Homework 3

Problem 1. *Prove the formula*

$$\binom{r}{r} + \binom{r+1}{r} + \binom{r+2}{r} + \cdots + \binom{n}{r} = \binom{n+1}{r+1}$$

Problem 2. *For natural numbers $m \le n$ calculate (i.e. express by a simple formula not containing a sum) $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k}$.*

Solution. $\binom{k}{m}\binom{n}{k} = \frac{k!}{m!(k-m)!} \cdot \frac{n!}{k!(n-k)!} = \frac{n!}{m!(n-m)!} \cdot \frac{(n-m)!}{(k-m)!(n-k)!} = \binom{n}{m}\binom{n-m}{n-k}$.

Thus $\sum_{k=m}^{n} \binom{k}{m}\binom{n}{k} = \sum_{k=m}^{n} \binom{n}{m}\binom{n-m}{n-k} = \binom{n}{m} \sum_{k=m}^{n} \binom{n-m}{n-k} = \binom{n}{m} 2^{n-m}$. □

Problem 3. *(a) Using* **Problem 1.** *for $r = 2$, calculate the sum $\sum_{i=2}^{n} i(i-1)$ and $\sum_{i=1}^{n} i^2$.*

(b) Use (a) and **Problem 1.** *for $r = 3$, calculate $\sum_{i=1}^{n} i^3$.*

Solution.

1. $$r = 2 : \qquad \binom{2}{2} + \binom{3}{2} + \cdots + \binom{i}{2} + \cdots + \binom{n}{2} = \binom{n+1}{3}$$

   Thus $\frac{\sum_{i=2}^{n} i(i-1)}{2!} = \binom{n+1}{3} \therefore \sum_{i=2}^{n} i(i-1) = 2\binom{n+1}{3}$

   $$r = 1 : \qquad \binom{1}{1} + \binom{2}{1} + \cdots + \binom{i}{1} + \cdots + \binom{n}{1} = \binom{n+1}{2}$$

   Thus $\therefore \sum_{i=1}^{n} i = \binom{n+1}{2}$.

   Finally, $\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} (i(i-1) + i) = \sum_{i=1}^{n} i(i-1) + \sum_{i=1}^{n} i = \frac{n(n+1)(2n+1)}{6}$.

2. $$r = 3 : \qquad \binom{3}{3} + \binom{4}{3} + \cdots + \binom{i}{3} + \cdots + \binom{n}{3} = \binom{n+1}{4}$$

   Thus $\frac{\sum_{i=3}^{n} i(i-1)(i-2)}{3!} = \binom{n+1}{4}$. $\therefore \sum_{i=3}^{n} i^3 - 3i^2 + 2i = 6\binom{n+1}{4}$,

   ...

   The final result is $\binom{n+1}{2}^2$.

$\square$

Problem 4. *1. Determine the coefficient of* x^{50} *in* $(x^7 + x^8 + x^9 + x^{10} + \cdots)^6$

2. Determine the coefficient of x^3 *in* $(2 + x)^{\frac{3}{2}}/(1 - x)$

Solution.

1.
$$\begin{aligned} &= x^{42}(1 + x + x^2 + x^3 + \cdots)^6 \\ &= x^{42} \textstyle\prod_{i=1}^{6}(1 + x + x^2 + x^3 + \cdots)_i \end{aligned}$$

The coefficient of x^{50} is $\binom{8+6-1}{6-1} = \binom{13}{5}$.

2.
$$\begin{aligned} &= (x + 2)^{3/2}(1 + x + x^2 + \cdots) \\ &= \textstyle\sum_{k=0}^{\infty} \binom{3/2}{k} x^k (2)^{3/2-k}(1 + x + x^2 + \cdots) \end{aligned}$$

The coefficient of x^3 is $\sum_{k=0}^{3} \binom{3/2}{k}(2)^{3/2-k}$. Then use the Newton formula ....

$\square$

Problem 5. *Find generating functions for the following sequences (express them in a closed form, without infinite series!):*

1. $0, 0, 0, 0, -6, 6, -6, 6, -6, \cdots$.

2. $1, 0, 1, 0, 1, 0, \cdots$.

3. $1, 2, 1, 4, 1, 8 \cdots$

Solution.

$\square$

Problem 6. *Let* a_n *be the number of ordered triples* $\langle i, j, k\rangle$ *of integer numbers such that* $i \geq 0, j \geq 1, k \geq 1$, *and* $i + 3j + 3k = n$. *Find the generating function of the sequence* $(a_0, a_1, a_2, \ldots$ *and calculate a formula for* a_n.

Solution.

$(1 + x + x^2 + x^3 + \cdots)(x^3 + x^6 + x^9 + \cdots)(x3 + x^6 + x^9 + \cdots)$
$= \frac{1}{1-x}\frac{x^3}{1-x^3}\frac{x^3}{1-x^3}$
$= \frac{x^6(1+x+x^2)}{(1-x^3)^3} = x^6(1 + x + x^2)(1 - x^3)^{-3}$.
Then use the generalized binomial theorem. $\square$

Sequence	Generating Function
$(1, 1, 1, 1, \ldots)$	$\frac{1}{1-x}$
$(1, -1, 1, -1, \ldots)$	$\frac{1}{1+x}$
$(-6, 6, -6, 6, \ldots)$	$\frac{-6}{1+x}$
$(0, 0, 0, 0, -6, 6, -6, 6, \ldots)$	$\frac{-6x^4}{1+x}$
$(1, 0, 1, 0, \ldots)$	$\frac{\frac{1}{1-x}+\frac{1}{1+x}}{2} = \frac{1}{1-x^2}$
$(0, 1, 0, 1, \ldots)$	$\frac{\frac{1}{1-x}-\frac{1}{1+x}}{2} = \frac{x}{1-x^2}$
$(1, 2, 4, 8, \ldots)$	$\frac{1}{1-2x}$
$(2, 4, 8, \ldots)$	$\frac{\frac{1}{1-2x}-1}{x} = \frac{2}{1-2x}$
$(1, 0, 2, 0, 4, 0, 8, \ldots)$	$\frac{1}{1-2x^2}$
$(1, 1, 2, 1, 4, 1, 8, \ldots)$	$\frac{1}{1-2x^2} + \frac{x}{1-x^2}$
$(1, 2, 1, 4, 1, 8, \ldots)$	$\frac{\frac{1}{1-2x^2}+\frac{x}{1-x^2}-1}{x} = -\frac{2x^3+2x^2-2x-1}{(1-2x^2)(1-x^2)}$