

IMU Preintegration

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1 Introduction

Quaternion-based derivation of the VINS-Mono IMU preintegration model. In this document I summarize what I've learned about the math behind IMU preintegration. All of the math in this document comes from [1] and [2]. In section 3 the propagation of pre-integration terms in (1), (2), and (3) are derived with the Euler method. In section 4 the error state kinematics are derived using the error-state Kalman filter (ESKF). In section 5, error state equations are developed using the Euler method and the expressions obtained from section 4.

2 Pre-integration Terms

From [1], we define our pre-integration terms as follows

$$\alpha_{b_{k+1}}^{b_k} = \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathbf{R}_t^{b_k} (\hat{\mathbf{a}}_t - \mathbf{b}_{a_t} - \mathbf{n}_a) dt dt \quad (1)$$

$$\beta_{b_{k+1}}^{b_k} = \int_{t_k}^{t_{k+1}} \mathbf{R}_t^{b_k} (\hat{\mathbf{a}}_t - \mathbf{b}_{a_t} - \mathbf{n}_a) dt \quad (2)$$

$$\gamma_{b_{k+1}}^{b_k} = \int_{t_k}^{t_{k+1}} \frac{1}{2} \boldsymbol{\Omega} (\hat{\boldsymbol{\omega}}_t - \mathbf{b}_{\omega_t} - \mathbf{n}_\omega) dt \quad (3)$$

$\alpha_{b_{k+1}}^{b_k}$, $\beta_{b_{k+1}}^{b_k}$, and $\gamma_{b_{k+1}}^{b_k}$ are the position, velocity, and orientation constraints between keyframes b_k and b_{k+1} .

$\hat{\mathbf{a}}_t$ and $\hat{\boldsymbol{\omega}}_t$ represent the measured IMU acceleration and angular velocity.

\mathbf{b}_{a_t} , \mathbf{b}_{ω_t} , \mathbf{n}_a and \mathbf{n}_ω represent accelerometer bias, gyroscope bias, accelerometer noise, and gyroscope noise.

3 Estimating Pre-integration Terms

When implementing (1), (2), and (3), we estimate the integrals using a numerical integration technique such as the Euler method, the mid-point method, or the RK4 method. Here, we derive the Euler method equations for estimating the pre-integration terms.

Given a differential equation of the form

$$\dot{\mathbf{x}} = f(t, \mathbf{x}) \quad (4)$$

Assuming the derivative f is constant over the interval $[n, n+1]$, we get the equation

$$\mathbf{x}_{n+1} = \mathbf{x}_n + \Delta t \cdot f(t_n, \mathbf{x}_n) \quad (5)$$

For the pre-integration estimation, we treat the noise terms \mathbf{n}_a and \mathbf{n}_ω as zero, and we take f to be $\mathbf{R}_t^{b_k}(\hat{\mathbf{a}}_t - \mathbf{b}_{a_t})$. Thus, we get

$$\boxed{\hat{\beta}_{i+1}^{b_k} = \hat{\beta}_i^{b_k} + \mathbf{R}(\hat{\gamma}_i^{b_k})(\hat{\mathbf{a}}_t - \mathbf{b}_{a_t})\delta t} \quad (6)$$

Where i is a discrete moment within the interval $[t_k, t_{k+1}]$, and δt is the time interval between two measurements i and $i+1$. Now, we integrate (6) with respect to δt as follows

$$\hat{\alpha}_{i+1}^{b_k} = \int \hat{\beta}_{i+1}^{b_k} d\delta t = \int (\hat{\beta}_i^{b_k} + \mathbf{R}(\hat{\gamma}_i^{b_k})(\hat{\mathbf{a}}_t - \mathbf{b}_{a_t})\delta t) d\delta t \quad (7)$$

$$\boxed{\hat{\alpha}_{i+1}^{b_k} = \hat{\alpha}_i^{b_k} + \hat{\beta}_i^{b_k} \delta t + \frac{1}{2} \mathbf{R}(\hat{\gamma}_i^{b_k})(\hat{\mathbf{a}}_t - \mathbf{b}_{a_t}) \delta t^2} \quad (8)$$

Next, we handle the propagation of the orientation quaternion $\hat{\gamma}_i^{b_k}$ using the following equivalence

$$\Omega(\hat{\omega}_i - \mathbf{b}_{w_i}) \longleftrightarrow \boldsymbol{\omega} = \begin{bmatrix} 0 \\ \omega_{i_x} - b_{w_{i_x}} \\ \omega_{i_y} - b_{w_{i_y}} \\ \omega_{i_z} - b_{w_{i_z}} \end{bmatrix} \quad (9)$$

Given the following differential equation

$$\frac{d\gamma}{dt} = \frac{1}{2} \Omega(\hat{\omega}_i - \mathbf{b}_{w_i}) \gamma \quad (10)$$

Using (9), we write (10) as

$$\frac{d\gamma}{dt} = \frac{1}{2} \boldsymbol{\omega} \gamma \quad (11)$$

Which has the following solution

$$\gamma(t) = \gamma(t_o) \exp\left(\frac{1}{2}\boldsymbol{\omega}(t - t_o)\right) \quad (12)$$

Substituting $t = t_o + \delta t$

$$\begin{aligned} \gamma(t_o + \delta t) &= \gamma(t_o) \exp\left(\frac{1}{2}\boldsymbol{\omega}\delta t\right) \\ &= \gamma(t_o) \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\boldsymbol{\omega}\delta t \end{bmatrix} \\ &= \gamma(t_o) \otimes \begin{bmatrix} 1 \\ \frac{1}{2}(\hat{\boldsymbol{\omega}}_{\mathbf{i}} - \mathbf{b}_{\omega_{\mathbf{i}}})\delta t \end{bmatrix} \end{aligned} \quad (13)$$

Thus, we arrive at the equation

$$\boxed{\hat{\gamma}_{i+1}^{b_k} = \hat{\gamma}_i^{b_k} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}(\hat{\boldsymbol{\omega}}_{\mathbf{i}} - \mathbf{b}_{\omega_{\mathbf{i}}})\delta t \end{bmatrix}} \quad (14)$$

We can now use (6), (8), and (14) to estimate our pre-integration terms (1), (2), and (3).

4 Error State Kinematics

In this section we use the error-state Kalman filter (ESKF) to estimate the error-state values of the position, velocity, quaternion, acceleration bias, and gyroscope bias variables. All derivations in this section are taken from [2]. We define the true-state variables as follows

$$\mathbf{p}_t = \mathbf{p} + \delta\mathbf{p} \quad (15)$$

$$\mathbf{v}_t = \mathbf{v} + \delta\mathbf{v} \quad (16)$$

$$\mathbf{q}_t = \mathbf{q} \otimes \delta\mathbf{q} \quad (17)$$

$$\mathbf{b}_{\mathbf{a}_t} = \mathbf{b}_{\mathbf{a}} + \delta\mathbf{b}_{\mathbf{a}} \quad (18)$$

$$\mathbf{b}_{\omega_t} = \mathbf{b}_{\omega} + \delta\mathbf{b}_{\omega} \quad (19)$$

Where \mathbf{p} , \mathbf{v} , \mathbf{q} , $\mathbf{a}_{\mathbf{b}}$, and $\omega_{\mathbf{b}}$ are nominal values, and the corresponding δ - terms are the error values.

We would like to derive the first order time-derivative of each error-state variable. During the process, we ignore any second-order infinitesimals that arise. Starting with (15), we rearrange for $\delta\mathbf{p}$ and take its time derivative.

$$\dot{\delta\mathbf{p}} = \dot{\mathbf{p}}_t - \dot{\mathbf{p}} \quad (20)$$

$$\dot{\delta\mathbf{p}} = \mathbf{v}_t - \mathbf{v} \quad (21)$$

$$\boxed{\delta \dot{\mathbf{p}} = \delta \mathbf{v}} \quad (22)$$

We do the same for (18) and (19). Recall that the derivative of the acceleration bias is $\mathbf{n}_{\mathbf{b}_a}$, and the derivative of the angular velocity bias is $\mathbf{n}_{\mathbf{b}_\omega}$. We also have $\dot{a}_b = 0$ and $\dot{b}_\omega = 0$.

$$\delta \mathbf{b}_a = \mathbf{b}_{a_t} - \mathbf{b}_a \quad (23)$$

$$\delta \dot{\mathbf{b}}_a = \dot{\mathbf{b}}_{a_t} - \dot{\mathbf{b}}_a \quad (24)$$

$$\boxed{\delta \dot{\mathbf{b}}_a = \mathbf{n}_{\mathbf{b}_a}} \quad (25)$$

The process is identical for the angular velocity.

$$\delta \mathbf{b}_\omega = \mathbf{b}_{\omega_t} - \mathbf{b}_\omega \quad (26)$$

$$\delta \dot{\mathbf{b}}_\omega = \dot{\mathbf{b}}_{\omega_t} - \dot{\mathbf{b}}_\omega \quad (27)$$

$$\boxed{\delta \dot{\mathbf{b}}_\omega = \mathbf{n}_{\mathbf{b}_\omega}} \quad (28)$$

Next, we show the non-trivial derivation of error terms $\delta \dot{\mathbf{v}}$ and $\delta \dot{\theta}$. We start with the velocity error. Consider the following equations

$$\mathbf{R}_t = \mathbf{R}(\mathbf{I} + [\delta \boldsymbol{\theta}]_\times) + \mathcal{O}(\|\delta \boldsymbol{\theta}\|^2) \quad (29)$$

$$\dot{\mathbf{v}} = \mathbf{R} \mathbf{a}_\beta + \mathbf{g} \quad (30)$$

where (29) is the small signal approximation of \mathbf{R}_t and we define \mathbf{a}_β and $\delta \mathbf{a}_\beta$ as such

$$\mathbf{a}_\beta = \hat{\mathbf{a}}_t - \mathbf{b}_a \quad (31)$$

$$\delta \mathbf{a}_\beta = \delta \mathbf{b}_a - \mathbf{n}_a \quad (32)$$

We can write the true acceleration taking into account small signal terms.

$$\mathbf{a}_t = \mathbf{R}_t(\mathbf{a}_\beta + \delta \mathbf{a}_\beta) + \mathbf{g} + \delta \mathbf{g} \quad (33)$$

We can use (16), (29), (30), and (33) to develop an expression for the velocity error $\delta \dot{\mathbf{v}}$. During this derivation we can ignore any terms involving \mathbf{g} because our pre-integration terms only depend on the IMU measurements.

$$\dot{\mathbf{v}} + \delta \dot{\mathbf{v}} = \dot{\mathbf{v}}_t = \mathbf{a}_t \quad (34)$$

$$\mathbf{R} \mathbf{a}_\beta + \delta \dot{\mathbf{v}} = \mathbf{R}(\mathbf{I} + [\delta \boldsymbol{\theta}]_\times)(\mathbf{a}_\beta + \delta \mathbf{a}_\beta) \quad (35)$$

Next, we rearrange and solve for $\delta \dot{\mathbf{v}}$. We get rid of any second order terms and use the relation $[\mathbf{a}]_\times \mathbf{b} = -[\mathbf{b}]_\times \mathbf{a}$ to simplify our expression.

$$\begin{aligned}
\delta \dot{\mathbf{v}} &= \mathbf{R}[\delta \boldsymbol{\theta}]_{\times} \mathbf{a}_{\beta} + \mathbf{R} \delta \mathbf{a}_{\beta} + \mathbf{R}[\delta \boldsymbol{\theta}]_{\times} \mathbf{a}_{\beta} \\
&= \mathbf{R}(\delta \mathbf{a}_{\beta} - [\mathbf{a}_{\beta}]_{\times} \delta \boldsymbol{\theta}) \\
&= \mathbf{R}(-[\hat{\mathbf{a}}_t - \mathbf{b}_a]_{\times} \delta \boldsymbol{\theta} - \delta \mathbf{b}_a - \mathbf{n}_a)
\end{aligned} \tag{36}$$

Thus, we end up with the following equality for the velocity error

$$\boxed{\delta \dot{\mathbf{v}} = -\mathbf{R}[\hat{\mathbf{a}}_t - \mathbf{b}_a]_{\times} \delta \boldsymbol{\theta} - \mathbf{R} \delta \mathbf{b}_a - \mathbf{R} \mathbf{n}_a} \tag{37}$$

Now, we develop the expression for the orientation error $\delta \dot{\boldsymbol{\theta}}$ using the true and nominal quaternion derivatives.

$$\dot{\mathbf{q}}_t = \frac{1}{2} \mathbf{q}_t \otimes \boldsymbol{\omega}_t \tag{38}$$

$$\dot{\mathbf{q}} = \frac{1}{2} \mathbf{q} \otimes \boldsymbol{\omega} \tag{39}$$

Similar to (31) and (32), we define $\boldsymbol{\omega}$ and $\delta \boldsymbol{\omega}$ as follows

$$\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}_t - \mathbf{b}_{\omega} \tag{40}$$

$$\delta \boldsymbol{\omega} = \delta \mathbf{b}_{\omega} - \mathbf{n}_{\omega} \tag{41}$$

which allows us to write $\boldsymbol{\omega}_t$ as

$$\boldsymbol{\omega}_t = \boldsymbol{\omega} + \delta \boldsymbol{\omega} \tag{42}$$

Using (17), we develop the expression for $\dot{\mathbf{q}}_t$

$$(\mathbf{q} \otimes \delta \mathbf{q}) = \dot{\mathbf{q}}_t = \frac{1}{2} \mathbf{q}_t \otimes \boldsymbol{\omega}_t \tag{43}$$

By chain rule and (38),

$$\dot{\mathbf{q}} \otimes \delta \mathbf{q} + \mathbf{q} \otimes \delta \dot{\mathbf{q}} = \frac{1}{2} \mathbf{q} \otimes \delta \mathbf{q} \otimes \boldsymbol{\omega}_t \tag{44}$$

$$2\delta \dot{\mathbf{q}} = 2 \left[\frac{\dot{1}}{\frac{\delta \boldsymbol{\theta}}{2}} \right] = \delta \mathbf{q} \otimes \boldsymbol{\omega}_t - \boldsymbol{\omega} \otimes \delta \mathbf{q} \tag{45}$$

$$\begin{aligned}
\begin{bmatrix} 0 \\ \delta \dot{\boldsymbol{\omega}} \end{bmatrix} &= [\mathbf{q}]_R(\boldsymbol{\omega}_t) - [\mathbf{q}]_L(\boldsymbol{\omega}) \delta \mathbf{q} \\
&= \begin{bmatrix} 0 & -(\boldsymbol{\omega}_t - \boldsymbol{\omega})^T \\ (\boldsymbol{\omega}_t - \boldsymbol{\omega}) & -[\boldsymbol{\omega}_t + \boldsymbol{\omega}]_{\times} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{\delta \boldsymbol{\theta}}{2} \end{bmatrix} + \mathcal{O}(\|\delta \boldsymbol{\theta}\|^2) \\
&= \begin{bmatrix} 0 & -\delta \boldsymbol{\omega}^T \\ \delta \boldsymbol{\omega} & -[2\boldsymbol{\omega} + \delta \boldsymbol{\omega}]_{\times} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{\delta \boldsymbol{\theta}}{2} \end{bmatrix} + \mathcal{O}(\|\delta \boldsymbol{\theta}\|^2)
\end{aligned} \tag{46}$$

From (46) we obtain two equations

$$0 = \delta\boldsymbol{\omega}^T \delta\boldsymbol{\theta} + \mathcal{O}(\|\delta\boldsymbol{\theta}\|^2) \quad (47)$$

$$\dot{\delta\boldsymbol{\theta}} = \delta\boldsymbol{\omega} - [\boldsymbol{\omega}]_{\times} \delta\boldsymbol{\theta} - \frac{1}{2}[\delta\boldsymbol{\omega}]_{\times} \delta\boldsymbol{\omega} + \mathcal{O}(\|\delta\boldsymbol{\theta}\|^2) \quad (48)$$

Taking (48), we ignore the second order terms and substitute (39) and (40), we get the angular error equation

$$\boxed{\dot{\delta\boldsymbol{\theta}} = -[\hat{\boldsymbol{\omega}}_t - \mathbf{b}_{\boldsymbol{\omega}}]_{\times} \delta\boldsymbol{\theta} - \delta\mathbf{b}_{\boldsymbol{\omega}} - \mathbf{n}_{\boldsymbol{\omega}}} \quad (49)$$

5 Euler Method Error State Equations

Using the general form of (5), we can develop error state equations for each state variable. Starting with the position error, we get

$$\delta\mathbf{p}_{k+1} = \delta\mathbf{p}_k + \delta t \dot{\delta\mathbf{p}} \quad (50)$$

$$\boxed{\delta\mathbf{p}_{k+1} = \delta\mathbf{p}_k + \delta\mathbf{v}\delta t} \quad (51)$$

For velocity, recall the velocity error term (37)

$$\begin{aligned} \delta\mathbf{v}_{k+1} &= \delta\mathbf{v}_k + \delta t \dot{\delta\mathbf{v}} \\ &= \delta\mathbf{v}_k + \delta t (-\mathbf{R}[\hat{\mathbf{a}}_t - \mathbf{b}_a]_{\times} \delta\boldsymbol{\theta} - \mathbf{R}\delta\mathbf{b}_a - \mathbf{R}\mathbf{n}_a) \end{aligned} \quad (52)$$

$$\boxed{\delta\mathbf{v}_{k+1} = \delta\mathbf{v}_k - \mathbf{R}[\hat{\mathbf{a}}_t - \mathbf{b}_a]_{\times} \delta\boldsymbol{\theta}\delta t - \mathbf{R}\delta\mathbf{b}_a\delta t - \mathbf{R}\mathbf{n}_a\delta t} \quad (53)$$

Similarly, we use (49) to develop the orientation error with Euler method. We denote the variable as $\delta\mathbf{q}_k$.

$$\begin{aligned} \delta\mathbf{q}_{k+1} &= \delta\mathbf{q}_k + \delta t \dot{\delta\mathbf{q}} \\ &= \delta\mathbf{q}_k + \delta t (-[\hat{\boldsymbol{\omega}}_t - \mathbf{b}_{\boldsymbol{\omega}}]_{\times} \delta\boldsymbol{\theta} - \delta\mathbf{b}_{\boldsymbol{\omega}} - \mathbf{n}_{\boldsymbol{\omega}}) \end{aligned} \quad (54)$$

$$\boxed{\delta\mathbf{q}_{k+1} = \delta\mathbf{q}_k - [\hat{\boldsymbol{\omega}}_t - \mathbf{b}_{\boldsymbol{\omega}}]_{\times} \delta\boldsymbol{\theta}\delta t - \delta\mathbf{b}_{\boldsymbol{\omega}}\delta t - \mathbf{n}_{\boldsymbol{\omega}}\delta t} \quad (55)$$

Finally, we handle the accelerometer bias and gyroscope bias

$$\delta\mathbf{b}_{a_{k+1}} = \delta\mathbf{b}_{a_k} + \delta t \dot{\delta\mathbf{b}_{a_k}} \quad (56)$$

$$\boxed{\delta\mathbf{b}_{a_{k+1}} = \delta\mathbf{b}_{a_k} + \mathbf{n}_{b_a}\delta t} \quad (57)$$

$$\delta\mathbf{b}_{g_{k+1}} = \delta\mathbf{b}_{g_k} + \delta t \dot{\delta\mathbf{b}_{g_k}} \quad (58)$$

$$\boxed{\delta\mathbf{b}_{g_{k+1}} = \delta\mathbf{b}_{g_k} + \mathbf{n}_{b_g}\delta t} \quad (59)$$

With these error equations we derive our \mathbf{F} and \mathbf{G} matrices

$$\begin{aligned}
\begin{bmatrix} \delta \mathbf{p}_{k+1} \\ \delta \mathbf{q}_{k+1} \\ \delta \mathbf{v}_{k+1} \\ \delta \mathbf{b}_{a_{k+1}} \\ \delta \mathbf{b}_{g_{k+1}} \end{bmatrix} &= \begin{bmatrix} \mathbf{I} & 0 & \delta t & 0 & 0 \\ 0 & \mathbf{I} & -[\hat{\boldsymbol{\omega}}_t - \mathbf{b}_\omega]_\times & 0 & -\delta t \\ 0 & -\mathbf{R}[\hat{\mathbf{a}}_t - \mathbf{b}_a]_\times & \mathbf{I} & -\mathbf{R}\delta t & 0 \\ 0 & 0 & 0 & \mathbf{I} & 0 \\ 0 & 0 & 0 & 0 & \mathbf{I} \end{bmatrix} \begin{bmatrix} \delta \mathbf{p}_k \\ \delta \mathbf{q}_k \\ \delta \mathbf{v}_k \\ \delta \mathbf{b}_{a_k} \\ \delta \mathbf{b}_{g_k} \end{bmatrix} \\
&+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\delta t \\ -\mathbf{R}\delta t & 0 & 0 & 0 \\ 0 & 0 & \delta t & 0 \\ 0 & 0 & 0 & \delta t \end{bmatrix} \begin{bmatrix} \mathbf{n}_a \\ \mathbf{n}_\omega \\ \mathbf{n}_{b_a} \\ \mathbf{n}_{b_\omega} \end{bmatrix} = \mathbf{F}_t \delta \mathbf{z}_{k+1} + \mathbf{G}_t \mathbf{n}_t
\end{aligned} \tag{60}$$

References

- [1] Tong Qin, Peiliang Li, Shaojie Shen. *VINS-Mono: A Robust and Versatile Monocular Visual-Inertial State Estimator*. IEEE, 2018.
- [2] Joan Sola. *Quaternion kinematics for the error-state Kalman filter*. 2017.