(16)

$$p(x) = \frac{\theta - 1}{x^{\theta}} \{x \geq 1\}, \quad \theta > 1$$

a) ОМП

$$\ln L(\theta) = \ln \left(\frac{\theta - 1}{x_i^{\theta}}\right)^n = n \ln(\theta - 1) - \theta \sum_{i=1}^{n} \ln x_i$$

$$\frac{\partial \ln L}{\partial \theta} = \frac{n}{\theta - 1} - \sum_{i=1}^{n} \ln x_i = 0$$

$$\hat{\theta} = \frac{n}{\sum_{i=1}^{n} \ln x_i} + 1$$

$$\frac{\partial^2 \ln L}{\partial \theta^2} = \frac{-n}{(\theta - 1)^2} < 0 \Rightarrow \max$$

в) $\int\limits_{-\infty}^{\tilde{x}} p(x)\,dx = \frac{1}{2}$

$$\int\limits_{1}^{\tilde{x}} \frac{\theta-1}{x^{\theta}}\,dx = -\frac{1}{x^{\theta-1}}\Bigg|_{1}^{\tilde{x}} = -\tilde{x}^{1-\theta}+1=\frac{1}{2}$$

$\tilde{x} = 2^{\frac{1}{\theta-1}}$ — медиана

$$I(\theta)=\int\limits_{1}^{+\infty}\left(\frac{1}{\theta-1}-\ln x\right)^{2}\frac{\theta-1}{x^{\theta}}\,dx = \int\limits_{1}^{\infty}\frac{1}{(\theta-1)x^{\theta}}\,dx -$$

$$-\int\limits_{1}^{\infty}\frac{2\ln x}{x^{\theta}}\,dx + \int\limits_{1}^{\infty}\frac{(\theta-1)\ln^{2} x}{x^{\theta}}\,dx =$$

$$=\frac{1}{(\theta-1)^{2}}-\frac{2}{(\theta-1)^{2}}+\frac{2}{(\theta-1)^{2}}=\frac{1}{(\theta-1)^{2}}$$

$$\frac{1}{\theta-1}\int\limits_{1}^{\infty}x^{-\theta}\,dx = \frac{1}{(\theta-1)^{2}}$$

$$(\theta-1)\int\limits_{1}^{\infty}\frac{\ln^{2} x}{x^{\theta}}\,dx = -\int\limits_{1}^{\infty}\ln^{2} x\,dx^{-\theta+1} =$$

$$= \cancelto{0}{-\ln^{2} x\, x^{-\theta+1}}\Bigg|_{1}^{\infty} + 2\int\limits_{1}^{\infty}\frac{\ln x}{x^{\theta}}\,dx = \frac{2}{(\theta-1)^{2}}$$

$g(\theta) = 2^{\frac{1}{\theta-1}}$

$$\sqrt{n}\,\frac{g(\hat{\theta})-g(\theta)}{\sqrt{\nabla^T g(\hat{\theta})\,I(\hat{\theta})\,\nabla g(\hat{\theta})}} \rightsquigarrow N(0,1)$$

$$\nabla g = 2^{\frac{1}{\theta-1}}\cdot \ln(2)\cdot\left(\frac{-1}{(\theta-1)^2}\right)$$

$$\sqrt{n}\,\frac{2^{\frac{1}{\hat{\theta}-1}}-2^{\frac{1}{\theta-1}}}{\frac{2^{\frac{1}{\hat{\theta}-1}}\ln 2}{\hat{\theta}-1}} \rightsquigarrow N(0,1)$$

$$-1,96 < \frac{\sqrt{n}(\hat{\theta}-1)\left(2^{\frac{1}{\hat{\theta}-1}}-2^{\frac{1}{\theta-1}}\right)}{2^{\frac{1}{\hat{\theta}-1}}\ln 2} < 1,96$$

$$\frac{-1,96\cdot \hat{med}\cdot\ln 2}{(\hat{\theta}-1)\sqrt{n}}+\hat{med} < med < \frac{+1,96\cdot\ln 2\,\hat{med}}{(\hat{\theta}-1)\sqrt{n}}+$$

$$+\hat{med}$$

c) $g(\theta)=\theta$

$$\sqrt{n}\,\frac{\hat{\theta}-\theta}{\hat{\theta}-1} \rightsquigarrow N(0,1)$$

$$\hat{\theta}-\frac{1,96(\hat{\theta}-1)}{\sqrt{n}} < \theta < \hat{\theta}+\frac{1,96(\hat{\theta}-1)}{\sqrt{n}}$$