1 Frame 29 – The Exponential Function

1.1 Definition

We define the **exponential function** e^z by writing

$$e^z = e^x e^{iy}$$

and we apply Euler's formula to get

$$e^z = e^x(\cos y + i \sin y)$$

Note that, when $y = 0$, e^z reduces to e^x.

Although we typically understand that $e^{1/n}$ would be the set of nth roots of e, here, we only use the real, positive root $\sqrt[n]{e}$.

1.2 Familar properties

First, in calculus, we know that

$$e^{x_1} e^{x_2} = e^{x_1 + x_2}$$

It is easy to verify that this holds true for complex numbers:

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

This also allows us to write

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

and, as a specific case,

$$\frac{1}{e^z} = e^{-z}$$

We showed earlier that e^z is differentiable everywhere in the complex plane, and that

$$\frac{d}{dz} e^z = e^z$$

We also know that e^z is never zero. This comes from the pair

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi$$

and since e^x is never zero, neither is e^z.

1.3 Unfamiliar properties

Since we can write

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

the exponential function is periodic with an imaginary period of $2\pi i$.

It is also possible for the complex exponential function to be negative. For an example, we know that Euler's identity states

$$e^{i\pi} = -1$$

In fact, e^z can be any given non-zero complex number.

Example: suppose we want solutions to the equation

$$e^z = 1 + i$$

The right side can be rewritten as

$$e^x e^{iy} = \sqrt{2} e^{i\pi/4}$$

and equating the parts of this equation gives

$$x = \ln \sqrt{2} = \frac{1}{2} \ln 2 \quad \textit{and} \quad y = \left(2n + \frac{1}{4}\right)\pi$$

so

$$z = \frac{1}{2} \ln 2 + \left(2n + \frac{1}{4}\right)\pi i$$

2 Frame 30 – The Logarithmic Function

2.1 Motivation

We said in the previous section that e^z can take on any non-zero complex value. To help us solve the equation

$$e^w = z$$

we will define a **logarithmic function**, such that

$$e^{\log z} = z \quad (z \neq 0)$$

We can solve for w by writing the two complex numbers in the form

$$z = re^{i\theta}$$
$$w = u + iv$$

Substituting these into the original equation gives

$$e^u e^{iv} = re^{i\theta}$$

so we get

$$w = \log z = \ln r + i(\theta + 2n\pi)$$

Note that this is a multi-valued function.

Example: if $z = -1 - i\sqrt{3}$, then $r = 2$ and $\theta = -2\pi/3$, so

$$\log(-1 - i\sqrt{3}) = \ln 2 + \left(n - \frac{1}{3}\right) 2\pi i$$

2.2 Precise definition

A more precise definition of the multi-valued logarithmic function is

$$\log z = \ln|z| + i \arg z$$

The **principal value** of $\log z$ is obtained by using the single-valued principal argument instead:

$$\operatorname{Log} z = \ln|z| + i\theta$$

Note that

$$\log z = \operatorname{Log} z + i2n\pi$$

2.3 Notes

The principal logarithmic function $\operatorname{Log} z$ reduces to the usual logarithm from calculus when z is positive and real – if $z = r$, then

$$\operatorname{Log} r = \ln r$$

However, we are now able to find the logarithm of negative real numbers, which we were unable to do in calculus.

Example: the logarithm of -1 *is*

$$\log(-1) = \ln 1 + (1 + 2n)i\pi = (2n + 1)i\pi$$

and

$$\operatorname{Log}(-1) = i\pi$$

3 Frame 31 – Branches & Derivatives of Logarithms

3.1 Limiting a logarithm's domain

We saw in the previous section that the multi-valued logarithm function of a complex number $z = re^{i\theta}$ can be written as

$$\log z = \ln r + i\theta$$

where θ can have any of the values

$$\theta = \text{Arg}(z) + 2n\pi$$

We can make the logarithmic function single-valued by restricting the value of θ to $\alpha < \theta < \alpha + 2\pi$ for any real value of θ. Then, the function is single-valued and is continuous everywhere in the domain of the function (ie: $r > 0$ and $\theta \in (\alpha, \alpha + 2\pi)$). Note that we cannot include in the ray $\theta = \alpha$ – the function would not be continuous here.

In this limited domain, the components of the log function also satisfy the polar Cauchy-Riemann equations

$$ru_r = 1 = v_\theta; \quad u_\theta = 0 = -rv_r$$

so the logarithmic function is analytic in this domain, with the derivative

$$\frac{d}{dz}\log z = e^{-i\theta}(u_r + iv_r) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

In particular, we can set $\alpha = -\pi$ and write

$$\frac{d}{dz}\,\text{Log}\, z = \frac{1}{z} \quad (|z| > 0, -\pi < \text{Arg}\, z < \pi)$$

Note that not all of the identities from calculus carry over to the complex plane.

Example: using the principal branch,

$$\text{Log}(i^3) = \text{Log}(-i) = -i\frac{\pi}{2}$$

but

$$3\,\text{Log}(i) = 3\left(i\frac{\pi}{2}\right) = i\frac{3\pi}{2}$$

so

$$\text{Log}(i^3) \neq 3\,\text{Log}(i)$$

3.2 Branches

A **branch** of a multi-valued function f is any single-valued, analytic function F such that $F(z)$ is one of the values of f at each point within the domain of F. For instance, our limited-domain logarithm is a branch of the multi-valued log function. The principal logarithm function

$$\operatorname{Log} z = \ln r + i\theta \quad (|z| > 0, -\pi < \operatorname{Arg} z < \pi)$$

is known as the **principal branch**.

A **branch cut** is a line/curve that is used to define a branch F of a multi-valued function f. Any point on the branch cut is a singular point of F. Any point that is common to all branch cuts of f is called a **branch point**. For example, the logarithmic function has a branch point at $z = 0$ and a branch cut on the ray $\theta = \alpha$. In particular, the ray $\theta = \pi$ is the branch cut for the principal logarithmic function.

4 Frame 32 – Logarithm Identities

We said earlier that arguments, which are multi-valued functions, can be compared in a special way – since each function is really a set of values, the sets will contain the same values. Specifically,

$$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$$

Now, we know that $|z_1 z_2| = |z_1||z_2|$, and from our knowledge of real-valued logarithms,

$$\ln|z_1 z_2| = \ln|z_1| + \ln|z_2|$$

Putting together these two statements, we see that

$$\log(z_1 z_2) = \log z_1 + \log z_2$$

which is to be understood as *set equality*, and does not necessarily apply to the principal values. In a similar manner,

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2$$

Two more properties will be useful in the next section. If z is any non-zero complex number, then

$$z^n = e^{n \log z}$$

for all values of $\log z$. When $n = 1$, this reduces to the familiar

$$z = e^{\log z}$$

Also, for any non-zero z, it is true that

$$z^{1/n} = e^{\frac{1}{n} \log z}$$

where both sides have n distinct values. To show this, we can write out the right side as

$$e^{\frac{1}{n} \log z} = e^{\frac{1}{n} \ln r + \frac{i(\theta + 2k\pi}{n}} = \sqrt[n]{r} e^{i(\theta/n + 2k\pi/n)}$$

which has n distinct values, for $k = 0, 1, \ldots, n-1$.

5 Frame 33 – Complex Exponents

5.1 Definition and basics

For non-zero z and complex c, we define the function z^c as

$$z^c = e^{c \log z}$$

Note that this definition uses the multi-valued log function.

We saw earlier that the exponential function has the property

$$\frac{1}{e^z} = e^{-z}$$

Now, for the general power equation, we have

$$\frac{1}{z^c} = \frac{1}{e^{c \log z}} = e^{-c \log z} = z^{-c}$$

Example: the values of i^{-2i} *can be found by first writing*

$$\log i = \ln 1 + i\left(\frac{\pi}{2} + 2n\pi\right) = i\left(2n + \frac{1}{2}\right)\pi$$

and so

$$i^{-2i} = e^{-2i \cdot i(2n+1/2)\pi} = e^{(4n+1)\pi}$$

Note that all of these powers are real numbers.

The **principal value** of z^c uses the single-valued log function:

$$\text{P.V. } z^c = e^{c \operatorname{Log} z}$$

Example: the principal value of $(-i)^i$ *is*

$$e^{i \operatorname{Log}(-i)} = e^{i(-i\pi/2)} = e^{\pi/2}$$

5.2 Other properties

To differentiate z^c, we can restrict the logarithmic function to a single branch

$$\log z = \ln + i\theta \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

Then, z^c is analytic in this domain. The derivative can be found through the chain rule:

$$\frac{d}{dz} z^c = \frac{d}{dz} e^{c \log z} = \frac{c}{z} e^{c \log z} = c z^{c-1}$$

Most of the laws of exponents remain valid in the complex plane. However, since the functions are multi-valued, we can only guarantee equality between sets – when using principal values, not all of the rules of real exponents work. For example, the law

$$z_1^c z_2^c = (z_1 z_2)^c$$

does not necessarily hold for all z_1, z_2 when using principal values.

5.3 Exponential functions with other bases

We can write the **exponential function** with a non-zero base c as

$$c^z = e^{z \log c}$$

Note that this function is, again, multi-valued: if $c = e$, then we don't recover our usual definition of e^z. However, if we use the principal value of the logarithm, the usual interpretation occurs.

This exponential function is an entire function for any non-zero c. It has the derivative

$$\frac{d}{dz} c^z = \frac{d}{dz} e^{z \log c} = e^{z \log c} \log c = c^z \log c$$

6 Frame 34 – Trigonometric Functions

6.1 Trigonometric functions – definitions

We know from Euler's formula that

$$e^{ix} = \cos x + i \sin x$$
$$e^{-ix} = \cos x - i \sin x$$

and we can rearrange this into

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i}$$
$$\cos x = \frac{e^{ix} + e^{-ix}}{2}$$

In a similar manner, we can define the complex **trigonometric functions**

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{iz}}{2}$$

6.2 Trigonometric functions – properties

First, the two trig functions are entire, since they are linear combinations of two entire functions. Thus, they are differentiable everywhere; their derivatives, from the complex exponential derivatives, are

$$\frac{d}{dz} \sin z = \cos z$$
$$\frac{d}{dz} \cos z = -\sin z$$

We can also see that the odd/even properties carry over:

$$\sin(-z) = -\sin z$$
$$\cos(-z) = \cos z$$

and Euler's formula also applies:

$$e^{iz} = \cos z + i \sin z$$

Many of the identities from trigonometry carry over. A sample of these identities is:

$$\begin{aligned}
\sin(z_1 + z_2) &= \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \\
\cos(z_1 + z_2) &= \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \\
\sin 2z &= 2 \sin z \cos z \\
\cos 2z &= \cos^2 z - \sin^2 z \\
\sin\left(z + \frac{\pi}{2}\right) &= \cos z \\
\sin\left(z - \frac{\pi}{2}\right) &= -\cos z \\
\sin(z + \pi) &= -\sin z \\
\sin(z + 2\pi) &= \sin z \\
\cos(z + \pi) &= -\cos z \\
\cos(z + 2\pi) &= \cos z \\
\sin^2 z + \cos^2 z &= 1
\end{aligned}$$

6.3 Using hyperbolic functions

The hyperbolic trig functions of a real number y are defined, from calculus, as

$$\begin{aligned}
\sinh y &= \frac{e^y - e^{-y}}{2} \\
\cosh y &= \frac{e^y + e^{-y}}{2}
\end{aligned}$$

We can use these definitions to write

$$\begin{aligned}
\sin(iy) &= i \sinh y \\
\cos(iy) &= \cosh y
\end{aligned}$$

Then, if $z = x + iy$ is a complex number, we can write

$$\begin{aligned}
\sin z &= \sin x \cosh y + i \cos x \sinh y \\
\cos z &= \cos x \cosh y - i \sin x \sinh y
\end{aligned}$$

These expressions allow us to write

$$\begin{aligned}
|\sin z|^2 &= \sin^2 x + \sinh^2 y \\
|\cos z|^2 &= \cos^2 x + \sinh^2 y
\end{aligned}$$

and, since $\sinh y$ is unbounded in y, the trigonometric functions are **unbounded** on the complex plane.

6.4 Extensions to other trigonometric functions

Since

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

we find that $\sin z$ only has zeroes at $x = n\pi$ and $y = 0$; ie:

$$\sin z = 0 \iff z = n\pi$$

Since $\cos z = -\sin(z - \pi/2$, we find that

$$\cos z = 0 \iff z = \left(\frac{1}{2} + n\right)\pi$$

With these zeroes in mind, we can define the four other trigonometric functions as expected:

$$\begin{aligned} \tan z &= \frac{\sin z}{\cos z} \\ \sec z &= \frac{1}{\cos z} \\ \cot z &= \frac{\cos z}{\sin z} \\ \csc z &= \frac{1}{\sin z} \end{aligned}$$

These functions are analytic everywhere except for the singularities caused by the denominators: tan and sec are analytic for all $z \neq (n + 1/2)\pi$, and cot and csc are analytic for all $z \neq n\pi$.

We can use our differentiation rules to find the expected differentiation formulas:

$$\begin{aligned} \frac{d}{dz}\tan z &= \sec^2 z \\ \frac{d}{dz}\sec z &= \sec z \tan z \\ \frac{d}{dz}\cot z &= -\csc^2 z \\ \frac{d}{dz}\csc z &= -\csc z \cot z \end{aligned}$$

7 Frame 35 – Hyperbolic Trigonometry

7.1 Definitions

Following suit from the previous section, we can define the **hyperbolic sine and cosine** of a complex variable as

$$\sinh z = \frac{e^z - e^{-z}}{2}$$
$$\cosh z = \frac{e^z + e^{-z}}{2}$$

As with the regular trigonometric functions, these are linear combinations of entire functions, so they are also entire, with the derivatives

$$\frac{d}{dz} \sinh z = \cosh z$$
$$\frac{d}{dz} \cosh z = \sinh z$$

7.2 Properties

Due to the similar definitions of the regular and hyperbolic trig functions, we can write the following relationships:

$$\begin{aligned}
\cosh(iz) &= \cos z \\
\cos(iz) &= \cosh z \\
-i \sinh(iz) &= \sin z \\
-i \sin(iz) &= \sinh z
\end{aligned}$$

Some common identities are:

$$\begin{aligned}
\sinh(-z) &= -\sinh z \\
\cosh(-z) &= \cosh z \\
\cosh^2 z - \sinh^2 z &= 1 \\
\sinh(z_1 + z_2) &= \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2 \\
\cosh(z_1 + z_2) &= \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2 \\
\sinh z &= \sinh x \cos y + i \cosh x \sin y \\
\cosh z &= \cosh x \cos y + i \sinh x \sin y \\
|\sinh z|^2 &= \sinh^2 x + \sin^2 y \\
|\cosh z|^2 &= \sinh^2 x + \cos^2 y
\end{aligned}$$

We can obtain most of these identities by converting the hyperbolic trig functions into regular trig (as above) and applying the identities discussed in the previous section.

7.3 Extensions

Since sin and cos are periodic with a period of 2π, it is clear that sinh and cosh are also periodic with a period of $2\pi i$. Extending this, we can find the zeroes of each function:

$$\begin{aligned} \sinh z = 0 &\iff z = n\pi i \\ \cosh z = 0 &\iff z = (n + 1/2)\pi i \end{aligned}$$

We can also define the remaining four hyperbolic trig functions as

$$\begin{aligned} \tanh z &= \frac{\sinh z}{\cosh z} \\ \operatorname{sech} z &= \frac{1}{\cosh z} \\ \coth z &= \frac{\cosh z}{\sinh z} \\ \operatorname{csch} z &= \frac{1}{\sinh z} \end{aligned}$$

Again, using the quotient rule, we find that their derivatives are

$$\begin{aligned} \frac{d}{dz} \tanh z &= \operatorname{sech}^2 z \\ \frac{d}{dz} \operatorname{sech} z &= -\operatorname{sech} z \tanh z \\ \frac{d}{dz} \coth z &= -\operatorname{csch}^2 z \\ \frac{d}{dz} \operatorname{csch} z &= -\operatorname{csch} z \coth z \end{aligned}$$