Homework 1

Problem 1. *Show the Venn-diagram representation for the following sets:*

(a) $(A \cup B) - C$

(b) $\overline{A \oplus (B \cap C)}$

Problem 2. *For any sets A, B and C, prove that*

$$A \cup B = A \cup C, A \cap B = A \cap C \textit{ implies } B = C.$$

Solution. (Proof by contradiction)

Suppose $B \neq C$. As B and C are symmetric, then without loss of generality, take $x \in B - C$:

1. if $x \in A$: then $x \in A \cap B$ and $x \notin A \cap C$;
2. if $x \notin A$: then $x \in A \cup B$ and $x \notin A \cup C$.

neither of the above case could be true. Thus the assumption $B \neq C$ is not correct, which leads to $B = C$. □

Problem 3. *1. Show that $\mathcal{R}$ is symmetric iff $\mathcal{R}^{-1} \subseteq \mathcal{R}$.*

2. Show that $\mathcal{R}$ is transitive iff $\mathcal{R} \circ \mathcal{R} \subseteq \mathcal{R}$.

Proof. We prove the the first statement and omit the second one.

$$\begin{array}{ll} & \mathcal{R} \text{ is symmetric} \\ \Longleftrightarrow & \forall x \forall y (x \mathcal{R} y \longrightarrow y \mathcal{R} x) \\ \Longleftrightarrow & \forall x \forall y (y \mathcal{R}^{-1} x \longrightarrow y \mathcal{R} x) \\ \Longleftrightarrow & \mathcal{R}^{-1} \subseteq \mathcal{R} \end{array}$$

□

Problem 4. *Prove that $\mathcal{P}(A) \approx 2^A$, where A is any set and $2^A = \{f \mid f : A \to \{0,1\}$ is a function.$\}$*

Solution. Check the slides. □

Problem 5. *A and B are countable sets. Prove that*

1. *$A \cup B$ is countable*
2. *$A \times B$ is countable*

Solution.(Hint) As $A \preceq \omega$ and $B \preceq \omega$, suppose $f : A \to \omega$, and $g : B \to \omega$ are both injective functions.

Then

1. $$h(x) = \begin{cases} 2 \cdot f(x) & x \in A \\ 2 \cdot g(x) + 1 & x \in B - A \end{cases}$$

   and then prove that $h(x)$ is injective.

2. $h(\langle x, y \rangle) = \langle f(x), g(y) \rangle$.

   Then prove that $h(x)$ is injective.

   Function h shows $A \times B \preceq \omega \times \omega$.

   As we know $\omega \times \omega \approx \omega$, we finally get $A \times B \preceq \omega$.

$\square$