Mean Body Initial Condition Implementation in the EIT Reconstruction Problem

QS Part (B) Requirement

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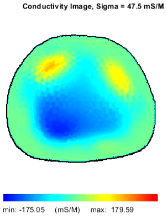
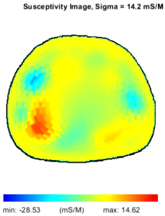
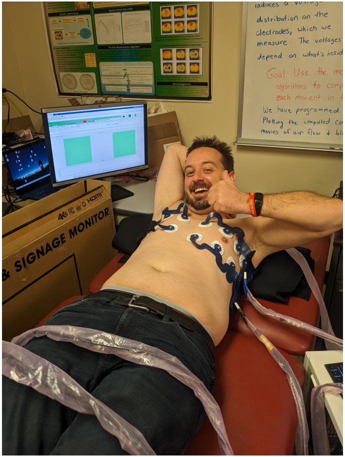
Electrical Impedance Tomography

- Real-time imaging method of internal body structures and functions using electrical properties
- Administer low-amplitude current to body surface and measure resulting voltages
- Solve inverse problem for conductivity distribution of internal body







Mathematical Formulation

- The governing partial differential equation for electric potential u given conductivity distribution $\sigma(\boldsymbol{p})$

$$\nabla \cdot \sigma(\boldsymbol{p}) \nabla u(\boldsymbol{p}) = 0 \qquad \text{for } \boldsymbol{p} \text{ in } B$$

$$\sigma(\boldsymbol{p}) \frac{\partial u(\boldsymbol{p})}{\partial \nu} = j(\boldsymbol{p}) \qquad \text{for } \boldsymbol{p} \text{ on } S$$

- Our boundary condition $j(\boldsymbol{p})$ are our applied current densities given by

$$j(\boldsymbol{p}) = \begin{cases} I_l/A_l, & \boldsymbol{p} \in e_l \\ 0, & \boldsymbol{p} \notin \cup_{l=1}^{L} e_l \end{cases}$$

- We apply trigonometric current patterns as our basis functions for current applied at each electrode
- The kth current pattern applies the following currents at the lth electrode

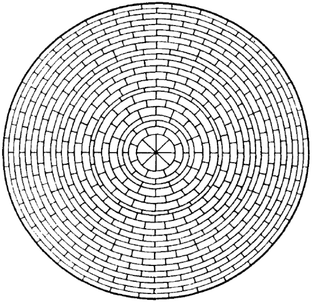
$$I_l^k = T_l^k = \begin{cases} \cos(k\theta_l), & k = 1, \ldots, L/2 \\ \sin((k - L/2)\theta_l), & k = L/2 + 1, \ldots, L - 1 \end{cases}$$

NOSER Reconstruction Algorithm

- The Newton One-Step Error Reconstructor — or NOSER — is one of the most commonly used algorithms for approximating solutions to this PDE
- Aims to find distribution of resistivities $\rho = \frac{1}{\sigma}$ on 496 elements of Joshua tree mesh to minimize square error of voltage estimate $\boldsymbol{U}^k(\boldsymbol{\rho})$

$$E(\boldsymbol{\rho}) = \sum_{k=1}^{L-1} \sum_{l=1}^{L} (V_l^k - U_l^k(\boldsymbol{\rho}))^2$$

- V_l^k and $U_l^k(\boldsymbol{\rho})$ are the measured and estimated voltages, respectively, at the lth electrode while applying the kth current pattern



Finding a Minimizer

- The problem reduces to finding $\boldsymbol{\rho}$ to satisfy the necessary conditions for a minimizer

$$0 = \frac{\partial E(\boldsymbol{\rho})}{\partial \rho_n} = -2 \sum_{k=1}^{L-1} \sum_{l=1}^{L} (V_l^k - U_l^k(\boldsymbol{\rho})) \frac{\partial U_l^k(\boldsymbol{\rho})}{\partial \rho_n}$$

- An approximate solution $\hat{\boldsymbol{\rho}}$ is found using one step of Gauss-Newton method with initial guess $\tilde{\boldsymbol{\rho}}$

$$\hat{\boldsymbol{\rho}} = \tilde{\boldsymbol{\rho}} - [\boldsymbol{F}'(\tilde{\boldsymbol{\rho}})]^{-1} \boldsymbol{F}(\tilde{\boldsymbol{\rho}})$$

$$F_n(\tilde{\boldsymbol{\rho}}) = -2 \sum_{k=1}^{L-1} \sum_{l=1}^{L} (V_l^k - U_l^k(\tilde{\boldsymbol{\rho}})) \frac{\partial U_l^k(\tilde{\boldsymbol{\rho}})}{\partial \rho_n}$$

$$F'_{n,m}(\tilde{\boldsymbol{\rho}}) \approx 2 \sum_{k=1}^{L-1} \sum_{l=1}^{L} \frac{\partial U_l^k(\tilde{\boldsymbol{\rho}})}{\partial \rho_n} \frac{\partial U_l^k(\tilde{\boldsymbol{\rho}})}{\partial \rho_m}$$

- This requires estimates for initial guess, voltage, and gradients

Computational Simplifications

- Initial resistivity assumed to be constant and forward equations are linear, so $U_l^k(\tilde{\boldsymbol{\rho}}) = \rho_b U_l^k(\mathbf{1})$ where ρ_b is resistivity of best least squares fit

$$\rho_b = \frac{\sum_{k=1}^{L-1} \sum_{l=1}^{L} V_l^k U_l^k(\mathbf{1})}{\sum_{k=1}^{L-1} \sum_{l=1}^{L} [U_l^k(\mathbf{1})]^2}$$

- Additionally we find

$$U_l^k(\mathbf{1}) = \begin{cases} \frac{r_0}{k} T_l^k, & k = 1, \ldots, L/2 \\ \frac{r_0}{k-L/2} T_l^k, & k = L/2+1, \ldots, L \end{cases}$$

- Gradients can be broken down with respect to trigonometric current bases and computed by

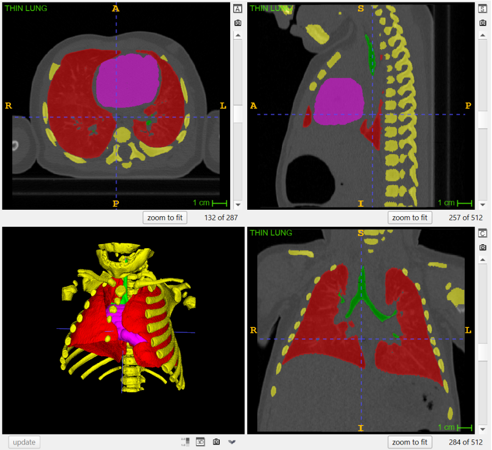
$$\frac{\partial \boldsymbol{U}^k(\tilde{\boldsymbol{\rho}})}{\partial \rho_n} = \rho_b \sum_{s=1}^{L-1} \frac{\left\langle \boldsymbol{T}^s, \frac{\partial \boldsymbol{U}^k(\mathbf{1})}{\partial \rho_n} \right\rangle}{\langle \boldsymbol{T}^s, \boldsymbol{T}^s \rangle}$$

Anatomically-Informed Initial Guess

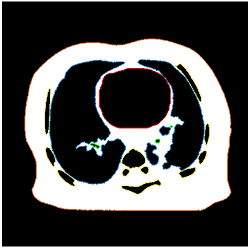
- We want to update NOSER with an anatomically-informed initial guess of conductivity distribution
- As we have seen previously we must generate the following
  - Initial guess distribution, $\tilde{\boldsymbol{\sigma}}$
  - Estimated voltage distribution, $\boldsymbol{U}^k(\tilde{\boldsymbol{\sigma}})$
  - Voltage gradients, $\frac{\partial \boldsymbol{U}^k(\tilde{\boldsymbol{\sigma}})}{\partial \sigma_n}$
- An Anatomical Atlas was developed and used in numerical generation of these components, allowing for an improvement to NOSER's initial conditions
- Additionally, as each component is numerically estimated, rather than explicitly derived, Gauss-Newton method can be performed with respect to $\boldsymbol{\sigma}$, rather than $\boldsymbol{\rho}$

SEGMENTATION OF CT SCANS

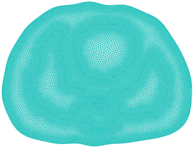
- Anatomical Atlas generated from a set if 8,171 2-D cross-sectional images of chest CT scans of infants aged 0 to 3 months
- CT scans were segmented into lung, trachea, bone, and soft tissue
- "Average" hearts were manually added



Conductivity and Voltage Modeling



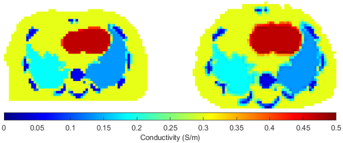
- Organ boundaries are then imported into MATLAB, with conductivity values assigned to each type of tissue
- Finite Element Method modeling used to numerically compute EIT output voltage distributions

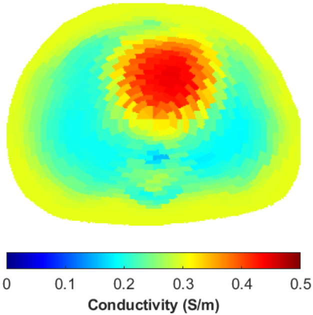


Tissue	Heart	Soft Tissue	Lung	Trachea	Bone
Conductivity base value (S/m)	0.5	0.3	0.15	0.15	0.05
Maximum perturbation (S/m)	$\pm$ 0.1	$\pm$ 0	$\pm$ 0.3	$\pm$ 0.05	$\pm$ 0.05

Mean Baby

- We computed the mean conductivity distribution $\bar{\sigma}$ to be our anatomically informed initial condition
- Prior to averaging, slices from anatomical atlas were conformally mapped to a common body shape to normalize body structure





Mean Baby Reconstructions

- We compute reconstruction using the following Gauss-Newton method

$$\hat{\boldsymbol{\sigma}} = \tilde{\boldsymbol{\sigma}} - [\boldsymbol{F}'(\tilde{\boldsymbol{\sigma}})]^{-1}\boldsymbol{F}(\tilde{\boldsymbol{\sigma}})$$

$$F_n(\tilde{\boldsymbol{\sigma}}) = -2\sum_{k=1}^{L-1}\sum_{l=1}^{L}(V_l^k - U_l^k(\tilde{\boldsymbol{\sigma}}))\frac{\partial U_l^k(\tilde{\boldsymbol{\sigma}})}{\partial \sigma_n}$$

$$F'_{n,m}(\tilde{\boldsymbol{\sigma}}) \approx 2\sum_{k=1}^{L-1}\sum_{l=1}^{L}\frac{\partial U_l^k(\tilde{\boldsymbol{\sigma}})}{\partial \sigma_n}\frac{\partial U_l^k(\tilde{\boldsymbol{\sigma}})}{\partial \sigma_m}$$

- To create reconstructions we also need estimates for $\boldsymbol{U}^k(\bar{\boldsymbol{\sigma}})$ and $\frac{\partial \boldsymbol{U}^k(\bar{\boldsymbol{\sigma}})}{\partial \sigma_n}$
- Voltages computed using finite element method, and gradients estimated using central difference method
- $\boldsymbol{F}'$ matrix is ill-conditioned, so we regularize

$$F'_{n,m}(\bar{\boldsymbol{\sigma}}) = \mathbf{A}_{n,m} + \gamma\mathbf{A}_{n,m}\delta_{n,m}$$
$$\boldsymbol{F}' = \boldsymbol{F}' + \alpha\max_n(F'_{n,n})I$$

Schur Complement Post-Processing

- Using Anatomical Atlas, we can compute a pixel-wise covariance between ground truth conductivity distributions $\boldsymbol{\sigma}$ and reconstructed conductivites $\hat{\boldsymbol{\sigma}}$
- Using Schur complement properties of covariance matrix

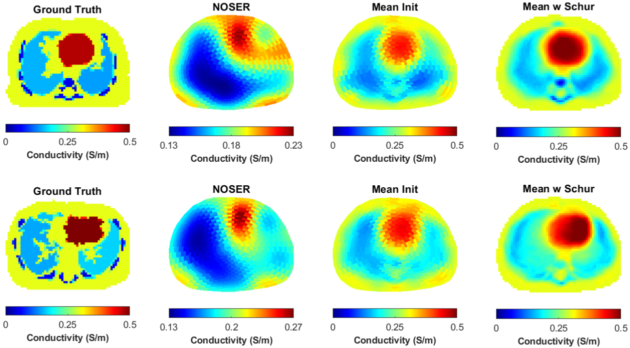
$$\Gamma = \begin{bmatrix} \Gamma_{\boldsymbol{\sigma}\boldsymbol{\sigma}} & \Gamma_{\boldsymbol{\sigma}\hat{\boldsymbol{\sigma}}} \\ \Gamma_{\hat{\boldsymbol{\sigma}}\boldsymbol{\sigma}} & \Gamma_{\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\sigma}}} \end{bmatrix}$$

- We can update approximation expected value for $\boldsymbol{\sigma}|\hat{\boldsymbol{\sigma}}$ and compute post-processed estimate

$$\begin{aligned} \hat{\boldsymbol{\sigma}}_S =& \boldsymbol{A}_\kappa \hat{\boldsymbol{\sigma}} + \boldsymbol{b}_\kappa \\ \text{where } \boldsymbol{A}_\kappa &= \Gamma_{\boldsymbol{\sigma}\hat{\boldsymbol{\sigma}}} (\Gamma_{\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\sigma}}} + \kappa I)^{-1} \\ \boldsymbol{b}_\kappa &= \bar{\boldsymbol{\sigma}} + \Gamma_{\boldsymbol{\sigma}\hat{\boldsymbol{\sigma}}} (\Gamma_{\hat{\boldsymbol{\sigma}}\hat{\boldsymbol{\sigma}}} + \kappa I)^{-1} \boldsymbol{\mu}_{\hat{\boldsymbol{\sigma}}} \end{aligned}$$

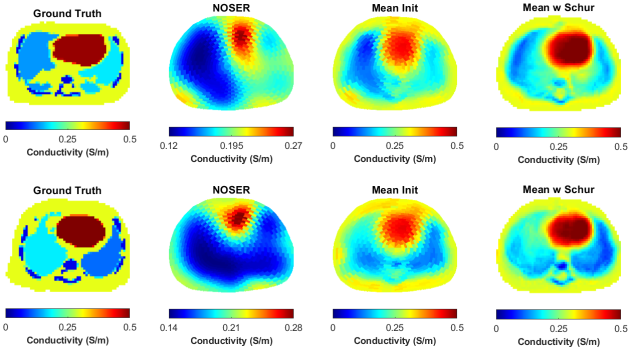
Typical Anatomy Cases

- NOSER appears to create more "blobby" reconstructions
- Mean Baby captures spinal column and division between lungs
  - Note conductivity scaling
- Schur complement provides additional resolution



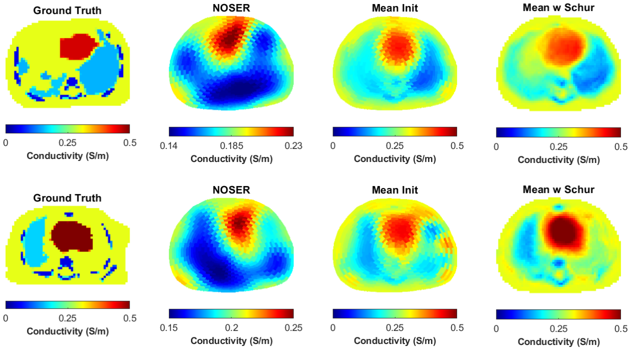
Asymmetrical Lungs

- Each method captures asymmetry with lower conductivity in right lung
- For lower conductivity left lung, Mean Baby approach mitigates size of right lung, but maintains uniform conductivity.



Missing Lung

- Some impact of lungs still shown in each reconstruction method
- Boundaries between tissue and lung blurred for Mean Baby and Schur complement approaches



Quantitative Comparison Metrics

- Performance of each algorithm compared to ground truth recorded using normalized L2 error, SSIM, and Correlation metrics.
- Normalized L2 Error measures a square root of sum of squares of our pixel-wise error as a proportion of the norm of our reconstruction

$$L^2 \text{ Norm Error} = \frac{\|\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}\|_2}{\|\hat{\boldsymbol{\sigma}}\|_2}$$

- Structural Similarity Index Measure (SSIM) measures similarity between two images as a product of terms comparing luminance, contrast, and structure

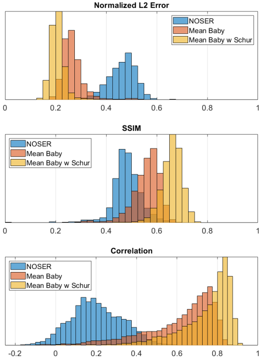
$$\text{SSIM} = \frac{(2\boldsymbol{\mu}_{\boldsymbol{\sigma}}\boldsymbol{\mu}_{\hat{\boldsymbol{\sigma}}} + C_1)(2\text{Cov}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}}) + C_2)}{(\boldsymbol{\mu}_{\boldsymbol{\sigma}}^2 + \boldsymbol{\mu}_{\hat{\boldsymbol{\sigma}}}^2 + C_1)(\text{Var}(\hat{\boldsymbol{\sigma}}) + \text{Var}(\boldsymbol{\sigma}) + C_2)}$$

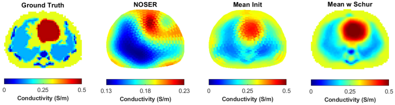
- Correlation measures degree to which a pixels from reconstruction are linearly related to equivalent pixel in ground truth

$$\text{Correlation} = \frac{\text{Cov}(\boldsymbol{\sigma}, \hat{\boldsymbol{\sigma}})}{\sqrt{\text{Var}(\hat{\boldsymbol{\sigma}})\text{Var}(\boldsymbol{\sigma})}}$$

Quantitative Comparison

- Statistics computed for each of the 8,171 cross-sectional ground truth slices
- Distributions show improved performance from Mean Baby approach
- Further improvement seen when applying Schur complement to Mean Baby reconstructions

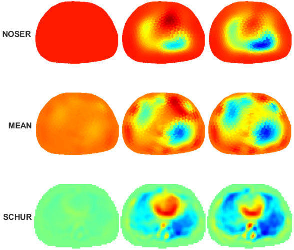




Reconstruction	L^2 Error	SSIM	Correlation
NOSER	0.4825	0.4607	0.3259
Mean Baby	0.2215	0.6101	0.8118
Mean w Schur	0.1830	0.6906	0.8674

Clinical Data Analysis

- For analysis of clinical data, we note two differences from simulated data
- Since ground truth is unknown we are limited to qualitative inspection
- We visualize difference images from some reference conductivity $\boldsymbol{\sigma}_{ref}$
- Analysis shows differences $\Delta\hat{\boldsymbol{\sigma}}_i = \hat{\boldsymbol{\sigma}}_i - \boldsymbol{\sigma}_{ref}$ for frames i at full expiration, during inspiration, and full inspiration from left to right, respectively.



This study was conducted in accordance with the amended Declaration of Helsinki. Data were collected at Children's Hospital Colorado (CHCO) in Aurora, Colorado, under the approval of the institutional review board (IRB) (approval number 18-1843). Informed parental consent was obtained prior to participation.

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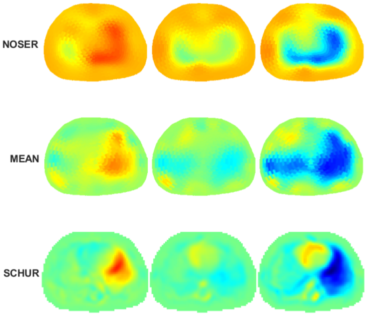
- We note that many of the general trends of data are consistent with existing NOSER imaging
- Applying Mean Baby approach helps visually separate each lung from each other
- Schur complement post-processing further helps increase resolution and produce realistic lung shapes



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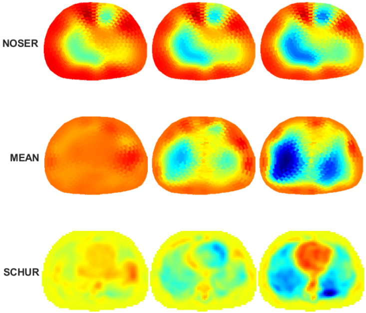
- Even in cases where NOSER creates lung separation, Mean Baby approach helps produce images more in line with anatomical expectations
- We see in some cases, Schur complement outputs may be sensitive to heartbeat when reference frame is selected



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- Once again NOSER and Mean Baby show analogous trends throughout respiratory cycle, with Mean Baby producing more defined lungs
- In some cases, Schur complement post-processing makes images less readable and produces unrealistic artifacts, such as a pulsating spine in this example