1 Frame 12 – Functions of Complex Variables

1(a) The function

$$f(z) = \frac{1}{z^2 + 1}$$

is defined everywhere except where $z^2 + 1 = 0$; ie:

$$z \neq \pm i$$

1(b) The function

$$f(z) = \text{Arg}\left(\frac{1}{z}\right)$$

is defined wherever $\frac{1}{z}$ is defined:

$$z \neq 0$$

1(c) The function

$$f(z) = \frac{z}{z + \bar{z}}$$

can be written as

$$f(x, y) = \frac{x + iy}{(x + iy) + (x - iy)} = \frac{x + iy}{2x} = \frac{1}{2} + i\frac{y}{x}$$

so the domain is

$$\text{Re}(z) \neq 0$$

1(d) The function

$$f(z) = \frac{1}{1 - |z|^2}$$

is equivalent to

$$f(r, \theta) = \frac{1}{1 - r^2}$$

so the domain is

$$r \neq 1$$

2 Substituting $z = x + iy$ gives

$$\begin{aligned}
f(x, y) &= (x + iy)^3 + (x + iy) + 1 \\
&= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x + iy + 1 \\
&= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y)
\end{aligned}$$

so

$$\begin{aligned}
u(x, y) &= x^3 - 3xy^2 + x + 1 \\
v(x, y) &= 3x^2y - y^3 + y
\end{aligned}$$

3 Using the two expressions

$$x = \frac{z + \bar{z}}{2}$$
$$y = \frac{z - \bar{z}}{2i}$$

gives

$$\begin{aligned}
f(z) &= \Big(\frac{z+\bar{z}}{2}\Big)^2 - \Big(\frac{z-\bar{z}}{2i}\Big)^2 - 2\frac{z-\bar{z}}{2i} \\
&\quad + i\Big[2\frac{z+\bar{z}}{2}\Big(1 - \frac{z-\bar{z}}{2i}\Big)\Big] \\
&= \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2 + z^2 - 2z\bar{z} + \bar{z}^2) + iz - i\bar{z} \\
&\quad + i\Big[z + \bar{z} + \frac{iz^2}{2} - \frac{i\bar{z}^2}{2}\Big] \\
&= \frac{1}{2}(z^2 + \bar{z}^2) + 2iz - \frac{iz^2}{2} + \frac{\bar{z}^2}{2} \\
&= \bar{z}^2 + 2iz
\end{aligned}$$

4 Using

$$z = re^{i\theta}$$

the function can be written as

$$\begin{aligned}
f(z) &= re^{i\theta} + \frac{1}{re^{i\theta}} \\
&= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\
&= (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta
\end{aligned}$$