1 Frame 55 – Sequences and Convergence

1.1 Definitions

An **infinite sequence** of complex numbers,

$$z_1, z_2, \ldots, z_n, \ldots$$

has a **limit** if, for each positive number ϵ, there exists a positive integer n_0 such that

$$|z_n - z| < \epsilon \quad \text{whenever} \quad n > n_0$$

Geometrically, this limit implies that for all $n > n_0$, each number z_n in the sequence will be inside an ϵ neighbourhood of z.

A sequence can only have one limit, at most. When this limit exists, we say that the sequence **converges** to z, and we write

$$\lim_{n\to\infty} z_n = z$$

If a sequence has no limit, it **diverges**.

1.2 Components

Theorem: If we write $z_n = x_n + iy_n$ and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

This theorem allows us to write

$$\lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

as long as the limits on either side of this equation exist.

1.3 Examples

Example 1: we can evaluate the following limit easily:

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{n^3 + i} &= \lim_{n\to\infty} \frac{1}{n^3} + i \lim_{n\to\infty} 1 \\ &= 0 + i \cdot 1 \\ &= i \end{aligned}$$

Example 2: Polar coordinates require some extra care. Looking at the sequence

$$z_n = -2 + i\frac{(-1)^n}{n^2}$$

we can see that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(-2) + i\lim_{n\to\infty}\frac{(-1)^n}{n^2} = -2$$

However, we can find that the principal polar representation of these numbers is

$$r_n = \sqrt{4 + \frac{1}{n^2}}$$
$$\Theta_n = \operatorname{Arg} z_n = \tan^{-1}\left(\frac{(-1)^n}{-2n^2}\right)$$

Evaluating the first limit, we find that

$$\lim_{n\to\infty} r_n = \sqrt{4} = 2$$

which is fine. However, the second sequence does not converge. Looking at every second term, we see that

$$\lim_{n\to\infty}\Theta_{2n} = \pi$$

and

$$\lim_{n\to\infty}\Theta_{2n-1} = -\pi$$

so Θ_n *diverges.*

2 Frame 56 – Series Convergence

2.1 Definitions

An infinite **series** of complex numbers,

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \dots,$$

converges to the sum S if the sequence of partial sums,

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N,$$

converges to S. If this is the case, then we can write

$$\sum_{n=1}^{\infty} z_n = S$$

Note that, since a sequence can have at most one limit, a series can have at most one sum. If a series does not converge, it **diverges**.

2.2 Properties – Components

First, as with sequences, we can split a series into its real and imaginary components.

Theorem: If $z_n = x_n + iy_n$ and $S = X + iY$, then

$$\sum_{n=1}^{\infty} z_n = S$$

iff

$$\sum_{n=1}^{\infty} x_n = X \quad \textit{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

To prove this, we can write the partial sums S_N as

$$S_N = X_N + iY_N$$

where

$$X_N = \sum_{n=1}^{N} x_n \quad \text{and} \quad Y_N = \sum_{n=1}^{N} y_n s$$

Then, the series only converges to S if

$$\lim_{N\to\infty} X_N = X \quad \text{and} \quad \lim_{N\to\infty} Y_N = Y$$

due to the theorem on sequences in the previous chapter. Thus, the theorem is proved.

2.3 Properties – Boundedness

The following corollary is a consequence of the previous theorem:

Corollary 1: If a series of complex numbers converges, the summed terms z_n converge to zero.

This is due to the fact that

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

and, in order for these two terms to converge, x_n and y_n must converge to zero (from calculus). Thus,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0$$

This corollary implies that the terms within convergent series are bounded – that is, there exists a constant M such that $|z_n| < M$ for all n.

2.4 Properties – Absolute Convergence

A series is **absolutely convergent** if the related series

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$$

converges. This has a simple implication:

Corollary 2: If a series is absolutely convergent, it is convergent.

To show this, consider the real component of the series. It can be written as

$$\sum_{n=1}^{\infty} x_n \leq \left| \sum_{n=1}^{\infty} x_n \right| \leq \sum_{n=1}^{\infty} |x_n| \leq \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} = 0$$

so the real component must converge. The same is true of the imaginary component, so the corollary is proved.

2.5 Remainders

It is often helpful to define the sequence of **remainders** using the partial sums:

$$\rho_N = S - S_N$$

or $S = S_N + \rho_N$. Since we can write that

$$|S_N - S| = |\rho_N|$$

then a series is only convergent if the sequence of remainders tends to zero.

Example: using remainders, we can verify that

$$\sum_{n=0}^{\infty} z_n = \frac{1}{1-z} \quad \textit{whenver} \quad |z| < 1$$

To do this, we recall that

$$S_N(z) = 1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1-z}$$

so

$$\rho_N(z) = \frac{1}{1-z} - \frac{1 - z^{N+1}}{1-z} = \frac{z^N}{1-z}$$

The moduli of these remaiders are

$$|\rho_N(z)| = \frac{|z|^N}{|1-z|}$$

so $\rho_N(z)$ tends to zero when $|z| < 1$.

3 Frame 57 – Taylor Series

The following theorem is known as **Taylor's theorem**:

Theorem: If a function f is analytic throughout a disk $|z - z_0| < R_0$, then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

This series converges to $f(z)$ when z is in this disk.

Taylor's theorem allows us to write

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots$$

This is true for any function that is analytic at z_0: the requirement for analyticity states that f must be analytic in some neighbourhood of z_0, so the disk mentioned in the theorem exists. In particular, entire functions can use arbitrarily large disks, ie:

$$|z - z_0| < \infty$$

so the series is convergent for all z in the plane.

It can be shown that Taylor's series converges at every point inside the disk – no convergence tests are required. In fact, the smallest radius at which it does **not** converge is the nearest point where f is not analytic.

If $z_0 = 0$ in a Taylor series, it is known as a **Maclaurin series**. Then, it takes the form

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

4 Frame 59 – Examples of Taylor Series

In this section, we will use the formula

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

to find the Maclaurin expansions of some common functions.

4.1 Example 1

The function e^x is entire, so its Maclaurin expansion is valid for all z. Since

$$f^{(n)}(z) = e^z$$

each term is $a_n = 1/n!$, so we find that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that, if $z = x + i0$, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as expected.

We can use this result to find the Maclaurin series for the entire function z^2e^{3z}. By replacing z with $3z$ and multiplying through by z^2, we find

$$z^2e^{3z} = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n$$

4.2 Example 2

Using the expansion

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

we can find the Maclaurin series for $f(z) = \sin z$. To do this, we write

$$\begin{aligned} \sin z &= \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] \\ &= \frac{1}{2i}\sum_{n=0}^{\infty}[1-(-1)^n]\frac{i^n}{n!}z^n \end{aligned}$$

Then, $1-(-1)^n$ is zero for n even, so only taking odd terms gives

$$= \frac{1}{2i}\sum_{n=0}^{\infty} 2\frac{i^{2n+1}}{(2n+1)!}z^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}z^{2n+1}$$

This expansion can be used directly to find $\cos z$. Since

$$\cos z = \frac{d}{dz}\sin z$$

we can write

$$\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{d}{dz}z^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2n+1)z^{2n}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}z^{2n}$$

4.3 Example 3

Since

$$\sinh z = -i\sin(iz)$$

we can write

$$\sinh z = -i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(iz)^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(-1)^n z^{2n+1}$$
$$= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}z^{2n+1}$$

Similarly,

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!}z^{2n}$$

Also, note that $\cosh z = \cosh(z+2\pi i)$, so

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(z+2\pi i)^{2n}$$

4.4 Example 4

If $f(z) = \frac{1}{1-z}$, then

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$$

so

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

If we substitute $-z$ for this expression, we find that

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

and if we substitute $1-z$ instead, we find

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

Note that all three of these Taylor series have a radius of convergence of 1.

4.5 Example 5

Notice that the function

$$f(z)\frac{1}{z^3}\frac{1}{1+z^2}$$

does not have a Maclaurin series – it is not analytic at $z = 0$. However,

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots$$

so we can write

$$\begin{aligned} f(z) &= \frac{1}{z^3}(1 - z^2 + z^4 - z^6 + \dots) \\ &= z^{-3} - z^{-1} + z^1 - z^3 + \dots \end{aligned}$$

We refer to the first two terms as **negative powers** of z.

5 Frame 60 – Laurent Series

5.1 Definition

We saw in the previous section that we often able to find series representations of functions that are not analytic by using both positive and negative powers of z. These representations are known as **Laurent series**. The central theorem for these functions is Laurent's theorem:

Theorem: Suppose that a function f is analytic over an annular domain,

$$R_1 < |z - z_0| < R_2$$

Then, $f(z)$ has the series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz \\ b_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz \end{aligned}$$

and where C is any positively-oriented contour in this domain containing z_0.

5.2 Alternate Form

Note that we can write this series more simply by defining

$$c_n = \begin{cases} b_{-n}, & n < 0 \\ a_n, & n \geq 0 \end{cases}$$

and writing

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Looking at the expressions for a_n and b_n,

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

5.3 Observations

First, note that

$$b_n = \frac{1}{2\pi i} \int_C f(z)(z - z_0)^{n-1} dz$$

If $f(z)$ is analytic wherever $|z - z_0| < R_2$, then all of these coefficients will be zero, and the Laurent series reduces to a Taylor series.

The domain of definition could expand to several different possibilities:

- $|z - z_0| < R_2$, if f is analytic everywhere inside this disk (note that this is now a Taylor series)
- $0 < |z - z_0| < R_2$, if f is analytic everywhere except for z_0 inside the R_2 disk
- $R_1 < |z - z_0| < \infty$, if f is analytic everywhere outside the R_1 disk
- $0 < |z - z_0| < \infty$, if f is analytic everywhere in the plane except for z_0

6 Frame 62 – Examples of Laurent Series

This section will show some examples of Laurent series.

6.1 Example 1

First, using the expansion for e^z, the Laurent series for $e^{1/z}$ is

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$$

Note that the coefficient b_1 here is 1, which tells us that

$$\int_C e^{1/z} dz = 2\pi i$$

for any positively-oriented simple closed contour that contains the origin.

6.2 Example 2

The function

$$\frac{1}{(z-i)^2}$$

is already a Laurent series, where $c_n = 0$ for all n except $c_{-2} = 1$. This tells us that

$$\int_C \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0, & n \neq -2 \\ 2\pi i, & n = -2 \end{cases}$$

6.3 Example 3

Consider the function

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

in the domain $|z| < 1$. Since f is analytic in this domain, we can write the Maclaurin series

$$\begin{aligned} f(z) &= \frac{-1}{1-z} + \frac{1}{2} \frac{1}{1-z/2} \\ &= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\ &= \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \end{aligned}$$

6.4 Example 4

Consider the same function on the domain $1 < |z| < 2$. Here, we can write

$$|1/z| < 1 \quad \text{and} \quad |z/2| < 1$$

so we can write the function as

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1-1/z} + \frac{1}{2}\frac{1}{1-z/2} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty} \frac{1}{z^n} \end{aligned}$$

which is the Laurent series for f in this domain.

6.5 Example 5

Finally, consider this function on the domain $2 < |z| < \infty$. Now, we can write

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1-1/z} - \frac{1}{z}\frac{1}{1-2/z} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^{n+1}} - \sum_{n=0}^{\infty} \frac{2^n}{z^{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{1-2^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{1-2^{n-1}}{z^n} \end{aligned}$$

which is the Laurent series for f in this domain. Note that $a_n = 0$ for all n.

7 Frame 63 – Absolute & Uniform Convergence

This section will discuss several properties of power series.

7.1 Absolute Convergence

The first theorem will discuss when a power series is convergent.

Theorem: if a power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

converges at the point $z = z_1 \neq z_0$*, then it is absolutely convergent on the open disk* $|z-z_0| < |z_1 - z_0|$.

Proof: since the series converges, all of its terms must be bounded, so we can write

$$|a_n(z_1 - z_0)^n| \leq M$$

for some positive M. Then, for each z inside the open disk described above, we can write

$$|a_n(z-z_0)^n| = |a_n(z-z_1)^n| \left(\frac{|z-z_0|}{|z_1-z_0|}\right)^n \leq M\rho^n$$

where ρ must be less than one. Then, the series must be less than the convergent geometric series

$$\sum_{n=0}^{\infty} M\rho^n$$

so the power series is absolutely convergent on this open disk.

Note: we refer to the circle $|z-z_0| = |z_1 - z_0|$ *as the* ***circle of convergence*** *– it is the largest circle around* z_0 *such that the power series converges everywhere inside it.*

7.2 Uniform Convergence

Next, we will define uniform convergence. We said earlier that the remainder of a series is the infinite sum less the partial sum

$$\rho_N(z) = S(z) - S_N(z)$$

and that, in order to converge, these remainders must approach zero as N approaches infinity. We can write this as

$$|\rho_N(z)| < \epsilon \quad \text{whenever} \quad N > N_\epsilon$$

We say that a series is **uniformly convergent** in a region if our choice of N_ϵ depends only on ϵ and not on z.

Theorem: If z_1 is a point inside the circle of convergence of a power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

then that series must be uniformly convergent in the closed disk $|z - z_0| \leq |z_1 - z_0|$.