Untitled

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Ranking Probabilities

For an n-item lineup consisting of one target and $n-1$ foils, let t be the probability that the eyewitness will detect the target (as the target) and let f be the probability that they will detect a foil (as a foil). Let $[n] = \{1, \dots, n\}$.

Suppose for a given eyewitness that $k \in [n-1]$ foils are not detected. We assume that the target is never detected as a foil. Then the *effective lineup size* is $x = k + 1$. Let $s(x)$ be the probability that the effective lineup size is x. Then

$$s(x) = \binom{n-1}{x-1}(1-f)^{x-1} f^{n-x}.$$

Rank probability

Suppose that the target is not detected as a target and the effective lineup size is x items. Then the target has an equal probability of being assigned a rank from 1 to x. Let $p(r)$ be the probability that the target is assigned rank $r \in [n]$, given it is not detected. Then

$$p(r) = \sum_{x=r}^{n} \frac{s(x)}{x}. \tag{1}$$

The function $p(r)$:

```r
b <- function(r, n, f){
  for (i in r:n)
    x[i] <- choose(n-1, i-1)*(1-f)^(i-1)*f^(n-i)/i
  return(sum(x, na.rm = TRUE))
}

p <- function(n, f){
  for (i in 1:n)
    x[i] <- b(i, n, f)
  return(x)
}
```

We assume that if the target is detected it is assigned rank 1. Let $q(r)$ be the probability that a target is assigned rank $r \in [n]$. Then

$$q(r) : \begin{cases} t + (1-t)p(r), & r = 1 \\ (1-t)p(r), & r > 1 \end{cases} \tag{2}$$

Conditional Ranking Probabilities

Let $Q(r)$ be the cumulative sum,

$$Q(r) = \sum_{i=1}^{r} q(i). \tag{3}$$

Let $c(r)$ be the conditional probability that the target is assigned rank r given that it has not been assigned any rank less than r. Then

$$c(r) : \begin{cases} q(r),\ r = 1 \\ \dfrac{q(r)}{(1 - Q(r-1))},\ r > 1 \end{cases} \tag{4}$$

Theorem

We want to show that $c(2) < c(3)$. Suppose $c(2) < c(3)$. I thought I would try a brute force approach (although unsuccessfully). From Equations (3) and (4),

$$\frac{q(2)}{1 - q(1)} < \frac{q(3)}{1 - q(1) - q(2)}$$

From Equation (4),

$$\frac{(1-t)p(2)}{1 - t - (1-t)p(1)} < \frac{(1-t)q(3)}{1 - t - (1-t)p(1) - (1-t)p(2)}$$

For $t < 1$,

$$\frac{p(2)}{1 - p(1)} < \frac{p(3)}{1 - p(1) - p(2)}$$

If the theorem is true, then $c(2)/c(3)$ is < 1 for all values of n and f.

Demonstration of theorem

The function $c(2)/c(3)$:

```
c2c3ratio <- function(n, f){
  x <- p(n,f)
  x[n+1] <- (x[2]/(1-x[1]))/(x[3]/(1-x[1]-x[2]))
  return(x[n+1])
}
# Create a vector of values of f
s <- seq(0,.99999, by=.001)

# The function calculating c(2)/c(3) for each value in vector s
Ratio <- function(n){
  for (i in 1:length(s))
    x[i] <- c2c3ratio(n,s[i])
  return(x)
}
```

Draw the function $c(2)/c(3)$ for $n \in \{4, 5, 6, 7, 8, 10, 20, 50, 100, 250\}$:

```
n4 <- Ratio(4)
n5 <- Ratio(5)
n6 <- Ratio(6)
n7 <- Ratio(7)
```

```r
n8 <- Ratio(8)
n10 <- Ratio(10)
n20 <- Ratio(20)
n50 <- Ratio(50)
n100 <- Ratio(100)
n250 <- Ratio(250)
n4 <- as.data.frame(cbind(n4,s))
n5 <- as.data.frame(cbind(n5,s))
n6 <- as.data.frame(cbind(n6,s))
n7 <- as.data.frame(cbind(n7,s))
n8 <- as.data.frame(cbind(n8,s))
n10 <- as.data.frame(cbind(n10,s))
n20 <- as.data.frame(cbind(n20,s))
n50 <- as.data.frame(cbind(n50,s))
n100 <- as.data.frame(cbind(n100,s))
n250 <- as.data.frame(cbind(n250,s))

ggplot()+
  geom_line(data = n4, aes(y=n4, x=s))+
  geom_line(data = n5, aes(y=n5, x=s))+
  geom_line(data = n6, aes(y=n6, x=s))+
  geom_line(data = n7, aes(y=n7, x=s))+
  geom_line(data = n8, aes(y=n8, x=s))+
  geom_line(data = n10, aes(y=n10, x=s))+
  geom_line(data = n20, aes(y=n20, x=s))+
  geom_line(data = n50, aes(y=n50, x=s))+
  geom_line(data = n100, aes(y=n100, x=s))+
  geom_line(data = n250, aes(y=n250, x=s))+
  labs(subtitle="Ratio of conditional second to conditional third choices for all values of f",
  theme_bw()
```

Ratio of conditional second to conditional third choices for all values of f