

# 1 Frame 29 – The Exponential Function

## 1.1 Definition

We define the **exponential function**  $e^z$  by writing

$$e^z = e^x e^{iy}$$

and we apply Euler's formula to get

$$e^z = e^x (\cos y + i \sin y)$$

Note that, when  $y = 0$ ,  $e^z$  reduces to  $e^x$ .

Although we typically understand that  $e^{1/n}$  would be the set of  $n$ th roots of  $e$ , here, we only use the real, positive root  $\sqrt[n]{e}$ .

## 1.2 Familiar properties

First, in calculus, we know that

$$e^{x_1} e^{x_2} = e^{x_1+x_2}$$

It is easy to verify that this holds true for complex numbers:

$$e^{z_1} e^{z_2} = e^{z_1+z_2}$$

This also allows us to write

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1-z_2}$$

and, as a specific case,

$$\frac{1}{e^z} = e^{-z}$$

We showed earlier that  $e^z$  is differentiable everywhere in the complex plane, and that

$$\frac{d}{dz} e^z = e^z$$

We also know that  $e^z$  is never zero. This comes from the pair

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi$$

and since  $e^x$  is never zero, neither is  $e^z$ .

### 1.3 Unfamiliar properties

Since we can write

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

the exponential function is periodic with an imaginary period of  $2\pi i$ .

It is also possible for the complex exponential function to be negative. For an example, we know that Euler's identity states

$$e^{i\pi} = -1$$

In fact,  $e^z$  can be any given non-zero complex number.

*Example: suppose we want solutions to the equation*

$$e^z = 1 + i$$

*The right side can be rewritten as*

$$e^x e^{iy} = \sqrt{2} e^{i\pi/4}$$

*and equating the parts of this equation gives*

$$x = \ln \sqrt{2} = \frac{1}{2} \ln 2 \quad \text{and} \quad y = \left(2n + \frac{1}{4}\right) \pi$$

*so*

$$z = \frac{1}{2} \ln 2 + \left(2n + \frac{1}{4}\right) \pi i$$

## 2 Frame 30 – The Logarithmic Function

### 2.1 Motivation

We said in the previous section that  $e^z$  can take on any non-zero complex value. To help us solve the equation

$$e^w = z$$

we will define a **logarithmic function**, such that

$$e^{\log z} = z \quad (z \neq 0)$$

We can solve for  $w$  by writing the two complex numbers in the form

$$\begin{aligned} z &= re^{i\theta} \\ w &= u + iv \end{aligned}$$

Substituting these into the original equation gives

$$e^u e^{iv} = re^{i\theta}$$

so we get

$$w = \log z = \ln r + i(\theta + 2n\pi)$$

Note that this is a multi-valued function.

*Example: if  $z = -1 - i\sqrt{3}$ , then  $r = 2$  and  $\theta = -2\pi/3$ , so*

$$\log(-1 - i\sqrt{3}) = \ln 2 + \left(n - \frac{1}{3}\right) 2\pi i$$

### 2.2 Precise definition

A more precise definition of the multi-valued logarithmic function is

$$\log z = \ln |z| + i \arg z$$

The **principal value** of  $\log z$  is obtained by using the single-valued principal argument instead:

$$\text{Log } z = \ln |z| + i\theta$$

Note that

$$\log z = \text{Log } z + i2n\pi$$

## 2.3 Notes

The principal logarithmic function  $\text{Log } z$  reduces to the usual logarithm from calculus when  $z$  is positive and real – if  $z = r$ , then

$$\text{Log } r = \ln r$$

However, we are now able to find the logarithm of negative real numbers, which we were unable to do in calculus.

*Example: the logarithm of  $-1$  is*

$$\log(-1) = \ln 1 + (1 + 2n)i\pi = (2n + 1)i\pi$$

*and*

$$\text{Log } (-1) = i\pi$$

### 3 Frame 31 – Branches & Derivatives of Logarithms

#### 3.1 Limiting a logarithm's domain

We saw in the previous section that the multi-valued logarithm function of a complex number  $z = re^{i\theta}$  can be written as

$$\log z = \ln r + i\theta$$

where  $\theta$  can have any of the values

$$\theta = \text{Arg}(z) + 2n\pi$$

We can make the logarithmic function single-valued by restricting the value of  $\theta$  to  $\alpha < \theta < \alpha + 2\pi$  for any real value of  $\theta$ . Then, the function is single-valued and is continuous everywhere in the domain of the function (ie:  $r > 0$  and  $\theta \in (\alpha, \alpha + 2\pi)$ ). Note that we cannot include in the ray  $\theta = \alpha$  – the function would not be continuous here.

In this limited domain, the components of the log function also satisfy the polar Cauchy-Riemann equations

$$ru_r = 1 = v_\theta; \quad u_\theta = 0 = -rv_r$$

so the logarithmic function is analytic in this domain, with the derivative

$$\frac{d}{dz} \log z = e^{-i\theta}(u_r + iv_r) = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

In particular, we can set  $\alpha = -\pi$  and write

$$\frac{d}{dz} \text{Log } z = \frac{1}{z} \quad (|z| > 0, -\pi < \text{Arg } z < \pi)$$

Note that not all of the identities from calculus carry over to the complex plane.

*Example: using the principal branch,*

$$\text{Log } (i^3) = \text{Log } (-i) = -i\frac{\pi}{2}$$

*but*

$$3\text{Log } (i) = 3\left(i\frac{\pi}{2}\right) = i\frac{3\pi}{2}$$

*so*

$$\text{Log } (i^3) \neq 3\text{Log } (i)$$

### 3.2 Branches

A **branch** of a multi-valued function  $f$  is any single-valued, analytic function  $F$  such that  $F(z)$  is one of the values of  $f$  at each point within the domain of  $F$ . For instance, our limited-domain logarithm is a branch of the multi-valued log function. The principal logarithm function

$$\text{Log } z = \ln r + i\theta \quad (|z| > 0, -\pi < \text{Arg } z < \pi)$$

is known as the **principal branch**.

A **branch cut** is a line/curve that is used to define a branch  $F$  of a multi-valued function  $f$ . Any point on the branch cut is a singular point of  $F$ . Any point that is common to all branch cuts of  $f$  is called a **branch point**. For example, the logarithmic function has a branch point at  $z = 0$  and a branch cut on the ray  $\theta = \alpha$ . In particular, the ray  $\theta = \pi$  is the branch cut for the principal logarithmic function.