

1 Frame 56 – Sequences and Series

3 If

$$\lim_{n \rightarrow \infty} z_n = z$$

then, for some integer n_0 , all of the terms z_k ($k > n_0$) will be in some ϵ neighbourhood of z ; ie:

$$|z_k - z| < \epsilon$$

However,

$$||z_k| - |z|| \leq \epsilon$$

so all of the terms $|z_k|$ must be inside the same ϵ neighbourhood of $|z|$, and we can say that

$$\lim_{n \rightarrow \infty} |z_n| = |z|$$

4 Starting from the series

$$\sum_{n=1}^{\infty} z^n = \frac{1}{1-z} - 1 = \frac{z}{1-z}$$

the components of this expression can be written as

$$\begin{aligned} \frac{z}{1-z} &= \frac{r \cos \theta + ir \sin \theta}{1 - r \cos \theta - ir \sin \theta} \\ &= \frac{r \cos \theta - r^2 \cos^2 \theta - r^2 \sin^2 \theta}{(1 - r \cos \theta)^2 + r^2 \sin^2 \theta} + i \frac{r \sin \theta - r^2 \sin \theta \cos \theta + r^2 \sin \theta \cos \theta}{(1 - r \cos \theta)^2 + r^2 \sin^2 \theta} \\ &= \frac{r \cos \theta - r^2}{1 - 2r \cos \theta + r^2} + i \frac{r \sin \theta}{1 - 2r \cos \theta + r^2} \end{aligned}$$

so, equating the real and imaginary parts of the sum,

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r \cos \theta - r^2}{1 - 2r \cos \theta + r^2}$$

and

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r \sin \theta}{1 - 2r \cos \theta + r^2}$$

2 Frame 59 – Taylor Series

1 The Maclaurin series for $z \cosh(z^2)$ is

$$z \cosh(z^2) = z \cdot \sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$$

3 The Maclaurin series for this function is

$$\frac{z}{9} \frac{1}{1 + (z^4/9)} = \frac{z}{9} \sum_{n=0}^{\infty} (-1)^n (z^4/9)^n = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{9^{n+1}}$$

4 Starting with the Maclaurin series for $\sin z$, this function's expansion is

$$\sin(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}$$

Thus, a_k is only non-zero for $k = 2, 6, 10, 14, \dots$

10 The function $\tanh z$ has singularities wherever $\cosh z = 0$, which occurs at $z = (k + 1/2)\pi i$. Thus, the closest singularity has a radius of $\pi/2$, and this is the radius of convergence.

We can find some of the terms of the Taylor series. The constant is

$$\tanh(0) = 0$$

The first derivative is

$$\left. \frac{d}{dz} \tanh(z) \right|_{z=0} = \operatorname{sech}^2(0) = 1$$

The second derivative is

$$\left. \frac{d}{dz} \operatorname{sech}^2(z) \right|_{z=0} = 2 \operatorname{sech}(z) \cdot (-\operatorname{sech} z \tanh z) \Big|_{z=0} = -2 \operatorname{sech}^2 z \tanh z \Big|_{z=0} = 0$$

The third derivative is

$$\left. \frac{d}{dz} -2 \operatorname{sech}^2 z \tanh z \right|_{z=0} = -2 [-2 \operatorname{sech}^2 z \tanh^2 z + \operatorname{sech}^4 z] \Big|_{z=0} = -2$$

so the first few terms of the Taylor series are

$$\tanh z \approx z - \frac{z^3}{3}$$

11(a) The series for this function is

$$\frac{e^z}{z^2} = \frac{1}{z^2} \sum_{n=0}^{\infty} \frac{z^n}{n!} = \sum_{n=0}^{\infty} \frac{z^{n-2}}{n!} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+2)!}$$

11(b) The series for this function is

$$\begin{aligned}\frac{\sin(z^2)}{z^4} &= \frac{1}{z^4} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z^2)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n-2} \\ &= \frac{1}{z^2} + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+3)!} z^{4n+2}\end{aligned}$$