T3

$$p(x) = \begin{cases} e^{-\frac{x}{\theta}} \cdot \frac{1}{\theta}, & x \geq 0 \\ 0, & x < 0 \end{cases}, \quad \theta > 0 \qquad n = 3$$

$$\tilde{\theta}_1 = \overline{x}, \quad \tilde{\theta}_2 = x_{(2)}$$

$$M[\xi] = \int_{-\infty}^{\infty} e^{-\frac{x}{\theta}} \cdot \frac{x}{\theta} dx = \theta \int_0^{\infty} e^{-t} t \, dt = \theta$$

a)

$\tilde{\theta}_1$:

$$M\left[\frac{1}{n} \sum_{i=1}^{n} x_i\right] = \frac{1}{n} \cdot n \cdot M[\xi] = \theta \Rightarrow \tilde{\theta}_1 \text{ - несм.}$$

$\tilde{\theta}_2$:

$$æ(t) = n \cdot p(t) \cdot C_{n-1}^{k-1} \cdot (1 - F(t))^{n-k} \cdot (F(t))^{k-1} =$$

$$= 3 \cdot p(t) \cdot C_2^1 \cdot (1 - F(t))^1 \cdot (F(t))^1$$

$$F(t) = \int_0^t p(x) dx = 1 - e^{-\frac{t}{\theta}}$$

$$æ(t) = 6 \cdot \left(\frac{e^{-\frac{2x}{\theta}}}{\theta} - \frac{e^{-\frac{3x}{\theta}}}{\theta} \right)$$

$$M[\tilde{\theta}_2] = 6 \int_0^{\infty} x \left(\frac{e^{-\frac{2x}{\theta}}}{\theta} - \frac{e^{-\frac{3x}{\theta}}}{\theta} \right) dx$$

$$I_1 = \int_0^{\infty} x \frac{e^{-\frac{2x}{\theta}}}{\theta} dx = -\frac{1}{2} \int_0^{\infty} x \, d e^{-\frac{2x}{\theta}} = -\frac{1}{2} \left(\cancelto{0}{x e^{-\frac{2x}{\theta}} \Big|_0^{\infty}} - \right.$$

$$\left. - \int_0^{\infty} e^{-\frac{2x}{\theta}} dx \right) = -\frac{1}{2} \cdot \frac{\theta}{2} \cdot e^{-\frac{2x}{\theta}} \Big|_0^{\infty} = \frac{\theta}{4}$$

$$I_2 = \int_0^\infty x \cdot \frac{e^{-\frac{3x}{\theta}}}{\theta} dx = -\frac{1}{3}\left(x \cdot e^{-\frac{3x}{\theta}}\Big|_0^\infty - \right.$$

$$\left. - \int_0^\infty e^{-\frac{3x}{\theta}} dx\right) = -\frac{1}{9}\theta e^{-\frac{3x}{\theta}}\Big|_0^\infty = \frac{1}{9}\theta$$

$M[\hat{\theta}_2] = 6(I_1 - I_2) = \frac{5}{6}\theta \Rightarrow \hat{\theta}_2$ – смещ.

$\hat{\theta}_2' = \frac{6}{5} x_{(2)}$ – несмещ.

в) $D[\hat{\theta}_1] = D\left[\frac{1}{n}\sum_{i=1}^{\infty} x_i\right] = \frac{1}{h^2}\sum_{i=1}^{n} D[\xi] \textcircled{=}$

$$\left\{\begin{aligned} & D[\xi] = M[\xi^2] - M^2\xi = 2\theta^2 - \theta^2 = \theta^2 \\ & M[\xi^2] = \int_0^\infty \frac{e^{-\frac{x}{\theta}}}{\theta} \cdot x^2 dx = \theta^2 \int_0^\infty e^{-t} t^2 dt = \\ & = -\theta^2 t^2 e^{-t}\Big|_0^\infty + \theta^2 \int_0^\infty 2te^{-t} dt = 2\theta^2 \end{aligned}\right\}$$

$\textcircled{=} \frac{\theta^2}{n} = \frac{\theta^2}{3}$

$$M[\hat{\theta}_2'^2] = \frac{36}{25} \cdot 6 \cdot \int_0^\infty x^2 \left(\frac{e^{-\frac{2x}{\theta}} - e^{-\frac{3x}{\theta}}}{\theta}\right) dx =$$

А

$$I_1 = \int_0^\infty x^2 \frac{e^{-\frac{2x}{\theta}}}{\theta}\,dx = -\frac{1}{2}\left(\cancel{x^2 e^{-\frac{2x}{\theta}}}\Big|_0^\infty - 2\int_0^\infty x e^{-\frac{2x}{\theta}}\,dx\right) =$$

$$= -\frac{\theta}{2}\int_0^\infty x\,d e^{-\frac{2x}{\theta}} = -\frac{\theta}{2}\left(\cancel{x e^{-\frac{2x}{\theta}}}\Big|_0^\infty - \int_0^\infty e^{-\frac{2x}{\theta}}\,dx\right) =$$

$$= -\frac{\theta}{2}\cdot\frac{\theta}{2}\cdot e^{-\frac{2x}{\theta}}\Big|_0^\infty = \frac{\theta^2}{4}$$

$$I_2 = \int_0^\infty x^2 \frac{e^{-\frac{3x}{\theta}}}{\theta}\,dx = -\frac{1}{3}\left(\cancel{x^2 e^{-\frac{3x}{\theta}}}\Big|_0^\infty - 2\int_0^\infty x e^{-\frac{3x}{\theta}}\,dx\right) =$$

$$= -\frac{2\theta}{9}\int_0^\infty x\,d e^{-\frac{3x}{\theta}} = -\frac{2\theta}{9}\left(\cancel{x e^{-\frac{3x}{\theta}}}\Big|_0^\infty - \int_0^\infty e^{-\frac{3x}{\theta}}\,dx\right) =$$

$$= -\frac{2\theta}{9}\cdot\frac{\theta}{3}\cdot e^{-\frac{3x}{\theta}}\Big|_0^\infty = \frac{2\theta^2}{27}$$

$$M[\hat{\theta}_2'^{\,2}] = \frac{36}{25}\cdot 6\left(\frac{\theta^2}{4} - \frac{2\theta^2}{27}\right) = \frac{36\cdot 6\cdot 19\theta^2}{25\cdot 108} =$$

$$= \frac{38}{25}\theta^2$$

$$D[\hat{\theta}_2'] = \frac{38}{25}\theta^2 - \theta^2 = \frac{13}{25}\theta^2$$

$$\frac{\theta^2}{3} < \frac{13\theta^2}{25} \quad \forall \theta > 0 \Rightarrow \hat{\theta}_1 \text{ эффективнее } \hat{\theta}_2'$$

c)

1) регулярн. модели:

$p(x,\theta)$ - непр. дифф., $\theta > 0$

$$\frac{\partial}{\partial\theta}\int_A p(x,\theta)\,dx = 0$$

$$\int_0^\infty \frac{\partial}{\partial\theta} p(x,\theta)\,dx = \int_0^\infty \frac{\partial}{\partial\theta}\left(\frac{e^{-\frac{x}{\theta}}}{\theta}\right)dx =$$

$$= \int_0^\infty \frac{\frac{x}{\theta^2}\cdot e^{-\frac{x}{\theta}}\cdot\cancel{\theta} - e^{-\frac{x}{\theta}}}{\theta^2}\,dx = \int_0^\infty x\,\frac{e^{-\frac{x}{\theta}}}{\theta^3} -$$

$$-\int_0^\infty \frac{e^{-\frac{x}{\theta}}}{\theta^2}\,dx = 0$$

$$\int_0^\infty x\,\frac{e^{-\frac{x}{\theta}}}{\theta^3} = -\frac{\theta}{\theta^3}\int_0^\infty x\,d e^{-\frac{x}{\theta}} = \frac{1}{\theta^2}\int_0^\infty e^{-\frac{x}{\theta}}\,dx$$

$$I(\theta) = M\left[\left(\frac{\partial \ln p(x,\theta)}{\partial\theta}\right)^2\right] =$$

$$= M\left[\left(\frac{x^2}{\theta^4} - 2\frac{x}{\theta^3} + \frac{1}{\theta^2}\right)\right] = \int_0^\infty\left(\frac{x^2}{\theta^4} -\right.$$

$$\left. - 2\frac{x}{\theta^3} + \frac{1}{\theta^2}\right)\cdot\frac{e^{-\frac{x}{\theta}}}{\theta}\,dx = \frac{1}{\theta^2} > 0 \quad \forall\theta$$

$\Rightarrow$ модель регулярна

$$\int_0^\infty \frac{x^2}{\theta^5} \cdot e^{-\frac{x}{\theta}} dx = -\int_0^\infty \frac{x^2}{\theta^4} d e^{-\frac{x}{\theta}} = -\frac{x^2}{\theta^4} \cdot e^{-\frac{x}{\theta}} \Big|_0^\infty +$$

$$+ 2\int_0^\infty \frac{x}{\theta^4} e^{-\frac{x}{\theta}} dx = -2\int_0^\infty \frac{x}{\theta^3} d e^{-\frac{x}{\theta}} = 2\int_0^\infty \frac{e^{-\frac{x}{\theta}}}{\theta^3} dx =$$

$$= -2\frac{e^{-\frac{x}{\theta}}}{\theta^2}\Big|_0^\infty = \frac{2}{\theta^2}$$

$$\int_0^\infty \frac{x}{\theta^4} e^{-\frac{x}{\theta}} dx = \frac{1}{\theta^2}$$

$$\int_0^\infty \frac{e^{-\frac{x}{\theta}}}{\theta^3} dx = \frac{1}{\theta^2} \qquad I(\theta) = \left(\frac{2}{\theta^2} - \frac{2}{\theta^2} + \frac{1}{\theta^2}\right) = \frac{1}{\theta^2}$$

2) регулярн. оценок (

модель рег.

$\tilde{\theta}_1$ – несм. оценка θ

$D[\tilde{\theta}_1] = \frac{\theta^2}{3}$ – огр. на $\forall$ комп.

$\Rightarrow \tilde{\theta}_1$ – рег.

аналог. $\tilde{\theta}_2'$ – рег.

3) нер-во К-Р:

$\tilde{\theta}_1$:

$$\frac{\theta^2}{3} \geq \frac{\theta^2}{3} \Rightarrow \tilde{\theta}_1 \text{ – эффект.}$$

$\tilde{\theta}_1$ эффективнее $\tilde{\theta}_2' \Rightarrow \tilde{\theta}_2'$ – не эффект.