Lecture Summary 3

Change of Notation: We have seen that we need to sample a density function, call it $f(x)$, in order to implement Monte Carlo integration. We have to start somewhere and it is assumed we can sample uniform random variables from $(0, 1)$. Computers can do this via clever deterministic sequences.

Some easy distributions can then be sampled via a single uniform U;

$$X = F^{-1}(U).$$

It is then straightforward to show that $\mathrm{P}(X \leq x) = F(x)$, so this method provides a random variable with distribution F.

For harder distributions we need something more sophisticated and the first attempt is known as Rejection Sampling. The basic idea is to sample a density $g(x)$ somehow close to $f(x)$, and to see whether we can use this as a sample from f. Suppose we can write $f(x) = h(x)\, g(x)$ where $h(x) \leq M < \infty$. To this end, consider the joint density function

$$f(x, u) = M\, \mathbf{1}(0 < u < h(x)/M)\, \mathbf{1}(0 < u < 1)\, g(x).$$

The marginal density for x is $f(x)$ so we can sample (x, u) from $f(x, u)$ and keep the x sample as coming from f.

We can sample from $f(x, u)$ by keep sampling from the density

$$\widetilde{f}(x, u) = \mathbf{1}(0 < u < 1)\, g(x)$$

unitil we get a (x, u) satisfying $u < h(x)/M$. We then keep the x value as coming from f. We can clearly sample from $\widetilde{f}$ by taking u uniform from $(0, 1)$ and x from g.

The probability of acceptance is important; too small and this method won't work. Now by the law of total probability,

$$\mathrm{P}\Big(U < h(X)/M\Big) = \int h(x)\, g(x)\, dx/M = 1/M.$$

Note that $M \geq 1$ since M is the maximum value of $f(x)/g(x)$ which must be greater than 1 since one density can not sit under another as they both must integrate to 1. The closer g is to f the smaller M becomes and hence the larger the probability of acceptance.