1 Frame 38 – Derivatives and Integrals

1(b) Breaking the derivative into its complex components,

$$\begin{aligned}\frac{d}{dt}[w(t)]^2 &= \frac{d}{dt}[u(t)+iv(t)]^2\\ &= 2[u(t)+iv(t)][u(t)+iv(t)]'\\ &= 2w(t)w'(t)\end{aligned}$$

2(a) Evaluating the integral,

$$\begin{aligned}\int_1^2 \left(\frac{1}{t}-i\right)^2 dt &= \int_1^2 \frac{1}{t^2}-1-i\frac{2}{t}dt\\ &= -\frac{1}{t}-t-2i\ln t\Big|_1^2\\ &= -(\frac{1}{2}-1)-(2-1)-2i(\ln 2-0)\\ &= -\frac{1}{2}-i\ln 4\end{aligned}$$

2(b)

$$\begin{aligned}\int_0^{\pi/6} e^{i2t} &= \frac{1}{2i}e^{i2t}\Big|_0^{\pi/6}\\ &= \frac{1}{2i}(e^{i\pi/3}-1)\\ &= \frac{1}{2i}\left(i\frac{\sqrt{3}}{2}-\frac{1}{2}\right)\\ &= \frac{\sqrt{3}}{4}+i\frac{1}{4}\end{aligned}$$

2(c) Converting this improper integral into a limit,

$$\begin{aligned}\int_0^{\infty} e^{-zt} &= \lim_{L\to\infty}\int_0^L e^{-zt}\\ &= \lim_{L\to\infty} -\frac{1}{z}e^{-zt}\Big|_0^L\\ &= -\frac{1}{z}\lim_{L\to\infty} e^{-zL}-1\\ &= \frac{1}{z}\end{aligned}$$

4 Evaluating the left-side integral,

$$\begin{aligned}
\int_0^\pi e^{1+i} x dx &= \frac{1}{1+i} e^{1+i} x \Big|_0^\pi \\
&= \frac{1}{1+i}(e^{\pi+i\pi} - 1) \\
&= \frac{1-i}{2}(-e^\pi - 1) \\
&= -\frac{1}{2}(e^\pi + 1) + \frac{i}{2}(e^\pi + 1)
\end{aligned}$$

2 Frame 39 – Contours

2 First, the original parametrization can be written as

$$z(\theta) = 2e^{i\theta} = 2\cos\theta + 2i\sin\theta$$

If $\theta = \arctan\frac{y}{\sqrt{4-y^2}}$, then this becomes

$$z(y) = 2\cos\arctan\frac{y}{\sqrt{4-y^2}} + 2i\sin\arctan\frac{y}{\sqrt{4-y^2}}$$

Next, these terms can be simplified using basic geometry. The expression $\arctan\frac{y}{\sqrt{4-y^2}}$ represents a right-angled triangle with legs of lengths $\sqrt{4-y^2}$ and y, so the hypotenuse must have a length of 2. Then,

$$\cos\arctan\frac{y}{\sqrt{4-y^2}} = \frac{\sqrt{4-y^2}}{2}$$
$$\sin\arctan\frac{y}{\sqrt{4-y^2}} = \frac{y}{2}$$

so the arc is

$$z(y) = \sqrt{4-y^2} + iy$$

6 (a) First, the function

$$z(t) = t + iy(t) = t + it^3\sin(\pi/t)$$

intersects the real axis whenever $y(t) = 0$. If $t = 1/n$, then this expression becomes

$$y(1/n) = \frac{\sin\left(\frac{\pi}{1/n}\right)}{n^3} = \frac{\sin(n\pi)}{n^3} = 0$$

as predicted.

(b) An arc is smooth if the function $z(t)$ is continuous and its derivative is piecewise continuous.

First, $z(t)$ is continuous for $0 < x \leq 1$ because $x(t) = x$ and $y(t) = y(x)$ are both continuous on this interval. To show continuity at $t = 0$, we must show that

$$\lim_{t\to 0+} y(t) = 0$$

However, the magnitude of $y(t)$ must be in the range

$$0 \leq \left|t^3\sin\left(\frac{\pi}{t}\right)\right| \leq t^3$$

and the left- and right-hand limits are

$$\lim_{t \to 0+} 0 = 0$$
$$\lim_{t \to 0+} t^3 = 0$$

so, by the squeeze theorem, the original limit holds, and $y(t)$ is continuous at $t = 0$.

Finally, the derivative of $z(t)$ is

$$z'(t) = 1 + i\left[3t^2 \sin(\pi/t) - \pi t \cos(\pi/t)\right]$$

Using the same process as above, the limit as t goes to zero is

$$\lim_{t \to 0+} z'(t) = 1 + i0$$

How can I tell whether this is continuous? The derivative isn't defined at zero.

3 Frame 42 – Contour Integrals

1(a) The integrand on this circle is

$$f[z(\theta)] = \frac{2e^{i\theta} + 2}{2e^{i\theta}} = 1 + e^{-i\theta}$$

and the derivative of the contour is

$$z'(\theta) = 2ie^{i\theta}$$

so the first contour integral is

$$\begin{aligned}
\int_C f(z)dz &= \int_0^{\pi} (1 + e^{-i\theta})(2ie^{i\theta})d\theta \\
&= 2i \int_0^{\pi} e^{i\theta} + 1d\theta \\
&= 2i \left(\frac{e^{i\theta}}{i} + \theta \right)_0^{\pi} \\
&= 2i \left(\frac{-2}{i} + \pi \right) \\
&= -4 + 2\pi i
\end{aligned}$$

1(b) The integrand has not changed, so

$$\begin{aligned}
\int_C f(z)dz &= 2i \left(\frac{e^{i\theta}}{i} + \theta \right)_{\pi}^{2\pi} \\
&= 4 + 2\pi i
\end{aligned}$$

1(c) The integral along the entire circle is just the sum of the two previous results:

$$\begin{aligned}
\int_C f(z)dz &= (-4 + 2\pi i) + (4 + 2\pi i) \\
&= 4\pi i
\end{aligned}$$

2(a) The function is

$$f[z(\theta)] = 1 + e^{i\theta} - 1 = e^{i\theta}$$

and the contour's derivative is

$$z'(\theta) = ie^{i\theta}$$

so the integral is

$$\begin{aligned}\int_C f(z)dz &= \int_\pi^{2\pi} e^{i\theta} \cdot ie^{i\theta} \\ &= i\int_\pi^{2\pi} e^{2i\theta} \\ &= i\frac{e^{2i\theta}}{2i}\Big|_\pi^{2\pi} \\ &= \frac{e^{4i\pi} - e^{2i\pi}}{2} \\ &= 0\end{aligned}$$

2(b) Now, the function is

$$f[z(x)] = x - 1$$

and the path's derivative is

$$z'(x) = 1$$

so the integral is

$$\int_C f(z)dz = \int_0^2 x - 1dx = \frac{(x-1)^2}{2}\Big|_0^2 = \frac{1-1}{2} = 0$$

3 Along the first edge, the integral is

$$\int_0^1 \pi e^{\pi x}dx = e^{\pi x}\Big|_0^1 = e^\pi - 1$$

Along the second edge, the integral is

$$e^\pi \int_0^1 i\pi e^{-i\pi y}dy = e^\pi \left(-\frac{\pi}{\pi}e^{-i\pi y}\right)_0^1 = 2e^\pi$$

On the third edge, the integral is

$$-\int_0^1 \pi e^{\pi x} \cdot (-1)dx = e^\pi - 1$$

Finally, on the fourth edge, the integral is

$$-\int_0^1 i \cdot \pi e^{-i\pi y}dy = \left(e^{-i\pi y}\right)_0^1 = -2$$

so the whole integral is

$$\int_C f(z)dz = 2(e^\pi - 1) + 2e^\pi - 2 = 4(e^\pi - 1)$$

4 If the path is $y = x^3$, then the direction is

$$z'(x) = 1 + iy'(x) = 1 + i3x^2$$

Then, the integral can be done in two parts. First, from $x = -1$ to 0,

$$\int_{-1}^{0} 1 \cdot (1 + i3x^2)dx = \left(x + ix^3\right)_{-1}^{0} = 1 + i$$

Then, from $x = 0$ to 1,

$$\int_0^1 4x^3 \cdot (1 + i3x^2)dx = \int_0^1 4x^3 + i12x^5 dx = x^4 + i2x^6\Big|_0^1 = 1 + 2i$$

so the total contour integral is

$$\int_C f(z)dz = 2 + 3i$$

5 If $f(z) = 1$, then

$$\int_C f(z)dz = \int_a^b 1 \cdot z'(t)dt = z(b) - z(a) = z_2 - z_1$$

6 If $z = e^{i\theta}$, then the function $f(z)$ is

$$f[z(\theta)] = (e^{i\theta})^{-1+i} = e^{-i\theta}e^{-\theta}$$

and the contour integral is

$$\begin{aligned}
\int_C f(z)dz &= \int_0^{2\pi} e^{-i\theta}e^{-\theta} \cdot ie^{i\theta}d\theta \\
&= i\int_0^{2\pi} e^{-\theta}d\theta \\
&= -ie^{-\theta}\Big|_0^{2\pi} \\
&= i(1 - e^{-2\pi})
\end{aligned}$$

7 On this semicircle,

$$f[z(\theta)] = (e^{i\theta})^i = e^{-\theta}$$

so

$$\begin{aligned}
\int_C f(z)dz &= \int_0^\pi e^{-\theta} \cdot ie^{i\theta} d\theta \\
&= i\int_0^\pi e^{(-1+i)\theta} d\theta \\
&= \frac{i}{-1+i} e^{(-1+i)\theta}\Big|_0^\pi \\
&= \frac{i(-1-i)}{2}(-e^{-\pi}-1) \\
&= \frac{-1+i}{2}(e^{-\pi}+1)
\end{aligned}$$

4 Frame 43 – Bounding Contour Integrals

1 On the contour,

$$|z^2 - 1| \leq ||z^2| - 1| = 3$$

so

$$|f(z)| \leq \frac{1}{3}$$

and the integral must satisfy

$$\left| \int_C \frac{dz}{z^2 - 1} \right| \leq \frac{1}{3} \cdot \pi = \frac{\pi}{3}$$

2 On the line segment from $z = i$ to $z = 1$, the function z^4 is minimized at the midpoint:

$$|z^4| \geq \left| \frac{1+i}{2} \right|^4 = \left(\frac{1}{\sqrt{2}} \right)^4 = \frac{1}{4}$$

so

$$|f(z)| \leq \frac{1}{|z^4|} \leq 4$$

Then, since the line segment has a length of $L = \sqrt{2}$,

$$\left| \int_C \frac{dz}{z^4} \right| \leq 4 \cdot \sqrt{2}$$

3 On these three line segments, $|e^z|$ is maximized when $\Re z$ is maximized, so

$$|e^z| \leq e^0 = 1$$

Next, $|\overline{z}|$ is maximized when $|z|$ is maximized, so

$$|\overline{z}| \leq |-4| = 4$$

Thus,

$$|f(z)| = |e^z - \overline{z}| \leq |e^z| + |\overline{z}| \leq 1 + 4 = 5$$

Finally, the three line segments have a total length of 12, so

$$\left| \int_C (e^z - \overline{z}) dz \right| \leq 5 \cdot 12 = 60$$

5 Frame 45 – Antiderivatives

1 Every function z^n for n a non-negative integer has an antiderivative

$$\frac{1}{n+1}z^{n+1}$$

Thus, any contour integral of z^n can be written as

$$\int_{z_1}^{z_2} z^n dz = \frac{1}{n+1}z^{n+1}\Big|_{z_1}^{z_2} = \frac{1}{n+1}(z_2^{n+1} - z_1^{n+1})$$

2(a) We know that

$$\frac{d}{dz}\frac{1}{\pi}e^{\pi z} = e^{\pi z}$$

so

$$\int_i^{i/2} e^{\pi z}dz = \frac{1}{\pi}e^{\pi z}\Big|_i^{i/2} = \frac{1}{\pi}(e^{i\pi/2} - e^{i\pi}) = \frac{1+i}{\pi}$$

2(b) An antiderivative is

$$\frac{d}{dz}2\sin(z/2) = \cos(z/2)$$

so

$$\begin{aligned}\int_0^{\pi+2i} \cos(z/2)dz &= 2\sin(z/2)\Big|_0^{\pi+2i} \\ &= 2(\sin(\pi/2+i) - \sin(0)) \\ &= 2(\sin(\pi/2)\cosh(1) + i\cos(\pi/2)\sinh(1) - 0) \\ &= 2\cosh(1) \\ &= e + \frac{1}{e}\end{aligned}$$

2(c) An antiderivative is

$$\frac{d}{dz}\frac{1}{4}(z-2)^4 = (z-2)^3$$

so

$$\int_1^3 (z-2)^3 dz = \frac{1}{4}(z-2)^4\Big|_1^3 = \frac{1}{4}(1^4 - (-1)^4) = 0$$

3 The antiderivative of any function

$$f(z) = (z-z_0)^{n-1}$$

is the function

$$F(z) = \frac{1}{n}(z-z_0)^n$$

which is always continuous everywhere except possibly for the point $z = z_0$. Thus, according to the theorem, if C_0 is a closed contour on this domain,

$$\int_{C_0} (z - z_0)^{n-1} dz = 0$$

4 To evaluate this integral, we can use the branch

$$F_2(z) = \frac{2}{3} R^{3/2} e^{i3\theta/2} \quad \left(\frac{\pi}{2} < \theta < \frac{5\pi}{2} \right)$$

Then, the integral becomes

$$\int_{C_2} f(z)dz = \frac{2}{3} 3^{3/2}(-1 + i) = 2\sqrt{3}(-1 + i)$$

Adding the original contour integral, we find that

$$\int_{C_2 - C_1} f(z)dz = 2\sqrt{3}[(-1 + i) - (1 + i)] = -4\sqrt{3}$$

5 An antiderivative of z^i is

$$F(z) = \frac{1}{i+1} z^{i+1} = \frac{1}{i+1} e^{(i+1)(\ln|z| + i\theta} = \frac{1}{i+1} R e^{-\theta} e^{i(\ln R + \theta)}$$

Taking this at the points $(R, \theta) = (1, \pi)$ and $(1, 0)$,

$$\begin{aligned} \int_{-1}^{1} f(z)dz &= \frac{1}{1+i} \left[1e^{i0} - e^{-\pi} e^{i\pi} \right] \\ &= \frac{1-i}{2}(1 + e^{-\pi}) \\ &= \frac{1 + e^{-\pi}}{2}(1 - i) \end{aligned}$$

6 Frame 49 – The Cauchy-Goursat Theorem

1

1. This function only has a singularity at $z = 3$, so the integral around C is zero.
2. This function is entire, so this integral is zero.
3. This function has singularities at $z = -1 \pm i$, which are outside the unit circle, so the integral is zero.
4. $\operatorname{sech} z$ has singularities at

   $$z = (n + 1/2)\pi i$$

   Since $\pi/2 > 1$, all of these are outside of the unit circle, so the integral is zero.
5. $\tan z$ has singularities at

   $$z = (n + 1/2)\pi$$

   As above, these are all outside of the unit circle, and the integral is zero.
6. $\operatorname{Log}(z + 2)$ has a branch point at $z = -2$ and a branch cut on the arc at $\theta = \pi$ starting at this point. Thus, no singularities are inside the unit circle, so the integral is zero.

2

1. This function only has singularities at $z = \pm\frac{1}{\sqrt{3}}$. There are no singularities between C_1 and C_2, so these integrals are the same.
2. This function has singularities at

   $$z = 2n\pi$$

   Only one of these singularities is inside C_2, and it is also inside C_1, so the integrals are the same.
3. This function has a singularity at $1 - e^z = 0$ or $z = 0$, so the integral is zero.

4(a) The integral along the bottom leg is

$$\int_{-a}^{a} e^{-x^2} dx = 2\int_0^a e^{-x^2} dx$$

and the integral along the top leg is

$$-\left(\int_{-a}^{a} e^{-(x+ib)^2} dx\right) = -2\int_0^a e^{-(x^2-b^2+2xib)} dx = -2e^{b^2}\int_0^a e^{-x^2} \cos 2bx dx$$

The integral on the right leg is

$$\int_0^b e^{-(a+iy)^2} i dy = i \int_0^b e^{-(a^2-y^2+i2ay)} dy = ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} dy$$

and the integral on the left leg is

$$-\left(\int_0^b e^{-(-a+iy)^2} i dy \right) = -i \int_0^b e^{-(a^2-y^2-i2ay)} dy = -ie^{-a^2} \int_0^b e^{y^2} e^{i2ay} dy$$

Then, by Cauchy-Goursat,

$$2\int_0^a e^{-x^2} dx - 2e^{b^2} \int_0^a e^{-x^2} \cos 2bx dx + ie^{-a^2} \int_0^b e^{y^2} e^{-i2ay} dy - ie^{-a^2} \int_0^b e^{y^2} e^{i2ay} dy = 0$$

or

$$\int_0^a e^{-x^2} \cos 2bx \ dx = e^{b^2} \int_0^a e^{-x^2} \ dx + e^{-a^2} e^{-b^2} \int_0^b e^{y^2} \sin 2ay \ dy$$

4(b) Now, as $a \to \infty$, the first integral is $\frac{\sqrt{\pi}}{2}$, so

$$\int_0^\infty e^{-x^2} \cos 2bx \ dx = e^{-b^2} \frac{\sqrt{\pi}}{2} + \lim_{a \to \infty} e^{-a^2} e^{-b^2} \int_0^b e^{y^2} \sin 2ay \ dy$$

Finding this second term, we see that

$$\begin{aligned}
\lim_{a \to \infty} \left| e^{-a^2} e^{-b^2} \int_0^b e^{y^2} \sin 2ay \ dy \right| &\leq \lim_{a \to \infty} e^{-a^2} e^{-b^2} \int_0^b |e^{y^2} \sin 2ay| \ dy \\
&\leq \lim_{a \to \infty} e^{-a^2} e^{-b^2} \int_0^b |e^{y^2}| \ dy \\
&= \lim_{a \to \infty} e^{-a^2} \cdot M \\
&= 0
\end{aligned}$$

where M is some positive number. Thus,

$$\int_0^\infty e^{-x^2} \cos 2bx \ dx = e^{-b^2} \frac{\sqrt{\pi}}{2}$$