Homework 8

Problem 1. *Show that, for constant $p \in (0,1)$, almost no graph in $\mathcal{G}(n,p)$ has a separating complete subgraph.*

Solution. The probability of k-subgraph exist

$$P = \sum_{i=1}^{k-1} \binom{n}{k} p^{\binom{k}{2}} (1-p)^{\binom{n}{2}-\binom{k}{2}}$$

We assume $r = \max\{p, 1-p\}$, then

$$P \leq \sum_{i=1}^{k-1} \binom{n}{k} r^{\binom{n}{2}} \leq 2^n r^{\binom{n}{2}} \to 0 \; as \; n \; \to \infty$$

Which means almost no graph in $\mathcal{G}(n,p)$ has a separating complete subgraph.

Problem 2. *What is the expected number of trees with k vertices in $G \in \mathcal{G}(n,p)$?*

Solution. For n different nodes, there are n^{n-2} types of tree. Refering to G(n,p) model, we can give the Expectation:

$$n^{(n-2)} \times C_n^k p^{k-1} (1-p)^{\binom{k}{2}-k+1}$$

Problem 3. *Show that if almost all $G \in \mathcal{G}(n,p)$ have a graph property $\mathcal{P}_1$ and almost all $G \in \mathcal{G}(n,p)$ have a graph property $\mathcal{P}_2$, then almost all $G \in \mathcal{G}(n,p)$ have both properties.*

Solution.

Once almost all $G \in \mathcal{G}(n,p)$ have both properties, it means G holds $\mathcal{P}_1 \cap \mathcal{P}_2$. Therefore, we will show that the property $\mathcal{P}_1 \cap \mathcal{P}_2$ holds for almost all graphs.

We show that for any $\epsilon > 0$, there exists $n \in N$ such that $P_{G(n,p)}(G \in P_1 \cap P_2) \geq 1-\epsilon$. For both $i = 1, 2$, Since P_i is a graph property that holds for almost all graphs in $G(n,p)$, there exists $n_i \in N$ such that $P_{G(n,p)}(G \in Pi) > 1 - \frac{\epsilon}{2}$. Therefore, for $n \geq \max\{n_1, n_2\}$, we have $P_{G(n,p)}(G \in \overline{P_1 \cap P_2}) = P_{G(n,p)}(G \in \overline{P1} \cup \overline{P2}) \leq \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus, $P_{G(n,p)}(G \in P1 \cap P2) = 1 - P_{G(n,p)}(G \in \overline{P1 \cap P2}) \geq 1 - \epsilon$.