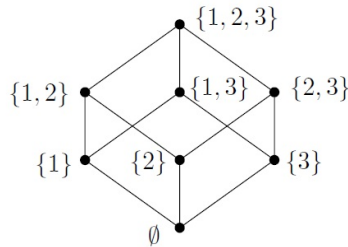


Homework 2

Problem 1. Draw the Hasse diagram of the set of all subsets of $\{1, 2, 3\}$ ordered by inclusion.

Solution.

□



Problem 2. Let (X, \leq_1) , (Y, \leq_2) be (partially) ordered sets. We say that they are isomorphic if there exists a bijection $f : X \rightarrow Y$ such that for every $x, y \in X$, we have $x \leq_1 y$ if and only if $f(x) \leq_2 f(y)$.

1. Draw Hasse diagrams for all nonisomorphic 3-element posets.
2. Prove that any two n -element linearly ordered sets are isomorphic.
3. Prove that (\mathbb{N}, \leq) and (\mathbb{Q}, \leq) are not isomorphic. (where \mathbb{N} is the set of natural numbers, \mathbb{Q} is the set of rational numbers, \leq is the usual 'less or equal to' between numbers).

Solution.

2. Hint: Always map the minimal/least element in one structure to the other.

3. Suppose there is such an isomorphism function $f : \mathbb{N} \rightarrow \mathbb{Q}$. $f(0) = a$, $f(1) = b$. We have $a < b$ for $0 < 1$. Then consider $f^{-1}(\frac{a+b}{2})$. □

Problem 3. Prove or disprove: If a partially ordered set (X, \leq) has a single minimal element, then it is a smallest element as well.

Solution. Wrong. Consider $(\{a\}, \langle a, a \rangle) \cup (\mathbb{Z}, \leq)$. □

Problem 4. Let (X, \leq) and (X', \leq') be partially ordered sets. A mapping $f : X \rightarrow X'$ is called an embedding of (X, \leq) into (X', \leq') if the following conditions hold:

- f is an injective mapping;
- $f(x) \leq' f(y)$ if and only if $x \leq y$.

Now consider the following problem

- Describe an embedding of the set $\{1, 2\} \times \mathbb{N}$ with the lexicographic ordering into the ordered set (\mathbb{Q}, \leq) .
- Solve the analog of a) with the set $\mathbb{N} \times \mathbb{N}$ (ordered lexicographically) instead of $\{1, 2\} \times \mathbb{N}$.

Solution.

- $f(i, n) = i \cdot y$ where $i \in \{1, 2\}$ and $y = \frac{1}{e^{1/x}}$.
- Similar.

□

Problem 5. Prove the following strengthening of the **Erdős-Szekeres Lemma**: Let κ, ℓ be natural numbers. Then every sequence of real numbers of length $\kappa\ell + 1$ contains a nondecreasing subsequence of length $\kappa + 1$ or a decreasing subsequence of length $\ell + 1$.

Solution. Hint: $\alpha(P) \cdot \omega(P) \geq \kappa\ell$. Then similar to the proof of Erdős-Szekeres Lemma: either $\omega(P) > \kappa$, which implies the existence of a nondecreasing subsequence of length $\kappa + 1$, or $\omega(P) < \kappa$, then $\omega(P) > \ell$, which implies the other case. □