1 Frame 55 – Sequences and Convergence

1.1 Definitions

An **infinite sequence** of complex numbers,

$$z_1, z_2, \ldots, z_n, \ldots$$

has a **limit** if, for each positive number ϵ, there exists a positive integer n_0 such that

$$|z_n - z| < \epsilon \quad \text{whenever} \quad n > n_0$$

Geometrically, this limit implies that for all $n > n_0$, each number z_n in the sequence will be inside an ϵ neighbourhood of z.

A sequence can only have one limit, at most. When this limit exists, we say that the sequence **converges** to z, and we write

$$\lim_{n\to\infty} z_n = z$$

If a sequence has no limit, it **diverges**.

1.2 Components

Theorem: If we write $z_n = x_n + iy_n$ and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

This theorem allows us to write

$$\lim_{n\to\infty}(x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

as long as the limits on either side of this equation exist.

1.3 Examples

Example 1: we can evaluate the following limit easily:

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n^3 + i} &= \lim_{n\to\infty} \frac{1}{n^3} + i \lim_{n\to\infty} 1 \\ &= 0 + i \cdot 1 \\ &= i\end{aligned}$$

Example 2: Polar coordinates require some extra care. Looking at the sequence

$$z_n = -2 + i\frac{(-1)^n}{n^2}$$

we can see that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(-2) + i\lim_{n\to\infty}\frac{(-1)^n}{n^2} = -2$$

However, we can find that the principal polar representation of these numbers is

$$r_n = \sqrt{4 + \frac{1}{n^2}}$$
$$\Theta_n = \operatorname{Arg} z_n = \tan^{-1}\left(\frac{(-1)^n}{-2n^2}\right)$$

Evaluating the first limit, we find that

$$\lim_{n\to\infty} r_n = \sqrt{4} = 2$$

which is fine. However, the second sequence does not converge. Looking at every second term, we see that

$$\lim_{n\to\infty}\Theta_{2n} = \pi$$

and

$$\lim_{n\to\infty}\Theta_{2n-1} = -\pi$$

so Θ_n *diverges.*

2 Frame 56 – Series Convergence

2.1 Definitions

An infinite **series** of complex numbers,

$$\sum_{n=1}^{\infty} z_n = z_1 + z_2 + z_3 + \cdots + z_n + \ldots,$$

converges to the sum S if the sequence of partial sums,

$$S_N = \sum_{n=1}^{N} z_n = z_1 + z_2 + \cdots + z_N,$$

converges to S. If this is the case, then we can write

$$\sum_{n=1}^{\infty} z_n = S$$

Note that, since a sequence can have at most one limit, a series can have at most one sum. If a series does not converge, it **diverges**.

2.2 Properties – Components

First, as with sequences, we can split a series into its real and imaginary components.

Theorem: If $z_n = x_n + iy_n$ and $S = X + iY$, then

$$\sum_{n=1}^{\infty} z_n = S$$

iff

$$\sum_{n=1}^{\infty} x_n = X \quad \textit{and} \quad \sum_{n=1}^{\infty} y_n = Y$$

To prove this, we can write the partial sums S_N as

$$S_N = X_N + iY_N$$

where

$$X_N = \sum_{n=1}^{N} x_n \quad \text{and} \quad Y_N = \sum_{n=1}^{N} y_n s$$

Then, the series only converges to S if

$$\lim_{N\to\infty} X_N = X \quad \text{and} \quad \lim_{N\to\infty} Y_N = Y$$

due to the theorem on sequences in the previous chapter. Thus, the theorem is proved.

2.3 Properties – Boundedness

The following corollary is a consequence of the previous theorem:

Corollary 1: If a series of complex numbers converges, the summed terms z_n converge to zero.

This is due to the fact that

$$\sum_{n=1}^{\infty} z_n = \sum_{n=1}^{\infty} x_n + i\sum_{n=1}^{\infty} y_n$$

and, in order for these two terms to converge, x_n and y_n must converge to zero (from calculus). Thus,

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n = 0$$

This corollary implies that the terms within convergent series are bounded – that is, there exists a constant M such that $|z_n| < M$ for all n.

2.4 Properties – Absolute Convergence

A series is **absolutely convergent** if the related series

$$\sum_{n=1}^{\infty} |z_n| = \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2}$$

converges. This has a simple implication:

Corollary 2: If a series is absolutely convergent, it is convergent.

To show this, consider the real component of the series. It can be written as

$$\sum_{n=1}^{\infty} x_n \le \left|\sum_{n=1}^{\infty} x_n\right| \le \sum_{n=1}^{\infty} |x_n| \le \sum_{n=1}^{\infty} \sqrt{x_n^2 + y_n^2} = 0$$

so the real component must converge. The same is true of the imaginary component, so the corollary is proved.

2.5 Remainders

It is often helpful to define the sequence of **remainders** using the partial sums:

$$\rho_N = S - S_N$$

or $S = S_N + \rho_N$. Since we can write that

$$|S_N - S| = |\rho_N|$$

then a series is only convergent if the sequence of remainders tends to zero.

Example: using remainders, we can verify that

$$\sum_{n=0}^{\infty} z_n = \frac{1}{1-z} \quad \textit{whenver} \quad |z| < 1$$

To do this, we recall that

$$S_N(z) = 1 + z + z^2 + \cdots + z^N = \frac{1 - z^{N+1}}{1-z}$$

so

$$\rho_N(z) = \frac{1}{1-z} - \frac{1 - z^{N+1}}{1-z} = \frac{z^N}{1-z}$$

The moduli of these remaiders are

$$|\rho_N(z)| = \frac{|z|^N}{|1-z|}$$

so $\rho_N(z)$ tends to zero when $|z| < 1$.