Homework 4

Problem 1. *Express the n^{th} term of the sequences given by the following recurrence relations*

1. $a_0 = 2, a_1 = 3, a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \dots)$.
2. $a_0 = 1, a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \dots)$.

Solution.

1. Characteristic function is $x^2 + 2x - 3 = (x+3)(x-1) = 0$.

   Let $f_n = a(-3)^n + b \cdot 1^n$. Then $\begin{cases} 2 &= a + b \\ 3 &= -3a + b \end{cases} \Rightarrow a = -1/4, b = 9/4$.

   $\therefore$ the $n-$th term is f_n.

2. Characteristic function for the homogeneous part is $x = 2$. Take $a_n = p2^n + \lambda$

   $a_0 = 1, a_1 = 5$. Now $\begin{cases} 1 &= p + \lambda \\ 5 &= 2p + \lambda \end{cases} \Rightarrow p = 4, \lambda = -3$.

$\square$

Problem 2. *Solve the recurrence relation $a_{n+2} = \sqrt{a_{n+1}a_n}$ with initial conditions $a_0 = 2, a_1 = 8$ and find $\lim_{n\to\infty} a_n$.*

Solution. Consider the sequence $b_n = \log_2 a_n$. Then

$$2\log_2 a_{n+2} = \log_2 a_{n+1} + \log_2 a_n$$

i.e. $2b_{n+2} = b_{n+1} + b_n$. $b_0 = 1, b_1 = 3$. One can find $b_n = (-\frac{4}{3})(-\frac{1}{2})^n + \frac{7}{3}$. $\therefore a_n = 2^{(-\frac{4}{3})(-\frac{1}{2})^n + \frac{7}{3}}$. $\lim_{n\to\infty} a_n = 2^{\frac{7}{3}}$. $\square$

Problem 3. *Fill in the blanks with either true (✓) or false (×)*

$f(n)$	$g(n)$	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$
$2n^3 + 3n$	$100n^2 + 2n + 100$	×	✓	×
$50n + \log n$	$10n + \log\log n$	✓	✓	✓
$50n \log n$	$10n \log\log n$	×	✓	×
$\log n$	$\log^2 n$	✓	×	×
$n!$	5^n	×	✓	×

Problem 4.

1. *Find two functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.*
2. *Furthermore, we say a function $h : \mathbb{R} \to \mathbb{R}$ is* monotonically increasing *if it satisfies the property '$x \leq y \implies h(x) \leq h(y)$'.*
   Find two monotonically increasing functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

(Please give the detailed proof that your functions satisfy the requirements.)

Solution.

1. $\begin{cases} f(x) = \sin(x); \\ g(x) = \cos(x). \end{cases}$
2. $\begin{cases} f(x) = x^{\sin(x)+x}; \\ g(x) = x^{\cos(x)+x}. \end{cases}$

The detailed proof are omitted. Just stick to the definition of $O(-)$. □

Problem 5.

a) *Show that the product of all primes p with $m < p \leq 2m$ is at most $\binom{2m}{m}$.*

b) *Using a), prove the estimate $\pi(x) = O(\frac{x}{\ln x})$, where $\pi(x)$ denote the number of primes not exceeding the number x.*

Solution.

1. $B = \binom{2m}{m} = \frac{(m+1)\times(m+2)\times\cdots\times(2m)}{1\times 2\times\cdots\times m}$. It is easy to find that if p is a prime number and $p \in (m, 2m]$ then $p|B$. Thus $\prod\limits_{m<p\leq 2m} p \mid B$. It follows that the upper bound of the products of prime numbers between m and $2m$ is B.

2. There are several ways to prove the second problem.

First proof: Combing a), w.l.o.g. assume n is even and $n = 2m$. It is obvious that

$$B \leq \sum_{i=0}^{2m} \binom{2m}{i} = 4^m$$

With a) we have $\prod_{m<p\leq 2m} p \leq B \leq 4^m$ (p is prime, as above). It follows that

$$\sum_{m<p\leq 2m} \log p \leq m \log 4 = 2m \qquad (\star)$$

Then count the number of primes between m and $2m$, i.e. the number of $p \in (m, 2m]$,

$$\pi(2m) - \pi(m) = \sum_{m<p\leq 2m} 1 \leq \sum_{m<p\leq 2m} \frac{\log p}{\log m} = \frac{1}{\log m}\left(\sum_{m<p\leq 2m} \log p\right) \overset{(\star)}{\leq} \frac{2m}{\log m}.$$

For any given x, there exists $k \geq 1$ such that $x \in (2^{k-1}, 2^k]$.
Finally with the above analysis

$$\pi(x) \leq \pi(2^k) = \sum_{i=1}^{k} \left(\pi(2^i) - \pi(2^{i-1})\right) = O\left(\sum_{j=1}^{k} \frac{2^j}{j}\right) = O\left(\frac{2^k}{k}\right) = O\left(\frac{x}{\ln x}\right).$$

Second proof: Proof by contradiction

□