1 Frame 29 – The Exponential Function

1(a) The value is
$$e^{2\pm 3\pi i} = e^2 e^{\pm 3\pi i} = e^2(-1) = e^2$$

1(b) The value is
$$e^{1/2+i\pi/4} = e^{1/2}e^{i\pi 4} = \sqrt{e}\frac{1+i}{\sqrt{2}} = \sqrt{\frac{e}{2}}(1+i)$$

1(c) This expression can be split up as
$$e^{z+\pi i} = e^z e^{i\pi} = e^z \cdot -1 = -e^z$$

3 The function can be written as
$$e^{\overline{z}} = e^x e^{-iy} = e^x \cos y - ie^x \sin y$$

so the partial derivatives are

$$\begin{aligned} u_x &= e^x \cos y \\ u_y &= -e^x \sin y \\ v_x &= -e^x \sin y \\ v_y &= -e^x \cos y \end{aligned}$$

These don't satisfy Cauchy-Riemann, so the function is differentiable nowhere and thus analytic nowhere.

8(a) The equation $e^z = -2$ can be written as
$$e^z = 2e^{i\pi}$$

so

$$\begin{aligned} e^x = 2 &\to x = \ln 2 \\ e^{iy} = e^{i\pi} &\to y = (2n+1)\pi \end{aligned}$$

and
$$z = \ln 2 + (2n+1)i\pi$$

8(b) The equation $e^z = 1 + i\sqrt{3}$ can be written as
$$e^z = 2e^{i\pi/3}$$

so

$$\begin{aligned} x &= \ln 2 \\ y &= \left(\frac{1}{3} + 2n\right)\pi \end{aligned}$$

and

$$z = \ln 2 + \left(\frac{1}{3} + 2n\right) i\pi$$

8(c) If

$$e^{2z-1} = 1$$

then

$$2z - 1 = i2n\pi$$

or, solving for z,

$$z = \frac{1}{2} + in\pi$$

2 Frame 31 – Logarithms

1(a) Evaluating the logarithm,

$$\text{Log}(-ei) = \ln(e) + \text{Arg}(-ei) = 1 - i\frac{\pi}{2}$$

1(b) As above,

$$\text{Log}(1-i) = \ln(\sqrt{2}) + \text{Arg}(1-i) = \frac{1}{2}\ln(2) - i\frac{\pi}{4}$$

2(a) The set of values is

$$\log(e) = \ln(e) + \arg(e) = 1 + i2n\pi$$

2(b) As above,

$$\log(i) = \ln(1) + \arg(i) = 0 + i\left(2n + \frac{1}{2}\right)\pi$$

2(c) As above,

$$\log(-1+i\sqrt{3}) = \ln(\sqrt{4}) + \arg(-1+i\sqrt{3}) = \ln 2 + i\left(2n + \frac{2}{3}\right)\pi$$

3(a) The left side is

$$\text{Log}(1+i)^2 = \ln(2) + i\frac{\pi}{2}$$

and the right side is

$$2\,\text{Log}(1+i) = 2\left[\frac{1}{2}\ln(2) + i\frac{\pi}{4}\right] = \ln(2) + i\frac{\pi}{2} = \text{Log}(1+i)^2$$

3(b) The left side is

$$\text{Log}(-1+i)^2 = \ln(2) - i\frac{\pi}{2}$$

and the right side is

$$2\,\text{Log}(-1+i) = 2\left[\frac{1}{2}\ln(2) + i\frac{3\pi}{4}\right] = \ln(2) + i\frac{3\pi}{2}$$

5(a) Since $i^{1/2}$ can be written as the set

$$i^{1/2} = e^{\pi/4+\pi k} \quad (k = 0, \pm 1, \pm 2, \dots)$$

it has the logarithm

$$\log(i^{1/2}) = \ln(1) + i\arg(i^{1/2}) = i\left(n + \frac{1}{4}\right)\pi$$

Then,

$$\frac{1}{2}\log i = \frac{1}{2}\left[\ln(1) + i\left(2n + \frac{1}{2}\right)\pi\right] = i\left(n + \frac{1}{4}\right)\pi = \log(i^{1/2})m$$

6 Differentiating and using the chain rule, the identity

$$z = e^{\log z}$$

becomes

$$1 = e^{\log z} \cdot \frac{d}{dz}\log z$$

or

$$\frac{d}{dz}\log z = \frac{1}{e^{\log z}} = \frac{1}{z}$$

7 To solve the equation

$$\log z = i\frac{\pi}{2}$$

we write

$$z = e^{i\pi/2} = i$$

3 Frame 33 – Complex Exponents

1(a) The values of $(1+i)^i$ are

$$(1+i)^i = e^{i\cdot(\ln\sqrt{2}+i(2n+1/4)\pi)} = e^{i\ln(2)/2}e^{-\pi/4+2n\pi}$$

1(b) The values of $(-1)^{1/\pi}$ are

$$(-1)^{1/pi} = e^{\frac{1}{\pi}\cdot i(2n+1)\pi} = e^{(2n+1)i}$$

2(a) The principal value of i^i is

$$i^i = e^{i\,\mathrm{Log}\,i} = e^{i(i\pi/2)} = e^{-\pi/2}$$

2(b) The principal value of $[\frac{e}{2}(-1-i\sqrt{3})]^{3\pi i}$ is

$$[\frac{e}{2}(-1-i\sqrt{3})]^{3\pi i} = \exp\left[3\pi i\left(1-i\frac{2\pi}{3}\right)\right] = e^{3\pi i}e^{-2\pi^2} = -e^{-2\pi^2}$$

2(c) The principal value of $(1-i)^{4i}$ is

$$(1-i)^{4i} = e^{4i(\frac{1}{2}\ln 2-i\pi/4)} = e^{i2\ln(2)}e^{\pi} = e^{\pi}[\cos(2\ln 2)+i\sin(2\ln 2)]$$

3 The values of $(-1+i\sqrt{3})^{3/2}$ are

$$(-1+i\sqrt{3})^{3/2} = e^{3/2\cdot(\ln 2+i\pi(2n+2/3))} = 2^{3/2}e^{i(\pi+3n\pi)} = \pm 2\sqrt{2}$$