IMU Preintegration

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1 Introduction

Quaternion-based derivation of the VINS-Mono IMU preintegration model. In this document I summarize what I've learned about the math behind IMU preintegration. All of the math in this document comes from [1] and [2]. In section 3 the propagation of pre-integration terms in (1), (2), and (3) are derived with the Euler method. In section 4 the error state kinematics are derived using the error-state Kalman filter (ESKF). In section 5, error state equations are developed using the Euler method and the expressions obtained from section 4.

2 Pre-integration Terms

From [1], we define our pre-integration terms as follows

$$\alpha^{\mathrm{b}_k}_{\mathrm{b}_{k+1}} = \int_{t_k}^{t_{k+1}} \int_{t_k}^{t_{k+1}} \mathbf{R}^{\mathbf{b_k}}_{\mathbf{t}}(\hat{\mathbf{a}}_{\mathbf{t}} - \mathbf{b}_{\mathbf{a_t}} - \mathbf{n_a})\, dt\, dt \tag{1}$$

$$\beta^{\mathrm{b}_k}_{\mathrm{b}_{k+1}} = \int_{t_k}^{t_{k+1}} \mathbf{R}^{\mathbf{b_k}}_{\mathbf{t}}(\hat{\mathbf{a}}_{\mathbf{t}} - \mathbf{b}_{\mathbf{a_t}} - \mathbf{n_a})\, dt \tag{2}$$

$$\gamma^{\mathrm{b}_k}_{\mathrm{b}_{k+1}} = \int_{t_k}^{t_{k+1}} \frac{1}{2}\mathbf{\Omega}(\hat{\omega}_t - \mathbf{b}_{\omega_\mathbf{t}} - \mathbf{n}_\omega)\, dt \tag{3}$$

$\alpha^{\mathrm{b}_k}_{\mathrm{b}_{k+1}}$, $\beta^{\mathrm{b}_k}_{\mathrm{b}_{k+1}}$, and $\gamma^{\mathrm{b}_k}_{\mathrm{b}_{k+1}}$ are the position, velocity, and orientation constraints between keyframes b_k and b_{k+1}.

$\hat{\mathbf{a}}_{\mathbf{t}}$ and $\hat{\boldsymbol{\omega}}_t$ represent the measured IMU acceleration and angular velocity.

$\mathbf{b}_{\mathbf{a_t}}$, $\mathbf{b}_{\omega_\mathbf{t}}$, $\mathbf{n_a}$ and $\mathbf{n}_\omega$ represent accelerometer bias, gyroscope bias, accelerometer noise, and gyroscope noise.

3 Estimating Pre-integration Terms

When implementing (1), (2), and (3), we estimate the integrals using a numerical integration technique such as the Euler method, the mid-point method, or the RK4 method. Here, we derive the Euler method equations for estimating the pre-integration terms.

Given a differential equation of the form

$$\dot{\mathbf{x}} = f(t, \mathbf{x}) \tag{4}$$

Assuming the derivative f is constant over the interval $[n, n+1]$, we get the equation

$$\mathbf{x_{n+1}} = \mathbf{x_n} + \Delta t \cdot f(t_n, \mathbf{x_n}) \tag{5}$$

For the pre-integration estimation, we treat the noise terms $\mathbf{n_a}$ and $\mathbf{n}_\omega$ as zero, and we take f to be $\mathbf{R_t^{b_k}}(\hat{\mathbf{a}}_\mathbf{t} - \mathbf{b_{a_t}})$. Thus, we get

$$\boxed{\hat{\beta}^{b_k}_{i+1} = \hat{\beta}^{b_k}_i + \mathbf{R}(\hat{\gamma}^{b_k}_i)(\hat{\mathbf{a}}_\mathbf{t} - \mathbf{b_{a_t}})\delta t} \tag{6}$$

Where i is a discrete moment within the interval $[t_k, t_{k+1}]$, and δt is the time interval between two measurements i and $i+1$. Now, we integrate (6) with respect to δt as follows

$$\hat{\alpha}^{b_k}_{i+1} = \int \hat{\beta}^{b_k}_{i+1}\, d\delta t = \int (\hat{\beta}^{b_k}_i + \mathbf{R}(\hat{\gamma}^{b_k}_i)(\hat{\mathbf{a}}_\mathbf{t} - \mathbf{b_{a_t}})\delta t) d\delta t \tag{7}$$

$$\boxed{\hat{\alpha}^{b_k}_{i+1} = \hat{\alpha}^{b_k}_i + \hat{\beta}^{b_k}_i \delta t + \frac{1}{2}\mathbf{R}(\hat{\gamma}^{b_k}_i)(\hat{\mathbf{a}}_\mathbf{t} - \mathbf{b_{a_t}})\delta t^2} \tag{8}$$

Next, we handle the propagation of the orientation quaternion $\hat{\gamma}^{b_k}_i$ using the following equivalence

$$\mathbf{\Omega}(\hat{\boldsymbol{\omega}}_{\boldsymbol{i}} - \mathbf{b_{w_i}}) \longleftrightarrow \boldsymbol{\omega} = \begin{bmatrix} 0 \\ \omega_{i_x} - b_{w_{i_x}} \\ \omega_{i_y} - b_{w_{i_y}} \\ \omega_{i_z} - b_{w_{i_z}} \end{bmatrix} \tag{9}$$

Given the following differential equation

$$\frac{d\gamma}{dt} = \frac{1}{2}\mathbf{\Omega}(\hat{\boldsymbol{\omega}}_{\boldsymbol{i}} - \mathbf{b_{w_i}})\gamma \tag{10}$$

Using (9), we write (10) as

$$\frac{d\gamma}{dt} = \frac{1}{2}\boldsymbol{\omega}\gamma \tag{11}$$

Which has the following solution

$$\gamma(t) = \gamma(t_o)\exp\left(\frac{1}{2}\boldsymbol{\omega}(t - t_o)\right) \tag{12}$$

Substituting $t = t_o + \delta t$

$$\begin{aligned}
\gamma(t_0 + \delta t) &= \gamma(t_0)\exp\left(\frac{1}{2}\boldsymbol{\omega}\delta t\right) \\
&= \gamma(t_0) \otimes \begin{bmatrix} 1 \\ \frac{1}{2}\boldsymbol{\omega}\delta t \end{bmatrix} \\
&= \gamma(t_0) \otimes \begin{bmatrix} 1 \\ \frac{1}{2}(\hat{\boldsymbol{\omega}}_{\boldsymbol{i}} - \mathbf{b}_{\omega_{\mathbf{i}}})\delta t \end{bmatrix}
\end{aligned} \tag{13}$$

Thus, we arrive at the equation

$$\boxed{\hat{\gamma}^{b_k}_{i+1} = \hat{\gamma}^{b_k}_{i} \otimes \begin{bmatrix} 1 \\ \frac{1}{2}(\hat{\boldsymbol{\omega}}_{\boldsymbol{i}} - \mathbf{b}_{\omega_{\mathbf{i}}})\delta t \end{bmatrix}} \tag{14}$$

We can now use (6), (8), and (14) to estimate our pre-integration terms (1), (2), and (3).

4 Error State Kinematics

In this section we use the error-state Kalman filter (ESKF) to estimate the error-state values of the position, velocity, quaternion, acceleration bias, and gyroscope bias variables. All derivations in this section are taken from [2]. We define the true-state variables as follows

$$\mathbf{p_t} = \mathbf{p} + \delta\mathbf{p} \tag{15}$$

$$\mathbf{v_t} = \mathbf{v} + \delta\mathbf{v} \tag{16}$$

$$\mathbf{q_t} = \mathbf{q} \otimes \delta\mathbf{q} \tag{17}$$

$$\mathbf{b_{a_t}} = \mathbf{b_a} + \delta\mathbf{b_a} \tag{18}$$

$$\mathbf{b}_{\omega_\mathbf{t}} = \mathbf{b}_\omega + \delta\mathbf{b}_\omega \tag{19}$$

Where $\mathbf{p}$, $\mathbf{v}$, $\mathbf{q}$, $\mathbf{a_b}$, and $\omega_\mathbf{b}$ are nominal values, and the corresponding $\delta-$ terms are the error values.

We would like to derive the first order time-derivative of each error-state variable. During the process, we ignore any second-order infinitesimals that arise. Starting with (15), we rearrange for $\delta\mathbf{p}$ and take its time derivative.

$$\dot{\delta\mathbf{p}} = \dot{\mathbf{p_t}} - \dot{\mathbf{p}} \tag{20}$$

$$\dot{\delta\mathbf{p}} = \mathbf{v_t} - \mathbf{v} \tag{21}$$

$$\boxed{\dot{\delta \mathbf{p}} = \delta \mathbf{v}} \tag{22}$$

We do the same for (18) and (19). Recall that the derivative of the acceleration bias is $\boldsymbol{n_{b_a}}$, and the derivative of the angular velocity bias is $\boldsymbol{n_{b_\omega}}$. We also have $\dot{a_b} = 0$ and $\dot{b_\omega} = 0$.

$$\delta \mathbf{b_a} = \mathbf{b_{a_t}} - \mathbf{b_a} \tag{23}$$

$$\delta \dot{\mathbf{b_a}} = \dot{\mathbf{b_{a_t}}} - \dot{\mathbf{b_a}} \tag{24}$$

$$\boxed{\delta \dot{\mathbf{b_a}} = \mathbf{n_{b_a}}} \tag{25}$$

The process is identical for the angular velocity.

$$\delta \mathbf{b}_\omega = \mathbf{b}_{\omega_\mathbf{t}} - \mathbf{b}_\omega \tag{26}$$

$$\delta \dot{\mathbf{b}_\omega} = \dot{\mathbf{b}_{\omega_\mathbf{t}}} - \dot{\mathbf{b}_\omega} \tag{27}$$

$$\boxed{\delta \dot{\mathbf{b}_\omega} = \mathbf{n}_{\mathbf{b}_\omega}} \tag{28}$$

Next, we show the non-trivial derivation of error terms $\dot{\delta \mathbf{v}}$ and $\dot{\delta \theta}$. We start with the velocity error. Consider the following equations

$$\mathbf{R_t} = \mathbf{R}(\mathbf{I} + [\delta \boldsymbol{\theta}]_\times) + \mathcal{O}(\|\delta \boldsymbol{\theta}\|^2) \tag{29}$$

$$\dot{\mathbf{v}} = \mathbf{R}\mathbf{a}_\beta + \boldsymbol{g} \tag{30}$$

where (29) is the small signal approximation of $\boldsymbol{R_t}$ and we define $\mathbf{a}_\beta$ and $\delta \mathbf{a}_\beta$ as such

$$\mathbf{a}_\beta = \hat{\mathbf{a}}_\mathbf{t} - \mathbf{b_a} \tag{31}$$

$$\delta \mathbf{a}_\beta = \delta \mathbf{b_a} - \mathbf{n_a} \tag{32}$$

We can write the true acceleration taking into account small signal terms.

$$\mathbf{a_t} = \mathbf{R_t}(\mathbf{a}_\beta + \delta \mathbf{a}_\beta) + \mathbf{g} + \delta \boldsymbol{g} \tag{33}$$

We can use (16), (29), (30), and (33) to develop an expression for the velocity error $\delta \dot{\boldsymbol{v}}$. During this derivation we can ignore any terms involving $\boldsymbol{g}$ because our pre-integration terms only depend on the IMU measurements.

$$\dot{\boldsymbol{v}} + \delta \dot{\boldsymbol{v}} = \dot{\boldsymbol{v}}_{\boldsymbol{t}} = \boldsymbol{a_t} \tag{34}$$

$$\mathbf{R}\mathbf{a}_\beta + \delta \dot{\boldsymbol{v}} = \mathbf{R}(\mathbf{I} + [\delta \boldsymbol{\theta}]_\times)(\mathbf{a}_\beta + \delta \mathbf{a}_\beta) \tag{35}$$

Next, we rearrange and solve for $\delta \dot{\boldsymbol{v}}$. We get rid of any second order terms and use the relation $[\boldsymbol{a}]_\times \boldsymbol{b} = -[\boldsymbol{b}]_\times \boldsymbol{a}$ to simplify our expression.

$$
\begin{aligned}
\delta\dot{\boldsymbol{v}} &= \mathbf{R}[\delta\boldsymbol{\theta}]_\times \mathbf{a}_\beta + \mathbf{R}\delta\mathbf{a}_\beta + \mathbf{R}[\delta\boldsymbol{\theta}]_\times \mathbf{a}_\beta \\
&= \mathbf{R}(\delta\boldsymbol{a_\beta} - [\boldsymbol{a_\beta}]_\times \delta\boldsymbol{\theta}) \\
&= \mathbf{R}(-[\hat{\boldsymbol{a}}_{\boldsymbol{t}} - \boldsymbol{b_a}]_\times \delta\boldsymbol{\theta} - \delta\boldsymbol{b_a} - \boldsymbol{n_a})
\end{aligned} \tag{36}
$$

Thus, we end up with the following equality for the velocity error

$$
\boxed{\delta\dot{\boldsymbol{v}} = -\mathbf{R}[\hat{\boldsymbol{a}}_{\boldsymbol{t}} - \boldsymbol{b_a}]_\times \delta\boldsymbol{\theta} - \mathbf{R}\delta\boldsymbol{b_a} - \mathbf{R}\boldsymbol{n_a}} \tag{37}
$$

Now, we develop the expression for the orientation error $\dot{\delta\boldsymbol{\theta}}$ using the true and nominal quaternion derivatives.

$$
\dot{\boldsymbol{q}}_{\boldsymbol{t}} = \frac{1}{2}\boldsymbol{q_t} \otimes \boldsymbol{\omega_t} \tag{38}
$$

$$
\dot{\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q} \otimes \boldsymbol{\omega} \tag{39}
$$

Similar to (31) and (32), we define $\boldsymbol{\omega}$ and $\delta\boldsymbol{\omega}$ as follows

$$
\boldsymbol{\omega} = \hat{\boldsymbol{\omega}}_{\boldsymbol{t}} - \boldsymbol{b_\omega} \tag{40}
$$

$$
\delta\boldsymbol{\omega} = \delta\boldsymbol{b_\omega} - \boldsymbol{n_\omega} \tag{41}
$$

which allows us to write $\boldsymbol{\omega_t}$ as

$$
\boldsymbol{\omega_t} = \boldsymbol{\omega} + \delta\boldsymbol{\omega} \tag{42}
$$

Using (17), we develop the expression for $\dot{\boldsymbol{q}}_{\boldsymbol{t}}$

$$
(\boldsymbol{q} \dot{\otimes} \delta\boldsymbol{q}) = \dot{\boldsymbol{q}}_{\boldsymbol{t}} = \frac{1}{2}\boldsymbol{q_t} \otimes \boldsymbol{\omega_t} \tag{43}
$$

By chain rule and (38),

$$
\dot{\boldsymbol{q}} \otimes \delta\boldsymbol{q} + \boldsymbol{q} \otimes \dot{\delta\boldsymbol{q}} = \frac{1}{2}\boldsymbol{q} \otimes \delta\boldsymbol{q} \otimes \boldsymbol{\omega_t} \tag{44}
$$

$$
2\dot{\delta\boldsymbol{q}} = 2\dot{\begin{bmatrix} 1 \\ \frac{\delta\boldsymbol{\theta}}{2} \end{bmatrix}} = \delta\boldsymbol{q} \otimes \boldsymbol{\omega_t} - \boldsymbol{\omega} \otimes \delta\boldsymbol{q} \tag{45}
$$

$$
\begin{aligned}
\begin{bmatrix} 0 \\ \dot{\delta\boldsymbol{\omega}} \end{bmatrix} &= [\boldsymbol{q}]_R(\boldsymbol{\omega_t}) - [\boldsymbol{q}]_L(\boldsymbol{\omega})\delta\boldsymbol{q} \\
&= \begin{bmatrix} 0 & -(\boldsymbol{\omega_t} - \boldsymbol{\omega})^T \\ (\boldsymbol{\omega_t} - \boldsymbol{\omega}) & -[\boldsymbol{\omega_t} + \boldsymbol{\omega}]_\times \end{bmatrix} \begin{bmatrix} 1 \\ \frac{\delta\boldsymbol{\theta}}{2} \end{bmatrix} + \mathcal{O}(\|\delta\boldsymbol{\theta}\|^2) \\
&= \begin{bmatrix} 0 & -\delta\boldsymbol{\omega}^T \\ \delta\boldsymbol{\omega} & -[2\boldsymbol{\omega} + \delta\boldsymbol{\omega}]_\times \end{bmatrix} \begin{bmatrix} 1 \\ \frac{\delta\boldsymbol{\theta}}{2} \end{bmatrix} + \mathcal{O}(\|\delta\boldsymbol{\theta}\|^2)
\end{aligned} \tag{46}
$$

From (46) we obtain two equations

$$0 = \delta\boldsymbol{\omega}^T \delta\boldsymbol{\theta} + \mathcal{O}(\|\delta\boldsymbol{\theta}\|^2) \tag{47}$$

$$\delta\dot{\boldsymbol{\theta}} = \delta\boldsymbol{\omega} - [\boldsymbol{\omega}]_\times \delta\boldsymbol{\theta} - \frac{1}{2}[\delta\boldsymbol{\omega}]_\times \delta\boldsymbol{\omega} + \mathcal{O}(\|\delta\boldsymbol{\theta}\|^2) \tag{48}$$

Taking (48), we ignore the second order terms and substitute (39) and (40), we get the angular error equation

$$\boxed{\delta\dot{\boldsymbol{\theta}} = -[\hat{\boldsymbol{\omega}}_t - \boldsymbol{b}_{\boldsymbol{\omega}}]_\times \delta\boldsymbol{\theta} - \delta\boldsymbol{b}_{\boldsymbol{\omega}} - \boldsymbol{n}_{\boldsymbol{\omega}}} \tag{49}$$

5 Euler Method Error State Equations

We wish to derive a first-order discrete-time covariance matrix $\boldsymbol{P}_{\boldsymbol{b_{k+1}}}^{\boldsymbol{b^k}}$ and the first-order jacobian matrix $\boldsymbol{J}_{\boldsymbol{b_{k+1}}}$. First, consider the error state vector $\boldsymbol{\delta x}$ and its linearized dynamics $\dot{\boldsymbol{\delta x}}$ derived in section 4.

$$\boldsymbol{\delta x} = \begin{bmatrix} \delta\boldsymbol{p} \\ \delta\boldsymbol{q} \\ \delta\boldsymbol{v} \\ \delta\boldsymbol{b_a} \\ \delta\boldsymbol{b_g} \end{bmatrix} \tag{50}$$

and

$$\dot{\boldsymbol{\delta x}} = \begin{bmatrix} \delta\dot{\boldsymbol{p}} \\ \delta\dot{\boldsymbol{q}} \\ \delta\dot{\boldsymbol{v}} \\ \delta\dot{\boldsymbol{b}}_{\boldsymbol{a}} \\ \delta\dot{\boldsymbol{b}}_{\boldsymbol{g}} \end{bmatrix} \tag{51}$$

Using the Euler method, we can propagate (50) as follows

$$\delta\boldsymbol{p} \leftarrow \delta\boldsymbol{p} + \delta\boldsymbol{v}\delta t + \frac{1}{2}\delta\dot{\boldsymbol{v}}\delta t^2 \tag{52}$$

$$\delta\boldsymbol{v} \leftarrow \delta\boldsymbol{v} + \delta\dot{\boldsymbol{v}}\delta t \tag{53}$$

$$\delta\boldsymbol{q} \leftarrow \delta\boldsymbol{q} + \delta\dot{\boldsymbol{q}}\delta t \tag{54}$$

$$\delta\boldsymbol{b_a} \leftarrow \delta\boldsymbol{b_a} + \delta\dot{\boldsymbol{b}}_{\boldsymbol{a}}\delta t \tag{55}$$

$$\delta\boldsymbol{b_g} \leftarrow \delta\boldsymbol{b_g} + \delta\dot{\boldsymbol{b}}_{\boldsymbol{g}}\delta t \tag{56}$$

where (52) comes from integrating (53) with respect to δt. We substitute the error state equations from section 4 into (52) - (56), and adopt the notation $\delta \boldsymbol{x_{k+1}}$ to represent the error term at keyframe $k+1$.

$$\delta \boldsymbol{p_{k+1}} = \delta \boldsymbol{p_k} + \delta \boldsymbol{v_k} \delta t + \frac{1}{2}(-\mathbf{R}[\boldsymbol{\hat{a}_t} - \boldsymbol{b_a}]_\times \delta \boldsymbol{q_k} - \mathbf{R}\delta \boldsymbol{b_a} - \mathbf{R}\boldsymbol{n_a})\delta t^2 \tag{57}$$

$$\delta \boldsymbol{v_{k+1}} = \delta \boldsymbol{v_k} + (-\mathbf{R}[\boldsymbol{\hat{a}_t} - \boldsymbol{b_a}]_\times \delta \boldsymbol{q_k} - \mathbf{R}\delta \boldsymbol{b_a} - \mathbf{R}\boldsymbol{n_a})\delta t \tag{58}$$

$$\delta \boldsymbol{q_{k+1}} = \delta \boldsymbol{q_k} + (-[\hat{\boldsymbol{\omega}}_t - \boldsymbol{b_\omega}]_\times \delta \boldsymbol{q_k} - \delta \boldsymbol{b_\omega} - \boldsymbol{n_\omega})\delta t \tag{59}$$

$$\delta \boldsymbol{b_{a_{k+1}}} = \delta \boldsymbol{b_{a_k}} + \boldsymbol{n_{b_a}} \delta t \tag{60}$$

$$\delta \boldsymbol{b_{g_{k+1}}} = \delta \boldsymbol{b_{g_k}} + \boldsymbol{n_{b_\omega}} \delta t \tag{61}$$

Using (57) - (61), we can derive the jacobian and covariance matrices. First, we calculate the following matrix $\boldsymbol{F}$

$$\begin{aligned}
\boldsymbol{F} = \frac{\partial \delta \boldsymbol{x_{k+1}}}{\partial \delta \boldsymbol{x_k}} &= \begin{bmatrix}
\frac{\partial \delta \boldsymbol{p_{k+1}}}{\partial \delta \boldsymbol{p_k}} & \frac{\partial \delta \boldsymbol{p_{k+1}}}{\partial \delta \boldsymbol{q_k}} & \frac{\partial \delta \boldsymbol{p_{k+1}}}{\partial \delta \boldsymbol{v_k}} & \frac{\partial \delta \boldsymbol{p_{k+1}}}{\partial \delta \boldsymbol{b_{a_k}}} & \frac{\partial \delta \boldsymbol{p_{k+1}}}{\partial \delta \boldsymbol{b_{g_k}}} \\
\frac{\partial \delta \boldsymbol{q_{k+1}}}{\partial \delta \boldsymbol{p_k}} & \frac{\partial \delta \boldsymbol{q_{k+1}}}{\partial \delta \boldsymbol{q_k}} & \frac{\partial \delta \boldsymbol{q_{k+1}}}{\partial \delta \boldsymbol{v_k}} & \frac{\partial \delta \boldsymbol{q_{k+1}}}{\partial \delta \boldsymbol{b_{a_k}}} & \frac{\partial \delta \boldsymbol{q_{k+1}}}{\partial \delta \boldsymbol{b_{g_k}}} \\
\frac{\partial \delta \boldsymbol{v_{k+1}}}{\partial \delta \boldsymbol{p_k}} & \frac{\partial \delta \boldsymbol{v_{k+1}}}{\partial \delta \boldsymbol{q_k}} & \frac{\partial \delta \boldsymbol{v_{k+1}}}{\partial \delta \boldsymbol{v_k}} & \frac{\partial \delta \boldsymbol{v_{k+1}}}{\partial \delta \boldsymbol{b_{a_k}}} & \frac{\partial \delta \boldsymbol{v_{k+1}}}{\partial \delta \boldsymbol{b_{g_k}}} \\
\frac{\partial \delta \boldsymbol{b_{a_{k+1}}}}{\partial \delta \boldsymbol{p_k}} & \frac{\partial \delta \boldsymbol{b_{a_{k+1}}}}{\partial \delta \boldsymbol{q_k}} & \frac{\partial \delta \boldsymbol{b_{a_{k+1}}}}{\partial \delta \boldsymbol{v_k}} & \frac{\partial \delta \boldsymbol{b_{a_{k+1}}}}{\partial \delta \boldsymbol{b_{a_k}}} & \frac{\partial \delta \boldsymbol{b_{a_{k+1}}}}{\partial \delta \boldsymbol{b_{g_k}}} \\
\frac{\partial \delta \boldsymbol{b_{g_{k+1}}}}{\partial \delta \boldsymbol{p_k}} & \frac{\partial \delta \boldsymbol{b_{g_{k+1}}}}{\partial \delta \boldsymbol{q_k}} & \frac{\partial \delta \boldsymbol{b_{g_{k+1}}}}{\partial \delta \boldsymbol{v_k}} & \frac{\partial \delta \boldsymbol{b_{g_{k+1}}}}{\partial \delta \boldsymbol{b_{a_k}}} & \frac{\partial \delta \boldsymbol{b_{g_{k+1}}}}{\partial \delta \boldsymbol{b_{g_k}}}
\end{bmatrix} \\
&= \begin{bmatrix}
\boldsymbol{I} & -\frac{1}{2}\boldsymbol{R}[\boldsymbol{\hat{a}_t} - \boldsymbol{b_a}]_\times \delta t & \delta t & -\frac{1}{2}\boldsymbol{R}\delta t^2 & 0 \\
0 & \boldsymbol{I} - [\boldsymbol{\hat{\omega}_t} - \boldsymbol{b_\omega}]_\times \delta t & 0 & 0 & -\delta t \\
0 & -\boldsymbol{R}[\boldsymbol{\hat{a}_t} - \boldsymbol{b_a}]_\times \delta t & \boldsymbol{I} & -\boldsymbol{R}\delta t & 0 \\
0 & 0 & 0 & \boldsymbol{I} & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{I}
\end{bmatrix}
\end{aligned} \tag{62}$$

Next, we calculate the following $\boldsymbol{G}$ matrix. First, consider the noise vector $\boldsymbol{n}$

$$\boldsymbol{n} = \begin{bmatrix} \boldsymbol{n_a} \\ \boldsymbol{n_\omega} \\ \boldsymbol{n_{b_a}} \\ \boldsymbol{n_{b_g}} \end{bmatrix} \tag{63}$$

$\boldsymbol{G}$ is defined as

$$
\boldsymbol{G} = \frac{\partial \delta \boldsymbol{x_{k+1}}}{\partial \boldsymbol{n}} = \begin{bmatrix}
\frac{\partial \delta \boldsymbol{p_{k+1}}}{\partial \boldsymbol{n_a}} & \frac{\partial \delta \boldsymbol{p_{k+1}}}{\partial \boldsymbol{n_\omega}} & \frac{\partial \delta \boldsymbol{p_{k+1}}}{\partial \boldsymbol{n_{b_a}}} & \frac{\partial \delta \boldsymbol{p_{k+1}}}{\partial \boldsymbol{n_{b_g}}} \\
\frac{\partial \delta \boldsymbol{q_{k+1}}}{\partial \boldsymbol{n_a}} & \frac{\partial \delta \boldsymbol{q_{k+1}}}{\partial \boldsymbol{n_\omega}} & \frac{\partial \delta \boldsymbol{q_{k+1}}}{\partial \boldsymbol{n_{b_a}}} & \frac{\partial \delta \boldsymbol{q_{k+1}}}{\partial \boldsymbol{n_{b_g}}} \\
\frac{\partial \delta \boldsymbol{v_{k+1}}}{\partial \boldsymbol{n_a}} & \frac{\partial \delta \boldsymbol{v_{k+1}}}{\partial \boldsymbol{n_\omega}} & \frac{\partial \delta \boldsymbol{v_{k+1}}}{\partial \boldsymbol{n_{b_a}}} & \frac{\partial \delta \boldsymbol{v_{k+1}}}{\partial \boldsymbol{n_{b_g}}} \\
\frac{\partial \delta \boldsymbol{b_{a_{k+1}}}}{\partial \boldsymbol{n_a}} & \frac{\partial \delta \boldsymbol{b_{a_{k+1}}}}{\partial \boldsymbol{n_\omega}} & \frac{\partial \delta \boldsymbol{b_{a_{k+1}}}}{\partial \boldsymbol{n_{b_a}}} & \frac{\partial \delta \boldsymbol{b_{a_{k+1}}}}{\partial \boldsymbol{n_{b_g}}} \\
\frac{\partial \delta \boldsymbol{b_{g_{k+1}}}}{\partial \boldsymbol{n_a}} & \frac{\partial \delta \boldsymbol{b_{g_{k+1}}}}{\partial \boldsymbol{n_\omega}} & \frac{\partial \delta \boldsymbol{b_{g_{k+1}}}}{\partial \boldsymbol{n_{b_a}}} & \frac{\partial \delta \boldsymbol{b_{g_{k+1}}}}{\partial \boldsymbol{n_{b_g}}}
\end{bmatrix}
= \begin{bmatrix}
-\frac{1}{2}\boldsymbol{R}\delta t^2 & 0 & 0 & 0 \\
0 & \boldsymbol{I} & 0 & 0 \\
\boldsymbol{R} & 0 & 0 & 0 \\
0 & 0 & \boldsymbol{I} & 0 \\
0 & 0 & 0 & \boldsymbol{I}
\end{bmatrix} \tag{64}
$$

Now, we can write the full output state equations

$$
\begin{bmatrix} \delta \boldsymbol{p_{k+1}} \\ \delta \boldsymbol{q_{k+1}} \\ \delta \boldsymbol{v_{k+1}} \\ \delta \boldsymbol{b_{a_{k+1}}} \\ \delta \boldsymbol{b_{g_{k+1}}} \end{bmatrix}
= \begin{bmatrix}
\boldsymbol{I} & -\frac{1}{2}\boldsymbol{R}[\boldsymbol{\hat{a}_t} - \boldsymbol{b_a}]_\times \delta t & \delta t & -\frac{1}{2}\boldsymbol{R}\delta t^2 & 0 \\
0 & \boldsymbol{I} - [\boldsymbol{\hat{\omega}_t} - \boldsymbol{b_\omega}]_\times \delta t & 0 & 0 & -\delta t \\
0 & -\boldsymbol{R}[\boldsymbol{\hat{a}_t} - \boldsymbol{b_a}]_\times \delta t & \boldsymbol{I} & -\boldsymbol{R}\delta t & 0 \\
0 & 0 & 0 & \boldsymbol{I} & 0 \\
0 & 0 & 0 & 0 & \boldsymbol{I}
\end{bmatrix}
\begin{bmatrix} \delta \boldsymbol{p_k} \\ \delta \boldsymbol{q_k} \\ \delta \boldsymbol{v_k} \\ \delta \boldsymbol{b_{a_k}} \\ \delta \boldsymbol{b_{g_k}} \end{bmatrix}
+ \begin{bmatrix}
-\frac{1}{2}\boldsymbol{R}\delta t^2 & 0 & 0 & 0 \\
0 & \boldsymbol{I} & 0 & 0 \\
\boldsymbol{R} & 0 & 0 & 0 \\
0 & 0 & \boldsymbol{I} & 0 \\
0 & 0 & 0 & \boldsymbol{I}
\end{bmatrix}
\begin{bmatrix} \boldsymbol{n_a} \\ \boldsymbol{n_\omega} \\ \boldsymbol{n_{b_a}} \\ \boldsymbol{n_{b_g}} \end{bmatrix} \tag{65}
$$

Finally, we have our jacobian and covariance matrix propagation equations.

$$
\mathbf{J_{t+\delta t}} = \mathbf{F}\mathbf{J_t} \tag{66}
$$

$$
\mathbf{P^{b_k}_{t+\delta t}} = \mathbf{F}\mathbf{P^{b_k}_t}\mathbf{F}^T + \mathbf{G}\mathbf{Q}\mathbf{G}^T \tag{67}
$$

References

[1] Tong Qin, Peiliang Li, Shaojie Shen. *VINS-Mono: A Robust and Versatile Monocular Visual-Inertial State Estimator*. IEEE, 2018.

[2] Joan Sola. *Quaternion kinematics for the error-state Kalman filter*. 2017.