

1 Frame 29 – The Exponential Function

1(a) The value is

$$e^{2 \pm 3\pi i} = e^2 e^{\pm 3\pi i} = e^2(-1) = e^2$$

1(b) The value is

$$e^{1/2 + i\pi/4} = e^{1/2} e^{i\pi/4} = \sqrt{e} \frac{1+i}{\sqrt{2}} = \sqrt{\frac{e}{2}}(1+i)$$

1(c) This expression can be split up as

$$e^{z+\pi i} = e^z e^{i\pi} = e^z \cdot -1 = -e^z$$

3 The function can be written as

$$e^{\bar{z}} = e^x e^{-iy} = e^x \cos y - ie^x \sin y$$

so the partial derivatives are

$$\begin{aligned}u_x &= e^x \cos y \\u_y &= -e^x \sin y \\v_x &= -e^x \sin y \\v_y &= -e^x \cos y\end{aligned}$$

These don't satisfy Cauchy-Riemann, so the function is differentiable nowhere and thus analytic nowhere.

8(a) The equation $e^z = -2$ can be written as

$$e^z = 2e^{i\pi}$$

so

$$\begin{aligned}e^x &= 2 \rightarrow x = \ln 2 \\e^{iy} &= e^{i\pi} \rightarrow y = (2n+1)\pi\end{aligned}$$

and

$$z = \ln 2 + (2n+1)i\pi$$

8(b) The equation $e^z = 1 + i\sqrt{3}$ can be written as

$$e^z = 2e^{i\pi/3}$$

so

$$\begin{aligned}x &= \ln 2 \\y &= \left(\frac{1}{3} + 2n\right)\pi\end{aligned}$$

and

$$z = \ln 2 + \left(\frac{1}{3} + 2n\right) i\pi$$

8(c) If

$$e^{2z-1} = 1$$

then

$$2z - 1 = i2n\pi$$

or, solving for z ,

$$z = \frac{1}{2} + in\pi$$

2 Frame 31 – Logarithms

1(a) Evaluating the logarithm,

$$\operatorname{Log}(-ei) = \ln(e) + \operatorname{Arg}(-ei) = 1 - i\frac{\pi}{2}$$

1(b) As above,

$$\operatorname{Log}(1-i) = \ln(\sqrt{2}) + \operatorname{Arg}(1-i) = \frac{1}{2}\ln(2) - i\frac{\pi}{4}$$

2(a) The set of values is

$$\log(e) = \ln(e) + \arg(e) = 1 + i2n\pi$$

2(b) As above,

$$\log(i) = \ln(1) + \arg(i) = 0 + i\left(2n + \frac{1}{2}\right)\pi$$

2(c) As above,

$$\log(-1 + i\sqrt{3}) = \ln(\sqrt{4}) + \arg(-1 + i\sqrt{3}) = \ln 2 + i\left(2n + \frac{2}{3}\right)\pi$$

3(a) The left side is

$$\operatorname{Log}(1+i)^2 = \ln(2) + i\frac{\pi}{2}$$

and the right side is

$$2\operatorname{Log}(1+i) = 2\left[\frac{1}{2}\ln(2) + i\frac{\pi}{4}\right] = \ln(2) + i\frac{\pi}{2} = \operatorname{Log}(1+i)^2$$

3(b) The left side is

$$\operatorname{Log}(-1+i)^2 = \ln(2) - i\frac{\pi}{2}$$

and the right side is

$$2\operatorname{Log}(-1+i) = 2\left[\frac{1}{2}\ln(2) + i\frac{3\pi}{4}\right] = \ln(2) + i\frac{3\pi}{2}$$

5(a) Since $i^{1/2}$ can be written as the set

$$i^{1/2} = e^{\pi/4 + \pi k} \quad (k = 0, \pm 1, \pm 2, \dots)$$

it has the logarithm

$$\log(i^{1/2}) = \ln(1) + i \arg(i^{1/2}) = i \left(n + \frac{1}{4} \right) \pi$$

Then,

$$\frac{1}{2} \log i = \frac{1}{2} \left[\ln(1) + i \left(2n + \frac{1}{2} \right) \pi \right] = i \left(n + \frac{1}{4} \right) \pi = \log(i^{1/2})$$

6 Differentiating and using the chain rule, the identity

$$z = e^{\log z}$$

becomes

$$1 = e^{\log z} \cdot \frac{d}{dz} \log z$$

or

$$\frac{d}{dz} \log z = \frac{1}{e^{\log z}} = \frac{1}{z}$$

7 To solve the equation

$$\log z = i \frac{\pi}{2}$$

we write

$$z = e^{i\pi/2} = i$$

3 Frame 33 – Complex Exponents

1(a) The values of $(1+i)^i$ are

$$(1+i)^i = e^{i \cdot (\ln \sqrt{2} + i(2n+1/4)\pi)} = e^{i \ln(2)/2} e^{-\pi/4 + 2n\pi}$$

1(b) The values of $(-1)^{1/\pi}$ are

$$(-1)^{1/\pi} = e^{\frac{1}{\pi} \cdot i(2n+1)\pi} = e^{(2n+1)i}$$

2(a) The principal value of i^i is

$$i^i = e^{i \operatorname{Log} i} = e^{i(i\pi/2)} = e^{-\pi/2}$$

2(b) The principal value of $[\frac{e}{2}(-1-i\sqrt{3})]^{3\pi i}$ is

$$[\frac{e}{2}(-1-i\sqrt{3})]^{3\pi i} = \exp \left[3\pi i \left(1 - i\frac{2\pi}{3} \right) \right] = e^{3\pi i} e^{-2\pi^2} = -e^{-2\pi^2}$$

2(c) The principal value of $(1-i)^{4i}$ is

$$(1-i)^{4i} = e^{4i(\frac{1}{2} \ln 2 - i\pi/4)} = e^{i2 \ln(2)} e^{\pi} = e^{\pi} [\cos(2 \ln 2) + i \sin(2 \ln 2)]$$

3 The values of $(-1+i\sqrt{3})^{3/2}$ are

$$(-1+i\sqrt{3})^{3/2} = e^{3/2 \cdot (\ln 2 + i\pi(2n+2/3))} = 2^{3/2} e^{i(\pi+3n\pi)} = \pm 2\sqrt{2}$$

4 Frame 34 – Trigonometric Functions

15 To find all roots of the equation

$$\sin z = \cosh 4$$

we can write

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Then, equating real and imaginary parts, we find that

$$\begin{aligned}\sin x \cosh y &= \cosh 4 \\ \cos x \sinh y &= 0\end{aligned}$$

The second equation says that $x = (n + 1/2)\pi$. Since $\sin x$ here is one of ± 1 , the first equation becomes

$$\cosh y = \pm \cosh 4$$

However, $\cosh y > 0$ for all y and $\cosh y = \cosh(-y)$, so the only solutions are

$$z = \left(2n + \frac{1}{2}\right)\pi \pm 4i$$

16 To find all roots of the equation

$$\cos z = 2$$

we can write

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

so

$$\begin{aligned}\cos x \cosh y &= 2 \\ \sin x \sinh y &= 0\end{aligned}$$

The second equation states that $x = n\pi$ or $y = 0$. The second case has no solutions, so we must have $x = n\pi$. Then, $\cos x = \pm 1$. As above, we require that $\cos x = 1$ for any solutions to be found. This gives

$$x = 2n\pi$$

so

$$y = \cosh^{-1}(2)$$

or, putting these together,

$$z = 2n\pi + i \cosh^{-1}(2)$$

To simplify this, we can try to find a simpler expression for y . If $\cosh y = 2$, then

$$\frac{e^y + e^{-y}}{2} = 2$$

or

$$(e^y)^2 - 4(e^y) + 1 = 0$$

This has solutions when

$$e^y = 2 \pm \sqrt{3}$$

or

$$y = \ln(2 \pm \sqrt{3})$$

However,

$$\frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}$$

so

$$\ln(2 + \sqrt{3}) = -\ln(2 - \sqrt{3})$$

and the solutions are

$$z = 2n\pi \pm i \ln(2 + \sqrt{3})$$