# 1    Frame 12 – Functions of Complex Variables

## 1.1    Functions

If $S$ is a set of complex numbers, then a **function** $f$ is a rule that assigns a complex number $w$ to each $z$ in $S$. The number $w$ is called the **value** of $f$ at $z$. We denote it as

$$w = f(z)$$

The set $S$ is called the **domain of definition** of $f$. Note that we need both a rule ($f$) and a domain ($S$) for a function to be well defined.

Suppose that $w = u + iv$ and $z = x + iy$. Then,

$$u + iv = f(x + iy)$$

Then, we can express $f(z)$ as a pair of real functions of $x$ and $y$:

$$f(z) = u(x, y) + iv(x, y)$$

Alternatively, we could use polar coordinates to write

$$u + iv = f(re^{i\theta})$$

so

$$f(z) = u(r, \theta) + iv(r, \theta)$$

*Example: the function $f(z) = z^2$ can be written as*

$$f(x + iy) = (x + iy)^2$$
$$= (x^2 - y^2) + i2xy$$

*so*

$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$

*In polar coordinates,*

$$f(x + iy) = (re^{i\theta})^2$$
$$= r^2 e^{i2\theta}$$
$$= r^2 \cos 2\theta + ir^2 \sin 2\theta$$

*so*

$$u(r, \theta) = r^2 \cos 2\theta$$
$$v(r, \theta) = r^2 \sin 2\theta$$

## 1.2   Real-Valued Functions

We say that $f$ is a **real-valued function** if $v$ is zero everywhere.

*Example: one real-valued function is*

$$f(z) = |z|^2 = x^2 + y^2 + i0$$

## 1.3   Polynomials

If $n$ is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are complex numbers with $a_n \neq 0$, then the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a **polynomial** of degree $n$. Note that this sum has a finite number of terms and that the domain of definition is the entire $z$ plane.

As in real numbers, a **rational function** is a quotient of two polynomials:

$$R(z) = \frac{P(z)}{Q(z)}$$

A rational function is defined everywhere that $Q(z) \neq 0$.

## 1.4   Multi-Valued Functions

A generalization of a function is a rule that assigns more than one value to a point $z$. These **multiple-valued functions** are usually studied by taking one of the possible values at each point and constructing a single-valued function.

*Example: we know that we can write*

$$z^{1/2} = \pm\sqrt{r}e^{i\theta/2}$$

*where we denoted $-\pi < \theta \leq \pi$ as the **principal value** of $\arg z$. To turn this into a single valued function, we can choose the positive value of $r$ and write*

$$f(z) = \sqrt{r}e^{i\theta/2}$$

*Then, $f$ is well-defined on the entire plane.*

# 2  Frame 13 – Mappings

## 2.1  Definitions

There is no convenient way to graph the function $w = f(z)$ – each of these complex numbers are located on a plane instead of a line. Instead, we can draw pairs of corresponding points on separate $z$ and $w$ planes. When we think of $f$ this way, we call it a **mapping** or **transformation**.

If $f$ is defined on the domain of definition $S$, then the **image** of a point $z \in S$ is the point $w = f(z)$. If $T$ is a subset of $S$, then the set of the images of each point in $T$ are called the image of $T$. In particular, the image of the entire domain, $S$, is called the **range** of $f$. The **inverse image** of a point $w$ is the set of points $z$ in $S$ that map to $w$ (possibly zero, one, or many points).

## 2.2  Basic transformations

Using this geometric interpretation, we can describe mappings using terms such as **translation**, **rotation**, and **reflection**. For instance, the mapping

$$w = z + 1 = (x + 1) + iy$$

can be thought of as a translation of each point $z$ one unit to the right. Another example is the rotational mapping

$$w = iz$$

where, using $i = e^{i\pi/2}$ and $z = re^{i\theta}$, is

$$w = re^{i(\theta + \pi/2)}$$

or, in other words, a $90°$ rotation. Finally, the mapping

$$w = \bar{z} = x - iy$$

is a reflection across the real axis. Usually, it is more useful to sketch an image of a curve rather than a single point.

## 2.3  Mapping a curve

*For an example, consider the mapping $w = z^2$. We showed earlier that this can be written as*

$$u = x^2 - y^2, \quad v = 2xy$$

*To sketch the image, we will first set $u = c_1$, which requires that*

$$x^2 - y^2 = c_1, \quad c_1 > 0$$

*which is the equation for a hyperbola. This equation can then be used to solve for the image points:*

$$u = c_1, \quad v = \pm 2y\sqrt{y^2 + c_1}$$

*where the plus-minus is resolved depending on which side the image point is on. Simply put, as $z$ travels up the right-side hyperbola or down the left-side hyperbola, $w$ travels up the vertical line $u = c_1$.*

*Next, we can set $v = c_2$, which requires*

$$2xy = c_2, \quad c_2 > 0$$

*This gives us the image set*

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

*As $x \to \pm\infty$, $u \to \infty$; as $x \to 0$, $u \to -\infty$. Thus, this hyperbola traces out the straight line $v = c_2$ towards the right as $z$ travels towards the left.*

## 2.4 Mapping a region

We can use some of the details from the previous example to find the image of a region, rather than a single curve.

*Consider the domain $x > 0, y > 0, xy < 1$. This region consists of the upper branches of the hyperbolas*

$$2xy = c, \quad 0 < c < 2$$

*and we know from the previous example that these hyperbolas map to the straight lines*

$$v = c$$

*Thus, this region maps to the horizontal strip $0 < v < 2$.*

*We can also close the domain to contain the curves $x = 0$, $y = 0$, and $xy = 1$. From the function $w = z^2$, we know that the points $(0, y)$ and $(x, 0)$ map to the points $(-y^2, 0)$ and $(x^2, 0)$, so including the two straight lines simply extends the strip to include $v = 0$. Similarly, the hyperbola $xy = 1$ maps to the horizontal line $v = 2$.*

*Simply put, the image of the closed region $x \geq 0$, $y \geq 0$, $xy \leq 1$ is the closed region $0 \leq v \leq 2$.*

## 2.5 Mapping with polar coordinates

Finally, we can use polar coordinates to simplify some mappings.

*Again, consider the mapping $w = z^2$. If we write $z = re^{i\theta}$, then the image point can be written as*

$$w = r^2 e^{2i\theta}$$

*Looking at the magnitude of $w$, points on a circle $r = r_0$ are mapped onto a circle $r' = r_0^2$. Also, looking at the argument of $w$, the angle of the image is doubled. This means that the first quadrant, which is defined as*

$$r \geq 0, \quad 0 \leq \theta \leq \pi/2$$

*is in a one-to-one mapping with the top plane, $0 \leq \theta \leq \pi$. Simiarly, the top place is mapped onto the entire complex plane (although this is not one-to-one, since the inverse image of the positive real axis is both real axes).*

Note that any mapping $w = z^n$ for positive integer $n$ has a similar form, where each non-zero point in the $w$ plane is the image of $n$ distinct points in the $z$ plane.

# 3   Frame 14 – Mappings by the Exponential Function

Now, we will look at the exponential function

$$e^z = e^{x+iy} = e^x e^{iy}$$

We can again look at straight lines and find their images in this mapping.

*Consider the transformation*

$$w = e^z = \rho e^{i\phi}$$

*where*

$$p = e^x \quad \phi = y$$

*This means that the image of a vertical line $x = c_1$ is a circle with radius $p = e^{c_1}$. Each point on the circle is the image of infinitely many points, each spaced $2\pi$ units apart on the vertical line. Similarly, the horizontal line $y = c_2$ is a ray with an angle of $\phi = c_2$.*

With these images in mind, we know that vertical and horizontal line segments are mapped onto arcs and rays, respectively. We can then use this information to map regions:

*Now, consider the rectangular region*

$$a \leq x \leq b \quad c \leq y \leq d$$

*The image of this region under the mapping $w = e^z$ is*

$$e^a \leq \rho \leq e^b \quad c \leq \phi \leq d$$

*This is a one-to-one mapping if $d - c < 2\pi$. In particular, the region with $c = 0, d = \pi$ is mapped onto half of a circular ring.*

# 4 Frame 15 – Limits

## 4.1 Definitions

Suppose that a function $f$ is defined at all points $z$ in some deleted neighborhood of $z_0$. The statement that the number $w_0$ is the **limit** of $f(z)$ as $z$ approaches $z_0$ means that the point $w = f(z)$ can be made *arbitrarily close* to $w_0$ if we choose $z$ close enough to $z_0$. We write this as

$$\lim_{z \to z_0} f(z) = w_0$$

To be more precise, if this limit exists, then for each positive number $\epsilon$, there is a positive number $\delta$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Geometrically, this definition says that each $\epsilon$ neighbourhood around $w_0$ has a corresponding deleted $\delta$ neighbourhood around $z_0$ such that the image of each point in the $\delta$ neighbourhood maps to a point in the $\epsilon$ neighbourhood.

Note that the deleted neighbourhood will always exist if $z_0$ is internal to the domain of definition of $f$. We can extend the definition of a limit to include boundary points by ignoring all of the neighbourhood's points that are outside the domain.

Also note that this definition only allows a given point to be tested as a limit – it does not provide a method for finding the limit. This will be covered in the next section.

## 4.2 Uniqueness

If the limit of a function $f(z)$ exists at $z_0$, it must be unique. To show this, consider two limits:

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} f(z) = w_1$$

This implies that we can find $\delta_0$ and $\delta_1$ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_1$$

Now, suppose that $\delta$ is a positive number smaller than both $\delta_0$ and $\delta_1$. Then, for all $0 < |z - z_0| < \delta$, we find that the difference between the two limits is

$$
\begin{aligned}
|w_1 - w_0| = ||f(z) - w_0| - |f(z) - w_1|| \\
\leq |f(z) - w_0| + |f(z) - w_1| \\
< \epsilon + \epsilon \\
= 2\epsilon
\end{aligned}
$$

and since $\epsilon$ can be made arbitrarily small, we must have

$$
w_1 = w_0
$$

## 4.3    Example – basic limit

*Consider the function $f(z) = \frac{i\bar{z}}{2}$. We can show that the limit of this function as $z \to 1$ is*

$$
\lim_{z \to 1} f(z) = \frac{i}{2}
$$

*To do this, we observe that*

$$
\begin{aligned}
\left| f(z) - \frac{i}{2} \right| = \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| \\
= \frac{|z - 1|}{2}
\end{aligned}
$$

*Then, we can fulfill the limit definition by writing*

$$
\left| f(z) - \frac{i}{2} \right| < \epsilon \text{ whenever } |z - 1| < 2\epsilon
$$

## 4.4    Example – direction dependence

In order for $w_0$ to be a limit of $f$ at $z_0$, the limit conditions must hold if $z$ approaches $z_0$ in any arbitrary manner.

*Consider the function*

$$
f(z) = \frac{z}{\bar{z}}
$$

*Then, the limit*

$$
\lim_{z \to 0} f(z)
$$

*does not exist. To illustrate this, the function's value for any non-zero point $z = (x, 0)$ is*

$$
f(x, 0) = \frac{x + i0}{x - i0} = 1
$$

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but the value for any non-zero point $z = (0, y)$ is

$$f(0, y) = \frac{0 + iy}{0 - iy} = -1$$

so the limit would not be unique.

# 5 Frame 16 – Theorems on Limits

Next, it is helpful to connect limits of complex functions and real-valued functions, allowing us to use our knowledge of calculus to simplify the process of finding complex limits

## 5.1 Splitting into real functions

First, the following theorem is helpful:

**Theorem 1.** Suppose that

$$f(z) = u(x,y) + iv(x,y)$$

and

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0$$

Then, the limit

$$\lim_{z \to z_0} = w_0$$

holds iff

$$\lim_{(x,y) \to (x_0,y_0)} u(x,y) = u_0 \text{ and } \lim_{(x,y) \to (x_0,y_0)} v(x,y) = v_0$$

The two implications of this theorem can be proved by considering the definitions of the neighbourhoods as open disks.

## 5.2 Combining simple limits

**Theorem 2.** Suppose that

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} F(z) = W_0$$

Then, we can write the following three limits:

$$\lim_{z \to z_0} f(z) + F(z) = w_0 + W_0$$

$$\lim_{z \to z_0} f(z)F(z) = w_0 W_0$$

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \text{ if } W_0 \neq 0$$

These can be proved easily by applying Theorem 1 to each limit.

## 5.3 Polynomials

Using the basic limit definition from the previous section, it is simple to show that

$$\lim_{z \to z_0} c = c$$

and

$$\lim_{z \to z_0} z = z_0$$

for any complex numbers $c$ and $z_0$. Then, by the multiplication property,

$$\lim z \to z_0 z^n = z_0^n$$

for any positive integer $z$. These limits can be used to show that, for any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

the limit as $z$ approaches a point $z_0$ is the polynomial's value:

$$\lim_{z \to z_0} P(z) = P(z_0)$$

# 6 Frame 17 – Limits Involving Infinity

## 6.1 The point at infinity

Sometimes, it is useful to include the **point at infinity** with the complex plane. This point is denoted by $\infty$. In order to visualize it, the complex plane can be drawn with a unit sphere centered at the origin. Then, a line can be drawn from the top of the sphere (or the *north pole*, denoted by $N$) to any point on the plane; the line will pass through exactly one other point $P$ on the sphere. This correspondence (between points on the plane, $z$, and the sphere, $P$) is called a **stereographic projection**, and the sphere is known as the **Riemann sphere**.

No point in the plane corresponds to the point $N$. We can let $N$ correspond to the point at infinity, giving us a one-to-one mapping between points on the sphere and points in the extended complex plane.

We will make the distinction that a point $z$ is a point in the finite plane unless we specifically describe the point at infinity – we will specifically mention $\infty$.

## 6.2 Neighbourhoods around infinity

Next, we can define neighbourhoods around the point at infinity. Looking at the Riemann sphere, we notice that all of the points $P$ in the upper hemisphere project to points $z$ outside of the unit disk.

Further, if $\epsilon$ is a small, positive number, then points in the plane such that

$$|z| > \frac{1}{\epsilon}$$

correspond to points on the sphere close to $N$. Thus, we call the set $|z| > 1/\epsilon$ an $(\epsilon)$ **neighbourhood** of $\infty$.

## 6.3 Limits with infinity

With this new point at infinity, we can give meaning to the statement

$$\lim_{z \to z_0} f(z) = w_0$$

when $z_0$ or $w_0$ are infinity. We can then use the following theorems:

$$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

$$\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$

$$\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(1/z)} = 0$$

## 6.4   Examples

Three limits using these new properties follow.

- To find
$$\lim_{z \to -1} \frac{iz + 3}{z + 1}$$
  we notice that
$$\lim_{z \to -1} \frac{z + 1}{iz + 3} = 0$$
  so the limit is infinity.

- To find
$$\lim_{z \to \infty} \frac{2z + i}{z + 1}$$
  we evaluate
$$\lim_{z \to 0} \frac{(2/z) + i}{(1/z) + 1} = \lim_{z \to 0} \frac{2 + iz}{1 + z}$$
$$= 2$$

- To find
$$\lim_{z \to \infty} \frac{2z^3 - 1}{z^2 + 1}$$
  we evaluate
$$\lim_{z \to 0} \frac{(1/z^2) + 1}{(2/z^3) - 1} = \lim_{z \to 0} \frac{z + z^3}{2 - z^3}$$
$$= 0$$

so the original limit is infinity.

# 7  Continuity

## 7.1  Definitions

A function is **continuous** at a point $z_0$ if all of the three following conditions are true:

$$\lim_{z \to z_0} f(z) \text{ exists}$$
$$f(z_0) \text{ exists}$$
$$\lim_{z \to z_0} f(z) = f(z_0)$$

This final statement says that for each positive number $\epsilon$ there is a positive number $\delta$ such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } |z - z_0| < \delta$$

We say that a function is said to be continuous in a region $R$ if it is continuous at each point in $R$.

## 7.2  Theorems

The basic limit identities allow us to find the continuity of more complex functions. If two functions are continuous at a point, their sum, product, and quotients are also continuous (in the last case, provided that the denominator is non-zero). A polynomial is continuous in the entire plane.

A complex function that can be split into its real and imaginary components

$$f(z) = u(x, y) + iv(x, y)$$

is continuous at $z_0 = (x_0, y_0)$ iff $u$ and $v$ are continuous at $z_0$.

We can state three more theorems about continuity:

- A composition of continuous functions is continuous.

  *Suppose that $w = f(z)$ is defined in a neighbourhood of $z_0$ and $W = g(w)$ is defined in a neighbourhood of $f(z_0)$. Also, suppose that $f$ is continuous at $z_0$ and $g$ is continuous at $f(z_0)$. Then, the statement that the composition*

  $$g[f(z)]$$

  *is continuous is equivalent to the statement that*

  $$|g[f(z)] - g[f(z_0)]| < \epsilon \text{ whenever } |f(z) - f(z_0)| < \gamma$$

  *Then, since $f$ is continuous, we can find a $\delta$ such that the right side is satisfied, so $g \circ f$ is continuous.*

- If $f$ is continuous and non-zero at $z_0$, then there is some neighbourhood of $z_0$ where $f(z) \neq 0$.

  *Suppose that we choose $\epsilon = |f(z_0)|/2$. Then, if there is a point where $f(z) = 0$ in a $\delta$ neighbourhood around $z_0$, the limit inequality is*

  $$|f(z_0)| < \frac{|f(z_0)|}{2}$$

  *so we have a contradiction, and there must be a neighbourhood where $f(z) = 0$.*

- If $f$ is continuous in a closed and bounded region $R$, then there exists a non-negative real number $M$ such that

  $$|f(z)| \leq M$$

  where equality holds for one or more $z$.

# 8  Frame 19 – Derivatives

## 8.1  Definitions

Suppose that a function $f$ is defined on a neighbourhood $|z - z_0| < \epsilon$. We define the **derivative** of $f$ at $z_0$ as the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

or, if we write $\Delta z = z - z_0$, then the definition can be written as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

We say that $f$ is **differentiable** at $z_0$ when $f'(z_0)$ exists.

As an augmented notation, we often write

$$\Delta w = f(z + \Delta z) - f(z)$$

and we use the derivative notation

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

## 8.2  Examples

*Example: suppose $f(z) = z^2$. Then,*

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} 2z + \Delta z = 2z$$

*Example: suppose $f(z) = \overline{z}$. Then,*

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

*and we showed earlier that this limit does not exist – it has different values depending on the angle of approach.*

*Example: suppose $f(z) = |z|^2$. Then,*

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z + \Delta z} - z\overline{z})}{\Delta z} = \lim_{\Delta z \to 0} \overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}$$

*We see, once again, that the $\frac{\overline{\Delta z}}{\Delta z}$ term stops us from obtaining a derivative. However, this term is not present if $z = 0$ – at this point, $dw/dz$ is zero. Thus, the derivative exists only at $z = 0$, and $f'(0) = 0$.*

## 8.3   Notes

We've seen that it is possible for a function to be differentiable at a point but nowhere else in a neighbourhood of that point. We can even note that our function, $f(z) = |z|^2$, can be written as

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0$$

so we have no trouble finding the partial derivatives, but the function is still not differentiable aside from $z = 0$.

We've also seen that our troublesome function, $f(z) = |z|^2$, was continuous but not differentiable. However, we can state that **differentiability implies continuity** at a point. We can see this easily in the numerator of the derivative limit: $f(z)$ must approach the point $f(z_0)$ as $z$ approaches $z_0$, which implies continuity.

# 9 Frame 20: Differentiation Formulas

The derivative definition in the previous section is extremely similar to the definition for real-valued functions. Similar basic differentiation formulas can be derived and used for complex functions. This section will cover the most useful formulas.

If $c$ is a complex constant and $f$ is a function that is differentiable at $z$, then:

$$\frac{d}{dz}c = 0$$

$$\frac{d}{dz}z = 1$$

$$\frac{d}{dz}[cf(z)] = cf'(z)$$

If $n$ is a non-zero integer,

$$\frac{d}{dz}z^n = nz^{n-1}$$

If two functions $f$ and $g$ are differentiable at $z$, then

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$$

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$$

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - g'(z)f(z)}{g^2(z)} \text{ if } g(z) \neq 0$$

Finally, when composing functions (such as $F(z) = g[f(z)]$), the standard chain rule applies:

$$F'(z_0) = g'[f(z_0)]f'(z_0)$$

or, in a different form, if $w = f(z)$ and $W = F(z)$,

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}$$

# 10  Frame 21: Cauchy-Riemann Equations

## 10.1  Basic proof

We will obtain a formula in this chapter that a function must satisfy at a point if its derivative exists there.

First, we can split up the following points into real and imaginary parts:

$$z_0 = x_0 + iy_0, \quad \Delta z = \Delta x + i\Delta y$$

Then, the term $\Delta w$ becomes

$$\begin{aligned}
\Delta w &= f(z_0 + \Delta z) - f(z_0) \\
&= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]
\end{aligned}$$

Next, we know that we can split up the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

into its real and imaginary components

$$f'(z_0) = \lim_{\Delta z \to 0} \left( \Re \frac{\Delta w}{\Delta z} \right) + i \lim_{\Delta z \to 0} \left( \Im \frac{\Delta w}{\Delta z} \right)$$

We've seen several times that this expression must hold as $\Delta z$ tends to zero in any direction. In particular, we can use the test points $(\Delta x, 0)$ and $(0, \Delta y)$ to test this derivative. In the first case, $\Delta y = 0$, so the derivative is

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\
&= u_x(x_0, y_0) + iv_x(x_0, y_0)
\end{aligned}$$

where $u_x$ and $v_x$ are the partial derivatives with respect to $x$. Similarly, we can set $\Delta x = 0$ and obtain

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{i\Delta y} \\
&= \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta y} - i \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta y} \\
&= v_y(x_0, y_0) - iu_y(x_0, y_0)
\end{aligned}$$

where $u_y$ and $v_y$ are the partial derivatives with respect to $y$.

Now, since these two values must be equal, we require two conditions for the derivative to exist

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$

and

$$u_y(x_0, y_0) = -v_x(x_0, y_0)$$

These are the **Cauchy-Riemann equations**. They are necessary conditions for a derivative to exist – if they are not satisfied at a point, the derivative does not exist there. Note, however, that the derivative is not guaranteed to exist – the Cauchy-Riemann equations are not a sufficient condition.

## 10.2    Formal theorem

We can summarize the above result.

*Suppose that*

$$f(z) = u(x, y) + iv(x, y)$$

*and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then, the partial derivatives of $u$ and $v$ must exist at $(x_0, y_0)$, and they must satisfy the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

*there. Also, the derivative can be written as*

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

## 10.3    Examples

*Example: to illustrate the Cauchy-Riemann equations, consider the function*

$$f(z) = z^2 = (x^2 - y^2) + i(2xy)$$

*The real and imaginary functions are*

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

*and their partial derivatives are*

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x$$

*Finally, we can write*

$$f'(z) = u_x + iv_x = 2x + i2y = 2z$$

*as expected.*

*Example: for the function*

$$f(z) = |z|^2$$

*we have the real and imaginary components*

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0$$

*In order for the Cauchy-Riemann equations to hold at a point, we require that*

$$u_x = 2x = 0$$
$$u_y = 2y = 0$$

*or, in other words, the derivative does not exist anywhere except possibly for the point $(0, 0)$.*

# 11 Frame 22 – Sufficient Conditions for Differentiability

## 11.1 Extensions on Cauchy-Riemann

In the previous frame, we derived the Cauchy-Riemann equations, which are necessary for differentiability at a point. However, they are not sufficient. We can add some continuity conditions to improve the theorem as follows:

*Suppose that a function*

$$f(z) = u(x, y) + iv(x, y)$$

*is defined in some $\epsilon$ neighbourhood of a point $z_0$. If the partial derivatives $u_x$, $u_y$, $v_x$, and $v_y$ exist everywhere in the neighbourhood, are continuous at $z_0$, and satisfy the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

*then the derivative $f'(z_0)$ exists, and it has the value*

$$f'(z_0) = u_x(z_0) + iv_x(z_0)$$

## 11.2 Examples

*Example: suppose that*

$$f(z) = e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

*Then, the partial derivatives are*

$$u_x = e^x \cos y = v_y, \quad u_y = -e^x \sin y = -v_x$$

*These derivatives are continuous everywhere and the Cauchy-Riemann equations are satisfied, so $f'(z_0)$ exists everywhere, and it has the value*

$$f'(z) = e^x \cos y + ie^x \sin y = e^z$$

*Example: suppose that*

$$f(z) = |z|^2 = (x^2 + y^2) + i0$$

*We saw before that this function has a derivative at $z = 0$. However, at all other points, the Cauchy-RIemann equations are not satisfied, so the derivative does not exist anywhere else.*

# 12  Frame 23 – Polar Coordinates

## 12.1  Basic concepts

We can restate the theorem from the previous section using the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

If we are using this choice of coordinates, then our goal is to express the first-order derivatives $u_r, u_\theta, \ldots$ in terms of $u_x, u_y, \ldots$ to allow us to rewrite the Cauchy-Riemann equations. We can do this using the chain rule:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

or, more simply,

$$u_r = u_x \cos\theta + u_y \sin\theta, \quad u_\theta = -u_x r \sin\theta + u_y r \cos\theta$$

The same applies to $v$. If the Cauchy-Riemann equations hold, then we require that

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

at $z_0$. The reverse is also true – if these hold, then the Cauchy-Riemann equations hold too – so these equations are equivalent.

## 12.2  Formal theorem

The above concepts can be stated as follows:

*Suppose that the function*

$$f(z) = u(r, \theta) + iv(r, \theta)$$

*is defined in some $\epsilon$ neighbourhood of $z_0 = r_0 e^{i\theta_0}$ ($r_0 \neq 0$). If the partial derivatives of $u$ and $v$ are continuous throughout the neighbourhood and they satisfy the polar Cauchy-Riemann equations*

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

*then $f'(z_0)$ exists and has the value*

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

## 12.3   Examples

*Example: consider the function*

$$f(z) = \frac{1}{z} = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$$

*Since the derivatives are*

$$ru_r = -\frac{\cos\theta}{r} = v_\theta, \quad u_\theta = -\frac{\sin\theta}{r} = v_r$$

*the derivative of $f$ exists for all $z \neq 0$, with the value*

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}$$

*Example: suppose that $r > 0$ and $\alpha$ is a constant. Then, the function*

$$f(z) = r^{1/3}e^{i\theta/3} = r^{1/3}\cos\frac{\theta}{3} + ir^13\sin\frac{\theta}{3} \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

*has a derivative everywhere in the domain. Here,*

$$ru_r = \frac{r^{1/3}}{3}\cos\frac{\theta}{3} = v_\theta, \quad u\theta = -\frac{r^{1/3}}{3}\sin\frac{\theta}{3} = -rv_r$$

*so the derivative is*

$$f'(z) = e^{-i\theta}\left[\frac{1}{3r^{3/2}}\cos\frac{\theta}{3} + i\frac{1}{3r^{3/2}}\sin\frac{\theta}{3}\right]$$

$$= \frac{1}{3(r^{1/3}e^{i\theta/3})^2}$$

$$= \frac{1}{3f^2(z)}$$

# 13 Frame 24 – Analytic Functions

## 13.1 Definitions

A function $f$ is **analytic at a point** $z_0$ if it has a derivative at each point in some neighbourhood of $z_0$. By extension, if $f$ is analytic at $z_0$, it must be analytic at each point in some neighbourhood of $z_0$. A function is **analytic in an open set** if it has a derivative everywhere in that set. For example, $f(z) = 1/z$ is analytic at each non-zero point, and $f(z) = |z|^2$ is analytic nowhere.

An **entire** function is a function that is analytic at each point in the finite complex plane. For example, a polynomial's derivative exists everywhere, so every polynomial is analytic.

If a function is not analytic at a point $z_0$ but is analytic at some point in every neighbourhood of $z_0$, then $z_0$ is called a **singular point** (or **singularity**) of $f$. For example, $f(z) = 1/z$ has a singular point at $z = 0$.

## 13.2 Extensions

We can obtain some useful rules about analytic functions by looking at the conditions for differentiability.

Here, suppose $f$ and $g$ are two analytic functions in a domain $D$. Since their sum, $f + g$, their product, $fg$, and their quotient, $f/g$, are all differentiable (provided the denominator does not vanish in the last case), all three of these new functions are also analytic in $D$.

We also know that the chain rule states that the composition $g[f(z)]$ is differentiable. Thus, the composition of two analytic functions is also also analytic.

We can derive another useful property about analytic functions:

*If $f'(z) = 0$ everywhere in a domain $D$, then $f(z)$ must be constant through $D$.*

This can be shown by writing $f$ in the form

$$f(z) = u(x,y) + iv(x,y)$$

Since $f'(z) = u_x + iv_x = 0$, both of these partial derivatives must be zero. Then, from Cauchy-Riemann,

$$u_x = u_y = v_x = v_y = 0$$

at each point in $D$. Next, if we consider any line segment in $D$, we know that the derivative of $u$ along its arc length $s$ is

$$\frac{du}{ds} = (\text{grad } u) \cdot U$$

where $U$ is a unit vector along the line segment. However, since $u_x = u_y = 0$, grad $u$ is also zero, so $u$ is constant on any line segment in $D$. Then, since we can connect any points in $D$ with a finite number of line segments, $u$ must have a constant value in $D$: ie,

$$u(x,y) = a$$

We can apply the same logic to get $v(x,y) = b$. Then, we conclude that $f(z) = a + bi$ at each point in $D$.

# 14  Frame 25 – Examples of Analyticity

This section will give some examples of analytic functions.

**Example 1:** the function

$$f(z) = \frac{z^3 + 4}{(z^2 - 3)(z^2 + 1)}$$

is analytic everywhere except for the points where the denominator vanishes due to the differentiation formulas available for polynomials, products, and quotients. There are four singular points at $z = \pm\sqrt{3}$ and $z = \pm i$.

**Example 2:** the function

$$f(z) = \cosh x \cos y + i \sinh x \sin y$$

has the component functions

$$u(x, y) = \cosh x \cos y, \quad v(x, y) = \sinh x \sin y$$

and the partial derivatives

$$u_x = \sinh x \cos y = v_y, \quad u_y = -\cosh x \sin y = -v_x$$

so $f$ is an entire function.

**Example 3:** suppose that a function

$$f(z) = u(x, y) + iv(x, y)$$

and its conjugate

$$\overline{f(z)} = u(x, y) - iv(x, y)$$

are both analytic in a domain $D$. From applying the Cauchy-Riemann equations to $f$, we see that

$$u_x = v_y, \quad u_y = -v_x$$

and, applying them to the second function,

$$u_x = -v_y, \quad u_y = v_x$$

Solving these equations gives $u_x = v_x = 0$. Thus, the derivative of $f$ is

$$f'(z) = 0 + i0 = 0$$

so $f$ is constant throughout $D$.

**Example 4:** suppose that a function has a constant magnitude – ie,

$$|f(z)| = c$$

for all $z$ in $D$. We can show that $f(z)$ is constant in $D$. First, if $c$ is zero, then we are done – 0 is constant. Otherwise, we can write

$$f(z)\overline{f(z)} = c^2$$

Since $c \neq 0$, we know that $f(z)$ is never zero. Then, we can write

$$\overline{f(z)} = \frac{c^2}{f(z)}$$

and since the quotient of two analytic functions is also analytic, both $f$ and its conjugate are analytic, so $f$ must be constant (see previous example).

# 15 Frame 26: Harmonic Functions

## 15.1 Definition

A **harmonic** function $H$ of two real variables $x$ and $y$ is a function that has continuous first- and second-order derivatives that satisfy Laplace's equation:

$$H_{xx}(x,y) + H_{yy}(x,y) = 0$$

*Example: it is easy to verify that $T(x,y) = e^{-y} \sin x$ is harmonic over the entire xy plane.*

## 15.2 A source of harmonic functions

The following theorem provides a source of harmonic functions.

*Theorem: if a function $f(z) = u(x,y) = iv(x,y)$ is analytic in a domain $D$, then its component functions $u$ and $v$ are harmonic in $D$.*

This theorem can be proved by differentiating the Cauchy-Riemann equations with respect to $x$ and $y$, giving

$$u_{xx} = v_{yx}, \quad u_{yx} = -v_{xx}$$
$$u_{xy} = v_{yy}, \quad u_{yy} = -v_{xy}$$

Then, since $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$, it follows that

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$$

so $u$ and $v$ are harmonic in $D$.

*Example: the function*

$$f(z) = e^{-y} \sin x - ie^{-y} \cos x$$

*is entire, so its real component must be harmonic in the entire xy plane. This matches the result from the previous example.*

*Example: since the function*

$$f(z) = \frac{i}{z^2}$$

*is analytic for all $z \neq 0$, we can write the function as*

$$f(z) = \frac{i}{z^2} \frac{\overline{z}^2}{\overline{z}^2} = \frac{i\overline{z}^2}{|z|^4} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2}$$

*and we obtain the two harmonic functions*

$$u(x, y) = \frac{2xy}{(x^2 + y^2)^2}$$

$$v(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

## 15.3   Harmonic conjugates

If two functions $u$ and $v$ are harmonic in a domain $D$ and their first-order derivatives satisfy Cauchy-Riemann in $D$, then we say that $v$ is a **harmonic conjugate** of $u$. We can say that a function $f$ is analytic in $D$ iff $v$ is a harmonic conjugate of $u$.

*Example: if we have the functions*

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

*then we know that these functions are the components of $f(z) = z^2$, so v must be a harmonic conjugate of u. However, we verified earlier that $2xy + i(x^2 - y^2)$ is analytic nowhere, so u is not a harmonic conjugate of v.*

*Example: the component function*

$$u(x, y) = y^3 - 3x^2y$$

*is harmonic because $u_{xx} + u_{yy} = 0$ in the entire xy plane. To find a harmonic conjugate, we can use the Cauchy-Riemann equations*

$$u_x = v_y, \quad u_y = -v_x$$

*so we can write, using the first equation,*

$$v_y(x, y) = -6xy$$

*which integrates to give*

$$v(x, y) = -3xy^2 + \phi(x)$$

*Then, the other equation gives*

$$3x^2 - 3y^2 = -3y^2 + \phi'(x)$$

*so*

$$v(x, y) = -3xy^2 + x^3 + C$$

*These equations correspond to the form*

$$f(z) = i(z^3 + C)$$