Homework 6

Problem 1. Given a sequence $(d_1, d_2, \ldots, d_n)$ of positive integers (where $n \geq 1$):

(i) There exists a tree with score $(d_1, d_2, \ldots, d_n)$.

(ii) $\sum_{i=1}^{n} d_i = 2n - 2$.

Prove that (i) and (ii) are equivalent.

Solution.

- If a sequence $(d1, \cdots, dn)$ is a degree sequence of a tree $\mathrm{T} = (V, E)$, then $\sum_{i=1}^{n} d_i = 2|E|$, but in a tree, $|E| = n - 1$, so $\sum_{i=1}^{n} d_i = 2|E| = 2(n-1)$

- Now that the sequence have n item, we ca easily claim that there must exsit i satisefy $d_i = 1$, otherwise $\sum_{i=1}^{n} d_i >= 2n$, since d_i is an unordered sequence, we assume $d_n = 1$. Samely, we can assume $d_{n-1} >= 2$. Then we can find the $n-1$ item sequence $(d'_1 = d_1, d'_2 = d_2, \cdots, d'_{n-1} = d_{n-1} - 1)$ have $2\{(n-1)-1\}$ degree sum, which confirms to $\sum_{i=1}^{n} d_i = 2|E| = 2(n-1)$. For $n = 2$, is trivial to find the proof established, by mathematical induction, we can claim the total proof established.

Problem 2. Find an example to verify the claim that '(pairwise) independence does not imply mutual independence'. Pls give a detailed proof.

Solution.

- We first assume complete event space $\Omega = \{1, 2, 3, 4\}$, with each $\omega \in \Omega$ equally likely to occur: $\forall \omega \in \Omega : \Pr(\omega) = \frac{1}{4}$

  Consider these events: $A = \{1, 2\}, B = \{1, 3\}, C = \{1, 4\}$

  It is obvious to see that: $\Pr(A) = \Pr(B) = \Pr(C) = \frac{1}{2}$

  We also have that: $\Pr(A \cap B) = \Pr(A \cap C) = \Pr(B \cap C) = \Pr(\{1\}) = \frac{1}{4}$
  Thus:

$$\Pr(A)\Pr(B) = \Pr(A \cap B)$$

$$\Pr(A)\Pr(C) = \Pr(A \cap C)$$

$$\Pr(B)\Pr(C) = \Pr(B \cap C)$$

Thus the events A, B, C are pairwise independent.

Then, we consider:

$$\Pr(A \cap B \cap C) = \Pr(\{1\}) = \frac{1}{4}$$

But:

$$\Pr(A)\Pr(B)\Pr(C) = \frac{1}{8} \neq \Pr(A \cap B \cap C)$$

So, although $\mathcal{S}$ is pairwise independent, it is not independent.

Problem 3. Show that, if $E_1, E_2, \ldots, E_n$ are mutually independent, then so are $\overline{E_1}, \overline{E_2}, \ldots, \overline{E_n}$.

Solution.

We can now prove that $\overline{E_1}, \overline{E_2}$ are matually independent.

Since E_1 and E_2 are matually independent, $P(E_1E_2) = P(E_1)P(E_2) = 1 - P(\overline{E_1}) - P(\overline{E_2}) + P(\overline{E_1})P(\overline{E_2})$. Then for $P(\overline{E_1E_2}) = 1 - P(E_1 + E_2) = 1 - P(E_1) - P(E_2) + P(E_1E_2) = P(\overline{E_1})P(\overline{E_2})$. For n matually independent E_i ,we can use recurrsive induction to find $\overline{E_1}, \overline{E_2}, \ldots, \overline{E_n}$ are mutually independent, too.

Problem 4. The problem on the 37^{st} page of slide on 'Probability: a quick review' (i.e., the more complicated example). (What is $Pr(U|W)$?)

Solution.

According to

$$Pr(U|W) = \frac{Pr(UW)}{Pr(W)}$$

As we can get

$$Pr(UW) = Pr(UW|R)Pr(R)+Pr(UW|\neg R)Pr(\neg R) = 0.9\times0.7\times0.8+0.2\times0.4\times0.2 = 0.52$$

$$Pr(W) = Pr(W|R)Pr(R) + Pr(W|\neg R)Pr(\neg R) = 0.7 \times 0.8 + 0.4 \times 0.2 = 0.64$$

Therefore, $Pr(U|W) = \dfrac{13}{16}$

Problem 5. Suppose X and Y are two independent random variables, show that

$$E(X \cdot Y) = E[X] \cdot E[Y]$$

.

Solution.

According to definition of expectation, we have:

$$E(X{\cdot}Y) = \sum_x \sum_y xy{\cdot}p_{X,Y}(x,y) = \sum_x \sum_y xp_X(x){\cdot}yp_Y(y) = (\sum_x x{\cdot}p_X(x))(\sum_y y{\cdot}p_Y(y)) = E[X]{\cdot}E[Y]$$

Problem 6. A monkey types on a 26 -letter keyboard that has lowercase letters only. Each letter is chosen independently and uniformly at random from the alphabet. If the monkey types 1,000,000 letters. what is the expected number of times the sequence "proof" appears?

Solution.

$$Pr = \sum_i^k = \frac{C^i_{10000-5i+i}}{26^{1000000}}$$

In this problem, $k = \lfloor \frac{1000000}{26} \rfloor = 38461$.