1 Frame 29 – The Exponential Function

1.1 Definition

We define the **exponential function** e^z by writing

$$e^z = e^x e^{iy}$$

and we apply Euler's formula to get

$$e^z = e^x(\cos y + i \sin y)$$

Note that, when $y = 0$, e^z reduces to e^x.

Although we typically understand that $e^{1/n}$ would be the set of nth roots of e, here, we only use the real, positive root $\sqrt[n]{e}$.

1.2 Familar properties

First, in calculus, we know that

$$e^{x_1} e^{x_2} = e^{x_1+x_2}$$

It is easy to verify that this holds true for complex numbers:

$$e^{z_1} e^{z_2} = e^{z_1+z_2}$$

This also allows us to write

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

and, as a specific case,

$$\frac{1}{e^z} = e^{-z}$$

We showed earlier that e^z is differentiable everywhere in the complex plane, and that

$$\frac{d}{dz} e^z = e^z$$

We also know that e^z is never zero. This comes from the pair

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi$$

and since e^x is never zero, neither is e^z.

1.3 Unfamiliar properties

Since we can write

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

the exponential function is periodic with an imaginary period of $2\pi i$.

It is also possible for the complex exponential function to be negative. For an example, we know that Euler's identity states

$$e^{i\pi} = -1$$

In fact, e^z can be any given non-zero complex number.

Example: suppose we want solutions to the equation

$$e^z = 1 + i$$

The right side can be rewritten as

$$e^x e^{iy} = \sqrt{2} e^{i\pi/4}$$

and equating the parts of this equation gives

$$x = \ln\sqrt{2} = \frac{1}{2}\ln 2 \quad \textit{and} \quad y = \left(2n + \frac{1}{4}\right)\pi$$

so

$$z = \frac{1}{2}\ln 2 + \left(2n + \frac{1}{4}\right)\pi i$$

2 Frame 30 – The Logarithmic Function

2.1 Motivation

We said in the previous section that e^z can take on any non-zero complex value. To help us solve the equation

$$e^w = z$$

we will define a **logarithmic function**, such that

$$e^{\log z} = z \quad (z \neq 0)$$

We can solve for w by writing the two complex numbers in the form

$$z = re^{i\theta}$$
$$w = u + iv$$

Substituting these into the original equation gives

$$e^u e^{iv} = re^{i\theta}$$

so we get

$$w = \log z = \ln r + i(\theta + 2n\pi)$$

Note that this is a multi-valued function.

Example: if $z = -1 - i\sqrt{3}$, then $r = 2$ and $\theta = -2\pi/3$, so

$$\log(-1 - i\sqrt{3}) = \ln 2 + \left(n - \frac{1}{3}\right) 2\pi i$$

2.2 Precise definition

A more precise definition of the multi-valued logarithmic function is

$$\log z = \ln |z| + i \arg z$$

The **principal value** of $\log z$ is obtained by using the single-valued principal argument instead:

$$\operatorname{Log} z = \ln |z| + i\theta$$

Note that

$$\log z = \operatorname{Log} z + i2n\pi$$

2.3 Notes

The principal logarithmic function $\operatorname{Log} z$ reduces to the usual logarithm from calculus when z is positive and real – if $z = r$, then

$$\operatorname{Log} r = \ln r$$

However, we are now able to find the logarithm of negative real numbers, which we were unable to do in calculus.

Example: the logarithm of -1 *is*

$$\log(-1) = \ln 1 + (1 + 2n)i\pi = (2n + 1)i\pi$$

and

$$\operatorname{Log}(-1) = i\pi$$

3 Frame 31 – Branches & Derivatives of Logarithms

3.1 Limiting a logarithm's domain

We saw in the previous section that the multi-valued logarithm function of a complex number $z = re^{i\theta}$ can be written as

$$\log z = \ln r + i\theta$$

where θ can have any of the values

$$\theta = \text{Arg}(z) + 2n\pi$$

We can make the logarithmic function single-valued by restricting the value of θ to $\alpha < \theta < \alpha + 2\pi$ for any real value of θ. Then, the function is single-valued and is continuous everywhere in the domain of the function (ie: $r > 0$ and $\theta \in (\alpha, \alpha + 2\pi)$). Note that we cannot include in the ray $\theta = \alpha$ – the function would not be continuous here.

In this limited domain, the components of the log function also satisfy the polar Cauchy-Riemann equations

$$ru_r = 1 = v_\theta; \quad u_\theta = 0 = -rv_r$$

so the logarithmic function is analytic in this domain, with the derivative

$$\frac{d}{dz} \log z = e^{-i\theta}(u_r + iv_r) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

In particular, we can set $\alpha = -\pi$ and write

$$\frac{d}{dz} \text{Log}\, z = \frac{1}{z} \quad (|z| > 0, -\pi < \text{Arg}\, z < \pi)$$

Note that not all of the identities from calculus carry over to the complex plane.

Example: using the principal branch,

$$\text{Log}(i^3) = \text{Log}(-i) = -i\frac{\pi}{2}$$

but

$$3\,\text{Log}(i) = 3\left(i\frac{\pi}{2}\right) = i\frac{3\pi}{2}$$

so

$$\text{Log}(i^3) \neq 3\,\text{Log}(i)$$

3.2 Branches

A **branch** of a multi-valued function f is any single-valued, analytic function F such that $F(z)$ is one of the values of f at each point within the domain of F. For instance, our limited-domain logarithm is a branch of the multi-valued log function. The principal logarithm function

$$\operatorname{Log} z = \ln r + i\theta \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

is known as the **principal branch**.

A **branch cut** is a line/curve that is used to define a branch F of a multi-valued function f. Any point on the branch cut is a singular point of F. Any point that is common to all branch cuts of f is called a **branch point**. For example, the logarithmic function has a branch point at $z = 0$ and a branch cut on the ray $\theta = \alpha$. In particular, the ray $\theta = \pi$ is the branch cut for the principal logarithmic function.

4 Frame 32 – Logarithm Identities

We said earlier that arguments, which are multi-valued functions, can be compared in a special way – since each function is really a set of values, the sets will contain the same values. Specifically,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Now, we know that $|z_1 z_2| = |z_1||z_2|$, and from our knowledge of real-valued logarithms,

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2|$$

Putting together these two statements, we see that

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

which is to be understood as *set equality*, and does not necessarily apply to the principal values. In a similar manner,

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

Two more properties will be useful in the next section. If z is any non-zero complex number, then

$$z^n = e^{n \log z}$$

for all values of $\log z$. When $n = 1$, this reduces to the familiar

$$z = e^{\log z}$$

Also, for any non-zero z, it is true that

$$z^{1/n} = e^{\frac{1}{n} \log z}$$

where both sides have n distinct values. To show this, we can write out the right side as

$$e^{\frac{1}{n} \log z} = e^{\frac{1}{n} \ln r + \frac{i(\theta + 2k\pi}{n}} = \sqrt[n]{r} e^{i(\theta/n + 2k\pi/n)}$$

which has n distinct values, for $k = 0, 1, \ldots, n-1$.

5 Frame 33 – Complex Exponents

5.1 Definition and basics

For non-zero z and complex c, we define the function z^c as

$$z^c = e^{c \log z}$$

Note that this definition uses the multi-valued log function.

We saw earlier that the exponential function has the property

$$\frac{1}{e^z} = e^{-z}$$

Now, for the general power equation, we have

$$\frac{1}{z^c} = \frac{1}{e^{c \log z}} = e^{-c \log z} = z^{-c}$$

Example: the values of i^{-2i} can be found by first writing

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(2n + \frac{1}{2}\right)\pi$$

and so

$$i^{-2i} = e^{-2i \cdot i(2n+1/2)\pi} = e^{(4n+1)\pi}$$

Note that all of these powers are real numbers.

The **principal value** of z^c uses the single-valued log function:

$$\text{P.V. } z^c = e^{c \operatorname{Log} z}$$

Example: the principal value of $(-i)^i$ is

$$e^{i \operatorname{Log}(-i)} = e^{i(-i\pi/2)} = e^{\pi/2}$$

5.2 Other properties

To differentiate z^c, we can restrict the logarithmic function to a single branch

$$\log z = \ln + i\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

Then, z^c is analytic in this domain. The derivative can be found through the chain rule:

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c \log z} = \frac{c}{z} e^{c \log z} = cz^{c-1}$$

Most of the laws of exponents remain valid in the complex plane. However, since the functions are multi-valued, we can only guarantee equality between sets – when using principal values, not all of the rules of real exponents work. For example, the law

$$z_1^c z_2^c = (z_1 z_2)^c$$

does not necessarily hold for all z_1, z_2 when using principal values.

5.3 Exponential functions with other bases

We can write the **exponential function** with a non-zero base c as

$$c^z = e^{z \log c}$$

Note that this function is, again, multi-valued: if $c = e$, then we don't recover our usual definition of e^z. However, if we use the principal value of the logarithm, the usual interpretation occurs.

This exponential function is an entire function for any non-zero c. It has the derivative

$$\frac{d}{dz} c^z = \frac{d}{dz} e^{z \log c} = e^{z \log c} \log c = c^z \log c$$

6 Frame 34 – Trigonometric Functions

6.1 Trigonometric functions – definitions

We know from Euler's formula that

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

and we can rearrange this into

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

In a similar manner, we can define the complex **trigonometric functions**

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{iz}}{2}$$

6.2 Trigonometric functions – properties

First, the two trig functions are entire, since they are linear combinations of two entire functions. Thus, they are differentiable everywhere; their derivatives, from the complex exponential derivatives, are

$$\frac{d}{dz} \sin z = \cos z$$
$$\frac{d}{dz} \cos z = -\sin z$$

We can also see that the odd/even properties carry over:

$$\sin(-z) = -\sin z$$
$$\cos(-z) = \cos z$$

and Euler's formula also applies:

$$e^{iz} = \cos z + i \sin z$$

Many of the identities from trigonometry carry over. A sample of these identities is:

$$
\begin{aligned}
\sin(z_1 + z_2) &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\
\cos(z_1 + z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\
\sin 2z &= 2 \sin z \cos z \\
\cos 2z &= \cos^2 z - \sin^2 z \\
\sin\left(z + \frac{\pi}{2}\right) &= \cos z \\
\sin\left(z - \frac{\pi}{2}\right) &= -\cos z \\
\sin(z + \pi) &= -\sin z \\
\sin(z + 2\pi) &= \sin z \\
\cos(z + \pi) &= -\cos z \\
\cos(z + 2\pi) &= \cos z \\
\sin^2 z + \cos^2 z &= 1
\end{aligned}
$$

6.3 Using hyperbolic functions

The hyperbolic trig functions of a real number y are defined, from calculus, as

$$
\begin{aligned}
\sinh y &= \frac{e^y - e^{-y}}{2} \\
\cosh y &= \frac{e^y + e^{-y}}{2}
\end{aligned}
$$

We can use these definitions to write

$$
\begin{aligned}
\sin(iy) &= i \sinh y \\
\cos(iy) &= \cosh y
\end{aligned}
$$

Then, if $z = x + iy$ is a complex number, we can write

$$
\begin{aligned}
\sin z &= \sin x \cosh y + i \cos x \sinh y \\
\cos z &= \cos x \cosh y - i \sin x \sinh y
\end{aligned}
$$

These expressions allow us to write

$$
\begin{aligned}
|\sin z|^2 &= \sin^2 x + \sinh^2 y \\
|\cos z|^2 &= \cos^2 x + \sinh^2 y
\end{aligned}
$$

and, since $\sinh y$ is unbounded in y, the trigonometric functions are **unbounded** on the complex plane.

6.4 Extensions to other trigonometric functions

Since
$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$
we find that $\sin z$ only has zeroes at $x = n\pi$ and $y = 0$; ie:
$$\sin z = 0 \iff z = n\pi$$
Since $\cos z = -\sin(z - \pi/2$, we find that
$$\cos z = 0 \iff z = \left(\frac{1}{2} + n\right)\pi$$

With these zeroes in mind, we can define the four other trigonometric functions as expected:
$$\begin{aligned}\tan z &= \frac{\sin z}{\cos z}\\ \sec z &= \frac{1}{\cos z}\\ \cot z &= \frac{\cos z}{\sin z}\\ \csc z &= \frac{1}{\sin z}\end{aligned}$$
These functions are analytic everywhere except for the singularities caused by the denominators: tan and sec are analytic for all $z \neq (n + 1/2)\pi$, and cot and csc are analytic for all $z \neq n\pi$.

We can use our differentiation rules to find the expected differentiation formulas:
$$\begin{aligned}\frac{d}{dz}\tan z &= \sec^2 z\\ \frac{d}{dz}\sec z &= \sec z \tan z\\ \frac{d}{dz}\cot z &= -\csc^2 z\\ \frac{d}{dz}\csc z &= -\csc z \cot z\end{aligned}$$