1 Frame 38 – Derivatives and Integrals

1(b) Breaking the derivative into its complex components,

$$\begin{aligned}\frac{d}{dt}[w(t)]^2 &= \frac{d}{dt}[u(t)+iv(t)]^2 \\ &= 2[u(t)+iv(t)][u(t)+iv(t)]' \\ &= 2w(t)w'(t)\end{aligned}$$

2(a) Evaluating the integral,

$$\begin{aligned}\int_1^2 \left(\frac{1}{t}-i\right)^2 dt &= \int_1^2 \frac{1}{t^2} - 1 - i\frac{2}{t}dt \\ &= -\frac{1}{t} - t - 2i\ln t\Big|_1^2 \\ &= -(\frac{1}{2}-1) - (2-1) - 2i(\ln 2 - 0) \\ &= -\frac{1}{2} - i\ln 4\end{aligned}$$

2(b)

$$\begin{aligned}\int_0^{\pi/6} e^{i2t} &= \frac{1}{2i}e^{i2t}\Big|_0^{\pi/6} \\ &= \frac{1}{2i}(e^{i\pi/3}-1) \\ &= \frac{1}{2i}\left(i\frac{\sqrt{3}}{2} - \frac{1}{2}\right) \\ &= \frac{\sqrt{3}}{4} + i\frac{1}{4}\end{aligned}$$

2(c) Converting this improper integral into a limit,

$$\begin{aligned}\int_0^{\infty} e^{-zt} &= \lim_{L\to\infty}\int_0^L e^{-zt} \\ &= \lim_{L\to\infty} -\frac{1}{z}e^{-zt}\Big|_0^L \\ &= -\frac{1}{z}\lim_{L\to\infty} e^{-zL} - 1 \\ &= \frac{1}{z}\end{aligned}$$

4 Evaluating the left-side integral,

$$\begin{aligned}\int_0^\pi e^{1+i}xdx &= \frac{1}{1+i}e^{1+i}x\Big|_0^\pi \\ &= \frac{1}{1+i}(e^{\pi+i\pi}-1) \\ &= \frac{1-i}{2}(-e^\pi-1) \\ &= -\frac{1}{2}(e^\pi+1)+\frac{i}{2}(e^\pi+1)\end{aligned}$$

2 Frame 39 – Contours

2 First, the original parametrization can be written as

$$z(\theta) = 2e^{i\theta} = 2\cos\theta + 2i\sin\theta$$

If $\theta = \arctan\frac{y}{\sqrt{4-y^2}}$, then this becomes

$$z(y) = 2\cos\arctan\frac{y}{\sqrt{4-y^2}} + 2i\sin\arctan\frac{y}{\sqrt{4-y^2}}$$

Next, these terms can be simplified using basic geometry. The expression $\arctan\frac{y}{\sqrt{4-y^2}}$ represents a right-angled triangle with legs of lengths $\sqrt{4-y^2}$ and y, so the hypotenuse must have a length of 2. Then,

$$\begin{aligned}\cos\arctan\frac{y}{\sqrt{4-y^2}} &= \frac{\sqrt{4-y^2}}{2}\\ \sin\arctan\frac{y}{\sqrt{4-y^2}} &= \frac{y}{2}\end{aligned}$$

so the arc is

$$z(y) = \sqrt{4-y^2} + iy$$

6 (a) First, the function

$$z(t) = t + iy(t) = t + it^3\sin(\pi/t)$$

intersects the real axis whenever $y(t) = 0$. If $t = 1/n$, then this expression becomes

$$y(1/n) = \frac{\sin\left(\frac{\pi}{1/n}\right)}{n^3} = \frac{\sin(n\pi)}{n^3} = 0$$

as predicted.

(b) An arc is smooth if the function $z(t)$ is continuous and its derivative is piecewise continuous.

First, $z(t)$ is continuous for $0 < x \leq 1$ because $x(t) = x$ and $y(t) = y(x)$ are both continuous on this interval. To show continuity at $t = 0$, we must show that

$$\lim_{t\to 0+} y(t) = 0$$

However, the magnitude of $y(t)$ must be in the range

$$0 \leq \left|t^3\sin\left(\frac{\pi}{t}\right)\right| \leq t^3$$

and the left- and right-hand limits are

$$\lim_{t\to 0+} 0 = 0$$
$$\lim_{t\to 0+} t^3 = 0$$

so, by the squeeze theorem, the original limit holds, and $y(t)$ is continuous at $t = 0$.

Finally, the derivative of $z(t)$ is

$$z'(t) = 1 + i\left[3t^2 \sin(\pi/t) - \pi t \cos(\pi/t)\right]$$

Using the same process as above, the limit as t goes to zero is

$$\lim_{t\to 0+} z'(t) = 1 + i0$$

How can I tell whether this is continuous? The derivative isn't defined at zero.

3 Frame 42 – Contour Integrals

1(a) The integrand on this circle is

$$f[z(\theta)] = \frac{2e^{i\theta} + 2}{2e^{i\theta}} = 1 + e^{-i\theta}$$

and the derivative of the contour is

$$z'(\theta) = 2ie^{i\theta}$$

so the first contour integral is

$$\begin{aligned}
\int_C f(z)dz &= \int_0^{\pi} (1 + e^{-i\theta})(2ie^{i\theta})d\theta \\
&= 2i \int_0^{\pi} e^{i\theta} + 1 d\theta \\
&= 2i \left(\frac{e^{i\theta}}{i} + \theta \right)_0^{\pi} \\
&= 2i \left(\frac{-2}{i} + \pi \right) \\
&= -4 + 2\pi i
\end{aligned}$$

1(b) The integrand has not changed, so

$$\begin{aligned}
\int_C f(z)dz &= 2i \left(\frac{e^{i\theta}}{i} + \theta \right)_{\pi}^{2\pi} \\
&= 4 + 2\pi i
\end{aligned}$$

1(c) The integral along the entire circle is just the sum of the two previous results:

$$\begin{aligned}
\int_C f(z)dz &= (-4 + 2\pi i) + (4 + 2\pi i) \\
&= 4\pi i
\end{aligned}$$

2(a) The function is

$$f[z(\theta)] = 1 + e^{i\theta} - 1 = e^{i\theta}$$

and the contour's derivative is

$$z'(\theta) = ie^{i\theta}$$

so the integral is

$$\begin{aligned}\int_C f(z)dz &= \int_\pi^{2\pi} e^{i\theta} \cdot ie^{i\theta} \\ &= i\int_\pi^{2\pi} e^{2i\theta} \\ &= i\frac{e^{2i\theta}}{2i}\Big|_\pi^{2\pi} \\ &= \frac{e^{4i\pi} - e^{2i\pi}}{2} \\ &= 0\end{aligned}$$

2(b) Now, the function is

$$f[z(x)] = x - 1$$

and the path's derivative is

$$z'(x) = 1$$

so the integral is

$$\int_C f(z)dz = \int_0^2 x - 1dx = \frac{(x-1)^2}{2}\Big|_0^2 = \frac{1-1}{2} = 0$$

3 Along the first edge, the integral is

$$\int_0^1 \pi e^{\pi x}dx = e^{\pi x}\Big|_0^1 = e^\pi - 1$$

Along the second edge, the integral is

$$e^\pi \int_0^1 i\pi e^{-i\pi y}dy = e^\pi \left(-\frac{\pi}{\pi}e^{-i\pi y}\right)_0^1 = 2e^\pi$$

On the third edge, the integral is

$$-\int_0^1 \pi e^{\pi x} \cdot (-1)dx = e^\pi - 1$$

Finally, on the fourth edge, the integral is

$$-\int_0^1 i \cdot \pi e^{-i\pi y}dy = \left(e^{-i\pi y}\right)_0^1 = -2$$

so the whole integral is

$$\int_C f(z)dz = 2(e^\pi - 1) + 2e^\pi - 2 = 4(e^\pi - 1)$$

4 If the path is $y = x^3$, then the direction is

$$z'(x) = 1 + iy'(x) = 1 + i3x^2$$

Then, the integral can be done in two parts. First, from $x = -1$ to 0,

$$\int_{-1}^{0} 1 \cdot (1 + i3x^2)dx = \left(x + ix^3\right)_{-1}^{0} = 1 + i$$

Then, from $x = 0$ to 1,

$$\int_{0}^{1} 4x^3 \cdot (1 + i3x^2)dx = \int_{0}^{1} 4x^3 + i12x^5 dx = x^4 + i2x^6\Big|_0^1 = 1 + 2i$$

so the total contour integral is

$$\int_C f(z)dz = 2 + 3i$$

5 If $f(z) = 1$, then

$$\int_C f(z)dz = \int_a^b 1 \cdot z'(t)dt = z(b) - z(a) = z_2 - z_1$$

6 If $z = e^{i\theta}$, then the function $f(z)$ is

$$f[z(\theta)] = (e^{i\theta})^{-1+i} = e^{-i\theta}e^{-\theta}$$

and the contour integral is

$$\begin{aligned}\int_C f(z)dz &= \int_0^{2\pi} e^{-i\theta}e^{-\theta} \cdot ie^{i\theta}d\theta \\ &= i\int_0^{2\pi} e^{-\theta}d\theta \\ &= -ie^{-\theta}\Big|_0^{2\pi} \\ &= i(1 - e^{-2\pi})\end{aligned}$$

7 On this semicircle,

$$f[z(\theta)] = (e^{i\theta})^i = e^{-\theta}$$

so

$$
\begin{aligned}
\int_C f(z)dz &= \int_0^\pi e^{-\theta} \cdot ie^{i\theta} d\theta \\
&= i\int_0^\pi e^{(-1+i)\theta} d\theta \\
&= \frac{i}{-1+i} e^{(-1+i)\theta}\Big|_0^\pi \\
&= \frac{i(-1-i)}{2}(-e^{-\pi}-1) \\
&= \frac{-1+i}{2}(e^{-\pi}+1)
\end{aligned}
$$