

# 1 Frame 12 – Functions of Complex Variables

## 1.1 Functions

If  $S$  is a set of complex numbers, then a **function**  $f$  is a rule that assigns a complex number  $w$  to each  $z$  in  $S$ . The number  $w$  is called the **value** of  $f$  at  $z$ . We denote it as

$$w = f(z)$$

The set  $S$  is called the **domain of definition** of  $f$ . Note that we need both a rule ( $f$ ) and a domain ( $S$ ) for a function to be well defined.

Suppose that  $w = u + iv$  and  $z = x + iy$ . Then,

$$u + iv = f(x + iy)$$

Then, we can express  $f(z)$  as a pair of real functions of  $x$  and  $y$ :

$$f(z) = u(x, y) + iv(x, y)$$

Alternatively, we could use polar coordinates to write

$$u + iv = f(re^{i\theta})$$

so

$$f(z) = u(r, \theta) + iv(r, \theta)$$

*Example: the function  $f(z) = z^2$  can be written as*

$$\begin{aligned} f(x + iy) &= (x + iy)^2 \\ &= (x^2 - y^2) + i2xy \end{aligned}$$

so

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy \end{aligned}$$

*In polar coordinates,*

$$\begin{aligned} f(x + iy) &= (re^{i\theta})^2 \\ &= r^2 e^{i2\theta} \\ &= r^2 \cos 2\theta + ir^2 \sin 2\theta \end{aligned}$$

so

$$\begin{aligned} u(r, \theta) &= r^2 \cos 2\theta \\ v(r, \theta) &= r^2 \sin 2\theta \end{aligned}$$

## 1.2 Real-Valued Functions

We say that  $f$  is a **real-valued function** if  $v$  is zero everywhere.

*Example: one real-valued function is*

$$f(z) = |z|^2 = x^2 + y^2 + i0$$

## 1.3 Polynomials

If  $n$  is a non-negative integer and  $a_0, a_1, a_2, \dots, a_n$  are complex numbers with  $a_n \neq 0$ , then the function

$$P(z) = a_0 + a_1z + a_2z^2 + \dots + a_nz^n$$

is a **polynomial** of degree  $n$ . Note that this sum has a finite number of terms and that the domain of definition is the entire  $z$  plane.

As in real numbers, a **rational function** is a quotient of two polynomials:

$$R(z) = \frac{P(z)}{Q(z)}$$

A rational function is defined everywhere that  $Q(z) \neq 0$ .

## 1.4 Multi-Valued Functions

A generalization of a function is a rule that assigns more than one value to a point  $z$ . These **multiple-valued functions** are usually studied by taking one of the possible values at each point and constructing a single-valued function.

*Example: we know that we can write*

$$z^{1/2} = \pm \sqrt{r}e^{i\theta/2}$$

where we denoted  $-\pi < \theta \leq \pi$  as the **principal value** of  $\arg z$ . To turn this into a single valued function, we can choose the positive value of  $r$  and write

$$f(z) = \sqrt{r}e^{i\theta/2}$$

Then,  $f$  is well-defined on the entire plane.

## 2 Frame 13 – Mappings

### 2.1 Definitions

There is no convenient way to graph the function  $w = f(z)$  – each of these complex numbers are located on a plane instead of a line. Instead, we can draw pairs of corresponding points on separate  $z$  and  $w$  planes. When we think of  $f$  this way, we call it a **mapping** or **transformation**.

If  $f$  is defined on the domain of definition  $S$ , then the **image** of a point  $z \in S$  is the point  $w = f(z)$ . If  $T$  is a subset of  $S$ , then the set of the images of each point in  $T$  are called the image of  $T$ . In particular, the image of the entire domain,  $S$ , is called the **range** of  $f$ . The **inverse image** of a point  $w$  is the set of points  $z$  in  $S$  that map to  $w$  (possibly zero, one, or many points).

### 2.2 Basic transformations

Using this geometric interpretation, we can describe mappings using terms such as **translation**, **rotation**, and **reflection**. For instance, the mapping

$$w = z + 1 = (x + 1) + iy$$

can be thought of as a translation of each point  $z$  one unit to the right. Another example is the rotational mapping

$$w = iz$$

where, using  $i = e^{i\pi/2}$  and  $z = re^{i\theta}$ , is

$$w = re^{i(\theta+\pi/2)}$$

or, in other words, a  $90^\circ$  rotation. Finally, the mapping

$$w = \bar{z} = x - iy$$

is a reflection across the real axis. Usually, it is more useful to sketch an image of a curve rather than a single point.

### 2.3 Mapping a curve

*For an example, consider the mapping  $w = z^2$ . We showed earlier that this can be written as*

$$u = x^2 - y^2, \quad v = 2xy$$

*To sketch the image, we will first set  $u = c_1$ , which requires that*

$$x^2 - y^2 = c_1, \quad c_1 > 0$$

which is the equation for a hyperbola. This equation can then be used to solve for the image points:

$$u = c_1, \quad v = \pm 2y\sqrt{y^2 + c_1}$$

where the plus-minus is resolved depending on which side the image point is on. Simply put, as  $z$  travels up the right-side hyperbola or down the left-side hyperbola,  $w$  travels up the vertical line  $u = c_1$ .

Next, we can set  $v = c_2$ , which requires

$$2xy = c_2, \quad c_2 > 0$$

This gives us the image set

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

As  $x \rightarrow \pm\infty$ ,  $u \rightarrow \infty$ ; as  $x \rightarrow 0$ ,  $u \rightarrow -\infty$ . Thus, this hyperbola traces out the straight line  $v = c_2$  towards the right as  $z$  travels towards the left.

## 2.4 Mapping a region

We can use some of the details from the previous example to find the image of a region, rather than a single curve.

Consider the domain  $x > 0, y > 0, xy < 1$ . This region consists of the upper branches of the hyperbolas

$$2xy = c, \quad 0 < c < 2$$

and we know from the previous example that these hyperbolas map to the straight lines

$$v = c$$

Thus, this region maps to the horizontal strip  $0 < v < 2$ .

We can also close the domain to contain the curves  $x = 0$ ,  $y = 0$ , and  $xy = 1$ . From the function  $w = z^2$ , we know that the points  $(0, y)$  and  $(x, 0)$  map to the points  $(-y^2, 0)$  and  $(x^2, 0)$ , so including the two straight lines simply extends the strip to include  $v = 0$ . Similarly, the hyperbola  $xy = 1$  maps to the horizontal line  $v = 2$ .

Simply put, the image of the closed region  $x \geq 0, y \geq 0, xy \leq 1$  is the closed region  $0 \leq v \leq 2$ .

## 2.5 Mapping with polar coordinates

Finally, we can use polar coordinates to simplify some mappings.

Again, consider the mapping  $w = z^2$ . If we write  $z = re^{i\theta}$ , then the image point can be written as

$$w = r^2 e^{2i\theta}$$

Looking at the magnitude of  $w$ , points on a circle  $r = r_0$  are mapped onto a circle  $r' = r_0^2$ . Also, looking at the argument of  $w$ , the angle of the image is doubled. This means that the first quadrant, which is defined as

$$r \geq 0, \quad 0 \leq \theta \leq \pi/2$$

is in a one-to-one mapping with the top plane,  $0 \leq \theta \leq \pi$ . Similarly, the top plane is mapped onto the entire complex plane (although this is not one-to-one, since the inverse image of the positive real axis is both real axes).

Note that any mapping  $w = z^n$  for positive integer  $n$  has a similar form, where each non-zero point in the  $w$  plane is the image of  $n$  distinct points in the  $z$  plane.

### 3 Frame 14 – Mappings by the Exponential Function

Now, we will look at the exponential function

$$e^z = e^{x+iy} = e^x e^{iy}$$

We can again look at straight lines and find their images in this mapping.

*Consider the transformation*

$$w = e^z = \rho e^{i\phi}$$

*where*

$$p = e^x \quad \phi = y$$

*This means that the image of a vertical line  $x = c_1$  is a circle with radius  $p = e^{c_1}$ . Each point on the circle is the image of infinitely many points, each spaced  $2\pi$  units apart on the vertical line. Similarly, the horizontal line  $y = c_2$  is a ray with an angle of  $\phi = c_2$ .*

With these images in mind, we know that vertical and horizontal line segments are mapped onto arcs and rays, respectively. We can then use this information to map regions:

*Now, consider the rectangular region*

$$a \leq x \leq b \quad c \leq y \leq d$$

*The image of this region under the mapping  $w = e^z$  is*

$$e^a \leq \rho \leq e^b \quad c \leq \phi \leq d$$

*This is a one-to-one mapping if  $d - c < 2\pi$ . In particular, the region with  $c = 0, d = \pi$  is mapped onto half of a circular ring.*

## 4 Frame 15 – Limits

### 4.1 Definitions

Suppose that a function  $f$  is defined at all points  $z$  in some deleted neighborhood of  $z_0$ . The statement that the number  $w_0$  is the **limit** of  $f(z)$  as  $z$  approaches  $z_0$  means that the point  $w = f(z)$  can be made *arbitrarily close* to  $w_0$  if we choose  $z$  close enough to  $z_0$ . We write this as

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

To be more precise, if this limit exists, then for each positive number  $\epsilon$ , there is a positive number  $\delta$  such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Geometrically, this definition says that each  $\epsilon$  neighbourhood around  $w_0$  has a corresponding deleted  $\delta$  neighbourhood around  $z_0$  such that the image of each point in the  $\delta$  neighbourhood maps to a point in the  $\epsilon$  neighbourhood.

Note that the deleted neighbourhood will always exist if  $z_0$  is internal to the domain of definition of  $f$ . We can extend the definition of a limit to include boundary points by ignoring all of the neighbourhood's points that are outside the domain.

Also note that this definition only allows a given point to be tested as a limit – it does not provide a method for finding the limit. This will be covered in the next section.

### 4.2 Uniqueness

If the limit of a function  $f(z)$  exists at  $z_0$ , it must be unique. To show this, consider two limits:

$$\lim_{z \rightarrow z_0} f(z) = w_0 \text{ and } \lim_{z \rightarrow z_0} f(z) = w_1$$

This implies that we can find  $\delta_0$  and  $\delta_1$  such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_1$$

Now, suppose that  $\delta$  is a positive number smaller than both  $\delta_0$  and  $\delta_1$ . Then, for all  $0 < |z - z_0| < \delta$ , we find that the difference between the two limits is

$$\begin{aligned} |w_1 - w_0| &= ||f(z) - w_0| - |f(z) - w_1|| \\ &\leq |f(z) - w_0| + |f(z) - w_1| \\ &< \epsilon + \epsilon \\ &= 2\epsilon \end{aligned}$$

and since  $\epsilon$  can be made arbitrarily small, we must have

$$w_1 = w_0$$

### 4.3 Example – basic limit

Consider the function  $f(z) = \frac{i\bar{z}}{2}$ . We can show that the limit of this function as  $z \rightarrow 1$  is

$$\lim_{z \rightarrow 1} f(z) = \frac{i}{2}$$

To do this, we observe that

$$\begin{aligned} \left| f(z) - \frac{i}{2} \right| &= \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| \\ &= \frac{|z - 1|}{2} \end{aligned}$$

Then, we can fulfill the limit definition by writing

$$\left| f(z) - \frac{i}{2} \right| < \epsilon \text{ whenever } |z - 1| < 2\epsilon$$

### 4.4 Example – direction dependence

In order for  $w_0$  to be a limit of  $f$  at  $z_0$ , the limit conditions must hold if  $z$  approaches  $z_0$  in any arbitrary manner.

Consider the function

$$f(z) = \frac{z}{\bar{z}}$$

Then, the limit

$$\lim_{z \rightarrow 0} f(z)$$

does not exist. To illustrate this, the function's value for any non-zero point  $z = (x, 0)$  is

$$f(x, 0) = \frac{x + i0}{x - i0} = 1$$



but the value for any non-zero point  $z = (0, y)$  is

$$f(0, y) = \frac{0 + iy}{0 - iy} = -1$$

so the limit would not be unique.

## 5 Frame 16 – Theorems on Limits

Next, it is helpful to connect limits of complex functions and real-valued functions, allowing us to use our knowledge of calculus to simplify the process of finding complex limits

### 5.1 Splitting into real functions

First, the following theorem is helpful:

**Theorem 1.** Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0$$

Then, the limit

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

holds iff

$$\lim_{(x,y) \rightarrow (x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \rightarrow (x_0,y_0)} v(x, y) = v_0$$

The two implications of this theorem can be proved by considering the definitions of the neighbourhoods as open disks.

### 5.2 Combining simple limits

**Theorem 2.** Suppose that

$$\lim_{z \rightarrow z_0} f(z) = w_0 \text{ and } \lim_{z \rightarrow z_0} F(z) = W_0$$

Then, we can write the following three limits:

$$\lim_{z \rightarrow z_0} f(z) + F(z) = w_0 + W_0$$

$$\lim_{z \rightarrow z_0} f(z)F(z) = w_0W_0$$

$$\lim_{z \rightarrow z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \text{ if } W_0 \neq 0$$

These can be proved easily by applying Theorem 1 to each limit.

### 5.3 Polynomials

Using the basic limit definition from the previous section, it is simple to show that

$$\lim_{z \rightarrow z_0} c = c$$

and

$$\lim_{z \rightarrow z_0} z = z_0$$

for any complex numbers  $c$  and  $z_0$ . Then, by the multiplication property,

$$\lim_{z \rightarrow z_0} z^n = z_0^n$$

for any positive integer  $n$ . These limits can be used to show that, for any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

the limit as  $z$  approaches a point  $z_0$  is the polynomial's value:

$$\lim_{z \rightarrow z_0} P(z) = P(z_0)$$

## 6 Frame 17 – Limits Involving Infinity

### 6.1 The point at infinity

Sometimes, it is useful to include the **point at infinity** with the complex plane. This point is denoted by  $\infty$ . In order to visualize it, the complex plane can be drawn with a unit sphere centered at the origin. Then, a line can be drawn from the top of the sphere (or the *north pole*, denoted by  $N$ ) to any point on the plane; the line will pass through exactly one other point  $P$  on the sphere. This correspondence (between points on the plane,  $z$ , and the sphere,  $P$ ) is called a **stereographic projection**, and the sphere is known as the **Riemann sphere**.

No point in the plane corresponds to the point  $N$ . We can let  $N$  correspond to the point at infinity, giving us a one-to-one mapping between points on the sphere and points in the extended complex plane.

We will make the distinction that a point  $z$  is a point in the finite plane unless we specifically describe the point at infinity – we will specifically mention  $\infty$ .

### 6.2 Neighbourhoods around infinity

Next, we can define neighbourhoods around the point at infinity. Looking at the Riemann sphere, we notice that all of the points  $P$  in the upper hemisphere project to points  $z$  outside of the unit disk.

Further, if  $\epsilon$  is a small, positive number, then points in the plane such that

$$|z| > \frac{1}{\epsilon}$$

correspond to points on the sphere close to  $N$ . Thus, we call the set  $|z| > 1/\epsilon$  an  $(\epsilon)$  **neighbourhood** of  $\infty$ .

### 6.3 Limits with infinity

With this new point at infinity, we can give meaning to the statement

$$\lim_{z \rightarrow z_0} f(z) = w_0$$

when  $z_0$  or  $w_0$  are infinity. We can then use the following theorems:

$$\begin{aligned}\lim_{z \rightarrow z_0} f(z) = \infty &\iff \lim_{z \rightarrow z_0} \frac{1}{f(z)} = 0 \\ \lim_{z \rightarrow \infty} f(z) = w_0 &\iff \lim_{z \rightarrow 0} f\left(\frac{1}{z}\right) = w_0 \\ \lim_{z \rightarrow \infty} f(z) = \infty &\iff \lim_{z \rightarrow 0} \frac{1}{f(1/z)} = 0\end{aligned}$$

## 6.4 Examples

Three limits using these new properties follow.

- To find

$$\lim_{z \rightarrow -1} \frac{iz + 3}{z + 1}$$

we notice that

$$\lim_{z \rightarrow -1} \frac{z + 1}{iz + 3} = 0$$

so the limit is infinity.

- To find

$$\lim_{z \rightarrow \infty} \frac{2z + i}{z + 1}$$

we evaluate

$$\begin{aligned}\lim_{z \rightarrow 0} \frac{(2/z) + i}{(1/z) + 1} &= \lim_{z \rightarrow 0} \frac{2 + iz}{1 + z} \\ &= 2\end{aligned}$$

- To find

$$\lim_{z \rightarrow \infty} \frac{2z^3 - 1}{z^2 + 1}$$

we evaluate

$$\begin{aligned}\lim_{z \rightarrow 0} \frac{(1/z^2) + 1}{(2/z^3) - 1} &= \lim_{z \rightarrow 0} \frac{z + z^3}{2 - z^3} \\ &= 0\end{aligned}$$

so the original limit is infinity.

## 7 Continuity

### 7.1 Definitions

A function is **continuous** at a point  $z_0$  if all of the three following conditions are true:

$$\begin{aligned}\lim_{z \rightarrow z_0} f(z) \text{ exists} \\ f(z_0) \text{ exists} \\ \lim_{z \rightarrow z_0} f(z) = f(z_0)\end{aligned}$$

This final statement says that for each positive number  $\epsilon$  there is a positive number  $\delta$  such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } |z - z_0| < \delta$$

We say that a function is said to be continuous in a region  $R$  if it is continuous at each point in  $R$ .

### 7.2 Theorems

The basic limit identities allow us to find the continuity of more complex functions. If two functions are continuous at a point, their sum, product, and quotients are also continuous (in the last case, provided that the denominator is non-zero). A polynomial is continuous in the entire plane.

A complex function that can be split into its real and imaginary components

$$f(z) = u(x, y) + iv(x, y)$$

is continuous at  $z_0 = (x_0, y_0)$  iff  $u$  and  $v$  are continuous at  $z_0$ .

We can state three more theorems about continuity:

- A composition of continuous functions is continuous.

*Suppose that  $w = f(z)$  is defined in a neighbourhood of  $z_0$  and  $W = g(w)$  is defined in a neighbourhood of  $f(z_0)$ . Also, suppose that  $f$  is continuous at  $z_0$  and  $g$  is continuous at  $f(z_0)$ . Then, the statement that the composition*

$$g[f(z)]$$

*is continuous is equivalent to the statement that*

$$|g[f(z)] - g[f(z_0)]| < \epsilon \text{ whenever } |f(z) - f(z_0)| < \gamma$$

*Then, since  $f$  is continuous, we can find a  $\delta$  such that the right side is satisfied, so  $g \circ f$  is continuous.*

- If  $f$  is continuous and non-zero at  $z_0$ , then there is some neighbourhood of  $z_0$  where  $f(z) \neq 0$ .

*Suppose that we choose  $\epsilon = |f(z_0)|/2$ . Then, if there is a point where  $f(z) = 0$  in a  $\delta$  neighbourhood around  $z_0$ , the limit inequality is*

$$|f(z_0)| < \frac{|f(z_0)|}{2}$$

*so we have a contradiction, and there must be a neighbourhood where  $f(z) \neq 0$ .*

- If  $f$  is continuous in a closed and bounded region  $R$ , then there exists a non-negative real number  $M$  such that

$$|f(z)| \leq M$$

where equality holds for one or more  $z$ .

## 8 Frame 19 – Derivatives

### 8.1 Definitions

Suppose that a function  $f$  is defined on a neighbourhood  $|z - z_0| < \epsilon$ . We define the **derivative** of  $f$  at  $z_0$  as the limit

$$f'(z_0) = \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

or, if we write  $\Delta z = z - z_0$ , then the definition can be written as

$$f'(z_0) = \lim_{\Delta z \rightarrow 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

We say that  $f$  is **differentiable** at  $z_0$  when  $f'(z_0)$  exists.

As an augmented notation, we often write

$$\Delta w = f(z + \Delta z) - f(z)$$

and we use the derivative notation

$$\frac{dw}{dz} = \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z}$$

### 8.2 Examples

*Example: suppose  $f(z) = z^2$ . Then,*

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} 2z + \Delta z = 2z$$

*Example: suppose  $f(z) = \bar{z}$ . Then,*

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{z + \Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\bar{z} + \overline{\Delta z} - \bar{z}}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\overline{\Delta z}}{\Delta z}$$

*and we showed earlier that this limit does not exist – it has different values depending on the angle of approach.*

*Example: suppose  $f(z) = |z|^2$ . Then,*

$$\lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{(z + \Delta z)(\overline{z + \Delta z} - z\bar{z})}{\Delta z} = \lim_{\Delta z \rightarrow 0} \bar{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}$$

*We see, once again, that the  $\frac{\overline{\Delta z}}{\Delta z}$  term stops us from obtaining a derivative. However, this term is not present if  $z = 0$  – at this point,  $dw/dz$  is zero. Thus, the derivative exists only at  $z = 0$ , and  $f'(0) = 0$ .*



### 8.3 Notes

We've seen that it is possible for a function to be differentiable at a point but nowhere else in a neighbourhood of that point. We can even note that our function,  $f(z) = |z|^2$ , can be written as

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0$$

so we have no trouble finding the partial derivatives, but the function is still not differentiable aside from  $z = 0$ .

We've also seen that our troublesome function,  $f(z) = |z|^2$ , was continuous but not differentiable. However, we can state that **differentiability implies continuity** at a point. We can see this easily in the numerator of the derivative limit:  $f(z)$  must approach the point  $f(z_0)$  as  $z$  approaches  $z_0$ , which implies continuity.