Lecture Summary 7

For Markov chain theory we will consider a state space of $\Omega = \{1, 2, \ldots, k\}$, technically known as a finite state space. The transitions are represented by a $k \times k$ matrix P where $p_{ij} = P(X_{n+1} = j | X_n = i)$.

The stationary distribution $\pi' = (\pi_1, \ldots, \pi_k)$ is connected to P via

$$\pi' = \pi' P.$$

Here $'$ denotes transpose so strictly π is a $1 \times k$ column vector. In particular

$$\pi_j = \sum_{i=1}^{k} \pi_i \, p_{ij}.$$

If we start the chain at q_0; i.e. $P(X_0 = j) = q_{0j}$, then from the law of total probability it is that

$$q_1' = q_0' \, P,$$

where $P(X_1 = j) = q_{1j}$, and in general $q_n' = q_0' \, P^n$ where $q_{nj} = P(X_n = j)$.

If $(X_n)_{n \geq 0}$ is aperiodic and irreducible (to be explained in class) then for such a P the π exists and is unique and

$$q_{nj} \to \pi_j.$$

The eigenvalues of P are important as to the convergence of q_n to π. If we assume

$$\pi_i \, p_{ij} = \pi_j \, p_{ji}, \quad \text{for all} \quad i, j$$

then the eigenvalues are real. The eigenvalues lie between -1 and $+1$ and the largest eigenvalue is 1. That is

$$-1 \leq \lambda_k \leq \lambda_{k-1} \leq \cdots \leq \lambda_2 \leq \lambda_1 = 1$$

where (λ_k) are the eiqenvalues of P in decreasing order.

That 1 is an eigenvalue of P follows from $\pi' = \pi' \, P$. Further, the chain is aperiodic if $\lambda_k > -1$ and is irreducible if $\lambda_2 < 1$. With this we will show in class that $q_{n\,j} \to \pi$.

The basic idea is that we can write

$$q_{nj} = \pi + \sum_{j=2}^{k} \alpha_j \lambda_j^n v_j,$$

for some (α_j), and v_j is the left eigenvector for eigenvalue λ_j.