Lecture Summary 4

To summarize: we want to sample e.g.

$$f(x) \propto x^3\, e^{-2x}\, (1-e^{-x})^2, \quad x>0,$$

where the aim would be to perform a Monte Carlo integration of

$$I = \int l(x)\, f(x)\, dx.$$

We have looked at sampling f using rejection sampling. But this is inefficient as we throw a lot of samples away.

We can adapt the rejection sampling whenever the density to be sampled, $f(x)$, is such that $\log f$ is a concave function. This means that the second derivative of $\log f$ is negative everywhere; i.e.

$$\frac{d^2}{dx^2} \log f(x) \le 0 \quad \text{for all} \quad x.$$

The problem with rejection sampling is that the rejected samples are not helpful in any way. The idea behind adaptive rejection sampling is to use the rejected samples to help get a better proposal density.

If $g(x)$ represents the proposal density, not nonrmalized, and $f(x)$ is the target density, then a good image for determining a best choice of g is to construct g as an upper envelope for f. A diagram and further explanation for this will be given in the class.

When $f(x)$ is log concave, a number of exponential densities can nicely provide an upper envelope for $f(x)$. For example, if $f(x)$ is standard normal, $\log f(x)$ is a constant minus $\frac{1}{2}x^2$ which is concave. And this $-\frac{1}{2}x^2$ can be upper bounded by two lines which meet at $x=0$.

Another rejection type sampling idea is ratio of uniforms. Here we consider the joint density

$$f(x,y) \propto y\, \mathbf{1}\left(y < \sqrt{f(x)}\right).$$

If we integrate out y we recover

$$\int f(x,y)\, dy = f(x),$$

as required. To sample (X,Y) from $f(x,y)$ we use the transformation $U_1 = Y$ and $U_2 = U_1 X$, and under certain conditions, detailed in class, we can sample U_1 and U_2 from a uniform distribution.