Homework 5

Problem 1. *Which of the following statements about graph G and H are true?*

1. *G and H are isomorphic if and only if for every map $f : V(G) \to V(H)$ and for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*

2. *G and H are isomorphic if and only if there exists a bijection $f : E(G) \to E(H)$.*

3. *If there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$, then G and H are isomorphic.*

4. *If G and H are isomorphic, then there exists a bijection $f : V(G) \to V(H)$ such that every vertex $u \in V(G)$ has the same degree as $f(u)$.*

5. *If G and H are isomorphic, then there exists a bijection $f : E(G) \to E(H)$.*

6. *G and H are isomorphic if and only if there exists a map $f : V(G) \to V(H)$ such that for any two vertices $u, v \in V(G)$, we have $\{u, v\} \in E(G) \Leftrightarrow \{f(u), f(v)\} \in E(H)$.*

7. *Every graph on n vertices is isomorphic to some graph on the vertex set $\{1, 2, \ldots, n\}$.*

8. *Every graph on $n \geq 1$ vertices is isomorphic to infinitely many graphs.*

Solution. 4,5,7,8. □

Problem 2. *Two simple graphs $G = (V, E)$ and $G' = (V', E')$. A map $f : V \to V'$. Now if f satisfies:*

i) *It is a bijective function;*

ii) *$\{x, y\} \in E$ if and only if $\{f(x), f(y)\} \in E'$;*

Then we say that graph G and G′ are isomorphic *to each other. We use $G \cong G'$ to stand for the isomorphism relation.*

Consider the following questions:

1. *$G = K_n$ (Recall: K_n is a clique with n vertices), $g : V \to V'$ is a function which only satisfies requirement ii). Prove that G' must contain a subgraph which is a clique with n-vertices.*

2. *$G = K_{n,m}$ (Recall: $K_{n,m}$ is the so-called* complete bipartite graphs*), g is the same as in question 1. What will be the simplest G' that is related to G under the new relation.*

Solution.

1. Two different vertices in G must be mapped to different vertices in G'. For if $u, v \in V$ are mapped to the same vertex $\omega \in V'$, the edge $\{u, v\} \in E$ cannot be reflected in E', for G' is a simple graph.

2. G' can be just one edge with two incident vertices.

□

Problem 3. *How many graphs on the vertex set* $\{1, 2, \ldots, 2n\}$ *are isomorphic to the graph consisting of n vertex-disjoint edges (i.e. with edge set* $\{\{1,2\},\{3,4\},\ldots,\{2n-1,2n\}\}$*?*

Solution. $\frac{(2n\cdot(2n-1))((2n-2)\cdot(2n-3))\cdots(2\cdot 1)}{2^n\cdot n!} = (2n-1)(2n-3)\cdot\cdots 5\cdot 3.$ □

Problem 4. *Construct an example of a sequence of length n in which each term is some of the numbers* $1, 2, \ldots, n-1$ *and which has an even number of odd terms, and yet the sequence is not a graph score. Show why it is not a graph score.*

Solution. E.g. $(1, 1, 3, 3, 4)$. Use the *Score theorem* to prove that it cannot be a graph score. □

Problem 5. *Let G be a graph with 9 vertices, each of degree 5 or 6. Prove that it has at least 5 vertices of degree 6 or at least 6 vertices of degree 5.*

Solution. x be the number of vertex in G with $deg_G(x) = 6$. Obviously $x \geq 5$ or $x \leq 4$.

1. If $x \geq 5$ then the first part of the argument is true.

2. Otherwise ($x \leq 4$). As the other vertices in graph G are of degree 5, there are at least $9 - x \geq 5$ such vertices. According to the hand-shake lemma, there must be even number of odd-degree vertices. Thus there should be at least 6 vertices with degree 5.

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