1 Frame 55 – Sequences and Convergence

1.1 Definitions

An **infinite sequence** of complex numbers,

$$z_1, z_2, \ldots, z_n, \ldots$$

has a **limit** if, for each positive number ϵ, there exists a positive integer n_0 such that

$$|z_n - z| < \epsilon \quad \text{whenever} \quad n > n_0$$

Geometrically, this limit implies that for all $n > n_0$, each number z_n in the sequence will be inside an ϵ neighbourhood of z.

A sequence can only have one limit, at most. When this limit exists, we say that the sequence **converges** to z, and we write

$$\lim_{n\to\infty} z_n = z$$

If a sequence has no limit, it **diverges**.

1.2 Components

Theorem: If we write $z_n = x_n + iy_n$ and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

This theorem allows us to write

$$\lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

as long as the limits on either side of this equation exist.

1.3 Examples

Example 1: we can evaluate the following limit easily:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^3 + i} &= \lim_{n\to\infty} \frac{1}{n^3} + i \lim_{n\to\infty} 1 \\
&= 0 + i \cdot 1 \\
&= i
\end{aligned}$$

Example 2: Polar coordinates require some extra care. Looking at the sequence

$$z_n = -2 + i\frac{(-1)^n}{n^2}$$

we can see that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(-2) + i\lim_{n\to\infty}\frac{(-1)^n}{n^2} = -2$$

However, we can find that the principal polar representation of these numbers is

$$r_n = \sqrt{4 + \frac{1}{n^2}}$$
$$\Theta_n = \operatorname{Arg} z_n = \tan^{-1}\left(\frac{(-1)^n}{-2n^2}\right)$$

Evaluating the first limit, we find that

$$\lim_{n\to\infty} r_n = \sqrt{4} = 2$$

which is fine. However, the second sequence does not converge. Looking at every second term, we see that

$$\lim_{n\to\infty}\Theta_{2n} = \pi$$

and

$$\lim_{n\to\infty}\Theta_{2n-1} = -\pi$$

so Θ_n diverges.

2 Frame 56 – Series Convergence

2.1 Definitions

An infinite **series** of complex numbers,

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \ldots,$$

converges to the sum S if the sequence of partial sums,

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N,$$

converges to S. If this is the case, then we can write

$$\sum_{n=1}^{\infty} z_n = S$$

Note that, since a sequence can have at most one limit, a series can have at most one sum. If a series does not converge, it **diverges**.

2.2 Properties – Components

First, as with sequences, we can split a series into its real and imaginary components.

Theorem: If $z_n = x_n + iy_n$ and $S = X + iY$, then

$$\sum_{n=1}^{\infty} z_n = S$$

iff

$$\sum_{n=1}^{\infty} x_n = X \quad \textit{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

To prove this, we can write the partial sums S_N as

$$S_N = X_N + iY_N$$

where

$$X_N = \sum_{n=1}^{N} x_n \quad \text{and} \quad Y_N = \sum_{n=1}^{N} y_n s$$

Then, the series only converges to S if

$$\lim_{N\to\infty} X_N = X \quad \text{and} \quad \lim_{N\to\infty} Y_N = Y$$

due to the theorem on sequences in the previous chapter. Thus, the theorem is proved.

2.3 Properties – Boundedness

The following corollary is a consequence of the previous theorem:

Corollary 1: If a series of complex numbers converges, the summed terms z_n converge to zero.

This is due to the fact that

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i \sum_{n=1}^{\infty} y_n$$

and, in order for these two terms to converge, x_n and y_n must converge to zero (from calculus). Thus,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0$$

This corollary implies that the terms within convergent series are bounded – that is, there exists a constant M such that $|z_n| < M$ for all n.

2.4 Properties – Absolute Convergence

A series is **absolutely convergent** if the related series

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$$

converges. This has a simple implication:

Corollary 2: If a series is absolutely convergent, it is convergent.

To show this, consider the real component of the series. It can be written as

$$\sum_{n=1}^{\infty} x_n \leq \left| \sum_{n=1}^{\infty} x_n \right| \leq \sum_{n=1}^{\infty} |x_n| \leq \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} = 0$$

so the real component must converge. The same is true of the imaginary component, so the corollary is proved.

2.5 Remainders

It is often helpful to define the sequence of **remainders** using the partial sums:

$$\rho_N = S - S_N$$

or $S = S_N + \rho_N$. Since we can write that

$$|S_N - S| = |\rho_N|$$

then a series is only convergent if the sequence of remainders tends to zero.

Example: using remainders, we can verify that

$$\sum_{n=0}^{\infty} z_n = \frac{1}{1-z} \quad \textit{whenver} \quad |z| < 1$$

To do this, we recall that

$$S_N(z) = 1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1-z}$$

so

$$\rho_N(z) = \frac{1}{1-z} - \frac{1 - z^{N+1}}{1-z} = \frac{z^N}{1-z}$$

The moduli of these remaiders are

$$|\rho_N(z)| = \frac{|z|^N}{|1-z|}$$

so $\rho_N(z)$ *tends to zero when* $|z| < 1$*.*

3 Frame 57 – Taylor Series

The following theorem is known as **Taylor's theorem**:

Theorem: If a function f is analytic throughout a disk $|z - z_0| < R_0$, then $f(z)$ has the power series representation

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

This series converges to $f(z)$ when z is in this disk.

Taylor's theorem allows us to write

$$f(z) = f(z_0) + \frac{f'(z_0)}{1!}(z - z_0) + \frac{f''(z_0)}{2!}(z - z_0)^2 + \dots$$

This is true for any function that is analytic at z_0: the requirement for analyticity states that f must be analytic in some neighbourhood of z_0, so the disk mentioned in the theorem exists. In particular, entire functions can use arbitrarily large disks, ie:

$$|z - z_0| < \infty$$

so the series is convergent for all z in the plane.

It can be shown that Taylor's series converges at every point inside the disk – no convergence tests are required. In fact, the smallest radius at which it does **not** converge is the nearest point where f is not analytic.

If $z_0 = 0$ in a Taylor series, it is known as a **Maclaurin series**. Then, it takes the form

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} z^n$$

4 Frame 59 – Examples of Taylor Series

In this section, we will use the formula

$$a_n = \frac{f^{(n)}(z_0)}{n!}$$

to find the Maclaurin expansions of some common functions.

4.1 Example 1

The function e^x is entire, so its Maclaurin expansion is valid for all z. Since

$$f^{(n)}(z) = e^z$$

each term is $a_n = 1/n!$, so we find that

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

Note that, if $z = x + i0$, then

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

as expected.

We can use this result to find the Maclaurin series for the entire function $z^2 e^{3z}$. By replacing z with $3z$ and multiplying through by z^2, we find

$$z^2 e^{3z} = \sum_{n=0}^{\infty} \frac{3^n}{n!} z^{n+2} = \sum_{n=2}^{\infty} \frac{3^{n-2}}{(n-2)!} z^n$$

4.2 Example 2

Using the expansion

$$\sin z = \frac{1}{2i}(e^{iz} - e^{-iz})$$

we can find the Maclaurin series for $f(z) = \sin z$. To do this, we write

$$\begin{aligned} \sin z &= \frac{1}{2i}\left[\sum_{n=0}^{\infty} \frac{(iz)^n}{n!} - \sum_{n=0}^{\infty} \frac{(-iz)^n}{n!}\right] \\ &= \frac{1}{2i} \sum_{n=0}^{\infty} [1 - (-1)^n] \frac{i^n}{n!} z^n \end{aligned}$$

Then, $1-(-1)^n$ is zero for n even, so only taking odd terms gives

$$\begin{aligned}
&= \frac{1}{2i}\sum_{n=0}^{\infty} 2\frac{i^{2n+1}}{(2n+1)!} z^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1}
\end{aligned}$$

This expansion can be used directly to find $\cos z$. Since

$$\cos z = \frac{d}{dz}\sin z$$

we can write

$$\begin{aligned}
\cos z &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{d}{dz} z^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (2n+1) z^{2n} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n}
\end{aligned}$$

4.3 Example 3

Since

$$\sinh z = -i\sin(iz)$$

we can write

$$\begin{aligned}
\sinh z &= -i\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (iz)^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (-1)^n z^{2n+1} \\
&= \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} z^{2n+1}
\end{aligned}$$

Similarly,

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} z^{2n}$$

Also, note that $\cosh z = \cosh(z+2\pi i)$, so

$$\cosh z = \sum_{n=0}^{\infty} \frac{1}{(2n)!} (z+2\pi i)^{2n}$$

4.4 Example 4

If $f(z) = \frac{1}{1-z}$, then

$$f^{(n)}(z) = \frac{n!}{(1-z)^{n+1}}$$

so

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n$$

If we substitute $-z$ for this expression, we find that

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n$$

and if we substitute $1-z$ instead, we find

$$\frac{1}{z} = \sum_{n=0}^{\infty} (-1)^n (z-1)^n$$

Note that all three of these Taylor series have a radius of convergence of 1.

4.5 Example 5

Notice that the function

$$f(z) \frac{1}{z^3} \frac{1}{1+z^2}$$

does not have a Maclaurin series – it is not analytic at $z = 0$. However,

$$\frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \ldots$$

so we can write

$$\begin{aligned} f(z) &= \frac{1}{z^3}(1 - z^2 + z^4 - z^6 + \ldots) \\ &= z^{-3} - z^{-1} + z^1 - z^3 + \ldots \end{aligned}$$

We refer to the first two terms as **negative powers** of z.