1 Frame 79 – Evaluating Improper Integrals

1 First, along a semi-circular contour $|z| = R$,

$$\int_{C_R} \frac{dx}{x^2+1} \leq \pi R \cdot \frac{1}{R^2-1} = \pi \frac{R}{R^2-1}$$

and this vanishes as R approaches infinity.

Then, we can find the residue at $z = i$ as

$$\mathrm{Res}_{z=i} \frac{1}{z^2+1} = \left.\frac{1}{z+i}\right|_{z=i} = -\frac{i}{2}$$

and so

$$\int_0^\infty \frac{dz}{z^2+1} = \frac{1}{2}\left[2\pi i \cdot \frac{-i}{2}\right] = \frac{\pi}{2}$$

2 The integral along the upper semicircle is

$$\int_{C_R} \frac{dx}{(x^2+1)^2} \leq \pi R \cdot \frac{1}{(R^2-1)^2}$$

which vanishes for large R. Then,

$$\begin{aligned}
\int_0^\infty \frac{dx}{(x^2+1)^2} &= \frac{1}{2} 2\pi i \,\mathrm{Res}_{z=i} \frac{1}{(z^2+1)^2} \\
&= \pi i \left[\frac{d}{dz} \left.\frac{1}{(z+i)^2}\right|_{z=i}\right] \\
&= \pi i \left[\frac{-2}{(2i)^3}\right] \\
&= \frac{\pi}{4}
\end{aligned}$$

3 The integral along the upper semicircle vanishes as above. Then, there are singular points at $c_k = e^{i\pi/4}, e^{i3\pi/4}$. The residue at these points is

$$\mathrm{Res}_{z=c_k} \frac{1}{z^4+1} = \frac{1}{4c_k^3}$$

or, specifically,

$$\begin{aligned}
\mathrm{Res}_{z=e^{i\pi/4}} f(z) &= \frac{1}{4} e^{-i3\pi/4} \\
&= \frac{1}{4\sqrt{2}}(-1-i) \\
\mathrm{Res}_{z=e^{i3\pi/4}} f(z) &= \frac{1}{4} e^{-i\pi/4} \\
&= \frac{1}{4\sqrt{2}}(1-i)
\end{aligned}$$

so

$$\sum B_k = \frac{1}{4\sqrt{2}}(-2i) = \frac{-i}{2\sqrt{2}}$$

Finally,

$$\int_0^\infty \frac{dx}{x^4+1} = \frac{1}{2} 2\pi i \sum B_k = \pi i \cdot \frac{-i}{2\sqrt{2}} = \frac{\pi}{2\sqrt{2}}$$

4 The limit

$$\lim_{R\to\infty} \frac{R^3}{(R^2+1)}(R^2+4)$$

is zero, so the integral along the semicircular contour vanishes. Then, this function has simple poles at $z = i, 2i$; the residue at each is

$$\begin{aligned}
\mathrm{Res}_{z=i} \frac{z^2}{(z^2+1)(z^2+4)} &= \frac{z^2}{(z+i)(z^2+4)}\Big|_{z=i} \\
&= \frac{-1}{(2i)(3)} \\
&= \frac{i}{6} \\
\mathrm{Res}_{z=2i} \frac{z^2}{(z^2+1)(z^2+4)} &= \frac{z^2}{(z^2+1)(z+2i)}\Big|_{z=2i} \\
&= \frac{-4}{(-3)(4i)} \\
&= -\frac{i}{3}
\end{aligned}$$

so the sum of the residues is $\sum B_k = -i/6$. Then, the improper integral is

$$\int_0^\infty \frac{x^2}{(x^2+1)(x^2+4)} = \frac{1}{2} 2\pi i \frac{-i}{6} = \frac{\pi}{6}$$

5 The limit

$$\lim_{R\to\infty} \frac{R^3}{(R^2-9)(R^2-4)^2} = 0$$

allows us to neglect the semicircular contour. Then, the function has singular points at $z = 2i, 3i$. The first of these residues is

$$\begin{aligned}
B_1 &= \operatorname{Res}_{z=2i} \frac{z^2}{(z^2+9)(z^2+4)^2} \\
&= \frac{d}{dz} \frac{z^2}{(z^2+9)(z+2i)^2}\Big|_{z=2i} \\
&= \frac{d}{dz} \frac{z^2}{(z^2+9)(z^2+4iz-4)}\Big|_{z=2i} \\
&= \frac{d}{dz} \frac{z^2}{z^4+4iz^3+5z^2+36iz-36}\Big|_{z=2i} \\
&= \frac{2z(z^2+9)(z+2i)^2 - z^2(4z^3+12iz^2+10z+36i)}{(z^2+9)^2(z+2i)^4}\Big|_{z=2i} \\
&= \frac{4i(5)(-16) - (-4)(-32i-48i+20i+36i)}{(25)(256)} \\
&= \frac{-320i-96i}{6400} \\
&= \frac{-13i}{200}
\end{aligned}$$

and the second is

$$\begin{aligned}
B_2 &= \operatorname{Res}_{z=3i} \frac{z^2}{(z^2+9)(z^2+4)^2} \\
&= \frac{z^2}{(z+3i)(z^2+4)^2}\Big|_{z=3i} \\
&= \frac{-9}{(6i)(-5)^2} \\
&= \frac{12i}{200}
\end{aligned}$$

so this integral is

$$\int_0^\infty \frac{x^2}{(x^2+9)(x^2+4)^2} = \pi i \cdot \frac{-i}{200} = \frac{\pi}{200}$$

6 The function

$$f(z) = \frac{1}{z^2+2z+2}$$

has simple poles at $z = -1 \pm i$. The residue at the top point is

$$B = \operatorname{Res}_{z=-1+i} \frac{1}{z^2+2z+2} = \frac{1}{2(-1+i)+2} = \frac{1}{2i}$$

so the principal value of the integral is

$$\text{P.V.} \int_{-\infty}^\infty \frac{dx}{x^2+2x+2} = 2\pi i \cdot \frac{1}{2i} = \pi$$

2 Frame 81 – Fourier Integrals

1 The function

$$f(z) = \frac{1}{(x^2 + a^2)(x^2 + b^2)}$$

has singular poles at $z = \pm ia, \pm ib$. The residue of the function $f(z)e^{iz}$ at the upper two points is

$$\begin{aligned}
\mathrm{Res}_{z=ia}\, f(z)e^{iz} &= \frac{e^{-a}}{2ia(b^2 - a^2)} \\
&= \frac{-e^{-a}}{2ia(a^2 - b^2)} \\
\mathrm{Res}_{z=ib}\, f(z)e^{iz} &= \frac{e^{-b}}{(a^2 - b^2)2ib}
\end{aligned}$$

and the integral along the semicircular contour vanishes, leaving

$$\int_{-\infty}^{\infty} f(x)\cos x \; dx = \Re\left(2\pi i \cdot \frac{e^{-b}/b - e^{-a}/a}{2i(a^2 - b^2)}\right) = \frac{\pi}{a^2 - b^2}\left(\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right)$$

2 The function

$$f(z) = \frac{1}{z^2 + 1}$$

has a pole at $z = i$. The residue of $f(z)e^{iaz}$ here is

$$\mathrm{Res}_{z=i} \frac{e^{iaz}}{z^2 + 1} = \frac{e^{-a}}{2i}$$

so

$$\int_0^{\infty} \frac{\cos ax}{x^2 + 1} dx = \Re\left(\pi i \cdot \frac{e^{-a}}{2i}\right) = \frac{\pi}{2} e^{-a}$$

3 The function

$$f(z) = \frac{1}{(z^2 + b^2)^2}$$

has a singular point at $z = ib$. The residue of $f(z)e^{iaz}$ here is

$$\begin{aligned}
\mathrm{Res}_{z=ib} \frac{e^{iaz}}{(z^2 + b^2)^2} &= \frac{d}{dz} \frac{e^{iaz}}{(z + ib)^2}\Big|_{z=ib} \\
&= \frac{e^{iaz}[ia(z + ib) - 2]}{(z + ib)^3}\Big|_{z=ib} \\
&= \frac{-2e^{-ab}(1 + ab)}{-8ib^3} \\
&= \frac{(1 + ab)e^{-ab}}{4ib^3}
\end{aligned}$$

so

$$\int_0^\infty f(x)\cos ax\ dx = \Re\left(\pi i \cdot \frac{(1+ab)e^{-ab}}{4ib^3}\right) = \frac{\pi}{4b^3}(1+ab)e^{-ab}$$

4 The function

$$f(z) = \frac{z}{z^2+3}$$

tends to zero for large $|z|$. It has a singular point at $z = i\sqrt{3}$, where the residue of $f(z)e^{2iz}$ is

$$\mathrm{Res}_{z=i\sqrt{3}} \frac{ze^{2iz}}{z^2+3} = \frac{ze^{2iz}}{z+i\sqrt{3}}\Big|_{z=i\sqrt{3}} = \frac{i\sqrt{3}e^{-2\sqrt{3}}}{2i\sqrt{3}} = \frac{e^{-2\sqrt{3}}}{2}$$

so the integral evaluates to

$$\int_0^\infty \frac{x\sin 2x}{x^2+3} = \Im\left(\pi i \cdot \frac{e^{-2\sqrt{3}}}{2}\right) = \frac{\pi}{2}e^{-2\sqrt{3}}$$