1 Frame 78 – Evaluating Improper Integrals

1.1 Improper Integrals

The **improper integral** of a continuous function $f(x)$ over the interval $0 \leq x < \infty$ is defined as

$$\int_0^\infty f(x)\ dx = \lim_{R\to\infty} \int_0^R f(x)\ dx$$

If this limit exists, we say that the improper integral **converges** to this limit. The improper integral of f over the infinite interval $-\infty < x < \infty$ is

$$\int_{-\infty}^{\infty} = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\ dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\ dx$$

If both of these limits exist, we say that the integral converges to their sum.

1.2 The Cauchy Principal Value

We say that the **Cauchy Principal Value** of an indefinite integral is

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx$$

as long as this limit exists.

If the regular improper integral converges, then

$$\begin{aligned} \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx &= \lim_{R\to\infty} \left[\int_{-R}^{0} f(x)\ dx + \int_0^R f(x)\ dx \right] \\ &= \lim_{R\to\infty} \int_{-R}^{0} f(x)\ dx + \lim_{R\to\infty} \int_0^R f(x)\ dx \end{aligned}$$

so the principal value also exists. However, the converse is not true – the existence of the principal value does not imply the existence of the improper integral.

Next, suppose that $f(x)$ is an even function; ie:

$$f(-x) = f(x)$$

and assume that the principal value exists. Then, the evenness of f allows us to write

$$\begin{aligned} \int_{-R_1}^{0} f(x)\ dx &= \frac{1}{2} \int_{-R_1}^{R_1} f(x)\ dx \\ \int_0^{R_2} f(x)\ dx &= \frac{1}{2} \int_{-R_2}^{R_2} f(x)\ dx \end{aligned}$$

so we can convert both single-sided limits into double-sided limits, and

$$\int_{-\infty}^{\infty} f(x)\ dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx$$

Also, extending this formula,

$$\int_{0}^{\infty} f(x)\ dx = \frac{1}{2}\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx$$

1.3 Using Residues

Now, we apply our knowledge of residues to integrate $f(z) = p(z)/q(z)$ along the real axis when p and q are polynomials. In this discussion, suppose that q has at least one zero above the real axis and no zeroes on the real axis.

From our knowledge of polynomials, we know that q has a finite number of distinct zeroes, which we can label as $z_1, z_2, \ldots, z_n$. Then, we can integrate the function $f(z)$ along the contour:

- Along the real axis from $-R$ to R;
- Along the upper semicircle with a radius of R from $(R, 0$ to $(-R, 0)$, which we call C_R.

This contour allows us to write

$$\int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$$

or

$$\int_{-R}^{R} f(x)\ dx = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z) - \int_{C_R} f(z)\ dz$$

Using this expression, we can say that if

$$\lim_{R\to\infty} \int_{C_R} f(z)\ dz = 0$$

then the following three equations hold (the latter two if f is even):

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum k = 1^n \text{Res}_{z=z_k} f(z)$$
$$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum_{z=z_k} f(z)$$
$$\int_{0}^{\infty} f(x)\ dx = \pi i \sum_{z=z_k} f(z)$$

2 Frame 79 – Example

This section will show a sample integral that can be calculated using the method from the previous section.

The goal of this example is to calculate

$$\int_0^\infty \frac{x^2}{x^6+1}$$

To do this, we can define

$$f(z) = \frac{z^2}{z^6+1}$$

and note that this has isolated singularities at the sixth roots of -1, or

$$c_k = e^{i(1+2k)\pi/6}$$

Three of these roots lie on the upper half-plane, at

$$\begin{aligned} c_0 &= e^{i\pi/6} \\ c_1 &= e^{i\pi/2} = i \\ c_2 &= e^{i5\pi/6} \end{aligned}$$

We can find the residue at these three points through the formula

$$B_k = \mathrm{Res}_{z=c_k} \frac{z^2}{z^6+1} = \left.\frac{z^2}{6z^5}\right|_{z=c_k} = \left.\frac{1}{6z^3}\right|_{z=c_k} = \frac{1}{6c_k^3}$$

so we can write

$$2\pi i \sum_{k=1}^{n} B_k = 2\pi i \left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i} \right) = \frac{\pi}{3}$$

and, as long as $R > 1$,

$$\int_{-R}^{R} f(x)\ dx = \frac{\pi}{3} - \int_{C_R} f(z)\ dz$$

Next, when $|z| = R$, we can write

$$\frac{|z^2|}{|z^6+1|} < \frac{R^2}{R^6-1}$$

so

$$\left| \int_{C_R} f(z)\ dz \right| \leq \frac{R^2}{R^6-1} \pi R = \frac{R^3}{R^6-1}$$

and, as $R \to \infty$, this approaches 0. Thus,

$$\text{P.V.} \int_{-\infty}^{\infty} f(z)\ dz = \frac{\pi}{3}$$

so

$$\int_0^\infty \frac{x^2}{x^6+1} = \frac{\pi}{6}$$

3 Frame 80 – Improper Integrals from Fourier Analysis

In this section, we will look at integrals of the form

$$\int_{-\infty}^{\infty} f(x) \sin ax \ dx$$

and

$$\int_{-\infty}^{\infty} f(x) \cos ax \ dx$$

where $f(x) = p(x)/q(x)$ is a rational function with no poles on the real axis, but at least one above it.

Note that we can't apply the process from the previous section directly, since

$$|\sin az|^2 = \sin^2 ax + \sinh^2 ay$$

is unbounded as $y \to \infty$.

3.1 Example

Consider the integral

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{2\pi}{e^3}$$

We can begin by defining the function

$$f(z) = \frac{1}{(z^2+1)^2}$$

and using the regular semicircular contour from $-R$ to $+R$. Since $f(z)$ only has a single pole in the upper half-plane (at $z = i$), as long as $R > 1$, we can write

$$\int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2} dx = 2\pi i \operatorname{Res}_{z=i}[f(z)e^{i3z}] - \int_{C_R} f(z) e^{i3z} dz$$

Now, this pole at $z = i$ is a second-order pole, so

$$\begin{aligned}
B &= \frac{d}{dz} \frac{e^{i3z}}{(z+i)^2}\Big|_{z=i} \\
&= \frac{3ie^{3iz}(z+i)^2 - 2e^{3iz}(z+i)}{(z+i)^4}\Big|_{z=i} \\
&= \frac{e^{3iz}[3iz-5]}{(z+i)^3}\Big|_{z=i} \\
&= \frac{-8e^{-3}}{(2i)^3} \\
&= \frac{1}{ie^3}
\end{aligned}$$

Next, the integral on the semicircle satisfies

$$\begin{aligned}\left|\int_{C_R} f(z)e^{i3z}\right| &\leq \pi R \cdot \frac{|e^{-3y}|}{(R^2-1)^2} \\ &\leq \pi R \cdot \frac{1}{(R^2-1)^2}\end{aligned}$$

and this vanishes as R tends to infinity. Thus, equating real parts of the integral formula,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos 3x}{(x^2+1)^2} = 2\pi i \frac{1}{ie^3} = \frac{2\pi}{e^3}$$