1 Frame 68 – Isolated Singular Points

1.1 Definition

Earlier, we defined a **singular point** of a function f as a point z_0 where f is not analytic, but f is analytic at some point in every neighbourhood of z_0. Additionally, we will define an **isolated** singular point as such a point where there exists a deleted neighbourhood on which f is analytic.

1.2 Examples

The function

$$f(z) = \frac{z+1}{z^3(z^2+1)}$$

has three singular points at $z = 0$ *and* $z = \pm i$*. These three points are isolated singularities.*

The principal branch of the logarithm

$$\operatorname{Log} z = \ln r + i\Theta \quad (-\pi < \theta < \pi)$$

has a singular point at $z = 0$*. However, this is not an isolated singular point, since every neighbourhood of* 0 *also contains some points on the negative real axis, where the function is not analytic.*

The function

$$f(z) = \frac{1}{\sin(\pi/z)}$$

has singular points at $z = 0$ *and* $z = 1/n$ *for all integers* n*. All of these singular points are isolated except for the one at* $z = 0$ *– every neighbourhood of* 0 *also contains a point* $z = 1/m$ *because we can find such a point*

$$0 < 1/m < \epsilon$$

for each ϵ*.*

1.3 Important Points

Note that if a function has a finite number of singular points, then all of these must be isolated – we can make a neighbourhood around each singular point that does not contain any others.

Finally, note that we may also refer to the point at infinity as an isolated singular point. This happens if there exists a positive R_1 such that there are no singularities in the region

$$R_1 < |z| < \infty$$

2 Frame 69 – Residues

2.1 Definition

Suppose that f is a function with an isolated singular point at z_0. Then, there exists some deleted neighbourhood

$$0 < |z - z_0| < R_2$$

where f is analytic. On this domain, we can express $f(z)$ as the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots + \frac{b_n}{(z - z_0)^n} + \ldots$$

These coefficients come from various integral representations. In particular,

$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

where C is a positively oriented, simple, closed contour around z_0 in the deleted neighbourhood described above. When $n = 1$, this expression becomes

$$\int_C f(z) dz = 2\pi i b_1$$

We say that the complex number b_1 is called the **residue** of f at the isolated singular point z_0, and we write

$$b_1 = \operatorname{Res}_{z=z_0} f(z)$$

so the contour integral around z_0 becomes

$$\int_C f(z) dz = 2\pi i \operatorname{Res}_{z=z_0} f(z)$$

2.2 Examples

Example 1: We can evaluate the integral

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz$$

where C is the positively-oriented unit circle. First, we note that the integrand is analytic everywhere except the origin, so the Laurent series converges on $0 < |z| < \infty$. Then, we can write

$$\begin{aligned} z^2 \sin(1/z) &= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (1/z)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{1-2n} \end{aligned}$$

so the coefficient b_1 *is* $-1/3!$. *Thus,*

$$\int_C z^2 \sin\left(\frac{1}{z}\right) dz = 2\pi i \left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}$$

Example 2: We can repeat the previous problem for the integral

$$\int_C e^{1/z^2}$$

Since this series is

$$\begin{aligned} e^{1/z^2} &= \sum_{n=0}^{\infty} \frac{(1/z^2)^n}{n!} \\ &= \sum_{n=0}^{\infty} \frac{1}{n! z^{2n}} \\ &= 1 + \frac{1}{z^2} + \frac{1}{2z^4} + \dots \end{aligned}$$

the residue at 0 *is* 0, *so*

$$\int_C e^{1/z^2} = 0$$

Example 3: Finally, we can use residues to evaluate the integral

$$\int_C \frac{1}{z(z-2)^4} dz$$

around the positively-oriented circle $|z-2| = 1$. *The Laurent series is*

$$\begin{aligned} \frac{1}{z(z-2)^4} &= \frac{1}{(z-2)^4} \frac{1}{2+(z-2)} \\ &= \frac{1}{2(z-2)^4} \frac{1}{1 - \frac{-(z-2)}{2}} \\ &= \frac{1}{2(z-2)^4} \sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} (z-2)^n \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} (z-2)^{n-4} \end{aligned}$$

Thus, the coefficient of $1/(z-2)$ *is at* $n = 3$, *and*

$$b_1 = -\frac{1}{16}$$

so

$$\int_C \frac{1}{z(z-2)^4} dz = 2\pi i \left(-\frac{1}{16}\right) = -\frac{\pi i}{8}$$

3 Frame 70 – Cauchy's Residue Theorem

3.1 Theorem

If a function has a finite number of singularities inside a contour, then we can use residues to simplify these contour integrals. The following theorem is Cauchy's residue theorem:

Theorem: Suppose that C is a positively-oriented, simple, closed contour. If a function f is analytic on C and has a finite number of singularities at z_k (where $k = 1, 2, \ldots, n$) inside C, then

$$\int_C f(z)dz = 2\pi i \sum_{k=1}^{\infty} \operatorname{Res}_{z=z_k} f(z)$$

Proof: suppose that the contours C_k are small, positively-oriented circles centered at the points z_k. Then, the region $C - \sum C_k$ is a multiply connected domain over which f is analytic, so

$$\int_C f(z)\ dz - \sum_{k=1}^{\infty} \int_{C_k} f(z)\ dz = 0$$

Then, since

$$\int_{C_k} f(z)\ dz = 2\pi i \operatorname{Res}_{z=z_k} f(z)$$

the integral can be written as in the theorem.

3.2 Example

If C is the counterclockwise circular contour $|z| = 2$, then we can evaluate the integral

$$\int_C \frac{5z-2}{z(z-1)} dz$$

using Cauchy's residue theorem. First, we can find the residue B_1 at $z = 0$ by writing out the function as the series

$$\begin{aligned}
\frac{5z-2}{z(z-1)} &= \frac{5z-2}{z} \cdot \frac{-1}{1-z} \\
&= \left(5 - \frac{2}{z}\right)(-1 - z - z^2 - \ldots) \\
&= \frac{2}{z} - 3 - 3z - \ldots
\end{aligned}$$

so $B_1 = 2$. Then, we can find the residue B_2 at $z = 1$ by writing

$$\begin{aligned}
\frac{5z-2}{z(z-1)} &= \frac{5(z-1)+3}{z-1} \cdot \frac{1}{1+(z-1)} \\
&= \left(5 + \frac{3}{z-1}\right)\left(1-(z-1)+(z-1)^2 - \dots\right) \\
&= \frac{3}{z-1} + 2 - 2(z-1) + \dots
\end{aligned}$$

so $B_2 = 3$, and

$$\int_C \frac{5z-2}{z(z-1)} = 2\pi i(2+3) = 10\pi i$$

Alternatively, we could use partial fractions to write

$$\frac{5z-2}{z(z-1)} = \frac{2}{z} + \frac{3}{z-1}$$

and immediately discover that $B_1 = 2$ and $B_2 = 3$.

4 Frame 71 – Residue at Infinity

4.1 Definition

Next, suppose that a function f is analytic everywhere in the plane except for a finite number of singular points and C is a positively oriented, simple, closed contour containing all of these points. Then, we can make a circle containing all of these points ($|z| = R_1$), and f must be analytic in the domain $R_1 < |z| < \infty$.

We can define the residue at infinity by making another circular contour, this one in the **negative direction** (ie: keeping infinity on the left), with a radius $R_0 > R_1$. Then, the **residue at infinity** is

$$\int_{C_0} f(z)\ dz = 2\pi i \operatorname{Res}_{z=\infty} f(z)$$

To simplify this integral, we know that

$$\int_C f(z)\ dz = \int_{-C_0} f(z)\ dz = -\int_{C_0} f(z)\ dz$$

so

$$\int_C f(z)\ dz = -2\pi i \operatorname{Res}_{z=\infty} f(z)$$

4.2 Main Theorem

We can find the residue at infinity by writing

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n \quad (R_1 < |z| < \infty)$$

where

$$c_n = \frac{1}{2\pi i} \int_{-C_0} \frac{f(z)}{z^{n+1}} dz$$

Then, if we modify the series to represent $f(1/z)/z^2$, we can write

$$\frac{1}{z^2} f\left(\frac{1}{z}\right) = \sum_{n=-\infty}^{\infty} \frac{c_n}{z^{n+2}} = \sum_{n=-\infty}^{\infty} \frac{c_{n-2}}{z^n}$$

so the residue of this modified function at zero is nearly the same as the original residue at infinity. Symbolically,

$$\operatorname{Res}_{z=\infty} f(z) = -2\pi i \operatorname{Res}_{z=0} \left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

This result is shown in the following theorem:

Theorem: If a function f is analytic everywhere except a finite number of singular points, and all of these points are interior to C, then

$$\int_C f(z)\;dz = 2\pi i \operatorname{Res}_{z=0}\left[\frac{1}{z^2} f\left(\frac{1}{z}\right)\right]$$

4.3 Example

We can now re-do the problem from the previous section. Since all of the singular points were contained inside $|z| = 2$, we can write

$$\begin{aligned}
\frac{1}{z^2} f\left(\frac{1}{z}\right) &= \frac{1}{z^2}\frac{5/z - 2}{1/z \cdot (1/z - 1)} \\
&= \frac{5 - 2z}{z(1-z)} \\
&= \left(\frac{5}{z} - 2\right)(1 + z + z^2 + \dots) \\
&= \frac{5}{z} + 3 + 3z + \dots
\end{aligned}$$

so the residue at infinity is 5, and

$$\int_C \frac{5z-2}{z(z-1)} dz = 2\pi i(5) = 10\pi i$$

5 Frame 72 – The Three Singular Points

We saw in the previous section that a function f with an isolated singular point at z_0 can be written as the Laurent series

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_n}{(z-z_0)^n} + \ldots$$

on the domain $0 < |z - z_0| < R$. We refer to the negative powers as the **principal part** of f at z_0. We can use the qualities of this principal part to classify a singular point into one of three categories.

5.1 Poles

If the principal part has at least one non-zero term but a finite number of terms, as in

$$f(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n + \frac{b_1}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \cdots + \frac{b_m}{(z-z_0)^m}$$

then the isolated singular point is referred to as a **pole of order** m. A first-order pole is called a **simple pole**.

Example 1: the function

$$\frac{z^2-2z+3}{z-2} = \frac{z(z-2)+3}{z-2} = 2 + (z-2) + \frac{3}{z-2}$$

has a simple pole at $z = 2$. Its residue there is 3.

Example 2: the function

$$\frac{1}{z^2(1+z)} = \frac{1}{z^2}(1 - z + z^2 - z^3 + \ldots) = \frac{1}{z^2} - \frac{1}{z} + 1 - z + \ldots$$

has a pole of order 2 at the origin. Its residue there is -1.

Example 3: the function

$$\frac{\sinh z}{z^4} = \frac{1}{z^4}\left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \ldots\right) = \frac{1}{z^3} + \frac{1}{3! \cdot z} + \frac{z}{5!} + \ldots$$

has a third-order pole at the origin with a residue of $1/6$.

5.2 Removable Singular Points

If every term of the principal part is zero, then we say that z_0 is a **removable singular point**. This allows us to (re)define f at the point z_0 as

$$f(z_0) = a_0$$

so that the function becomes defined over the non-deleted disk, removing the singularity. Note that the residue at any removable singular point is zero.

Example 4: we can write

$$\begin{aligned}\frac{1-\cos z}{z^2} &= \frac{1}{z^2}\left[1-\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\frac{z^6}{6!}+\dots\right)\right]\\ &= \frac{1}{z^2}\left[\frac{z^2}{2!}-\frac{z^4}{4!}+\frac{z^6}{6!}-\dots\right]\\ &= \frac{1}{2}-\frac{z^2}{4!}+\frac{z^4}{6!}\end{aligned}$$

so if we assign the value $f(0) = 1/2$, then the function becomes entire.

5.3 Essential Singular Points

If the principal part has an infinite number of non-zero terms, we say that z_0 is an **essential singular point** of f.

Example 5: the function

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n! \cdot z^n} = 1 + \frac{1}{z} + \frac{1}{2! \cdot z^2} + \dots$$

has an essential singular point at the origin, with a residue of 1.

We can also briefly state **Picard's theorem**: *In every neighbourhood of an essential singular point, a function assumes every finite value (except possibly zero) an infinite number of times.*

6 Frame 73 – Residues at Poles

The following theorem makes it more convenient to find residues at poles.

Theorem: If $\phi(z)$ is a function that is analytic and non-zero at z_0, then the isolated singular point at z_0 of the function

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

is a pole of order m. Also,

$$\text{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

In particular, if $m = 1$,

$$\text{Res}_{z=z_0} f(z) = \phi(z_0)$$

To prove this, we can write the Taylor expansion of $\phi(z)$ as

$$\phi(z) = \phi(z_0) + \frac{\phi'(z_0)}{1!}(z-z_0) + \frac{\phi''(z_0)}{2!}(z-z_0)^2 + \cdots + \frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + \ldots$$

so the Laurent expansion of f is

$$f(z) = \frac{\phi(z_0)}{(z-z_0)^m} + \frac{\phi'(z_0)/1!}{(z-z_0)^{m-1}} + \frac{\phi''(z_0)/2!}{(z-z_0)^{m-2}} + \cdots + \frac{\phi^{(m-1)}(z_0)/(m-1)!}{z-z_0} + \ldots$$

and the residue is the expression theorized above.

7 Frame 74 – Examples of Poles

This section will contain several examples of the theorem from the previous section.

7.1 Example 1

The function

$$f(z) = \frac{z+1}{z^2+9}$$

has singular points at $z = \pm 3i$. At $z = 3i$, we can write the function as

$$f(z) = \frac{\phi(z)}{z-3i} \quad \text{where} \quad \phi(z) = \frac{z+1}{z+3i}$$

so the residue B_1 at this point is

$$B_1 = \phi(3i) = \frac{1+3i}{6i} = \frac{3-i}{6}$$

Similarly, the residue B_2 at $z = -3i$ is

$$B_2 = \frac{3+i}{6}$$

7.2 Example 2

The function

$$f(z) = \frac{z^3+2z}{(z-i)^3}$$

can be written as

$$f(z) = \frac{\phi(z)}{(z-i)^3} \quad \text{where} \quad \phi(z) = z^3 + 2z$$

Note that $\phi(i) \neq 0$, so f has a pole of order 3 at $z = i$. The residue here is

$$B = \frac{\phi''(i)}{2!} = \frac{6z|_{z=i}}{2} = 3i$$

7.3 Example 3

The function

$$f(z) = \frac{(\log z)^3}{z^2+1}$$

(where $\log z$ uses the branch $0 < \theta < 2\pi$) has simple poles at $z = \pm i$. To find the residue at $z = i$, we can write

$$f(z) = \frac{\frac{(\log z)^3}{z+i}}{z-i}$$

Then, since the numerator is analytic at $z = i$, we can find the residue as

$$B = \frac{(\log i)^3}{2i} = \frac{(0 + i\pi/2)^3}{2i} = -\frac{\pi^3}{16}$$

7.4 Example 4

Sometimes, it is easier to characterize poles and find residues using Laurent series. For example, the function

$$f(z) = \frac{\sinh z}{z^4}$$

can **not** be written as

$$f(z) = \frac{\phi(z)}{z^4}, \quad \phi(z) = \sinh z$$

because $\sinh 0 = 0$. This tells us that the origin is not, in fact, a pole of order 4. It is simpler to write out the Laurent series for this function.

7.5 Example 5

The origin is an isolated singular point of the function

$$f(z) = \frac{1}{z(e^z - 1)}.$$

We can write the Maclaurin series for $z(e^z - 1)$ as

$$\begin{aligned} z(e^z - 1) &= z\left(\frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) \\ &= z^2\left(1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots\right) \end{aligned}$$

which allows us to write the function as

$$f(z) = \frac{\phi(z)}{z^2}, \quad \phi(z) = \frac{1}{1 + z/2! + z^2/3! + \dots}$$

Then, the residue at the origin is $\phi'(0)$, which is

$$\begin{aligned}
B &= \phi'(0) \\
&= \frac{d}{dz}\phi(z)\Big|_{z=0} \\
&= \frac{-(1/2! + 2z/3! + \dots)}{(1 + z/2! + z^2/3! + \dots)^2}\Big|_{z=0} \\
&= \frac{-1/2}{1} \\
&= -1/2
\end{aligned}$$

8 Frame 75 – Zeros of Analytic Functions

8.1 Definition

Suppose that a function f is analytic at a point z_0, which means that its derivatives of all orders exist at this point. Then, if $f(z_0) = 0$, but eventually for some integer m, $f^{(m)}(z_0) \neq 0$, we say that f has a **zero of order** m at z_0.

We can characterize zeroes using the following theorem:

Theorem: Suppose that f is analytic at z_0. It has a zero of order m at the point z_0 iff there is an analytic, non-zero function g at z_0 such that

$$f(z) = (z - z_0)^m g(z)$$

Both conditions of this theorem can be derived from the Taylor series expansions for f and g.

8.2 Examples

Example 1: the polynomial

$$f(z) = z^3 - 8 = (z - 2)(z^2 + 2z + 4)$$

has a first-order zero at $z = 2$, since

$$z^2 + 2z + 4\Big|_{z=2} = 12 \neq 0$$

Example 2: the entire function

$$f(z) = z(e^z - 1)$$

satisfies $f(0) = f'(0) = 0$ and $f''(0) = 2 \neq 0$. This lets the function be written as

$$f(z) = z^2\,(g(z))$$

where

$$g(z) = \begin{cases} (e^z - 1)/z, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

Note that this function is entire, as we showed earlier.

8.3 Isolated Zeros

The next theorem states that zeroes of an analytic function are isolated.

Theorem: suppose that a function f is analytic and zero at a point z_0, but not identically zero in any neighbourhood of z_0. Then, there is a deleted neighbourhood of z_0 where $f(z) \neq 0$.

To prove this, notice that some derivative of f must be non-zero – otherwise, the function would be zero everywhere. Thus, f must have a zero of finite order m, and

$$f(z) = (z - z_0)^m g(z)$$

Then, since g must be analytic, it is continuous and non-zero in a neighbourhood of z_0. Therefore, $f(z) \neq 0$ in some neighbourhood of z_0.

Finally, we can look at what happens when zeroes are not isolated.

Theorem: suppose that f is analytic throughout a neighbourhood N_0 of a point z_0 and $f(z) = 0$ at each point in a domain D or line segment L which contains z_0. Then, $f(z) = 0$ in N_0.

Proof: first, there must be some neighbourhood N of z_0 where $f(z) = 0$. If there was not, then there could not be a domain or line segment with z_0 where $f(z) = 0$, according to the previous theorem.

Then, if there is some neighbourhood where $f(z) = 0$, its Taylor series is identically zero, so it must be zero everywhere in N_0 as well.

9 Frame 76 – Zeros and Poles

Next, we can use our knowledge of zeroes to study poles.

9.1 Reciprocals of Poles

Theorem: suppose that two functions p and q are analytic at z_0. Also, suppose that $p(z_0) \neq 0$ and $q(z_0)$ has a zero of order m at z_0. Then, the quotient

$$\frac{p(z)}{q(z)}$$

has a pole of order m at z_0.

This can easily be shown: if q has a zero of order m at z_0, this must be an isolated singularity. Then, we can write

$$q(z) = (z - z_0)^m g(z)$$

where g is analytic and non-zero at z_0. This allows us to write

$$\frac{p(z)}{q(z)} = \frac{\phi(z)}{(z - z_0)^m} \quad \text{where} \quad \phi(z) = \frac{p(z)}{g(z)}$$

so this quotient has a pole of order m.

The function

$$z(e^z - 1)$$

has a second-order zero at the origin, so the function

$$\frac{1}{z(e^z - 1)}$$

has a second-order pole there.

9.2 Residues

We can now use this quotient to simplify residue calculations.

Theorem: suppose that p and q are analytic at z_0. If

$$p(z_0) \neq 0$$
$$q(z_0) = 0$$
$$q'(z_0) \neq 0$$

then z_0 is a simple pole of the quotient, and

$$\text{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{q'(z_0)}$$

To prove this, we first know that z_0 is a first order zero of p – the derivative is non-zero. Then, we can write

$$q(z) = (z - z_0)g(z)$$

and

$$\frac{p(z)}{q(z)} = \frac{p(z)/g(z)}{z - z_0}$$

so

$$\mathrm{Res}_{z=z_0} \frac{p(z)}{q(z)} = \frac{p(z_0)}{g(z_0)} = \frac{p(z_0)}{q'(z_0)}$$

9.3 Examples

Example: the function

$$f(z) = \cot z = \frac{\cos z}{\sin z}$$

has simple poles at $z = n\pi$. Here, the residue is

$$B = \frac{\cos(n\pi)}{\cos(n\pi)} = 1$$

Example: the function

$$f(z) = \frac{\tanh z}{z^2} = \frac{\sinh z}{z^2 \cosh z}$$

has a simple pole at $z = i\pi/2$. Here,

$$p\left(\frac{i\pi}{2}\right) = \sinh\left(\frac{i\pi}{2}\right) = i\sin\left(\frac{\pi}{2}\right) = i$$

$$q'\left(\frac{i\pi}{2}\right) = \left(\frac{i\pi}{2}\right)^2 \sinh\left(\frac{i\pi}{2}\right) = -i\frac{\pi^2}{4}$$

so

$$B = \frac{p(i\pi/2)}{q'(i\pi/2)} = -\frac{4}{\pi^2}$$

Example: the function

$$q(z) = z^4 + 4$$

has a zero at $z_0 = 1 + i$. Thus, at this point, the function

$$f(z) = \frac{z}{z^4 + 4}$$

has a residue of

$$B = \frac{z_0}{4z_0^3} = \frac{1}{4z_0^2} = \frac{1}{8i} = -\frac{i}{8}$$