^2 + \partial_y^2 + \partial_z^2$ is the *Laplace operator*.

3. Determine the constant $\alpha \in \mathbb{R}$ such that the vector field

$$\mathbf{F}(x, y, z) := (x + 3y)\mathbf{i} + (y - 2z)\mathbf{j} + (x + \alpha z)\mathbf{k}$$

in incompressible, i.e., $\operatorname{div}(\mathbf{F}) = 0$.

4. Show that

(i) (additivity) $\operatorname{div}(\mathbf{F} + \mathbf{G}) = \operatorname{div}(\mathbf{F}) + \operatorname{div}(\mathbf{G})$.

(ii) (Leibniz rule) $\operatorname{div}(f\mathbf{F}) = (\nabla f) \cdot \mathbf{F} + f\operatorname{div}(\mathbf{F})$.

5. Let $\mathbf{F}(x, y) = 2x^2\mathbf{i} + \sin(y)\mathbf{j} + zy\mathbf{k}$.

(i) Compute $\operatorname{div}(\mathbf{F})$.

(ii) Write the 1–form $\omega_{\mathbf{F}}$ associated to $\mathbf{F}$.

(iii) Compute the exterior derivative of $\alpha = 2x^2 dy \wedge dz - \sin(y) dx \wedge dz + zy dx \wedge dy$.

(iv) How does the answer of (i) compare with that of (iii)?

6. Let $\mathbf{F}(x, y) = (1 - e^{-y})\mathbf{i} + (x^2 + y^2)\mathbf{j} + z^2\mathbf{k}$.

(i) Compute $\operatorname{div}(\mathbf{F})$.

(ii) Write the 1–form $\omega_{\mathbf{F}}$ associated to $\mathbf{F}$.

(iii) Compute the exterior derivative of $\alpha = (1-e^{-y})dy\wedge dz-(x^2+y^2)dx\wedge dz+z^2dx\wedge dy$.

(iv) How does the answer of (i) compare with that of (iii)?

7. Let $\mathbf{F}(x, y, z) = (1 - x^2)\mathbf{i} + (1 + y^2)\mathbf{j} + z\mathbf{k}$.

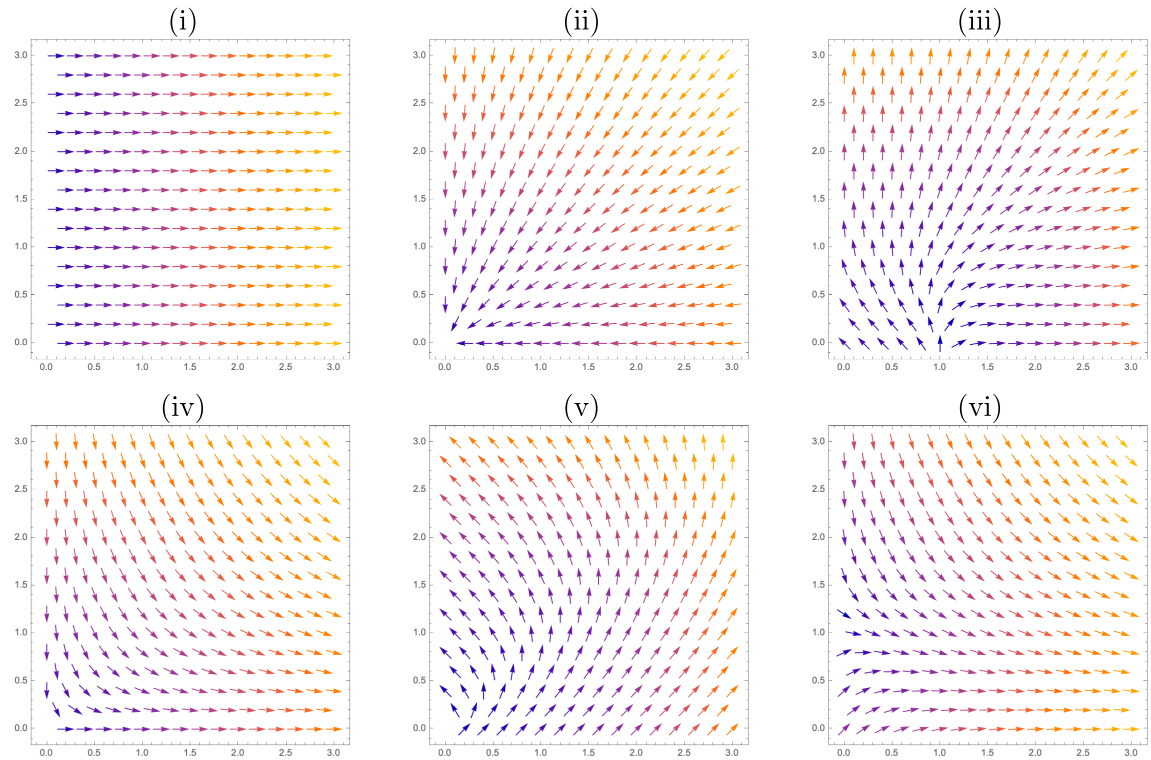
(i) Compute $\operatorname{curl}(\mathbf{F})$.

(ii) Compute $\operatorname{div}(\mathbf{F})$.

(iii) Compute $\operatorname{div}(\operatorname{curl}(\mathbf{F}))$.

(iv) Does one need to do the calculation explicitly in (iii) in order to know the result? Explain.

8. Determine whether the following vector fields have positive, negative, or zero divergence:



9. Determine whether the vector fields $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ defined below are given by the curl of some other vector field:

(i) $\mathbf{F}(x, y, z) = zy^2\mathbf{i} + 2xz^2\mathbf{j} - 3z^3\mathbf{k}$.

(ii) $\mathbf{F}(x, y, z) = yz\mathbf{i} + xyz\mathbf{j} + xy\mathbf{k}$.

10. Show that the vector field

$$\mathbf{F}(x, y, z) = (y^2 + z^2)\mathbf{i} + (x^2 - z^2)\mathbf{j} + (x^9 + y^9)\mathbf{k}$$

is incompressible.

11. Show that the divergence is a linear operator, i.e., for any vector fields $\mathbf{F}, \mathbf{G}$, and any real numbers α, β, show that

$$\mathrm{div}(\alpha\mathbf{F} + \beta\mathbf{G}) = \alpha\mathrm{div}(\mathbf{F}) + \beta\mathrm{div}(\mathbf{G}).$$

12. Compute the divergence and curl of the vector field

$$\mathbf{F}(x, y, z) = x^z\mathbf{i} + 2x\sin(y)\mathbf{j} + 2z\cos(y)\mathbf{k}.$$

13. Show that there is no vector field $\mathbf{F}$ such that

$$\mathrm{curl}(\mathbf{F}) = 2x\mathbf{i} + 3yz\mathbf{j} - xz^2\mathbf{k}.$$

14. A vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ is said to be a *complex lamellar* vector field if

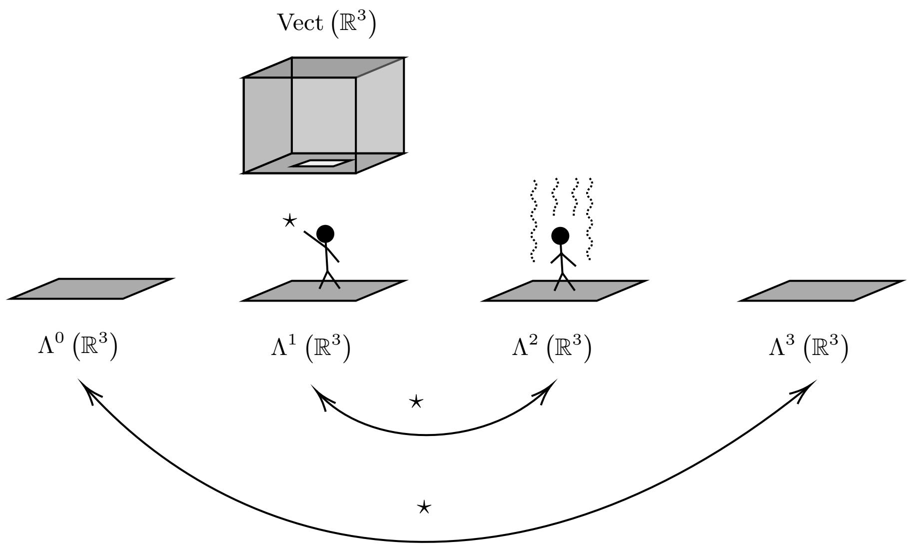
$$\mathbf{F} \cdot (\nabla \times \mathbf{F}) \;=\; 0.$$

(i) Show that if $d\omega_{\mathbf{F}} = 0$, then $\mathbf{F}$ is a complex lamellar flow.

(ii) Show that complex lamellar vector fields are precisely those that are normal to a family of surfaces.

2.3. The Hodge–$\star$ operator

In the previous sections, we have seen that the Hodge $\star$–operator (which has yet to be defined) has the property that it sends k-forms on $\mathbb{R}^3$ to $(3-k)$–forms on $\mathbb{R}^3$. In the Barry's world pictorial representation, this was represented by



The purpose of the present section is give to precise formulas for the Hodge $\star$–operator and, of course, give a definition.

Definition 2.3.1. (Hodge $\star$–operator). The Hodge $\star$–operator is the linear map

$$\star : \Lambda^k(\mathbb{R}^3) \to \Lambda^{3-k}(\mathbb{R}^3)$$

which sends a k–form $\alpha \in \Lambda^k(\mathbb{R}^3)$ to the $(3-k)$–form $\star\alpha$ which satisfies

$$\alpha \wedge (\star\alpha) \;=\; dx \wedge dy \wedge dz.$$

Example 2.3.2. Consider the 1–form $dx \in \Lambda^1(\mathbb{R}^3)$. Then $\star dx \in \Lambda^2(\mathbb{R}^3)$ is the 2–form such that

$$dx \wedge (\star dx) = dx \wedge dy \wedge dz.$$

In particular, we see that

$$\star dx = dy \wedge dz.$$

Example 2.3.3. Consider the 1–form $dy \in \Lambda^1(\mathbb{R}^3)$. Then $\star dy \in \Lambda^2(\mathbb{R}^3)$ is the 2–form such that

$$dy \wedge (\star dy) = dx \wedge dy \wedge dz.$$

In this case, we see that

$$dx \wedge dy \wedge dz \;=\; -dy \wedge dx \wedge dz \implies \star dy = -dx \wedge dz.$$

The remaining properties of the Hodge $\star$–operator are indicated in the following definition:

Definition 2.3.4. (Hodge $\star$–operator). The Hodge $\star$–operator is the linear map $\star : \Lambda^k(\mathbb{R}^3) \to \Lambda^{3-k}(\mathbb{R}^3)$ which maps a k–form on $\mathbb{R}^3$ to a form of type $(3-k)$, i.e., the Hodge $\star$–operator maps

(†) functions (i.e., 0–forms) to 3–forms:

$$\star f := f dx \wedge dy \wedge dz.$$

(†) 1–forms to 2–forms

$$\star(Pdx + Qdy + Rdz) = Pdy \wedge dz - Qdx \wedge dz + Qdx \wedge dy.$$

(†) 2–forms to 1–forms

$$\begin{aligned} \star(f dx \wedge dy) &= f dz \\ \star(f dy \wedge dz) &= f dx \\ \star(f dx \wedge dz) &= -f dy. \end{aligned}$$

(†) 3–forms to functions (i.e., 0–forms):

$$\star(f dx \wedge dy \wedge dz) = f.$$

Hodge $\star$–operator diagram. A more concise version of the Barry's world pictorial representation is given here:

$$
\begin{tikzcd}
 & \mathrm{Vect}\left(\mathbb{R}^3\right) \arrow[d, leftrightarrow] & & \\
\Lambda^0\left(\mathbb{R}^3\right) \arrow[r, "d"] \arrow[rrr, leftrightarrow, bend right, "\star"'] & \Lambda^1\left(\mathbb{R}^3\right) \arrow[r, "d"] \arrow[r, leftrightarrow, bend right, "\star"'] & \Lambda^2\left(\mathbb{R}^3\right) \arrow[r, "d"] & \Lambda^3\left(\mathbb{R}^3\right)
\end{tikzcd}
$$

Example 2.3.5. Compute $\star(2x^3dx + xydy + \sin(xy)dz)$.

SOLUTION. We have

$$
\begin{aligned}
\star(2x^3dx + xydy + \sin(xy)dz) &= 2x^3(\star dx) + xy(\star dy) + \sin(xy)(\star dz) \\
&= 2x^3dy \wedge dz - xydx \wedge dz + \sin(xy)dx \wedge dy.
\end{aligned}
$$

□

Example 2.3.6. Compute

$$\star(xydx \wedge dy - z\cos(y)dx \wedge dz + e^{-z}dy \wedge dz).$$

SOLUTION. We know that $\star(dx \wedge dy) = dz$, and $\star(dx \wedge dz) = -dy$, and $\star(dy \wedge dz) = dx$. Hence,

$$\star(xydx \wedge dy - z\cos(y)dx \wedge dz + e^{-z}dy \wedge dz) = xydz + z\cos(y)dy + e^{-z}dx$$

□

Example 2.3.7. Determine the vector field associated to

$$\star(2x^3dx \wedge dy + \log_e(x+y)dx \wedge dz).$$

SOLUTION. We know that $\star(dx \wedge dy) = dz$ and $\star(dx \wedge dz) = -dy$. Hence,

$$\star(2x^3dx \wedge dy + \log_e(x+y)dx \wedge dz) = 2x^3dz - \log_e(x+y)dy.$$

Since this is a 1–form, we can associate to it a vector field

$$\mathbf{F} = 0\mathbf{i} - \log_e(x+y)\mathbf{j} + 2x^3\mathbf{k}.$$

□

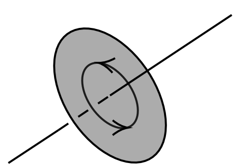
Divergence and Curl as derivatives. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field on $\mathbb{R}^3$. Let us write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ for smooth functions $P, Q, R : \mathbb{R}^3 \to \mathbb{R}$. We have seen that the curl of $\mathbf{F}$ is the *vector field*

$$\mathrm{curl}(\mathbf{F}) \quad = \quad \nabla \times \mathbf{F}.$$

The divergence, on the other hand, is the (scalar-valued) *function*

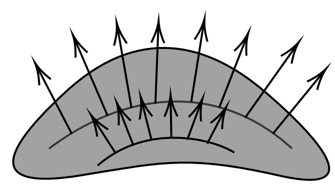
$$\mathrm{div}(\mathbf{F}) \quad = \quad \nabla \cdot \mathbf{F}.$$

I claim that both the curl and the divergence are notions of *derivatives* of the vector field $\mathbf{F}$.



curl measures the
variation
of a vector field
in two dimensions

$\mathrm{curl}(\mathbf{F}) = \star d\omega_{\mathbf{F}}$



divergence measures the
variation
of a vector field
in three dimensions

$\mathrm{div}(\mathbf{F}) = \star d \star \omega_{\mathbf{F}}$

We know that vector fields can be identified with 1–forms (and vice versa), so let $\omega_{\mathbf{F}}$ be the 1–form associated to $\mathbf{F}$. Applying the exterior derivative to $\omega_{\mathbf{F}}$ produces the 2–form $d\omega_{\mathbf{F}}$. Since 2–forms are not associated to vector fields (only 1–forms are), we would like an operator which allows us to change the type of the form, while maintaining the information it contains. This is achieved via the Hodge $\star$–operator:

Theorem 2.3.8. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field with associated 1–form ω. The curl of $\mathbf{F}$ is the vector field associated to the 1–form $\star d\omega$.

Proof. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field given by $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$. The associated 1–form is $\omega = Pdx + Qdy + Rdz$. Hence,

$$\begin{aligned} d\omega &= d(P)dx + d(Q)dy + d(R)dz \\ &= (P_x dx + P_y dy + P_z dz) \wedge dx + (Q_x dx + Q_y dy + Q_z dz) \wedge dy \\ &\qquad\qquad + (R_x dx + R_y dy + R_z dz) \wedge dz \\ &= P_y dy \wedge dx + P_z dz \wedge dx + Q_x dx \wedge dy + Q_z dz \wedge dy + R_x dx \wedge dz + R_y dy \wedge dz \\ &= (Q_x - P_y) dx \wedge dy + (R_x - P_z) dx \wedge dz + (R_y - Q_z) dy \wedge dz. \end{aligned}$$

The Hodge $\star$–operator applied to $d\omega$ is then

$$\begin{aligned}\star d\omega &= (Q_x - P_y)(\star dx \wedge dy) + (R_x - P_z)(\star dx \wedge dz) + (R_y - Q_z)(\star dy \wedge dz)\\ &= (Q_x - P_y)dz - (R_x - P_z)dy + (R_y - Q_z)dx.\end{aligned}$$

In particular, $\star d\omega$ is the 1–form associated to $\operatorname{curl}(\mathbf{F})$. $\square$

Let us now give an alternative proof of the fact that we saw previously; namely, that $\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0$, using our knowledge of the Hodge $\star$–operator:

Theorem. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field. Then

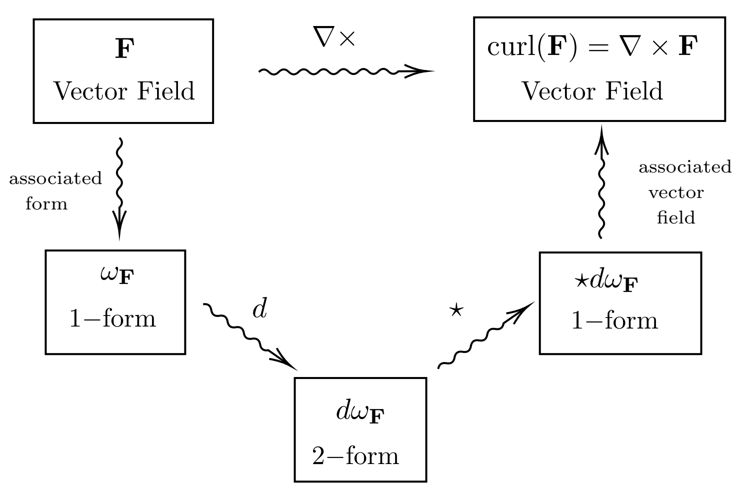
$$\operatorname{div}(\operatorname{curl}(\mathbf{F})) = 0.$$

Proof. Let $\omega_{\mathbf{F}}$ be the 1–form associated to $\mathbf{F}$. Then

$$\begin{aligned}\operatorname{div}(\operatorname{curl}(\mathbf{F})) &= \star d \star (\star d\omega_{\mathbf{F}})\\ &= \star d \star^2 d\omega_{\mathbf{F}}\\ &= \star d(d\omega_{\mathbf{F}}) \;=\; 0,\end{aligned}$$

where the second equality makes use of $\star^2 = \mathrm{id}$, and the third equality makes use $d^2 = 0$. $\square$

Summary of the above proof. A summary of the above proof is given by the diagram:



Essential properties of the Hodge $\star$–operator.

$$\begin{aligned}\star dx &= dy \wedge dz & \qquad \star(dx \wedge dy) &= dz\\ \star dy &= -dx \wedge dz & \star(dy \wedge dz) &= dx\\ \star dz &= dx \wedge dy & \star(dx \wedge dz) &= -dy.\end{aligned}$$

$$\begin{aligned} \star(1) &= dx \wedge dy \wedge dz \\ 1 &= \star(dx \wedge dy \wedge dz). \end{aligned}$$

Summary of the derivatives in terms of forms. If $\mathbf{F}$ is a vector field with associated 1–form $\omega_{\mathbf{F}}$, then the two notions of derivative are summarized here:

notion of derivative	vector field language	form language	type
curl	$\nabla \times \mathbf{F}$	$\star d\omega_{\mathbf{F}}$	vector field
divergence	$\nabla \cdot \mathbf{F}$	$\star d \star \omega_{\mathbf{F}}$	function

Aside: The Laplace–Beltrami Operator. One of the most important uses of the Hodge $\star$–operator is in its use to define the *codifferential* $d^* : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$.

Definition. Let Ω be a domain in $\mathbb{R}^n$. The *codifferential* $d^* : \Lambda^k(\Omega) \to \Lambda^{k-1}(\Omega)$ is defined by the formula

$$d^* \; := \; (-1)^{n(k-1)+1} \star d \star,$$

where d is the exterior derivative $d : \Lambda^k(\Omega) \to \Lambda^{k+1}(\Omega)$, and $\star : \Lambda^k(\Omega) \to \Lambda^{n-k}(\Omega)$ is the Hodge $\star$–operator.

Definition. The *Laplace–Beltrami operator* $\Delta_d : \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ is defined by

$$\Delta_d := d^*d + dd^*.$$

Definition. A k–form $\omega \in \Lambda^k(\Omega)$ is said to be *harmonic* if

$$\Delta_d \omega = 0.$$

That is, the kernel of $\Delta_d : \Lambda^k(\Omega) \to \Lambda^k(\Omega)$ defines the space of harmonic k–forms.

Exercises

1. Compute the Hodge $\star$–operator of the following forms on $\mathbb{R}^3$:

(i) $dx + dz$.
(ii) $xdy + \sin^3(y)dx$.
(iii) 4.
(iv) $7x^9 dx \wedge dy \wedge dz$.
(v) $x^3 dx + (x - y)dy + (x - z)dz$.
(vi) $zdx \wedge dy - xdy \wedge dz + ydx \wedge dz$.

2. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be the function given by

$$f(x, y, z) = xe^{-y} + z\cos(y).$$

(i) Compute df.
(ii) Compute $\star df$.

3. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = (2x^3 - y^2)\mathbf{i} + z^4 \sin(x)\mathbf{j} + 5z\mathbf{k}.$$

(i) Compute $d\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $\star d\omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{curl}(\mathbf{F})$ using part (ii).

4. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = y\cos(x)\mathbf{i} + e^{-z}\mathbf{j} + 2\sin(y)\mathbf{k}.$$

(i) Compute $d\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $\star d\omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{curl}(\mathbf{F})$ using part (ii).

5. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = e^{y-x}\mathbf{i} + (1 - xz)\mathbf{j} + 2e^{z}\mathbf{k}.$$

(i) Compute $d\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $\star d\omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{curl}(\mathbf{F})$ using part (ii).

6. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = \sin(xy)\mathbf{i} + \cos(xz)\mathbf{j} + \sin(yz)\mathbf{k}.$$

(i) Compute $\star\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $d \star \omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{div}(\mathbf{F})$ using part (ii).

7. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field given by

$$\mathbf{F}(x, y, z) = 2z^9\mathbf{i} + 4xy^4\mathbf{j} + 10\mathbf{k}.$$

(i) Compute $\star\omega_{\mathbf{F}}$, where $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$.
(ii) Compute $d \star \omega_{\mathbf{F}}$.
(iii) Compute $\mathrm{div}(\mathbf{F})$ using part (ii).

8. Determine (with justification) whether the following are true or false:

(i) The curl of a vector field is the vector field associated to the 1–form $\star d \star \omega_{\mathbf{F}}$.
(ii) The curl of a vector field is the vector field associated to the 1–form $\star d\omega_{\mathbf{F}}$.
(iii) The Hodge $\star$–operator satisfies $\star^2 = 1$.
(iv) The divergence of a vector field is given by $\star d \star \omega_{\mathbf{F}}$.
(v) For any smooth function $f : \mathbb{R}^3 \to \mathbb{R}$, we have $d(\star f) = 0$.

9. Determine (with justification) whether the following are true or false:

(i) If f is a smooth function, then f is a 0–form.
(ii) If $f : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function, then df is a 2–form.
(iii) If $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function, then $\star f$ is a 3–form.
(iv) If ω is a 2–form on $\mathbb{R}^3$, then $\star\omega$ can be associated to a vector field.
(v) If α is a k–form, then $\star(\star\alpha)$ is a k–form.
(vi) If α is a k–form, and β is a $(k + 1)$–form, then $\alpha \wedge \beta$ is a $(k + 2)$–form.

10. Determine (with justification) whether the following are true or false:

(i) If $\omega \in \Lambda^2(\mathbb{R}^3)$, then $\star\omega$ is a 1–form.
(ii) If $\eta \in \Lambda^1(\mathbb{R}^3)$, then $\star\eta$ is a 1–form.
(iii) If $\omega \in \Lambda^1(\mathbb{R}^3)$, then $\star d \star d\omega$ is a 1–form.
(iv) If $\omega \in \Lambda^2(\mathbb{R}^3)$, then $\star d \star d \star \omega$ is a 1–form.

11. Determine which of the following are scalars (**S**), vector fields (**VF**), 1–forms (**1**), 2–forms (**2**), 3–forms (**3**):

(i) $\mathrm{curl}(\mathbf{F}) \cdot \mathbf{k}$.
(ii) $\mathrm{curl}(\mathbf{F})$.
(iii) $\mathrm{div}(\mathbf{F})$.

(iv) df.
(v) $d\omega$, where ω is a 1–form.
(vi) $x^2 dx \wedge dy \wedge dz$.
(vii) $dx \wedge dx$.
(viii) $\omega_{\mathbf{F}}$.
(ix) $\star d\omega_{\mathbf{F}}$.
(x) $\star d \star \omega_{\mathbf{F}}$.

12.

(i) Define $\star : \Lambda^1(\mathbb{R}^2) \to \Lambda^1(\mathbb{R}^2)$ by $\star(dx) = dy$ and $\star(dy) = -dx$, extending by linearity. If f is a smooth function, show that

$$d \star df \;=\; (f_{xx} + f_{yy}) dx \wedge dy.$$

(ii) For the Hodge $\star$–operator defined on $\mathbb{R}^3$, show that

$$d \star df \;=\; (f_{xx} + f_{yy} + f_{zz}) dx \wedge dy \wedge dz.$$

13. Recall that a k–form ω is said to be closed if $d\omega = 0$. Moreover, we say that ω is exact if $\omega = d\alpha$ for some $(k-1)$–form α.

(i) Show that every exact form is closed.
(ii) Is every closed form exact?
(iii) Show that if ω and η are both closed, then $\omega \wedge \eta$ is closed.
(iv) If ω is exact and η is closed, then $\omega \wedge \eta$ is exact.

14. Suppose $k \leq n$. Let $\omega_1, ..., \omega_k$ be n–forms and suppose that $\sum_{i=1}^{k} dx_i \wedge \omega_i = 0$. Show that there are scalars $A_{ij} \in \mathbb{R}$ such that $A_{ij} = A_{ji}$ and $\omega_i = \sum_{j=1}^{k} A_{ij} dx^j$.

CHAPTER 3

Integration Theory – Curves

"Everything useful in mathematics has been devised for a purpose. Even if you don't know it, the guy who did it first, he knew what he was doing. Banach didn't just develop Banach spaces for the sake of it. He wanted to put many spaces under one heading. Without knowing the examples, the whole thing is pointless."

– Sir. Michael Atiyah

The integration theory of vector calculus is distinct from the familiar integration theory in that we are integrating of curves $\mathcal{C}$, surfaces $\mathcal{S}$, and solid regions $\mathcal{V}$. The fundamental theorem of calculus (FTC) tells us that

$$\int_a^b f'(x)dx = f(b) - f(a).$$

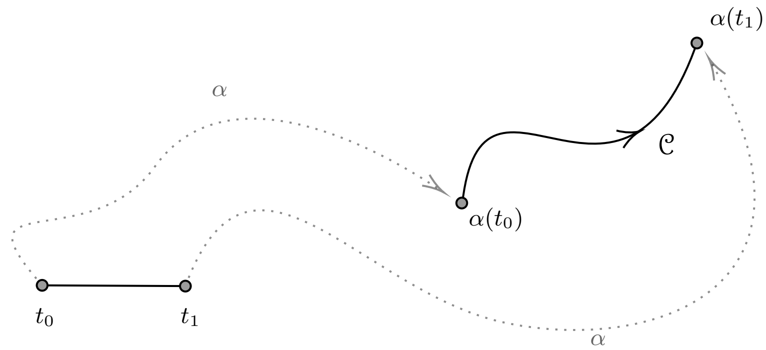
The purpose of the present chapter is to discuss the extension (or better, different incarnation) of the FTC vector fields (or really, 1–forms) on curves. For curves $\mathcal{C}$, the FTC will be the *fundamental theorem of line integrals*

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(b) - f(a).$$

This will then allow us to explore the notion of *path independence*, one of the fundamental properties of gradient fields.

3.1. Line Integrals

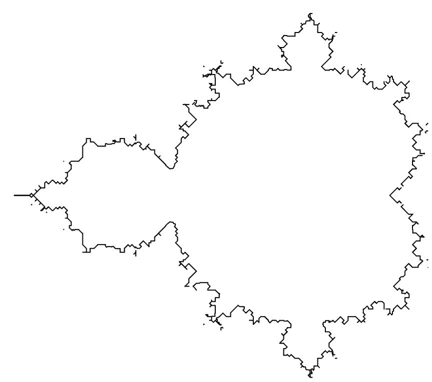
Definition 3.1.1. A *curve* $\mathcal{C} \subset \mathbb{R}^n$ is (locally) the image of a continuous map (which we call a *parametrization* for $\mathcal{C}$) $\alpha : [t_0, t_1] \to \mathbb{R}^n$, with $t_0, t_1 \in \mathbb{R} \cup \{-\infty, \infty\}$. A curve is said to be *smooth* if it affords a smooth parametrization. .



A smooth parametrization α of a curve $\mathcal{C}$.

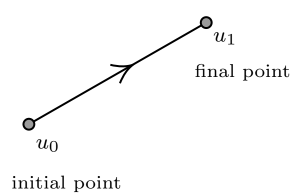
Example 3.1.2. The curve $\alpha : \mathbb{R} \to \mathbb{R}^2$ given by $\alpha(t) = (t, |t|)$ is continuous, but is not smooth.

Example 3.1.3. The Mandelbrot set is a highly non-smooth curve:



A highly non-smooth curve.

Formula for the parametrization of a line. The formula for parametrizing a line between two points u_0 and u_1 (in $\mathbb{R}^2$ or $\mathbb{R}^3$) is $\alpha(t) = (1-t)u_0 + tu_1$, where $0 \le t \le 1$.



$\alpha(t) = (1-t)u_0 + tu_1$

Example 3.1.4. Find a parametrization for the line passing between the points $(0, 1)$ and $(2, 3)$.

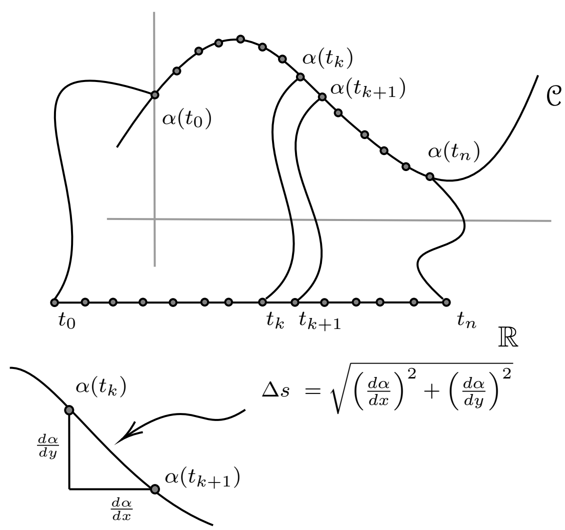
SOLUTION. The above formula tells us that $\alpha : [0,1] \to \mathbb{R}^2$ given by

$$\begin{aligned} \alpha(t) &= (1-t)(0,1) + t(2,3) \\ &= (0, 1-t) + (2t, 3t) \\ &= 2t\mathbf{i} + (1-t+3t)\mathbf{j} \ = \ 2t\mathbf{i} + (1+2t)\mathbf{j} \end{aligned}$$

yields the desired parametrization. □

Definition 3.1.5. Let $\mathcal{C}$ be a curve in $\mathbb{R}^3$. Let $\alpha(t) : [t_0, t_1] \to \mathbb{R}^3$ be a parametrization for $\mathcal{C}$. The *arc length* of $\mathcal{C}$ is given by

$$\int_{t_0}^{t_1} |\dot{\alpha}(t)| dt = \int_{t_0}^{t_1} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} dt$$



Example 3.1.6. Let $\mathcal{C}$ be the curve given by the straight line connecting the points $(1, -2, 4)$ and $(0, 5, -1)$. Find a parametrization of $\mathcal{C}$ and compute its arc length.

SOLUTION. We want a map $\alpha : [0,1] \to \mathbb{R}^3$ which is linear, and satisfies $\alpha(0) = (1, -2, 4)$ and $\alpha(1) = (0, 5, -1)$. The desired path is given by

$$\alpha(t) = (1-t) \cdot (1, -2, 4) + t \cdot (0, 5, -1) = (1-t)\mathbf{i} + (-2+7t)\mathbf{j} + (4-5t)\mathbf{k}.$$

To compute its arc length, we note that

$$\alpha'(t) = -\mathbf{i} + 7\mathbf{j} - 5\mathbf{k},$$

and therefore,

$$|\alpha'(t)| = \sqrt{(-1)^2 + (7)^2 + (-5)^2} = \sqrt{75}.$$

Hence, the arc length of $\mathcal{C}$ is given by

$$\int_0^1 \sqrt{75}dt \quad = \quad \sqrt{75}.$$

$\square$

Example 3.1.7. Let $\mathcal{C}$ be the curve given by the unit circle in $\mathbb{R}^2$, centered at the origin. Find a parametrization of $\mathcal{C}$ and compute its arc length.

SOLUTION. A parametrization of the unit circle is given by $\alpha : [0, 2\pi] \to \mathbb{R}^2$, where $\alpha(t) = (\cos(t), \sin(t))$. Here, $\dot{\alpha}(t) = (-\sin(t), \cos(t))$, and

$$|\dot{\alpha}(t)| = \sqrt{(-\sin(t))^2 + (\cos(t))^2} = \sqrt{\sin^2(t) + \cos^2(t)} = 1.$$

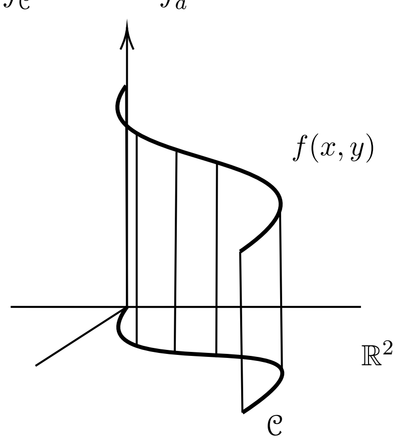
Hence, the arc length of $\mathcal{C}$ is given by

$$\int_0^{2\pi} dt \quad = \quad 2\pi.$$

$\square$

Definition 3.1.8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and let $\mathcal{C}$ be a smooth curve. Let $\alpha : [a, b] \to \mathbb{R}^n$ be a smooth function parametrizing $\mathcal{C}$. The *line integral of f along* $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} f ds \quad = \quad \int_a^b f(\alpha(t))|\alpha'(t)|dt.$$



Evaluating line integrals of functions. Let f be a smooth function and $\mathcal{C}$ a curve. The steps to evaluating a line integral are as follows:

(1) Parametrize the curve $\mathcal{C}$ by a smooth function $\alpha : [t_0, t_1] \to \mathbb{R}^n$.
(2) Compute the derivative $\alpha'(t)$ of the parametrization.
(3) Compute the norm $|\alpha'(t)|$.
(4) Express the function f in terms of the parametrization $\alpha(t)$.

(5) Evaluate the integral $\int_{t_0}^{t_1} f(\alpha(t)) \cdot |\alpha'(t)| dt$.

Example 3.1.9. Evaluate the line integral

$$\int_{\mathcal{C}} f ds,$$

where $f(x,y) = x$ and $\mathcal{C}$ is the arc of the parabola $y = x^2$ from $(1,1)$ to $(2,4)$.

SOLUTION. Parametrize the curve $\mathcal{C}$ by $\alpha : [1,2] \to \mathbb{R}^2$, given by $\alpha(t) = t\mathbf{i} + t^2\mathbf{j}$. Then $\alpha'(t) = \mathbf{i} + 2t\mathbf{j}$ and hence,

$$|\alpha'(t)| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}.$$

Now,

$$\int_{\mathcal{C}} f ds = \int_1^2 t\sqrt{1+4t^2} dt = \frac{1}{12}\left(17\sqrt{17} - 5\sqrt{5}\right).$$

□

Example 3.1.10. Find the mass of the wire in the shape of an arc given by $x^2 + y^2 = 1$ for $x \geq 0$ and $y \geq 0$, where the density function is $\rho(x,y) = x^2 + y$.

SOLUTION. The mass of the wire is given by the line integral of the density function. Hence, we need to evaluate

$$\int_{\mathcal{C}} \rho ds,$$

where $\mathcal{C}$ is the curve describing the wire.
Parametrize the wire using the function $\alpha : [0, \pi/2] \to \mathbb{R}^2$ given by $\alpha(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j}$. Then $\alpha'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j}$ and hence,

$$|\alpha'(t)| = \sqrt{\sin^2(t) + \cos^2(t)} = 1.$$

With respect to this parametrization,

$$\rho = \cos^2(t) + \sin(t),$$

and therefore, the mass of the wire is

$$\begin{aligned}
\int_{\mathcal{C}} \rho ds &= \int_0^{\pi/2} (\cos^2(t) + \sin(t)) dt \\
&= \int_0^{\pi/2} \left(\frac{1}{2}(\cos(2t) + 1) + \sin(t)\right) dt \\
&= \left[\frac{1}{4}\sin(2t) + \frac{1}{2}t - \cos(t)\right]_0^{\pi/2} \\
&= \frac{\pi}{4} + 1.
\end{aligned}$$

□

It becomes simpler to evaluate line integrals of 1–forms:

Example 3.1.11. Let $\mathcal{C}$ be the curve given by $\alpha : [0,1] \to \mathbb{R}^2$ be the curve parametrized by $\alpha(t) = t^2\mathbf{i} + t^3\mathbf{j}$. Evaluate the line integral

$$\int_{\mathcal{C}} \omega,$$

where $\omega = 2xdx + xydy$.

SOLUTION. The parametrization $\alpha(t) = t^2\mathbf{i} + t^3\mathbf{j}$ informs us that $x = t^2$ and $y = t^3$. Hence, $dx = 2tdt$ and $dy = 3t^2dt$. The 1–form ω then reads

$$\begin{aligned} \omega \; = \; 2xdx + xydy \; &= \; 2(t^2)(2tdt) + (t^2)(t^3)(3t^2dt) \\ &= \; 4t^3dt + 3t^7dt \; = \; (4t^3 + 3t^7)dt. \end{aligned}$$

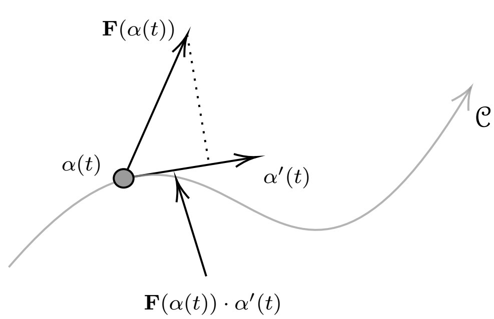
Integrating gives

$$\int_{\mathcal{C}} \omega \; = \; \int_0^1 (4t^3 + 3t^7)dt \; = \; \frac{11}{8}.$$

□

Definition 3.1.12. Let $\mathbf{F}$ be a smooth vector field on a smooth curve $\mathcal{C}$. Let $\alpha : [a,b] \to \mathbb{R}^n$ be a smooth function parametrizing $\mathcal{C}$. The *line integral of $\boldsymbol{F}$ along* $\mathcal{C}$ is defined by

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \; = \; \int_a^b \mathbf{F}(\alpha(t)) \cdot \alpha'(t)dt.$$



$\mathbf{F}(\alpha(t)) \cdot \alpha'(t)$ expresses the *amount of $\boldsymbol{F}$ which lies over* $\mathcal{C}$

Evaluating line integrals of vector fields. Let $\mathbf{F}$ be a vector field and $\mathcal{C}$ a curve. The steps to evaluating a line integral are as follows:

(1) Parametrize the curve $\mathcal{C}$ by a smooth function $\alpha : [t_0, t_1] \to \mathbb{R}^n$.
(2) Compute the derivative $\alpha'(t)$ of the parametrization.
(3) Express the vector field $\mathbf{F}$ in terms of the parametrization $\alpha(t)$.
(4) Compute the dot product $\mathbf{F}(\alpha(t)) \cdot \alpha'(t)$, producing a function of t.
(5) Evaluate the integral $\int_{t_0}^{t_1} \mathbf{F}(\alpha(t)) \cdot \alpha'(t)dt$.

Example 3.1.13. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) := z\mathbf{i} + x\mathbf{j} + y\mathbf{k}.$$

Let $\mathcal{C}$ be the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^3$, where $\alpha(t) := (t, t^2, 3)$. Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

Solution. The velocity of the parametrization is $\alpha'(t) = (1, 2t, 0)$. We also note that $x(t) = t$, $y(t) = t^2$ and $z(t) = 3$. Hence,

$$\mathbf{F}(x, y, z) \quad = \quad 3\mathbf{i} + t\mathbf{j} + t^2\mathbf{k} \; = \; (3, t, t^2),$$

and subsequently,

$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \quad &= \quad \int_0^1 (3, t, t^2) \cdot (1, 2t, 0)dt \\ &= \quad \int_0^1 \left[3 + 2t^2 + 0\right] dt \\ &= \quad \int_0^1 (3 + 2t^2)dt \; = \; \left[3t + \frac{2}{3}t^3\right]_0^1 \; = \; 3 + \frac{2}{3} \; = \; \frac{11}{3}. \end{aligned}$$

□

Example 3.1.14. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) \; := \; y\mathbf{i} - x\mathbf{j} + \mathbf{k}.$$

Let $\mathcal{C}$ be the curve traced out by $\alpha(t) = (\cos(t), -\sin(t), t/2\pi)$, where $0 \leq t \leq 2\pi$. Compute the work done by $\mathbf{F}$ along $\mathcal{C}$.

SOLUTION. The work done by $\mathbf{F}$ along $\mathcal{C}$ is simply $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$. Hence,

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin(t), -\cos(t), 1) \cdot (-\sin(t), -\cos(t), 1/2\pi) dt \\
&= \int_0^{2\pi} \left(\sin^2(t) + \cos^2(t) + \frac{1}{2\pi} \right) dt \\
&= \int_0^{2\pi} \left(1 + \frac{1}{2\pi} \right) dt \\
&= 2\pi \left(1 + \frac{1}{2\pi} \right).
\end{aligned}$$

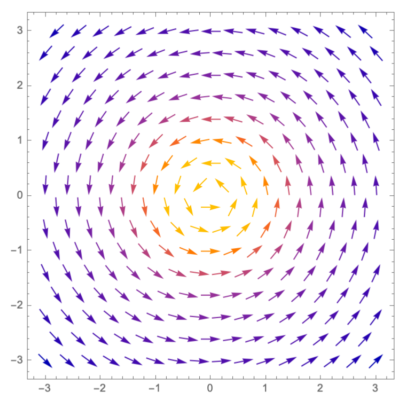
□

Example 3.1.15. Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x, y) = -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

Let $\mathcal{C} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$. Compute the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$



The vector field $\mathbf{F}(x, y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$.

SOLUTION. The curve $\mathcal{C}$ can be parametrized by the curve $\alpha : [0, 2\pi] \to \mathbb{R}^2$, $\alpha(t) = (\cos(t), \sin(t))$. Of course,

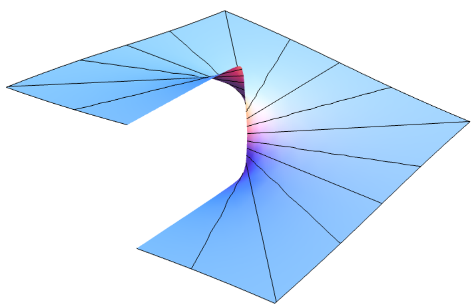
$$\alpha'(t) = (-\sin(t), \cos(t)),$$

and therefore

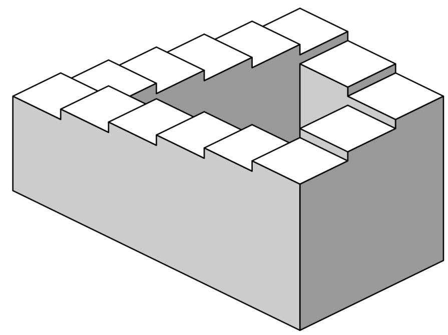
$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (-\sin(t), \cos(t)) \cdot (-\sin(t), \cos(t)) dt \\
&= \int_0^{2\pi} [\sin^2(t) + \cos^2(t)] dt = \int_0^{2\pi} dt = 2\pi.
\end{aligned}$$

□

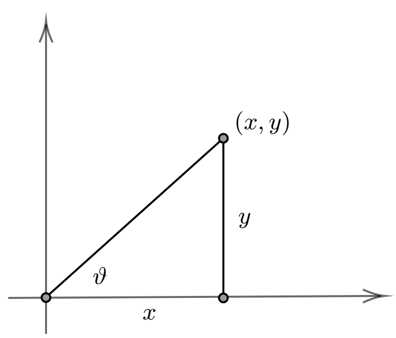
Example 3.1.16.

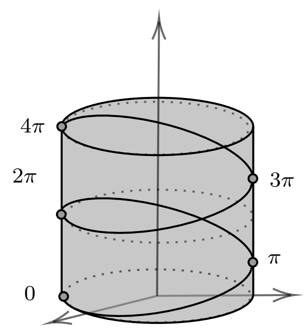


The function $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$.



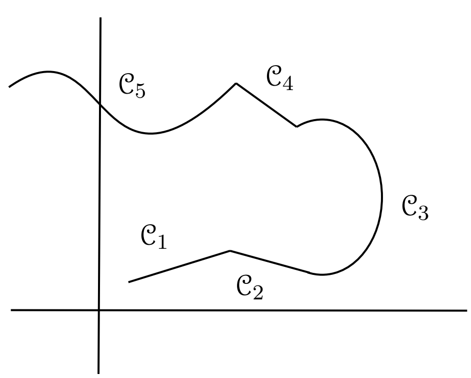
M. C. Escher's staircase gives a pictorial representation of $\tan^{-1}\left(\frac{y}{x}\right)$.





The function $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$ can be interpretted as the *angle function*, which is understood to be a function *over* the plane.

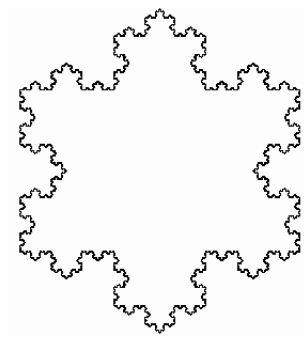
Definition 3.1.17. A curve $\mathcal{C}$ is said to be *piecewise smooth* if there is a finite number of smooth curves $\mathcal{C}_\alpha$, where $1 \le \alpha \le m$, such that $\mathcal{C} = \bigcup_{\alpha=1}^{m} \mathcal{C}_\alpha$ and the initial point of $\mathcal{C}_{\alpha+1}$ is the endpoint of $\mathcal{C}_\alpha$.



Definition 3.1.18. Let $\mathcal{C} = \bigcup_{\alpha=1}^m \mathcal{C}_\alpha$ be a piecewise smooth curve in $\Omega \subseteq \mathbb{R}^n$. Then, for a vector field $\mathbf{F} : \Omega \to \mathbb{R}^n$, we define

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \quad = \quad \sum_{\alpha=1}^{m} \int_{\mathcal{C}_\alpha} \mathbf{F} \cdot d\mathbf{r}.$$

Example 3.1.19. The Koch curve is not piecewise smooth



Key Properties of Line integrals of Functions. Let $\mathcal{C}$ be a smooth curve in $\Omega \subset \mathbb{R}^n$. Some important properties of line integrals include:

(i) (Linearity). If $f, g : \Omega \to \mathbb{R}$ are smooth functions and $\lambda, \mu \in \mathbb{R}$ are constants, then

$$\int_{\mathcal{C}} (\lambda f + \mu g) ds \quad = \quad \lambda \int_{\mathcal{C}} f ds + \mu \int_{\mathcal{C}} g ds.$$

(ii) (Independence of reparametrization). The value of

$$\int_{\mathcal{C}} f ds$$

is independent of the choice of parametrization $\alpha : [0,1] \to \Omega$ for $\mathcal{C}$.

Indeed, let $\beta : [0,1] \to [0,1]$ be a smooth function such that $\beta(0) = 0$ and $\beta(1) = 1$. Let $\alpha : [0,1] \to \Omega$ be a smooth parametrization, and $\widetilde{\alpha} : [0,1] \to \Omega$ a reparametrization defined by $\widetilde{\alpha}(t) := \alpha(\beta(t))$. Computing with respect to the reparametrization $\widetilde{\alpha}$, we have

$$\begin{aligned}
\int_{\mathcal{C}} f ds \quad &= \quad \int_0^1 f(\widetilde{\alpha}(t)) |\widetilde{\alpha}'(t)| dt \\
&= \quad \int_0^1 f(\alpha(\beta(t)) |\beta'(t) \alpha'(\beta(t))| dt \\
&= \quad \int_0^1 f(\alpha(\tau)) |\alpha'(\tau)| d\tau \\
&= \quad \int_{\mathcal{C}} f ds,
\end{aligned}$$

where we changed variables $\tau := \beta(t)$ in the second-last equality.

(iii) (Orientation-dependence). Let $\widehat{\alpha} : [0,1] \to \Omega$ be the parametrization of $\mathcal{C}$ given by $\widehat{\alpha}(t) := \alpha(1-t)$, i.e., traversing the curve in reverse. If $-\mathcal{C}$ is the curve parametrized by $\widehat{\alpha}$, then

$$\int_{-\mathcal{C}} f ds \quad = \quad -\int_{\mathcal{C}} f ds.$$

(iv) (Additive on paths). If $\mathcal{C}_1$ and $\mathcal{C}_2$ are two curves such that the endpoint of $\mathcal{C}_1$ is the initial point of $\mathcal{C}_2$, then

$$\int_{\mathcal{C}_1+\mathcal{C}_2} f ds \quad = \quad \int_{\mathcal{C}_1} f ds + \int_{\mathcal{C}_2} f ds.$$

Remark 3.1.20. The reader should compare the above properties with the properties familiar to Riemann integration:

(i) (Linearity). If $f, g : [a,b] \to \mathbb{R}$ are smooth functions, then

$$\int_a^b (\lambda f(x) + \mu g(x))dx = \lambda \int_a^b f(x)dx + \mu \int_a^b g(x)dx$$

for all $\lambda, \mu \in \mathbb{R}$.

(ii) (Independence of reparametrization). The value of

$$\int_a^b f(x)dx$$

(iii) (Orientation-dependence).

$$\int_a^b f(x)dx = -\int_b^a f(x)dx.$$

(iv) (Additive on paths).

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx.$$

Exercises

1. Find a parametrization for the following curves:

(i) The line connecting $(0, 3)$ to $(-1, 4)$.
(ii) The line connecting $(-1, 4, 3)$ to $(6, 2, -3)$.
(iii) The circle centered at the origin of radius 4.
(iv) The sphere centered at the point $(0, 3, 1)$ of radius 9.

2. Evaluate the following line integrals:

(i) $\int_{\mathcal{C}} xyds$, where $\mathcal{C}$ is the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^2$, $\alpha(t) = (t^2 - 1)\mathbf{i} + 6t\mathbf{j}$.
(ii) $\int_{\mathcal{C}} y/(x^2+1)ds$, where $\mathcal{C}$ is the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^2$, $\alpha(t) = t\mathbf{i} + t\mathbf{j}$.
(iii) $\int_{\mathcal{C}} xe^y ds$, where $\mathcal{C}$ is the line segment which connects $(0, 1)$ to $(-3, 5)$.
(iv) Let $\mathcal{C}$ consist of the line segments connecting $(0, 0)$ to $(-1, 0)$ and $(-1, 0)$ to $(3, 1)$. Compute $\int_{\mathcal{C}} x\sqrt{y+1}dx + (1 - x)ydy$.

3. Evaluate the following line integrals $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where

(i) $\mathbf{F}(x, y) = (x - 3y^2)\mathbf{i} + y\mathbf{j}$ and $\mathcal{C}$ is the unit circle in $\mathbb{R}^2$ centered at the origin.
(ii) $\mathbf{F}(x, y) = (1 - \sin(x))\mathbf{i} + y\mathbf{j}$ and $\mathcal{C}$ is the curve parametrized by $\alpha(t) = t^2\mathbf{i} + (1 - t^2)\mathbf{j}$, for $0 \le t \le 1$.
(iii) $\mathbf{F}(x, y) = \sin(x)\mathbf{i} - \sin(y)\mathbf{j}$ and $\mathcal{C}$ is the curve parametrized by $\alpha(t) = t^3\mathbf{i} + t^2\mathbf{j}$ for $0 \le t \le 2$.

4. Let $f = 2xyz^2$ and $\mathbf{F}(x, y, z) = xy\mathbf{i} - z\mathbf{j} + x^2\mathbf{k}$. Let $\mathcal{C}$ be parametrized by $\alpha : [0, 1] \to \mathbb{R}^3$, $\alpha(t) = (t^2, 2t, t^3)$.

(i) Evaluate $\int_{\mathcal{C}} f d\mathbf{r}$.
(ii) Evaluate $\int_{\mathcal{C}} \mathbf{F} d\mathbf{r}$.

5. A piece of wire is bent into the shape of a semi-circle $x^2 + y^2 = 9$, where $y \ge 0$. Suppose the density of the wire is constant, equal to ρ.

(i) Find the mass of the wire.
(ii) Find the center of mass of the wire.

6. Find the work done by the force field $\mathbf{F}(x, y) = x\mathbf{i} + (2 - y)\mathbf{j}$ as an object moves along the curve parametrized by $\alpha : [0, 2\pi] \to \mathbb{R}^2$, $\alpha(t) = (t - \sin(t))\mathbf{i} + (1 - \cos(t))\mathbf{j}$.

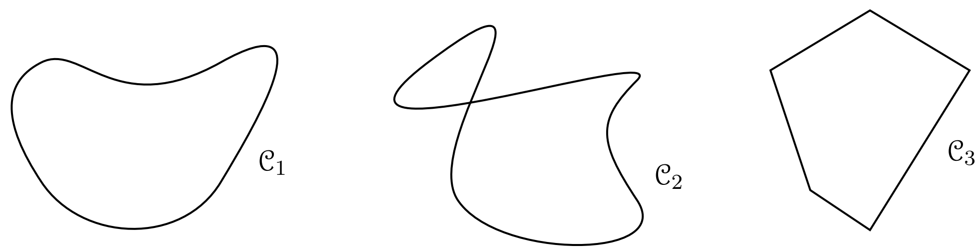
7. Find the work done by the force field $\mathbf{F}(x, y, z) = yx\mathbf{i} + zy\mathbf{j} + zx\mathbf{k}$ as an object moves along the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^2$, $\alpha(t) = t^2\mathbf{i} + t\mathbf{j} + t^3\mathbf{k}$.

8. Let $\mathbf{F}(x, y) = 3xy\mathbf{i} - y^2\mathbf{j}$. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ if $\mathcal{C}$ is the curve given by $y = 2x^2$, from the origin to $(1, 2)$.

9. Find the work done in moving a particle once around the circle $\mathcal{C}$ of radius 3 centered at the origin if

$$\mathbf{F}(x, y, z) = (2x - y + z)\mathbf{i} + (x + y - z^2)\mathbf{j} + (3x - 2y + 4z)\mathbf{k}.$$

10. Determine which of the following curves are piecewise smooth:



3.2. Path Dependence of Line Integrals

The fundamental theorem of calculus states that

$$\int_a^b f'(x)dx = f(b) - f(a),$$

where $f : \mathbb{R} \to \mathbb{R}$ is a smooth function. We now recognize the 1–form $f'(x)dx$ as the exterior derivative $df = f'(x)dx$. Hence, the fundamental theorem of calculus can be written as

$$\int_a^b df = f(b) - f(a).$$

In particular, the integral of the exterior derivative depends only on the endpoints. This extends to line integrals:

Theorem 3.2.1. (Fundamental theorem of line integrals). Let $\mathcal{C}$ be a smooth curve described by a smooth map $\alpha : [a, b] \to \mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} \quad = \quad f(\alpha(b)) - f(\alpha(a)).$$

Proof. It suffices to prove the theorem for $n = 2$, since the case for general $n \in \mathbb{N}$ is the same. To this end, let $\alpha : [a, b] \to \mathbb{R}^2$ be a smooth parametrization for $\mathcal{C}$. Then

$$\begin{aligned}
\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} &= \int_a^b \nabla f \cdot \alpha'(t)dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j}\right) \cdot \left(\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right) dt \\
&= \int_a^b \left(\frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\right) dt \\
&= \int_a^b \frac{d}{dt} f(\alpha(t))dt \;=\; f(\alpha(b)) - f(\alpha(a)),
\end{aligned}$$

where the last equality follows from the fundamental theorem of calculus (in one variable). □

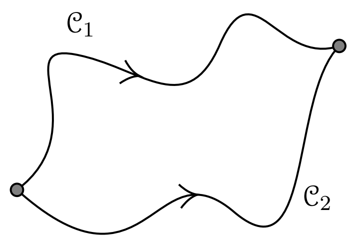
Definition 3.2.2. Let $\mathbf{F}$ be a smooth vector field on (some domain in) $\mathbb{R}^n$. We say that the line integral

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

is *independent of path* if, for all smooth paths $\mathcal{C}_1, \mathcal{C}_2$ which share their endpoints, we have

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} \quad = \quad \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}.$$

Figure 3.2.3. Two smooth paths which share their endpoints:



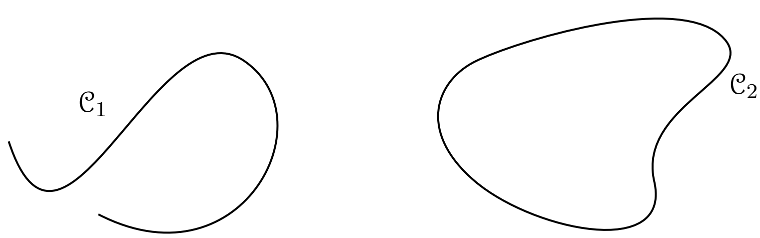
Immediate from the fundamental theorem of line integrals is the following:

Corollary 3.2.4. If $\mathbf{F}$ is a gradient field, then $\mathbf{F}$ has the path independence property.

$$\underset{\text{Gradient field}}{\mathbf{F} = \nabla f} \longrightarrow \underset{\text{path independent}}{\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}}$$

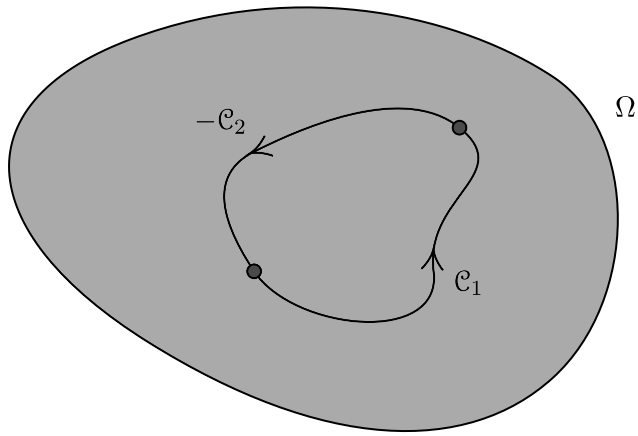
The following shows that if $\mathbf{F} : \Omega \to \mathbb{R}^2$ is a vector field which fails to be *smooth* in Ω, then $\mathbf{F} = df$ does not imply that $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is *not* independent of path.

Definition 3.2.5. A curve $\mathcal{C}$ in $\mathbb{R}^n$ is said to be *closed* if the endpoints coincide. That is, if $\mathcal{C}$ is described by a curve $\alpha : [a, b] \to \mathbb{R}^n$, then the curve is closed if $\alpha(a) = \alpha(b)$.



not closed closed

Theorem 3.2.6. Let $\Omega \subseteq \mathbb{R}^n$ be a domain (i.e., a path-connected open subset of $\mathbb{R}^n$). Let $\mathbf{F} : \Omega \to \mathbb{R}^n$ be a smooth vector field on Ω. The line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path in Ω if and only if $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths $\mathcal{C}$ in Ω.



A closed loop can be decomposed into two paths

Example 3.2.7. Let $\mathbf{F}(x, y) : \mathbb{R}^2 \backslash \{(0, 0)\} \to \mathbb{R}^2$ be the vector field

$$\mathbf{F}(x, y) := -\frac{y}{x^2 + y^2}\mathbf{i} + \frac{x}{x^2 + y^2}\mathbf{j}.$$

Then $\mathbf{F} = \nabla f$, where $f = \tan^{-1}(y/x)$.
This vector field does not satisfy the path independence property, however. Indeed, expressing $\mathbf{F}$ in polar coordinates $x = r\cos(\vartheta)$, $y = r\sin(\vartheta)$, we see that

$$\begin{aligned}
\omega_{\mathbf{F}} \;=\; -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy \;&=\; -\frac{r\sin(\vartheta)}{r^2}(\cos(\vartheta)dr - r\sin(\vartheta)d\vartheta) \\
&\quad\; + \frac{r\cos(\vartheta)}{r^2}(\sin(\vartheta)dr + r\cos(\vartheta)d\vartheta) \\
&=\; -\frac{1}{r}\sin(\vartheta)\cos(\vartheta)dr + \sin^2(\vartheta)d\vartheta \\
&\quad\; + \frac{1}{r}\cos(\vartheta)\sin(\vartheta)dr + \cos^2(\vartheta)d\vartheta \\
&=\; d\vartheta.
\end{aligned}$$

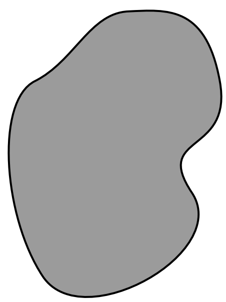
Hence, the line integral over the curve given by the unit circle centered at origin is

$$\int_0^{2\pi} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_0^{2\pi} d\vartheta \;=\; 2\pi.$$

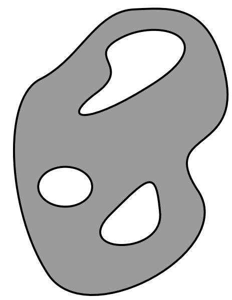
Since this is not zero, we see that $\mathbf{F}$ does not satisfy the path independence property.
In the previous sections, we have seen that gradient fields satisfy the path-independence property. Further, this is equivalent to $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed loops $\mathcal{C}$. It is natural to ask whether a vector field satisfying the path-independence property is necessarily a gradient field. On $\mathbb{R}^3$ this is true (see *Theorem 3.2.16* below), but in general it depends on the presence

of holes in the domain of the vector field. A region in $\mathbb{R}^n$ which has no holes, is called a simply connection domain:

Definition 3.2.8. Let $\Omega \subseteq \mathbb{R}^2$ be a region in the plane. Denote by $\partial\Omega$ the boundary of Ω. If $\partial\Omega$ is path connected, we say that Ω is *simply connected*.

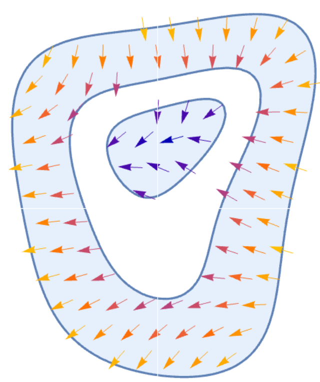


simply connected



Not simply connected

Example 3.2.9.



A vector field on a non-simply connected region.

Example 3.2.10. Euclidean space $\mathbb{R}^n$ is simply connected.

Example 3.2.11. Let

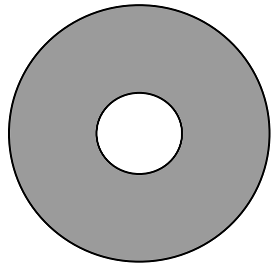
$$\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$$

denote the (closed) unit disk in the plane. The boundary $\partial\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ is connected. Hence, Ω is simply connected.

Example 3.2.12. Let $\Omega := \{(x, y) \in \mathbb{R}^2 : 1 \leq x^2 + y^2 \leq 2\}$ denote the (closed) annulus. The boundary is

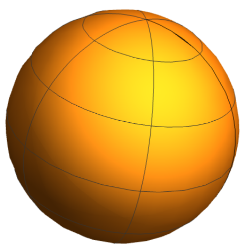
$$\partial\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\} \cup \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 2\}.$$

Since $\partial\Omega$ is not connected, the annulus is not simply connected.

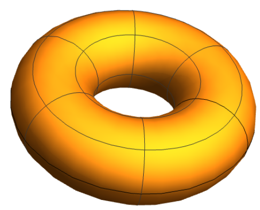


Example 3.2.13.

The sphere is simply connected



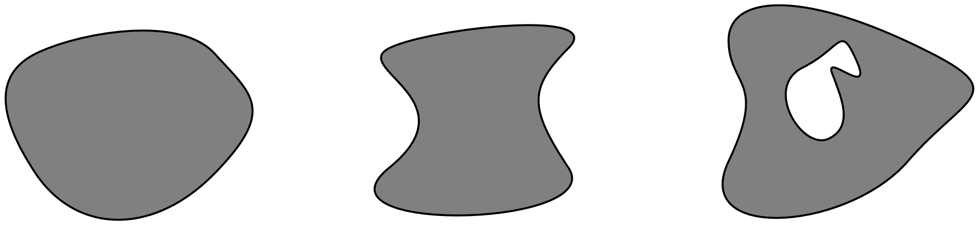
The torus is not simply connected



Definition 3.2.14. Let Ω be a region in $\mathbb{R}^n$. We say that Ω is *convex* if, for any two points $p, q \in \Omega$, the line connecting them is contained in Ω.

The following result is clear:

Proposition 3.2.15. A convex set in $\mathbb{R}^n$ is simply connected.

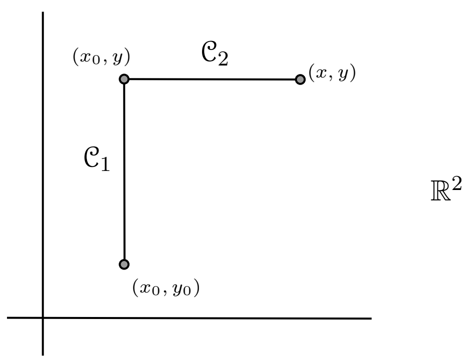


Theorem 3.2.16. Let $\mathbf{F}$ be a vector field which is smooth on a simply connected region $\Omega \subseteq \mathbb{R}^3$. If $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path in Ω, then $\mathbf{F}$ is a gradient vector field.

Proof. For simplicity, we consider the case when $\Omega = \mathbb{R}^3$. We need to produce a function f such that $\mathbf{F} = \nabla f$. Fix a point $(x_0, y_0) \in \mathbb{R}^2$, I claim that

$$f(x,y) := \int_{(x_0,y_0)}^{(x,y)} \mathbf{F} \cdot d\mathbf{r}$$

is the desired potential. To see this, observe that since $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is path independent, we can choose the linear paths which connect (x_0, y_0) to (x_0, y), followed by the linear path which connects (x_0, y) to (x, y):



Hence,

$$f(x,y) \quad = \quad \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}.$$

The curve $\mathcal{C}_1$ does not vary in x, while $\mathcal{C}_2$ does not vary in y. Therefore,

$$f_x \quad = \quad \frac{\partial}{\partial x} \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \frac{\partial}{\partial x} \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} \quad = \quad \frac{\partial}{\partial x} \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}, \tag{3.2.1}$$

and

$$f_y \quad = \quad \frac{\partial}{\partial y} \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} + \frac{\partial}{\partial x} \int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} \quad = \quad \frac{\partial}{\partial y} \int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}. \tag{3.2.2}$$

Let us write $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, and write $\omega_{\mathbf{F}} = Pdx + Qdy$ for the associated 1–form. We may write

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_{\mathcal{C}} Pdx + Qdy.$$

Since $\mathcal{C}_1$ does not vary in x, along $\mathcal{C}_1$ we have $dx = 0$. Therefore,

$$\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_{\mathcal{C}_1} Pdx + Qdy \;=\; \int_{\mathcal{C}_1} Qdy,$$

and from (3.2.2), we have:

$$f_y \;=\; \frac{\partial}{\partial y}\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} \;=\; \frac{\partial}{\partial y}\int_{\mathcal{C}_1} Qdy \;=\; \frac{\partial}{\partial y}\int_{(x_0,y_0)}^{(x_0,y)} Q(x,t)dt.$$

By the (one-variable) fundamental theorem of calculus, we have $f_y = Q$. Similarly, since $\mathcal{C}_2$ does not vary in y, along $\mathcal{C}_2$ we have $dy = 0$. Therefore,

$$\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_{\mathcal{C}_2} Pdx + Qdy \;=\; \int_{\mathcal{C}_2} Pdx,$$

and from (3.2.1) we have

$$f_x \;=\; \frac{\partial}{\partial x}\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r} \;=\; \frac{\partial}{\partial x}\int_{\mathcal{C}_2} Pdx \;=\; \frac{\partial}{\partial x}\int_{(x_0,y)}^{(x,y)} P(t,y)dt.$$

By the (one variable) fundamental theorem of calculus, we have $f_x = P$. Hence,

$$\mathbf{F} \;=\; P\mathbf{i} + Q\mathbf{j} \;=\; f_x\mathbf{i} + f_y\mathbf{j} \;=\; \nabla f.$$

□

Recall that in *Theorem 2.1.12*, we saw that all gradient fields are irrotational:

$$\underset{\text{gradient field}}{\mathbf{F} = \nabla f} \quad\longrightarrow\quad \underset{\text{irrotational}}{\operatorname{curl}(\mathbf{F}) = \mathbf{0}}$$

On simply connected regions, the converse is true:

Theorem 3.2.17. Let $\mathbf{F} : \Omega \to \mathbb{R}^3$ be a smooth vector field on a simply connected region $\Omega \subseteq \mathbb{R}^3$. If $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$, then $\mathbf{F}$ is a gradient vector field.

A direct proof can be given, but a more transparent proof will be given in *Theorem 4.4.8*. The direct proof of the above is analogous to the proof of the following theorem:

Theorem 3.2.18. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Suppose $\operatorname{div}(\mathbf{F}) = 0$. Then $\mathbf{F} = \operatorname{curl}(\mathbf{G})$ for some $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$.

PROOF. □

Remark 3.2.19. The above theorem is not true, in general, not even for simply connected regions. For instance, consider the vector field $\mathbf{F} : \mathbb{S}^2 \to \mathbb{R}^3$ defined by

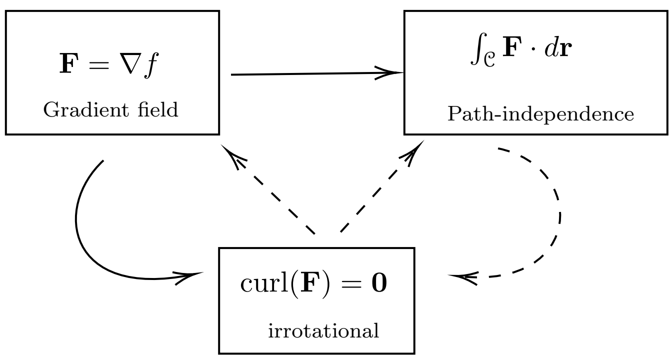
$$\mathbf{F}(x, y, z) := \frac{x}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2 + y^2 + z^2)^{3/2}}\mathbf{k}.$$

This vector field is incompressible but is not solenoidal, i.e., $\mathbf{F}$ is not the curl of a vector field $\mathbf{G}$.

It is clear from direct calculation that $\mathbf{F}$ is incompressible. We will see in the next chapter that $\mathbf{F}$ is indeed not solenoidal.

Remark 3.2.20. Determining whether an irrotational vector field is a gradient field is measured by the first de Rham cohomology group $H^1_{\mathrm{DR}}(\mathcal{S})$. On the other hand, determining whether an incompressible vector field is a solenoidal vector field is measured by the second de Rham cohomology group $H^2_{\mathrm{DR}}(\mathcal{S})$. *Remark 3.2.19* implies that $H^1_{\mathrm{DR}}(\mathbb{S}^2) = 0$ but $H^2_{\mathrm{DR}}(\mathbb{S}^2) \neq 0$. On the other hand, *Theorem 3.2.18* implies that $H^2_{\mathrm{DR}}(\mathbb{R}^3) = 0$.

Summary 3.2.21. In terms of the vector field $\mathbf{F}$ alone, we have the following diagram:



solid lines = always true; dashed lines = true on simply connected domains

Exercises

1. Evaluate the line integrals $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ by first determining whether $\mathbf{F}$ is a gradient field, and if so, using the fundamental theorem of line integrals. If the vector field is not a gradient field, evaluate the line integral directly.

(i) $\mathcal{C}$ is the unit circle in $\mathbb{R}^2$ centered at the origin, and

$$\mathbf{F}(x, y) = x\mathbf{i} + y\mathbf{j}.$$

(ii) $\mathcal{C}$ is the curve parametrized by $\alpha : [0, 1] \to \mathbb{R}^2$, $\alpha(t) = \sqrt{t}\mathbf{i} + (1 + t^3)\mathbf{j}$ and

$$\mathbf{F}(x, y) = x^2 y\mathbf{i} - yx^4\mathbf{j}.$$

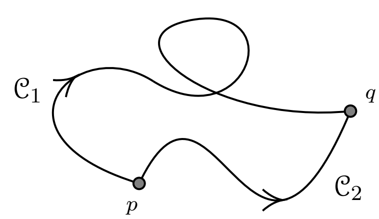
(iii) $\mathcal{C}$ is the line segment from $(0, 1, -3)$ to $(2, -4, 1)$ and

$$\mathbf{F}(x, y, z) = x\mathbf{i} - 2y\mathbf{j} + 4z\mathbf{k}.$$

(iv) $\mathcal{C}$ is the curve parametrized by $\alpha : [0, 2\pi] \to \mathbb{R}^3$, $\alpha(t) = \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + t\mathbf{k}$ and

$$\mathbf{F}(x, y, z) = (2xz + \sin(y))\mathbf{i} + x\cos(y)\mathbf{j} + x^2\mathbf{k}.$$

2. Let $p, q \in \mathbb{R}^2$ be two points. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two paths from p to q as indicated in the following diagram below:



Suppose $\mathbf{F}$ is a gradient field on $\mathbb{R}^2$, and $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r} = 10$. Evaluate $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$.

3. Determine (with justification) whether the following regions in $\mathbb{R}^2$ are simply connected:

(i) $\mathbb{R}^2$.
(ii) $\mathbb{S}^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.
(iii) $A(0, 1) := \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 < 1\}$.

4. Determine (with justification) whether the following statements are true or false:

(i) The union of simply connected regions is simply connected.
(ii) The intersection of simply connected regions is simply connected.
(iii) If X is a simply connection region in $\mathbb{R}^2$, then $X \backslash \{p\}$ is simply connected, where $p \in X$ is some point.

5. Let $\mathbf{F}$ be a smooth vector field on a region $\Omega \subseteq \mathbb{R}^n$. Suppose, moreover, that $\mathbf{F}$ is irrotational, i.e., $\text{curl}(\mathbf{F}) = \mathbf{0}$.

(i) If $\Omega = \mathbb{R}^2$, is $\mathbf{F} = \nabla f$ for some function $f \in \mathcal{C}^\infty(\mathbb{R}^2)$? Justify your answer.
(ii) If Ω is a convex set in $\mathbb{R}^3$, is $\mathbf{F} = \nabla f$ for some smooth function $f : \Omega \to \mathbb{R}$? Justify your answer.
(iii) If Ω is a simply connected domain, is $\mathbf{F}$ a gradient field?

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $f'(x) = 0$ for all $x \in \mathbb{R}$.

(i) Show that $f(x) = c$ for some $c \in \mathbb{R}$.
(ii) Suppose $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ is such that $f'(x) = 0$ for all $x \in \mathbb{R}\backslash\{0\}$. Is f constant?

7. Determine (with justification) whether the following statements are true or false:

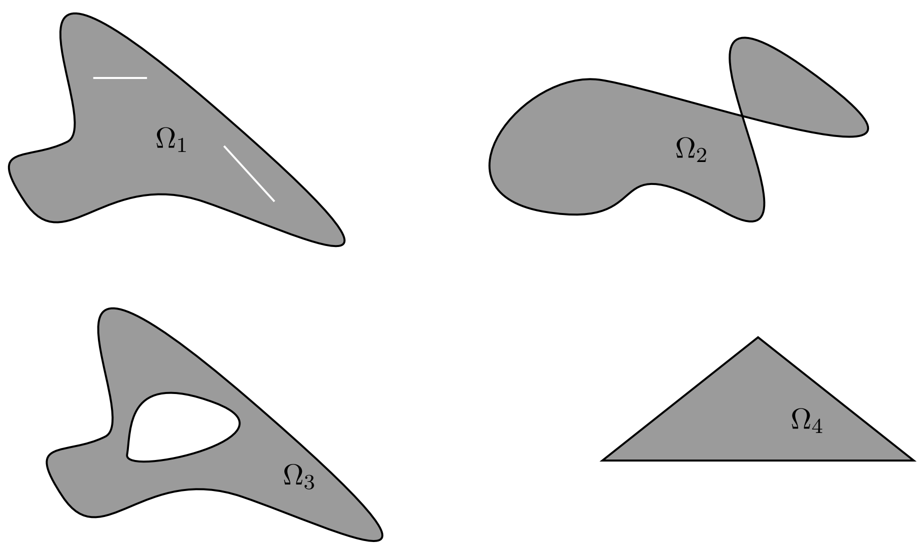
(i) Every irrotational vector field (i.e., a vector field $\mathbf{F}$ such that $\text{curl}(\mathbf{F}) = \mathbf{0}$) is a gradient field.
(ii) An irrotational vector field on a convex set is a gradient field.
(iii) The line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path if $\mathbf{F}$ is an irrotational vector field on the annulus $\{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}$.
(iv) The line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path if and only if $\mathbf{F}$ is irrotational.

8. Is every vector field on a simply connected domain in $\mathbb{R}^2$, the curl of another vector field?

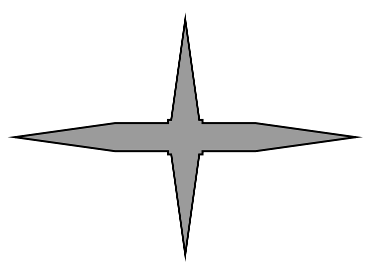
9. State (with justification) whether the following statements are true or false:

(i) The intersection of simply connected sets is simply connected.
(ii) The union of convex sets is convex.

10. Determine which of the following regions are (i) simply connected, or (ii) convex:



11. Is the region given below simply connected?



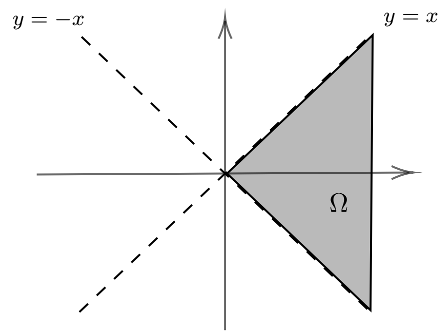
12. Consider the vector field

$$\mathbf{F}(x,y) = -\frac{y}{x^2\sqrt{1-\frac{y^2}{x^2}}}\mathbf{i} + \frac{1}{x\sqrt{1-\frac{y^2}{x^2}}}\mathbf{j}.$$

(i) Show that $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$.

(ii) Show that $\mathbf{F} = \nabla f$ for some f. State the function f explicitly.

(iii) Is $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ path independent for all curves $\mathcal{C} \subset \Omega$, where Ω is the region described below:



13. Let $\mathbf{F}$ be the vector field

$$\mathbf{F}(x,y) = (x+y)\mathbf{i} + y\mathbf{j}.$$

(i) Let $\mathcal{C}_1$ be the line connecting the points $(0,0)$ and $(1,1)$. Determine a parametrization for $\mathcal{C}_1$.
(ii) Compute $\int_{\mathcal{C}_1} \mathbf{F} \cdot d\mathbf{r}$ using the parametrization for $\mathcal{C}_1$ given by part (i).
(iii) Let $\mathcal{C}_2$ be the curve parametrized by $\alpha : [0,1] \to \mathbb{R}^2$, $\alpha(t) = (t, t^2)$. Compute $\int_{\mathcal{C}_2} \mathbf{F} \cdot d\mathbf{r}$.
(iv) Does $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ satisfy the path independence property?
(v) Without doing any computation, is $\mathbf{F}$ a gradient field?
(vi) Is $\mathbf{F}$ an irrotational vector field?

14. In each of the following cases, show that $\mathbf{F}$ satisfies the path independence property, and use the fundamental theorem of line integrals to compute the value of

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r},$$

where

(i) $\omega_{\mathbf{F}} = (2xy^3 + 4x)dx + (3x^2y^2 - 9y^2)dy$ and $\mathcal{C}$ is the path from $(2,1)$ to $(4,3)$.
(ii) $\omega_{\mathbf{F}} = (6x^2 - 3y)dx + (8y - 3x)dy$ and $\mathcal{C}$ is the path from $(1,1)$ to $(2,2)$.
(iii) $\omega_{\mathbf{F}} = 2xzdx + 4yz^2dy + (x^2 + 4y^2z - 9z^2)dz$ and $\mathcal{C}$ is the path from $(-1,1,2)$ to $(4,0,1)$.
(iv) $\omega_{\mathbf{F}} = -\sin(\pi x)\sin(\pi(y-z))dx + \cos(\pi x)\cos(\pi(y-z))dy - \cos(\pi x)\cos(\pi(y-z))dz$ and $\mathcal{C}$ is the path from $\left(\frac{1}{4}, 1, \frac{1}{4}\right)$ to $\left(0, 1, \frac{1}{2}\right)$.

15. In each of the following cases, compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

by showing that $\mathbf{F}$ satisfies the path independence property computing the line integral by using a "simpler" curve $\mathcal{C}'$, where

(i) $\mathbf{F} = 2xy\mathbf{i} + x^2\mathbf{j}$ and $\mathcal{C}$ is the curve parametrized by $\alpha : [0,1] \to \mathbb{R}^2$,

$$\alpha(t) \ := \ (t^3 - t^2 + t + 1)\mathbf{i} + (t^3 - 2t^2 + 2t + 1)\mathbf{j}.$$

(ii) $\mathbf{F} = (4x^3 + 6x^2y^2z)\mathbf{i} + (4x^3yz + 3z^2 - 4y^3)\mathbf{j} + (2x^3y^2 + 6yz - 9z^2)\mathbf{k}$ and $\mathcal{C}$ is the curve parametrized by $\alpha : [-2,2] \to \mathbb{R}^3$,

$$\alpha(t) = (t^4 + t^2 + 3)\mathbf{i} + (t^6 - 2t^2 + 7)\mathbf{j} + (t^8 - 4t^4 + 9)\mathbf{k}.$$

16.

(i) Let $\mathcal{C}$ be any simple closed curve bounding a region having area A. Prove that if $a_1, a_2, a_3, b_1, b_2, b_3 \in \mathbb{R}$ are are constants, then

$$\int_{\mathcal{C}} (a_1x + a_2y + a_3)dx + (b_1x + b_2y + b_3)dy \ = \ (b_1 - a_2)A.$$

(i) Under what conditions will the line integral around any path $\mathcal{C}$ be zero?

CHAPTER 4

Integration Theory – Surfaces and Beyond

"Good mathematics is proving things that should be true. Great mathematics is proving things that shouldn't."

– Misha Gromov

The previous chapter extended the fundamental theorem of calculus (FTC) to line integrals. In this chapter, we will extend this to regions in $\mathbb{R}^2$, yielding *Green's theorem*. The two-dimensional version of line-integrals, namely, *surface integrals*, are then presented, and the corresponding version of the FTC is given. In this guise, it is referred to as *Stokes' theorem*. Finally, for solid surfaces in $\mathbb{R}^3$, the FTC is arises under the name of the *divergence theorem*. For the reader's convenience, we will begin by recalling some known-results concerning the computation of double and triple integrals.

4.1. Multiple integrals

We assume the reader has some familiarity with multiple integrals, we will recall the results here for their convenience.

Iterated integrals. Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be a continuous function. The iterated integral is defined

$$\int_a^b \int_c^d f(x, y)dydx \;=\; \int_a^b \left(\int_c^d f(x, y)dy \right) dx. \tag{4.1.1}$$

The first (or inner) integral on the right-hand side of (4.1.1) is evaluated by fixing x and viewing $f(x, y)$ as a function of the variable y alone. This is analogous to how partial derivatives are computed. The result of this (definite) integration is a function $F(x)$ which is independent of y. The resulting iterated integral is obtained by evaluating this integral of a one-variable function $\int_a^b F(x)dx$ in the familiar manner:

Example 4.1.1. Evaluate the iterated integral

$$\int_0^1 \int_2^5 xy^2 dydx.$$

SOLUTION. From (4.1.1), we have

$$\begin{aligned}\int_0^1\int_2^5 xy^2dydx &= \int_0^1\left(\int_2^5 xy^2dy\right)dx\\ &= \int_0^1\left[\frac{1}{3}xy^3\right]_2^5 dx\\ &= \int_0^1\left[\frac{1}{3}x(5^3-2^3)\right]dx\\ &= \int_0^1\frac{117}{3}xdx = \left[\frac{117}{6}x^2\right]_0^1 = \frac{117}{6}.\end{aligned}$$

□

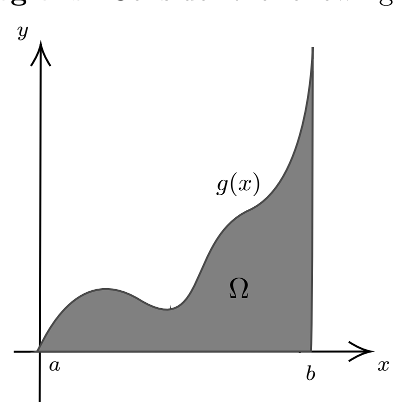
One can observe that in these cases, the order of integration does not matter, more formally:

Fubini's Theorem. If $f:[a,b]\times[c,d]\to\mathbb{R}$ is a continuous function, then

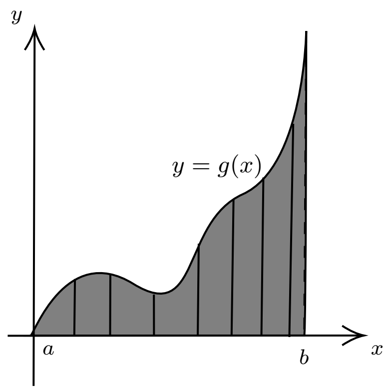
$$\int_a^b\int_c^d f(x,y)dydx = \int_c^d\int_a^b f(x,y)dxdy.$$

Remark 4.1.2. Fubini's theorem applies to functions which are continuous on *rectangles* and where the integration is taken over *rectangles*. The situation is more delicate when considering integrals over more general regions:

Integration over general regions. Consider the following region in $\mathbb{R}^2$:

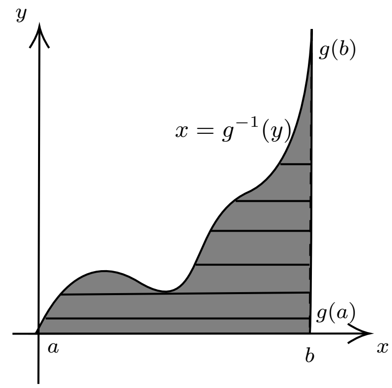


If we want to evaluate the integral of a function $f:\Omega\to\mathbb{R}$, then the order of integration will matter:



integrating with respect to y then x

$a \leq x \leq b \qquad 0 \leq y \leq g(x)$



integrating with respect to x then y

$g^{-1}(y) \leq x \leq b \qquad g(a) \leq y \leq g(b)$

Example 4.1.3. Let Ω be the region bounded between the parabola $y = x^2 + 1$, the coordinate axes, and the line $x = 1$. Evaluate

$$\iint_\Omega ye^{-x} dA.$$

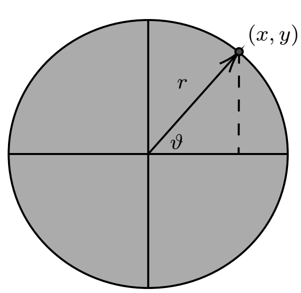
Solution. The region is most simply described by $0 \leq x \leq 1$ and $0 \leq y \leq x^2 + 1$. Hence

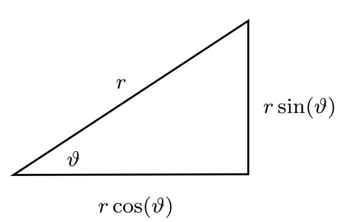
$$\begin{aligned}
\iint_\Omega ye^{-x} dA &= \int_0^1 \int_0^{x^2+1} ye^{-x} dy dx \\
&= \int_0^1 \frac{1}{2}(x^2+1)^2 e^{-x} dx \;=\; \frac{29}{2} - \frac{38}{e}.
\end{aligned}$$

□

Polar coordinates: For circular regions of integration (such as a circle), it is often convenient to pass to *polar coordinates*:

$$x = r\cos(\vartheta) \qquad y = r\sin(\vartheta)$$





Proposition 4.1.4. In polar coordinates $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$, the area form is

$$\boxed{dA = r\ dr\ d\vartheta}$$

PROOF. If $x = r\cos(\vartheta)$, then $dx = \cos(\vartheta)dr - r\sin(\vartheta)d\vartheta$. If $y = r\sin(\vartheta)$, then $dy = \sin(\vartheta)dr + r\cos(\vartheta)d\vartheta$. Then

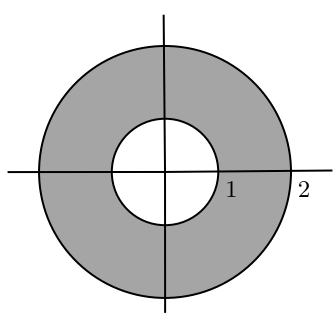
$$\begin{aligned} dx \wedge dy &= (\cos(\vartheta)dr - r\sin(\vartheta)d\vartheta) \wedge (\sin(\vartheta)dr + r\cos(\vartheta)d\vartheta) \\ &= \cos(\vartheta)\sin(\vartheta)dr \wedge dr + r\cos^2(\vartheta)dr \wedge d\vartheta \\ &\qquad -r\sin^2(\vartheta)d\vartheta \wedge dr - r^2\sin(\vartheta)\cos(\vartheta)d\vartheta \wedge d\vartheta \\ &= r\cos^2(\vartheta)dr \wedge d\vartheta - r\sin^2(\vartheta)d\vartheta \wedge dr \\ &= r(\cos^2(\vartheta) + \sin^2(\vartheta))dr \wedge d\vartheta \\ &= r dr \wedge d\vartheta. \end{aligned}$$

□

Example 4.1.5. Compute the double integral

$$\iint_\Omega (x^2 + y^2) dA,$$

where Ω is the annulus



$$\Omega = \{(x, y) \in \mathbb{R}^2 : 1 < x^2 + y^2 < 4\}.$$

SOLUTION. Because the region Ω is circular, we can use polar coordinates: $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$. The associated volume form is $dxdy = rdrd\vartheta$. Hence, using the fact that

$x^2 + y^2 = r^2$,

$$\begin{aligned}\iint_\Omega (x^2+y^2)dA &= \int_0^{2\pi}\int_1^2 r^2 \cdot r dr d\vartheta \\ &= \int_0^{2\pi}\int_1^2 r^3 dr d\vartheta \\ &= 2\pi \int_1^2 r^3 dr \\ &= 2\pi \left[\frac{1}{4}r^4\right]_1^2 = \frac{15\pi}{2}.\end{aligned}$$

□

Example 4.1.6. Let Ω be the region bounded by the semi-circle $x = \sqrt{9-y^2}$ and the y–axis. Compute

$$\iint_\Omega e^{-x^2-y^2} dA.$$

SOLUTION. The region Ω is circular, so can be described via polar coordinates $x = r\cos(\vartheta)$ and $y = r\sin(\vartheta)$. The region Ω is described in polar coordinates by $-\frac{\pi}{2} \le \vartheta \le \frac{\pi}{2}$ and $0 \le r \le 3$. Now,

$$\begin{aligned}\iint_\Omega e^{-x^2-y^2} dA &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^3 e^{-r^2} r dr d\vartheta \\ &= \pi \int_0^3 re^{-r^2} dr.\end{aligned}$$

Let $u = -r^2$, then $du = -2rdr$ and

$$\begin{aligned}\iint_\Omega e^{-x^2-y^2} dA &= \pi \int_0^3 re^{-r^2} dr \\ &= \pi \int_0^{-9} re^u \frac{du}{-2r} \\ &= -\frac{\pi}{2}\int_0^{-9} e^u du \\ &= \frac{\pi}{2}\int_{-9}^0 e^u du \\ &= \frac{\pi}{2}(1-e^{-9}).\end{aligned}$$

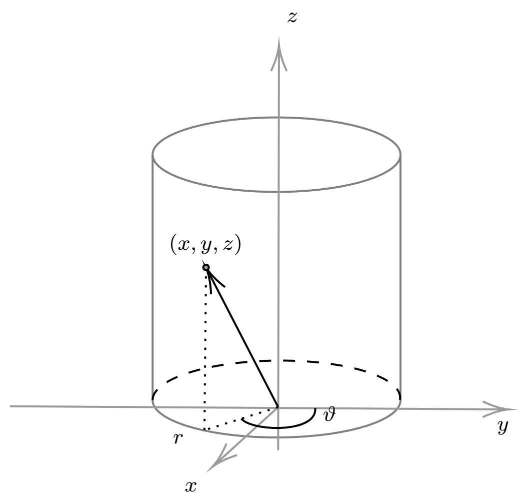
□

Cylindrical coordinates: In cylindrical coordinates

$$x = r\cos(\vartheta), \qquad y = r\sin(\vartheta), \qquad z = z.$$

The associated volume form is

$$\boxed{dV = r\ dr\ d\vartheta\ dz}$$



Example 4.1.7. Evaluate the integral

$$\iiint_\Omega x(x^2+y^2)dV,$$

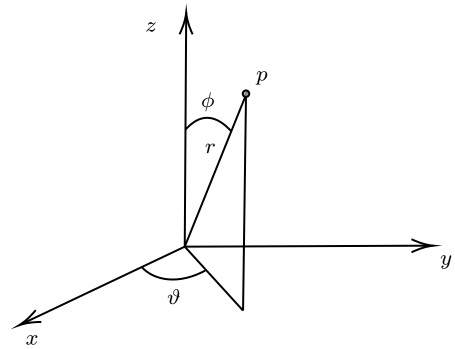
where Ω is the region inside the cylinder $x^2+y^2 = 9$ and between the planes $z = 0$ and $z = 3$.

Solution. Passing to cylindrical coordinates, we have $x = r\cos(\vartheta)$, $y = r\sin(\vartheta)$, and $z = z$, for $r \in [0,3]$, $\vartheta \in [0,2\pi]$, and $z \in [0,3]$. Hence,

$$\begin{aligned}
\iiint_\Omega x(x^2+y^2)dV &= \int_0^3\int_0^{2\pi}\int_0^3 r\cos(\vartheta)(r^2)rdrd\vartheta dz \\
&= 3\int_0^{2\pi}\int_0^3 r^4\cos(\vartheta)drd\vartheta \\
&= 3\int_0^{2\pi}\cos(\vartheta)d\vartheta\int_0^3 r^4dr \\
&= 3\left[\sin(\vartheta)\right]_0^{2\pi}\left[\frac{1}{4}r^4\right]_0^3 \\
&= \frac{3}{4}(\sin(2\pi)-\sin(0))(3^4-0) \;=\; 0.
\end{aligned}$$

□

Spherical coordinates. Spherical coordinates express a point in $\mathbb{R}^3$ in terms of a radial parameter r, and two angle parameters ϑ and ϕ. Here, ϑ is the angle formed with the x–axis and ϕ is the angle formed with the z–axis:



In spherical coordinates

$$x = r\cos(\vartheta)\sin(\phi), \qquad y = r\sin(\vartheta)\sin(\phi), \qquad z = r\cos(\phi).$$

The associated *spherical volume form* (i.e., the volume form in spherical coordinates) is

$$\boxed{dV = r^2 \sin(\phi)\ dr\ d\phi\ d\vartheta}$$

PROOF. We have

$$\begin{aligned} dx &= \sin(\phi)\cos(\vartheta)dr + r\cos(\phi)\cos(\vartheta)d\phi - r\sin(\phi)\sin(\vartheta)d\vartheta \\ dy &= \sin(\phi)\sin(\vartheta)dr + r\cos(\phi)\sin(\vartheta)d\phi + r\sin(\phi)\cos(\vartheta)d\vartheta \\ dz &= \cos(\phi)dr - r\sin(\phi)d\phi. \end{aligned}$$

First compute

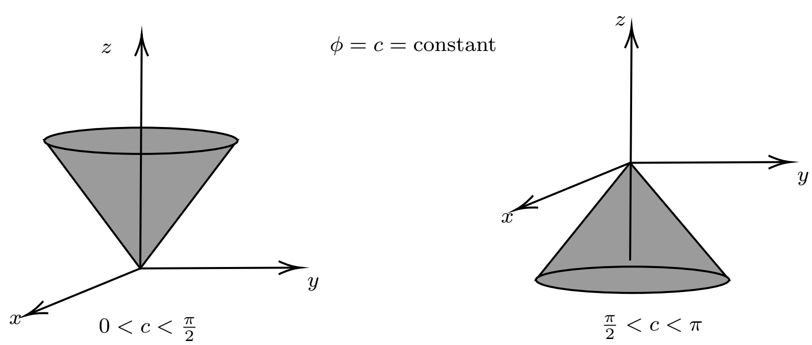
$$\begin{aligned} dy \wedge dz &= (\sin(\phi)\sin(\vartheta)dr + r\cos(\phi)\sin(\vartheta)d\phi + r\sin(\phi)\cos(\vartheta)d\vartheta) \wedge (\cos(\phi)dr - r\sin(\phi)d\phi) \\ &= -r\sin^2(\phi)\sin(\vartheta)dr \wedge d\phi + r\cos^2(\phi)\sin(\vartheta)d\phi) \wedge dr \\ &\quad +r\sin(\phi)\cos(\phi)\cos(\vartheta)d\vartheta \wedge dr - r^2\sin^2(\phi)\cos(\vartheta)d\vartheta \wedge d\phi \\ &= -r\sin(\vartheta)dr \wedge d\phi - r\sin(\phi)\cos(\phi)\cos(\vartheta)dr \wedge d\vartheta + r^2\sin^2(\phi)\cos(\vartheta)d\phi \wedge d\vartheta. \end{aligned}$$

Finally, compute

$$\begin{aligned}
dx\wedge dy\wedge dz &= dx\wedge\big(-r\sin(\vartheta)dr\wedge d\phi - r\sin(\phi)\cos(\phi)\cos(\vartheta)dr\wedge d\vartheta + r^2\sin^2(\phi)\cos(\vartheta)d\phi\wedge d\vartheta\big)\\
&= (\sin(\vartheta)\cos(\phi)dr + r\cos(\vartheta)\cos(\phi)d\vartheta - r\sin(\vartheta)\sin(\phi)d\phi)\\
&\quad\wedge\big(-r\sin(\vartheta)dr\wedge d\phi - r\sin(\phi)\cos(\phi)\cos(\vartheta)dr\wedge d\vartheta + r^2\sin^2(\phi)\cos(\vartheta)d\phi\wedge d\vartheta\big)\\
&= r^2\sin^3(\phi)\cos^2(\vartheta)dr\wedge d\phi\wedge d\vartheta - r^2\sin(\phi)\cos^2(\phi)\cos^2(\vartheta)d\phi\wedge dr\wedge d\vartheta\\
&\quad + r^2\sin(\phi)\sin^2(\vartheta)d\vartheta\wedge dr\wedge d\phi\\
&= r^2\left(\sin^3(\phi)\cos^2(\vartheta) + \sin(\phi)\cos^2(\phi)\cos^2(\vartheta) + \sin(\phi)\sin^2(\vartheta)\right)dr\wedge d\phi\wedge d\vartheta\\
&= r^2\sin(\phi)\left(\sin^2(\phi)\cos^2(\vartheta) + \cos^2(\phi)\cos^2(\vartheta) + \sin^2(\vartheta)\right)dr\wedge d\phi\wedge d\vartheta\\
&= r^2\sin(\phi)(\cos^2(\vartheta) + \sin^2(\vartheta))dr\wedge d\phi\wedge d\vartheta\\
&= r^2\sin(\phi)dr\wedge d\phi\wedge d\vartheta.
\end{aligned}$$

□

Example 4.1.8.



Example 4.1.9. Let $\mathcal{V}$ be the hemi-sphere given by

$$0 \le z \le \sqrt{1-x^2-y^2}.$$

Compute

$$\iiint_{\mathcal{V}} (x^2+y^2)dV.$$

Solution. In spherical coordinates, $x = r\cos(\vartheta)\sin(\phi)$, $y = r\sin(\vartheta)\sin(\phi)$, and $z = r\cos(\phi)$. The volume form is given by $dV = r^2\sin(\phi)dr\wedge d\phi\wedge d\vartheta$. The hemi-sphere is described by $0 \le r \le 1$, $0 \le \vartheta \le 2\pi$, and $0 \le \phi \le \frac{\pi}{2}$. Hence,

$$\begin{aligned}
\iiint_{\mathcal{V}} (x^2+y^2)dV &= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 [(r\cos(\vartheta)\sin(\phi))^2 + (r\sin(\vartheta)\sin(\phi))^2]r^2\sin(\phi)drd\phi d\vartheta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 [r^2\sin^2(\phi)]r^2\sin(\phi)drd\phi d\vartheta \\
&= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^1 r^4\sin^3(\phi)drd\phi d\vartheta \\
&= 2\pi\left(\int_0^{\frac{\pi}{2}}\sin^3(\phi)d\phi\right)\left(\int_0^1 r^4dr\right) \\
&= \frac{4\pi}{15}.
\end{aligned}$$

□

Example 4.1.10. Let $\mathcal{V}$ be the region between the spheres of radius 1 and radius 3, in the first octant. Evaluate

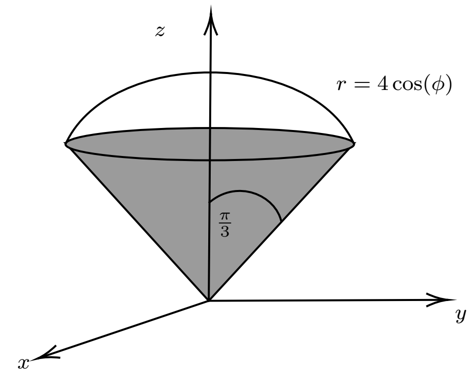
$$\iiint_{\mathcal{V}} (xyz+4)dV.$$

Solution. In spherical coordinates, $x = r\cos(\vartheta)\sin(\phi)$, $y = r\sin(\vartheta)\sin(\phi)$, $z = r\cos(\phi)$, and the volume form is $dV = r^2\sin(\phi)dr\wedge d\phi\wedge d\vartheta$. For the region $\mathcal{V}$, we have $1 \le r \le 3$, $0 \le \vartheta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{2}$. Hence,

$$\begin{aligned}
\iiint_{\mathcal{V}} (xyz+4)dV &= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^3 (r\cos(\phi)\sin(\phi)r\sin(\phi)\sin(\vartheta)r\cos(\phi)+4)r^2\sin(\phi)drd\phi d\vartheta \\
&= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^3 [r^5\sin^3(\phi)\sin(\vartheta)\cos^2(\phi)+4r^2\sin(\phi)]drd\phi d\vartheta \\
&= \left(\int_0^{\frac{\pi}{2}}\sin(\vartheta)d\vartheta\right)\left(\int_0^{\frac{\pi}{2}}\sin^3(\phi)\cos^2(\phi)d\phi\right)\left(\int_1^3 r^5dr\right) \\
&\quad +2\pi\left(\int_0^{\frac{\pi}{2}}\sin(\phi)d\phi\right)\left(\int_1^3 r^2dr\right) \\
&= \frac{2}{15}\frac{364}{3} + 2\pi\frac{26}{3} \\
&= \frac{52}{45}(14+15\pi).
\end{aligned}$$

□

Example 4.1.11. Find the volume of the solid $\mathcal{V}$ that lies above the cone $\phi = \frac{\pi}{3}$ and below the sphere $r = 4\cos(\phi)$.



SOLUTION. $x = r\cos(\vartheta)\sin(\phi)$, $y = r\sin(\vartheta)\sin(\phi)$, and $z = r\cos(\phi)$. We know that $0 \leq \phi \leq \frac{\pi}{3}$, $0 \leq \vartheta \leq 2\pi$, and $0 \leq r \leq 4\cos(\phi)$. Hence,

$$\begin{aligned}
\mathrm{vol}(\mathcal{V}) &= \int_0^{2\pi}\int_0^{\frac{\pi}{3}}\int_0^{4\cos(\phi)} r^2\sin(\phi)drd\phi d\vartheta \\
&= 2\pi\int_0^{\frac{\pi}{3}}\left[\frac{1}{3}r^3\right]_0^{4\cos(\phi)}\sin(\phi)d\phi \\
&= \frac{2\pi}{3}\int_0^{\frac{\pi}{3}} 4^3\cos^3(\phi)\sin(\phi)d\phi \\
&= 10\pi.
\end{aligned}$$

$\square$

Exercises

1. Evaluate the following iterated integrals

(i) $\int_0^1 \int_0^2 (x^2 - y^2) dx dy$.

(ii) $\int_0^1 \int_0^{\pi} (\sqrt{x} + \sin(y)) dy dx$.

(iii) $\int_0^1 \int_0^v \sqrt{1-v^2} du dv$.

2. Evaluate

$$\int_{\Omega} x^2 e^{x^2+y^2} dx dy,$$

where $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 4,\ y \geq 0\}$.

3. Let Ω be the region bounded by $y = x^2$, $x = 2$ and $y = 0$.

(i) Evaluate $\int_{\Omega} xy dA$ by first integrating with respect to x.
(ii) Evaluate $\int_{\Omega} xy dA$ by first integrating with respect to y.

4. Evaluate

$$\int_0^1 \int_x^{2x} e^{x+y} dy dx.$$

5. Use spherical coordinates to compute the integrals

(i)

$$\iiint_{\Omega} 4(x^2 + y^2 + z^2)^2 dV,$$

where Ω is the ball $x^2 + y^2 + z^2 \leq 4$.

(ii)

$$\iiint_{\Omega} z e^{(x^2+y^2+z^2)} dV,$$

where Ω is the region bounded between $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 9$.

6. Evaluate the double integrals $\iint_{\Omega} f(x,y) dA$, where

(i) $f(x,y) = x^3 y^6$ and $\Omega := \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 3,\ -x \leq y \leq x\}$.
(ii) $f(x,y) = \frac{5y}{x^2+4}$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq 1,\ 0 \leq y \leq \sqrt{x}\}$.
(iii) $f(x,y) = 1 - e^{-y^2}$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1, 0 \leq x \leq y\}$.
(iv) $f(x,y) = 4x\sqrt{y^2 - x^2}$ and $\Omega = \{(x,y) \in \mathbb{R}^2 : 0 \leq y \leq 1,\ 0 \leq x \leq y\}$.

7. Evaluate the double integrals $\iint_\Omega f(x,y)dA$, where

(i) $f(x,y) = x\sin(y)$ and Ω is the region bounded by $y = 0$, $y = x^2$, and $x = 1$.
(ii) $f(x,y) = \frac{1}{3}(x+y)$ and Ω is the region bounded by $y = \sqrt{x}$ and $y = x^2$.
(iii) $f(x,y) = y^2 - x$ and Ω is the region bounded by $x = y^2$ and $x = 3 - 2y^2$.
(iv) $f(x,y) = ye^x$ and Ω is the triangular region with vertices $(0,0)$, $(1,1)$, and $(5,3)$.

8. Sketch the region of integration and change the order of integration for the following double integrals:

(i)
$$\int_0^2 \int_0^{1-x} (x^2 - y)dydx.$$
(ii)
$$\int_1^2 \int_0^{\ln(x)} e^{-\ln(x^3)}dydx.$$
(iii)
$$\int_0^{\pi/9} \int_0^{\sin(x)} e^{\cos(x^2)-9}dydx.$$
(iv)
$$\int_0^1 \int_{\tan^{-1}(x)}^{\pi/3} (x-y)dydx.$$

9. Evaluate the following double integrals by reversing the order:

(i)
$$\int_0^1 \int_{2x}^2 e^{x^2}dxdy.$$
(ii)
$$\int_0^4 \int_{\sqrt{y}}^1 \sqrt{x^3+1}dxdy.$$
(iii)
$$\int_0^3 \int_y^9 y\cos(x^2)dxdy.$$
(iv)
$$\int_0^1 \int_x^1 x^3\sin(y^3)dydx.$$

10. Evaluate
$$\iint_\Omega (x^2\tan(x) + y^3 + 4)dA,$$
where $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2\}$.

11. Use polar coordinates to find the area of the following regions:

(i) two loops of the rose $r = \cos(3\vartheta)$.
(ii) the region enclosed by the lemniscate $r^2 = 4\cos(2\vartheta)$.
(iii) the region enclosed by the cardioid $r = 1 - \sin(\vartheta)$.
(iv) the region bounded by the paraboloids $z = 3x^2 + 3y^2$ and $z = 4 - x^2 - y^2$.

12. Evaluate the following integrals using cylindrical coordinates

(i)
$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz dz dx dy.$$

(ii)
$$\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2+x^2+y^2} (x^2+y^2)^{\frac{3}{2}} dz dy dx.$$

13. For the region $\Omega := \{(x, y) \in \mathbb{R}^2 : 1 \le x^2 + y^2 \le 4\}$, compute
$$\iint_\Omega \sqrt{1 + x^2 + y^2} dx dy.$$

14. The sphere $x^2 + y^2 + z^2 = 25$ has a hole bored through it by the cylinder $x^2 + y^2 = 4$. Find the volume of that part of the sphere that is removed.

15. Sketch the region of integration and evaluate
$$\int_0^1 \int_x^{2x} e^{x+2y} dy dx.$$

16. Evaluate the following integrals using spherical coordinates:

(i) $\iiint_{\mathcal{V}} (16 - x^2 - y^2) dV$, where $\mathcal{V}$ is the solid hemi-sphere $x^2 + y^2 + z^2 = 16$, $z \le 0$.
(ii) Let $\mathcal{V}$ be the region enclosed by the sphere $x^2 + y^2 + z^2 = 4$ in the first octant. Compute $\iiint_{\mathcal{V}} e^{x^2+y^2+z^2} dV$.

17. Find the volume of the following regions:

(i) Inside the cone $z = \sqrt{x^2 + y^2}$ and inside the sphere $x^2 + y^2 + z^2 = r^2$.
(ii) Between the paraboloids $z = 10 - x^2 - y^2$ and $z = 2(x^2 + y^2 - 1)$.
(iii) Above the xy–plane, inside the cone $z = 2a - \sqrt{x^2 + y^2}$ and inside the cylinder $x^2 + y^2 = 2ay$.

18. Find the volume of the region inside the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$
and above the plane $z = b - y$.

19. Show that in cylindrical coordinates, the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \vartheta^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

20. Show that in spherical coordinates the Laplace equation is given by

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{\cot(\phi)}{r^2}\frac{\partial u}{\partial \phi} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2 \sin^2(\phi)}\frac{\partial^2 u}{\partial \vartheta^2} \;=\; 0.$$

4.2. Green's theorem

The fundamental theorem of line integrals informs us that if $\mathbf{F} : \Omega \to \mathbb{R}^2$ is a gradient field, i.e., $\mathbf{F} = \nabla f$, where $f : \Omega \to \mathbb{R}$ is a smooth function, then

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = f(\alpha(1)) - f(\alpha(0)),$$

for any smooth parametrization $\alpha : [0,1] \to \mathbb{R}^2$ of $\mathcal{C}$. We have seen that it is more natural to express this in terms of forms:

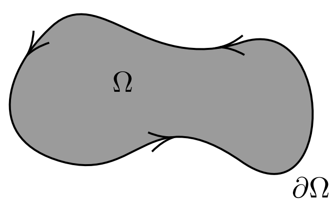
$$\int_{\mathcal{C}} df = f(\alpha(1)) - f(\alpha(0)).$$

In particular, we have

$$\int_{\mathcal{C}} d(0\text{–form}) \quad = \quad \text{values of 0–form on boundary of } \mathcal{C} \; = \; \int_{\partial\mathcal{C}} (0\text{–form}).$$

Now, if we want to integrate the 2–form $d(1\text{–form})$, then we integrate over a region $\Omega \subseteq \mathbb{R}^2$ in the plane. In particular, we would expect that

$$\iint_{\Omega} d(1\text{–form}) \quad = \quad \text{values of 1–form on boundary of } \Omega \; = \; \int_{\partial\Omega} (1\text{–form}).$$



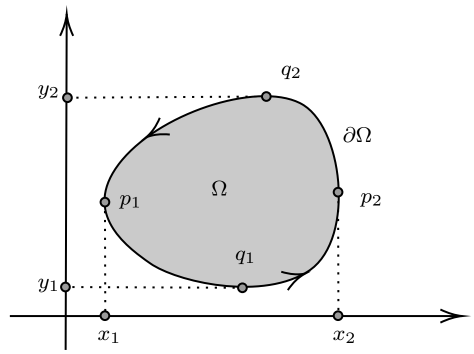
This is exactly Green's theorem:

Theorem 4.2.1. (Green's theorem). Let $\Omega \subset \mathbb{R}^2$ be a connected region with smooth boundary $\partial\Omega$, oriented positively. Then

$$\iint_{\Omega} \omega \quad = \quad \int_{\partial\Omega} d\omega. \tag{4.2.1}$$

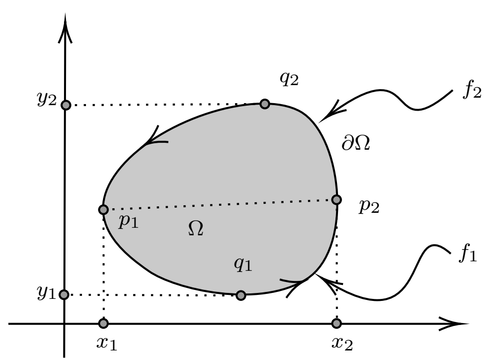
Proof. The proof is broken up into three cases, distinguished by proving (4.2.1) in successively greater generality.

Proof for regions of type I. Consider first regions $\Omega \subset \mathbb{R}^2$ where the boundary $\partial\Omega$ can be expressed as the graph of four functions. Another way of formulating the definition of such regions is the following: We require that the intersection of $\partial\Omega$ with a line parallel to the coordinates axes consists of *at most* two points:



Let us mark points p_1, p_2, q_1, q_2 on $\partial\Omega$. Let x_1 denote the x–coordinate of p_1, x_2 denote the x–coordinate of p_2, y_1 denote the y–cordinate of q_1, and y_2 denote the y–coordinate of q_2, as depicted above.

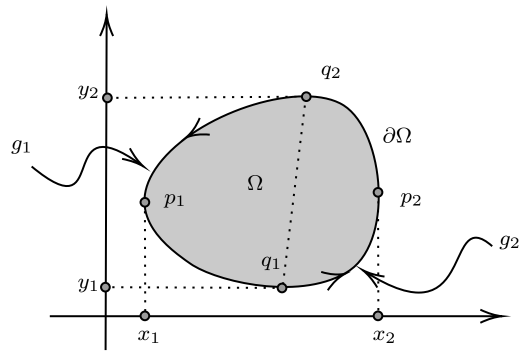
Let $f_1 : [x_1, x_2] \to \mathbb{R}$ be the function whose graph is given by the part of $\partial\Omega$ connecting p_1 to p_2, passing through q_1, and let $f_2 : [x_1, x_2] \to \mathbb{R}$ be the function whose graph is given by the part of $\partial\Omega$ connecting p_1 to p_2, passing through q_2:



Let now $\omega = Pdx + Qdy$. Then

$$\begin{aligned}
\iint_\Omega P_y dxdy &= \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} P_y dydx \\
&= \int_{x_1}^{x_2} [P(x,y)]_{f_1(x)}^{f_2(x)} dx \\
&= \int_{x_1}^{x_2} [P(x, f_2(x)) - P(x, f_1(x))] dx \\
&= -\int_{x_2}^{x_1} P(x, f_2(x)) dx - \int_{x_1}^{x_2} P(x, f_1(x)) dx \;=\; -\int_{\partial\Omega} P dx.
\end{aligned}$$

Similarly, let $g_1 : [y_1, y_2] \to \mathbb{R}$ be the function whose graph is given by the part of $\partial\Omega$ connecting q_1 and q_2, passing through p_1, and let $g_2 : [y_1, y_2] \to \mathbb{R}$ be the function whose graph is given by the part of $\partial\Omega$ connecting q_1 and q_2, passing through p_2:



Now,

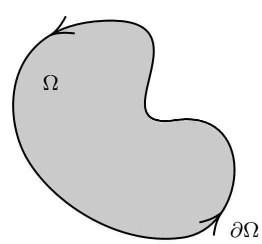
$$\begin{aligned}
\iint_\Omega Q_x dxdy &= \int_{y_1}^{y_2} \int_{g_1(y)}^{g_2(y)} Q_x dxdy \\
&= \int_{y_1}^{y_2} [Q(x,y)]_{g_1(y)}^{g_2(y)} dy \\
&= \int_{y_1}^{y_2} [Q(g_2(y), y) - Q(g_1(y), y)] dy \\
&= \int_{y_1}^{y_2} Q(g_2(y), y) dy + \int_{y_2}^{y_1} Q(g_1(y), y)
\end{aligned}$$

Summing over the two results obtained, we see that

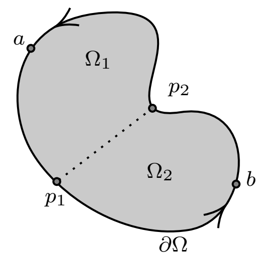
$$\iint_\Omega (Q_x - P_y) dx \wedge dy = \int_{\partial\Omega} P dx + Q dy.$$

□

Proof for simply connected regions. Consider now a simply connected region Ω, e.g., the region shown below:



Decompose Ω into regions of type I:



From the proof of Green's theorem for type I regions, we know that

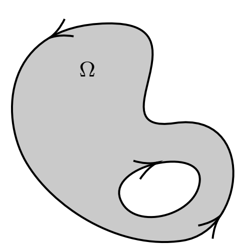
$$\int_{\partial\Omega_1} \omega = \iint_{\Omega_1} d\omega, \qquad \int_{\partial\Omega_2} \omega = \iint_{\Omega_2} d\omega.$$

We have

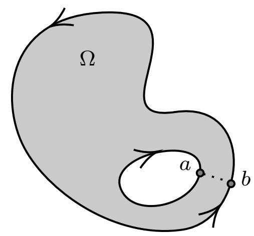
$$\begin{aligned}
\iint_{\Omega} d\omega = \iint_{\Omega_1} d\omega + \iint_{\Omega_2} d\omega &= \int_{\partial\Omega_1} \omega + \int_{\partial\Omega_2} \omega \\
&= \int_{p_1p_2ap_1} \omega + \int_{p_1bp_2p_1} \omega \\
&= \int_{p_1p_2} \omega + \int_{p_2ap_1} \omega + \int_{p_1bp_2} \omega + \int_{p_2p_1} \omega \\
&= \int_{p_1p_2} \omega + \int_{p_2p_1} \omega + \int_{p_2ap_1bp_2} \omega \\
&= \int_{p_1p_2} \omega - \int_{p_1p_2} \omega + \int_{\partial\Omega} \omega \\
&= \int_{\partial\Omega} \omega.
\end{aligned}$$

□

Proof for multiply-connected domains. If Ω is a multiply-connected domain, e.g., the domain shown below:



Sever Ω into simply connected domains by connecting the boundaries via a line:



We can then apply Green's theorem on each simply connected piece of Ω. ☐

☐

Example 4.2.2. Let $\mathcal{C} \subset \mathbb{R}^2$ be the circle $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 9\}$. Compute

$$\int_{\mathcal{C}} (3y - e^{\sin(x)})dx + (7x + \sqrt{y^4 + 1})dy.$$

Solution. Let $\Omega := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 9\}$ be the region with boundary $\partial\Omega = \mathcal{C}$. By Green's theorem,

$$\begin{aligned} & \int_{\mathcal{C}} (3y - e^{\sin(x)})dx + (7x + \sqrt{y^4 + 1})dy \\ = & \int_{\Omega} \left[\frac{\partial}{\partial x}(7x + \sqrt{y^4 + 1}) - \frac{\partial}{\partial y}(3y - e^{\sin(x)}) \right] dxdy \\ = & \int_{\Omega} (7 - 3)\, dxdy \\ = & \ 4 \int_{\Omega} dxdy \\ = & \ 4\text{area}(\Omega) \ = \ 4(9\pi) \ = \ 36\pi. \end{aligned}$$

☐

Example 4.2.3. Find the area of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

Solution. Parametrize the ellipse by $x = a\cos(\theta)$ and $y = b\sin(\theta)$, for $0 \leq \theta \leq 2\pi$. Green's theorem applied to $1 = Q_x - P_y$ gives

$$\begin{aligned} \text{Area} \ = \int_D 1 dx \wedge dy \ & = \ \frac{1}{2} \int_{\mathcal{C}} xdy - ydx \\ & = \ \frac{1}{2} \int_0^{2\pi} a\cos(\theta)(b\cos(\theta))d\theta - (b\sin(\theta))(-a\sin(\theta))d\theta \\ & = \ \frac{ab}{2} \int_0^{2\pi} d\theta \\ & = \ \pi ab. \end{aligned}$$

□

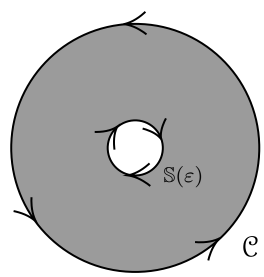
Example 4.2.4. Let $\mathbf{F} : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x,y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

Let $\mathcal{C} \subset \mathbb{R}^2 \backslash \{0\}$ be any closed path which circles the origin. Show that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \quad = \quad 2\pi.$$

Solution. Let $\varepsilon > 0$. Let $\mathbb{S}(\varepsilon) := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = \varepsilon^2\}$. Consider the region formed from removing $\mathbb{S}(\varepsilon)$ from the region bounded by $\mathcal{C}$. The positively-oriented boundary is given by $\mathcal{C} \cup (-\mathbb{S}(\varepsilon))$.



A parametrization of $\mathbb{S}(\varepsilon)$ is given by $x = \varepsilon\cos(\theta)$, $y = \varepsilon\sin(\theta)$. Green's theorem tells us that

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} + \int_{-\mathbb{S}(\varepsilon)} \mathbf{F} \cdot d\mathbf{r} \quad = \quad \int_{D(\varepsilon)} (Q_x - P_y) dx \wedge dy,$$

where $P = \frac{-y}{x^2+y^2}$ and $Q = \frac{x}{x^2+y^2}$. Compute

$$P_y \;=\; \frac{y^2-x^2}{(x^2+y^2)^2}, \qquad Q_x \;=\; \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Hence,

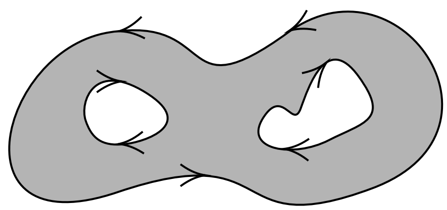
$$\int_{D(\varepsilon)} (Q_x - P_y) dx \wedge dy \;=\; \int_{D(\varepsilon)} \left(\frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} \right) dx \wedge dy \;=\; 0,$$

and so

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; -\int_{-\mathbb{S}(\varepsilon)} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_{\mathbb{S}(\varepsilon)} \mathbf{F} \cdot d\mathbf{r}.$$

□

Orientation of Boundary. When attempting to determine the orientation of the boundary curves, the standard heuristic is the following: If you image walking along the boundary curves, the orientation is positive if the interior region is always to the left. For instance, if the boundary of Ω has multiple components, then a positive orientation is given by:



Aside for experts. The validity of Green's theorem under various relaxations of the regularity can be found in [**11**] and the references therein.

Exercises

1. Use Green's theorem to compute the line integrals $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where

(i) $\mathbf{F}(x, y) = y\mathbf{i} + x\mathbf{j}$ and $\mathcal{C}$ is the unit circle.

(ii) $\mathbf{F}(x, y) = x\cos(y)\mathbf{i} + y\sin(x)\mathbf{j}$ and $\mathcal{C}$ is the boundary of the region between the circles $\mathbb{S}^1(1)$ and $\mathbb{S}^1(2)$.

(iii) $\mathbf{F}(x, y) = -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}$ and $\mathcal{C}$ is the boundary of the square with vertices $(-1, -1)$, $(-1, 1)$, $(1, 1)$, and $(1, -1)$.

(iv) $\mathbf{F}(x, y) = (x\sin(y^2) - y^2)\mathbf{i} + (x^2 y\cos(y^2) + 3x)\mathbf{j}$ and $\mathcal{C}$ is the trapezoid with vertices $(0, -2)$, $(1, -1)$, $(1, 1)$, and $(0, 2)$.

2. Use Green's theorem to evaluate

$$\int_{\mathcal{C}} -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy,$$

where

(i) $\mathcal{C}$ is the arc of the parabola $y = \frac{1}{4}x^2 + 1$ from $(-2, 2)$ to $(2, 2)$.

(ii) $\mathcal{C}$ is the arc of the parabola $y = x^2 - 2$ from $(-2, 2)$ to $(2, 2)$.

(iii) What do the results of part (i) and (ii) indicate about the path independence or path dependence of $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$?

3. Find the area of the region bounded by the curve parametrized by $\alpha(t) = \sin^3(t)\mathbf{i} + \cos^3(t)\mathbf{j}$, where $0 \le t \le 2\pi$.

4. Let $\mathbf{F} : \mathbb{R}^2 \backslash \{(1, 0)\} \to \mathbb{R}^2$ be the vector field

$$\mathbf{F}(x, y) := -\frac{y}{(x-1)^2+y^2}\mathbf{i} + \frac{x-1}{(x-1)^2+y^2}\mathbf{j}.$$

Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where $\mathcal{C}$ is the ellipse $\frac{x^2}{25} + \frac{y^2}{36} = 1$.

5. Let $\mathcal{C}$ be the triangle with vertices $(0, 0)$, $(1, 0)$, and $(1, 3)$ oriented anti-clockwise. Use Green's theorem to evaluate

$$\int_{\mathcal{C}} \sqrt{9 + 3x^3}dx + 4xydy.$$

6. Use Green's theorem to evaluate

$$\int_{\mathcal{C}} x^2ydx + yx^3dy,$$

where $\mathcal{C}$ is the circle $(x-1)^2 + (y+1)^2 = 9$ oriented clockwise.

7. Let $\mathbf{F} : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$ be the vector field defined by

$$\mathbf{F}(x,y) := \frac{y^3}{(x^2+y^2)^2}\mathbf{i} - \frac{xy^2}{(x^2+y^2)^2}\mathbf{j}.$$

(i) Let $\mathcal{C}$ be the unit circle. Evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.
(ii) Evaluate $\int_{\mathcal{C}_0} \mathbf{F} \cdot d\mathbf{r}$, where $\mathcal{C}_0$ is the ellipse $\frac{x^2}{9} + \frac{y^2}{36} = 1$.

8. Find a smooth, simple, closed, counterclockwise oriented curve $\mathcal{C} \subset \mathbb{R}^2$ which maximizes the line integral

$$\int_{\mathcal{C}} (y^3 - y)dx - 2x^3 dy.$$

Is this maximum unique?

9. Evaluate the following line integrals using Green's theorem:

(i) $\int_{\mathcal{C}} (x^2+y)dx - y^2 dy$, where $\mathcal{C}$ is the rectangle with vertices $(0,0)$, $(0,-2)$, $(2,2)$, and $(-1,1)$.
(ii) $\int_{\mathcal{C}} (x-y)dx + (x+y^4)dy$, where $\mathcal{C}$ is the triangle with vertices $(0,0)$, $(0,2)$, and $(1,1)$.
(iii) $\int_{\mathcal{C}} e^{-xy}dx + xdy$, where $\mathcal{C}$ is the circle centered at the origin with radius 4.
(iv) $\int_{\mathcal{C}} \sin(y)dx - \cos(x)dy$, where $\mathcal{C}$ is the parallelogram formed from the vectors $\mathbf{u} = (1,1)$ and $\mathbf{v} = (-1,2)$.
(v) $\int_{\mathcal{C}} (y^2 - \tan^{-1}(x))dx + (4x + \cos(y))dy$, where $\mathcal{C}$ is the boundary of the region enclosed by the parabola and the line $y = 3$.

10. Evaluate the line integrals $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ using Green's theorem, where

(i) $\mathbf{F}(x,y) = y^6\mathbf{i} + xy^5\mathbf{j}$ and $\mathcal{C}$ is the ellipse $4x^2 + y^2 = 1$.
(ii) $\mathbf{F}(x,y) = -x(x+y)\mathbf{i} + x^2y\mathbf{j}$ and $\mathcal{C}$ is the curve given by the line connecting $(0,0)$ to $(1,0)$, then the line connecting $(1,0)$ to $(0,1)$, then back to the origin.

11. Use Green's theorem to determine the area of the following regions:

(i) The region bounded by the hypocycloid $\alpha : [0, 2\pi] \to \mathbb{R}^2$,

$$\alpha(t) = \cos^3(t)\mathbf{i} + \sin^3(t)\mathbf{j}.$$

(ii) The region bounded by the curve parametrized by $\alpha : [0, 2\pi] \to \mathbb{R}$,

$$\alpha(t) = \cos(t)\mathbf{i} + \sin^3(t)\mathbf{j}.$$

(iii) The region bounded by the ellipse given by $2x^2 + 3y^2 = 4y$.
(iv) The region bounded by the ellipse given by $9x^2 + 4y^2 = 3x$.

12. Green's theorem requires the component functions of the vector field $\mathbf{F}$ to have continuous first-order partial derivatives. Let $\Omega = [0,1] \times [0,1]$ denote the unit square in $\mathbb{R}^2$. Let $\mathbf{F}(x,y) = 0\mathbf{i} + Q(x,y)\mathbf{j}$, where

$$Q(x,y) := \begin{cases} x^2 y \sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0. \end{cases}$$

(i) Show that the partial derivative Q_x exists.
(ii) Show that Q_x is not continuous on $\{(x,y) : x = 0,\ 0 < y \leq 1\}$.
(iii) Verify that Green's theorem holds for $\mathbf{F} : \Omega \to \mathbb{R}^2$.

13. Produce an example of a vector field for which Green's theorem does not hold.

14. Verify the statement of Green's theorem for

$$\int_{\mathcal{C}} xy dx + x^2 y dy,$$

where $\mathcal{C}$ is the triangle with vertices $(0,0)$, $(1,0)$, and $(1,2)$.

15. Use Green's theorem to evaluate $\int_{\mathcal{C}} \omega$, where

$$\omega = (1 + \tan x)dx + (x^2 + e^{-y})dy,$$

where $\mathcal{C}$ is the positively-oriented boundary of the region enclosed by the curves $y = \sqrt{x}$, $x = 1$ and $y = 0$.

16. Let $\mathcal{C}$ be the boundary of the square given by $0 \leq x \leq \frac{\pi}{3}$ and $0 \leq y \leq \frac{\pi}{3}$. Compute

$$\int_{\mathcal{C}} (\cos(x)\cos(y) + 3^{x^2})dx + (\sin(x)\sin(y) + \sqrt{y^4 + 1})dy.$$

17. Let $\mathcal{C}$ be the boundary of the region enclosed by the ellipse $x^2 + 4y^2 = 9$, lying above the line $x + 2y = 3$. Compute

$$\int_{\mathcal{C}} -2x^2 y^2 dx + 4x^3 y dy.$$

18. Let Ω be the triangular region bounded by the curves $y = 1$, $x = 2$, and $y = x$. Let $\alpha = -ye^x dx + e^x dy$. Compute

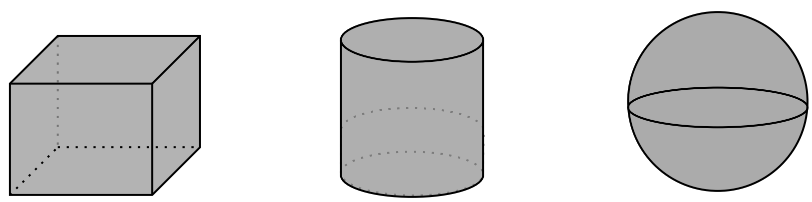
(i) the exterior derivative $d\alpha$.
(ii) the double integral $\iint_{\Omega} d\alpha$.
(iii) the line integral $\int_{\partial\Omega} \alpha$.

4.3. Surface integrals

Green's theorem applies to regions Ω in the plane. There is no reason to restrict to regions in $\mathbb{R}^2$, however. There is a more general Green's theorem which holds for vector fields defined on surfaces on $\mathbb{R}^2$. This version of the theorem is commonly referred to as the *Kelvin–Stokes theorem* and requires us to develop the machinery of surface integrals – two-dimensional versions of line integrals.

Reminder 4.3.1. Recall that a curve $\mathcal{C}$ is said to be *smooth* if there is a smooth parametrization $\alpha : I \to \mathbb{R}^n$, where $I \subseteq \mathbb{R}$ is an interval. In a similar manner, we have:

Definition 4.3.2. A surface $\mathcal{S}$ is said to be *smooth* if there is a smooth parametrization $\alpha : I \times J \to \mathbb{R}^n$, for intervals $I, J \subset \mathbb{R}$. If $\mathcal{S}$ can be expressed as the (finite) disjoint union of smooth surfaces, then $\mathcal{S}$ is said to be a *piecewise smooth surface*.



piecewise smooth   piecewise smooth   smooth

Remark 4.3.3. Throughout, we will use $\mathcal{S}$ to denote a surface in $\mathbb{R}^3$. Moreover, unless otherwise stated, all surfaces are assumed to be piecewise smooth.

Formula for the parametrization of a sphere. The formula for parametrizing a sphere of (fixed) radius $r > 0$ is given by the spherical coordinate parametrization: $\alpha : [0, 2\pi] \times [0, \phi] \to \mathbb{R}^3$,

$$\alpha(\vartheta, \phi) \ := \ r\cos(\vartheta)\sin(\phi)\mathbf{i} + r\sin(\vartheta)\sin(\phi)\mathbf{j} + r\cos(\phi)\mathbf{k}.$$

Formula for the parametrization of a cylinder. The formula for parametrizing a cylinder of (fixed) radius $r > 0$ is given by the cylindrical coordinate parametrization: $\alpha : [0, 2\pi] \times [z_0, z_1] \to \mathbb{R}^3$,

$$\alpha(\vartheta, z) \ := \ r\cos(\vartheta)\mathbf{i} + r\sin(\vartheta)\mathbf{j} + z\mathbf{k}.$$

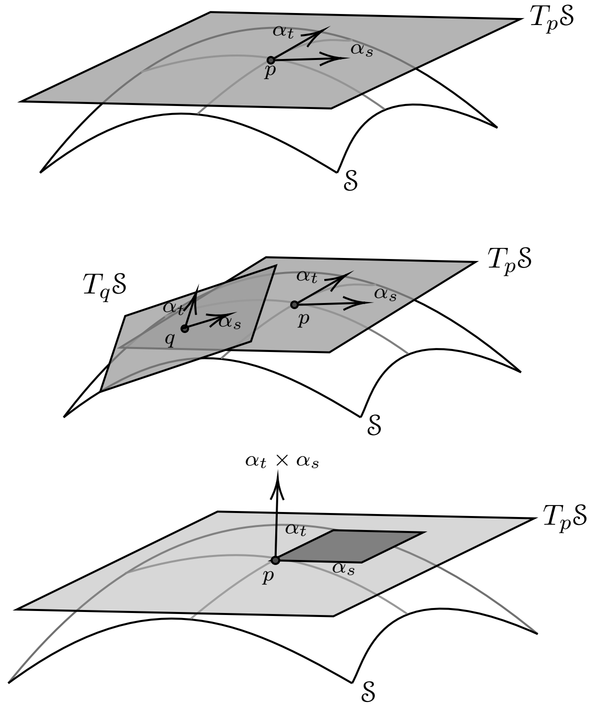
Formula for the parametrization of a torus. The formula for parametrizing a torus of fixed radii $\rho > 0$ and $\rho > 0$ is given by: $\alpha : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$,

$$\alpha(\phi, \vartheta) \ := \ (\rho + r\cos(\vartheta))\cos(\phi)\mathbf{i} + (\rho + r\cos(\vartheta))\sin(\phi)\mathbf{j} + r\sin(\vartheta)\mathbf{k}.$$

Reminder: Arc length. In defining the line integral of a (scalar) function f and subsequently, a vector field $\mathbf{F}$, we first obtained an expression for the arc length. The arc length of the curve $\mathcal{C}$ was given by adding up (i.e., integrating) the norms of the tangent vectors of a smooth parametrization $\alpha : [a, b] \to \mathbb{R}^n$:

$$\text{arc length}(\mathcal{C}) \;:=\; \int_{\mathcal{C}} ds \;:=\; \int_a^b |\alpha'(t)| dt.$$

In a similar manner, the surface area of a surface $\mathcal{S}$ will be given by adding up (i.e., integrating) the area of tangent planes formed by the two coordinate partial derivatives of a smooth parametrization $\alpha : [a, b] \times [c, d] \to \mathbb{R}^n$ (viewed as tangent vectors):



With $\alpha : [a, b] \times [c, d] \to \mathbb{R}^3$, $(s, t) \mapsto \alpha(s, t)$ denoting the parametrization, we write α_s and α_t for the tangent vectors. The area of the parallelogram $\alpha_s \wedge \alpha_t$ is given by $|\alpha_s \times \alpha_t|$.

Definition 4.3.4. Let $\mathcal{S}$ be a surface parametrized by $\alpha : [s_0, s_1] \times [t_0, t_1] \to \mathbb{R}^3$. Then the *surface area* of $\mathcal{S}$ is given by

$$\text{surface area}(\mathcal{S}) \quad := \quad \int_{t_0}^{t_1} \int_{s_0}^{s_1} |\alpha_s \times \alpha_t| ds dt.$$

Example 4.3.5. Let $\mathbb{S}^2_r := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2\}$ denote the sphere of radius r in $\mathbb{R}^3$. Compute the surface area of $\mathbb{S}^2_r$.

Solution. Using the spherical coordinate parametrization of $\mathbb{S}^2_r$, namely $\alpha : [0, 2\pi] \times [0, \phi] \to \mathbb{R}^3$,

$$\alpha(\vartheta, \phi) \;:=\; r\cos(\vartheta)\sin(\phi)\mathbf{i} + r\sin(\vartheta)\sin(\phi)\mathbf{j} + r\cos(\phi)\mathbf{k},$$

we see that

$$\begin{aligned}
\alpha_\vartheta &= -r\sin(\vartheta)\sin(\phi)\mathbf{i} + r\cos(\vartheta)\sin(\phi) + 0\mathbf{k}, \\
\alpha_\phi &= r\cos(\vartheta)\cos(\phi)\mathbf{i} + r\sin(\vartheta)\cos(\phi)\mathbf{j} - r\sin(\phi)\mathbf{k}.
\end{aligned}$$

The cross product is then computed to be

$$\begin{aligned}
\alpha_\phi \times \alpha_\vartheta &= \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ r\cos(\vartheta)\cos(\phi) & r\sin(\vartheta)\cos(\phi) & -r\sin(\phi) \\ -r\sin(\vartheta)\sin(\phi) & r\cos(\vartheta)\sin(\phi) & 0 \end{bmatrix} \\
&= r^2\sin^2(\phi)\cos(\vartheta)\mathbf{i} + r^2\sin(\vartheta)\sin^2(\phi)\mathbf{j} \\
&\quad +(r^2\cos^2(\vartheta)\cos(\phi)\sin(\phi) + r^2\sin^2(\vartheta)\cos(\phi)\sin(\phi))\mathbf{k} \\
&= r^2\sin^2(\phi)\cos(\vartheta)\mathbf{i} + r^2\sin(\vartheta)\sin^2(\phi)\mathbf{j} + r^2\cos(\phi)\sin(\phi)\mathbf{k}.
\end{aligned}$$

Computing the norm of this vector (field), we see that

$$\begin{aligned}
|\alpha_\phi \times \alpha_\vartheta| &= \sqrt{r^4\sin^4(\phi)\cos^2(\vartheta) + r^4\sin^2(\vartheta)\sin^4(\phi) + r^4\cos^2(\phi)\sin^2(\phi)} \\
&= \sqrt{r^4\sin^4(\phi) + r^4\cos^2(\phi)\sin^2(\phi)} \\
&= r^2\sin(\phi).
\end{aligned}$$

Hence, the surface area of $\mathbb{S}^2_r$ is

$$\int_0^{2\pi}\int_0^{\pi} r^2\sin(\phi)d\phi d\vartheta \;=\; 2\pi r^2\int_0^{\pi}\sin(\phi)d\phi \;=\; 4\pi r^2.$$

□

Example 4.3.6. Let $\mathbb{T}^2$ denote the torus, which affords the parametrization $\alpha : [0, 2\pi] \times [0, 2\pi] \to \mathbb{R}^3$,

$$\alpha(\vartheta, \phi) \;:=\; (\rho + r\cos(\vartheta))\cos(\phi)\mathbf{i} + (\rho + r\cos(\vartheta))\sin(\phi)\mathbf{j} + r\sin(\vartheta)\mathbf{k}.$$

Compute the surface area of $\mathbb{T}^2$.

Solution. We compute the partial derivatives of the parametrization:

$$\begin{aligned}
\alpha_\vartheta &= -r\sin(\vartheta)\cos(\phi)\mathbf{i} - r\sin(\vartheta)\sin(\phi)\mathbf{j} + r\cos(\vartheta)\mathbf{k} \\
\alpha_\phi &= -(\rho + r\cos(\vartheta))\sin(\phi)\mathbf{i} + (\rho + r\cos(\vartheta))\cos(\phi)\mathbf{j} + 0\mathbf{k}.
\end{aligned}$$

Compute the cross product

$$\begin{aligned}
\alpha_\phi \times \alpha_\vartheta &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -(\rho + r\cos(\vartheta))\sin(\phi) & (\rho + r\cos(\vartheta))\cos(\phi) & 0 \\ -r\sin(\vartheta)\cos(\phi) & -r\sin(\vartheta)\sin(\phi) & r\cos(\vartheta) \end{bmatrix} \\
&= r\cos(\vartheta)(\rho + r\cos(\vartheta))\cos(\phi)\mathbf{i} + r(\rho + r\cos(\vartheta))\sin(\phi)\cos(\vartheta)\mathbf{j} \\
&\quad + [r\sin(\vartheta)(\rho + r\cos(\vartheta))\sin^2(\phi) + r(\rho + r\cos(\vartheta))\cos^2(\phi)\sin(\vartheta)]\mathbf{k} \\
&= r\cos(\vartheta)\cos(\phi)(\rho + r\cos(\vartheta))\mathbf{i} + r\cos(\vartheta)\sin(\phi)(\rho + r\cos(\vartheta))\mathbf{j} \\
&\quad + r\sin(\vartheta)(\rho + r\cos(\vartheta))\mathbf{k}.
\end{aligned}$$

Computing the norm of this vector (field), we have

$$\begin{aligned}
|\alpha_\phi \times \alpha_\vartheta|^2 &= r^2\cos^2(\vartheta)\cos^2(\phi)(\rho + r\cos(\vartheta))^2 + r^2\cos^2(\vartheta)\sin^2(\phi)(\rho + r\cos(\vartheta))^2 \\
&\quad + r^2\sin^2(\vartheta)(\rho + r\cos(\vartheta))^2 \\
&= r^2(\rho + r\cos(\vartheta))^2(\cos^2(\vartheta)\cos^2(\phi) + \cos^2(\vartheta)\sin^2(\phi) + \sin^2(\vartheta)) \\
&= r^2(\rho + r\cos(\vartheta))^2.
\end{aligned}$$

Therefore, taking the square root of both sides recovers:

$$|\alpha_\phi \times \alpha_\vartheta| = r(\rho + r\cos(\vartheta)).$$

The surface area is then

$$\begin{aligned}
\text{surface area}(\mathbb{T}^2) &= \int_0^{2\pi}\int_0^{2\pi} r(\rho + r\cos(\vartheta))d\phi d\vartheta \\
&= 2\pi r\int_0^{2\pi}(\rho + r\cos(\vartheta))d\vartheta \\
&= 4\pi^2 r\rho.
\end{aligned}$$

$\square$

Reminder: Line integrals of functions. Recall that to compute the line integral $\int_{\mathcal{C}} f ds$, where $\mathcal{C}$ is a curve, and f is a smooth function, we determined a parametrization $\alpha : [t_0, t_1] \to \mathbb{R}^n$ for the curve $\mathcal{C}$, computed $|\alpha'(t)|$, expressed f in terms of α, and evaluate

$$\int_{\mathcal{C}} f ds = \int_{t_0}^{t_1} f(\alpha(t))|\alpha'(t)|dt.$$

In a similar manner, surface integrals of functions are defined as follows:

Definition 4.3.7. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function and let $\mathcal{S}$ be a smooth surface. Let $\alpha : [s_0, s_1] \times [t_0, t_1] \to \mathbb{R}^3$ be a smooth parametrization for $\mathcal{S}$. The *surface integral of f over* $\mathcal{S}$ is defined by

$$\iint_{\mathcal{S}} f dS \;:=\; \int_{t_0}^{t_1} \int_{s_0}^{s_1} f(\alpha(s,t))|\alpha_s \times \alpha_t| ds dt.$$

Example 4.3.8. Let $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}$ be the unit sphere in $\mathbb{R}^3$. Compute the surface integral

$$\iint_{\mathcal{S}} y^2 dS.$$

Solution. A parametrization for the unit sphere is given by $\alpha : [0, 2\pi] \times [0, \phi] \to \mathbb{R}^3$,

$$\alpha(\vartheta, \phi) \;:=\; \cos(\vartheta)\sin(\phi)\mathbf{i} + \sin(\vartheta)\sin(\phi)\mathbf{j} + \cos(\phi)\mathbf{k}.$$

Moreover, we know that

$$|\alpha_\phi \times \alpha_\vartheta| \;=\; \sin(\phi).$$

Hence,

$$\begin{aligned}\iint_{\mathcal{S}} y^2 dS \;&=\; \int_0^{2\pi} \int_0^{\pi} \sin^2(\vartheta)\sin^2(\phi)\sin(\phi) d\phi d\vartheta \\ &=\; \left(\int_0^{2\pi} \sin^2(\vartheta) d\vartheta\right)\left(\int_0^{\pi} \sin^3(\phi) d\phi\right) \;=\; \frac{4}{3}\pi.\end{aligned}$$

□

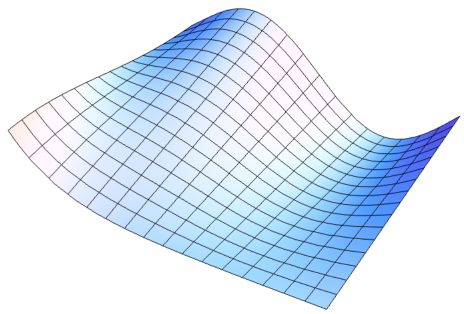
Evaluating surface integrals of functions. Let f be a smooth function and $\mathcal{S}$ a surface. The steps to evaluate the surface integral

$$\iint_{\mathcal{S}} f dS$$

are the following:

(1) Parametrize the surface $\mathcal{S}$ by a smooth map $\alpha : [s_0, s_1] \times [t_0, t_1] \to \mathbb{R}^3$.
(2) Compute the derivatives $\alpha_s = \frac{\partial \alpha}{\partial s}$ and $\alpha_t = \frac{\partial \alpha}{\partial t}$.
(3) Compute the norm of the cross product $|\alpha_s \times \alpha_t|$.
(4) Express the function f in terms of the parametrization $\alpha(s, t)$.
(5) Evaluate the double integral $\int_{t_0}^{t_1} \int_{s_0}^{s_1} f(\alpha(s,t))|\alpha_s \times \alpha_t| ds dt$.

Surface integrals of graphs of functions. Suppose our surface $\mathcal{S} \subset \mathbb{R}^3$ is given by the graph of a (smooth) function $f : \mathbb{R}^2 \to \mathbb{R}$.



The graph of a smooth function.

Proposition. Let $\mathcal{S}$ be a smooth surface given by the graph of a function $f : [a,b]\times[c,d] \to \mathbb{R}$. A parametrization for $\mathcal{S}$ is given by $\alpha : [a,b] \times [c,d] \to \mathbb{R}^3$,

$$\alpha(s,t) \ := \ s\mathbf{i} + t\mathbf{j} + f(s,t)\mathbf{k}.$$

Moreover, if $g : \mathcal{S} \to \mathbb{R}$ is a smooth function on $\mathcal{S}$, then

$$\iint_{\mathcal{S}} g dS \ = \ \int_c^d \int_a^b g(s,t)\sqrt{1 + f_s^2 + f_t^2}dsdt.$$

Proof. The previous ideas can easily be applied to give a formula for the surface integral over the graph of f. Indeed, if $\mathcal{S}$ is the graph of f, then a parametrization is given by

$$\alpha : \mathbb{R}^2 \to \mathbb{R}^3, \qquad \alpha(s,t) = s\mathbf{i} + t\mathbf{j} + f(s,t)\mathbf{k}.$$

Hence, the coordinate partial derivatives of α are $\alpha_s = \mathbf{i} + 0\mathbf{j} + f_s\mathbf{k}$ (where $f_s = \frac{\partial f}{\partial s}$) and $\alpha_t = 0\mathbf{i} + \mathbf{j} + f_t\mathbf{k}$ (where $f_t = \frac{\partial f}{\partial t}$). The cross product is computed:

$$\alpha_s \times \alpha_t \ = \ \det\begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_s \\ 0 & 1 & f_t \end{pmatrix} \ = \ -f_s\mathbf{i} - f_t\mathbf{j} + \mathbf{k}.$$

The norm is then

$$|\alpha_s \times \alpha_t| \ = \ \sqrt{1 + f_s^2 + f_t^2}.$$

Hence, by the standard formula for the surface integral, this proves the result.

$\square$

Example 4.3.9. Evaluate $\iint_{\mathcal{S}} y dS$, where $\mathcal{S} = \{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le 1, 0 \le y \le 2, z = x + y^2\}$.

Solution. Let $\alpha : [0, 1] \times [0, 2] \to \mathbb{R}^3$ be a parametrization for $\mathcal{S}$, where

$$\alpha(s, t) \ := \ s\mathbf{i} + t\mathbf{j} + (s + t^2)\mathbf{k}.$$

Then if $f(s, t) := s + t^2$, we have $f_s = 1$ and $f_t = 2t$. In particular,

$$|\alpha_s \times \alpha_t| = \sqrt{1 + 1^2 + (2t)^2} \ = \ \sqrt{2 + 4t^2},$$

and we evaluate

$$\begin{aligned} \iint_{\mathcal{S}} y dS &= \int_0^2 \int_0^1 t\sqrt{2 + 4t^2} ds dt \\ &= \int_0^2 t\sqrt{2 + 4t^2} dt \\ &= \frac{13\sqrt{2}}{3}. \end{aligned}$$

□

Example 4.3.10. Let $\mathcal{S}$ be the part of the cone $x^2 + y^2 = z^2$ with $0 \le z \le 1$. Compute

$$\iint_{\mathcal{S}} x^2 y^2 z^2 dS.$$

Solution. We can write $\mathcal{S}$ as the graph of the function $f : \Omega \to \mathbb{R}$, $f(x, y) = \sqrt{x^2 + y^2}$, where $\Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \le 1\}$. We have

$$f_x \ = \ \frac{x}{\sqrt{x^2 + y^2}}, \qquad f_y \ = \ \frac{y}{\sqrt{x^2 + y^2}}.$$

We then have

$$\begin{aligned} |f_x \times f_y| &= \sqrt{1 + f_x^2 + f_y^2} \\ &= \sqrt{1 + \frac{x^2}{x^2 + y^2} + \frac{y^2}{x^2 + y^2}} \\ &= \sqrt{2}. \end{aligned}$$

The integral is then

$$\begin{aligned}
\iint_{\mathcal{S}} x^2y^2z^2 dS &= \sqrt{2}\iint_{\Omega} x^2y^2(x^2+y^2)dxdy \\
&= \sqrt{2}\int_0^{2\pi}\int_0^1 (r\cos(\vartheta))^2(r\sin(\vartheta))^2 r^3 dr d\vartheta \\
&= \sqrt{2}\left(\int_0^{2\pi}\sin^2(\vartheta)\cos^2(\vartheta)d\vartheta\right)\left(\int_0^1 r^7 dr\right) \\
&= \frac{\pi\sqrt{2}}{32}.
\end{aligned}$$

□

Reminder: Line integrals of vector fields. Recall that to compute the line integral $\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r}$, where $\mathcal{C}$ is a curve and $\mathbf{F}$ is a smooth vector field, we determined a parametrization $\alpha : [t_0, t_1] \to \mathbb{R}^n$ for the curve, compute $\alpha'(t)$, expressed $\mathbf{F}$ in terms of α, and evaluated

$$\int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{r} = \int_{t_0}^{t_1} \mathbf{F}(\alpha(t))\cdot\alpha'(t)dt.$$

In a similar manner, surface integrals of vector fields are defined as follows:

Definition 4.3.11. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field and let $\mathcal{S}$ be a smooth surface. Let $\alpha : [s_0, s_1] \times [t_0, t_1] \to \mathbb{R}^3$ be a smooth parametrization for $\mathcal{S}$. The *surface integral of* $\boldsymbol{F}$ *over* $\mathcal{S}$ is defined by

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \int_{t_0}^{t_1}\int_{s_0}^{s_1} \mathbf{F}(\alpha(s,t))\cdot(\alpha_s\times\alpha_t)dsdt.$$

Evaluating surface integrals of vector fields. Let $\mathbf{F}$ be a vector field and $\mathcal{S}$ a surface. The steps to evaluate the surface integral

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}$$

are the following:

(1) Parametrize the surface $\mathcal{S}$ by a smooth map $\alpha : [s_0, s_1] \times [t_0, t_1] \to \mathbb{R}^3$.
(2) Compute the derivatives $\alpha_s = \frac{\partial\alpha}{\partial s}$ and $\alpha_t = \frac{\partial\alpha}{\partial t}$.
(3) Compute the cross product $\alpha_s \times \alpha_t$.
(4) Express $\mathbf{F}$ in terms of the parametrization.
(5) Evaluate the double integral

$$\int_{t_0}^{t_1}\int_{s_0}^{s_1} \mathbf{F}(\alpha(s,t))\cdot(\alpha_s\times\alpha_t)dsdt.$$

Example 4.3.12. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$\mathbf{F}(x, y, z) = -y\mathbf{i} + x\mathbf{j} + 3z\mathbf{k}.$$

If $\mathcal{S}$ is the upper-hemisphere of radius 1, compute

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

SOLUTION. We have $\alpha(\vartheta, \phi) = \sin(\phi)\cos(\vartheta)\mathbf{i} + \sin(\phi)\sin(\vartheta)\mathbf{j} + \cos(\phi)\mathbf{k}$, where $0 \le \phi \le \frac{\pi}{2}$ and $0 \le \vartheta \le 2\pi$. We know that

$$\alpha_\phi \times \alpha_\vartheta = \sin^2(\phi)\cos(\vartheta)\mathbf{i} + \sin^2(\phi)\sin(\vartheta)\mathbf{j} + \sin(\phi)\cos(\phi)\mathbf{k},$$

and moreover,

$$\mathbf{F}(\vartheta, \phi) = -\sin(\phi)\sin(\vartheta)\mathbf{i} + \sin(\phi)\cos(\vartheta)\mathbf{j} + 3\sin(\phi)\cos(\phi)\mathbf{k}.$$

Hence,

$$\begin{aligned} \mathbf{F}(\vartheta, \phi) \cdot (\alpha_\phi \times \alpha_\vartheta) &= (-\sin(\phi)\sin(\vartheta)\mathbf{i} + \sin(\phi)\cos(\vartheta)\mathbf{j} + 3\sin(\phi)\cos(\phi)\mathbf{k}) \\ &\quad \cdot \left(\sin^2(\phi)\cos(\vartheta)\mathbf{i} + \sin^2(\phi)\sin(\vartheta)\mathbf{j} + \sin(\phi)\cos(\phi)\mathbf{k}\right) \\ &= -\sin^3(\phi)\sin(\vartheta)\cos(\vartheta) + \sin^3(\phi)\cos(\vartheta)\sin(\vartheta) + 3\sin^2(\phi)\cos^2(\phi) \\ &= 3\sin^2(\phi)\cos^2(\phi). \end{aligned}$$

Computing the surface integral now gives

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} &= \int_0^{\frac{\pi}{2}} \int_0^{2\pi} 3\sin^2(\phi)\cos^2(\phi) d\vartheta d\phi \\ &= 6\pi \int_0^{\frac{\pi}{2}} \sin^2(\phi)\cos^2(\phi) d\phi \\ &= \frac{3\pi^2}{8}. \end{aligned}$$

□

Example 4.3.13. Let $\mathbf{F} : \mathbb{R}^3 \backslash \{0\} \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y) = \frac{2x}{x^2 + y^2}\mathbf{i} + \frac{2y}{x^2 + y^2}\mathbf{j} + \mathbf{k}.$$

Compute

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathcal{S}$ is the surface parametrized by $\alpha : [0, 1] \times [0, 2\pi] \to \mathbb{R}^3$, where

$$\alpha(s, t) = s\cos(t)\mathbf{i} + s\sin(t)\mathbf{j} + s^2\mathbf{k}.$$

Solution. We have the parametrization, hence we need to compute the partial derivatives of α. To this end, we see that

$$\begin{aligned}\alpha_s &= \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + 2s\mathbf{k},\\ \alpha_t &= -s\sin(t)\mathbf{i} + s\cos(t)\mathbf{j} + 0\mathbf{k}.\end{aligned}$$

The cross product is given by

$$\begin{aligned}\alpha_s \times \alpha_t &= \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos(t) & \sin(t) & 2s \\ -s\sin(t) & s\cos(t) & 0 \end{bmatrix}\\ &= -2s^2\cos(t)\mathbf{i} - 2s^2\sin(t)\mathbf{j} + s\mathbf{k}.\end{aligned}$$

Writing $\mathbf{F}$ in terms of α, we see that

$$\mathbf{F}(\alpha(s,t)) = \frac{2\cos(t)}{s}\mathbf{i} + \frac{2\sin(t)}{s}\mathbf{j} + \mathbf{k}.$$

Hence

$$\begin{aligned}\iint_{\mathcal{S}} \mathbf{F}\cdot(\alpha_s\times\alpha_t) &= \int_0^{2\pi}\int_0^1 \left(\frac{2\cos(t)}{s}\mathbf{i} + \frac{2\sin(t)}{s}\mathbf{j} + \mathbf{k}\right)\cdot\left(-2s^2\cos(t)\mathbf{i} - 2s^2\sin(t)\mathbf{j} + s\mathbf{k}\right)\\ &= \int_0^{2\pi}\int_0^1 \left(-4s\cos^2(t) - 4s\sin^2(t) + s\right) dsdt\\ &= \int_0^{2\pi}\int_0^1 (-3s)dsdt\\ &= -6\pi\int_0^1 sds = -3\pi.\end{aligned}$$

□

Formula for the surface integral of a vector field over the graph of a function. If $\mathbf{F} : \mathcal{S} \to \mathbb{R}^3$ is a smooth vector field on the surface $\mathcal{S}$ (given by the graph of f), then

$$\int_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\Omega} \mathbf{F}\cdot(\alpha_s\times\alpha_t)dsdt.$$

If $\mathbf{F} = P(s,t)\mathbf{i} + Q(s,t)\mathbf{j} + R(s,t)\mathbf{k}$, then since $\alpha_s \times \alpha_t = -f_s\mathbf{i} - f_t\mathbf{j} + \mathbf{k}$, we have

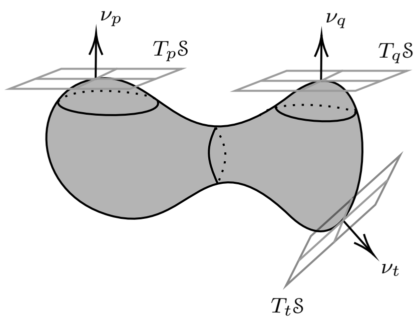
$$\mathbf{F}\cdot(\alpha_s\times\alpha_t) = -P(s,t)f_s - Q(s,t)f_t + R(s,t).$$

In other words,

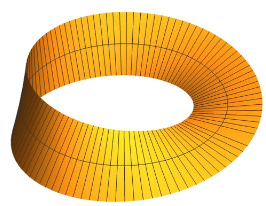
$$\int_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} = \iint_{\Omega} (R(s,t) - P(s,t)f_s - Q(s,t)f_t)dsdt.$$

Remarks on orientation.

Definition 4.3.14. Let $\mathcal{S}$ be a smooth surface in $\mathbb{R}^3$. We say that $\mathcal{S}$ is *orientable* if, for every point $p \in \mathcal{S}$, there is a vector field $\nu : \mathcal{S} \to \mathbb{R}^3$ which is everywhere orthogonal to the tangent plane $T_p\mathcal{S}$.

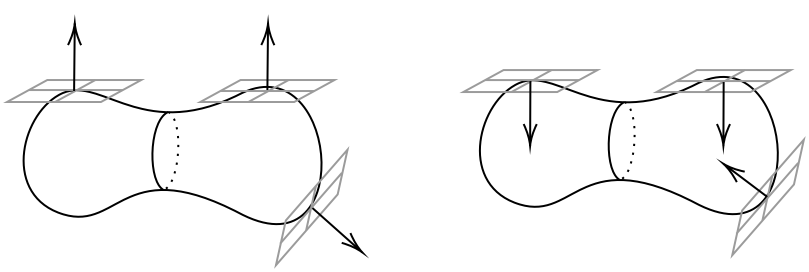


Example 4.3.15. The Möbius strip is the prototypical example of a non-orientable surface:



The Möbius strip.

Remark 4.3.16. We understand a surface to be positively oriented if the normal is outward-pointing, and negatively oriented if the normal is inward pointing. By the right-hand rule, positively oriented surface have boundary curves which are oriented anti-clockwise (i.e., positively oriented boundary curves).



positively oriented

negatively oriented

Historical remarks. As early as 1760 Lagrange had given an explicit expression for the element of surface dS in the process of calculating surface areas. It was not until 1811, however, in the second edition of his Mécanique analytique, that Lagrange introduced the general notion of a surfae integral.

Exercises

1. Compute the area of the following regions:

(i) The helicoid surface parametrized by the function $f : [0,1] \times [0,2\pi] \to \mathbb{R}$, defined by $f(x,y) := x\cos(y)\mathbf{i} + x\sin(y)\mathbf{j} + y\mathbf{k}$.
(ii) The part of the paraboloid $y = z^2 + x^2$ which lies in the sphere $x^2 + y^2 + z^2 = 9$.
(iii) The hyperbolic surface $z = y^2 - x^2$ that lies between the cylinders $x^2 + y^2 = 9$ and $x^2 + y^2 = 16$.
(iv) The surface given by rotating $z = e^{-y}$, for $0 \le y \le 1$, around the y–axis.
(v) The part of the sphere $x^2 + y^2 + z^2 = 1$ which lies between $z = -\frac{1}{2}$ and $z = \frac{1}{2}$.

2. Compute the following surface integrals $\int_{\mathcal{S}} f dS$, where

(i) $f(x,y,z) = x+y+z$ and $\mathcal{S}$ is the hemi-sphere $\mathcal{S} = \{(x,y,z) : x^2+y^2+z^2 = 1, z \ge 0\}$.
(ii) $f(x,y,z) = xy+yz$ and $\mathcal{S}$ is the triangular region with vertices $(0,1,0)$, $(1,0,2)$ and $(0,0,3)$.
(iii) $f(x,y,z) = z + \sqrt{x+y}$ and $\mathcal{S} = \{(x,y,z) : x^2 + y^2 = 9,\ -1 \le z \le 1\}$.

3. Find the center of mass of the hemisphere $x^2+(y-1)^2+(z+1)^2 = 4$ assuming the density is constant.

4. Find the mass of the cone $z = 2\sqrt{x^2+y^2}$, where $0 \le z \le 6$, if $\rho(x,y,z) := 1 - z$ is the density function.

5. Let $\rho = 10$ denote the density of a fluid. Let $\mathbf{v} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$ denote the velocity of the fluid. Determine the rate of flow upward through the paraboloid

$$z = 9 - \frac{1}{4}(x^2 + y^2)$$

if $x^2 + y^2 \le 49$.

6. Let $\mathbb{S}^2 \subset \mathbb{R}^3$ denote the unit sphere. Evaluate the surface integral

$$\int_{\mathbb{S}^2} (x^2 + y + z)dA.$$

7. Evaluate

$$\int_{\Omega} (x^3 - 3xy^2)dxdy,$$

where

$$\Omega = \{(x,y) \in \mathbb{R}^2 : (x+1)^2 + y^2 \le 9,\ (x-1)^2 + y^2 \ge 1\}.$$

8. Let $\mathcal{S}$ be the part of the surface $x^2 + y^2 + z = 2$ that lies above the square $-1 \le x \le 1$ and $-1 \le y \le 1$.

(i) Compute $\int_{\mathcal{S}} \frac{x^2+y^2}{\sqrt{1+x^2+y^2}} dS$.

(ii) Compute the flux of $\mathbf{F} = -x\mathbf{i} + y\mathbf{j} - z\mathbf{k}$ upward through $\mathcal{S}$.

9. Let $\mathcal{S}$ be the part of the surface $f(x, y) = xy(x + 1)$ that lies over the square $(x, y) \in [0, 3] \times [0, 3]$.

(i) Compute $\int_{\mathcal{S}} \frac{x^2(y+1)}{\sqrt{1+x^2+y^2}} dS$.

(ii) Compute the flux of $\mathbf{F} = x^2\mathbf{i} + (y - x)\mathbf{j} + (z + 3)\mathbf{k}$ upward through $\mathcal{S}$.

10. Find the area of the part of the surface $z = \sqrt{y^3 + 1}$ that lies above the square $(x, y) \in [-1, 0] \times [1, 0]$.

11. Let $\mathcal{S}$ be the surface given by the equation

$$x^2 + y^2 = \cos^2(z),$$

lying between the planes $z = 0$ and $z = \frac{\pi}{2}$. Evaluate

$$\int_{\mathcal{S}} \sqrt{1 + \sin^2(z)} dS.$$

12. Evaluate the surface integrals

$$\iint_{\mathcal{S}} \omega,$$

where

(i) $\mathcal{S}$ is the graph of $f : [-1, 1] \times [-2, 2] \to \mathbb{R}$, $f(x, y) := x^3 + 3xy^2$, and $\omega = dz \wedge dy + dz \wedge dx$.

(ii) $\mathcal{S}$ is the graph of $f : [0, 1] \times [0, 2] \to \mathbb{R}$, $f(x, y) = x^2 + y^3$, and $\omega = 5xzdy \wedge dz + 7yzdz \wedge dx + 31x^2y^3dx \wedge dy$.

(iii) $\mathcal{S}$ is the graph of $f : \Omega \to \mathbb{R}$, $f(x, y) := x^2 - 2xy$, where Ω is the triangular region bounded by the lines $y = 0$, $x = 3$, and $y = 2x$, and $\omega = dy \wedge dz + xdz \wedge dx + (y^2 + 4)dx \wedge dy$.

(iv) $\mathcal{S}$ is the graph of $z = xy$ over the region Ω, where Ω is the top half of the unit disk $x^2 + y^2 \le 1$, i.e., where $y \ge 0$, and $\omega = x^2dy \wedge dz + xydz \wedge dx + 7xzdx \wedge dy$.

(v) $\mathcal{S}$ is the portion of the plane $x + 2y + 3z = 6$ that lies in the first octant and $\omega = xdy \wedge dz - ydx \wedge dz + 3zdx \wedge dy$.

(vi) $\mathcal{S}$ is the graph of $z = 7 + x^2 + y^2$ over the disk $x^2 + y^2 \le 9$, and $\omega = z^2dx \wedge dy$.

(vii) $\mathcal{S}$ is the graph of $z = 25 - (x^2 + y^2)$ over the disk $x^2 + y^2 \le 9$, and $\omega = z^2dx \wedge dy$.

(viii) $\mathcal{S}$ is the surface parametrized by $\alpha : [0,1] \times [0,2] \to \mathbb{R}^3$,

$$\alpha(s,t) := (s+t)\mathbf{i} + (s-t^2)\mathbf{j} + st\mathbf{k},$$

and $\omega = 6z dz \wedge dx$.

(ix) $\mathcal{S}$ is the surface parametrized by $\alpha : [1,2] \times [0,2\pi] \to \mathbb{R}^3$,

$$\alpha(s,t) \;:=\; s^3\cos(t)\mathbf{i} + s^3\sin(t)\mathbf{j} + s^2\mathbf{k},$$

and $\omega = -x dy \wedge dz + y dx \wedge dz + 2z dx \wedge dy$.

(x) $\mathcal{S}$ is the surface parametrized by $\alpha : \Omega \to \mathbb{R}^3$,

$$\alpha(s,t) \;:=\; t^2\mathbf{i} + s^3\mathbf{j} + (2s+t)\mathbf{k},$$

where Ω is the region in the first quadrant bounded by the axes and the curve $t = 4 - s^2$, and $\omega = 2z dy \wedge dz + -3y dx \wedge dz + dx \wedge dy$.

(xi) $\mathcal{S}$ is the sphere of radius a centered at the origin, given by $x^2+y^2+z^2 = a^2$, and $\omega = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$.

(xii) $\mathcal{S}$ is the closed surface surface whose "top" part is $z = x^2y^2 - x^2 - y^2$ and whose "bottom" part is $z = -1$ and $\omega = y^3 dz \wedge dx + 2y^2 z dx \wedge dy$.

(xiii) $\mathcal{S}$ is the surface of the cube with vertices $(0,0,0)$, $(0,0,1)$, $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, $(1,0,1)$, $(1,1,0)$, and $(1,1,1)$, and $\omega = x dy \wedge dz - y^2 dz \wedge dx + 2xz dx \wedge dy$.

13. Evaluate

$$\iint_{\mathcal{S}} \mathbf{F} \cdot \nu dS,$$

where

(i) $\mathcal{S}$ is the graph of $z = x^2 - y^2$ over $0 \le x^2 + y^2 \le 1$, $y \ge 0$, and $\mathbf{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} - 2y^2\mathbf{k}$.

(ii) $\mathcal{S}$ is the graph of $z = x^4 + 6x^2y^2$ over the region in the first quadrant bounded by $y = 0$, $x = 2$, and $y = \sqrt{x}$, and $\mathbf{F}(x,y,z) = xy\mathbf{i} + 2y^2\mathbf{j} + 4yz\mathbf{k}$.

4.4. Stokes' theorem

The fundamental theorem of calculus

$$\int_a^b f'(x)dx \;=\; f(b) - f(a)$$

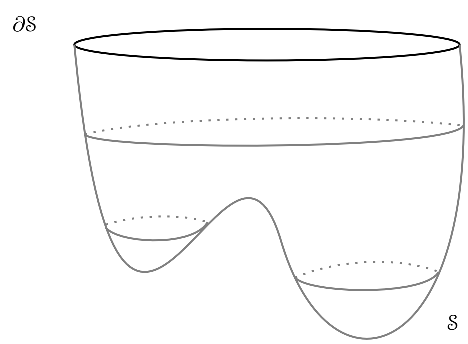
relates the integral of $f'(x)$ over $[a,b] \subseteq \mathbb{R}$ to the behaviour of the $f(x)$ on the boundary $\{a,b\}$. The fundamental theorem of line integrals

$$\int_{\mathcal{C}} \nabla f \cdot d\mathbf{r} = \nabla f(b) - \nabla f(a)$$

relates the integral of a gradient field $\mathbf{F} = \nabla f$ over $\mathcal{C}$ to the behaviour of its potential f on the boundary of $\mathcal{C}$. Green's theorem

$$\iint_{\Omega} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{k} dA \;=\; \int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{r}$$

relates the curl of $\mathbf{F}$, or equivalently $d\omega_{\mathbf{F}}$, over $\Omega \subset \mathbb{R}^2$, to the behaviour of $\mathbf{F}$, or equivalently $\omega_{\mathbf{F}}$, over the boundary of Ω. Stokes' theorem relates the surface integral of $\mathrm{curl}(\mathbf{F})$, or equivalently $d\omega_{\mathbf{F}}$, over a surface $\mathcal{S}$, to the behaviour of $\mathbf{F}$, or equivalently $\omega_{\mathbf{F}}$, over the boundary of $\mathcal{S}$:



Theorem 4.4.1. (Stokes' theorem). Let $\mathcal{S}$ be an oriented piecewise-smooth surface, with smooth boundary $\partial\mathcal{S}$. Let $\mathbf{F} : \mathcal{S} \to \mathbb{R}^3$ be a smooth vector field on $\mathcal{S}$. Then

$$\int_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

Proof. Let $\omega_{\mathbf{F}} = Pdx + Qdy + Rdz$ be the 1–form associated to $\mathbf{F}$. Then from **??**, we know that $\text{curl}(\mathbf{F})dV = d\omega_{\mathbf{F}}$. Hence, if

$$\int_{\partial \mathcal{S}} \omega_{\mathbf{F}} \;=\; \iint_{\mathcal{S}} d\omega_{\mathbf{F}}, \tag{4.4.1}$$

then

$$\int_{\partial \mathcal{S}} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_{\partial \mathcal{S}} \omega_{\mathbf{F}} \;=\; \iint_{\mathcal{S}} d\omega_{\mathbf{F}} \;=\; \iint_{\mathcal{S}} \text{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

It remains to (4.4.1), which will be treated later. □

Example 4.4.2. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) \;=\; (2z + \sin(x^{146}))\mathbf{i} - 5z\mathbf{j} - 5y\mathbf{k}.$$

Let $\mathcal{C}$ be the circle $x^2 + y^2 = 4$, $z = 1$, oriented counter clockwise when viewed from above. Compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

Solution. Let $\mathcal{S} = \{(x, y, z) : x^2 + y^2 \leq 4,\ z = 1\}$. This is a surface with boundary $\mathcal{C}$. We can use Stokes' theorem. Let us compute the curl. To this end, let

$$\omega_{\mathbf{F}} = (2z + \sin(x^{146}))dx - 5zdy - 5ydz$$

be the 1–form associated to $\mathbf{F}$. Then

$$\begin{aligned} d\omega_{\mathbf{F}} &= 2dz \wedge dx - 5dz \wedge dy - 5dy \wedge dz \\ &= -2dx \wedge dz. \end{aligned}$$

Since $\mathcal{S}$ has no variation in the z parameter,

$$\iint_{\mathcal{S}} d\omega_{\mathbf{F}} = 0.$$

□

Remark 4.4.3. (Green's theorem). Observe that if $\mathcal{S}$ is a region $\Omega \subset \mathbb{R}^2$, and $\mathbf{F} : \Omega \to \mathbb{R}^2$, then $\text{curl}(\mathbf{F}) \cdot d\mathbf{S} = \text{curl}(\mathbf{F}) \cdot \mathbf{k}dA$, and we recover Green's theorem

$$\int_{\partial \Omega} \mathbf{F} \cdot d\mathbf{r} = \iint_{\Omega} \text{curl}(\mathbf{F}) \cdot \mathbf{k}dA.$$

Example 4.4.4. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) := 2z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$. Let $\mathbb{S}^2_+ := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1,\ z \geq 0\}$. Verify Stokes' theorem in this case.

Solution. We want to verify that the result obtained from computing the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ coincides with the result obtained from evaluating the surface integral $\int_{\mathbb{S}^2_+} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S}$. Let us first compute the line integral: The boundary of $\mathbb{S}^2_+$ is given by the curve $\mathcal{C} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = 1,\ z = 0\}$. This is parametrized by the curve $\alpha : [0, 2\pi] \to \mathbb{R}^3$ defined by $\alpha(t) := (\cos(t), \sin(t), 0)$. Hence,

$$\begin{aligned}\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} (0, \cos(t), \sin(t)) \cdot (-\sin(t), \cos(t), 0) dt \\ &= \int_0^{2\pi} \cos^2(t) dt \;=\; \pi.\end{aligned}$$

Let us now compute the surface integral: The curl of $\mathbf{F}$ is $\mathrm{curl}(\mathbf{F}) = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$. In spherical coordinates, $\alpha : [0, 2\pi] \times [0, \pi/2] \to \mathbb{R}^2$, where $\alpha(\vartheta, \phi) = \sin(\vartheta)\cos(\phi)\mathbf{i} + \sin(\vartheta)\sin(\phi)\mathbf{j} + \cos(\vartheta)\mathbf{k}$. Hence,

$$\alpha_\phi \times \alpha_\vartheta = \sin^2(\phi)\cos(\vartheta)\mathbf{i} + \sin^2(\phi)\sin(\vartheta)\mathbf{j} + \sin(\phi)\cos(\phi)\mathbf{k},$$

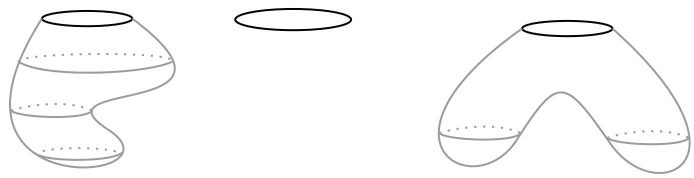
and we compute

$$\begin{aligned}\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) d\mathbf{S} &= \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \big(\sin^2(\phi)(\cos(\vartheta) + 2\sin(\vartheta)) + \sin(\phi)\cos(\phi)\big)\, d\phi d\vartheta \\ &= \int_0^{2\pi} \left(\frac{1}{4}(2 + \pi\cos(\vartheta) + 2\pi\sin(\vartheta)\right) d\vartheta \;=\; \pi.\end{aligned}$$

□

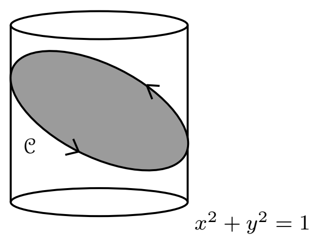
Computational use of Stokes' theorem. Green's theorem permits us to compute a double integral of a 2–form (i.e., $d\omega_{\mathbf{F}}$) in place of a line integral of a 1–form (i.e., $\omega_{\mathbf{F}}$), and conversely. This has useful implications for computations, especially if $d\omega_{\mathbf{F}} = 0$, or $d\omega_{\mathbf{F}}$ is much simpler to integrate than $\omega_{\mathbf{F}}$. The same is true for Stokes' theorem.

What often makes Stokes' theorem more difficult, however, in contrast with Green's theorem, is the choice of surface. Indeed, if $\mathcal{C}$ is a curve in $\mathbb{R}^2$, Green's theorem is applied to the region Ω with $\partial\Omega = \mathcal{C}$; there is no flexibility here. For Stokes' theorem, however, given a curve $\mathcal{C}$ in $\mathbb{R}^3$, there are many surfaces $\mathcal{S}$ with $\partial\mathcal{S} = \mathcal{C}$:



The problem remains: *What surface do we choose?* This question often has an easy answer: Choose the surface which is simplest to compute with.

Example 4.4.5. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field $\mathbf{F}(x, y, z) := -y^2\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$, and let $\mathcal{C}$ be the curve given by the intersection of $y + z = 2$ and $x^2 + y^2 = 1$. Orient $\mathcal{C}$ to be counterclockwise when viewed from above.



Evaluate

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

using Stokes' theorem.

SOLUTION. The curl of $\mathbf{F}$ is calculated to be

$$\mathrm{curl}(\mathbf{F}) \quad = \quad (1 + 2y)\mathbf{k}.$$

There are many surfaces with boundary $\mathcal{C}$, but the most convenient choice is the elliptical region $\mathcal{S}$ given in the plane $y + z = 2$ that is bounded by $\mathcal{C}$. If we orient S upward, then $\mathcal{C}$ has the induced positive orientation. The projection Ω of $\mathcal{S}$ on the (x, y)–plane is the disk $x^2 + y^2 \leq 1$. Hence, with $z = g(x, y) = 2 - y$, we have

$$\begin{aligned}
\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S} \\
&= \iint_{\Omega} (1 + 2y) dA \\
&= \int_0^{2\pi} \int_0^1 (1 + 2r \sin\vartheta) r dr d\vartheta \\
&= \int_0^{2\pi} \left[\frac{1}{2} r^2 + \frac{2}{3} r^3 \sin\vartheta \right]_0^1 d\vartheta \;=\; \pi.
\end{aligned}$$

□

Example 4.4.6. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) = -y^3\mathbf{i} + x^3\mathbf{j} - z^3\mathbf{k}.$$

Let $\mathcal{C}$ be the curve given by intersecting the cylinder $x^2+y^2 = 1$ and the plane $2x+2y+z = 3$. Compute

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$$

using Stokes' theorem.

Solution. Let $\omega_{\mathbf{F}} = -y^3dx + x^3dy - z^3dz$ be the 1–form associated to $\mathbf{F}$. Then

$$d\omega_{\mathbf{F}} \;=\; -3y^2dy \wedge dx + 3x^2dx \wedge dy \;=\; (3x^2 + 3y^2)dx \wedge dy.$$

Hence, by Stokes' theorem,

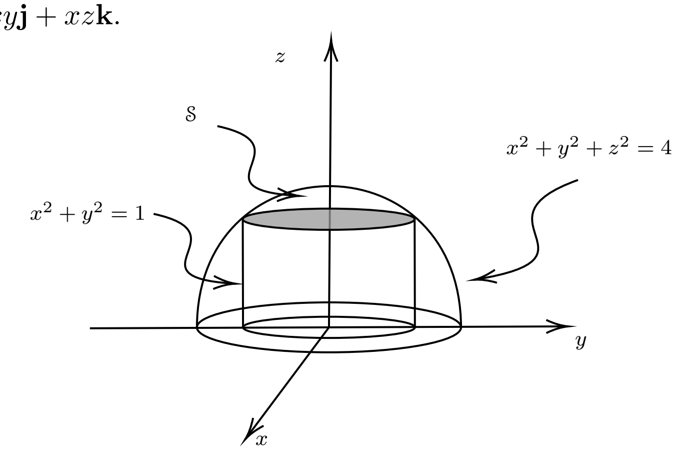
$$\begin{aligned} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} &= \iint_{\mathcal{S}} 3(x^2 + y^2)dx \wedge dy \\ &= \int_0^{2\pi} \int_0^1 3(r^2)rdrd\vartheta \;=\; \frac{3\pi}{2}. \end{aligned}$$

□

Example 4.4.7. Let $\mathcal{S}$ be the surface given by the region of the ball $x^2 + y^2 + z^2 \leq 4$ which lies over the cylinder $x^2 + y^2 = 1$. Compute

$$\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S},$$

where $\mathbf{F} = xy\mathbf{i} + xy\mathbf{j} + xz\mathbf{k}$.



Solution. The boundary curve $\mathcal{C}$ is given by solving the equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$. This is easily seen to yield $z = \sqrt{3}$ (note that we take $z > 0$ since we want to region over the cylinder). Hence, $\mathcal{C}$ is the curve given by $x^2 + y^2 = 1$, $z = \sqrt{3}$. A parametrization for $\mathcal{C}$ is given by $\alpha : [0, 2\pi] \to \mathbb{R}^3$,

$$\alpha(t) \;:=\; \cos(t)\mathbf{i} + \sin(t)\mathbf{j} + \sqrt{3}\mathbf{k}.$$

The derivative is then $\alpha'(t) = -\sin(t)\mathbf{i} + \cos(t)\mathbf{j} + 0\mathbf{k}$. In terms of the parametrization, the vector field reads

$$\mathbf{F}(\alpha(t)) \;=\; \cos(t)\sin(t)\mathbf{i} + \cos(t)\sin(t)\mathbf{j} + \sqrt{3}\cos(t)\mathbf{k}.$$

Hence, by Stokes' theorem

$$\begin{aligned}
\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} &= \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{2\pi} \mathbf{F}(\alpha(t)) \cdot \alpha'(t) dt \\
&= \int_0^{2\pi} [-\cos(t)\sin^2(t) + \cos^2(t)\sin(t)] dt \;=\; 0.
\end{aligned}$$

□

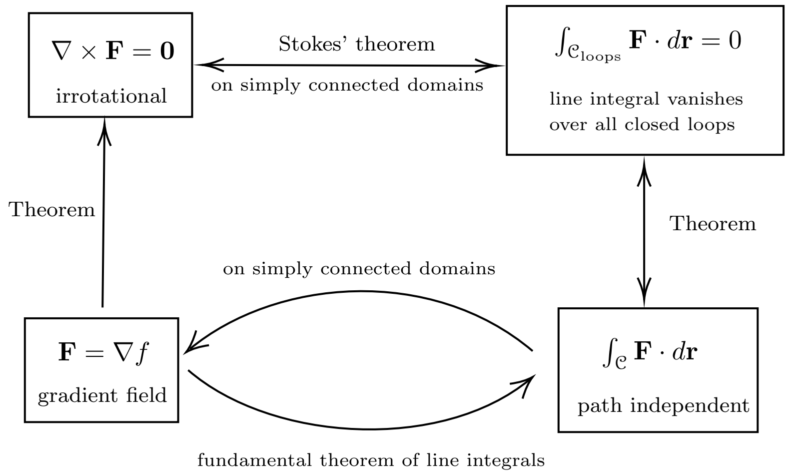
Theorem 4.4.8. Let $\mathbf{F}$ be an irrotational vector field on a simply connected region Ω. Then, for any curve $\mathcal{C}$ in Ω, the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path. Moreover, $\mathbf{F}$ is a gradient field.

PROOF. Suppose $\mathbf{F}$ is an irrotational vector field. Then $\operatorname{curl}(\mathbf{F}) = \mathbf{0}$. By Stokes' theorem *Theorem 4.4.1*, for any closed loop $\mathcal{C}$, we have

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F}) \cdot d\mathbf{S} \;=\; 0.$$

In particular, the line integral over $\mathbf{F}$ over all closed loops $\mathcal{C}$ is zero. By *Theorem 3.2.6*, it follows that the line integral $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ is independent of path. By *Theorem 3.2.16*, it follows that $\mathbf{F}$ is a gradient field. □

Irrotational Vector Fields are Gradient Fields on Simply Connected Regions.



Remark 4.4.9. Recall that in *Remark 3.2.19*, we claimed that the vector field $\mathbf{F} : \mathbb{S}^2 \to \mathbb{R}^3$ defined by

$$\mathbf{F}(x, y, z) := \frac{x}{(x^2+y^2+z^2)^{3/2}}\mathbf{i} + \frac{y}{(x^2+y^2+z^2)^{3/2}}\mathbf{j} + \frac{z}{(x^2+y^2+z^2)^{3/2}}\mathbf{k}$$

is incompressible but not solenoidal.
Computing the surface integral, we see that

$$\iint_{\mathbb{S}^2} \mathbf{F} \cdot d\mathbf{S} \;=\; 4\pi.$$

On the other hand, if $\mathbf{F} = \mathrm{curl}(\mathbf{G})$, then by Stokes' theorem, the surface integral over a closed surface (such as $\mathbb{S}^2$!) vanishes identically. Hence, $\mathbf{F}$ is not solenoidal.

We end the chapter verifying that the definition of curl given earlier, can indeed be interpreted as a measure of the rotation of a vector field:

Theorem 4.4.10. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. Then for any point $p \in \mathbb{R}^3$, we have

$$\mathrm{curl}(\mathbf{F})(p) \;=\; \lim_{|\mathcal{S}| \searrow 0} \frac{1}{|\mathcal{S}|} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}, \tag{4.4.2}$$

where $\mathcal{S}$ is a smooth surface (with surface area $|\mathcal{S}|$) in $\mathbb{R}^3$ containing p, and whose boundary is $\mathcal{C}$.

Proof. Let $A_{\mathcal{S}}(\mathbf{F}) := \frac{1}{2}(\max_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) - \min_{\mathcal{S}} \mathrm{curl}(\mathbf{F}))$ denote the average of the curl on $\mathcal{S}$. By the mean value theorem for integrals, we have

$$\iint_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S} \;=\; A_{\mathcal{S}}(\mathbf{F})(p) \cdot |\mathcal{S}|.$$

Hence, by Stokes' theorem, we have

$$A_{\mathcal{S}}(\mathbf{F})(p) = \frac{1}{|\mathcal{S}|} \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}.$$

A standard argument involving ε's and δ's completes the proof. $\square$

Exercises

1. Use Stokes' theorem to evaluate $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$, where

(i) $\mathbf{F}(x, y, z) = (x - y)\mathbf{i} + (y - z)\mathbf{j} + z^2\mathbf{k}$ and $\mathcal{C}$ is the triangle with vertices $(1, 0, 0)$, $(0, 1, 3)$ and $(0, 2, -1)$.

(ii) $\mathbf{F}(x, y, z) = e^{-x}\mathbf{i} - e^{-y}\mathbf{j} + e^{-z}\mathbf{k}$ and $\mathcal{C}$ is the boundary of the paraboloid $z = 2 - x^2 - y^2$ in the first orthant.

2. Let $\mathbf{F}(x, y, z) := xe^y\mathbf{i} + ye^z\mathbf{j} + ze^x\mathbf{k}$ and let $\mathbb{H} := \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1, z \geq 0\}$. Evaluate the surface integral

$$\int_{\mathbb{H}} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

3. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. For any sphere $\mathbb{S}^2(r) \subset \mathbb{R}^3$, compute the surface integral

$$\int_{\mathbb{S}^2} \mathrm{curl}(\mathbf{F}) \cdot d\mathbf{S}.$$

4. Let $\Omega \subseteq \mathbb{R}^3$ be a surface with boundary $\partial\Omega$. Let $u, v : \mathbb{R}^3 \to \mathbb{R}$ be two smooth functions. Show that

$$\int_{\partial\Omega} (u\nabla v) \cdot d\mathbf{r} = \int_{\Omega} (\nabla u \times \nabla v) \cdot d\mathbf{S}.$$

5. Let $\mathcal{C}$ be a smooth curve in $\mathbb{R}^3$. Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function. Show that

$$\int_{\mathcal{C}} (f\nabla f) \cdot d\mathbf{r} \quad = \quad 0.$$

6. Let $\mathcal{C}$ be a smooth curve in $\mathbb{R}^3$. Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be two smooth functions. Show that

$$\int_{\mathcal{C}} (f\nabla g + f\nabla f) \cdot d\mathbf{r} = 0.$$

7. Let $\mathcal{C}$ be the curve given by the intersection of the plane whose normal vector is $\mathbf{n} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, with the cylinder $x^2 + y^2 = 4$. Orient this curve anti-clockwise, when viewed from above.

(i) Determine the equation of the plane.

(ii) Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$ using Stokes' theorem.

8. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field

$$\mathbf{F}(x, y, z) := (x^3 - axz^2)\mathbf{i} + (x^2y + bz)\mathbf{j} + cy^2z\mathbf{k}.$$

Let $\mathcal{C}$ be the curve given by the intersection of the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ with the plane $x + y + z = 0$. Determine the values of $a, b, c \in \mathbb{R}$ such that

$$\int_S \mathbf{F} \cdot d\mathbf{S}$$

is independent of the surface S whose boundary is $\partial S = \mathcal{C}$.

9. Determine (with justification) whether the following statements are true or false:

(i) Gradient fields are irrotational on simply connected domains.
(ii) Irrotational vector fields are gradient fields on convex domains.
(iii) If $\mathbf{F}$ is an irrotational vector field, then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} = 0$ for all curves $\mathcal{C}$.

10. Use Stokes' theorem to evaluate

$$\int_{\mathcal{C}} ydx + zdy + xdz,$$

where $\mathcal{C}$ is the intersection of $x^2 + y^2 + z^2 = 1$ and the plane $x + y + z = 0$.

11. Let $\mathcal{C}$ be the triangle with vertices $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) \ := \ (x + y^2)\mathbf{i} + (y + z^2)\mathbf{j} + (z + x^2)\mathbf{k}.$$

Compute $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}$.

12. Use Stokes' theorem to evaluate $\int_{\mathcal{C}} \omega$, where

$$\omega \ := \ x^2zdx + xy^2dy + z^2dz,$$

and $\mathcal{C}$ is the curve of intersection of the plane $x + y + z = 1$ and the cylinder $x^2 + y^2 = 9$.

13. Let $\mathcal{S}$ be the surface given by the part of the paraboloid which lies above the plane $z = 1$, oriented upward. Compute $\int_{\partial \mathcal{S}} \omega$, where $\omega = y^2dx + xdy + z^2dz$.

14. Compute the work done by the force

$$\mathbf{F}(x, y, z) = z^2\mathbf{i} + 2xy\mathbf{j} + 4y^2\mathbf{k}$$

to move a particle along the triangular path $\mathcal{C}$ whose vertices are $(1, 0, 0)$, $(1, 2, 1)$, and $(0, 2, 1)$.

15. Suppose $\mathcal{S}$ is a smooth surface satisfying the assumptions of Stokes' theorem. Let f and g be smooth functions on $\mathbb{R}^3$. Show that

(a)
$$\int_{\mathcal{C}} (f\nabla g) \cdot d\mathbf{r} \;=\; \iint_{\mathcal{S}} (\nabla f \times \nabla g) \cdot d\mathbf{S}.$$

(b)
$$\int_{\mathcal{C}} (f\nabla f) \cdot d\mathbf{r} \;=\; 0.$$

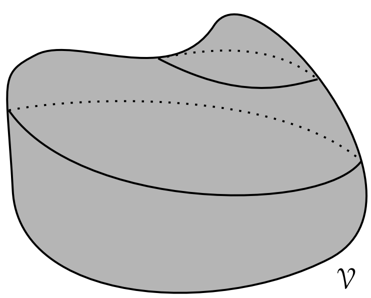
(c)
$$\int_{\mathcal{C}} (f\nabla g + g\nabla f) \cdot d\mathbf{r} \;=\; 0.$$

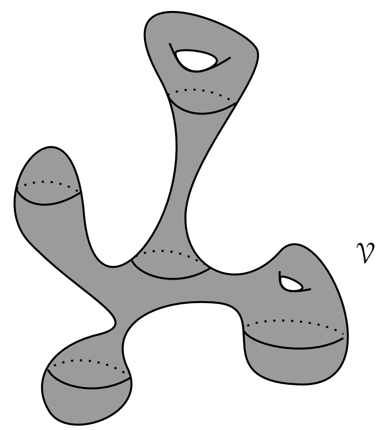
4.5. The Divergence Theorem and Surface Independence

Green's theorem related the double integral of the curl of a two-dimension vector field $\mathbf{F}$, i.e., $\iint_\Omega \operatorname{curl}(\mathbf{F})dA$, to the line integral of $\mathbf{F}$ over the curve which is the boundary of Ω, i.e., $\int_{\partial\Omega} \mathbf{F}\cdot d\mathbf{r}$. The Stokes' theorem from the previous section extends this (more or less) without change: $\iint_{\mathcal{S}} \operatorname{curl}(\mathbf{F})\cdot d\mathbf{S} = \int_{\partial\mathcal{S}} \mathbf{F}\cdot d\mathbf{r}$. Of course, Green's theorem relates the line integral to a double integral, while Stokes' theorem relates the surface integral to a line integral, but this change is minor – Green's theorem is merely a specific instance of Stokes' theorem.

We saw that the curl of a vector field is a particular notion of derivative for a vector field. Indeed, if $\omega_{\mathbf{F}}$ is the 1–form associated to $\mathbf{F}$, then the 1–form associated to $\operatorname{curl}(\mathbf{F})$ just $\star d\omega_{\mathbf{F}}$. Stokes' theorem then merely translates to

$$\iint_{\mathcal{S}} d\omega_{\mathbf{F}} = \int_{\partial\mathcal{S}} \omega_{\mathbf{F}}.$$





We have seen another notion of derivative for a vector field, however, namely, the divergence $\operatorname{div}(\mathbf{F})$. In terms of forms, the divergence is merely the vector field associated to the 1–form $\star d \star \omega_{\mathbf{F}}$. In the same way that we obtained Stokes' theorem (and hence, Green's theorem), we obtain the divergence theorem:

Theorem 4.5.1. (Divergence theorem). Let $\mathcal{V} \subseteq \mathbb{R}^3$ be a region in $\mathbb{R}^3$ with boundary $\partial\mathcal{V}$. Let $\mathbf{F} : \mathcal{V} \to \mathbb{R}^3$ be a smooth vector field on Ω. Then

$$\iint_{\partial\mathcal{V}} \mathbf{F}\cdot d\mathbf{S} \quad = \quad \iiint_{\mathcal{V}} \operatorname{div}(\mathbf{F})dV.$$

PROOF. Let $\omega_{\mathbf{F}} = Pdx + Qdy + Rdz$ be the 1–form associated to $\mathbf{F}$. Then from **??**, we know that $\operatorname{div}(\mathbf{F})dV = d\star\omega_{\mathbf{F}}$. Hence, if

$$\iint_{\partial\mathcal{V}} \omega_{\mathbf{F}} \;=\; \iiint_{\mathcal{V}} d\omega_{\mathbf{F}}, \tag{4.5.1}$$

then

$$\iint_{\partial\mathcal{V}} \mathbf{F}\cdot d\mathbf{S} \;=\; \iint_{\partial\mathcal{V}} \omega_{\mathbf{F}} \;=\; \iiint_{\mathcal{V}} d\omega_{\mathbf{F}} \;=\; \iiint_{\mathcal{V}} \operatorname{div}(\mathbf{F})dV$$

It remains to (4.5.1), which will be treated later. □

Example 4.5.2. Compute the flux of the vector field $\mathbf{F}:\mathbb{R}^3\to\mathbb{R}^3$, where

$$\mathbf{F}(x,y,z) := z\mathbf{i} + y\mathbf{j} + x\mathbf{k},$$

over the unit sphere $\mathbb{S}^2 := \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 = 1\}$.

SOLUTION. The divergence of $\mathbf{F}$ is

$$\operatorname{div}(\mathbf{F}) \;=\; \partial_x(z) + \partial_y(y) + \partial_z(x) \;=\; 1.$$

The sphere $\mathbb{S}^2$ is the boundary of the ball $\mathbb{B}^2 := \{(x,y,z)\in\mathbb{R}^3 : x^2+y^2+z^2 \le 1\}$. Hence,

$$\begin{aligned}\iint_{\mathbb{S}^2} \mathbf{F}\cdot d\mathbf{S} \;=\; \iiint_{\mathbb{B}^2} \operatorname{div}(\mathbf{F})dV \;&=\; \iiint_{\mathbb{B}^2} 1dV \\ &=\; \operatorname{vol}(\mathbb{B}^2) \;=\; \frac{4\pi}{3}.\end{aligned}$$

□

Example 4.5.3. Let $\mathbf{F}:\mathbb{R}^3\to\mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x,y,z) = (4x + y^{102} - z^{23})\mathbf{i} + (y^2 + \sin(xe^{z^{10}}))\mathbf{j} + (z+1)\mathbf{k}.$$

Suppose $\mathcal{V} := \{(x,y,z)\in\mathbb{R}^3 : 0\le x\le 1,\ 0\le y\le 3,\ 0\le z\le 2\}$. Compute

$$\iint_{\partial\mathcal{V}} \mathbf{F}\cdot d\mathbf{S}.$$

SOLUTION. Compute the divergence:

$$\operatorname{div}(\mathbf{F}) \;=\; 5 + 2y.$$

By the divergence theorem,

$$\begin{aligned}
\iint_{\partial\mathcal{V}} \mathbf{F}\cdot d\mathbf{S} &= \iiint_{\mathcal{V}} \operatorname{div}(\mathbf{F})dV \\
&= \int_0^1\int_0^3\int_0^2 (5+2y)dxdydz \\
&= 2\int_0^3 (5+2y)dy \\
&= 2\left[5y+y^2\right]_0^3 \\
&= 2(15+9) \;=\; 48.
\end{aligned}$$

□

Example 4.5.4. Let $\mathbf{F}:\mathbb{R}^3\to\mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x,y,z) \;:=\; xy^2\mathbf{i}+x^2y\mathbf{j}+(x^2+y^2)z^2\mathbf{k}.$$

Let $\mathcal{S}$ be the surface given by the boundary of the solid cylinder $\mathcal{V}$, specified by the inequalities $x^2+y^2\le 9$ and $0\le z\le 1$. Compute

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}.$$

Solution. The divergence of $\mathbf{F}$ is $\operatorname{div}(\mathbf{F}) = y^2+x^2+2z(x^2+y^2) = (x^2+y^2)(1+2z)$. Hence, by the divergence theorem

$$\begin{aligned}
\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S} &= \iiint_{\mathcal{V}} \operatorname{div}(\mathbf{F})dV \\
&= \int_0^1\int_0^{2\pi}\int_0^3 r^2(1+2z)rdrd\vartheta dz \\
&= 2\pi\left(\int_0^1 (1+2z)dz\right)\left(\int_0^3 r^3dr\right) \\
&= 81\pi.
\end{aligned}$$

□

Example 4.5.5. Let $\mathbf{F}:\mathbb{R}^3\to\mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x,y,z) \;=\; xz^2\mathbf{i}+(x^2y-z^3)\mathbf{j}+(2xy+y^2z+e^{\cos(y)})\mathbf{k}.$$

Let $\mathcal{V} := \{(x,y,z)\in\mathbb{R}^3 : z^2+y^2+z^2\le 1,\ z\ge 0\}$. Evaluate

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}.$$

SOLUTION. The divergence of $\mathbf{F}$ is

$$\operatorname{div}(\mathbf{F}) = z^2 + x^2 + y^2.$$

By the divergence theorem,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \quad = \quad \iiint_{\mathcal{V}} (x^2 + y^2 + z^2) dV.$$

Using spherical coordinates, we have $x^2+y^2+z^2 = r^2$, and $dV = r^2 \sin(\phi) dr d\phi d\vartheta$. Moreover, the region is given by $0 \leq r \leq 1$, $0 \leq \vartheta \leq 2\pi$, and $0 \leq \phi \leq \frac{\pi}{2}$. Therefore,

$$\begin{aligned} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \quad &= \quad \iiint_{\mathcal{V}} (x^2 + y^2 + z^2) dV \\ &= \quad \int_0^{2\pi} \int_0^{\frac{\pi}{2}} \int_0^1 r^4 \sin(\phi) dr d\phi d\vartheta \quad = \quad \frac{2\pi}{5}. \end{aligned}$$

□

Theorem 4.5.6. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field. For any point $p \in \mathbb{R}^3$, let $\mathcal{V}$ be a solid region enclosing the point p. Let $\operatorname{vol}(\mathcal{V}) := \iiint_{\mathcal{V}} dV$ denote the volume $\mathcal{V}$. Then

$$\operatorname{div}(\mathbf{F})(p) \quad = \quad \lim_{\operatorname{vol}(\mathcal{V}) \searrow 0} \frac{1}{\operatorname{vol}(\mathcal{V})} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}, \tag{4.5.2}$$

where $\mathcal{S}$ is the boundary of the solid region $\mathcal{V}$.

PROOF. Let $A_{\mathcal{V}}(\mathbf{F}) := \frac{1}{2}(\max_{\mathcal{V}} \operatorname{div}(\mathbf{F}) - \min_{\mathcal{V}} \operatorname{div}(\mathbf{F}))$ denote the average value of the divergence on $\mathcal{V}$. By the mean value theorem for integrals, we can write

$$\iiint_{\mathcal{V}} \operatorname{div}(\mathbf{F}) dV \quad = \quad A_{\mathcal{V}}(\mathbf{F}) \cdot \operatorname{vol}(\mathcal{V}).$$

Hence, by the divergence theorem,

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} \quad = \quad \iiint_{\mathcal{V}} \operatorname{div}(\mathbf{F}) dV \quad = \quad A_{\mathcal{V}}(\mathbf{F}) \cdot \operatorname{vol}(\mathcal{V}).$$

Therefore,

$$A_{\mathcal{V}}(\mathbf{F}) \quad = \quad \frac{1}{\operatorname{vol}(\mathcal{V})} \iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

A standard ε–δ argument furnishes the proof. □

Remark 4.5.7. The equality (4.5.2) can be used to define the divergence. Observe that this is intuitively what is understood to be the divergence in physics. Indeed, the quantity $\frac{1}{\mathrm{vol}(\mathcal{V})}\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ represents the flux or net outflow per unit volume of the vector field $\mathbf{F}$ over the surface $\mathcal{S}$.
If the divergence is positive in a neighborhood of the point p, then the outflow is positive – the point p is then referred to as a *source*. On the other hand, if the divergence is negative in a neighborhood of the point p, the outflow is negative, and the point p is referred to as a *sink*.

Gauss' law. Let $\mathbf{E}$ be an electric field with vacuum permittivity ε_0. Denote by ρ the volume charge density. Gauss' law states that the surface integral of $\mathbf{E}$ over a closed surface $\mathcal{S}$ is given by

$$\iint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\varepsilon_0},$$

where Q is the charge. The divergence theorem offers a different expression for Gauss' law. Indeed, let $\mathcal{V} \subseteq \mathbb{R}^3$ be the region such that $\partial\mathcal{V} = \mathcal{S}$. By the divergence theorem,

$$\iiint_{\mathcal{V}} \mathrm{div}(\mathbf{E}) dV \quad = \quad \iint_{\mathcal{S}} \mathbf{E} \cdot d\mathbf{S} \; = \; \frac{Q}{\varepsilon_0}.$$

The charge Q is given by $Q = \iiint_{\mathcal{V}} \rho dV$, and therefore,

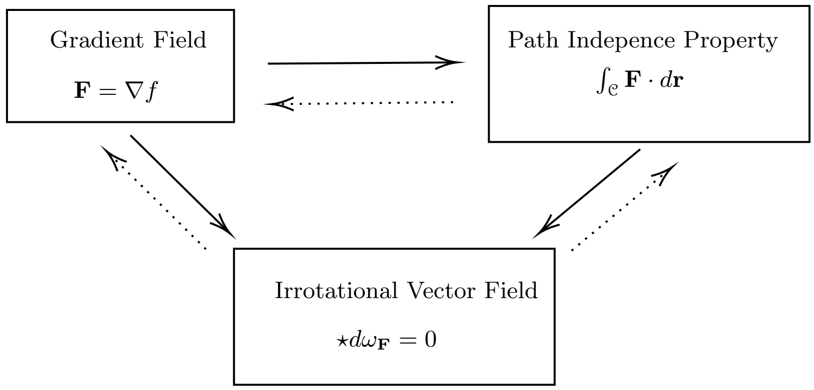
$$\iiint_{\mathcal{V}} \mathrm{div}(\mathbf{E}) dV = \iiint_{\mathcal{V}} \frac{\rho}{\varepsilon_0} dV.$$

Since this holds for all regions $\mathcal{V}$ with boundary $\partial\mathcal{S}$, the integrands must coincide, and therefore,

$$\mathrm{div}(\mathbf{E}) = \frac{\rho}{\varepsilon_0}.$$

Reminder 4.5.8. Recall that in chapter 3 we showed that gradient fields could be characterized by the path independence property. Moreover, if the region on which the vector field is defined is simply connected, then these notions are, moreover, equivalent to being irrotational. We have firmly established that curl($\mathbf{F}$) is the vector field associated to the 1–form $\star d\omega_{\mathbf{F}}$. From the perspective of Stokes' and Green's theorem, however, it is clear that the use of the Hodge $\star$–operator in this definition is to ensure that it coincides with the classical notion of curl. In other words, one can make the argument that curl should really be defined as $d\omega_{\mathbf{F}}$, which is a 2–form on $\mathbb{R}^3$.

The question of whether an irrotational vector field is a gradient field is then equivalent to asking: If $d\omega_{\mathbf{F}} = 0$, does there exist a smooth function $f \in \Lambda^0(\mathbb{R}^3)$ such that $\omega_{\mathbf{F}} = df$? Moreover, this question can be addressed by looking at whether $\int_{\mathcal{C}} \omega_{\mathbf{F}} = 0$ for all closed curves $\mathcal{C}$.



Dashed arrows indicate that the implication holds on simply connected domains.

On the other hand, we have seen the divergence $\operatorname{div}(\mathbf{F})$ is given by $\star d \star \omega_{\mathbf{F}}$. In the same way that we can think of curl simply as $d\omega_{\mathbf{F}}$, we can think of the divergence as simply $d \star \omega_{\mathbf{F}}$. The question of whether an incompressible vector field is a solenoidal vector field is then equivalent to asking: If $d \star \omega_{\mathbf{F}} = 0$, does there exist a smooth 1–form $\alpha \in \Lambda^1(\mathbb{R}^3)$ such that $\star\omega_{\mathbf{F}} = d\alpha$?

Following this string of analogies, we would not expect this question to be addressed by looking at whether $\int_{\mathcal{C}} \omega_{\mathbf{F}} = 0$ for all closed curves $\mathcal{C}$. Rather, we would expect that $\iint_{\mathcal{S}} \star\omega_{\mathbf{F}} = 0$ for all closed surfaces $\mathcal{S}$.

Theorem 4.5.9. Let $\mathbf{F} : \mathcal{S} \to \mathbb{R}^3$ be a solenoidal vector field on a closed surface $\mathcal{S}$. Then

$$\iint_{\mathcal{S}} \star\omega_{\mathbf{F}} = 0.$$

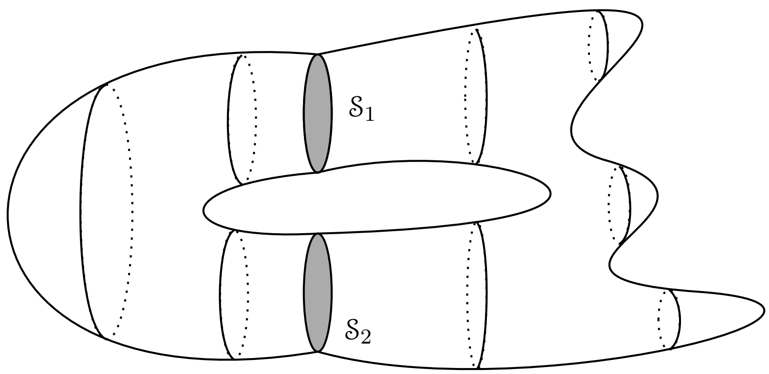
Proof. If $\mathbf{F}$ is a solenoidal vector field, then $\mathbf{F}$ is incompressible. Hence, by the divergence theorem

$$\iint_{\mathcal{S}} \star\omega_{\mathbf{F}} \quad = \quad \iiint_{\mathcal{V}} d \star \omega_{\mathbf{F}} \;=\; 0.$$

$\square$

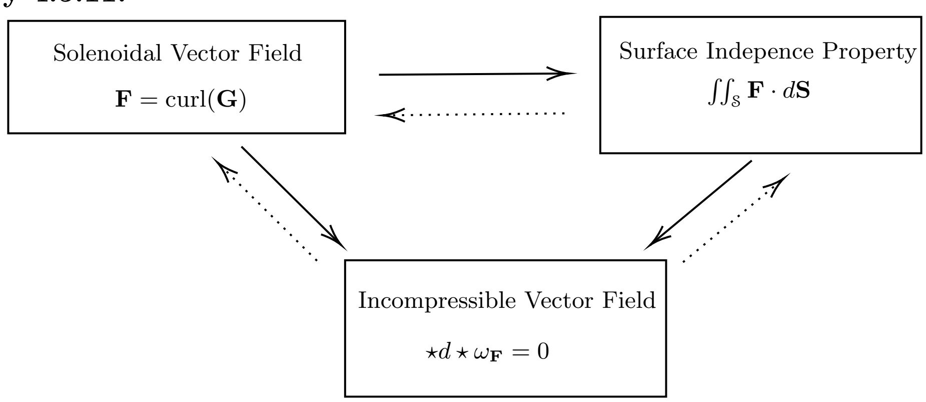
Definition 4.5.10. We say that a vector field $\mathbf{F}$ has the *surface independence property* if, for all smooth surfaces $\mathcal{S}_1, \mathcal{S}_2$, which have the same boundaries, we have

$$\iint_{\mathcal{S}_1} \star\omega_{\mathbf{F}} \;=\; \iint_{\mathcal{S}_2} \star\omega_{\mathbf{F}}.$$



Two regions in $\mathbb{R}^3$ with boundary $\mathcal{S}_1 \cup \mathcal{S}_2$.

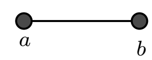
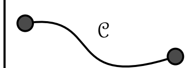
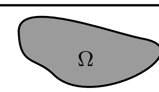
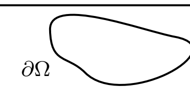
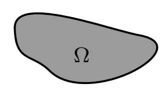
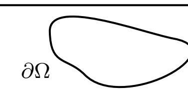
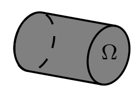
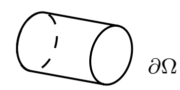
Summary 4.5.11.



Dashed arrows indicate that the implication holds on $\mathbb{R}^3$.

Remark 4.5.12. Let us emphasize that it is not the case that every incompressible vector field is solenoidal on a simply connected region. It is a little more challenging to describe the type of regions for which an incompressible vector field is solenoidal (other than, of course, $\mathbb{R}^3$). To give some detail, however, one can define a region Ω to be simply connected if every loop in Ω can be contracted to a point. In more precise language, every continuous map $\mathbb{S}^1 \to \Omega$ is homotopic to a constant map. To describe the types of regions Ω in which every incompressible vector field is solenoidal, we would require that every continuous map $\mathbb{S}^2 \to \Omega$ is homotopic to a constant map. This is beyond the scope of the present course, outside of the two examples: every incompressible vector field on $\mathbb{R}^3$ is solenoidal; there is a vector field on $\mathbb{S}^2$ which is incompressible, but not solenoidal.

Summary of Main Theorems.

Theorem	Statement	Notion of derivative	Region of integration	Boundary
Fundamental theorem of calculus	$\int_a^b f'(x)dx = f(b) - f(a)$	$f'(x)$		
Fundamental theorem of line integrals	$\int_{\mathcal{C}} \nabla f dr = f(b) - f(a)$	∇f		
Green's theorem	$\int_{\partial\Omega} \mathbf{F} \cdot d\mathbf{r} = \int_{\Omega} \mathrm{curl}(\mathbf{F}) \cdot \mathbf{k} dA$	$\mathrm{curl}(\mathbf{F}) \cdot \mathbf{k}$		
Stokes' theorem	$\int_{\partial\mathcal{S}} \mathbf{F} \cdot d\mathbf{r} = \int_{\mathcal{S}} \mathrm{curl}(\mathbf{F}) dA$	$\mathrm{curl}(\mathbf{F})$		
Divergence theorem	$\int_{\partial\mathcal{V}} \mathbf{F} \cdot d\mathbf{S} = \int_{\mathcal{V}} \mathrm{div}(\mathbf{F}) dV$	$\mathrm{div}(\mathbf{F})$		

Exercises

1. Verify the divergence theorem for the vector fields $\mathbf{F}$ and regions $\mathcal{V}$ given by

(i) $\mathbf{F}(x, y, z) = 2x\mathbf{i} + (x - y)\mathbf{j} + (1 - xz)\mathbf{k}$ and $\mathcal{V}$ is the solid cylinder $x^2 + y^2 \leq 9$, $-1 \leq z \leq 1$.

(ii) $\mathbf{F}(x, y, z) = x^2\mathbf{i} - y^2\mathbf{j} + z^2\mathbf{k}$ and $\mathcal{V}$ is the unit ball $x^2 + y^2 + z^2 \leq 1$.

(iii) $\mathbf{F}(x, y, z) = xy^2\mathbf{i}+yx^2\mathbf{j}+yz^2\mathbf{k}$ and $\mathcal{V}$ is the cube $-1 \leq x \leq 1$, $-1 \leq y \leq 2$, $0 \leq z \leq 3$.

2. Use the divergence theorem to calculate the following surface integrals:

(i) $\mathbf{F}(x, y, z) = e^x \cos(y)\mathbf{i} - e^{-x} \sin(y)\mathbf{j} + z\mathbf{k}$, where $\mathcal{S}$ is the surface of the rectangle bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 2$, $z = 0$ and $z = 4$.

(ii) $\mathbf{F}(x, y, z) = x^3\mathbf{i}+y^3\mathbf{j}+z^3\mathbf{k}$, and $\mathcal{S}$ is the surface bounded by the cylinder $x^2+y^2 = 1$ and the planes $z = -1$ and $z = 3$.

(iii) $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathcal{S}$ is the ellipsoid $4x^2 + 9y^2 + 6z^2 = 36$.

(iv) $\mathbf{F}(x, y, z) = (x^2+y)\mathbf{i}+2yz\mathbf{j}+(1-e^{-z})\mathbf{k}$ and $\mathcal{S}$ is the cylinder $y^2+z^2 = 4$, $-1 \leq x \leq 1$.

3. Let $\mathbf{F} : \Omega \to \mathbb{R}^3$ be a vector field which is always normal to $\partial\Omega$. Show that

$$\int_\Omega \operatorname{curl}(\mathbf{F}) dV = 0.$$

4. Suppose $\int_{\mathcal{C}} \mathbf{E} \cdot d\mathbf{r} = -\frac{1}{c}\frac{\partial}{\partial t} \int_\Omega \mathbf{H} \cdot d\mathbf{S}$, where Ω is any surface bounded by the closed curve $\mathcal{C}$. Show that

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{H}}{\partial t}.$$

5. Let $\mathbf{F} : \mathbb{S}^2 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) = (x - 3y)\mathbf{i} + (4 + 9z^3)\mathbf{j} + (y^2 - 10z)\mathbf{k}.$$

(i) Compute $\operatorname{div}(\mathbf{F})$.

(ii) Compute $\int_{\mathbb{S}^2} \mathbf{F} \cdot d\mathbf{S}$.

6. Let Σ be some closed surface in $\mathbb{R}^3$. Let ρ denote the density of some fluid in Σ. The mass of the fluid in Σ is then given by

$$\int_\Sigma \rho dV,$$

while the total mass of the fluid flowing out of Σ in unit time is given by

$$\int_{\partial\Sigma} \rho\mathbf{v} \cdot d\mathbf{S}.$$

Show that

$$\frac{\partial\rho}{\partial t} + \operatorname{div}(\rho\mathbf{v}) = 0.$$

7. Let Σ be some closed surface in $\mathbb{R}^3$. Let p denote the pressure of a fluid inside Σ. The total force acting on the volume is given by

$$-\int_{\partial\Sigma} pd\mathbf{S}.$$

(i) Determine the function φ such that $-\int_{\partial\Sigma} pd\mathbf{S} = -\int_{\Sigma} \varphi dV$.

(ii) Use Newton's second law of motion to show that

$$\rho\frac{d\mathbf{v}}{dt} = -\nabla p,$$

where $\mathbf{v}$ is the velocity of the fluid, and ρ is the density.

(iii) Show that

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v}\cdot\nabla\mathbf{v} = -\frac{1}{\rho}\nabla p.$$

(iv) Determine the corresponding equation when the fluid experiences a graviational force $\mathbf{g}$.

8. Let $\mathcal{V}$ be a region in $\mathbb{R}^3$ for which the divergence theorem holds. Show that the volume of $\mathcal{V}$ is given by

$$\mathrm{vol}(\mathcal{V}) = \frac{1}{3}\iint_{\partial\mathcal{V}} xdy\wedge dz + ydz\wedge dx + zdx\wedge dy.$$

9. Let $\mathbf{F}(x,y,z) = 2x\mathbf{i} + 3xy\mathbf{j} - yz^2\mathbf{k}$ and let $\mathcal{S}$ be the surface bounded by the planes $x = 0$, $x + y = 2$, $y = 0$, $z = 0$, and $z = 2$. Compute the surface integral

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S}.$$

10. Find the flux of $\mathbf{F}(x,y,z) = xz^2\mathbf{i} + (x^2y - z^3)\mathbf{j} + (2xy + y^2z)\mathbf{k}$ outwards, across the entire surface of the hemisphere bounded by $z = \sqrt{r^2 - x^2 - y^2}$ and $z = 0$.

11. Compute

$$\int_{\mathcal{C}} x^2y^3dx + dy + zdz,$$

where $\mathcal{C}$ is the curve given by traversing along the circle $x^2 + y^2 = r^2$ anti-clockwise.

12. Use the divergence theorem to evaluate the following surface integrals

$$\iint_{\mathcal{S}} \mathbf{F}\cdot d\mathbf{S},$$

where

(i) $\mathbf{F}(x,y,z) = 3y^2z^3\mathbf{i} + 9x^2yz^2\mathbf{j} - 4xy^2\mathbf{k}$ and $\mathcal{S}$ is the surface of the region bounded by the planes $x = 0$, $x = 3$, $y = 0$, $y = 2$, $z = 0$, and $z = 1$.

(ii) $\mathbf{F}(x,y,z) = -xz\mathbf{i} - yz\mathbf{j} + z^2\mathbf{k}$ and $\mathcal{S}$ is the ellipsoid $\frac{x^2}{9} + \frac{y^2}{4} + z^2 = 1$.

(iii) $\mathbf{F}(x,y,z) = 3xy\mathbf{i} + y^2\mathbf{j} - x^2y^4\mathbf{k}$ and $\mathcal{S}$ is the surface of the tetrahedron with vertices $(0,1,0)$, $(0,0,0)$, $(1,0,0)$, and $(0,0,1)$.

(iv) $\mathbf{F}(x, y, z) = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$ and $\mathcal{S}$ is the unit sphere centered at the origin in $\mathbb{R}^3$.

13. Let $\mathbf{F}(x, y, z) = y^2\mathbf{i} + xz\mathbf{j} - (x + y)\mathbf{k}$.

(i) Find the vector field $\mathbf{G}$ such that $\mathbf{F} = \text{curl}(\mathbf{G})$.

(ii) Hence, evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem, where $\mathcal{S}$ is any closed surface.

14. Let $\mathcal{S}$ be the surface of the solid that lies above the xy–plane and below the surface $z = 2 - x^4 - y^4$, where $-1 \le x \le 1$ and $-1 \le y \le 1$. Evaluate

$$\iint_{\mathcal{S}} e^y \tan(z) dx + y\sqrt{3 - y^2} dy + x \sin(y) dz.$$

15. Let $\mathcal{S}$ be the surface of the solid bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z - y + 3 = 0$. Evaluate

$$\iint_{\mathcal{S}} ye^{z^2} dx + y^2 dy + e^{xy} dz.$$

16. Let $\mathbf{F}$ be a vector field such that $\mathbf{F} \ne \text{curl}(\mathbf{G})$ for any vector field $\mathbf{G}$.

(i) What can be concluded about the divergence of $\mathbf{F}$?

(ii) Hence, what can be concluded about the surface integrals $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$?

17. Let $\mathcal{S}$ be the surface $z = 1 - x^2 - y^2$ for $x^2 + y^2 \le 1$, and $\mathbf{F}(x, y, z) = x\mathbf{i} + y\mathbf{j} + \mathbf{k}$.

(i) Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ directly.

(ii) Evaluate the surface integral $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem.

18. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a smooth vector field such that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \pi(r^3 + 2r^4),$$

for every $r > 0$, where $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = r^2$. Determine the value of $\nabla \cdot \mathbf{F}$ at the origin.

19. State which of the following theorem: (**G**) Green's theorem, (**S**) Stokes' theorem, (**D**) Divergence theorem, would be most appropriate to address the following problems:

(i) Computing a line integral by computing a surface integral.

(ii) Compute a double integral by computing a line integral.

(iii) Compute a surface integral by computing a triple integral.

(iv) Showing that a line integral of an irrotational vector field on a simply connected domain is independent of path.

20. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field such that

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S} = \pi(\lambda^3 + 2\lambda^4),$$

where $\mathcal{S}$ is the sphere $x^2 + y^2 + z^2 = \lambda^2$ for all $\lambda > 0$. Determine the value of $\star d \star \omega_{\mathbf{F}}$ at the origin.

21. Let $\mathcal{V}$ be a solution region in $\mathbb{R}^3$ with boundary $\mathcal{S}$. Let $u : \mathbb{R}^3 \to \mathbb{R}$ be a function such that

$$u_{xx} + u_{yy} + u_{zz} = 0$$

everywhere. Show that $\iint_{\mathcal{S}} \nabla u \cdot d\mathbf{S} = 0$.

22. Let $\mathcal{S}$ be the surface of the box $[0, 1] \times [0, 1] \times [0, 2]$. Compute

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

using the divergence theorem, where $\mathbf{F}(x, y, z) = e^x \sin(y)\mathbf{i} + e^x \cos(y)\mathbf{j} + yz^2\mathbf{k}$.

23. Let $\mathcal{S}$ be the surface bounded by the solid cylinder $y^2 + z^2 = 1$ and the planes $x = -1$ and $x = 2$. Compute

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$$

using the divergence theorem, where $\mathbf{F}(x, y, z) = 3xy^2\mathbf{i} + xe^z\mathbf{j} + z^3\mathbf{k}$.

24. Let $\mathcal{S}$ be the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ and

$$\mathbf{F}(x, y, z) \;:=\; xy \sin(z)\mathbf{i} + \cos(xz)\mathbf{j} + 2xyz\mathbf{k}.$$

Compute $\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}$ using the divergence theorem.

25. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) \;:=\; xz^2\mathbf{i} + \left(\frac{1}{3}y^3 + \tan(z)\right)\mathbf{j} + (zx^2 + y^2)\mathbf{k}.$$

Let $\mathcal{S}$ be the top half of the sphere $x^2 + y^2 + z^2 = 1$. Compute

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

26. Let $\mathcal{S}$ be a smooth surface satisfying the assumptions of the divergence theorem. Let $\mathbf{u} \in \mathbb{R}^3$ be a vector and let ν denote the vector normal to the surface $\mathcal{S}$. Show that

$$\iint_{\mathcal{S}} \mathbf{u} \cdot \nu dS \;=\; 0.$$

27. Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F}(x, y, z) \ := \ x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Let $\mathcal{V}$ be a solid region satisfying the assumptions of the divergence theorem. Show that

$$\mathrm{vol}(\mathcal{V}) \ = \ \frac{1}{3}\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S}.$$

28. Let $\mathcal{V}$ be a solid region in $\mathbb{R}^3$ satisfying the assumptions of the divergence theorem. Show that for a smooth function $f : \mathcal{V} \to \mathbb{R}$, we have

$$\iint_{\partial\mathcal{V}} D_\nu f dS \ = \ \iiint_{\mathcal{V}} \nabla^2 f dV.$$

29. Let $\mathcal{V}$ be a solid region in $\mathbb{R}^3$ satisfying the assumptions of the divergence theorem. Show that for smooth functions $f, g : \mathcal{V} \to \mathbb{R}$, we have

$$\iint_{\partial\mathcal{V}} (f\nabla g) \cdot \nu dS \ = \ \iiint_{\mathcal{V}} (f\nabla^2 g + \nabla f \cdot \nabla g) dV.$$

30. Let $\mathcal{V}$ be a solid region in $\mathbb{R}^3$ satisfying the assumptions of the divergence theorem. Show that for smooth functions $f, g : \mathcal{V} \to \mathbb{R}$, we have

$$\iint_{\partial\mathcal{V}} (f\nabla g - g\nabla f) \cdot \nu dS \ = \ \iiint_{\mathcal{V}} (f\nabla^2 g - g\nabla^2 f) dV.$$

31. Let $\mathcal{V}$ be the region bounded by $2x + 2y + z = 6$, $x = 0$, $y = 0$, and $z = 0$. Compute

$$\iint_{\partial\mathcal{V}} \mathbf{F} \cdot d\mathbf{S}$$

using the divergence theorem.

32. Evaluate

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = (z^2 - x)\mathbf{i} - xy\mathbf{j} + 3z\mathbf{k}$ and $\mathcal{S}$ is the surface of the region bounded by $z = 4 - y^2$, $x = 0$, $x = 3$, and the xy–plane.

33. Let $\mathcal{S}$ be the surface of the sphere of radius 3 centered at $(3, -1, 2)$. Compute

$$\iint_{\mathcal{S}} \mathbf{F} \cdot d\mathbf{S},$$

where $\omega_{\mathbf{F}} = (2x + 3z)dx - (xz + y)dy + (y^2 + 2z)dz$ is the 1–form associated to $\mathbf{F}$.

34. Let $\mathcal{S}$ be the surface of the region bounded by the cylinder $x^2 + y^2 = 9$ and the planes $z = 0$ and $z = 3$. Compute

$$\iint_{\mathcal{S}} x dy \wedge dz + y dz \wedge dx + z dx \wedge dy.$$

CHAPTER 5

The High Road to Vector Calculus

"If there is one thing in mathematics that fascinates me more than anything else (and doubtless always has), it is neither 'number' nor 'size,' but always form."

– Alexander Grothendieck

5.1. The Differentiation Theory

Throughout the course, we have looked at the calculus of vector fields and differential forms. The appropriate notion of derivative is given by the exterior derivative

$$d : \Lambda^k \to \Lambda^{k+1},$$

which has the property that the composition

$$\Lambda^k \xrightarrow{d} \Lambda^{k+1} \xrightarrow{d} \Lambda^{k+2}$$

vanishes identically: $d^2 = 0$.

Reminder 5.1.1. The equation $d^2 = 0$ manifests in the fact that the curl of a gradient field vanishes, and the divergence of a solenoidal vector field vanishes. Indeed, recall that the 1–form of a gradient field $\mathbf{F} = \nabla f$ is given by $d\omega_{\mathbf{F}} = df$ and the curl of $\mathbf{F}$ is the vector field associated to the 1–form $\star d\omega_{\mathbf{F}}$. Hence,

$$\operatorname{curl}(\nabla f) \ \sim \ \star d(df) \ = \ \star d^2 f \ = \ 0,$$

where $\sim$ signifies the association between vector fields and 1–forms. Similarly, the 1–form associated to a solenoidal vector field $\mathbf{F} = \operatorname{curl}(\mathbf{G})$ is given by $\omega_{\mathbf{F}} = \star d\omega_{\mathbf{G}}$, and the divergence of a vector field $\mathbf{F}$ is given by $\operatorname{div}(\mathbf{F}) = \star d \star \omega_{\mathbf{F}}$. Therefore,

$$\operatorname{div}(\operatorname{curl}(\mathbf{G})) \ = \ \star d \star (\star d\omega_{\mathbf{G}}) \ = \ \star d \star^2 d\omega_{\mathbf{G}} \ = \ \star d^2 \omega_{\mathbf{G}} \ = \ 0.$$

Example 5.1.2. In general, however, we have seen that irrotational vector fields are *not* necessarily gradient fields. The standard example of an irrotational vector field which fails to be a gradient field is the vector field $\mathbf{F} : \mathbb{R}^2 \backslash \{(0,0)\} \to \mathbb{R}^2$,

$$\mathbf{F}(x,y) \ := \ -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

Such vector fields only exist, however, because $\mathbb{R}^2\backslash\{(0,0)\}$ is not simply connected. That is, if $\mathbf{F} : \Omega \to \mathbb{R}^n$ is a smooth irrotational vector field on a simply connected domain Ω, then $\mathbf{F}$ is a gradient field.

In fact, for regions $\Omega \subset \mathbb{R}^2$, simply connectedness of Ω can be measured by looking at whether every irrotational vector field happens to be a gradient field:

Theorem 5.1.3. Suppose every irrotational vector field $\mathbf{F} : \Omega \to \mathbb{R}^2$ on a region $\Omega \subseteq \mathbb{R}^2$ is a gradient field. Then Ω is simply connected.

Remark 5.1.4. The above theorem fails in higher-dimensions, illustrated by the Alexander horned sphere.

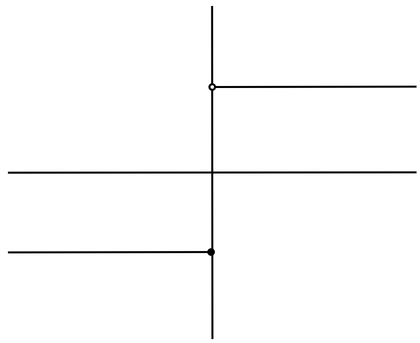
Remark 5.1.5. The idea of using vector fields (or more precisely, differential forms) to measure the properties of a space leads to the theory of *de Rham cohomology.*

The idea is the following: We want to understand the properties of a space Ω by looking at certain invariants of the space which are (preferably) straightforward to calculate.

Example 5.1.6. For instance, suppose we consider an interval $I \subset \mathbb{R}$. An invariant of I is the number of connected components. The interval $(0,3)$ has one connected component, while the interval $(0,1) \cup (2,3) \cup (4,6)$ has three connected components:

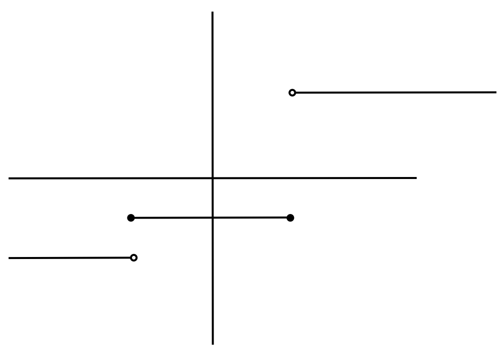
Of course, if we know the interval already, then calculating the number of connected components is trivial. But in general (and this is often the case), we may not know the space explicitly. Invariants are detection mechanisms that allow us to understand something about the space.

Consider now the following problem: Let $f : \Omega \to \mathbb{R}$ be a smooth function which satisfies $f'(x) = 0$ for all $x \in \Omega$. What can be said about f? Any student with a rudimentary understanding of calculus knowns that stationary points of f are given by $f'(x) = 0$, hence, f must be locally constant. It is not necessarily the case, however, that f is a constant function, however, as illustrated by the heaviside function $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$, defined by $f(x) := 1$ for $x > 0$ and $f(x) := -1$ for $x < 0$:



The heaviside function satisfies $f'(x) = 0$ for all $x \in \mathbb{R} \backslash \{0\}$, but is not constant.

Similarly, if $f : \mathbb{R}\backslash\{-1, 1\} \to \mathbb{R}$ satisfies $f'(x) = 0$ for all $x \in \mathbb{R}\backslash\{-1, 1\}$, then f is permitted to assume three distinct values:



Hence, we can measure the number of connected components of an interval $I \subseteq \mathbb{R}$ (or a space Ω more generally) by looking at the failure of locally constant functions (functions $f : I \to \mathbb{R}$ with $f'(x) = 0$ for all $x \in I$) to be constant functions.

The measurement is made more precise as follows: Let $\mathcal{Z}^0(\Omega)$ denote the space of locally constant functions on Ω, and denote by $\mathcal{B}^0(\Omega)$ the subset of constant functions on Ω. From the above discussion, we see that if $\mathcal{B}^0(\Omega) = \mathcal{Z}^0(\Omega)$, i.e., every locally constant function is constant, then Ω is connected. The failure of connectedness is therefore measured by the size of the complement $\mathcal{Z}^0(\Omega)\backslash\mathcal{B}^0(\Omega)$.

It may be the case that both $\mathcal{Z}^0(\Omega)$ and $\mathcal{B}^0(\Omega)$ are infinite sets, so interpretting the "size" of the complement in terms of the cardinality of the set $\mathcal{Z}^0(\Omega)\backslash\mathcal{B}^0(\Omega)$ will result in meaningless expressions of the form $\infty - \infty$. More ground is covered by recognising that both $\mathcal{Z}^0(\Omega)$ and $\mathcal{B}^0(\Omega)$ are (real) vector spaces (with $\mathcal{B}^0(\Omega)$ a vector subspace of $\mathcal{Z}^0(\Omega)$). Vector spaces have a meaningful notion of "size" attached to them, namely, their dimension (the number of vectors in a basis).

If $\mathcal{Z}^0(\Omega)$ has dimension n and $\mathcal{B}^0(\Omega)$ has dimension k, then the complementary subspace of $\mathcal{B}^0(\Omega)$ in $\mathcal{Z}^0(\Omega)$ has dimension $n-k$. This complementary subspace of dimension $n-k$ is the *zeroth de Rham cohomology group*

$$H^0_{\mathrm{DR}}(\Omega) := \mathcal{Z}^0(\Omega)/\mathcal{B}^0(\Omega).$$

More precisely, $H^0_{\mathrm{DR}}(\Omega)$ is a *quotient vector space*, but for our purposes here, we need only understand that the dimension $b_0(\Omega) := \dim H^0_{\mathrm{DR}}(\Omega) \geq 0$ is a measure of the failure of locally constant functions on Ω to be constant functions.

Remark 5.1.7. It may be the case that $\dim \mathcal{Z}^0(\Omega) = \infty$ and $\dim \mathcal{B}^0(\Omega) = \infty$. We will not consider such regions Ω here.

We can repeat the above construction to look at other properties of the space Ω. Let $\mathcal{Z}^1(\Omega)$ denote the space of irrotational vector fields[1] $\mathbf{F} : \Omega \to \mathbb{R}^n$, and let $\mathcal{B}^1(\Omega)$ denote the space of gradient fields on Ω. The failure of every irrotational vector field on Ω to be a gradient field is measured by the *first de Rham cohomology group*

$$H^1_{\mathrm{DR}}(\Omega) \ := \ \mathcal{Z}^1(\Omega)/\mathcal{B}^1(\Omega).$$

Since every irrotational vector field on a simply connected region is a gradient field, we have:

Proposition 5.1.8. Let Ω be a simply connected region. Then $H^1_{\mathrm{DR}}(\Omega) = 0$.

Repeating the construction again, let $\mathcal{Z}^2(\Omega)$ be the space of incompressible vector fields $\mathbf{F} : \Omega \to \mathbb{R}^n$, and let $\mathcal{B}^2(\Omega)$ denote the space of solenoidal vector fields. The failure of an incompressible vector field to be solenoidal is given by the *second de Rham cohomology group*

$$H^2_{\mathrm{DR}}(\Omega) \ := \ \mathcal{Z}^2(\Omega)/\mathcal{B}^2(\Omega).$$

Example 5.1.9. From **??**, we see that $H^2_{\mathrm{DR}}(\mathbb{R}^3) = 0$. From **??**, we see that $H^2_{\mathrm{DR}}(\mathbb{S}^2) \neq 0$.

Remark 5.1.10. One of the important features of these de Rham cohomology groups is that they form just one example of a *cohomology theory*. That is, throughout the notes we have implicitly calculated the de Rham cohomology groups of certain spaces, but these could be calculated in other ways. Hence, we can deduce facts about vector calculus by using calculations of de Rham cohomology groups made by mathematicians in completely different

[1] Observe that to define the curl for vector fields on $\mathbb{R}^n$, one requires the more general definition using forms, namely, $d\omega_{\mathbf{F}}$.

areas of mathematics!

A discussion of other cohomology theories will take us far afield, so we remark only that such theories exist in abundance. To name just a few examples, there is simplicial cohomology, cell cohomology, sheaf cohomology, Čech cohomology, ℓ–adic cohomology, and so on.

Let us proceed more generally, and define the kth de Rham cohomology groups. To this end, for a region $\Omega \subset \mathbb{R}^n$, let

$$\mathcal{Z}^k(\Omega) \ := \ \{\alpha \in \Lambda^k(\Omega) : d\alpha = 0\}$$

denote the set of closed k–forms on Ω. Let

$$\mathcal{B}^k(\Omega) \ := \ \{\beta \in \Lambda^k(\Omega) : \beta = d\gamma \text{ for some } \gamma \in \Lambda^{k-1}(\Omega)\}$$

denote the set of exact k–forms on Ω.

Example 5.1.11. Observe that closed 1–forms correspond to irrotational vector fields since

$$\operatorname{curl}(\mathbf{F}) \ \sim \ \star d\omega_{\mathbf{F}} = 0 \quad \Longleftrightarrow \quad d\omega_{\mathbf{F}}.$$

Exact 1–forms correspond to gradient fields, since $\mathbf{F} = \nabla f$ is equivalent to $\omega_{\mathbf{F}} = df$. Similarly, closed 2–forms correspond to incompressible vector fields since

$$\operatorname{div}(\mathbf{F}) \ = \ \star d \star \omega_{\mathbf{F}} \ = \ 0 \quad \Longleftrightarrow \quad d \star \omega_{\mathbf{F}} \ = \ 0.$$

Exact 2–forms correspond to solenoidal vector fields, since

$$\mathbf{F} \ = \ \operatorname{curl}(\mathbf{G}) \quad \Longleftrightarrow \quad \omega_{\mathbf{F}} \ = \ \star d\omega_{\mathbf{G}} \quad \Longleftrightarrow \quad \star\omega_{\mathbf{F}} \ = \ d\omega_{\mathbf{G}}.$$

In general, the extent to which closed k–forms fail to be exact is measured by the quotient vector spaces:

Definition 5.1.12. Let $\Omega \subset \mathbb{R}^n$ be a region. The *kth de Rham cohomology group*, $0 \leq k \leq n$, is defined to be

$$H^k_{\mathrm{DR}}(\Omega) \ := \ \mathcal{Z}^k(\Omega)/\mathcal{B}^k(\Omega).$$

In particular, if $H^k_{\mathrm{DR}}(\Omega) = 0$, then every closed k–form is an exact k–form.

Example 5.1.13. The de Rham cohomology groups of the n–sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ are

$$H^k_{\mathrm{DR}}(\mathbb{S}^n) \quad = \quad \begin{cases} \mathbb{R}, & k \in \{0, n\}, \\ 0, & \text{otherwise.} \end{cases}$$

In particular, since $H^2_{\mathrm{DR}}(\mathbb{S}^2) \neq 0$, there is a closed 2–form which is not an exact 2–form. This is precisely the statement that not every incompressible vector field on $\mathbb{S}^2$ is solenoidal.

Example 5.1.14. The de Rham cohomology groups of the n–torus $\mathbb{T}^n = \mathbb{S}^1 \times \cdots \times \mathbb{S}^1$ (n–times) are

$$H^k_{\mathrm{DR}}(\mathbb{T}^n) \;=\; \mathbb{R}^{\binom{n}{k}}.$$

In particular, $H^1_{\mathrm{DR}}(\mathbb{T}^2) = \mathbb{R}^2 \neq 0$ implies that that there is an irrotational vector field on $\mathbb{T}^2$ which is not a gradient field. Moreover, $H^2_{\mathrm{DR}}(\mathbb{T}^2) = \mathbb{R} \neq 0$ implies that there is an incompressible vector field on $\mathbb{T}^2$ which is not solenoidal.

5.2. The Integration Theory

One aspect of the central narrative that we have thus far omitted in our recollection is the integration theory, specifically the path and surface independence properties. Indeed, recall that a vector field $\mathbf{F} : \Omega \to \mathbb{R}^n$ is said to have the path independence property if

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; 0$$

for any closed curve $\mathcal{C} \subset \Omega$. In terms of the associated 1–form, this is equivalent to

$$\int_{\mathcal{C}} \omega_{\mathbf{F}} = 0$$

for all closed curves $\mathcal{C} \subset \Omega$.

Remark 5.2.1. Before proceeding, let us observe that the de Rham cohomology groups of a space Ω, were constructed from a series of vector spaces $\Lambda^k(\Omega)$ (the space of k–forms on Ω) and an operator $d : \Lambda^k \to \Lambda^{k+1}$ which has the property that $d^2 = 0$. This is made precise by the following notion:

Definition 5.2.2. A *cochain complex* $(\mathcal{C}_\bullet, d_\bullet)$ is a sequence of vector spaces $\mathcal{C}_0, \mathcal{C}_1, \ldots$, connected by maps

$$d_k : \mathcal{C}_k \to \mathcal{C}_{k+1}$$

such that the composition $d_{k+1} \circ d_k \equiv 0$ for all $k \geq 0$.

If the arrows occur in the reverse direction, we obtain:

Definition 5.2.3. A *chain complex* $(\mathscr{A}_\bullet, \partial_\bullet)$ is a sequence of vector spaces $\mathscr{A}_0, \mathscr{A}_1, \ldots$, connected by maps

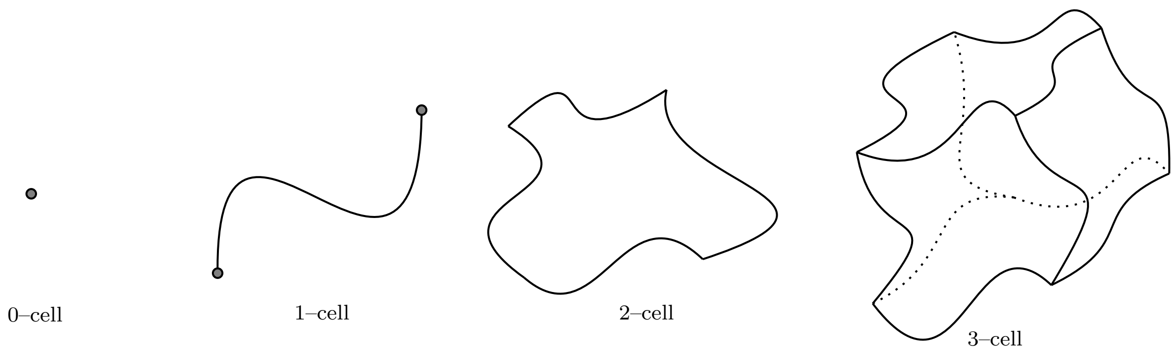
$$\partial_k : \mathscr{A}_{k+1} \to \mathscr{A}_k$$

such that the composition $\partial_k \circ \partial_{k+1} \equiv 0$ for all $k \geq 0$.

We have already seen an example of a chain complex, but less explicitly. Indeed, recall that a smooth curve $\mathcal{C} \subset \mathbb{R}^n$ is the image of a smooth map $\alpha : [t_0, t_1] \to \mathbb{R}^n$. For simplicity, we will take the interval $[t_0, t_1]$ to simpy be the interval $[0, 1]$.

Definition 5.2.4. A smooth map $\alpha : [0,1]^k \to \mathbb{R}^n$ is called a *k–cell.*

Example 5.2.5. Observe that a 0–cell is a point, a 1–cell is a curve, a 2–cell is a surface, etc:



To achieve a sequence of vector spaces, we will consider objects which are a little more general:

Definition 5.2.6. A *k–chain* is a formal linear combination of k–cells $\alpha_\nu : [0,1]^k \to \mathbb{R}^n$ with integer coefficients n_ν:

$$\sigma \;:=\; \sum_{\nu=1}^{N} n_\nu \alpha_\nu.$$

Remark 5.2.7. Let $\mathcal{Q}_\nu$ be the image of the k–cell $\alpha_\nu : [0,1]^k \to \mathbb{R}^n$. If $\Sigma = \sum_{\nu=1}^{N} n_\nu \mathcal{Q}_\nu$ is a k–chain, and $\omega \in \Lambda^k(\mathbb{R}^n)$ is a k–form on $\mathbb{R}^n$, we define

$$\int_\Sigma \omega \;:=\; \sum_{\nu=1}^{N} n_\nu \int_{\mathcal{Q}_\nu} \omega.$$

Remark 5.2.8. If $\Sigma = \sum_{\nu=1}^{N} n_\nu \mathcal{Q}_\nu$ and $\Theta = \sum_{\nu=1}^{N} m_\nu \mathcal{Q}_\nu$ are two k–chains, we define their sum via the formulae

$$\Sigma + \Theta \;:=\; \sum_{\nu=1}^{N} (n_\nu + m_\nu)\mathcal{Q}_\nu.$$

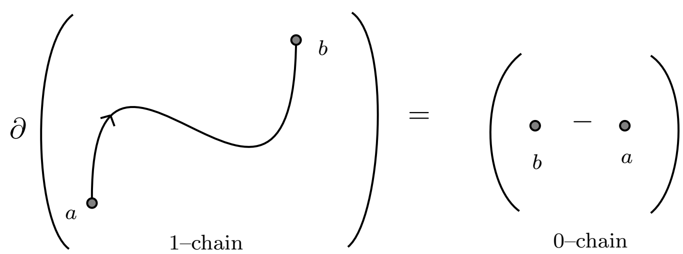
Definition 5.2.9. We define $\mathcal{F}_k$ to be the group of k–chains with addition defined as above.

Definition 5.2.10. Let $\mathcal{Q}$ be a k–cell given by $\alpha : [0,1]^k \to \mathbb{R}^n$. Define $\partial\mathcal{Q}$ to be the $(k-1)$–cell given by

$$\partial\mathcal{Q} \;:=\; \sum_{i=1}^{k}(-1)^{i-1}[\alpha|_{\mathcal{Q}^{k-1}_{i,1}} - \alpha|\mathcal{Q}^{k-1}_{i,0}],$$

where $\mathcal{Q}^{k-1}_{i,0} := \{t \in [0,1]^k : t_i = 0\}$ and $\mathcal{Q}^{k-1}_{i,1} := \{t \in [0,1]^k : t_i = 1\}$.

The formula is made more transparent through the following illustrations:

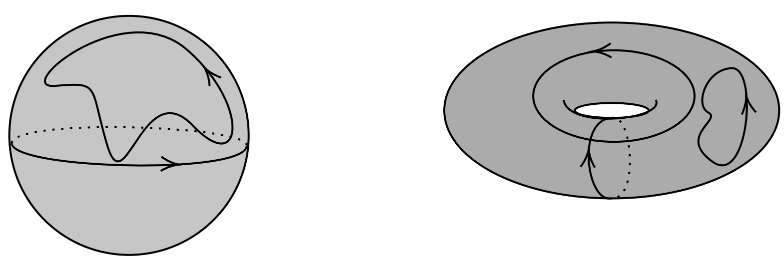


The boundary operator $\partial : \mathcal{F}_k \to \mathcal{F}_{k-1}$ is then defined on a k–chain $\Sigma := \sum_{\nu=1}^{N} n_\nu \mathcal{Q}_\nu$ by

$$\partial\Sigma \;:=\; \sum_{\nu=1}^{N} n_\nu \partial\mathcal{Q}_\nu.$$

Definition 5.2.11. We say that a k–chain Σ is a k*–cycle* if $\partial\Sigma = 0$. The space of k–cycles is denoted by $\mathcal{Z}_k$. Further, we say that a k–chain Σ is a k*–boundary* if $\Sigma = \partial\Theta$, for some $(k-1)$–chain Θ. The space of k–boundaries is denoted by $\mathcal{B}_k$.

Example 5.2.12.



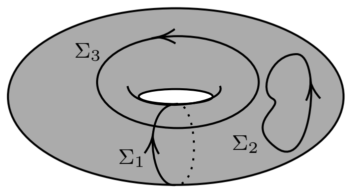
1-cycles on $\mathbb{S}^2$ 1-cycles on $\mathbb{T}^2$

Remark 5.2.13. Since the boundary of a boundary is empty, it follows that a k–boundary is a k–cycle.

Definition 5.2.14. The *kth homology* group of a space Ω is defined

$$H_k(\Omega, \mathbb{Z}) \ := \ \mathcal{Z}_k(\Omega)/\mathcal{B}_k(\Omega).$$

Example 5.2.15. Observe that we can readily compute the first homology group of the torus $\mathbb{T}^2$. Indeed, if we consider the cycles depicted below:

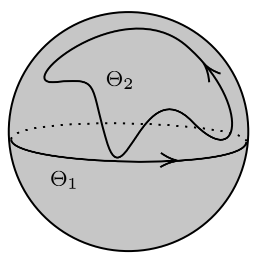


1-cycles on $\mathbb{T}^2$

We observe that the cycle Σ_2 can be retracted to a point, while Σ_1 and Σ_3 cannot be retracted to a point (due to the presence of a hole in the torus). The first homology group $H_1(\mathbb{T}^2, \mathbb{Z})$ is generated by the cycles Σ_1 and Σ_3, and therefore given by

$$H_1(\mathbb{T}^2, \mathbb{Z}) \ \simeq \ \mathbb{Z} \times \mathbb{Z}.$$

If we compare this with the cycles on $\mathbb{S}^2$:



1-cycles on $\mathbb{S}^2$

Observe that the cycles Θ_1 and Θ_2 can both retract to points. Hence, the first homology group of $\mathbb{S}^2$ vanishes:

$$H_1(\mathbb{S}^2, \mathbb{Z}) = 0.$$

Remark 5.2.16. Observe that we have now associated two series of invariants to a space Ω: the de Rham cohomology groups $H^k_{\mathrm{DR}}(\Omega)$, and the homology groups $H_k(\Omega, \mathbb{Z})$. Knowledge of the de Rham cohomology groups is important from our perspective, since they inform of us of whether all irrotational vector fields are gradient fields (measured by $H^1_{\mathrm{DR}}(\Omega)$) and whether all incompressible vector fields are solenoidal (measured by $H^2_{\mathrm{DR}}(\Omega)$).

On the other hand, in comparison with the homology groups $H_k(\Omega, \mathbb{Z})$, the de Rham cohomology groups may be difficult to compute. For instance, computing the first homology group of a torus (namely, $H_1(\mathbb{T}^2, \mathbb{Z}) \simeq \mathbb{Z} \times \mathbb{Z}$) is simpler than computing the first de Rham cohomology group $H^1_{\mathrm{DR}}(\mathbb{T}^2) \simeq \mathbb{R} \times \mathbb{R}$.

In a sense that we will continue to refine with more precision, these invariants are two sides of the same coin. That is, if we know the homology groups of Ω, we can compute the de Rham cohomology groups of Ω, and conversely.

Proposition 5.2.17.

(i) Let Σ_k and Σ'_k be two homologous k–cycles. Then, for any closed k–form ω,

$$\int_{\Sigma_k} \omega \;=\; \int_{\Sigma'_k} \omega.$$

(ii) Let ω and η be two cohomologous k–forms. Then for any k–cycle Σ_k,

$$\int_{\Sigma_k} \omega \;=\; \int_{\Sigma_k} \eta.$$

Proof. We first prove (i): If Σ_k and Σ'_k are homologous, then there is a $(k+1)$–cycle Θ_{k+1} such that

$$\Sigma'_k = \Sigma_k + \partial\Theta_{k+1}.$$

We compute

$$\int_{\Sigma'_k} \omega \;=\; \int_{\Sigma_k + \partial\Theta_{k-1}} \omega \;=\; \int_{\Sigma_k} \omega + \int_{\partial\Theta_{k-1}} \omega \;=\; \int_{\Sigma_k} \omega + \int_{\Theta_{k+1}} d\omega \;=\; \int_{\Sigma_k} \omega,$$

where the second equality uses Stokes' theorem, and the last equality uses the fact that ω is closed (i.e., $d\omega = 0$).

For the proof of (ii): If ω and η are cohomologous, there is a $(k-1)$–form φ such that $\omega = \eta + d\varphi$. We compute

$$\int_{\Sigma_k} \omega \;=\; \int_{\Sigma_k} (\eta + d\varphi) \;=\; \int_{\Sigma_k} \eta + \int_{\Sigma_k} d\varphi \;=\; \int_{\Sigma_k} \eta + \int_{\partial\Sigma_k} \varphi \;=\; \int_{\Sigma_k} \eta,$$

where the second equality uses Stokes' theorem, and the last equality uses the fact that Σ_k is a k–cycle (i.e., $\partial\Sigma_k = 0$). $\square$

Remark 5.2.18. The above proposition shows that the integral $\int : \mathcal{Z}_k(\Omega) \times \mathcal{Z}^k(\Omega) \to \mathbb{R}$ gives a well-defined function

$$\int : H_k(\Omega) \times H^k_{\mathrm{DR}}(\Omega) \longrightarrow \mathbb{R}.$$

In fact, the integral is most importantly, the *dual pairing* between homology and de Rham cohomology. Let us give the definition:

Definition 5.2.19. Let V, W be two (real) vector spaces. A map $\Phi : V \times W \to \mathbb{R}$ is said to be a *dual pairing* if it is bilinear in the sense that

(i) $\Phi(v, \cdot) : W \to \mathbb{R}$ is a linear map for all $v \in V$,
(ii) $\Phi(\cdot, w) : V \to \mathbb{R}$ is a linear map for all $w \in W$.

And moreover, Φ is non-degenerate in the sense that

(i) if $\Phi(v, w) = 0$ for all $w \in W$, then $v = 0$,
(ii) if $\Phi(v, w) = 0$ for all $v \in V$, then $w = 0$.

Example 5.2.20. The integral $\int : H_k(\Omega) \times H^k_{\mathrm{DR}}(\Omega) \longrightarrow \mathbb{R}$ yields a dual pairing between the homology groups $H_k(\Omega)$ and the cohomology groups $H^k_{\mathrm{DR}}(\Omega)$. Indeed, for any k–cycle Σ_k, the map

$$\omega \quad \longmapsto \quad \int_{\Sigma_k} \omega$$

is linear by standard properties of integrals. Similarly, for any closed k–form ω, the map

$$\Sigma_k \quad \longmapsto \quad \int_{\Sigma_k} \omega$$

is linear by the definition of integration over cycles. The non-degeneracy is clear.

Remark 5.2.21. Duality is one of the central notions in modern mathematics, and physics, so let us give some details concerning its meaning. Following [2], duality is not a theorem, but a *principle*; fundamentally, duality gives two different points of view when looking at the same thing (two sides of a coin).

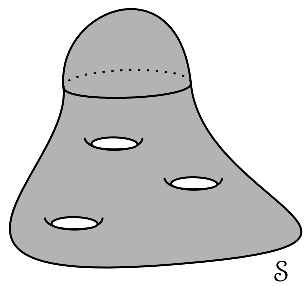
Other examples of dualities include: the duality between vector fields and 1–forms; the duality between k–forms and $(n-k)$–forms on $\mathbb{R}^n$ given by the Hodge $\star$–operator. The fundamental example coming from physics is Maxwell's discovery of the duality between electricity and magnetism; specifically, the electric field $\mathbf{E}$ is dual to the magnetic field $\mathbf{B}$.

The power of having a duality (e.g., the duality between homology and de Rham cohomology) is that one can pass from one side of the duality to the other, interchangably. Hence, if one cannot solve problems from one perspective, one can use duality to try to solve the corresponding (hopefully simpler) dual problem. More precisely, we have the following theorem:

Theorem 5.2.22. (de Rham theorem). The homology groups $H_k(\Omega, \mathbb{Z})$ are isomorphic to the de Rham cohomology groups $H^k_{\mathrm{DR}}(\Omega)$.

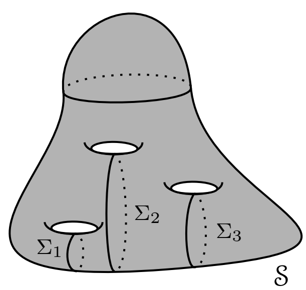
The proof of the above theorem is far beyond the scope of the present manuscript. Let us, however, exhibit the utility of the above theorem:

Example 5.2.23. Consider the surface $\mathcal{S}$ given below



Determine whether every irrotational vector field $\mathbf{F}$ on $\mathcal{S}$ is a gradient field.

SOLUTION. The problem can be stated in terms of whether the first de Rham cohomology group $H^1_{\mathrm{DR}}(\mathcal{S})$ vanishes. Using the de Rham theorem, it suffices to compute the first homology group $H_1(\mathcal{S}, \mathbb{Z})$. To this end, let us observe that $H_1(\mathcal{S}, \mathbb{Z})$ has 1–cycles which do not retract to points:



In particular, $H_1(\mathcal{S}, \mathbb{Z}) \neq 0$, and by de Rham's theorem, $H^1_{\mathrm{DR}}(\mathcal{S}) \neq 0$. It follows that there are irrotational vector fields on $\mathcal{S}$ which are not gradient fields. $\square$

CHAPTER 6

The Hard Road to Vector Calculus

The aim of the present chapter is to provide a more detailed (albeit, significantly more advanced) treatment of the contents within the first four chapters. The notions of differential forms and vector fields are properly defined as sections of certain vector bundles. The regions Ω, surfaces $\mathcal{S}$, and solid regions $\mathcal{V}$ are given their proper definitions as manifolds (possibly with boundary). This will require a brief discussion of topological spaces and basic differential geometry. We will, moreover, discuss Riemannian metrics, illucidating their hidden use in the main part of the book. Finally, we give full details of the proof of the generalized Stokes theorem – the central core of the entire text.

6.1. Linear Algebra

To give the proper definition of a differential form, one requires the notion of a vector bundle. A vector bundle is a family of vector spaces, parametrized by a manifold, and satisfying a certain local triviality condition, allowing us to compare nearby vector spaces. In particular, we need to begin with some reminders concerning basic linear algebra.

Definition 6.1.1. A (real) vector space is a set V together with two operations:

(i) (vector addition). If $u, v \in V$, then $u + v \in V$.
(ii) (scalar multiplication). If $v \in V$ and $\lambda \in \mathbb{R}$, then $\lambda v \in V$.

These operations satisfy a number of standard axioms: associativity, commutativity, distributive laws, existence of identity elements, and so forth, which we assume the reader to be familiar with.

Example 6.1.2. For any $n \in \mathbb{N}$, Euclidean space $\mathbb{R}^n$ is an example of a vector space.

We assume the reader has familiarity with the following notions:

Reminder 6.1.3. A set of vectors $v_1, ..., v_k \in V$ is said to be

(i) *linearly independent* if the map $f : \mathbb{R}^k \to V$ defined by

$$f(a_1, ..., a_k) \ := \ c_1 v_1 + \cdots + c_k v_k$$

is injective.

(ii) *span* V if the map $f : \mathbb{R}^k \to V$ defined by

$$f(a_1, ..., a_k) \ := \ c_1 v_1 + \cdots + c_k v_k$$

is surjective.

(iii) a *basis for* V if the map $f : \mathbb{R}^k \to V$ defined by

$$f(a_1, ..., a_k) \ := \ c_1 v_1 + \cdots + c_k v_k$$

is bijective.

Example 6.1.4. The vectors $\mathbf{e}_1 = \mathbf{i} + 0\mathbf{j}$ and $\mathbf{e}_2 = 0\mathbf{i} + \mathbf{j}$ form the *standard basis* of $\mathbb{R}^2$.

Reminder 6.1.5. Let V be a vector space. The *dimension of* V is the number of vectors in a basis for V.

Remark 6.1.6. We will always assume that vector spaces are finite-dimensional. Many vector spaces one encounters in the wild, however, are certainly not finite-dimensional, e.g., the vector space of continuous functions $f : [0,1] \to \mathbb{R}$ on $[0,1]$.

Definition 6.1.7. Let V and W be two vector spaces. A map $f : V \to W$ is said to be *linear* if

(i) (additivity). $f(u + v) = f(u) + f(v)$, for all $u, v \in V$.
(ii) (homogeneity). $f(\lambda v) = \lambda f(v)$, for all $\lambda \in \mathbb{R}$, $v \in V$.

Example 6.1.8. Every linear map $f : \mathbb{R} \to \mathbb{R}$ is given by $f(x) = ax$ for some $a \in \mathbb{R}$. Of course, the map $f(x) = ax + b$ for $b \in \mathbb{R}\backslash\{0\}$ is *not* a linear map.

Remark 6.1.9. Some elementary remarks worth making explicit:

(i) A *matrix* is a representation of a linear map in terms of a choice of basis.
(ii) A linear map is determined by its values on basis vectors.

Definition 6.1.10. A linear map $f : V \to W$ between vector spaces is said to be an *isomorphism* if f is injective and surjective.
We say that two vector spaces V, W are *isomorphic* if and only if there is a linear isomorphism between them.

Theorem 6.1.11. Two vector spaces V, W are isomorphic if and only if $\dim(V) = \dim(W)$.

Corollary 6.1.12. Every vector space of dimension n is isomorphic to $\mathbb{R}^n$.

Definition 6.1.13. Let V be a vector space. Let $V^k := V \times \cdots \times V$ (k–times) be the k–fold cartesian product. A map $f : V^k \to \mathbb{R}$ is said to be *multi-linear* if it is linear in each of its arguments.

Example 6.1.14. Let A be an $n \times n$ matrix, acting on a vector space V of dimension n. The map $f : V \times V \to \mathbb{R}$ defined by

$$f(\mathbf{u}, \mathbf{v}) \;:=\; \mathbf{u}^t A \mathbf{v},$$

is multilinear.

Remark 6.1.15. When $k = 2$, a multilinear map is said to be *bilinear form*.

Example 6.1.16. The dot product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a (symmetric) bilinear form.

The dot product is an example of a more general object:

Example 6.1.17. Let V be a vector space. A symmetric bilinear form $(\cdot, \cdot) : V \times V \to \mathbb{R}$ is said to be an *inner product* if $(\cdot, \cdot)$ is positive-definite in the sense that

$$\|v\|^2 \;:=\; (v, v) > 0,$$

for all $v \in V - \{0\}$.

Remark 6.1.18. It is common to refer to an inner product as a symmetric positive-definite quadratic form. We will not treat the beautiful theory of quadratic forms here.

Definition 6.1.19. Let V be a vector space. We say that a function $f : V \to \mathbb{R}$ is a *norm* if

(i) f is positive homogeneous of degree one, i.e.,

$$f(\alpha v) \;=\; |\alpha| f(\mathbf{v}),$$

for all $\alpha \in \mathbb{R}$ and $\mathbf{v} \in V$.

(ii) f satisfies the triangle inequality:

$$f(\mathbf{u} + \mathbf{v}) \;\leq\; f(\mathbf{u}) + f(\mathbf{v}),$$

for all $\mathbf{u}, \mathbf{v} \in V$.

(iii) f is non-degenerate in the sense that

$$f(\mathbf{v}) = 0 \iff \mathbf{v} = \mathbf{0}.$$

Example 6.1.20. If $(\cdot, \cdot)$ is an inner product on a vector space V, a norm is given by

$$\|\mathbf{v}\| \;:=\; \sqrt{(\mathbf{v}, \mathbf{v})}.$$

Remark 6.1.21. Every inner product produces a norm, but the converse is not true. The uniform norm

$$\|f\|_\infty := \max_{x \in [0,1]} |f(x)|$$

on the space of continuous functions $f : [0, 1] \to \mathbb{R}$ is a norm which does not come from an inner product.

Quotient vector spaces. Quotient vector spaces are not commonly treated within a first course in linear algebra, but they are very important (and for us, essential). Before looking at the notion of a quotient vector space, let us consider the following example:

Example 6.1.22. Suppose we have an analogue clock, with times $1, 2, 3, ..., 12$ on the face of the clock. Suppose it is 2pm and we want to know what the time will be in 3 hours. This is an easy exercise in arithmetic, $2+3 = 5$, so the time will be 5pm. Suppose it is 6pm, however, and we want to know the time 8 hours from now. In this case, kindergarten arithmetic tells us that it $6 + 8 = 14$, but this is not we understand to be the time: we understand 14 to be 2am. What do we do here? Well, we know that the clock is grouped in terms of 12 hour increments, so we perform the arithmetic in the familiar way – namely, calculate $6 + 8 = 14$ – and then proceed to determine the remainder after dividing by 12:

$$6 + 8 \ = \ 14 \ = \ 12 + 2.$$

This idea of looking only at objects, up to some equivalence, is made precise through the notion of a quotient space. This specific instance of *clock arithmetic* is formalized by the *integers modulo 12*, denoted $\mathbb{Z}/12\mathbb{Z}$.
Here,

$$\mathbb{Z}/12\mathbb{Z} = \{0, 1, 2, ..., 11\}$$

(identifying 12 with 0) but addition is understood to be the remainder of the ordinary addition after dividing by 12.

Example 6.1.23. In $\mathbb{Z}/12\mathbb{Z}$, we have

(i) $11 + 4 = 15 = 12 + 3 \equiv 3$.
(ii) $9 + 11 = 20 = 12 + 8 \equiv 8$.

Of course, the fact that the number 12 is present is just to make transparent the ordinary familiar nature of this construction. We could equally well consider $\mathbb{Z}/5\mathbb{Z} = \{0, 1, 2, ..., 4\}$:

Example 6.1.24. In $\mathbb{Z}/5\mathbb{Z}$, we have

(i) $4 + 4 = 8 = 5 + 3 \equiv 3$.
(ii) $2^4 = 16 = 3 \cdot 5 + 1 \equiv 1$.

Remark 6.1.25. Quotients formalize the notion of looking at what remains after identifying all the parts of the structure one does not want to look at. We saw in Chapter 5 that the de Rham cohomology groups $H^k_{\mathrm{DR}}(X, \mathbb{R})$ were defined as quotients. This is because we wanted to understand the obstruction to every irrotational vector field being a gradient field. In more detail, let $\mathcal{Z}^1(X)$ denotes the vector space of irrotational vector fields, and let $\mathcal{B}^1(X)$ denote the vector space of gradient fields. If we want to measure the extent to which every irrotational vector field is a gradient field, we want to understand what is left when we take remainders modulo gradient fields, i.e., the size of $\mathcal{Z}^1(X)/\mathcal{B}^1(X)$.

Let us give a more precise discussion:

Definition 6.1.26. Let U be a subspace of a vector space V. We declare two vectors $\mathbf{v}_1, \mathbf{v}_2 \in V$ to be *equivalent* and write $\mathbf{v}_1 \sim \mathbf{v}_2$ if $\mathbf{v}_1 - \mathbf{v}_2 \in U$.

Remark 6.1.27. The above definition defines an equivalence relation, i.e.,

(i) $\mathbf{v}_1 \sim \mathbf{v}_2$ if and only if $\mathbf{v}_2 \sim \mathbf{v}_1$.
(ii) If $\mathbf{v}_1 \sim \mathbf{v}_2$ and $\mathbf{v}_2 \sim \mathbf{v}_3$, then $\mathbf{v}_1 \sim \mathbf{v}_3$.

Example 6.1.28. Let $f : V \to W$ be a linear map. The isomorphism theorem asserts that

$$V \;\simeq\; \ker(f) \oplus \mathrm{Im}(f).$$

In particular,

$$\mathrm{Im}(f) \;\simeq\; V/\ker(f).$$

Definition 6.1.29. A W–coset is a set of the form

$$v + W \;:=\; \{v + w : w \in W\}.$$

Remark 6.1.30. The reader can check that if $u - v \in W$, then the cosets $v + W$ and $u + W$ coincide. On the other hand, if $u - v \notin W$, then the cosets $u + W$ and $v + W$ are disjoint. Hence, the W–cosets decompose V into a disjoint collection of subsets of V. We denote this collection of subsets by V/W.

Definition 6.1.31. The *quotient vector space* V/W is defined as a set to be the disjoint collection of W–cosets of V. Vector addition is defined by

$$(u + W) + (v + W) \;=\; u + v + W.$$

Scalar multiplication is defined by

$$\lambda \cdot (v + W) \;=\; \lambda v + W.$$

Definition 6.1.32. Let V be a finite-dimensional real vector space. We define a *covector* on V to be a linear map $\omega : V \to \mathbb{R}$. The space of all covectors forms a vector space – the *dual space* to V – and is denoted by V^*.

Definition 6.1.33. Let V be a (real) vector space. The dual space V^* is the set of all linear maps $V \to \mathbb{R}$.

Remark 6.1.34. If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a basis for V. A basis for the dual space V^* of V is given by the linear maps $\varepsilon^k : V \to \mathbb{R}$ such that

$$\varepsilon^i(\mathbf{v}_j) \;=\; \delta^i_j \;=\; \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

We refer to $\varepsilon^1, ..., \varepsilon^n$ as the *dual basis* of $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$.

Remark 6.1.35. From **??**, we know that a (finite-dimensional) vector space is determined (up to isomorphism) by its dimension. In particular, a vector space V and its dual space V^* are isomorphic. But there is no *natural isomorphism* between V and its dual V^*.

Remark 6.1.36. Up to this point, we have looked exclusively at the subject of linear algebra – the study of (finite-dimensional) vector spaces and linear maps between them. We are exposed, however, to objects that not strictly within the confines of linear algebra, in standard introductory courses.

For instance, the dot product $\cdot : V \times V \to \mathbb{R}$ is not a linear map between vector spaces. The dot product is an example of a more general object, more precisely considered to be the subject of multilinear algebra. The dot product is an example of the following general object:

Remark 6.1.37. Just as any linear map $f : V \to V$ can be expressed as a matrix in terms of a basis: $f(\mathbf{v}) = A\mathbf{v}$, any bilinear form $B : V \times V \to \mathbb{R}$ can be expressed as a matrix in terms of a basis: $B(\mathbf{u}, \mathbf{v}) = \mathbf{u}^t A\mathbf{v}$.

Definition. A *multilinear map...*

Example. The cross product on $\mathbb{R}^3$ is a bilinear map $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.

The dot product is, of course, not just a bilinear form. The dot product is the model example of an inner product:

Definition. Let V be a real vector space. An *inner product* is a bilinear form $B : V \times V \to \mathbb{R}$ which is

(i) symmetric in the sense that $B(\mathbf{u}, \mathbf{v}) = B(\mathbf{v}, \mathbf{u})$.

non-degenerate in the sense that $B(\mathbf{v}, \mathbf{w}) = 0$ for all $\mathbf{w} \in V$ if and only if $\mathbf{v} = \mathbf{0}$.

It remains to justify why the dot product, or more generally, inner products, appear in linear algebra. The miracle of the subject is that there is a distinguished multi-linear map $\otimes$ which permits us to realize any *multilinear map* $f : V_1 \times \cdots \times V_k \to W$ as a *linear map*

$\widehat{f} : V_1 \otimes \cdots \otimes V_k \to W$ for some vector space $V_1 \otimes \cdots \otimes V_k$ which is independent of f.

More precisely, the tensor product $\otimes$ is defined via the following universal property:

Definition. Let V and W be two vector spaces. The *tensor product* $V \otimes W$ is a vector space, equipped with a bilinear map $\otimes : V \times W \to V \otimes W$, defined by $\otimes(v, w) := v \otimes w$ such that, for any bilinear map $f : V \times W \to Z$, there is a unique linear map $\widehat{f} : V \otimes W \to Z$, such that $f = \widehat{f} \circ \otimes$.

A diagram is most fitting here:

$$\begin{tikzcd} V \times W \arrow[r, "f"] \arrow[d, "\otimes"'] & Z \\ V \otimes W \arrow[ur, dotted, "\widehat{f}"'] & \end{tikzcd}$$

Remark. If $v_1, ..., v_n$ is a basis for V and $w_1, ..., w_m$ is a basis for W, then a basis for $V \otimes W$ is given by $\{v_i \otimes w_j\}_{1 \leq i \leq n, 1 \leq j \leq m}$. In particular, $\dim(V \otimes W) = \dim(V) \cdot \dim(W)$.

The definition given above can remain quite mysterious. In geometry, however, we will understand the tensor product as follows. First, let us make the following definition:

Definition. Let V be a vector space. A *covariant* k*–tensor* on V is a multi-linear map $f : V^k \to \mathbb{R}$, where $V^k := V \otimes \cdots \otimes V$ (k–times). The space of covariant k–tensors is denoted $\mathcal{T}_k(V)$.

Remark. Covariant k–tensors on V are just k–multilinear forms on V. Hence, covariant 2–tensors on V are bilinear forms on V.

Definition. Let V be a vector space. A *convariant* ℓ*–tensor* on V is a multilinear map $f : (V^*)^{\otimes \ell} \to \mathbb{R}$, where $(V^*)^{\otimes \ell} := V^* \otimes \cdots \otimes V^*$ (ℓ–times). The space of contravariant ℓ–tensors on V is denoted $\mathcal{T}^\ell(V)$.

Example.

Definition. Let V be a vector space. A (k, ℓ)–tensor on V is a multilinear map

$$f : (V^*)^{\otimes k} \otimes V^{\otimes \ell} \to \mathbb{R}.$$

The space of (k, ℓ)–tensors on V is denoted by $\mathcal{T}^k_\ell(V)$.

Example. If V is a vector space, the space of $(1, 1)$–tensors on V is isomorphic to the space of linear maps $V \to V$.

Remark. The tensor product $\otimes : V^{\otimes k} \times V^{\otimes \ell} \to V^{\otimes (k+\ell)}$ is a multiplication on the vector space

$$\mathfrak{T}(V) := \bigoplus_{k=0}^{\infty} V^{\otimes k}.$$

This gives $\mathfrak{T}(V)$ the structure of an *algebra*, and as a consequence, we refer to $\mathfrak{T}(V)$ as the *tensor algebra of* V.

We will want to build the *exterior algebra* $\Lambda(V)$ from the tensor algebra. This is given by taking a suitable quotient of $\mathfrak{T}(V)$. We have defined the quotient of a vector space, but we will want to extend this to define the quotient of an algebra. To this end, let us introduce the following terminology:

Definition. A *group* is a set G, together with an associative binary operator $\cdot : G \times G \to G$ such that:

(i) there exists a distinguished *identity element* $e \in G$ such that

$$e \cdot g = g \cdot e = g,$$

for all $g \in G$.

(ii) for all $g \in G$, there exists an element $h \in G$ such that $g \cdot h = h \cdot g = e$.

Example. The integers $\mathbb{Z}$ form a group with respect to addition $+ : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$. The identity element is 0.

Definition. Let $(A, +, \cdot)$ be an algebra. A subset $\mathfrak{I} \subset A$ is called a *left ideal* of A if

(i) $(\mathfrak{I}, +)$ is a group.

(ii) For every $a \in A$ and every $x \in \mathfrak{I}$, we have $a \cdot x \in \mathfrak{I}$.

We say $\mathfrak{I} \subset A$ is a *two-sided ideal* of A if, in addition, $x \cdot a \in \mathfrak{I}$ for all $a \in A$ and $x \in \mathfrak{I}$.

Definition. The *exterior algebra* of a vector space V is the quotient algebra $(\Lambda(V), \wedge)$ given by quotienting the tensor algebra $\mathfrak{T}(V)$ by the two-sided ideal $\mathfrak{I}$ generated by $v \otimes v$. If $\pi : \mathfrak{T}(V) \longrightarrow \Lambda(V)$ denotes the quotient map, then the induced multiplication is called the *wedge product*

$$\wedge : \Lambda(V) \times \Lambda(V) \longrightarrow \Lambda(V), \qquad \wedge(\alpha \wedge \beta) \ := \ \pi(A \otimes B),$$

where $\pi(A) = \alpha$ and $\pi(B) = \beta$.

Remark A.1.14. The spaces $T^k(V)$ and $T_\ell(V)$ form real vector spaces with respect to the obvious operations.

Examples A.1.15.

(i) A covariant 1–tensor is a covector $\omega : V \to \mathbb{R}$. Hence, $T^1(V)$ coincides with the dual vector space V^*.

(ii) A covariant 2–tensor is a real-valued bilinear form $B : V \times V \to \mathbb{R}$. The dot product on $\mathbb{R}^n$ is an example of a covariant 2–tensor.

(iii) If we view the determinant of an $n \times n$ matrix as a multi-linear map on the columns of the matrix (viewed as vectors) then the determinant is a covariant n–tensor on $\mathbb{R}^n$.

Proposition A.1.16. Let V be a real vector space of dimension n. Let $\{v_1, ..., v_n\}$ be a basis for V with dual basis $\{\varepsilon^1, ..., \varepsilon^n\}$. The covariant k–tensors of the form

$$\varepsilon^{i_1} \otimes \cdots \otimes \varepsilon^{i_k},$$

for $1 \le i_1, ..., i_k \le n$ is a basis for $T^k(V)$. Hence, $\dim T^k(V) = n^k$.

Proposition A.1.17. Let V and W be finite-dimensional real vector spaces. If $A : V \times W \to Z$ is a bilinear map into any vector space Z, there is a unique linear map $\widehat{A} : V \otimes W \to X$ such that the diagram

$$\begin{tikzcd}
V \times W \arrow[r, "A"] \arrow[d, "\pi"'] & Z \\
V \otimes W \arrow[ur, "\hat{A}"'] &
\end{tikzcd}$$

Definition A.1.18. Let V be a finite-dimensional vector space. The *tensor algebra* of V is defined

$$T(V) \; := \; \bigoplus_{k=0}^{\infty} T^k(V).$$

The multiplication on $T(V)$ is defined via the tensor product:

$$T^k(V) \otimes T^\ell(V) \longrightarrow T^{k+\ell}(V),$$

extended linearly to all of $T(V)$.

Definition A.1.19. Let $T(V)$ denote the tensor algebra of a finite-dimensional vector space V. The quotient of $T(V)$ by the two-sided ideal generated by $v \otimes v$, for $v \in V$, defines the *exterior algebra*.

Definition A.1.20. Let $\pi : T(V) \to \Lambda(V)$ denote the quotient map. We define the *wedge product* of two elements $\alpha, \beta \in \Lambda(V)$ by

$$\alpha \wedge \beta \ := \ \pi(A \otimes B),$$

where $\pi(A) = \alpha$ and $\pi(B) = \beta$.

Remark A.1.21. The reader may easily verify that the wedge product is well-defined, independent of the choice of representatives.

Remark A.1.22. The exterior algebra affords the grading

$$\Lambda(V) \ = \ \bigoplus_{k=0}^{\infty} \Lambda^k(V),$$

where $\Lambda^k(V)$ denotes the *kth exterior algebra*. The kth exterior algebra forms a subspace $\Lambda^k(V) \subseteq \Lambda(V)$ spanned by the wedge of k elements of V.

Remark A.1.23. If $\{\varepsilon^1, ..., \varepsilon^n\}$ is a basis for V^*, then a basis for $\Lambda^k(V^*)$ is given by

$$\{\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k} : 1 \leq i_1 < i_2 < \cdots < i_k \leq n\}.$$

In particular, $\dim \Lambda^k(V^*) = \binom{n}{k}$.

Let V be an n–dimensional real vector space. The top exterior power $\Lambda^n(V^*)$ is a one-dimensional vector space, in particular, it is isomorphic to $\mathbb{R}$. Hence, $\Lambda^n(V^*) - \{0\}$ is disconnected: it is a union of two components.

Definition A.1.24. An *orientation* on a vector space V is a choice of one of the components of $\Lambda^n(V^*) - \{0\}$.

Remark A.1.25.

(i) If $\{\varepsilon_1, ..., \varepsilon_n\}$ is an ordered basis for V^*, the orientation on V is determined to be the component of $\Lambda^n(V^*) - \{0\}$ in which $\varepsilon_1 \wedge \cdots \wedge \varepsilon_n$ lies.

(ii) Let $\{\varepsilon_1, ..., \varepsilon_n\}$ and $\{\delta_1, ..., \delta_n\}$ be two ordered bases for V^*. Let $A = (A_{ij})$ be the invertible matrix mapping $\{\varepsilon_1, ..., \varepsilon_n\}$ to $\{\delta_1, ..., \delta_n\}$. Then

$$\delta_i \ = \ \sum_{j=1}^{n} A_{ij} \varepsilon_j,$$

for each $1 \leq i \leq n$, and hence,

$$\delta_1 \wedge \cdots \wedge \delta_n \ = \ \det(A_{ij}) \varepsilon_1 \wedge \cdots \wedge \varepsilon_n.$$

From this, we see that two ordered bases determine the same orientation if and only if the change of basis matrix has positive determinant.

6.2. Topological Spaces

Our aim in this chapter is to provide the appropriate formalism behind the concepts which are treated in the earlier chapters. The previous section treated the linear algebraic theory, and our end goal is to speak of vector bundles. Vector bundles are families of vector spaces over a space. The present section begins the formalism behind the notion of *space*.

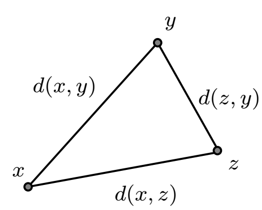
The model example that we are most acquainted with is Euclidean space $\mathbb{R}^n$. This has a number of special features, and we wish to progressively drop such special features to produce classes of more general objects which still support enough structure to be useable.

Besides the vector space structure that $\mathbb{R}^n$ posesses, it also supports a notion of *distance*. This is formalized through the following:

Reminder 6.2.1. A space X is said to be a *metric space* if there exists a function $d : X \times X \to \mathbb{R}$ such that

(i) (non-degeneracy). $d(x, y) = 0$ if and only if $x = y$.
(ii) (positivity). $d(x, y) > 0$ for all $x \neq y$.
(iii) (symmetry). $d(x, y) = d(y, x)$.
(iv) (triangle inequality). $d(x, z) \leq d(x, y) + d(y, z)$, for all $x, y, z \in X$.

The function d is referred to as a *metric* or, more precisely, a *distance function*.



Example 6.2.2. The standard Euclidean distance on $\mathbb{R}^n$ given by

$$d(x, y) \ := \ \sqrt{(x_1 - y_1)^2 + \cdots + (x_n - y_n)^2}$$

is a metric.

Example 6.2.3. Let $L^1([0, 1])$ denote the space of integrable functions on $[0, 1] \subset \mathbb{R}$. The function $d : L^1([0, 1]) \times L^1([0, 1]) \to \mathbb{R}$ defined by

$$d(f, g) \ := \ \int_0^1 |f(x) - g(x)| dx$$

defines a metric.

In the presence of a metric, one can make sense of a function being continuous:

Reminder 6.2.4. Let (X, d_X) and (Y, d_Y) be metric spaces. We say that a map $f : X \to Y$ is continuous if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $d_X(x, y) < \delta$, then

$$d_Y(f(x), f(y)) < \varepsilon.$$

Remark 6.2.5. Metric spaces form an important class of spaces, and when a space admits a metric, it is often a very useful structure to make use of. Not all spaces admit metrics, however, and moreover, it is not so easy to construct new metric spaces from old ones – quotients of metric spaces do not produce metric spaces, for instance.

It turns out that a more general notion of a space is more fruitful to consider. This can be motivated from a number of directions, but one reason is that continuity is a *deeper* notion than distance. The right framework for speaking of spaces is given by a topology:

Definition 6.2.6. A *topological space* is a pair (X, τ), where X is a non-empty set and τ is a family of subsets which satisfy:

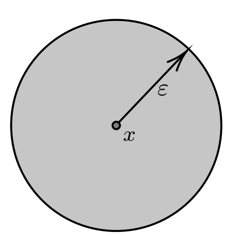
(i) $\emptyset \in \tau$ and $X \in \tau$.
(ii) τ is closed under arbitrary unions in the sense that if $\mathcal{U}_\alpha \in \tau$, then
$$\bigcup_\alpha \mathcal{U}_\alpha \in \tau.$$
(iii) τ is closed under finite intersections in the sense that if $\mathcal{U}_\alpha \in \tau$, then
$$\bigcap_\alpha \mathcal{U}_\alpha \in \tau.$$

The family of subsets τ is called a *toplogy*, and the elements of a topology are called *open sets*. The complement of an open set is called a *closed set*.

Example 6.2.7. The *discrete topology* on a set X is given by declaring all subsets of X to be open (and hence, all subsets to be closed).

Example 6.2.8. The *metric topology* on a metric space (X, d) is the coarsest topology on M relative to which the metric d is a continuous map. The open sets in the metric topology are given by open balls

$$B_\varepsilon(x) := \{y \in X : d(x, y) < \varepsilon\}.$$



Definition 6.2.9. Let (X, τ) be a topological space. We say that (X, τ) is a *Hausdorff space* if for every pair of distinct points $x, y \in X$, there exists $\mathcal{U}, \mathcal{V} \in \tau$ such that $x \in \mathcal{U}$, $y \in \mathcal{V}$, $x \notin \mathcal{V}$, and $y \notin \mathcal{U}$.

Remarks 6.2.10. Most spaces one considers in calculus are Hausdorff. For instance, all metric spaces are Hausdorff. But there are a number of natural topologies which are not Hausdorff, the most notable being the Zariski topology on an algebraic variety.

Remark 6.2.11. In a Hausdorff topological space, a sequence of points can have at most one limit.

Definition 6.2.12. Let (X, τ) and (Y, σ) be two topological spaces. A map $f : X \to Y$ is said to be *continuous* if for every $V \in \sigma$, we have $f^{-1}(V) \in \tau$.

In other words, a map is continuous if the preimage of an open set is open.

Remark 6.2.13. If (X, τ) and (Y, σ) are both metric spaces, with τ and σ both being the metric topology, then the above definition of continuity coincides with the definition of continuity for metric spaces.

Definition 6.2.14. A continuous map $f : (X, \tau) \to (Y, \sigma)$ is a *homeomorphism* if f is invertible with $f^{-1} : (Y, \sigma) \to (X, \tau)$ continuous.

Example 6.2.15. Let $f : (\mathbb{R}, \sigma) \to (\mathbb{R}, \tau)$ be the identity map $f(x) = x$, where σ is the discrete topology and τ is the standard metric topology. Then f is continuous and bijective, but f^{-1} is not continuous. In particular, a continuous bijection is not necessarily a homeomorphism.

An important consequence of the theory of topological spaces, is that we can easily make sense of what it means for a space to be *connected*:

Definition 6.2.16. Let X be a topological space. A *separation* of X is a pair $\mathcal{U}, \mathcal{V}$ of non-empty open subsets of X whose union is X. We say that X is *connected* if X does not support a separation.

Example 6.2.17. The interval $(0, 1) \subset \mathbb{R}$ is connected, but $(0, 1) \cup (2, 3)$ is not connected.

Remark 6.2.18. The property of being connected is described entirely in terms of open sets, hence is a topological property: any topological space homeomorphic to a connected topological space is connected.

Proposition 6.2.19. The continuous image of a connected set is connected.

Let us introduce the following stronger notion of connectedness:

Definition 6.2.20. Let X be a topological space, and let $x, y \in X$ be two points. A *path from x to y* is a continuous map $\alpha : [a, b] \to X$ such that $\alpha(a) = x$ and $\alpha(b) = y$. A topological space X is said to be *path connected* if every pair of points of X can be joined by a path in X.

Remark 6.2.21. It is clear that a path connected space is connected. The converse, however, is false, as illustrated by the topologist's sine curve

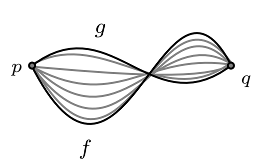
$$\{(0,0)\} \cup \left\{ \left(x, \sin\left(\frac{1}{x}\right)\right) : x \in (0,1] \right\},$$

with the Euclidean topology induced by the inclusion in $\mathbb{R}^2$ (see, e.g., [19, p. 156–157] for details).

Throughout the main component of the course, we have assumed, unless stated otherwise, that the spaces we consider are connected. Further, we saw in Chapter 3 that an important concept, central to theory, was the notion of simply connectedness. For pedagogical reasons, we gave the non-sense heuristic definition that a space is simply connected if it has no holes, or a little better – a space is simply connected if its boundary is connected.

The appropriate language to speak of such notions is homotopy theory (a sub-branch of algebraic topology). To give the appropriate definition, let us introduce the following:

Definition 6.2.22. Let $f, g : X \to Y$ be continuous maps between topological spaces. A *homotopy* between f and g is a continuous map $H : X \times [0,1] \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$.



Example 6.2.23. Let $f : \mathbb{R} \to \mathbb{R}$ be the map given by $f(x) = x$, and let $g : \mathbb{R} \to \mathbb{R}$ be the map given by $g(x) = x^2$. A homotopy between f and g is given by $H : \mathbb{R} \times [0,1] \to \mathbb{R}$,

$$H(x,t) \ := \ (1-t)x + tx^2.$$

Definition 6.2.24. Let X be a topological space. Declare two paths $\alpha, \gamma : [0,1] \to X$ from $p \in X$ to $q \in X$ to be *homotopically equivalent* if there is a homotopy $H : [0,1] \times [0,1] \to X$ between α and γ such that $H(0,t) = p$ and $H(1,t) = q$ for all $t \in [0,1]$.

Notation 6.2.25. Write $[\alpha]$ for the homotopy equivalence class of α.

Definition 6.2.26. Let X be a topological space. A path $\alpha : [0,1] \to X$ is called a *loop* at x if $\alpha(0) = \alpha(1) = x$. The set of homotopy equivalence classes of loops at x is denoted $\pi_1(X,x)$ and is called the *fundamental group* of X.

Remark 6.2.27. The reader should be demanding that the above definition defines a *set*, but not necessarily a *group*, since there is no specification of the group operation. The operation is given by concatenating loops:

$$(\gamma \circ \alpha)(t) \ := \ \begin{cases} \alpha(2t), & 0 \leq t \leq \frac{1}{2}, \\ \gamma(2t-1), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

The inverse operation is given by $\alpha^{-1}(t) := \alpha(1-t)$.

Example 6.2.28. The fundamental group of $\mathbb{R}$ (at the point 0) is

$$\pi_1(\mathbb{R}, 0) \ = \ \{0\}.$$

It suffices to show that every loop $\alpha : [0,1] \to \mathbb{R}$ in $\mathbb{R}$ based at 0 is homotopic to a point. Indeed, simply define the homotopy $H : \mathbb{R} \times [0,1] \to \mathbb{R}$ by $H(x,t) := (1-t)\alpha(x)$.

Example 6.2.29. Let $\mathbb{S}^1$ denote the unit circle in $\mathbb{R}^2$, centered at the origin. Then

$$\pi_1(\mathbb{S}^1, 1) \ \simeq \ \mathbb{Z}.$$

To prove this statement, let us recall that one of the central examples in the main theory was the vector field $\mathbf{F} : \mathbb{R}^2 \backslash \{0\} \to \mathbb{R}^2 \backslash \{0\}$ defined by

$$\mathbf{F}(x,y) \ := \ -\frac{y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j}.$$

In polar coordinates, the 1–form associated to $\mathbf{F}$ is given by $\omega_{\mathbf{F}} = d\vartheta$.

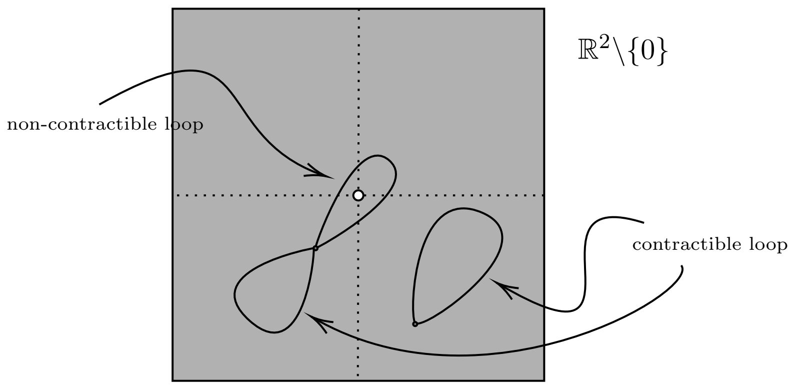
Definition 6.2.30. Let $\mathcal{C}$ be a loop in $\mathbb{R}^2$. The *winding number* of $\mathcal{C}$ is defined to be

$$w(\mathcal{C}) \ := \ \frac{1}{2\pi} \int_{\mathcal{C}} d\vartheta.$$

Theorem 6.2.31. The winding number gives a well-defined isomorphism $w : \pi_1(\mathbb{S}^1) \to \mathbb{Z}$.

PROOF. https://math.stackexchange.com/questions/3311034/elaborating-why-the-fundamental- https://math.i-learn.unito.it/pluginfile.php/82982/mod_resource/content/1/Kosniowski_p144-151.pdf □

Example 6.2.32. The punctured Euclidean plane $\mathbb{R}^2 \backslash \{0\}$ is not simply connected.



Definition 6.2.33. A topological space X is said to be *simply connected* if $\pi_1(X) = 0$.

Example 6.2.34. From *Example 6.2.28*, we know that $\mathbb{R}$ is simply connected. From *Example 6.2.29*, we know that $\mathbb{S}^1$ is not simply connected, and hence, $\mathbb{R}^2\backslash\{0\}$ is not simply connected.

Definition 6.2.35. Let X be a topological space with $\mathcal{U}$ be an open covering of X. We say that $\mathcal{U}$ is *locally finite* if, for each $x \in X$, there is an open neighborhood $\mathcal{V}_x$ containing x such that $\mathcal{V}_x$ intersections (non-trivially) a finite number of open sets in $\mathcal{U}$.

Definition 6.2.36. A topological space X is *paracompact* if every open covering of X has a locally finite refinement. That is, if for every open covering $\mathcal{U}$, there exists a locally finite open covering $\mathcal{V}$ such that for any $\mathcal{V}_\alpha \in \mathcal{V}$ there is some $\mathcal{U}_\alpha \in \mathcal{U}$ such that $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$.

Example 6.2.37. A metric space is paracompact.

Definition 6.2.38. Let X be a smooth manifold. A smooth *partition of unity* of X is a pair $(\mathcal{V}, \mathcal{F})$, where $\mathcal{V} = (\mathcal{V}_\alpha)$ is a locally finite covering of X and $\mathcal{F} = \{f_\alpha\}$ is a collection of smooth $\mathbb{R}$–valued functions on X such that

(i) $f_\alpha \geq 0$ for each α.
(ii) For each $\mathcal{V}_\alpha \in \mathcal{V}$, the support of f_α, i.e., the closure of the set $\{x \in X : f_\alpha(x) \neq 0\}$ is contained in $\mathcal{V}_\alpha$.
(iii) $\sum_\alpha f_\alpha = 1$.

Remark 6.2.39. Note that the sum in (iii) makes sense since for each $x \in X$, we have $f_\alpha(x) = 0$ for all but finitely many $\mathcal{V}_\alpha \in \mathcal{V}$.

6.3. Smooth Manifolds

Let M be a connected Hausdorff topological space with a countable base of open sets. For any arbitrary indexing set A, we assume M admits a covering $\mathcal{U} := (\mathcal{U}_\alpha)_{\alpha \in A}$ by connected open sets $\mathcal{U}_\alpha \subset M$ which are homeomorphic to balls $\mathbb{B}_\alpha \subset \mathbb{R}^n$. The pair $(\mathcal{U}_\alpha, \varphi_\alpha)$, where $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{B}_\alpha$ is a homeomorphism, is called a *chart*, and the set of charts $\mathscr{A} := \{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is said to be the *atlas* of the covering $\mathcal{U}$.

Remark 6.3.1. The charts permit one to locally identify a neighborhood of a point in M with a neighborhood of the origin in some Euclidean space $\mathbb{R}^n$. In particular, if $(x_1, ..., x_n)$ denote the coordinates on $\mathbb{R}^n$, these coordinates can be pulled back via the homeomorphism $\varphi_\alpha : \mathcal{U}_\alpha \to \mathbb{B}_\alpha \subset \mathbb{R}^n$ to furnish *local coordinates* on M, and hence, $\mathcal{U}_\alpha$ is sometimes called a *coordinate chart*.

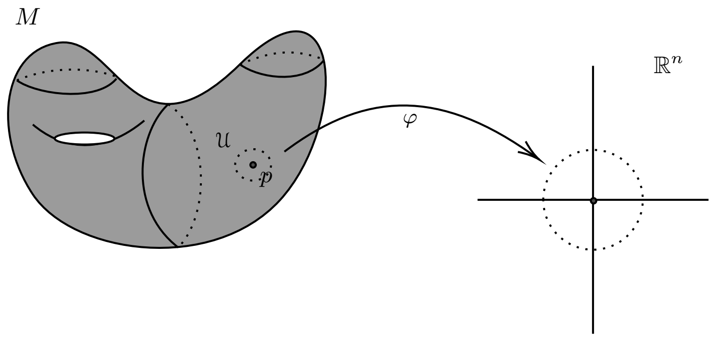
On any overlap of $\mathcal{U}_{\alpha\beta} := \mathcal{U}_\alpha \cap \mathcal{U}_\beta$, the composition

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(\mathcal{U}_{\alpha\beta}) \longrightarrow \varphi_\beta(\mathcal{U}_{\alpha\beta})$$

defines a homeomorphism between open subsets of $\mathbb{R}^n$, which we call *transition maps*. These transition maps allow one to make sense of the *regularity* of M. Namely, if the transition maps are of class $\mathcal{C}^k$, for some $k \in \mathbb{N}$, we say that the atlas $\mathscr{A}$ is a $\mathcal{C}^k$*–atlas*. If $\mathscr{A}$ is a $\mathcal{C}^k$–atlas for all $k \in \mathbb{N}$, we say $\mathscr{A}$ is a $\mathcal{C}^\infty$*–atlas* or a *smooth atlas*.

Remark 6.3.2. To remove the dependence on the choice, let us declare two $\mathcal{C}^k$–atlases $\mathscr{A}$ and $\mathscr{B}$ to be *equivalent* if their union is a $\mathcal{C}^k$–atlas. This defines an equivalence relation on the $\mathcal{C}^k$–atlases of M and ensures that the transition maps from the charts of one atlas to the charts of the other atlas have the same regularity as the regularity of the constituent transition maps for each atlas separately.

Definition 6.3.3. A $\mathcal{C}^k$*–manifold* is a connected Hausdorff topological space M endowed with an equivalence class of $\mathcal{C}^k$–atlases. The dimension of the balls to which the domains of the covering $\mathcal{U}$ are homeomorphic is called the (real) *dimension* of M, and is denoted $\dim_{\mathbb{R}} M$.



Example 6.3.4. Let $\mathbb{S}^2$ denote the unit sphere in $\mathbb{R}^3$. Let $\mathcal{U}_1 := \mathbb{S}^2 - \{(0,0,-1)\}$ and $\mathcal{U}_2 := \mathbb{S}^2 - \{(0,0,1)\}$. Define $\varphi_1 : \mathcal{U}_1 \to \mathbb{R}^2$ and $\varphi_2 : \mathcal{U}_2 \to \mathbb{R}^2$ by

$$\varphi_1(x,y,z) := \left(\frac{x}{1+z}, \frac{y}{1+z}\right), \qquad \varphi_2(x,y,z) := \left(\frac{x}{1-z}, \frac{y}{1-z}\right).$$

The reader may easily compute $\varphi_1 \circ \varphi_2^{-1}$ and $\varphi_2 \circ \varphi_1^{-1}$, verifying that these transition maps are smooth.

Definition 6.3.5. Let M be a manifold of class $\mathcal{C}^k$. A function $f : M \to \mathbb{R}$ is said to be of class $\mathcal{C}^\ell$, for some $\ell \leq k$, if the composite map $f \circ \varphi_\alpha^{-1}$ is of class $\mathcal{C}^\ell$ on the open set $\varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$.

Similarly, if M and N are two $\mathcal{C}^k$ manifolds, a map $f : M \to N$ is said to be of class $\mathcal{C}^\ell$, for some $\ell \leq k$, if the composite map $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is of class $\mathcal{C}^\ell$ on the open set $\varphi_\alpha(\mathcal{U}_\alpha) \subset \mathbb{R}^n$. These notions are clearly well-defined.

The tangent space. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. To any point $x \in \Omega$, we can assign a *tangent space* to Ω at x, which we denote by $T_x\Omega$. We identify $T_x\Omega = \{(x,v) : x \in \Omega,\ v \in \mathbb{R}^n\}$, and define a *tangent vector* to Ω at x to be an element of $T_x\Omega$. The tangent space $T_x\Omega$ is a (real) vector space of dimension n. This coincides with our familiar understanding of tangent vectors to functions, which is seen as follows:

Let $v_x \in T_x\Omega$ be a tangent vector. Then we can define a map $D_v|_x : \mathcal{C}^\infty(\Omega) \to \mathbb{R}$ which is defined to be the directional derivative of f at x in the direction of v:

$$D_v|_x f \;:=\; D_v f(x) \;=\; \left.\frac{d}{dt}\right|_{t=0} f(x+tv).$$

This operation is linear and satisfies the Leibniz rule:

$$D_v|_x(fg) = f(x)D_v|_x g + g(x) D_v|_x f.$$

If we write $v_p = \sum_{k=1}^n v^k e_k|_p$ in terms of the standard basis of $\mathbb{R}^n$ (restricted to Ω), then, by the chain rule, $D_v|_x f$ can be written as

$$D_v|_p f \;=\; v^k \frac{\partial f}{\partial x_k}(p).$$

Definition 6.3.6. Let $p \in M$. A linear map $X : \mathcal{C}^\infty(M) \to \mathbb{R}$ is said to be a *derivation at* p if it satisfies the Leibniz rule:

$$X(fg) \;=\; f(p)X(g) + g(p)X(f).$$

We let $\mathrm{Der}_p(M)$ denote the set of all derivations of $\mathcal{C}^\infty(M)$ at p.

Remark 6.3.7. It is an elementary exercise to show that $\mathrm{Der}_p(M)$ forms a (real) vector space under the operations $(X+Y)(f) = X(f)+Y(f)$, and $(\lambda X)(f) = \lambda(X(f))$ for all $\lambda \in \mathbb{R}$, $f \in \mathcal{C}^\infty(M)$, and $X, Y \in \mathrm{Der}_p(M)$.

Theorem 6.3.8. For any $p \in M$, the map $v_p \mapsto D_v|_p$ is an isomorphism from T_pM onto $\mathrm{Der}_p(M)$.

Corollary 6.3.9. For any $p \in M$, the derivations

$$\left.\frac{\partial}{\partial x_1}\right|_p, \cdots, \left.\frac{\partial}{\partial x_n}\right|_p,$$

defined by

$$\left.\frac{\partial}{\partial x_k}\right|_p f \;=\; \frac{\partial f}{\partial x_k}(p),$$

form a basis for T_pM.

Definition 6.3.10. Let M be a smooth manifold with T_pM the tangent space to M at $p \in M$. The *cotangent space* at p, denoted by T_p^*M, is the dual space to T_pM:

$$T_p^*M \;:=\; (T_pM)^*.$$

Example 6.3.11. Let $\partial_{x_1}|_p, ..., \partial_{x_n}|_p$ denote the coordinate partial derivatives at a point $p \in \Omega$, which we can view as either tangent vectors or as derivations. These provide a basis for the tangent space $T_x\Omega$, and the corresponding dual basis is defined $dx^1, ..., dx^n$.

6.4. Vector Bundles

This linear algebraic picture works if we keep the point $p \in \Omega$ fixed. But, of course, this is very restrictive – this would demand that all functions are constants we do not allow p to vary. If we allow p to vary, however, then we need to permit the tangent spaces $T_p\Omega$, and the cotangent spaces $T_p^*\Omega$ to vary.
This leads to the notion of a vector bundle. Let $\Omega \subseteq \mathbb{R}^n$ be a domain. For each $p \in \Omega$, let V_p be a (real) vector space of dimension k. Set $V = \coprod_{p\in\Omega} V_p$ and let $\pi : V \to \Omega$ be the map which projects V_p to p.

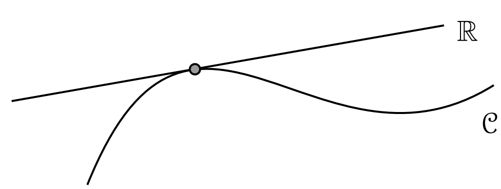
Definition 6.4.1. Let $\pi : \mathcal{E} \to M$ be a smooth map of smooth manifolds. We say that π is a *rank k vector bundle* if the fibers $\mathcal{E}_p := \pi^{-1}(p)$ are vector spaces of rank k and for any point $p \in M$, there is an open neighbourhood $\mathcal{U} \subset X$ of p such that $\pi^{-1}(\mathcal{U}) \simeq \mathcal{U} \times \mathbb{R}^k$.

Example 6.4.2. A vector bundle $\mathcal{E} \to X$ is said to be *trivial* if $\mathcal{E} = X \times \mathbb{R}^k$ for some $k \in \mathbb{N}_0$.

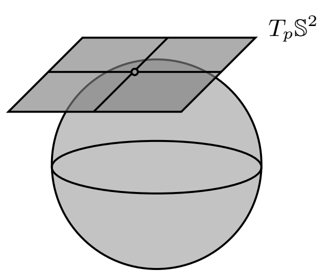
Example 6.4.3. Let X be a smooth manifold with T_pX the tangent space at $p \in X$. The *tangent bundle* is the smooth vector bundle of rank $\dim(X)$ given by $\pi : TX \to X$, where

$$TX = \coprod_{p\in X} T_pX.$$

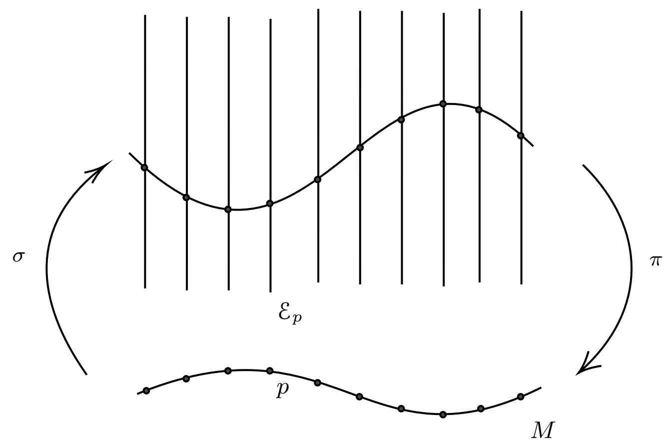
Example 6.4.4. The tangent to a curve gives an example of the tangent bundle:



Example 6.4.5. The tangent bundle to the sphere $\mathbb{S}^2$ is given by a family of tangent planes $T_p\mathbb{S}^2$ orthogonal to $p \in \mathbb{S}^2$, for each point $p \in \mathbb{S}^2$.



Definition 6.4.6. Let $\pi : \mathcal{E} \to \Omega$ be a vector bundle. A section of $\mathcal{E}$ is a smooth map $\sigma : \Omega \to \mathcal{E}$ such that $\pi \circ \sigma = \mathrm{id}$.



Example 6.4.7. A function $f : X \to \mathbb{R}$ is a section of the trivial bundle $\mathcal{E} = X \times \mathbb{R} \to X$.

Example 6.4.8. A vector field is a section of the tangent bundle TX.

Proliferation of Vector Bundles. Many of the natural operations from vector spaces carry over to vector bundles. If $\mathcal{E} \to X$ and $\mathcal{F} \to X$ are smooth vector bundles of rank k and rank ℓ respectively, then

(i) the *direct sum* $\mathcal{E} \times \mathcal{F} \to X$ is a smooth vector bundle of rank $k + \ell$.
(ii) the *tensor product* $\mathcal{E} \otimes \mathcal{F} \to X$ is a smooth vector bundle of rank $k\ell$.
(iii) the *dual bundle* $\mathcal{E}^* \to X$ is a smooth vector bundle of rank k.

Example 6.4.9. Let X be a smooth manifold. The vector bundle dual to the tangent bundle is called the *cotangent bundle*, and is denoted by T^*X. Sections of T^*X are called *differential* 1*–forms* or 1*–forms*.

Example 6.4.10. Let X be a smooth manifold. The *bundle of* (k, ℓ)*–tensors* is the vector bundle $\mathcal{T}^\ell_k(X) \to X$ given by

$$\mathcal{T}^\ell_k(X) \; := \; \coprod_{p \in X} \mathcal{T}^\ell_k(T_pX) \; = \; \coprod_{p \in X} \otimes^k T^*_p X \otimes^\ell T_pX.$$

Sections of $\mathcal{T}^\ell_k(X)$ are called (k, ℓ)*–tensors* or (k, ℓ)*–tensor fields*.

Definition 6.4.11. Let $\pi_{\mathcal{E}} : \mathcal{E} \to X$ and $\pi_{\mathcal{F}} : \mathcal{F} \to X$ be smooth vector bundles over a smooth manifold X. We say that $\mathcal{F}$ is a *subbundle* of $\mathcal{E}$ if for each $p \in X$, the fibers $\mathcal{F}_p := \pi^{-1}_{\mathcal{F}}(p)$ is a vector subspace of $\mathcal{E}_p := \pi^{-1}_{\mathcal{E}}(p)$.

Definition 6.4.12. If $\mathcal{F} \to X$ is a subbundle of $\mathcal{E} \to X$, then we can form the *quotient bundle* $\mathcal{E}/\mathcal{F} \to X$, whose fibers are given by the quotient vector spaces $\mathcal{E}_p/\mathcal{F}_p$.

Example 6.4.13. The *kth exterior power* $\Lambda^k(X)$ of the cotangent bundle T^*X of a smooth manifold X is the vector bundle $\Lambda^k(X) \to X$, where

$$\Lambda^k(X) \ := \ \Lambda^k(T^*X) \ := \ \coprod_{p\in X} \Lambda^k(T_p^*X).$$

Smooth sections of $\Lambda^k(X)$ are called *differential k–forms* or *k–form*.

Theorem 6.4.14. Let X be a smooth manifold. There exists a unique linear map

$$d : \Lambda^\bullet(X) \longrightarrow \Lambda^\bullet(X)$$

such that

(i) $d : \Lambda^k(X) \longrightarrow \Lambda^{k+1}(X)$.
(ii) $d(f) = df$ (the ordinary differential) for $f \in \Lambda^0(X)$.
(iii) (Leibniz rule). If $\sigma \in \Lambda^k(X)$ and $\tau \in \Lambda^\bullet(X)$, then

$$d(\sigma \wedge \tau) \ = \ (d\sigma) \wedge \tau + (-1)^k \sigma \wedge d\tau.$$

(iv) (Nilpotence). $d^2 = 0$.

Before proving the above theorem, we first establish the following lemma showing that for any exterior differentiation operator d, the value $(d\omega)(x)$ depends only on the behavior of ω in a small neighborhood of x. In particular, exterior differentiation operators are *local* in nature.

Lemma 6.4.15. Let d be an exterior differentiation operator, i.e., a linear map satisfying conditions (i)–(iv) above. Let ω be a differential form such that $\omega|_{\mathcal{U}} = 0$ for some open set $\mathcal{U} \subset X$. Then $(d\omega)|_{\mathcal{U}} = 0$. In particular, if ω and τ are differential forms such that $\omega|_{\mathcal{U}} = \tau|_{\mathcal{U}}$ for some open set $\mathcal{U} \subset X$, then $(d\omega)|_{\mathcal{U}} = (d\tau)|_{\mathcal{U}}$.

Proof. Suppose ω vanishes identically on the open set $\mathcal{U}$. Let $x_0 \in \mathcal{U}$. Take $f : X \to \mathbb{R}$ to be a smooth function such that $f(x_0) = 1$ and $f(x) = 0$ for all $x \notin \mathcal{U}$. The differential form $f\omega$ then vanishes identically on X. Hence, by condition (iii), i.e., the Leibniz rule, we have

$$0 \ = \ d(f\omega) \ \ = \ \ (df) \wedge \omega + f d\omega.$$

Evaluating at x_0 shows that $(d\omega)(x_0) = 0$, and since this holds for all $x_0 \in \mathcal{U}$, we see that $(d\omega)|_{\mathcal{U}} = 0$. If $\omega|_{\mathcal{U}} = \tau|_{\mathcal{U}}$, then $(\omega - \tau)|_{\mathcal{U}} = 0$, and therefore,

$$0 \ = \ (d(\omega - \tau))|_{\mathcal{U}} \ \ = \ \ (d\omega - d\tau)\ |_{\mathcal{U}},$$

which implies that $(d\omega)|_{\mathcal{U}} = (d\tau)|_{\mathcal{U}}$. $\square$

We are now ready to prove the theorem:

Proof of Uniqueness. Suppose an exterior differentiation operator d exists. Let us show that there is only one of them. To this end, let $x \in X$ be a point, contained in a chart $\mathcal{U} \subset X$, in which we have local coordinates $(x_1, ..., x_n)$. Let ω be a smooth k–form on X. Restricting ω to the coordinate chart $\mathcal{U}$ permits us to write

$$\omega|_{\mathcal{U}} = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \tag{6.4.1}$$

where $a_{i_1 \cdots i_k} \in \mathcal{C}^\infty(\mathcal{U}, \mathbb{R})$. Since the right-hand side of (6.4.1) is not a differential form on X, we cannot apply an exterior differentiation operator to it. To deal with this, let $\mathcal{U}_1$ be an open ball containing x such that the closure $\overline{\mathcal{U}}_1 \subset \mathcal{U}$ is contained in $\mathcal{U}$. Let $f \in \mathcal{C}^\infty(X, \mathbb{R})$ be a smooth function defined such that $f(x) = 1$ for all $x \in \mathcal{U}_1$, and $f(x) = 0$ for all $x \notin \mathcal{U}$. Then

$$\widetilde{\omega} := \sum_{i_1 < \cdots < i_k} (f a_{i_1 \cdots i_k}) d(f x_{i_1}) \wedge \cdots \wedge d(f x_{i_k})$$

is a smooth k–form on X. Compute

$$\begin{aligned} d\widetilde{\omega} &= \sum_{i_1 < \cdots < i_k} d(f a_{i_1 \cdots i_k} d(f x_{i_1}) \wedge \cdots \wedge d(f x_{i_k})) \\ &= \sum_{i_1 < \cdots < i_k} d(f a_{i_1 \cdots i_k}) \wedge d(f x_{i_1}) \wedge \cdots \wedge d(f x_{i_k}) \\ &\qquad\qquad + \sum_{i_1 < \cdots < i_k} (f a_{i_1 \cdots i_k}) d(d(f x_{i_1}) \wedge \cdots \wedge d(f x_{i_k})) \\ &= \sum_{i_1 < \cdots < i_k} d(f a_{i_1 \cdots i_k}) \wedge d(f x_{i_1}) \wedge \cdots \wedge d(f x_{i_k}), \end{aligned}$$

where the first equality follows from linearity, the second from the Leibniz rule, and third equality from the Leibniz rule and nilpotence. From the lemma, $\widetilde{\omega}|_{\mathcal{U}_1} = \omega|_{\mathcal{U}_1}$ implies that $(d\widetilde{\omega})|_{\mathcal{U}_1} = (d\omega)|_{\mathcal{U}_1}$. Since f is identically 1 on $\mathcal{U}_1$, we see that

$$(d\omega)|_{\mathcal{U}_1} = \sum_{i_1 < \cdots < i_k} \partial_{x_j}(a_{i_1 \cdots i_k}) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Hence, if an exterior differentiation operator d exists, its value at x must be given by the above formula. Since the point x was arbitrary, this establishes uniqueness. $\square$

Proof of Existence. Let $\mathcal{U}$ be a coordinate chart on X, in which we have local coordinates $(x_1, ..., x_n)$. We first define d locally on $\mathcal{U}$. To this end, let $\omega \in \Lambda^k(\mathcal{U})$ be a smooth

k–form given by

$$\omega \;:=\; \sum_{i_1<\cdots<i_k} a_{i_1\cdots i_k} dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Define

$$d_{\mathcal{U}}\omega \;=\; \sum_{i_1<\cdots<i_k}\sum_{j=1}^{n} \partial_{x_j}(a_{i_1\cdots i_k})dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

We extend the definition of $d_{\mathcal{U}}$ to any differential form on $\mathcal{U}$ by forcing $d_{\mathcal{U}}$ to be linear. Properties (i) and (ii) are then immediate. It remains to verify the Liebniz rule and the nilpotence property. First note that any differential form is a sum of forms of the type $a_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}$. Since $d_{\mathcal{U}}$ is linear, and the wedge product is distributive, we need only verify (iii) and (iv) on forms of this type.

Let us verify (iii) for $d_{\mathcal{U}}$. Write $\sigma := a_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}$ and $\tau := b_{j_1\cdots j_\ell}dx^{j_1}\wedge\cdots\wedge dx^{j_\ell}$. Then

$$\sigma\wedge\tau \;=\; a_{i_1\cdots i_k}b_{j_1\cdots j_\ell}dx^{i_1}\wedge\cdots\wedge dx^{i_k}\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_\ell},$$

and hence,

$$\begin{aligned}
d_{\mathcal{U}}(\sigma\wedge\tau) &= \sum_{r=1}^{n}\left[\partial_{x_r}(a_{i_1\cdots i_k})b_{j_1\cdots j_\ell} + a_{i_1\cdots i_k}\partial_{x_r}(b_{j_1\cdots j_\ell})\right]dx^r\wedge dx^{i_1}\wedge\cdots\wedge dx^{j_\ell}\\
&= \left(\sum_{r=1}^{n}\partial_{x_r}(a_{i_1\cdots i_k})dx^r\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\wedge\left(b_{j_1\cdots j_\ell}dx^{j_1}\wedge\cdots\wedge dx^{j_\ell}\right)\\
&\quad +(-1)^k(a_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k})\wedge\left(\sum_{r=1}^{n}\partial_{x_r}(b_{j_1\cdots j_\ell}dx^r\wedge dx^{j_1}\wedge\cdots\wedge dx^{j_\ell}\right)\\
&= (d_{\mathcal{U}}\sigma)\wedge\tau + (-1)^k\mu\wedge(d_{\mathcal{U}}\tau).
\end{aligned}$$

This verifies property (iii) for $d_{\mathcal{U}}$. For property (iv), if $\sigma = a_{i_1\cdots i_k}dx^{i_1}\wedge\cdots\wedge dx^{i_k}$, then

$$d_{\mathcal{U}}\sigma \;=\; \sum_{r=1}^{n}\partial_{x_r}(a_{i_1\cdots i_k})dx^r\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

Applying the exterior derivative $d_{\mathcal{U}}$ again,

$$d_{\mathcal{U}}(d_{\mathcal{U}}\sigma) \;=\; \sum_{s=1}^{n}\sum_{r=1}^{n}\partial_{x_s}(\partial_{x_r}(a_{i_1\cdots i_k}))dx^s\wedge dx^r\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}.$$

The $r=s$ terms in the above expression vanish by the nilpotence of the wedge product, while the $r\neq s$ terms vanish by Clairaut's theorem.

This shows that $d_{\mathcal{U}}$ satisfies the properties of the exterior differentiation operator (i.e., properties (i)–(iv)). From the uniqueness proof given before, every linear operator satisfying properties (i)–(iv) is given by the formula specifying $d_{\mathcal{U}}$. In particular, if $\mathcal{U}_1$ is any open subset of $\mathcal{U}$, the coordinates on $\mathcal{U}$ restrict to coordinates on $\mathcal{U}_1$, and the formula for $d_{\mathcal{U}_1}$ coincides with the formula for $(d_{\mathcal{U}})|_{\mathcal{U}_1}$. Hence, we can define d globally by declaring $(d\omega)|_{\mathcal{U}} = d_{\mathcal{U}}(\omega|_{\mathcal{U}})$ for all differential forms ω, where $\mathcal{U}$ is any coordinate neighborhood. It remains to check that d is well-defined; but this is elementary, since for any pair of coordinate charts $\mathcal{U}$ and $\mathcal{V}$, we have

$$(d_{\mathcal{U}}(\omega|_{\mathcal{U}}))|_{\mathcal{U}\cap\mathcal{V}} \;=\; d_{\mathcal{U}\cap\mathcal{V}}(\omega|_{\mathcal{U}\cap\mathcal{V}}) \;=\; (d_{\mathcal{V}}(\omega|_{\mathcal{V}}))|_{\mathcal{U}\cap\mathcal{V}}.$$

It is clear that d has properties (i)–(iv), since $d_{\mathcal{U}}$ has these properties for all $\mathcal{U}$. □

Pullback of differential forms. Recall from our first course in one-variable calculus that if we are given an integral of the form

$$\int_1^3 x\sqrt{x^2+1}dx,$$

for instance, we evaluate the integral by making a substitution $u = x^2+1$. In this case, we see that

$$\int_1^3 x\sqrt{x^2+1}dx \;=\; \frac{1}{2}\int_2^{10}\sqrt{u}du \;=\; \frac{2\sqrt{2}}{3}(5\sqrt{5}-1).$$

More generally, if we are given an integral of the form

$$\int_a^b f(g(x))g'(x)dx,$$

we set $u = g(x)$, and write $du = g'(x)dx$. Then

$$\int_a^b f(g(x))g'(x)dx \;=\; \int_{g(a)}^{g(b)} f(u)du.$$

This is formalized through pulling back differential forms:

Definition 6.4.16. Let X and Y be smooth manifolds, and let $\varphi : X \to Y$ be a smooth map. We define the *pullback of a k–form* as follows:

(i) If $f : Y \to \mathbb{R}$ is a 0–form (i.e., a function), then $\varphi^*(f) = f \circ \varphi$.

(ii) If $\omega \in \Lambda^k(Y)$, then

$$(\varphi^*\omega)(x)(v_1, ..., v_k) \;=\; \omega(\varphi(x))(d\varphi(v_1), ..., d\varphi(v_k)),$$

where $v_1, ..., v_k \in T_xX$, and $x \in X$.

Remark 6.4.17. The following facts are straightforward:

(i) If ω is a smooth differential form, then $\varphi^*\omega$ is easily observed to be a smooth differential form.

(ii) The pullback of a k–form is a k–form.

(iii) The pullback is linear, and moreover, is compatible with the wedge product in the sense that

$$\varphi^*(\sigma \wedge \tau) \;=\; (\varphi^*\sigma) \wedge (\varphi^*\tau).$$

This can be formulated as stating that $\varphi^* : \Lambda^k(Y) \to \Lambda^k(X)$ is an algebra homomorphism.

Theorem 6.4.18. Let $\varphi : X \to Y$ be a smooth map between smooth manifolds. Then

$$d \circ \varphi^* \;=\; \varphi^* \circ d.$$

PROOF. We first prove the statement on 0–forms. To this end, let $f : Y \to \mathbb{R}$ be a smooth function. Then, for $v \in T_xX$, we compute

$$(d \circ \varphi^*)(f)(v) \;=\; d(f \circ \varphi)(v) \;=\; (df \circ d\varphi)(v) \;=\; \varphi^*(df)(v) \;=\; (\varphi^* \circ d)(f)(v).$$

Suppose now that $\omega \in \Lambda^1(Y)$ is a 1–form given by $\omega = df$. Then

$$(d \circ \varphi^*)(\omega) \;=\; d(\varphi^*(df)) \;=\; d(\varphi^* \circ d(f)) \;=\; d(d \circ \varphi^*(f)) \;=\; 0.$$

Similarly,

$$(\varphi^* \circ d)(\omega) \;=\; \varphi^*(d\omega) \;=\; \varphi^*(d^2 f) \;=\; \varphi^*(0) \;=\; 0.$$

From these two cases, and the fact that φ^* is an algebra homomorphism, the result is established in general by checking it locally on k–forms ω restricted to local coordinate neighborhoods

$$\omega|_{\mathcal{U}} \;=\; \sum_{1<i_1<\cdots<i_k} a_{i_1\cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

This task is left to the reader. □

Example 6.4.19. Let $\varphi : [0, 2\pi] \to \mathbb{R}^2$ be the smooth map $\varphi(\vartheta) := \cos(\vartheta)\mathbf{i} + \sin(\vartheta)\mathbf{j}$. Let $\omega \in \Lambda^1(\mathbb{R}^2\backslash\{0\})$ be the 1–form given by

$$\omega \;:=\; -\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy.$$

Then

$$\begin{aligned}
\varphi^*\omega \;&=\; -\frac{\sin(\vartheta)}{\sin^2(\vartheta)+\cos^2(\vartheta)}d(\cos(\vartheta)) + \frac{\cos(\vartheta)}{\cos^2(\vartheta)+\sin^2(\vartheta)}d(\sin(\vartheta)) \\
&=\; -\sin(\vartheta)d(\cos(\vartheta)) + \cos(\vartheta)d(\sin(\vartheta)) \\
&=\; (-\sin(\vartheta))(-\sin(\vartheta)d\vartheta) + \cos(\vartheta)(\cos(\vartheta)d\vartheta) \\
&=\; (\sin^2(\vartheta) + \cos^2(\vartheta))d\vartheta \\
&=\; d\vartheta.
\end{aligned}$$

Let us show that $H^k_{\mathrm{DR}}(\mathbb{R}^n, \mathbb{R}) = 0$ for all $k > 0$. Since $\mathbb{R}^n$ is diffeomorphic to the unit ball centered at the origin in $\mathbb{R}^n$, it suffices to show that $H^k_{\mathrm{DR}}(\mathbb{B}^n, \mathbb{R}) = 0$. For this, we need the following lemma:

Lemma 6.4.20. Let X be a smooth manifold. Then for each k, consider the maps

$$\Lambda^{k-1}(X) \xrightarrow{d} \Lambda^k(X) \xrightarrow{d} \Lambda^{k+1}(X).$$

Suppose there exist linear maps

$$H_{k-1} : \Lambda^k(X) \longrightarrow \Lambda^{k-1}(X), \qquad H_k : \Lambda^{k+1}(X) \longrightarrow \Lambda^k(X)$$

such that

$$H_k \circ d + d \circ H_{k-1} \;=\; \mathrm{id}_k,$$

where id_k denotes the identity map on $\Lambda^k(X)$. Then $H^k_{\mathrm{DR}}(X, \mathbb{R}) = 0$.

Proof. Let $\omega \in \Lambda^k(X)$ be a closed k–form. Then

$$\omega \;=\; \mathrm{id}(\omega) \;=\; (H_k \circ d + d \circ H_{k-1})(\omega) \;=\; H_k(d\omega) + d(H_{k-1}(\omega)) \;=\; d(H_{k-1}(\omega)).$$

□

Definition 6.4.21. Let X be a smooth manifold. A sequence of linear maps

$$H_k : \Lambda^{k+1}(X) \longrightarrow \Lambda^k(X),$$

where $k \in \mathbb{N}_0$, satisfying

$$H_k \circ d + d \circ H_{k-1} = \mathrm{id}_k$$

for all k, is called a *homotopy operator*.

Theorem 6.4.22. (the Poincaré lemma). Let $\mathbb{B}^n \subset \mathbb{R}^n$ denote the unit ball centered at the origin in $\mathbb{R}^n$. Then for all $k > 0$,

$$H^k_{\mathrm{DR}}(\mathbb{B}^n, \mathbb{R}) = 0.$$

Proof. From the previous lemma, it suffices to construct a homotopy operator. For each k, the maps will be required to be linear, it suffices to define H_{k-1} on forms

$$\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

With ω defined as above, set

$$H_{k-1}(\omega)(x) \;:=\; \left(\int_0^1 t^{k-1} f(tx) dt \right) \sigma,$$

where

$$\sigma \;:=\; x_{i_1} dx^{i_2} \wedge \cdots \wedge dx^{i_k} - x_{i_2} dx^{i_1} \wedge dx^{i_3} \wedge \cdots \wedge dx^{i_k} + \cdots + (-1)^{k-1} x_{i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_{k-1}}.$$

Note that σ is precisely the $(k-1)$–form such that $d\sigma = k dx^{i_1}\wedge\cdots\wedge dx^{i_k}$. It suffices to show that

$$H_k\circ d + d\circ H_{k-1} = \mathrm{id}_k.$$

To this end, compute

$$\begin{aligned}
(d\circ H_{k-1})(\omega)(x) &= d\left[\left(\int_0^1 t^{k-1}f(tx)dt\right)\sigma\right]\\
&= \sum_{j=1}^n \partial_{x_j}\left(\int_0^1 t^{k-1}f(tx)dt\right)dx^j\wedge\sigma + \left(\int_0^1 t^{k-1}f(tx)dt\right)d\sigma\\
&= \sum_{j=1}^n \left(\int_0^1 t^{k-1}\partial_{x_j}(f(tx))dt\right)dx^j\wedge\sigma + \left(\int_0^1 t^{k-1}f(tx)dt\right)d\sigma\\
&= \sum_{j=1}^n \left(\int_0^1 t^k f_{x_j}(tx)dt\right)dx^j\wedge\sigma + k\left(\int_0^1 t^{k-1}f(tx)dt\right)dx^{i_1}\wedge\cdots\wedge dx^{i_k}.
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
(H_k\circ d)(\omega)(x) &= H_k\left(\sum_{j=1}^n f_{x_j}dx^j\wedge dx^{i_1}\wedge\cdots\wedge dx^{i_k}\right)\\
&= \sum_{j=1}^n\left(\int_0^1 t^k f_{x_j}(tx)dt\right)(x_j dx^{i_1}\wedge\cdots\wedge dx^{i_k} - dx^j\wedge\sigma).
\end{aligned}$$

Combining these expressions yields

$$\begin{aligned}
&(d\circ H_{k-1} + H_k\circ d)(\omega)(x)\\
=\ &\left[k\left(\int_0^1 t^{k-1}f(tx)dt\right) + \sum_{j=1}^n\left(\int_0^1 t^k f_{x_j}(tx)x_j dt\right)\right]dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
=\ &\left[\int_0^1\left(kt^{k-1}f(tx) + t^k\frac{d}{dt}f(tx)\right)dt\right]dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
=\ &\left(\int_0^1 \frac{d}{dt}(t^k f(tx))dt\right)dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
=\ &\Big[t^k f(tx)\Big]_0^1 dx^{i_1}\wedge\cdots\wedge dx^{i_k}\\
=\ &f(x)dx^{i_1}\wedge\cdots\wedge dx^{i_k} \ = \ \omega(x),
\end{aligned}$$

for all $x\in\mathbb{B}^n$. $\square$

Remark 6.4.23. The homotopy operator constructed in the proof of the Poincaré lemma are not plucked out of thin air. To illuminate their definition, let us observe that for a vector v in vector space V, we can define a map

$$i_v : \Lambda^k(V^*) \longrightarrow \Lambda^{k-1}(V^*), \qquad i_v(\omega)(u_1, ..., u_{k-1}) \;=\; \omega(v, u_1, ..., u_{k-1}).$$

The map $i_\cdot : V \otimes \Lambda^k(V^*) \to \Lambda^{k-1}(V^*)$ is bilinear.

Definition 6.4.24. The bilinear map $i_\cdot : V \otimes \Lambda^k(V^*) \longrightarrow \Lambda^{k-1}(V^*)$ defined by

$$i_v : \Lambda^k(V^*) \longrightarrow \Lambda^{k-1}(V^*), \qquad i_v(\omega)(u_1, ..., u_{k-1}) \;=\; \omega(v, u_1, ..., u_{k-1})$$

is called the *interior product*.

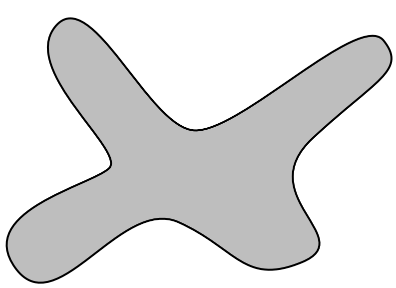
Remark 6.4.25. The $(k-1)$th homotopy operator H_{k-1} is given by applying i_x to ω and averaging over the line through the origin in the direction of x.

Remark 6.4.26. The above theorem is, in fact, a special case of a more general result: Let $\mathcal{U}$ be a smooth manifold. Suppose there exists a smooth map $\Psi : \mathcal{U} \times (-\varepsilon, 1+\varepsilon) \to \mathcal{U}$, where $\Phi(u, 1) = u$ for all $u \in \mathcal{U}$, and $\Phi(u, 0) = u_0$ for all $u \in \mathcal{U}$, and some $u_0 \in \mathcal{U}$. Then $H^k_{\mathrm{DR}}(\mathcal{U}) = 0$ for all $k \in \mathbb{N}$.

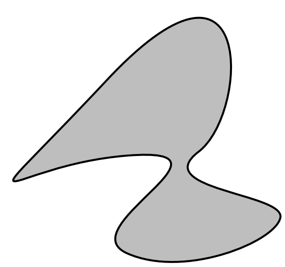
The map Ψ is a *smooth homotopy.* The theorem asserted here states that if $\mathcal{U}$ is smoothly homotopic to a point, then the cohomology of $\mathcal{U}$ is the cohomology of a point. In the theorem we proved above, the smooth homotopy is given by

$$\Phi(x, t) \;:=\; tx, \qquad t \in (-\varepsilon, 1+\varepsilon),\; x \in \mathbb{B}^n.$$

Remark 6.4.27. The proof of the Poincaré lemma given above works equally well for domains which are *star-shaped*, i.e., there is a point $x_0 \in \mathcal{U}$ such that the line segment joining x_0 to any other point in $\mathcal{U}$ is contained in $\mathcal{U}$.



star-shaped



not star-shaped

Proof of Stokes' theorem. Let (ρ_α) be a partition of unity for M, with each ρ_α supported in a chart $\varphi_\alpha : \mathcal{U}_\alpha \to \mathcal{V}_\alpha$. Then

$$\begin{aligned}\int_M d\omega &= \sum_\alpha \int_M d(\rho_\alpha \omega) \\ &= \sum_\alpha \int_{\mathcal{V}_\alpha} ((\varphi_\alpha^{-1})_*(d(\rho_\alpha\omega)))(e_1, ..., e_n) dx^1 \wedge \cdots \wedge dx^n \\ &= \sum_\alpha \int_{\mathcal{V}_\alpha} d((\varphi_\alpha^{-1})_*(\rho_\alpha\omega))(e_1, ..., e_n) dx^1 \wedge \cdots \wedge dx^n.\end{aligned}$$

For each α, there are two possibilities: φ_α is a standard chart, or it is a boundary chart. If φ_α is a standard chart, with $\mathcal{V}_\alpha$ an open set in $\mathbb{R}^n$, we can write

$$\omega = \sum_{k=1}^n \omega_k dx^1 \wedge dx^{k-1} \wedge dx^{k+1} \wedge \cdots \wedge dx^n.$$

The integrand in the corresponding integral becomes

$$\sum_{k=1}^n \partial_{x_k}(\rho_\alpha \omega_k).$$

By Fubini's theorem and the fundamental theorem of calculus, the resulting integral vanishes since ρ_α vanishes on the boundary of the chart.

If φ_α happens to be a boundary chart, however, then we have

$$\begin{aligned}\sum_{k=1}^n \int_{\mathcal{V}_\alpha} \partial_{x_j}(\rho_\alpha\omega_k) dx^1 \wedge \cdots \wedge dx^n &= \int_{\mathbb{R}^{n-1}\times\{0\}} \int_{-\infty}^0 \partial_{x_n}(\rho_\alpha\omega_n) dx^n \wedge dx^1 \wedge \cdots \wedge dx^{n-1} \\ &= \int_{\mathbb{R}^{n-1}\times\{0\}} \rho_\alpha \omega_n dx^1 \wedge \cdots \wedge dx^{n-1} \\ &= \int_{\partial M} \rho_\alpha \omega,\end{aligned}$$

where the last line is obtained by ensuring that ∂M has the correct orientation. Summing over α, remembering that $\sum_\alpha \rho_\alpha = 1$, proves the theorem.

6.5. Riemannian metrics

We have seen that if X is a smooth manifold, the natural objects to integrate are given by differential forms. Throughout the course, however, we have encountered (likely, without recognition) objects which required additional structure.

For instance, recall that we exhibited three types of line integrals in Chapter 3:

(i) line integrals of functions:

$$\int_{\mathcal{C}} f ds \;=\; \int_{t_0}^{t_1} f(\alpha(t))|\alpha'(t)|dt,$$

(ii) line integrals of vector fields

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r} \;=\; \int_{t_0}^{t_1} \mathbf{F}(\alpha(t)) \cdot \alpha'(t)dt,$$

(iii) line integrals of 1–forms

$$\int_{\mathcal{C}} \omega.$$

where $\mathcal{C}$ is parametrized by $\alpha : [t_0, t_1] \to \mathbb{R}^n$.

Line integrals of type (iii), i.e., of 1–forms, are the most straightforward to evaluate. There is an important distinction to be made between line integrals of type (i) and the line integrals of type (ii)–(iii), however.

Remark 6.5.1. Let us first observe that line integrals of type (ii) and type (iii) are equivalent: Let $\alpha : [t_0, t_1] \to \mathbb{R}^2$ be a parametrization for a curve $\mathcal{C}$, given by $\alpha(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$. If $\mathbf{F} = P\mathbf{i} + Q\mathbf{j}$, then the 1–form associated to $\mathbf{F}$ is $\omega_{\mathbf{F}} = Pdx + Qdy$, and we see that

$$\begin{aligned} \int_{\mathcal{C}} \omega_{\mathbf{F}} &= \int_{t_0}^{t_1} P(x(t), y(t))x'(t)dt + Q(x(t), y(t))y'(t)dt \\ &= \int_{t_0}^{t_1} (P(x(t), y(t))\mathbf{i} + Q(x(t), y(t))\mathbf{j}) \cdot (x'(t)\mathbf{i} + y'(t)\mathbf{j})dt \\ &= \int_{t_0}^{t_1} \mathbf{F}(\alpha(t)) \cdot \alpha'(t)dt \;=\; \int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{r}. \end{aligned}$$

On the other hand, the line integral of type (i), i.e., of a function, is not of this type. The line integrals of type (i) require an additional level of structure in order to compute $|\alpha'(t)|$. Indeed, to compute $|\alpha'(t)|$, we need to understand *where* we are computing the norm. Observe that the vector $\alpha'(t)$ is a vector in the tangent line to $\mathcal{C}$ at the point t, which we denote by $T_t\mathcal{C}$. Hence, we need to have a norm on $T_t\mathcal{C}$, for each $t \in \mathcal{C}$, which varies (at least continuously in t). This is given by a *Riemannian metric*.

Definition 6.5.2. Let X be a smooth manifold. A *Riemannian metric* g on X is a smooth family of inner products $g_p : T_pX \times T_pX \to \mathbb{R}$ on the tangent spaces T_pX.

Example 6.5.3. A Riemannian metric on $\mathbb{R}^n$ is given by the dot product $\cdot : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

Example 6.5.4. Again, consider the smooth manifold $M = \mathbb{R}^n$. Let A be an $n \times n$ positive-definite matrix. A Riemannian metric $g : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ can be defined by

$$g(\mathbf{u}, \mathbf{v}) \ := \ \mathbf{u}^t A \mathbf{v}.$$

Remark 6.5.5. A Riemannian metric g allows us to compute the lengths of tangent vectors $v \in T_pM$. In particular, if $\alpha : [0,1] \to M$ is a curve in M, then the length of the tangent vector $\alpha'(t)$ is given by $|\alpha'(t)|^2 := g(\alpha'(t), \alpha'(t))$. From our knowledge of arc length, we see that the Riemannian metric subsequently permits us to define the length of the curve α:

Definition 6.5.6. Let M be a smooth manifold with a Riemannian metric g. The *length function* associated to g is given by

$$\mathrm{length}_g(\alpha) \ := \ \int_{t_0}^{t_1} \sqrt{g(\alpha'(t), \alpha'(t))}dt,$$

where $\alpha : [t_0, t_1] \to M$ is a smooth curve.

Example 6.5.7. If $M = \mathbb{R}^n$ and g is the dot product, this yields the familiar formula for arc length:

$$\mathrm{length}(\alpha) \ = \ \int_{t_0}^{t_1} |\alpha'(t)|dt.$$

Remark 6.5.8. Observe that we can now give a meaningful description of the ds term appearing in the definition of a line integral of a function: Let g be a Riemannian metric on a smooth manifold M. The Riemannian metric takes in two tangent vectors, and outputs a real number. In the language of tensors, the metric g defines a $(2,0)$–tensor. Let $(x_1, ..., x_n)$ denote local coordinates near a point $p \in M$. Near p, there is a matrix $(g_{ij})_{1 \leq i,j \leq n}$ such that

$$g = \sum_{i,j=1}^{n} g_{ij} dx_i \otimes dx_j.$$

For instance, the Riemannian metric on $\mathbb{R}^2$ given by the dot product on $\mathbb{R}^2$ is given by

$$g = dx \otimes dx + dy \otimes dy.$$

This is sometimes abbreviated to

$$g = (dx)^2 + (dy)^2.$$

The length of a curve α in $\mathbb{R}^2$ is then given by

$$\int_{t_0}^{t_1} \sqrt{g(\alpha'(t), \alpha'(t))}dt = \int_{t_0}^{t_1} \sqrt{(dx)^2 + (dy)^2} \ =: \ \int_{\mathcal{C}} ds.$$

Hodge $\star$–operator. Another construction used in the course which requires an additional level of structure is the Hodge $\star$–operator. We can now present a more formal definition.

Definition. Let X be an oriented Riemannian manifold of dimension n. The *Hodge $\star$–operator* $\star : \Lambda^k(X) \to \Lambda^{n-k}(X)$ by

$$\star(\varepsilon^{i_1} \wedge \cdots \wedge \varepsilon^{i_k}) \;=\; \varepsilon^{i_{k+1}} \wedge \cdots \wedge \varepsilon^{i_n},$$

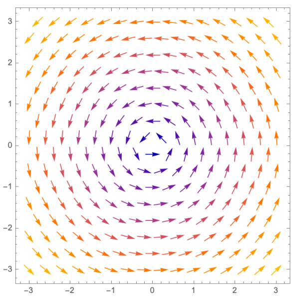
where $\varepsilon^{i_1}, ..., \varepsilon^{i_n}$ is a positive basis.

Remark. In local coordinates, we have

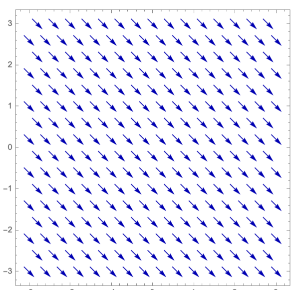
$$\star(1) \;=\; \sqrt{\det(g_{ij})}dx^1 \wedge \cdots \wedge dx^n.$$

Answers

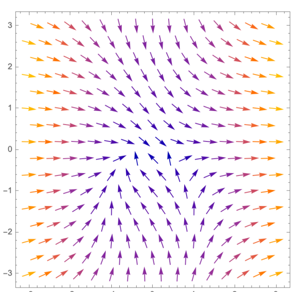
Chapter 1.1.



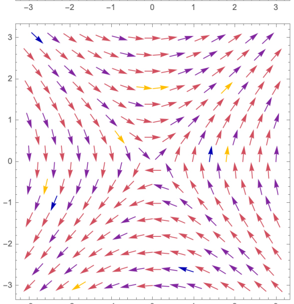
1. **(i)**



(ii)



(iii)



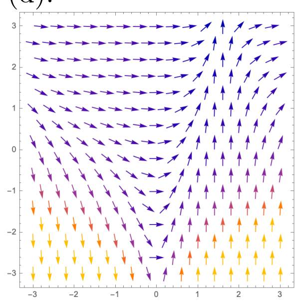
(iv)

2. **(i)** (c).

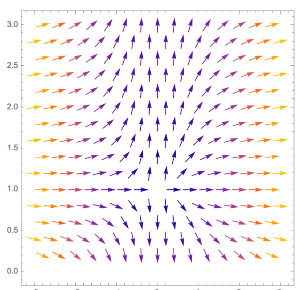
(ii) (a).

(iii) (b).

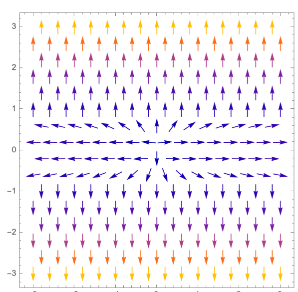
(iv) (d).



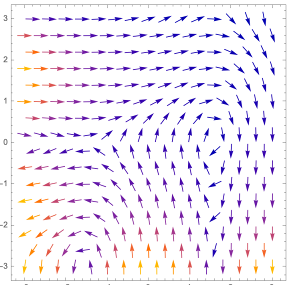
3. **(i)**

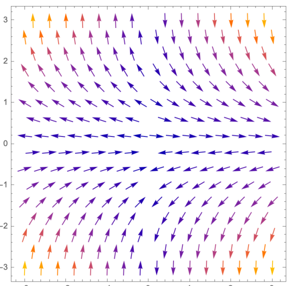


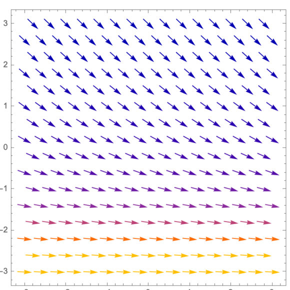
(ii)



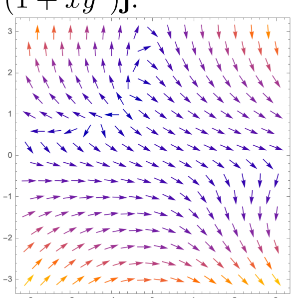
(iii)

(iv) 

4. (i) 

(ii) 

(iii) $\mathbf{F}+\mathbf{G} = (1+e^{-y}+2x\sin(y))\mathbf{i} - (1+xy^2)\mathbf{j}$.

(iv) 

5. (i) $\omega_{\mathbf{F}} = (1-x^2)dx + 6y^3dy$.
(ii) $\omega_{\mathbf{F}} = e^{-x}dx + e^y\sin(x^2)dy$.
(iii) $\omega_{\mathbf{F}} = (3x-y^2)dx + (y+1)^2dy$.
(iv) $\omega_{\mathbf{F}} = (e^{1-zx} + e^{x+y})dx + (x^3 + z^2)dy - xdz$.

6. (i) $\mathbf{F} = xy^3\mathbf{i} - 3yz\mathbf{j} + 2z^3\mathbf{k}$
(ii) $\mathbf{F} = xz\mathbf{i} + 2x^3\mathbf{j} + y\mathbf{k}$
(iii) $\mathbf{F} = 2z^2\mathbf{i} + (y-2z^2)\mathbf{j} + (z+4xy)\mathbf{k}$
(iv) $\mathbf{F} = (1+e^{-xz})\mathbf{i} + 2x\cos(z-y)\mathbf{j} + (x^2+7y)\mathbf{k}$
(v) $\mathbf{F} = (2x+z^2)\mathbf{i} + (xy - \sin(xz)\mathbf{j} - \cos(z)\mathbf{k}$

7. (i) $(2\sqrt{x^2+1} + 2 - \tan(x))dx + (15y^2 + 1 + e^x)dy$.
(ii) $(6\sqrt{x^2+1} + 4\tan(x) - 8)dx + (45y^2 + 3 - 4e^x)dy$.
(iii) $(2x^2\sqrt{x^2+1} + 3\sin(y)\tan(x) - 6\sin(y))dx + (15x^2y^2 + x^2 - 3\sin(y)e^x)dy$.

8. (i) $(\cos(x-z) - \sin(x))dx - \cos(x-z)dz$.
(ii) $-\frac{yz}{x^2\left(1+\frac{y^2}{x^2}\right)}dx + \frac{z}{x\left(1+\frac{y^2}{x^2}\right)}dy + \tan^{-1}\left(\frac{y}{x}\right)dz$.
(iii) $\Big(x + (1-y)\cos(x-z) + (y-1)\sin(x) + \frac{yz(2-\log_e}{x^2(1+y^2/}$ $\left(\frac{z(\log_e(z)-2)}{x(1+y^2/x^2)} + 3\sin(y)\right)dy + \big((\log_e(z)-2)\tan^{-1}(y/x) + (y-1)\cos(x-z)\big)dz.$

9. (i) $\frac{z}{\sqrt{1+z^2}}dz$.
(ii) $\frac{dx}{x-y} - \frac{dy}{x-y} + 2ze^{z^2}dz$.
(iii)

10. (i)
(ii)

11. (i)
(ii)

12. (i) $(3x^2e^{-y}+y)dx + (x - e^{-y}x^3)dy$.
(ii) $(-z + y\cos(xy+z))dx + x\cos(xy+z)dy + (\cos(xy+z) - x)dz$.
(iii) $\frac{dx}{x} - \frac{dy}{y} + \frac{dz}{z}$.
(iv) $\frac{e^{-z^2}}{2\sqrt{x+y}}dx + \frac{e^{-z^2}}{2\sqrt{x+y}}dy - 2e^{-z^2}z\sqrt{x+y}dz$.

13. (i) Not a gradient field.
(ii) Yes.

(**iii**) Not a gradient field.
(**iv**) Not a gradient field.
(**v**) Not a gradient field.
(**vi**) Not a gradient field.
(**vii**) Not a gradient field.
(**viii**) Not a gradient field.
(**ix**) Not a gradient field.
(**x**) Not a gradient field.

14. (**i**) Not a gradient field.
(**ii**) $f = x + y - z$.
(**iii**) Not a gradient field.
(**iv**) Not a gradient field.
(**v**) Not a gradient field.
(**vi**) Not a gradient field.
(**vii**) $f = x\sqrt{1+x^2} + \sinh^{-1}(x) + y + 5y^3$.
(**viii**) Not a gradient field.
(**ix**) Not a gradient field.
(**x**) Not a gradient field.
(**xi**) Not a gradient field.
(**xii**) Not a gradient field.

15. $\mathbf{F} = \nabla \tan^{-1}(y/x)$.

16.

17.

18.

Chapter 1.2.

1. (i) $dy \wedge dx$.
(ii) -1.
(iii) $dx \wedge dy$.
(iv) 1.

2. (i) $(4x^2y + z)dx \wedge dy + (6x - 2xz - 2y)dx \wedge dz + (z^2 - 3z - 4xy^2)dy \wedge dz$.
(ii) $(z^2 - 3z - 4xy^2)\mathbf{i} - (6x - 2xz - 2y)\mathbf{j} + (4x^2y + z)\mathbf{k}$.
(iii) $(z^2 - 3z - 4xy^2)\mathbf{i} - (6x - 2xz - 2y)\mathbf{j} + (4x^2y + z)\mathbf{k}$.

3. (i) 11.
(ii) $6xy^2 + 2xy - 2y$.

4. (i) 2–form.
(ii) 1–form.
(iii) 0–form.

5. (i) $(x^3 - y^3)dx \wedge dy$.
(ii) 0.

6. (i) $-2ydx \wedge dy$.
(ii) $\cos(x)dx \wedge dz$.
(iii) 0.
(iv) $-zdx \wedge dy + (x - y^2)dy \wedge dz$.
(v) 0.

7. (i) $(-3y^2z - 2z)dx \wedge dy + (4x^3 - y^3)dx \wedge dz + (3y^2 + 2x)dy \wedge dz$.
(ii) $(2x + 3y^2x^2z^4)dx \wedge dy \wedge dz$.

8. (i) $2zdy \wedge dz$.
(ii) $-dx \wedge dy - dx \wedge dz$.
(iii) $-x^3dx\wedge dz + yz^2dx\wedge dy + xz^2dy\wedge dz$.

9. (i) $dx \wedge dz + 2ydy \wedge dz$.
(ii) $-dx \wedge dz - 2ydy \wedge dz$.
(iii) $12dx \wedge dz + 24ydy \wedge dz$.
(iv) 0.
(v) 0.
(vi) 0.
(vii) 0.
(viii) 0.

10. (i) $2xdx - 3dy - \sin(z)dz$.
(ii) 0.
(iii) No.

11. (i) $\omega_{\mathbf{F}} = (\sin(x) - \cos(y))dx + 2xzdy - \tan(z^2)dz$.
(ii) $(2z - \sin(y))dx \wedge dy - 2xdy \wedge dz$.

12. (i) $\omega_{\mathbf{F}} = (x^3 + 2xe^y)dx + 2x^4e^{1-y}dy$.

(ii) $(8x^3e^{1-y} - 2xe^y)dx \wedge dy$.

13. (i) F.
(ii) T.
(iii) T.
(iv) F.
(v) T.
(vi) F.
(vii) T.

16. (i) $\cos(\vartheta)dr - r\sin(\vartheta)d\vartheta$.
(ii) $\sin(\vartheta)dr + r\cos(\vartheta)d\vartheta$.
(iii) $r\cos(\vartheta)\sin(\vartheta)dr + r^2\cos^2(\vartheta)d\vartheta$.
(iv) $rd\vartheta$.
(v) $\frac{dr}{r}$.
(vi) $d\vartheta$.

17. (i) T.
(ii) F.
(iii) F.
(iv) F.

19. (i) $(4x^3 - 4x^4)dx \wedge dy + (4x^3y - 4x^3z)dx \wedge dz + [(y-z)\sin(z) + (1-x)(y^2-1)]dy \wedge dz$.
(ii) $(-2y - \cos(z))dy \wedge dz$.
(iii) $-dx \wedge dy + dy \wedge dz$.
(iv) $(4x^3 + y^2 - 1)dx \wedge dy \wedge dz$.
(v) 0.
(vi) 0.

20. (i) T.
(ii) F.
(iii) T.
(iv) T.
(v) F.

21. (i) 0.
(ii) $\left(1 - e^{-x}\tan(y) + \frac{y}{\sqrt{y^2+1}}\right) dx \wedge dy \wedge dz$.
(iii) 0.
(iv) 0.

22. (i) 2.
(ii) $\mathbf{F}_\omega = -y\mathbf{i} + x\mathbf{j}$.
(iii) $2\mathbf{k}$.

Chapter 2.1.

1. (i) $(6y+1)\mathbf{k}$.
(ii) $\mathbf{k}$.
(iii) $\sec^2(x)\mathbf{k}$.
(iv) $-(2z + e^{-z})\mathbf{j} - 3x^2\mathbf{k}$.
(v) $-2z\mathbf{i} + \mathbf{j} - \sin(x-y)\mathbf{k}$.
(vi) $-x(y+z)\mathbf{i} + z(y - 2\sin(z^2))\mathbf{j} + yz\mathbf{k}$.
(vii) $-\frac{3z^2}{2\sqrt{z^3+1}}\mathbf{i} + 2ze^{z^2-x}\mathbf{j}$.

4. (i) $z(x+2)\mathbf{i} - yz\mathbf{j}$.
(ii) $-3z\mathbf{j} + y\mathbf{k}$.
(iii) $(x^2 + 1 - z^2)\mathbf{i} + (y - z^2 + xy)\mathbf{j} + xz(y+1)\mathbf{k}$.
(iv) $z(x+2)\mathbf{i} - (y+3)z\mathbf{j} + y\mathbf{k}$.

7. (i) $-2z\mathbf{i} + (x\cos(xy) + 2x - 1)\mathbf{k}$.
(ii) $-6z^2\mathbf{j} + (2xy - 1)\mathbf{k}$.

8. (i) $-dx \wedge dy$.
(ii) $dx \wedge dy$.
(iii) $(x\sin(y) - \cos(x))dx \wedge dy$.
(iv) $(6x^2 - 1)dx \wedge dy$.

11. (i) Irrotational.
(ii) Irrotational.
(iii) Irrotational.
(iv) Irrotational.

12. (i) Not a gradient field.

13. (i) $(e^{-y^2} + \sin(x))dx \wedge dy$.
(ii) $-\left(2x\sin(x^2) + \frac{3y}{\sqrt{x^2+3y^2}}\right) dx \wedge dy$.

14. 0.

15. (i) T.

(ii) F.
(iii) T.
(iv) T.

16. (i) T.
(ii) T.
(iii) F.

18. (i) An irrotational vector field is a Beltrami vector field.
(ii) $\frac{z}{(z^2+1)^{\frac{3}{2}}}\mathbf{i} - \frac{1}{(z^2+1)^{\frac{3}{2}}}\mathbf{j}$.

20. (i) $(x + 2e^{-y})dx \wedge dy$.
(ii) $-2x \log_e(y) dx \wedge dy$.
(iii) $-(y + x\cos(xy))dx \wedge dy$.

21. (i) $(2x + 2y)\mathbf{k}$.
(ii) $(x^3 - y^2)dx + (x^2 - \sin(y^3))dy$.
(iii) $(2x + 2y)dx \wedge dy$.

22. (i) $\left(-\frac{x}{y} - \frac{1}{x+y} + \cos(x) - \sin(y)\right)\mathbf{k}$.
(ii) $(x\log_e(y) - \cos(y))dx + (\sin(x) - \log_e(x + y))dy$.
(iii) $\left(-\frac{x}{y} - \frac{1}{x+y} + \cos(x) - \sin(y)\right) dx \wedge dy$.

23. (i) Not a gradient field.
(ii) Gradient field.
(iii) Gradient field.

Chapter 2.2.

1. (i) -1.
(ii) -19.

5. (i) $4x + \cos(y) + y$.
(ii) $2x^2 dx + \sin(y)dy + zydz$.
(iii) $(4x + \cos(y) + y)dx \wedge dy \wedge dz$.

6. (i) $2y + 2z$.
(ii) $(1 - e^{-y})dx + (x^2 + y^2)dy + z^2 dz$.
(iii) $(2y + 2z)dx \wedge dy \wedge dz$.

7. (i) $\mathbf{0}$.
(ii) $-2x + 2y + 1$.
(iii) 0.

8. (i) zero.
(ii) negative.
(iii) positive.
(iv) zero.
(v) positive.
(vi) zero.

9. (i) Yes.
(ii) No.

Chapter 2.3.

1. (i) $dy \wedge dz + dx \wedge dy$.
(ii) $-xdx \wedge dz + \sin^3(y)dy \wedge dz$.
(iii) $4dx \wedge dy \wedge dz$.
(iv) $7x^9$.
(v) $x^3 dy \wedge dz + (y - x)dx \wedge dz + (x - z)dx \wedge dy$.
(vi) $zdz - xdx - ydy$.

2. (i) $e^{-y}dx - (xe^{-y} + z\sin(y))dy + \cos(y)dz$.
(ii) $e^{-y}dy \wedge dz + (xe^{-y} + z\sin(y))dx \wedge dz + \cos(y)dx \wedge dy$.

3. (i) $-4z^3 \sin(x)dy \wedge dz + (z^4 \cos(x) + 2y)dx \wedge dy$.
(ii) $-4z^3 \sin(x)dx + (z^4 \cos(x) + 2y)dz$.
(iii) $-4z^3 \sin(x)\mathbf{i} + (z^4 \cos(x) + 2y)\mathbf{k}$.

4. (i) $-\cos(x)dx \wedge dy + (e^{-z} + 2\cos(y))dy \wedge dz$.
(ii) $-\cos(x)dz + (e^{-z} + 2\cos(y))dx$.
(iii) $(2\cos(y) + e^{-z})\mathbf{i} - \cos(x)\mathbf{k}$.

5. (i) $(-z - e^{y-x})dx \wedge dy + xdy \wedge dz$.
(ii) $(-z - e^{y-x})dz + xdx$.
(iii) $x\mathbf{i} - (z + e^{y-x})\mathbf{k}$.

6. (i) $\sin(xy)dy\wedge dz-\cos(xz)dx\wedge dz+\sin(yz)dx\wedge dy$.
(ii) $(y\cos(xy)+y\cos(yz))dx\wedge dy\wedge dz$.
(iii) $y\cos(xy)+y\cos(yz)$.

7. (i) $2z^9dy\wedge dz-4xy^4dx\wedge dz+10dx\wedge dy$.
(ii) $16xy^3dx\wedge dy\wedge dz$.
(iii) $16xy^3$.

8. (i) F.
(ii) T.
(iii) T.
(iv) T.
(v) T.

9. (i) T.
(ii) F.
(iii) F.
(iv) F.
(v) T.
(vi) F.

10. (i) T.
(ii) F.
(iii) T.
(iv) F.

11. (i) **S**.
(ii) **VF**.
(iii) **S**.
(iv) 1–form.
(v) 2–form.
(vi) 3–form.
(vii) 2–form.
(viii) 1–form.
(ix) 1–form.
(x) **S**.

Chapter 3.1.

1. (i) $-t\mathbf{i}+(3+t)\mathbf{j}$.
(ii) $(7t-1)\mathbf{i}+(4-2t)\mathbf{j}+(3-6t)\mathbf{k}$.
(iii) $4\cos(t)\mathbf{i}+4\sin(t)\mathbf{j}$.
(iv) $9\cos(\vartheta)\sin(\phi)\mathbf{i}+(3+9\sin(\vartheta)\sin(\phi))\mathbf{j}+(1+9\cos(\phi))\mathbf{k}$.

2. (i)
(ii)
(iii)
(iv)

3. (i)
(ii)
(iii)

4. (i)
(ii)

5. (i)
(ii)

6.

7.

8.

9.

10. (i)
(ii)
(iii)

Chapter 3.2.

1. (i)

2. -10.

3. (i) simply connected.
(ii) not simply connected.
(iii) not simply connected.

4. (i)
(ii)
(iii)

5. (i) Y.
(ii) Y.
(iii) Y.

6. (ii) No.
7. (i)
(ii)
(iii)
(iv)
8.
9. (i)
(ii)
10. (i) Ω_2, Ω_4.
(ii) Ω_4.
11. Yes.
12. (ii)
(iii)
(iv)
13. (i)
(ii)
(iii)
(iv)
(v)
(vi)
14. (i) 374.
(ii) 17.
(iii) 27.
(iv) $-\frac{1}{2\pi}$.
15. (i) 7.
(ii) 0.
16. (ii) $b_1 = a_2$.

Chapter 4.1.

1. (i)
(ii)
(iii)
2.
3. (i)
(ii)
4.
5. (i)
(ii)
6. (i)
(ii)
(iii)
(iv)
7. (i)
(ii)
(iii)
(iv)
8. (i)
(ii)
(iii)
(iv)
9. (i)
(ii)
(iii)
(iv)
10.
11. (i)
(ii)
(iii)
(iv)
12. (i)
(ii)
13.
14.
15.

Chapter 4.2.

1. (i) 0.
(ii) 0.
(iii)
(iv) 9.
2. (i)
(ii)

(iii)

3. $-\frac{3}{8}\pi$.

4. 0.

5. 6.

6. 117π.

7. (i) $-\pi$.

(ii) 0.

8.

9. (i)

(ii)

(iii)

(iv)

(v)

10. (i)

(ii)

11. (i)

(ii)

(iii)

(iv)

13.

15.

16.

17.

18. (i)

(ii)

(iii)

Chapter 4.3.

1. (i)

(ii)

(iii)

(iv)

(v)

2. (i)

(ii)

(iii)

3.

4.

5.

6.

7.

8. (i)

(ii)

9. (i)

(ii)

10.

11.

12. (i) 40.

(ii) -388.

(iii) 171.

(iv) $\frac{2}{3}$.

(v) 30.

(vi) 1251π.

(vii) 3843π.

(viii) 4.

(ix) 510π.

(x) 128.

(xi) $4\pi a^3$.

(xii) 16/9.

(xiii) 1.

13. (i) $-\frac{\pi}{4}$.

(ii) $-\frac{96\pi}{5}$.

Chapter 4.4.

1. (i)

(ii)

2.

3.

7. (i)

(ii)

8.

9. (i) T.

(ii) T.

(iii) F.

10.

11. -1.

Chapter 4.5.

1.

3. (i)

(ii)

5. (i)

(iv)

7.

8.

9.

10. (i)

(ii)

(iii)

(iv)

11. (i)

(ii)

12.

13.

12. 80π.

13. $\frac{81\pi}{2}$.

14. 3.

14. (i)

(ii)

15. (i)

(ii)

16.

17. (i)

(ii)

(iii)

(iv)

22. 2.

23. $\frac{9\pi}{2}$.

24. 0.

25. $\frac{13\pi}{20}$.

31. 27.

32. 16.

33. 108π.

34. 81π.

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