Homework 1

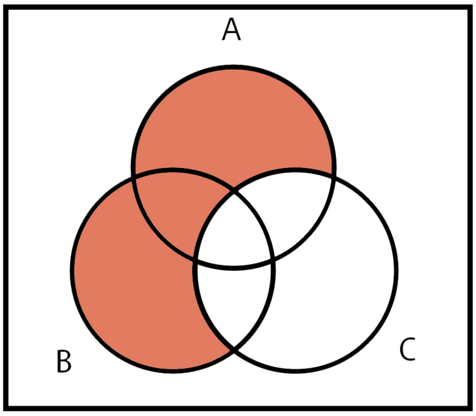
Problem 1. *Show the Venn-diagram representation for the following sets:*

(a) $(A \cup B) - C$

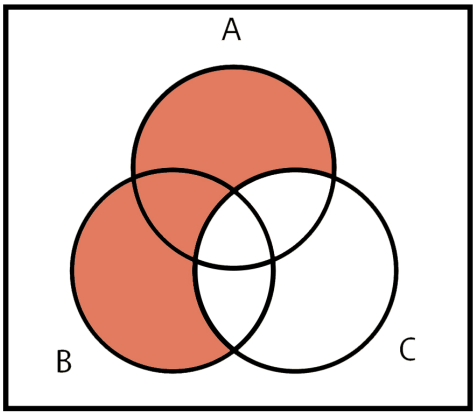
(b) $\overline{A \oplus (B \cap C)}$

Solution. We first assume that any two sets have intersection.

then (a) can be as follow:



then (a) can be as follow:



Problem 2. *For any sets A, B and C, prove that*

$$A \cup B = A \cup C, A \cap B = A \cap C \textit{ implies } B = C.$$

Proof. We first assume $B \neq C$, so that

$$\exists x \in B,\ x \notin C$$

$$x \in B \Rightarrow x \in (A \cup B), A \cup B = A \cup C \Rightarrow x \in (A \cup C)$$

$$since\ x \notin C \Rightarrow x \in A$$

$$since\ x \in A, x \in B \Rightarrow x \in (A \cap B), since\ (A \cap B) = (A \cap C) \Rightarrow x \in (A \cap C)$$

Therefore,$x \in C$it have a contradiction with previous assumption. We can claim $B = C$.

Problem 3. *1. Show that $\mathcal{R}$ is symmetric iff $\mathcal{R}^{-1} \subset \mathcal{R}$.*

2. Show that $\mathcal{R}$ is transitive iff $\mathcal{R} \circ \mathcal{R} \subset \mathcal{R}$.

Proof.

1).

necessity: If $\mathcal{R}^{-1} \subset \mathcal{R}$, for any relation $x \to y \in \mathcal{R}$, we can claim $y \to x \in \mathcal{R}$, therefore $\mathcal{R}$ is *symmetric.*

sufficiency: If $\mathcal{R}$ is *symmetric*, $\mathcal{R}^{-1} = \mathcal{R}$ Therefore $\mathcal{R}^{-1} \subset \mathcal{R}$.

2).

necessity: For the definition, $\mathcal{R} \circ \mathcal{R} \subset \mathcal{R}$, aRb, $bRc \Rightarrow aRc$. therefore, R is *transitive.*

sufficiency: If R is *transitive*, we can obviously see that combination of two relation $\mathcal{R}$ is satisfied to $\mathcal{R}$. it says that, $\mathcal{R} \circ \mathcal{R} \subset \mathcal{R}$.

Problem 4. *Prove that $\mathcal{P}(A) \approx 2^A$, where A is any set and $2^A = \{f \mid f : A \to \{0,\ 1\}$ is a function.$\}$*

Proof. Define a function from $P(A)$ onto 2^A *as* :

For any subset B of A, H(B) is the characteristic function of B:

$$f_B(x) = \begin{cases} 1 & if\ x \in B, \\ 0 & if\ x \in A - B. \end{cases}$$

H is one-to-one and onto.

Problem 5. *A and B are countable sets. Prove that*

1. $A \cup B$ is countable

2. $A \times B$ is countable

Proof.

1). We can assume set $A = \{a_1, a_2, a_3, a_4, \cdots\}$, and $B = \{b_1, b_2, b_3, b_4, \cdots\}$, therefore we can get $A \cup B = \{a_1, b_1, a_2, b_2, a_3, b_3, a_4, b_4, \cdots\}$

We can define a one-to-one mapping from A to $(A \cup B)$, it folows:

$$a_1 \Rightarrow a_1$$
$$a_2 \Rightarrow b_1$$
$$a_3 \Rightarrow a_2$$
$$a_4 \Rightarrow b_2$$
$$a_5 \Rightarrow a_3$$
$$\cdots$$

Since A is a countable set, We can claim $(A \cup B)$ is countable.

2). Since A is countable, we can first assume $\mid A \mid= k$, $\mid B \mid= p$

Then we can define a one-to-one mapping from number set ω to $A \times B$.

For arithmetic progression $\omega_1 = \{c_1, c_2, c_3, \cdots c_p\} = \{2k, 4k \cdots 2pk\}$, ω_1 is countable, and

$$c_1 \Rightarrow (a_1, b_1)$$
$$c_2 \Rightarrow (a_1, b_2)$$
$$c_3 \Rightarrow (a_1, b_3)$$
$$\cdots$$
$$c_i \Rightarrow (a_1, b_i)$$
$$\cdots$$
$$c_p \Rightarrow (a_1, b_p)$$

Then, we can define another countable set,

$$\omega_2 = \{c_{11}, c_{12}, c_{13}, \cdots c_{1(k-1)}, \cdots c_{21}, c_{22}, c_{23} \cdots c_{2(k-1)} \cdots c_{p1}, c_{p2}, c_{p3} \cdots c_{p(k-1)}\}$$

and we can define one-to-one mapping:

$$c_{11} \Rightarrow (a_2, b_1)$$
$$c_{12} \Rightarrow (a_3, b_1)$$
$$c_{13} \Rightarrow (a_4, b_1)$$
$$\cdots$$
$$c_{ij} \Rightarrow (a_{j+1}, b_i)$$

Then, we let $\omega = \omega_1 \cup \omega_2$, for the conclusion in (1), we can claim ω is countable.Therefore, $A \times B$ is countable.