1 Frame 12 – Functions of Complex Variables

1(a) The function

$$f(z) = \frac{1}{z^2+1}$$

is defined everywhere except where $z^2 + 1 = 0$; ie:

$$z \neq \pm i$$

1(b) The function

$$f(z) = \text{Arg}\left(\frac{1}{z}\right)$$

is defined wherever $\frac{1}{z}$ is defined:

$$z \neq 0$$

1(c) The function

$$f(z) = \frac{z}{z+\bar{z}}$$

can be written as

$$f(x,y) = \frac{x+iy}{(x+iy)+(x-iy)} = \frac{x+iy}{2x} = \frac{1}{2} + i\frac{y}{x}$$

so the domain is

$$\text{Re}(z) \neq 0$$

1(d) The function

$$f(z) = \frac{1}{1-|z|^2}$$

is equivalent to

$$f(r,\theta) = \frac{1}{1-r^2}$$

so the domain is

$$r \neq 1$$

2 Substituting $z = x + iy$ gives

$$\begin{aligned} f(x,y) &= (x+iy)^3 + (x+iy) + 1 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x + iy + 1 \\ &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) \end{aligned}$$

so

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2 + x + 1 \\ v(x,y) &= 3x^2y - y^3 + y \end{aligned}$$

3 Using the two expressions

$$x = \frac{z + \bar{z}}{2}$$
$$y = \frac{z - \bar{z}}{2i}$$

gives

$$\begin{aligned}
f(z) &= \Big(\frac{z+\bar{z}}{2}\Big)^2 - \Big(\frac{z-\bar{z}}{2i}\Big)^2 - 2\frac{z-\bar{z}}{2i} \\
&\quad + i\Big[2\frac{z+\bar{z}}{2}\Big(1 - \frac{z-\bar{z}}{2i}\Big)\Big] \\
&= \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2 + z^2 - 2z\bar{z} + \bar{z}^2) + iz - i\bar{z} \\
&\quad + i\Big[z + \bar{z} + \frac{iz^2}{2} - \frac{i\bar{z}^2}{2}\Big] \\
&= \frac{1}{2}(z^2 + \bar{z}^2) + 2iz - \frac{iz^2}{2} + \frac{\bar{z}^2}{2} \\
&= \bar{z}^2 + 2iz
\end{aligned}$$

4 Using

$$z = re^{i\theta}$$

the function can be written as

$$\begin{aligned}
f(z) &= re^{i\theta} + \frac{1}{re^{i\theta}} \\
&= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\
&= (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta
\end{aligned}$$

2 Frame 14 – Mappings by the Exponential Function

1 We saw earlier that the hyperbolas

$$x^2 - y^2 = c_1$$

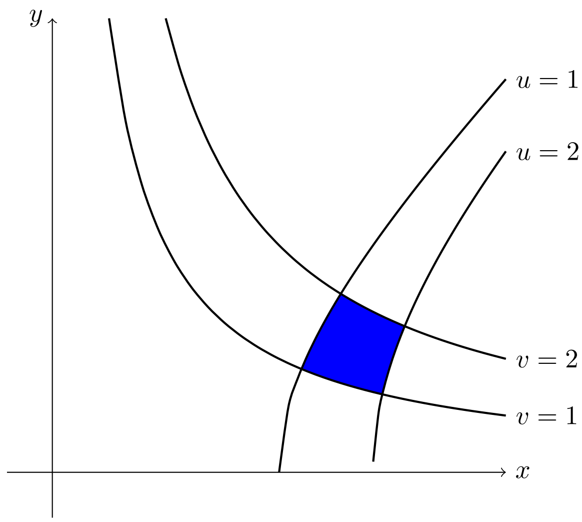
map onto the horizontal lines $u = c_1$ and the hyperbolas

$$2xy = c_2$$

map onto the vertical lines $v = c_2$. Thus, a domain on the z-plane that maps onto $1 \leq u \leq 2$ and $1 \leq v \leq 2$ is

$$1 \leq x^2 - y^2 \leq 2 \quad 1 \leq 2xy \leq 2$$

A sketch of this region is:



2 The first hyperbola can be written as

$$y^2 - x^2 = |c_1|$$

Then, substitution into the v equation gives

$$u = c_1, \quad v = \begin{cases} 2x\sqrt{x^2 + |c_1|}, & y > 0 \\ -2x\sqrt{x^2 + |c_1|}, & y < 0 \end{cases}$$

This maps out the entire v line as x moves right (on the top branch) or left (on the bottom branch).

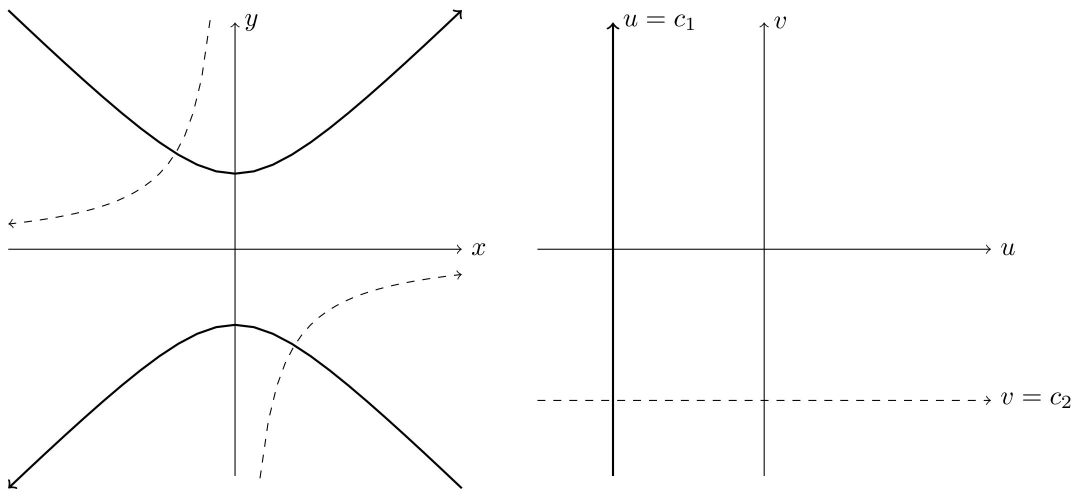
The second hyperbola can be written as

$$2xy = -|c_2|$$

and substituting this into the u equation gives

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

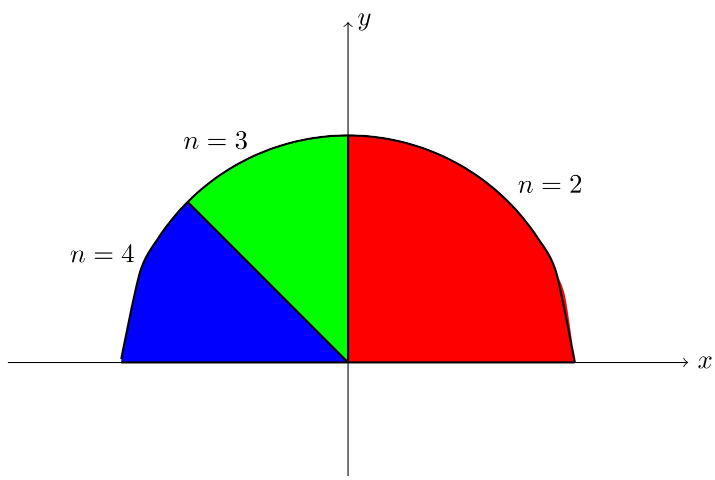
This maps out the entire u line: as x gets large in magnitude, so too does u. A sketch of these mappings is:



3 The image of the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ under the mapping $w = z^n$ is

$$\rho \leq 1, \quad 0 \leq \theta \leq n\frac{\pi}{4}$$

A sketch of these images for $n = 2, 3, 4$ is:



4 If z follows the straight line $ay = x$, then the mapping $w = e^z$ is

$$\begin{aligned} w &= e^{x+iy} \\ &= e^{ay} e^{iy} \\ &= e^{a\phi} e^{i\phi} \\ &= \rho e^{i\phi} \end{aligned}$$

where $\rho = a\phi$.

5 The rectangular region $a \leq x \leq b$, $c \leq y \leq d$ is made up of the horizontal line segments

$$x = t, \quad y = c_1$$

where t is a parameter running from a to b and c_1 is a constant in the range $[c, d]$. These horizontal lines have the images

$$\rho = e^t, \quad \phi = c_1$$

Since t starts at a and ends at b, these images have a radius in the range $[e^a, e^b]$. Then, the entire image is the set of these lines, which range from $\phi = c$ to $\phi = d$. Thus, the entire image is

$$e^a \leq \rho \leq e^b, \quad c \leq \phi \leq d$$

6 Looking at the z plane, the initial set is the infinite strip

$$x \leq 0, \quad 0 \leq y \leq \pi$$

This maps to the image set

$$\lim_{a \to -\infty} e^a \leq \rho \leq e^0, \quad 0 \leq \phi \leq \pi$$

or

$$0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \pi$$

This is the upper half of the unit disk, as shown in the figure.

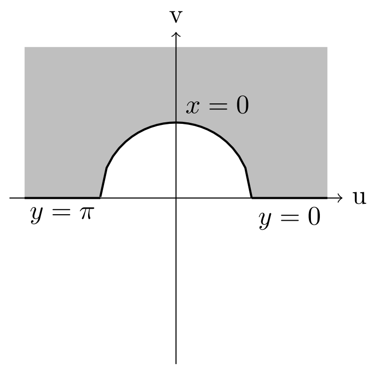
7 In a similar manner to the previous problem, the image of the strip

$$x \geq 0, \quad 0 \leq y \leq \pi$$

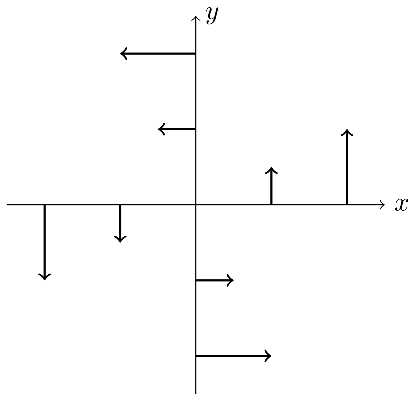
is the upper half-plane with a unit disk cut out:

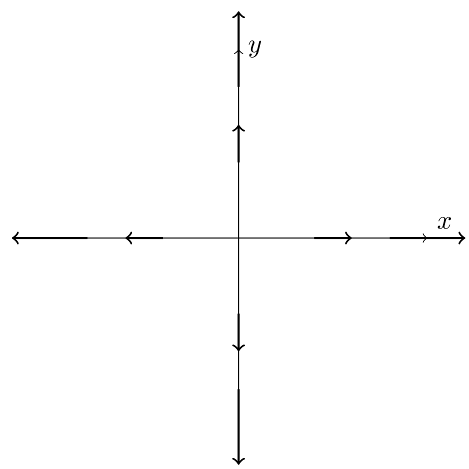
$$\rho \geq 1, \quad 0 \leq \phi \leq \pi$$

A sketch of this region is:



8 Some sample vectors in these two fields are:





3 Frame 18 – Limits and Continuity

1(a) The left side of the limit is

$$|\Re(z) - \Re(z_0)| = |\Re(z - z_0)| < |z - z_0|$$

so

$$|\Re(z) - \Re(z_0)| < \delta \text{ whenever } |z - z_0| < \delta$$

1(b) The left side of the limit is

$$|\bar{z} - \bar{z_0}| = |z \bar{-} z_0| = |z - z_0|$$

so the limit holds.

1(c) The limit expression is

$$|\frac{\bar{z}^2}{z}| < \epsilon \text{ whenever } |z| < \delta$$

For $z \neq 0$, the left side expression is $|z|$, so the limit holds where $\epsilon = \delta$.

2(a) The left side of the limit is

$$|(az + b) - (az_0 + b)| = |a(z - z_0)| = |a||z - z_0|$$

so the limit holds for $\delta = a\epsilon$.

2(b) The left side is

$$|(z^2 + c) - (z_0^2 + c)| = |z^2 - z_0^2| = |z + z_0||z - z_0| \approx 2|z_0||z - z_0|$$

for $\delta << |z_0|$. Thus, the limit holds for $\delta = 2z_0\epsilon$. Note that if $z_0 = 0$, then this reduces to the limit of a constant value c, which is trivial.

2(c) The right side is

$$\begin{aligned} |z - (1 - i)| &= |[x - 1] + i[y + 1]| \\ &= \sqrt{(x - 1)^2 + (y + 1)^2} \end{aligned}$$

The left side is

$$\begin{aligned} |[x + i(2x + y)] - [1 + i]| &= |[x - 1] + i[2x + y - 1]| \\ &= |[x - 1] + i[2(x - 1) + (y + 1)]| \end{aligned}$$

Not sure how to prove limits – it appears as the right side goes to 0, the left side must too.

3(a) If $z_0 \neq 0$, the limit must be

$$\lim_{z \to z_0} \frac{1}{z^n} = \frac{\lim_{z \to z_0} 1}{\lim_{z \to z_0} z^n} = \frac{1}{z_0^n}$$

3(b)

$$\lim_{z \to i} \frac{iz^3 - 1}{z + i} = \frac{\lim z \to iiz^3 - 1}{\lim z \to iz + i} = \frac{i(i^3) - 1}{(i) + i} = 0$$

3(c)

$$\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{\lim_{z \to z_0} P(z)}{\lim_{z \to z_0} Q(z)} = \frac{P(z_0)}{Q(z_0)}$$

4 The base case is

$$\lim_{z \to z_0} z = z_0$$

which was shown earlier.

If it is known that

$$\lim_{z \to z_0} z^k = z_0^k$$

then the following limit is

$$\lim_{z \to z_0} z^{k+1} = \left(\lim_{z \to z_0} z^k \right) \left(\lim_{z \to z_0} z \right) = (z_0^k)(z_0) = z_0^{k+1}$$

so, by induction, we are finished.

5 First, for any point $z = (x, 0)$, the function is

$$f(x, 0) = \left(\frac{x}{x} \right)^2 = (1)^2 = 1$$

so the limit as $x \to 0$ is 1.

Then, for any point $z = (x, x)$, the function is

$$f(x, x) = \left(\frac{x + ix}{x - ix} \right)^2 = (i)^2 = -1$$

so the limit as $x \to 0$ is -1. This conflicts with the previous result, so the limit does not exist.

6(a) *The statement to be proved is*

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0$$

If f and F are split into their real and imaginary parts (u, v, U, and V), then we know that

$$\begin{aligned}
\lim_{z\to z_0} u(x,y) &= u_0 \\
\lim_{z\to z_0} v(x,y) &= v_0 \\
\lim_{z\to z_0} U(x,y) &= U_0 \\
\lim_{z\to z_0} V(x,y) &= V_0
\end{aligned}$$

Then, the sum $f(z)+F(z)$ has a real and imaginary part, which have the limits

$$\begin{aligned}
\lim_{z\to z_0} u(x,y) + U(x,y) &= u_0 + U_0 \\
\lim_{z\to z_0} v(x,y) + V(x,y) &= v_0 + V_0
\end{aligned}$$

so the result is

$$\lim_{z\to z_0} f(z) + F(z) = u_0 + v_0 + U_0 + V_0 = w_0 + W_0$$

6(b) The left side of the limit expression is

$$\begin{aligned}
|f(z) + F(z) - w_0 - W_0| &= |f(z) - w_0 + F(z) - W_0| \\
&\leq |f(z) - w_0| + |F(z) - W_0| \\
&< \epsilon + \epsilon \\
&= 2\epsilon
\end{aligned}$$

so the statement is true, and the limit is $w_0 + W_0$.

7 The left hand side of the expression is

$$||f(z)| - |w_0|| \leq |f(z) - w_0|$$

which is the standard limit expression. Since $\lim_{z\to z_0} f(z) = w_0$, this expression is true, and the limit is $|w_0|$.

10(a) Making the replacement in the expression, this limit is

$$\begin{aligned}
\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} &= \lim_{z\to 0} \frac{4(1/z)^2}{(1-(1/z))^2} \\
&= \lim_{z\to 0} \frac{4}{(z-1)^2} \\
&= \frac{4}{1} = 4
\end{aligned}$$

10(b) The reciprocal of this expression has the limit

$$\lim_{z\to 1}(z-1)^3 = 0$$

so the limit is infinity.

10(c) Taking both reciprocals,

$$\begin{aligned}\lim_{z\to\infty}\frac{z-1}{z^2+1} &= \lim_{z\to 0}\frac{(1/z)-1}{(1/z)^2+1}\\ &= \lim_{z\to 0}\frac{z-z^2}{1+z^2}\\ &= \frac{0}{1} = 0\end{aligned}$$

so the limit is infinity.

4 Frame 20 – Differentiation

1 These four derivatives are:

- (a)
$$\frac{d}{dz}(3z^2 - 2z + 4) = 6z - 2$$

- (b)
$$\frac{d}{dz}(1-4z^2)^3 = 3(1-4z^2)^2(-8z) = -24z(1-4z^2)^2$$

- (c)
$$\frac{d}{dz}\frac{z-1}{2z+1} = \frac{(1)(2z+1)-(2)(z-1)}{(2z+1)^2} = \frac{-1}{(2z+1)^2}$$

- (d)
$$\begin{aligned}\frac{d}{dz}\frac{(1+z^2)^4}{z^2} &= \frac{4(1+z^2)^3(2z)(z^2) - 2z(1+z^2)^4}{z^4} \\ &= \frac{8z^3(1+z^2)^3 - (2z+2z^3)(1+z^2)^3}{z^4} \\ &= \frac{2z(3z^2-1)(1+z^2)^3}{z^4}\end{aligned}$$

2 First, each term of the polynomial $P_k(z)$ is

$$P_k(z) = a_k z^k$$

All of these terms are differentiable everywhere, and the derivative is

$$P_k'(z) = k a_k z^{k-1}, \quad k \neq 0$$

where the derivative of a_0 is simply zero. Then, we proved that the derivative of a sum is the sum of two derivatives, so we can apply this repeatedly to find

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$$

Notice that the function's value at zero is $P(0) = a_0$. The derivative's value at zero is similarly $P'(0) = a_1$. Applying the same process again, we find that

$$P''(0) = 2a_2$$

and

$$P^{(k)}(0) = k! \cdot a_k$$

Rearranging these terms, we can write

$$\begin{aligned}
a_0 &= P(0) \\
a_1 &= \frac{P'(0)}{1!} \\
a_2 &= \frac{P''(0)}{2!} \\
&\dots \\
a_n &= \frac{P^{(n)}(0)}{n!}
\end{aligned}$$

3 We can find the derivative of $f(z) = \frac{1}{z}$ by using the definition:

$$\frac{d}{dz}\frac{1}{z} = \lim_{\Delta z \to 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z) - (z+\Delta z)}{z\Delta z(z+\Delta z)} = \lim_{\Delta z \to 0} \frac{-1}{z(z+\Delta z)} = \frac{-1}{z^2}$$

4 Applying the definition of a derivative, we can see that

$$\begin{aligned}
\lim_{z \to z_0} \frac{f(z)}{g(z)} &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} \\
&= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \frac{z - z_0}{g(z) - g(z_0)} \\
&= \frac{f'(z_0)}{g'(z_0)}
\end{aligned}$$

5 Following the proof shown in the chapter, the derivative of a sum makes the term Δw into

$$\Delta w = f(z+\Delta z) - f(z) + g(z+\Delta z) - g(z)$$

Then, the derivative is

$$\begin{aligned}
\frac{d}{dz}[f(z) + g(z)] &= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z) + g(z+\Delta z) - g(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \to 0} \frac{g(z+\Delta z) - g(z)}{\Delta z} \\
&= f'(z) + g'(z)
\end{aligned}$$

6 *Base case:* the derivative of z^1 is 1.

$k+1$ *case:* if the derivative of z^k is kz^{k-1}, then the derivative of z^{k+1} is

$$\frac{d}{dz}z^{k+1} = \frac{d}{dz}(z^k)(z) = (kz^{k-1})(z) + (z^k) = (k+1)z^k$$

Therefore, by induction, we are done.

7 When n is a negative integer, we can write $m = -n$ and rewrite the function as

$$f(z) = \frac{1}{z^m}$$

Then, using the quotient rule, the derivative of this function is

$$f'(z) = \frac{(0)(z^m) - (1)(mz^{m-1})}{z^{2m}} = \frac{nz^{-n-1}}{z^{-2n}} = nz^{n-1}$$

so the derivative formula is still valid.