1 Frame 37 – Derivatives with Real Variables

1.1 Definition

In the previous chapter, we looked at derivatives of complex functions of a complex variable z. Now, we look at the derivatives of a complex-valued function of a real variable t. If we write our function as

$$w(t) = u(t) + iv(t)$$

where u and v are real-valued, then we can define the derivative of w at a point t as

$$w'(t) = \frac{d}{dt}w(t) = u'(t) + iv'(t)$$

provided that u' and v' exist at t.

1.2 Properties

If $z_0 = x_0 + iy_0$ is a complex constant, then we can show that

$$\begin{aligned}
\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u(t) + iv(t)]' \\
&= [x_0 u(t) - y_0 v(t)]' + i[y_0 u(t) + x_0 v(t)]' \\
&= [x_0 u'(t) - y_0 v'(t)] + i[y_0 u'(t) + x_0 v'(t)] \\
&= z_0 w'(t)
\end{aligned}$$

as we expect.

Next, if z_0 is still a complex constant, the derivative of $e^{z_0 t}$ is

$$\begin{aligned}
\frac{d}{dt}e^{z_0 t} &= \frac{d}{dt}e^{x_0 t}(\cos y_0 t + isiny_0 t) \\
&= \frac{d}{dt}e^{x_0 t}\cos y_0 t + i\frac{d}{dt}e^{x_0 t}\sin y_0 t \\
&= (x_0 + iy_0)(e^{x_0 t}\cos y_0 t + ie^{x_0 t}\sin y_0 t) \\
&= z_0 e^{z_0 t}
\end{aligned}$$

Many other rules carry over from standard calculus. However, some rules no longer apply. For instance, in calculus, the mean value theorem for derivatives states that

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

for some c in the interval $a \le c \le b$ as long as w is continuous. However, this is easily disproved by the function

$$w(t) = e^{it}$$

If $a = 0$ and $b = 2\pi$, then $w(a) = w(b) = 1$ and we expect to find a point c in $[0, 2\pi]$ such that $w'(c) = 0$. However, no such points exist – the magnitude of the derivative is always 1.