# Contents

# 1 Frame 55 – Sequences and Convergence

## 1.1 Definitions

An **infinite sequence** of complex numbers,

$$z_1, z_2, \ldots, z_n, \ldots$$

has a **limit** if, for each positive number $\epsilon$, there exists a positive integer $n_0$ such that

$$|z_n - z| < \epsilon \quad \text{whenever} \quad n > n_0$$

Geometrically, this limit implies that for all $n > n_0$, each number $z_n$ in the sequence will be inside an $\epsilon$ neighbourhood of $z$.

A sequence can only have one limit, at most. When this limit exists, we say that the sequence **converges** to $z$, and we write

$$\lim_{n\to\infty} z_n = z$$

If a sequence has no limit, it **diverges**.

## 1.2 Components

*Theorem: If we write $z_n = x_n + iy_n$ and $z = x + iy$, then*

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

This theorem allows us to write

$$\lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

as long as the limits on either side of this equation exist.

## 1.3 Examples

*Example 1: we can evaluate the following limit easily:*

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^3 + i} &= \lim_{n\to\infty} \frac{1}{n^3} + i \lim_{n\to\infty} 1 \\
&= 0 + i \cdot 1 \\
&= i
\end{aligned}$$

*Example 2: Polar coordinates require some extra care. Looking at the sequence*

$$z_n = -2 + i\frac{(-1)^n}{n^2}$$

*we can see that*

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(-2) + i \lim_{n\to\infty} \frac{(-1)^n}{n^2} = -2$$

*However, we can find that the principal polar representation of these numbers is*

$$r_n = \sqrt{4 + \frac{1}{n^2}}$$
$$\Theta_n = \operatorname{Arg} z_n = \tan^{-1}\left(\frac{(-1)^n}{-2n^2}\right)$$

*Evaluating the first limit, we find that*

$$\lim_{n\to\infty} r_n = \sqrt{4} = 2$$

*which is fine. However, the second sequence does not converge. Looking at every second term, we see that*

$$\lim_{n\to\infty} \Theta_{2n} = \pi$$

*and*

$$\lim_{n\to\infty} \Theta_{2n-1} = -\pi$$

*so* $\Theta_n$ *diverges.*

# 2 Frame 56 – Series Convergence

## 2.1 Definitions

An infinite **series** of complex numbers,

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \ldots,$$

**converges** to the sum $S$ if the sequence of partial sums,

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N,$$

converges to $S$. If this is the case, then we can write

$$\sum_{n=1}^{\infty} z_n = S$$

Note that, since a sequence can have at most one limit, a series can have at most one sum. If a series does not converge, it **diverges**.

## 2.2 Properties – Components

First, as with sequences, we can split a series into its real and imaginary components.

*Theorem: If $z_n = x_n + iy_n$ and $S = X + iY$, then*

$$\sum_{n=1}^{\infty} z_n = S$$

*iff*

$$\sum_{n=1}^{\infty} x_n = X \quad \textit{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

To prove this, we can write the partial sums $S_N$ as

$$S_N = X_N + iY_N$$

where

$$X_N = \sum_{n=1}^{N} x_n \quad \text{and} \quad Y_N = \sum_{n=1}^{N} y_n s$$

Then, the series only converges to $S$ if

$$\lim_{N\to\infty} X_N = X \quad \text{and} \quad \lim_{N\to\infty} Y_N = Y$$

due to the theorem on sequences in the previous chapter. Thus, the theorem is proved.

## 2.3 Properties – Boundedness

The following corollary is a consequence of the previous theorem:

*Corollary 1: If a series of complex numbers converges, the summed terms $z_n$ converge to zero.*

This is due to the fact that

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

and, in order for these two terms to converge, $x_n$ and $y_n$ must converge to zero (from calculus). Thus,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0$$

This corollary implies that the terms within convergent series are bounded – that is, there exists a constant $M$ such that $|z_n| < M$ for all $n$.

## 2.4 Properties – Absolute Convergence

A series is **absolutely convergent** if the related series

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$$

converges. This has a simple implication:

*Corollary 2: If a series is absolutely convergent, it is convergent.*

To show this, consider the real component of the series. It can be written as

$$\sum_{n=1}^{\infty} x_n \le \left| \sum_{n=1}^{\infty} x_n \right| \le \sum_{n=1}^{\infty} |x_n| \le \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} = 0$$

so the real component must converge. The same is true of the imaginary component, so the corollary is proved.

## 2.5 Remainders

It is often helpful to define the sequence of **remainders** using the partial sums:

$$\rho_N = S - S_N$$

or $S = S_N + \rho_N$. Since we can write that

$$|S_N - S| = |\rho_N|$$

then a series is only convergent if the sequence of remainders tends to zero.

*Example: using remainders, we can verify that*

$$\sum_{n=0}^{\infty} z_n = \frac{1}{1-z} \quad \textit{whenver} \quad |z| < 1$$

*To do this, we recall that*

$$S_N(z) = 1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1-z}$$

*so*

$$\rho_N(z) = \frac{1}{1-z} - \frac{1 - z^{N+1}}{1-z} = \frac{z^N}{1-z}$$

*The moduli of these remaiders are*

$$|\rho_N(z)| = \frac{|z|^N}{|1-z|}$$

*so $\rho_N(z)$ tends to zero when $|z| < 1$.*

## 3 Frame 57 – Taylor Series

The following theorem is known as **Taylor's theorem**:

*Theorem: If a function $f$ is analytic throughout a disk $|z - z_0| < R_0$, then $f(z)$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

*where*

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

*This series converges to $f(z)$ when $z$ is in this disk.*

Taylor's theorem allows us to write

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \ldots$$

This is true for any function that is analytic at $z_0$: the requirement for analyticity states that $f$ must be analytic in some neighbourhood of $z_0$, so the disk mentioned in the theorem exists. In particular, entire functions can use arbitrarily large disks, ie:

$$|z - z_0| < \infty$$

so the series is convergent for all $z$ in the plane.

It can be shown that Taylor's series converges at every point inside the disk – no convergence tests are required. In fact, the smallest radius at which it does **not** converge is the nearest point where $f$ is not analytic.

If $z_0 = 0$ in a Taylor series, it is known as a **Maclaurin series**. Then, it takes the form

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

# 4 Frame 59 – Examples of Taylor Series

In this section, we will use the formula

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

to find the Maclaurin expansions of some common functions.

## 4.1 Example 1

The function $e^x$ is entire, so its Maclaurin expansion is valid for all $z$. Since

$$f^{(n)}(z) = e^z$$

each term is $a_n = 1/n!$, so we find that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that, if $z = x + i0$, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as expected.

We can use this result to find the Maclaurin series for the entire function $z^2 e^{3z}$. By replacing $z$ with $3z$ and multiplying through by $z^2$, we find

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n$$

## 4.2 Example 2

Using the expansion

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

we can find the Maclaurin series for $f(z) = \sin z$. To do this, we write

$$\begin{aligned}\sin z &= \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] \\ &= \frac{1}{2i}\sum_{n=0}^{\infty}[1 - (-1)^n]\frac{i^n}{n!} z^n\end{aligned}$$

Then, $1-(-1)^n$ is zero for $n$ even, so only taking odd terms gives

$$\begin{aligned}
&= \frac{1}{2i}\sum_{n=0}^{\infty} 2\frac{i^{2n+1}}{(2n+1)!}z^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}z^{2n+1}
\end{aligned}$$

This expansion can be used directly to find $\cos z$. Since

$$\cos z = \frac{d}{dz}\sin z$$

we can write

$$\begin{aligned}
\cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\frac{d}{dz}z^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(2n+1)z^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}z^{2n}
\end{aligned}$$

## 4.3 Example 3

Since

$$\sinh z = -i\sin(iz)$$

we can write

$$\begin{aligned}
\sinh z &= -i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(iz)^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(-1)^n z^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}z^{2n+1}
\end{aligned}$$

Similarly,

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!}z^{2n}$$

Also, note that $\cosh z = \cosh(z+2\pi i)$, so

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!}(z+2\pi i)^{2n}$$

### 4.4 Example 4

If $f(z) = \frac{1}{1-z}$, then

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$$

so

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

If we substitute $-z$ for this expression, we find that

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

and if we substitute $1-z$ instead, we find

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

Note that all three of these Taylor series have a radius of convergence of 1.

### 4.5 Example 5

Notice that the function

$$f(z) \frac{1}{z^3} \frac{1}{1+z^2}$$

does not have a Maclaurin series – it is not analytic at $z = 0$. However,

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \dots$$

so we can write

$$\begin{aligned} f(z) &= \frac{1}{z^3}(1 - z^2 + z^4 - z^6 + \dots) \\ &= z^{-3} - z^{-1} + z^1 - z^3 + \dots \end{aligned}$$

We refer to the first two terms as **negative powers** of $z$.

# 5 Frame 60 – Laurent Series

## 5.1 Definition

We saw in the previous section that we often able to find series representations of functions that are not analytic by using both positive and negative powers of $z$. These representations are known as **Laurent series**. The central theorem for these functions is Laurent's theorem:

*Theorem: Suppose that a function $f$ is analytic over an annular domain,*

$$R_1 < |z - z_0| < R_2$$

*Then, $f(z)$ has the series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=0}^{\infty} \frac{b_n}{(z - z_0)^n}$$

*where*

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$
$$b_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{-n+1}} dz$$

*and where $C$ is any positively-oriented contour in this domain containing $z_0$.*

## 5.2 Alternate Form

Note that we can write this series more simply by defining

$$c_n = \begin{cases} b_{-n}, & n < 0 \\ a_n, & n \geq 0 \end{cases}$$

and writing

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

Looking at the expressions for $a_n$ and $b_n$,

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

## 5.3 Observations

First, note that

$$b_n = \frac{1}{2\pi i} \int_C f(z)(z - z_0)^{n-1} dz$$

If $f(z)$ is analytic wherever $|z - z_0| < R_2$, then all of these coefficients will be zero, and the Laurent series reduces to a Taylor series.

The domain of definition could expand to several different possibilities:

- $|z - z_0| < R_2$, if $f$ is analytic everywhere inside this disk (note that this is now a Taylor series)
- $0 < |z - z_0| < R_2$, if $f$ is analytic everywhere except for $z_0$ inside the $R_2$ disk
- $R_1 < |z - z_0| < \infty$, if $f$ is analytic everywhere outside the $R_1$ disk
- $0 < |z - z_0| < \infty$, if $f$ is analytic everywhere in the plane except for $z_0$

# 6 Frame 62 – Examples of Laurent Series

This section will show some examples of Laurent series.

## 6.1 Example 1

First, using the expansion for $e^z$, the Laurent series for $e^{1/z}$ is

$$e^{1/z} = \sum_{n=0}^{\infty} \frac{1}{n!z^n} = 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots$$

Note that the coefficient $b_1$ here is 1, which tells us that

$$\int_C e^{1/z} dz = 2\pi i$$

for any positively-oriented simple closed contour that contains the origin.

## 6.2 Example 2

The function

$$\frac{1}{(z-i)^2}$$

is already a Laurent series, where $c_n = 0$ for all $n$ except $c_{-2} = 1$. This tells us that

$$\int_C \frac{dz}{(z-i)^{n+3}} = \begin{cases} 0, & n \neq -2 \\ 2\pi i, & n = -2 \end{cases}$$

## 6.3 Example 3

Consider the function

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2}$$

in the domain $|z| < 1$. Since $f$ is analytic in this domain, we can write the Maclaurin series

$$\begin{aligned} f(z) &= \frac{-1}{1-z} + \frac{1}{2}\frac{1}{1-z/2} \\ &= -\sum_{n=0}^{\infty} z^n + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}} \\ &= \sum_{n=0}^{\infty} (2^{-n-1} - 1) z^n \end{aligned}$$

## 6.4 Example 4

Consider the same function on the domain $1 < |z| < 2$. Here, we can write

$$|1/z| < 1 \quad \text{and} \quad |z/2| < 1$$

so we can write the function as

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1-1/z} + \frac{1}{2}\frac{1}{1-z/2} \\ &= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} + \sum_{n=1}^{\infty}\frac{1}{z^n} \end{aligned}$$

which is the Laurent series for $f$ in this domain.

## 6.5 Example 5

Finally, consider this function on the domain $2 < |z| < \infty$. Now, we can write

$$\begin{aligned} f(z) &= \frac{1}{z}\frac{1}{1-1/z} - \frac{1}{z}\frac{1}{1-2/z} \\ &= \sum_{n=0}^{\infty}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} \\ &= \sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty}\frac{1-2^{n-1}}{z^n} \end{aligned}$$

which is the Laurent series for $f$ in this domain. Note that $a_n = 0$ for all $n$.

# 7 Frame 63 – Absolute & Uniform Convergence

This section will discuss several properties of power series.

## 7.1 Absolute Convergence

The first theorem will discuss when a power series is convergent.

*Theorem: if a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

*converges at the point $z = z_1 \neq z_0$, then it is absolutely convergent on the open disk $|z - z_0| < |z_1 - z_0|$.*

Proof: since the series converges, all of its terms must be bounded, so we can write

$$|a_n (z_1 - z_0)^n| \leq M$$

for some positive $M$. Then, for each $z$ inside the open disk described above, we can write

$$|a_n (z - z_0)^n| = |a_n (z - z_1)^n| \left( \frac{|z - z_0|}{|z_1 - z_0|} \right)^n \leq M \rho^n$$

where $\rho$ must be less than one. Then, the series must be less than the convergent geometric series

$$\sum_{n=0}^{\infty} M \rho^n$$

so the power series is absolutely convergent on this open disk.

*Note: we refer to the circle $|z - z_0| = |z_1 - z_0|$ as the* ***circle of convergence*** *– it is the largest circle around $z_0$ such that the power series converges everywhere inside it.*

## 7.2 Uniform Convergence

Next, we will define uniform convergence. We said earlier that the remainder of a series is the infinite sum less the partial sum

$$\rho_N(z) = S(z) - S_N(z)$$

and that, in order to converge, these remainders must approach zero as $N$ approaches infinity. We can write this as

$$|\rho_N(z)| < \epsilon \quad \text{whenever} \quad N > N_\epsilon$$

We say that a series is **uniformly convergent** in a region if our choice of $N_\epsilon$ depends only on $\epsilon$ and not on $z$.

*Theorem: If $z_1$ is a point inside the circle of convergence of a power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

*then that series must be uniformly convergent in the closed disk $|z - z_0| \leq |z_1 - z_0|$.*

# 8 Frame 64 – Continuity of Power Series

## 8.1 Continuity

The following theorem is a consequence of uniform convergence.

*Theorem: A power series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

*represents a continuous function $S(z)$ at each point inside its circle of convergence.*

To show this, we can write $S(z)$ as a partial sum plus the remainder:

$$S(z) = S_N(z) + \rho_N(z)$$

Then, we wish to show that

$$|S(z) - S(z_1)| < \epsilon \quad \text{whenever} \quad |z - z_1| < \delta$$

but

$$\begin{aligned} |S(z) - S(z_1)| &= |S_N(z) + \rho_N(z) - S_N(z_1) - \rho_N(z_1)| \\ &\leq |S_N(z) - S_N(z_1)| + |\rho_N(z)| + |\rho_N(z_1)| \end{aligned}$$

Now, the first term is a difference of two polynomials, which are continuous, so we must be able to choose a $\delta$ small enough to satisfy

$$|S_N(z) - S_N(z_1)| < \frac{\epsilon}{3} \quad \text{whenever} \quad |z - z_1| < \delta$$

Then, due to uniform convergence, there exists an $N_\epsilon$ such that

$$|\rho_N(z)| < \frac{\epsilon}{3} \quad \text{whenever} \quad N > N_\epsilon$$

which takes care of the latter two terms. Thus, for all $N > N_\epsilon$,

$$|S(z) - S(z_1)| < 3 \cdot \frac{\epsilon}{3} \quad \text{whenever} \quad |z - z_1| < \delta$$

so $S(z)$ is continuous.

## 8.2 Adjustments for Laurent series

For series of the form

$$\sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

we can write the complex number

$$w = \frac{1}{z - z_0}$$

to turn the series into the form

$$\sum_{n=1}^{\infty} b_n w^n$$

which converges absolutely to a continuous function whenever $|w| < \frac{1}{|z_1 - z_0|}$. This implies that the function converges on the **outside** of the circle

$$|z - z_0| > |z_1 - z_0|$$

rather than the inside.

# 9 Frame 65 – Integration and Differentiation

## 9.1 Integration

Next, we will look at integration of the power series

$$S(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

which represents a continuous function on its circle of convergence.

*Theorem: Suppose that $C$ is a contour that is inside the circle of convergence of this power series, and let $g(z)$ be any function that is continuous on $C$. Then,*

$$\int_C g(z)S(z)dz = \sum_{n=0}^{\infty} a_n \int_C g(z)(z - z_0)^n$$

Since both $g(z)$ and $S(z)$ are continuous on $C$, their product can be written as

$$g(z)S(z) = \sum_{n=0}^{N-1} a_n g(z)(z - z_0)^n + g(z)\rho_N(z)$$

so the integral can be written as

$$int_C g(z)S(z)dz = \sum_{n=0}^{N-1} a_n \int_C g(z)(z - z_0)^n dz + \int_C g(z)\rho_N(z)dz$$

Then, uniform convergence tells us that there exists an integer $N_\epsilon$ such that

$$|\rho_N(z)| < \epsilon \quad \text{whenever} \quad N > N_\epsilon$$

so

$$\left| \int_C g(z)\rho_N(z)dz \right| < M\epsilon L \quad \text{whenever} \quad N > N_\epsilon$$

which implies that this integral vanishes as $N$ approaches infinity, and the theorem is proved.

## 9.2 Analyticity

Using the previous theorem, consider the special case $g(z) = 1$. Then, each of the integrals evaluate to

$$\int_C g(z)(z - z_0)^n dz = \int_C (z - z_0)^n dz = 0$$

since these polynomials are entire. Thus,

$$\int_C S(z)dz = 0$$

which implies the following corollary:

*Corollary: the sum of a power series, $S(z)$, is analytic at each point interior to the circle of convergence.*

*Example: Consider the function*

$$f(z) = \begin{cases} (e^z - 1)/z, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

*This function has the Maclaurin series expansion*

$$f(z) = \sum_{n=1}^{\infty} \frac{z^{n-1}}{n!} = 1 + \frac{z}{2!} + \frac{z^2}{3!} + \dots$$

*When $z = 0$ in this series, it evaluates to $1$, so it represents both of the cases defined in $f(z)$. Since this sum of polynomial terms is clearly entire, $f(z)$ must also be entire. The continuity of $f$ could also be evaluated by confirming that*

$$\lim_{z \to 0} \frac{e^z - 1}{z} = f(0) = 1$$

Note that this corollary confirms that we have chosen the largest possible circle of convergence. We picked a circle centered at $z_0$ and intersecting the nearest non-analytic point $z_1$; if we chose a bigger circle, then it would contain $z_1$, incorrectly implying that $f$ is analytic at $z_1$.

## 9.3 Differentiability

Next, we can look at differentiability of power series:

*Theorem: a power series can be differentiated term-by-term. That is,*

$$S'(z) = \sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

To prove this, suppose that $C$ is, as above, a contour interior to the circle of convergence and $z$ is any point inside $C$. We can define the function

$$g(s) = \frac{1}{2\pi i} \frac{1}{(s - z)^2}$$

which is continuous on $C$. Then, we can use Cauchy's integral formulas to write

$$\int_C g(s)S(s)ds = \frac{1}{2\pi i}\int_C \frac{S(s)}{(s-z)^2}ds = S'(z)$$

However, we can alternatively write

$$\int_C g(s)S(s)ds = \sum_{n=0}^{\infty} a_n \int Cg(s)(s-z_0)^n ds$$

where each term of this sum is

$$\int_C g(s)(s-z_0)^n = \frac{1}{2\pi i}\int_C \frac{(s-z_0)^n}{(s-z)^2}ds = \frac{d}{dz}(z-z_0)^n$$

so

$$S'(z) = \sum_{n=0}^{\infty} a_n \frac{d}{dz}(z-z_0)^n$$

*Example: we saw earlier that*

$$\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

*We can differentiate both sides to write*

$$-\frac{1}{z^2} = \sum_{n=1}^{\infty}(-1)^n n(z-1)^{n-1}$$

*or*

$$\frac{1}{z^2} = \sum_{n=0}^{\infty}(-1)^n(n+1)(z-1)^n$$

# 10 Frame 66 – Uniqueness of Power Series

This section will discuss the uniqueness of the Taylor and Laurent series expansions.

## 10.1 Taylor Series

*Theorem: If a series*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

*converges to $f(z)$ at each point within a circle $|z - z_0| < R$, then it is* ***the*** *Taylor series expansion for $f$ in powers of $z - z_0$.*

First, according to the theorem, we should write

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n = \sum_{m=0}^{\infty} a_m (z - z_0)^m$$

Then, if we define $g(z)$ as any of the functions ($n \geq 0$)

$$g(z) = \frac{1}{2\pi i} \frac{1}{(z - z_0)^{n+1}}$$

we can write

$$\begin{aligned} \int_C g(z) f(z) dz &= \sum_{m=0}^{\infty} a_m \int_C g(z) (z - z_0)^m dz \\ &= \sum_{m=0}^{\infty} a_m \frac{1}{2\pi i} \int_C \frac{dz}{(z - z_0)^{n-m+1}} \\ &= a_n \end{aligned}$$

However, we also know that

$$\int_C g(z) f(z) dz = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz = \frac{f^{(n)}(z_0)}{n!}$$

so

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

and the Taylor series is unique.

## 10.2 Laurent series

*Theorem: If a series*

$$\sum_{n=-\infty}^{\infty} c_n (z - z_0)^n$$

*converges to $f(z)$ at all points in an annular domain centered at $z_0$, then it is* ***the*** *Laurent series expansion for $f$ in powers of $z - z_0$ in that domain.*

The proof of this theorem has a similar structure to the previous proof. The result is that

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$