# Homework 8

**Problem 1.** *Show that, for constant $p \in (0,1)$, almost no graph in $\mathcal{G}(n,p)$ has a separating complete subgraph.*

*Solution.* This is a simple application of the 'almost always true of property $P_{i,j}$'.

Detail:

- **Separating subgraph**: Given $G = (V, E)$, and some $X \subseteq V \cup E$, we call $X$ a separating subgraph if there exists two vertices $u, v \in V(G - X)$ such that $u, v$ are in the some component of $G$, while $u, v$ lie in two disconnected components of $G - X$ (i.e., $X$ separates $u$ and $v$).
- **Separating complete subgraph**: If the above subgraph $X$ is also a complete graph.

Now consider a graph $G = (V, E)$ with property $\mathcal{P}_{2,1}$. We claim that a graph with property $\mathcal{P}_{2,1}$ has the following property: For any pair of vertices $u, v \in G$, there exists a pair of vertices $w_1, w_2$ such that

$$(w_1, u) \in E, \quad (w_1, v) \in E$$

$$(w_2, u) \in E, \quad (w_2, v) \in E$$

$$(w_1, w_2) \notin E.$$

To prove the claim: consider vertices $u, v$ and an arbitrary vertex $x$. By property $\mathcal{P}_{2,1}$, there exists a vertex $w_1$ which is neighbor to $u$ and $v$, but not to $x$. Now using property $\mathcal{P}_{2,1}$ again (with $x$ replaced by $w_1$) it follows that there exists a vertex $w_2$ which is neighbor to $u$ and $v$, but not to $w_1$. Thus the claim holds.

Finally, consider a complete subgraph $H \subset G$ and two arbitrary vertices $u$ and $v$ in $G - V(H)$. By the claim above, there are two non-adjacent vertices $w_1$ and $w_2$ in $G$ which are both neighbors of both $u$ and $v$. Since $H$ is complete, it follows that $w_1$ and $w_2$ cannot both belong to $H$, therefore remove $H$ will not separate $u$ and $v$. In another word, $H$ does not separate $G$. The statement now follows since almost all graphs in $\mathcal{G}(n,p)$ have property $\mathcal{P}_{2,1}$ for any constant $p \in (0,1)$.

□

**Problem 2.** *What is the expected number of trees with k vertices in $G \in \mathcal{G}(n,p)$?*

*Solution*. By *Cayley's formula* and the linearity of expectation, it is $\binom{n}{k}k^{k-2}p^{k-1}$ $\square$

**Problem 3.** *Show that if almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_1$ and almost all $G \in \mathcal{G}(n, p)$ have a graph property $\mathcal{P}_2$, then almost all $G \in \mathcal{G}(n, p)$ have both properties.*

*Solution*. The portion of the graphs have both properties equals 1 minus the portion of the graphs which does not have property $\mathcal{P}_1$ or $\mathcal{P}_2$. However the portion of the graph does not have property $\mathcal{P}_1$ or $\mathcal{P}_2$ is bounded by the sum of the portion of the graphs does not have property $\mathcal{P}_1$ and the the portion of the graphs does not have property $\mathcal{P}_2$, which both tend to 0 as $n$ approaches $\infty$. The claim in the question then follows. $\square$

**Problem 4.** *(Optional)*

1. *Prove that the threshold for the existence of cycles in $\mathcal{G}(n, p)$ is $p = \frac{1}{n}$.*

2. *Search the* World Wide Web *to find some real world graphs in machine readable form or data bases that could automatically be converted to graphs.*

   (a) *Plot the degree distribution of each graph.*

   (b) *Compute the average degree of each graph.*

   (c) *Count the number of connected components of each size in each graph.*

   (d) *Describe what you find.*

3. *Create a simulation (an animation) to show the evolution of the $\mathcal{G}(n, p)$ (Erdös-Rényi) random graph as its density $p$ is gradually increased. Observe the phase transitions for trees of increasing orders, followed by the emergence of the giant component, etc.*