1 Frame 12 – Functions of Complex Variables

1.1 Functions

If S is a set of complex numbers, then a **function** f is a rule that assigns a complex number w to each z in S. The number w is called the **value** of f at z. We denote it as

$$w = f(z)$$

The set S is called the **domain of definition** of f. Note that we need both a rule (f) and a domain (S) for a function to be well defined.

Suppose that $w = u + iv$ and $z = x + iy$. Then,

$$u + iv = f(x + iy)$$

Then, we can express $f(z)$ as a pair of real functions of x and y:

$$f(z) = u(x, y) + iv(x, y)$$

Alternatively, we could use polar coordinates to write

$$u + iv = f(re^{i\theta})$$

so

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Example: the function $f(z) = z^2$ can be written as

$$\begin{aligned} f(x + iy) &= (x + iy)^2 \\ &= (x^2 - y^2) + i2xy \end{aligned}$$

so

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy \end{aligned}$$

In polar coordinates,

$$\begin{aligned} f(x + iy) &= (re^{i\theta})^2 \\ &= r^2 e^{i2\theta} \\ &= r^2 \cos 2\theta + ir^2 \sin 2\theta \end{aligned}$$

so

$$\begin{aligned} u(r, \theta) &= r^2 \cos 2\theta \\ v(r, \theta) &= r^2 \sin 2\theta \end{aligned}$$

1.2 Real-Valued Functions

We say that f is a **real-valued function** if v is zero everywhere.

Example: one real-valued function is

$$f(z) = |z|^2 = x^2 + y^2 + i0$$

1.3 Polynomials

If n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are complex numbers with $a_n \neq 0$, then the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a **polynomial** of degree n. Note that this sum has a finite number of terms and that the domain of definition is the entire z plane.

As in real numbers, a **rational function** is a quotient of two polynomials:

$$R(z) = \frac{P(z)}{Q(z)}$$

A rational function is defined everywhere that $Q(z) \neq 0$.

1.4 Multi-Valued Functions

A generalization of a function is a rule that assigns more than one value to a point z. These **multiple-valued functions** are usually studied by taking one of the possible values at each point and constructing a single-valued function.

Example: we know that we can write

$$z^{1/2} = \pm\sqrt{r}e^{i\theta/2}$$

*where we denoted $-\pi < \theta \leq \pi$ as the **principal value** of $\arg z$. To turn this into a single valued function, we can choose the positive value of r and write*

$$f(z) = \sqrt{r}e^{i\theta/2}$$

Then, f is well-defined on the entire plane.

1.5 Exercises

1(a) The function

$$f(z) = \frac{1}{z^2+1}$$

is defined everywhere except where $z^2 + 1 = 0$; ie:

$$z \neq \pm i$$

1(b) The function

$$f(z) = \mathrm{Arg}\Big(\frac{1}{z}\Big)$$

is defined wherever $\frac{1}{z}$ is defined:

$$z \neq 0$$

1(c) The function

$$f(z) = \frac{z}{z+\bar{z}}$$

can be written as

$$f(x,y) = \frac{x+iy}{(x+iy)+(x-iy)} = \frac{x+iy}{2x} = \frac{1}{2} + i\frac{y}{x}$$

so the domain is

$$\mathrm{Re}(z) \neq 0$$

1(d) The function

$$f(z) = \frac{1}{1-|z|^2}$$

is equivalent to

$$f(r,\theta) = \frac{1}{1-r^2}$$

so the domain is

$$r \neq 1$$

2 Substituting $z = x + iy$ gives

$$\begin{aligned} f(x,y) &= (x+iy)^3 + (x+iy) + 1 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x + iy + 1 \\ &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) \end{aligned}$$

so

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2 + x + 1 \\ v(x,y) &= 3x^2y - y^3 + y \end{aligned}$$

3 Using the two expressions

$$x = \frac{z + \bar{z}}{2}$$
$$y = \frac{z - \bar{z}}{2i}$$

gives

$$\begin{aligned}
f(z) &= \Big(\frac{z + \bar{z}}{2}\Big)^2 - \Big(\frac{z - \bar{z}}{2i}\Big)^2 - 2\frac{z - \bar{z}}{2i} \\
&\quad + i\Big[2\frac{z + \bar{z}}{2}\Big(1 - \frac{z - \bar{z}}{2i}\Big)\Big] \\
&= \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2 + z^2 - 2z\bar{z} + \bar{z}^2) + iz - i\bar{z} \\
&\quad + i\Big[z + \bar{z} + \frac{iz^2}{2} - \frac{i\bar{z}^2}{2}\Big] \\
&= \frac{1}{2}(z^2 + \bar{z}^2) + 2iz - \frac{iz^2}{2} + \frac{\bar{z}^2}{2} \\
&= \bar{z}^2 + 2iz
\end{aligned}$$

4 Using

$$z = re^{i\theta}$$

the function can be written as

$$\begin{aligned}
f(z) &= re^{i\theta} + \frac{1}{re^{i\theta}} \\
&= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\
&= (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta
\end{aligned}$$