Homework 4

Problem 1. *Express the n^{th} term of the sequences given by the following recurrence relations*

1. $a_0 = 2, a_1 = 3, a_{n+2} = 3a_n - 2a_{n+1}$ $(n = 0, 1, 2, \ldots)$.
2. $a_0 = 1, a_{n+1} = 2a_n + 3$ $(n = 0, 1, 2, \ldots)$.

Solution.

(1) As we can get the characteristic equation $x^2+2x-3 = 0$*, so* $x_1 = -3$ *and* $x_2 = 1$*, we can assume* $a_n = c_1(-3)^n + c_2$.

Then, we make $n = 0$ *and* $n = 1$*, get* $c_1 + c_2 = 2$; $-3c_1 + c_2 = 3$*, so* $c_1 = -\frac{1}{4}, c_2 = \frac{9}{4}$*, it means* $a_n = -\frac{1}{4}(-3)^n + \frac{9}{4}$.

(2) For special solution, we assume its form folows $a'_n = c_1$*, then we can get* $c_1 = -3$*. For homogeneous part,* $a_n = c_2 2^n$*, for* $a_0 = 1$*, we can get* $c_2 = 1$*. As is mentioned above,* $a_n = -3 + 2^n$.

Problem 2. *Solve the recurrence relation* $a_{n+2} = \sqrt{a_{n+1}a_n}$ *with initial conditions* $a_0 = 2, a_1 = 8$ *and find* $\lim_{n\to\infty} a_n$.

Solution.

(1) We can use logarithm to simplise the equotion as $\log(a_{n+2}) = \frac{1}{2}\log(a_{n+1}) + \frac{1}{2}\log(a_n)$*, we define* $b_n = \log(a_n)$.

we can get the characteristic equotion as $2x^2-x-1 = 0 \Rightarrow x_1 = -\frac{1}{2},\ x_2 = 1$*, it means the solution have the form* $b_n = c_1 + c_2(-\frac{1}{2})^n$*, as* $b_0 = \log(2), b_1 = 3\log(2)$*, we can get* $b_n = \frac{7}{3}\log(2) - \frac{4}{3}\log(2)(-\frac{1}{2})^n$*. Therefore, if we choose* $e = 2.718281828\cdots$ *as logarithm base,* $a_n = exp(\frac{7}{3}\ln(2) - \frac{4}{3}\ln(2)(-\frac{1}{2})^n)$.

(2) As $n \to \infty$*,* $a_n \to 2^{\frac{7}{3}}$.

Problem 3. *Fill in the blanks with either true (✓) or false (×)*

Problem 4. *1. Find two functions* $f(x)$ *and* $g(x)$ *such that* $f(x) \neq O(g(x))$ *and* $g(x) \neq O(f(x))$.

$f(n)$	$g(n)$	$f = O(g)$	$f = \Omega(g)$	$f = \Theta(g)$
$2n^3 + 3n$	$100n^2 + 2n + 100$	$\times$	$\checkmark$	$\times$
$50n + \log n$	$10n + \log\log n$	$\checkmark$	$\checkmark$	$\checkmark$
$50n \log n$	$10n \log\log n$	$\times$	$\checkmark$	$\checkmark$
$\log n$	$\log^2 n$	$\checkmark$	$\times$	$\times$
$n!$	5^n	$\checkmark$	$\times$	$\times$

2. *Furthermore, we say a function $h : \mathbb{R} \to \mathbb{R}$ is* monotonically increasing *if it satisfies the property '$x \leq y \Rightarrow h(x) \leq h(y)$'.*
   Find two monotonically increasing functions $f(x)$ and $g(x)$ such that $f(x) \neq O(g(x))$ and $g(x) \neq O(f(x))$.

(Please give the detailed proof that your functions satisfy the requirements.)

Solution.

(1) As we can define a pair of functions to satisfy this situation as folows:

$$f(x) = x, x \in N$$
$$g(x) = \begin{cases} x+1 & , x \in Even, \\ x-1 & , x \in Odd. \end{cases}$$

For this pair of function, we can find there not exists a specific relation between them, which can satisfy origin situation.

(2) We can give a pair of functions as follows:

$$f(x) = x, x \in N$$
$$g(x) = \begin{cases} x+1 & , x \in Even, \\ x-1 & , x \in Odd. \end{cases}$$

For $f(x)$ and $g(x)$, they are both monotonically increasing functions, but we can find:

- *if $x \in Odd$, $g(x) < f(x)$, therefore, $g(x) \neq O(f(x))$.*
- *if $x \in Even$, $g(x) > f(x)$, therefore, $f(x) \neq O(g(x))$.*

As is mentioned above, this pair of functions satisfy this situation.

Problem 5.

a) Show that the product of all primes p with $m + 1 < p \leq 2m$ is at most $\binom{2m}{m}$.

b) Using a), prove the estimate $\pi(n) = O(\frac{n}{\ln n})$, where $\pi(n)$ denotes the number of primes not exceeding the number n.

Solution.

(a) $C_{m+1}^{2m+1} = \binom{2m+1}{m+1} = \frac{(2m+1)!}{(m+1)!m!} = \frac{(2m+1)2m\cdots(m+2)}{m(m-1)\cdots 2\cdot 1}$. *As we all know this number is an integer, and all the prime number between $m+1$ and $2m$ can't be divided by denominator $m, (m-1), \cdots 2, 1$, it means once C_{m+1}^{2m+1} be diveded by all the prime between $m+1$ and $2m$, it still remains a positive integer(positive integer must be greater than 1). Therefore, the product of all primes between $m+1$ and $2m$ is at most C_{m+1}^{2m+1}.*

(b) According to a), we have $\Pi_{\frac{n}{2} \leq p \leq n, prime} < C_{\frac{n}{2}}^{n} < 4^n$. *We can define t as a constant number which is smaller than $\frac{n}{2}$. So we have* $t^{\pi(n)-\pi(\frac{n}{2})} < \Pi_{\frac{n}{2} \leq p \leq n, prime} < C_{\frac{n}{2}}^{n} < 4^n$. *So we can get* $\pi(n) - \pi(\frac{n}{2}) \leq \frac{n \ln(4)}{\ln(t)}$, *with iteration, we can get* $\pi(n) \leq \frac{n \ln(4)}{\ln(t)} \times (2 - \frac{2}{n}) < \frac{2n \ln(4)}{\ln(t)}$. *We let $n = \frac{1}{2}n$, it makes* $\pi(n) \leq \frac{2n \ln(4)}{\ln(n) - \ln(2)} \leq \frac{4n \ln(4)}{\ln(n)}$, *it means* $\pi(n) = O(\frac{n}{\ln n})$.