Homework 5

Problem 1. *Which of the following statements about graph G and H are true?*

1. *G and H are isomorphic if and only if for every map $f : V(G) \to V(H)$ and for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*
2. *G and H are isomorphic if and only if there exists a bijection $f : E(G) \to E(H)$.*
3. *If there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$, then G and H are isomorphic.*
4. *If G and H are isomorphic, then there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$.*
5. *If G and H are isomorphic, then there exists a bijection $f : E(G) \to E(H)$.*
6. *G and H are isomorphic if and only if there exists a map $f : V(G) \to V(H)$ such that for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*
7. *Every graph on n vertices is isomorphic to some graph on the vertex set $\{1, 2, \ldots, n\}$.*
8. *Every graph on $n \geq 1$ vertices is isomorphic to infinitely many graphs.*

Solution.

(1) False, not for every map, definition only require that there exist a biejective function.

(2) False, because the number od vertices may not equal.

(3) False, definition require edges' mapping relation.

(4) True, isomorphic graphs must have the same degree sequence.

(5) True, according to origin definition.

(6) True, according to origin definition.

(7) False, definition require edges' mapping relation.

(8) True, we can re-define all the graphs' vertices to get a totally new isomorphic graph, so it's infinite for all the possible isomorphic graphs.

Problem 2. *Two simple graphs $G = (V, E)$ and $G' = (V', E')$. A map $f : V \to V'$. Now if f satisfies:*

i) It is a bijective function;

ii) $\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$;

Then we say that graph G and G' are isomorphic *to each other. We use $G \cong G'$ to stand for the isomorphism relation.*

Consider the following questions:

1. *$G = K_n$ (Recall: K_n is a clique with n vertices), $g : V \to V'$ is a function which only satisfies requirement ii). Prove that G' must contain a subgraph which is a clique with n-vertices.*

2. *$G = K_{n,m}$ (Recall: $K_{n,m}$ is the so-called* complete bipartite graphs*), g is the same as in question 1. What will be the simplest G' that is related to G under the new relation.*

Solution.

(1) Now that we have n points and $\frac{n(n-1)}{2}$ edges in G. acoording to ii, we will know G' have $\frac{n(n-1)}{2}$ edges at most. So our aim can be to prove G' have n points. According to ii, we know for every point in G', it's linked to all other points. It means, for a $k-$point clique in G', it has $E'(k) = 1 + 2 + 3 + \cdots + k - 1 = \frac{k(k-1)}{2}$.Therefore, G' must have a clique with n point.

(2) Once we define, for $x \in V$, there exist $\{x, x\} \in E$. Therefore, we have a simplest graph G' only have two separate points.

Problem 3. *How many graphs on the vertex set* $\{1, 2, \ldots, 2n\}$ *are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set* $\{\{1,2\},\{3,4\},\ldots,\{2n\text{-}1,2n\}\}$*?*

Solution. *There is a one to one correspondence between graphs which are isomorphic to the given graph and the number of possible partitions of [1,2n]=*$\{1, 2, 3 \ldots 2n\}$ *into n sets each of size 2.*

To create such a partition, we can choose the first two elements in $\binom{2n}{2}$ *ways*

The next two elements can be chosen in $\binom{2n-2}{2}$ *ways and so on till* $\binom{2n-(2n-2)}{2}$ *is the number of ways of choosing the last pair of adjacent edges*

Thus, the number of such sets is $\prod_{i=0}^{(n-1)} \binom{2n-2i}{2}$

But this also accounts for the relative order of the 2-sets themselves which is irrelevant

Thus, the correct number of partitions is

$$\frac{1}{n!} \prod_{i=0}^{(n-1)} \binom{2n-2i}{2}$$

And so, we have

$$\frac{1}{n!} \prod_{i=0}^{(n-1)} \binom{2n-2i}{2} = \prod_{i=0}^{n-1} (2n-2i-1) = (2n-1) \cdot (2n-3) \ldots 1 = (2n-1)!!$$

Which can also be written as

$$\prod_{i=0}^{n-1} (2n-2i-1) = \frac{(2n)!}{2^n (n)!}$$

Problem 4. *Construct an example of a sequence of length n in which each term is some of the numbers* $1, 2, \ldots, n-1$ *and which has an even number of odd terms, and yet the sequence is not a graph score. Show why it is not a graph score.*

Solution.

- *If n is odd, we can get a sequence with* $n-1$ *of* 1 *and* 1 *of* 3*, therefore, the sum of all vertice' degree is odd, which is impossible.*

- *If n is even, we can get a sequence with* $n-2$ *of* 1, 2 *of* $n-2$*, therefore,* 2 *of* $n-2$ *can't be satisfied because there only have* $n-2$ *of* 1 *vertices.*

Problem 5. *Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.*

Solution.

Define G a graph with k vertices of degree 6*.*

Then we can suppose $k < 5$ *,it means either* $k = 4$ *or* $k \leq 3$*. For the former case, it implies that there exist* $94 = 5$ *vertices of degree 5 in G, it means sum of degree is odd, which is contradictory to fundamental theory. For the latter case, it implies that G contains six or more vertices of degree* 5*, as original problem required.*