Lecture Summary 13

We will start by looking at a Gibbs sampler which needs a Metropolis step. The model is

$$y_i = \frac{\alpha}{\beta + x_i} + \sigma\varepsilon_i,$$

where $\theta = (\alpha, \beta, \sigma)$ and the (ε_i) are i.i.d. standard normal variables. Here β is positive so it would have gamma prior, for example. If $\lambda = 1/\sigma^2$ has a gamma prior and the α has a normal prior, then the conditionals for α and λ are normal and gamma, respectively.

We will then look at a random effects model;

$$y_{ij} = x_{ij}'\beta_i + \sigma\varepsilon_{ij} \quad \text{and} \quad \beta_i \sim N_p(\mu, \Sigma) \quad \text{independently},$$

for $j = 1, \ldots, m$ and $i = 1, \ldots, n$. Here x_{ij} is a $p \times 1$ column vector and β_i a $p \times 1$ column vector. The parameters are $\theta = (\sigma, \mu, \Sigma)$.

The priors for σ and μ are straighforward to set, a gamma prior for $\lambda = 1/\sigma^2$ and a normal prior for μ, say $N_p(0, \Omega)$.

The issue is a prior for the covariance matrix Σ. The common choice here is an inverse–Wishart prior; the density function being

$$f(\Sigma) \propto |\Sigma|^{-k/2} \exp\left\{-\tfrac{1}{2}\text{trace}(\nu\nu'\Sigma^{-1})\right\}$$

where ν is a $p \times 1$ column vector.

A key result to get the conditional density for Σ is that for z a $p \times 1$ column vector and A a $p \times p$ matrix, we have

$$\text{trace}(zz'A) = z'Az.$$

It can then be shown that $f(\Sigma|\beta_1, \ldots, \beta_n, \mu)$ is also an inverse–Wishart density. Note then we need the (β_i) to be incorporated into the Gibbs sampler, hence we need

$$f(\beta_i|\mu, \Sigma, \sigma\text{data})$$

as well as the usual ones;

$$f(\sigma|\beta_1, \ldots, \beta_n, \text{data}) \quad \text{and} \quad f(\mu|\Sigma, \beta_1, \ldots, \beta_n).$$

Although it is possible to integrate out the β_i from the model; to get

$$k(y_i|\mu, \Sigma, \sigma),$$

we would then have a problem with the sampling of the Σ since it would appear in a non–helpful way for sampling.