1 Frame 68 – Isolated Singular Points

1.1 Definition

Earlier, we defined a **singular point** of a function f as a point z_0 where f is not analytic, but f is analytic at some point in every neighbourhood of z_0. Additionally, we will define an **isolated** singular point as such a point where there exists a deleted neighbourhood on which f is analytic.

1.2 Examples

The function

$$f(z) = \frac{z+1}{z^3(z^2+1)}$$

has three singular points at $z = 0$ and $z = \pm i$. These three points are isolated singularities.

The principal branch of the logarithm

$$\operatorname{Log} z = \ln r + i\Theta \quad (-\pi < \theta < \pi)$$

has a singular point at $z = 0$. However, this is not an isolated singular point, since every neighbourhood of 0 also contains some points on the negative real axis, where the function is not analytic.

The function

$$f(z) = \frac{1}{\sin(\pi/z)}$$

has singular points at $z = 0$ and $z = 1/n$ for all integers n. All of these singular points are isolated except for the one at $z = 0$ – every neighbourhood of 0 also contains a point $z = 1/m$ because we can find such a point

$$0 < 1/m < \epsilon$$

for each ϵ.

1.3 Important Points

Note that if a function has a finite number of singular points, then all of these must be isolated – we can make a neighbourhood around each singular point that does not contain any others.

Finally, note that we may also refer to the point at infinity as an isolated singular point. This happens if there exists a positive R_1 such that there are no singularities in the region

$$R_1 < |z| < \infty$$

2 Frame 69 – Residues

2.1 Definition

Suppose that f is a function with an isolated singular point at z_0. Then, there exists some deleted neighbourhood

$$0 < |z - z_0| < R_2$$

where f is analytic. On this domain, we can express $f(z)$ as the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \ldots$$

These coefficients come from various integral representations. In particular,

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

where C is a positively oriented, simple, closed contour around z_0 in the deleted neighbourhood described above. When $n = 1$, this expression becomes

$$\int_C f(z) dz = 2\pi i b_1$$

We say that the complex number b_1 is called the **residue** of f at the isolated singular point z_0, and we write

$$b_1 = \operatorname{Res}_{z=z_0} f(z)$$

so the contour integral around z_0 becomes

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=z_0} f(z)$$

2.2 Examples

Example 1: We can evaluate the integral

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz$$

where C is the positively-oriented unit circle. First, we note that the integrand is analytic everywhere except the origin, so the Laurent series converges on $0 < |z| < \infty$. Then, we can write

$$\begin{aligned} z^2 \sin(1/z) &= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (1/z)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{1-2n} \end{aligned}$$

so the coefficient b_1 *is* $-1/3!$. *Thus,*

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz = 2\pi i \left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}$$

Example 2: We can repeat the previous problem for the integral

$$\int_C e^{1/z^2}$$

Since this series is

$$\begin{aligned} e^{1/z^2} &= \sum_{n=0}^{\infty} \frac{(1/z^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n! z^{2n}} \\ &= 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \dots \end{aligned}$$

the residue at 0 *is* 0, *so*

$$\int_C e^{1/z^2} = 0$$

Example 3: Finally, we can use residues to evaluate the integral

$$\int_C \frac{1}{z(z-2)^4} dz$$

around the positively-oriented circle $|z-2| = 1$. *The Laurent series is*

$$\begin{aligned} \frac{1}{z(z-2)^4} &= \frac{1}{(z-2)^4} \frac{1}{2+(z-2)} \\ &= \frac{1}{2(z-2)^4} \frac{1}{1 - \frac{-(z-2)}{2}} \\ &= \frac{1}{2(z-2)^4} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (z-2)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-4} \end{aligned}$$

Thus, the coefficient of $1/(z-2)$ *is at* $n = 3$, *and*

$$b_1 = -\frac{1}{16}$$

so

$$\int_C \frac{1}{z(z-2)^4} dz = 2\pi i \left(-\frac{1}{16}\right) = -\frac{\pi i}{8}$$

3 Frame 70 – Cauchy's Residue Theorem

3.1 Theorem

If a function has a finite number of singularities inside a contour, then we can use residues to simplify these contour integrals. The following theorem is Cauchy's residue theorem:

Theorem: Suppose that C is a positively-oriented, simple, closed contour. If a function f is analytic on C and has a finite number of singularities at z_k (where $k = 1, 2, \ldots, n$) inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{\infty} \operatorname{Res}_{z=z_k} f(z)$$

Proof: suppose that the contours C_k are small, positively-oriented circles centered at the points z_k. Then, the region $C - \sum C_k$ is a multiply connected domain over which f is analytic, so

$$\int_C f(z)\ dz - \sum_{k=1}^{\infty} \int_{C_k} f(z)\ dz = 0$$

Then, since

$$\int_{C_k} f(z)\ dz = 2\pi i \operatorname{Res}_{z=z_k} f(z)$$

the integral can be written as in the theorem.

3.2 Example

If C is the counterclockwise circular contour $|z| = 2$, then we can evaluate the integral

$$\int_C \frac{5z-2}{z(z-1)} dz$$

using Cauchy's residue theorem. First, we can find the residue B_1 at $z = 0$ by writing out the function as the series

$$\begin{aligned} \frac{5z-2}{z(z-1)} &= \frac{5z-2}{z} \cdot \frac{-1}{1-z} \\ &= \left(5 - \frac{2}{z}\right)(-1 - z - z^2 - \ldots) \\ &= \frac{2}{z} - 3 - 3z - \ldots \end{aligned}$$

so $B_1 = 2$. Then, we can find the residue B_2 at $z = 1$ by writing

$$\begin{aligned}\frac{5z-2}{z(z-1)} &= \frac{5(z-1)+3}{z-1} \cdot \frac{1}{1+(z-1)} \\ &= \left(5 + \frac{3}{z-1}\right)\left(1-(z-1)+(z-1)^2 - \dots\right) \\ &= \frac{3}{z-1} + 2 - 2(z-1) + \dots\end{aligned}$$

so $B_2 = 3$, and

$$\int_C \frac{5z-2}{z(z-1)} = 2\pi i(2+3) = 10\pi i$$

Alternatively, we could use partial fractions to write

$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{z-1}$$

and immediately discover that $B_1 = 2$ and $B_2 = 3$.