Lecture Summary 5

For the ratio of uniforms method, we considered

$$f(x,u) \propto u\,\mathbf{1}\Big(0 < u < \sqrt{f(x)}\Big).$$

The marginal density for X is $f(x)$. Using the transformation $U_1 = U$ and $U_2 = X\,U$ we have

$$f(u_1,u_2) \propto u_1\,\mathbf{1}\Big(0 < u_1 < \sqrt{f(u_2/u_1)}\Big)\,|J|,$$

where J is the Jacobian;

$$J = \begin{pmatrix} \partial x/\partial u_1 & \partial x/\partial u_2 \\ \partial u/\partial u_1 & \partial u/\partial u_2 \end{pmatrix}$$

and so $|J| = 1/u_1$. Hence, we need to sample (U_1, U_2) uniformly from the interval

$$\Big\{(u_1,u_2) : \, u_1 < \sqrt{f(u_2/u_1)}\Big\}.$$

Clearly U_1 lies between 0 and $\max_x \sqrt{f(x)}$. Also

$$(u_1^2/u_2^2)\,u_2^2 \le f(u_2/u_1) \quad \text{so} \quad u_2^2 \le x^2\,f(x) \quad \text{for all} \quad x.$$

Thus $\min_x x\sqrt{f(x)} < U_2 < \max_x x\sqrt{f(x)}$. Hence, we sample U_1 and U_2 uniformly from their respective intervals, and accept $X = U_2/U_1$ as coming from $f(x)$ if $U_1 < \sqrt{f(U_2/U_1)}$. In class we look at how this works for a gamma density.

We will then go on and look at importance sampling when the expectation we are interested in is with resepct to a density for which we do not know the normalizing constant and neither can sample from. So we want to estimate $\int l(x)\,f(x)\,dx$ but do not know f in full and can not sample from it. Hence, we sample from density g and write the integral as

$$I = \int [l(x)\,f(x)/g(x)]\,g(x)\,dx.$$

But we only know $f^*(x)$ which is proportional to $f(x)$. Hence, we really need $\int f^*(x)\,dx$ as well. The solution is to estimate I via

$$\widehat{I}_N = \frac{N^{-1}\sum_{i=1}^N l(X_i)\,f^*(X_i)/g(X_i)}{N^{-1}\sum_{i=1}^N f^*(X_i)/g(X_i)}.$$

The numerator converges to $\int l(x)\,f^*(x)\,dx$ and the denominator to $\int f^*(x)\,dx$; so overall $\widehat{I}_N$ converges to I.