

1 Frame 12 – Functions of Complex Variables

1(a) The function

$$f(z) = \frac{1}{z^2 + 1}$$

is defined everywhere except where $z^2 + 1 = 0$; ie:

$$z \neq \pm i$$

1(b) The function

$$f(z) = \operatorname{Arg}\left(\frac{1}{z}\right)$$

is defined wherever $\frac{1}{z}$ is defined:

$$z \neq 0$$

1(c) The function

$$f(z) = \frac{z}{z + \bar{z}}$$

can be written as

$$f(x, y) = \frac{x + iy}{(x + iy) + (x - iy)} = \frac{x + iy}{2x} = \frac{1}{2} + i\frac{y}{x}$$

so the domain is

$$\operatorname{Re}(z) \neq 0$$

1(d) The function

$$f(z) = \frac{1}{1 - |z|^2}$$

is equivalent to

$$f(r, \theta) = \frac{1}{1 - r^2}$$

so the domain is

$$r \neq 1$$

2 Substituting $z = x + iy$ gives

$$\begin{aligned} f(x, y) &= (x + iy)^3 + (x + iy) + 1 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x + iy + 1 \\ &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) \end{aligned}$$

so

$$\begin{aligned} u(x, y) &= x^3 - 3xy^2 + x + 1 \\ v(x, y) &= 3x^2y - y^3 + y \end{aligned}$$

3 Using the two expressions

$$x = \frac{z + \bar{z}}{2}$$
$$y = \frac{z - \bar{z}}{2i}$$

gives

$$\begin{aligned} f(z) &= \left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 - 2\frac{z - \bar{z}}{2i} \\ &\quad + i\left[2\frac{z + \bar{z}}{2}\left(1 - \frac{z - \bar{z}}{2i}\right)\right] \\ &= \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2 + z^2 - 2z\bar{z} + \bar{z}^2) + iz - i\bar{z} \\ &\quad + i\left[z + \bar{z} + \frac{iz^2}{2} - \frac{i\bar{z}^2}{2}\right] \\ &= \frac{1}{2}(z^2 + \bar{z}^2) + 2iz - \frac{iz^2}{2} + \frac{\bar{z}^2}{2} \\ &= \bar{z}^2 + 2iz \end{aligned}$$

4 Using

$$z = re^{i\theta}$$

the function can be written as

$$\begin{aligned} f(z) &= re^{i\theta} + \frac{1}{re^{i\theta}} \\ &= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\ &= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta \end{aligned}$$

2 Frame 14 – Mappings by the Exponential Function

1 We saw earlier that the hyperbolas

$$x^2 - y^2 = c_1$$

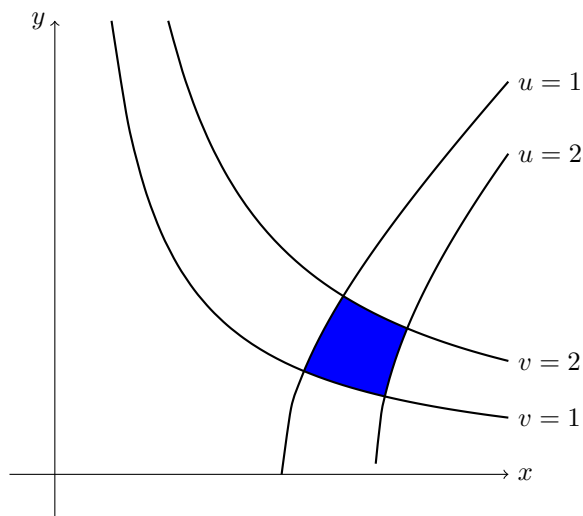
map onto the horizontal lines $u = c_1$ and the hyperbolas

$$2xy = c_2$$

map onto the vertical lines $v = c_2$. Thus, a domain on the z -plane that maps onto $1 \leq u \leq 2$ and $1 \leq v \leq 2$ is

$$1 \leq x^2 - y^2 \leq 2 \quad 1 \leq 2xy \leq 2$$

A sketch of this region is:



2 The first hyperbola can be written as

$$y^2 - x^2 = |c_1|$$

Then, substitution into the v equation gives

$$u = c_1, \quad v = \begin{cases} 2x\sqrt{x^2 + |c_1|}, & y > 0 \\ -2x\sqrt{x^2 + |c_1|}, & y < 0 \end{cases}$$

This maps out the entire v line as x moves right (on the top branch) or left (on the bottom branch).

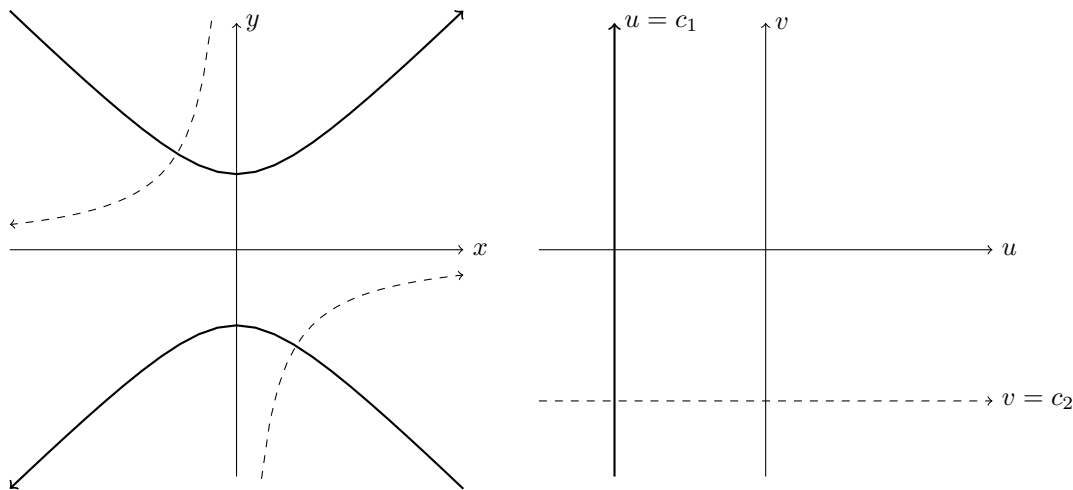
The second hyperbola can be written as

$$2xy = -|c_2|$$

and substituting this into the u equation gives

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

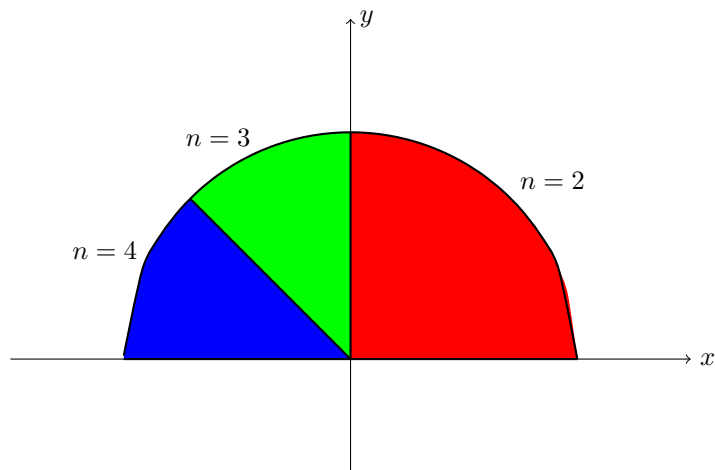
This maps out the entire u line: as x gets large in magnitude, so too does u . A sketch of these mappings is:



3 The image of the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ under the mapping $w = z^n$ is

$$\rho \leq 1, \quad 0 \leq \theta \leq n \frac{\pi}{4}$$

A sketch of these images for $n = 2, 3, 4$ is:



4 If z follows the straight line $ay = x$, then the mapping $w = e^z$ is

$$\begin{aligned} w &= e^{x+iy} \\ &= e^{ay} e^{iy} \\ &= e^{a\phi} e^{i\phi} \\ &= \rho e^{i\phi} \end{aligned}$$

where $\rho = a\phi$.

5 The rectangular region $a \leq x \leq b$, $c \leq y \leq d$ is made up of the horizontal line segments

$$x = t, \quad y = c_1$$

where t is a parameter running from a to b and c_1 is a constant in the range $[c, d]$. These horizontal lines have the images

$$\rho = e^t, \quad \phi = c_1$$

Since t starts at a and ends at b , these images have a radius in the range $[e^a, e^b]$. Then, the entire image is the set of these lines, which range from $\phi = c$ to $\phi = d$. Thus, the entire image is

$$e^a \leq \rho \leq e^b, \quad c \leq \phi \leq d$$

6 Looking at the z plane, the initial set is the infinite strip

$$x \leq 0, \quad 0 \leq y \leq \pi$$

This maps to the image set

$$\lim_{a \rightarrow -\infty} e^a \leq \rho \leq e^0, \quad 0 \leq \phi \leq \pi$$

or

$$0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \pi$$

This is the upper half of the unit disk, as shown in the figure.

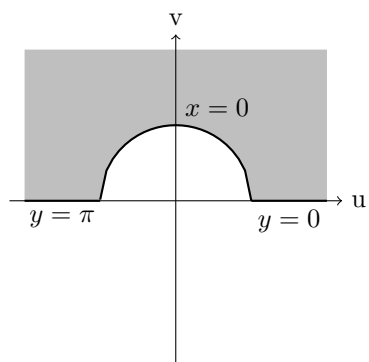
7 In a similar manner to the previous problem, the image of the strip

$$x \geq 0, \quad 0 \leq y \leq \pi$$

is the upper half-plane with a unit disk cut out:

$$\rho \geq 1, \quad 0 \leq \phi \leq \pi$$

A sketch of this region is:



8 Some sample vectors in these two fields are:

