Lecture Summary 11

In this lecture we will look in detail at a particular Markov chain, known as the Metropolis–Hastings algorithm. Suppose the target stationary density is a one dimensional density $f(x)$.

Introduce the density $q(x)$ which will assist the algorithm, and define

$$\alpha(x,x') = \min\left\{1, \frac{f(x')\,q(x)}{f(x)\,q(x')}\right\}$$

and

$$r(x) = \int \alpha(x,x')\,q(x')\,dx'.$$

Now define the transition density, for $x \to x'$, as

$$p(x'|x) = \alpha(x,x')\,q(x') + (1 - r(x))\,\mathbf{1}(x' = x).$$

It is straightforward to show that

$$p(x'|x)\,f(x) = p(x|x')\,f(x')$$

so f is the stationary density for this transition density.

In fact we should write

$$p(x'|x) = r(x)\,\frac{\alpha(x,x')\,q(x')}{r(x)} + (1 - r(x))\,\mathbf{1}(x' = x)$$

and so with probability $r(x)$ we take x' from the density

$$\tilde{q}(x'|x) = \frac{\alpha(x',x)\,q(x')}{r(x)}$$

else with probability $1 - r(x)$ we set $x' = x$. But we do not know $r(x)$ and so some trick is needed to sample $p(x'|x)$, which we will see in class. The idea is that we somehow need to find an event we can sample for which occurs with probability $r(x)$.

The algorithm can be made more general and useful by using a q which depends on the current x; i.e. of the form $q(x'|x)$.

The algorithm is as follows: take x' from $q(x'|x_n)$; take u from uniform on $(0,1)$, if $u < \alpha(x_n, x')$ put $x_{n+1} = x'$ else $x_{n+1} = x_n$.