Lecture Summary 20

The Bayes factor (BF) is used for calculating the posterior probability of a model. A Bayesian model, say $\mathcal{M}$, is given by

$$\mathcal{M} = \{k(x|\theta),\ f(\theta)\};$$

i.e. the model and prior.

The BF for comparing models $\mathcal{M}_1$ and $\mathcal{M}_2$ is given by

$$BF = \frac{\int \prod_{i=1}^n k_1(x_i|\theta_1)\, f_1(\theta_1)\, d\theta_1}{\int \prod_{i=1}^n k_2(x_i|\theta_2)\, f_2(\theta_2)\, d\theta_2}.$$

Hence, it is necessary to compute *marginal likelihoods*

$$I = \int \prod_{i=1}^n k(x_i|\theta)\, f(\theta)\, d\theta.$$

This is actually a hard problem in general and the idea is to see it as

$$I = \frac{k(\mathbf{x}|\tilde{\theta})\, f(\tilde{\theta})}{f(\tilde{\theta}|\mathbf{x})}$$

for some $\tilde{\theta}$.

But while it is easy to compute $k(\mathbf{x}|\theta)$, it is not so easy to compute $f(\theta|\mathbf{x})$ due to the lack of the marginal likelihood which appears in the denominator of the posterior.

However, there is a clever Monte Carlo approach to getting $f(\tilde{\theta}|\mathbf{x})$. Define

$$p(\theta,\theta') = \alpha(\theta,\theta')\, q(\theta'|\theta) \quad \text{where} \quad \alpha(\theta,\theta') = \min\left\{1, \frac{f(\theta'|\mathbf{x})\, q(\theta|\theta')}{f(\theta|\mathbf{x})\, q(\theta'|\theta)}\right\}$$

so that

$$p(\theta,\theta')\, f(\theta|\mathbf{x}) = p(\theta',\theta)\, f(\theta'|\mathbf{x}).$$

Hence,

$$f(\theta|\mathbf{x}) = \frac{\int p(\theta',\theta)\, f(\theta'|\mathbf{x})\, d\theta'}{\int \alpha(\theta,\theta')\, q(\theta'|\theta)\, d\theta'}.$$

Both the numerator and denominatr here can be evaluated using Monte Carlo methods. We now use this to estimate $f(\tilde{\theta}|\mathbf{x})$ for a chosen $\tilde{\theta}$.