Homework 7

Problem 1. 1. *Prove that, for every integer n, there exists a coloring of the edges of the complete graph K_n by two colors so that the total number of monochromatic copies of K_4 is at most $\binom{n}{4}2^{-5}$.*

2. *Give a randomized algorithm for finding a coloring with at most $\binom{n}{4}2^{-5}$ monochromatic (i.e. single-color) copies of K_4 that runs in expected time polynomial in n.*

Solution.

(a) X is the random variable denoting the number of monochromatics copies of K_4. The probability that a certain $4-subset$ forms a monochromatic K_4 is 2×2^{-6}, where 2 is for the two different colours. Then,

$$E[X] = \binom{n}{4} \times 2 \times 2^{-6} = \binom{n}{4}2^{-5}$$

(b)) Colour edges independently and uniformly. Let $p = Pr(X \leq \binom{n}{4}2^{-5})$. Then we have

$$\binom{n}{4}2^{-5} = E[X] = \sum_{i \leq \binom{n}{4}2^{-5}} iPr(X = i) + \sum_{i \geq \binom{n}{4}2^{-5}} iPr(X = i) \geq p + (1-p)(\binom{n}{4}2^{-5} + 1)$$

which implies that

$$\frac{1}{p} \leq \binom{n}{4}2^{-5}$$

Thus, the expected number of samples is at most $\binom{n}{4}2^{-5}$. Testing this to see if $X \leq \binom{n}{4}2^{-5}$ can be done in $O(n^4)$ time. As such the algorithm can be done in polynomial time.

Problem 2. *Use the Lovasz local lemma to show that if*

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1$$

then it is possible to color the edges of K_n with two colors so that it has no monochromatic(i.e. single color) K_k subgraph.

Solution. Consider a random 2-coloring of the graph. Let E_i be the case that the ith $k-set$ is a monochromatic clique.

$$Pr(E_i) = 2 \cdot (\frac{1}{2})^{\binom{k}{2}} = 2^{1-\binom{k}{2}}$$

Then we have the number of $k-clique$ is bounded by $\binom{k}{2}\binom{n}{k-2}$. Thus, whenever

$$4\binom{k}{2}\binom{n}{k-2}2^{1-\binom{k}{2}} \leq 1$$

we can apply the general Lovasz local lemma and show that there exists a coloring satisfys the problem requirement.