Lecture Summary 12

We will continue to look at the Metropolis–Hastings algorithm with transition density $p(x'|x) = \alpha(x,x')\,q(x'|x) + (1-r(x))\,\mathbf{1}(x'=x)$ where

$$\alpha(x,x') = \min\left\{1, \frac{f(x')\,q(x|x')}{f(x)\,q(x'|x)}\right\} \quad \text{and} \quad r(x) = \int_{x'} \alpha(x,x')\,q(x'|x)\,dx'.$$

It is easy to see that $f(x)\,p(x'|x) = f(x')\,p(x|x')$.
The algorithm for sampling from $p(x'|x)$ is as follows: From x_n,
1. Take x' from $q(x'|x_n)$.
2. Take u from the uniform distribution on $(0,1)$.
3. If $u < \alpha(x,x')$ then $x_{n+1} = x'$, else $x_{n+1} = x_n$.

In class we will look at a couple of illustrations. The first involves $f(x) \propto x^a\,e^{-x}, \quad x > 0$ with $a > 0$. Since $x > 0$ the normal proposal is problematic, but we can use a log normal distribution,

$$q(x'|x) \propto (x')^{-1}\,\exp\left\{-\tfrac{1}{2}(\log x' - \log x)^2/\sigma^2\right\}.$$

It is usual to center the proposal on the current value.

The second example involes sampling the Poisson density,

$$f(x) \propto \frac{\theta^x}{x!}, \quad x \in \{0,1,2,\ldots\}.$$

One idea for a proposal here, which we will consider, is

$$q(x'|x) = \left\{\begin{array}{ll} \frac{1}{2} & x' = x+1 \\ \frac{1}{2} & x' = x-1 \end{array}\right.$$

when $x \geq 1$ and $q(1|0) = 1$ when $x = 0$. In this case $\alpha(0,1)$ and $\alpha(1,0)$ need some attention.

Finally, we will look again at the density

$$f(a,b) \propto a^n\,b^n\,S^b\,\exp\left\{-a\sum_{i=1}^{n} y_i^b\right\}, \quad S = \prod_{i=1}^{n} y_i.$$

Within a Gibbs framework we obtain $f(a|b)$ and $f(b|a)$. We can not sample $f(b_{n+1}|a_n)$ directly so we introduce a Metropolis step to do this. That is we sample $p_M(b_{n+1}|a_n)$ instead, where this satisfies

$$f(b_{n+1}|a_n)\,p_M(b_n|b_{n+1},a_n) = f(b_n|a_n)\,p_M(b_{n+1}|b_n,a_n).$$

More details in class, where we show the new transition density

$$p(a_{n+1},b_{n+1}|a_n,b_n) = f(a_{n+1}|b_{n+1})\,p_M(b_{n+1}|b_n,a_n)$$

satisfies the reversible/stationary equation with respect to $f(a,b)$.