Lecture Summary 10

In this lecture we will formally look at the Gibbs sampler, though we have already come accross it. Suppose $f(x)$ is the target density with dimension $d$ for $x$, so $x = (x_1, \ldots, x_d)$.

Consider the $d$ conditional densities given by

$$f_j(x_j|x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_d)$$

as the full conditional density for $x_j$ given $x_{-j}$; shortened to $f_j(x_j|x_{-j})$. We will look at examples of how to get these in class.

Then define the transition density as

$$p(x'|x) = \prod_{j=1}^{d} f_j(x_j|x'_1, \ldots, x'_{j-1}, x_{j+1}, \ldots, x_d).$$

It can be shown (exercise) that

$$f(x') = \int p(x'|x)\, f(x)\, dx.$$

The transition density can be sampled easily as long as each full conditional can be sampled easily. We take $x'_1$ from $f_1(\cdot|x_{-1})$ and so on up to taking $x'_d$ from $f_d(\cdot|x'_{-d})$.

So suppose we have a two dimensional density $f(x, y)$ (i.e. $x_1 = x$ and $x_2 = y$ to avoid overuse of notation) and we have $f_1(x|y)$ and $f_2(y|x)$. Starting at say $y_0$ we take $x_1$ from $f_1(\cdot|y_0)$ and $y_1$ from $f_2(\cdot|x_1)$ to complete an iteration. Then we get $x_2$ from $f_1(\cdot|y_1)$ and $y_2$ from $f_2(\cdot|x_2)$ to complete another iteration.

After running for $N$ iterations we will have

$$(x_1, y_1),\ (x_2, y_2), \ldots, (x_N, y_N).$$

As $N$ gets bigger so $(x_N, y_N)$ become more as if they came from $f(x, y)$; this is the stationary condition. Moreover, for any function $l$,

$$\lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} l(x_n, y_n) = \int l(x, y)\, f(x, y)\, dx\, dy, \quad \text{a.s.}$$