

1 Frame 79 – Evaluating Improper Integrals

1 First, along a semi-circular contour $|z| = R$,

$$\int_{C_R} \frac{dx}{x^2 + 1} \leq \pi R \cdot \frac{1}{R^2 - 1} = \pi \frac{R}{R^2 - 1}$$

and this vanishes as R approaches infinity.

Then, we can find the residue at $z = i$ as

$$\text{Res}_{z=i} \frac{1}{z^2 + 1} = \frac{1}{z + i} \Big|_{z=i} = -\frac{i}{2}$$

and so

$$\int_0^\infty \frac{dz}{z^2 + 1} = \frac{1}{2} \left[2\pi i \cdot \frac{-i}{2} \right] = \frac{\pi}{2}$$

2 The integral along the upper semicircle is

$$\int_{C_R} \frac{dx}{(x^2 + 1)^2} \leq \pi R \cdot \frac{1}{(R^2 - 1)^2}$$

which vanishes for large R . Then,

$$\begin{aligned} \int_0^\infty \frac{dx}{(x^2 + 1)^2} &= \frac{1}{2} 2\pi i \text{Res}_{z=i} \frac{1}{(z^2 + 1)^2} \\ &= \pi i \left[\frac{d}{dz} \frac{1}{(z + i)^2} \Big|_{z=i} \right] \\ &= \pi i \left[\frac{-2}{(2i)^3} \right] \\ &= \frac{\pi}{4} \end{aligned}$$

3 The integral along the upper semicircle vanishes as above. Then, there are singular points at $c_k = e^{i\pi/4}, e^{i3\pi/4}$. The residue at these points is

$$\text{Res}_{z=c_k} \frac{1}{z^4 + 1} = \frac{1}{4c_k^3}$$

or, specifically,

$$\begin{aligned} \text{Res}_{z=e^{i\pi/4}} f(z) &= \frac{1}{4} e^{-i3\pi/4} \\ &= \frac{1}{4\sqrt{2}} (-1 - i) \\ \text{Res}_{z=e^{i3\pi/4}} f(z) &= \frac{1}{4} e^{-i\pi/4} \\ &= \frac{1}{4\sqrt{2}} (1 - i) \end{aligned}$$

so

$$\sum B_k = \frac{1}{4\sqrt{2}}(-2i) = \frac{-i}{2\sqrt{2}}$$

Finally,

$$\int_0^\infty \frac{dx}{x^4 + 1} = \frac{1}{2} 2\pi i \sum B_k = \pi i \cdot \frac{-i}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

4 The limit

$$\lim_{R \rightarrow \infty} \frac{R^3}{(R^2 + 1)}(R^2 + 4)$$

is zero, so the integral along the semicircular contour vanishes. Then, this function has simple poles at $z = i, 2i$; the residue at each is

$$\begin{aligned} \text{Res}_{z=i} \frac{z^2}{(z^2 + 1)(z^2 + 4)} &= \frac{z^2}{(z + i)(z^2 + 4)} \Big|_{z=i} \\ &= \frac{-1}{(2i)(3)} \\ &= \frac{i}{6} \\ \text{Res}_{z=2i} \frac{z^2}{(z^2 + 1)(z^2 + 4)} &= \frac{z^2}{(z^2 + 1)(z + 2i)} \Big|_{z=2i} \\ &= \frac{-4}{(-3)(4i)} \\ &= -\frac{i}{3} \end{aligned}$$

so the sum of the residues is $\sum B_k = -i/6$. Then, the improper integral is

$$\int_0^\infty \frac{x^2}{(x^2 + 1)(x^2 + 4)} = \frac{1}{2} 2\pi i \frac{-i}{6} = \frac{\pi}{6}$$

5 The limit

$$\lim_{R \rightarrow \infty} \frac{R^3}{(R^2 - 9)(R^2 - 4)^2} = 0$$

allows us to neglect the semicircular contour. Then, the function has singular

points at $z = 2i, 3i$. The first of these residues is

$$\begin{aligned}
B_1 &= \text{Res}_{z=2i} \frac{z^2}{(z^2+9)(z^2+4)^2} \\
&= \frac{d}{dz} \frac{z^2}{(z^2+9)(z+2i)^2} \Big|_{z=2i} \\
&= \frac{d}{dz} \frac{z^2}{(z^2+9)(z^2+4iz-4)} \Big|_{z=2i} \\
&= \frac{d}{dz} \frac{z^2}{z^4+4iz^3+5z^2+36iz-36} \Big|_{z=2i} \\
&= \frac{2z(z^2+9)(z+2i)^2 - z^2(4z^3+12iz^2+10z+36i)}{(z^2+9)^2(z+2i)^4} \Big|_{z=2i} \\
&= \frac{4i(5)(-16) - (-4)(-32i-48i+20i+36i)}{(25)(256)} \\
&= \frac{-320i-96i}{6400} \\
&= \frac{-13i}{200}
\end{aligned}$$

and the second is

$$\begin{aligned}
B_2 &= \text{Res}_{z=3i} \frac{z^2}{(z^2+9)(z^2+4)^2} \\
&= \frac{z^2}{(z+3i)(z^2+4)^2} \Big|_{z=3i} \\
&= \frac{-9}{(6i)(-5)^2} \\
&= \frac{12i}{200}
\end{aligned}$$

so this integral is

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2} = \pi i \cdot \frac{-i}{200} = \frac{\pi}{200}$$

6 The function

$$f(z) = \frac{1}{z^2+2z+2}$$

has simple poles at $z = -1 \pm i$. The residue at the top point is

$$B = \text{Res}_{z=-1+i} \frac{1}{z^2+2z+2} = \frac{1}{2(-1+i)+2} = \frac{1}{2i}$$

so the principal value of the integral is

$$\text{P.V.} \int_{-\infty}^\infty \frac{dx}{x^2+2x+2} = 2\pi i \cdot \frac{1}{2i} = \pi$$