1 Frame 12 – Functions of Complex Variables

1.1 Functions

If S is a set of complex numbers, then a **function** f is a rule that assigns a complex number w to each z in S. The number w is called the **value** of f at z. We denote it as

$$w = f(z)$$

The set S is called the **domain of definition** of f. Note that we need both a rule (f) and a domain (S) for a function to be well defined.

Suppose that $w = u + iv$ and $z = x + iy$. Then,

$$u + iv = f(x + iy)$$

Then, we can express $f(z)$ as a pair of real functions of x and y:

$$f(z) = u(x, y) + iv(x, y)$$

Alternatively, we could use polar coordinates to write

$$u + iv = f(re^{i\theta})$$

so

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Example: the function $f(z) = z^2$ can be written as

$$\begin{aligned} f(x + iy) &= (x + iy)^2 \\ &= (x^2 - y^2) + i2xy \end{aligned}$$

so

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy \end{aligned}$$

In polar coordinates,

$$\begin{aligned} f(x + iy) &= (re^{i\theta})^2 \\ &= r^2 e^{i2\theta} \\ &= r^2 \cos 2\theta + ir^2 \sin 2\theta \end{aligned}$$

so

$$\begin{aligned} u(r, \theta) &= r^2 \cos 2\theta \\ v(r, \theta) &= r^2 \sin 2\theta \end{aligned}$$

1.2 Real-Valued Functions

We say that f is a **real-valued function** if v is zero everywhere.

Example: one real-valued function is

$$f(z) = |z|^2 = x^2 + y^2 + i0$$

1.3 Polynomials

If n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are complex numbers with $a_n \neq 0$, then the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a **polynomial** of degree n. Note that this sum has a finite number of terms and that the domain of definition is the entire z plane.

As in real numbers, a **rational function** is a quotient of two polynomials:

$$R(z) = \frac{P(z)}{Q(z)}$$

A rational function is defined everywhere that $Q(z) \neq 0$.

1.4 Multi-Valued Functions

A generalization of a function is a rule that assigns more than one value to a point z. These **multiple-valued functions** are usually studied by taking one of the possible values at each point and constructing a single-valued function.

Example: we know that we can write

$$z^{1/2} = \pm\sqrt{r}e^{i\theta/2}$$

*where we denoted $-\pi < \theta \leq \pi$ as the **principal value** of $argz$. To turn this into a single valued function, we can choose the positive value of r and write*

$$f(z) = \sqrt{r}e^{i\theta/2}$$

Then, f is well-defined on the entire plane.

2 Frame 13 – Mappings

2.1 Definitions

There is no convenient way to graph the function $w = f(z)$ – each of these complex numbers are located on a plane instead of a line. Instead, we can draw pairs of corresponding points on separate z and w planes. When we think of f this way, we call it a **mapping** or **transformation**.

If f is defined on the domain of definition S, then the **image** of a point $z \in S$ is the point $w = f(z)$. If T is a subset of S, then the set of the images of each point in T are called the image of T. In particular, the image of the entire domain, S, is called the **range** of f. The **inverse image** of a point w is the set of points z in S that map to w (possibly zero, one, or many points).

2.2 Basic transformations

Using this geometric interpretation, we can describe mappings using terms such as **translation**, **rotation**, and **reflection**. For instance, the mapping

$$w = z + 1 = (x + 1) + iy$$

can be thought of as a translation of each point z one unit to the right. Another example is the rotational mapping

$$w = iz$$

where, using $i = e^{i\pi/2}$ and $z = re^{i\theta}$, is

$$w = re^{i(\theta+\pi/2)}$$

or, in other words, a 90° rotation. Finally, the mapping

$$w = \bar{z} = x - iy$$

is a reflection across the real axis. Usually, it is more useful to sketch an image of a curve rather than a single point.

2.3 Mapping a curve

For an example, consider the mapping $w = z^2$. We showed earlier that this can be written as

$$u = x^2 - y^2, \quad v = 2xy$$

To sketch the image, we will first set $u = c_1$, which requires that

$$x^2 - y^2 = c_1, \quad c_1 > 0$$

which is the equation for a hyperbola. This equation can then be used to solve for the image points:

$$u = c_1, \quad v = \pm 2y\sqrt{y^2 + c_1}$$

where the plus-minus is resolved depending on which side the image point is on. Simply put, as z travels up the right-side hyperbola or down the left-side hyperbola, w travels up the vertical line $u = c_1$.

Next, we can set $v = c_2$, which requires

$$2xy = c_2, \quad c_2 > 0$$

This gives us the image set

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

As $x \to \pm\infty$, $u \to \infty$; as $x \to 0$, $u \to -\infty$. Thus, this hyperbola traces out the straight line $v = c_2$ towards the right as z travels towards the left.

2.4 Mapping a region

We can use some of the details from the previous example to find the image of a region, rather than a single curve.

Consider the domain $x > 0, y > 0, xy < 1$. This region consists of the upper branches of the hyperbolas

$$2xy = c, \quad 0 < c < 2$$

and we know from the previous example that these hyperbolas map to the straight lines

$$v = c$$

Thus, this region maps to the horizontal strip $0 < v < 2$.

We can also close the domain to contain the curves $x = 0$, $y = 0$, and $xy = 1$. From the function $w = z^2$, we know that the points $(0, y)$ and $(x, 0)$ map to the points $(-y^2, 0)$ and $(x^2, 0)$, so including the two straight lines simply extends the strip to include $v = 0$. Similarly, the hyperbola $xy = 1$ maps to the horizontal line $v = 2$.

Simply put, the image of the closed region $x \geq 0$, $y \geq 0$, $xy \leq 1$ is the closed region $0 \leq v \leq 2$.

2.5 Mapping with polar coordinates

Finally, we can use polar coordinates to simplify some mappings.

Again, consider the mapping $w = z^2$. If we write $z = re^{i\theta}$, then the image point can be written as

$$w = r^2 e^{2i\theta}$$

Looking at the magnitude of w, points on a circle $r = r_0$ are mapped onto a circle $r' = r_0^2$. Also, looking at the argument of w, the angle of the image is doubled. This means that the first quadrant, which is defined as

$$r \geq 0, \quad 0 \leq \theta \leq \pi/2$$

is in a one-to-one mapping with the top plane, $0 \leq \theta \leq \pi$. Simiarly, the top place is mapped onto the entire complex plane (although this is not one-to-one, since the inverse image of the positive real axis is both real axes).

Note that any mapping $w = z^n$ for positive integer n has a similar form, where each non-zero point in the w plane is the image of n distinct points in the z plane.

3 Frame 14 – Mappings by the Exponential Function

Now, we will look at the exponential function

$$e^z = e^{x+iy} = e^x e^{iy}$$

We can again look at straight lines and find their images in this mapping.

Consider the transformation

$$w = e^z = \rho e^{i\phi}$$

where

$$p = e^x \quad \phi = y$$

This means that the image of a vertical line $x = c_1$ is a circle with radius $p = e^{c_1}$. Each point on the circle is the image of infinitely many points, each spaced 2π units apart on the vertical line. Similarly, the horizontal line $y = c_2$ is a ray with an angle of $\phi = c_2$.

With these images in mind, we know that vertical and horizontal line segments are mapped onto arcs and rays, respectively. We can then use this information to map regions:

Now, consider the rectangular region

$$a \leq x \leq b \quad c \leq y \leq d$$

The image of this region under the mapping $w = e^z$ is

$$e^a \leq \rho \leq e^b \quad c \leq \phi \leq d$$

This is a one-to-one mapping if $d - c < 2\pi$. In particular, the region with $c = 0, d = \pi$ is mapped onto half of a circular ring.

4 Frame 15 – Limits

4.1 Definitions

Suppose that a function f is defined at all points z in some deleted neighborhood of z_0. The statement that the number w_0 is the **limit** of $f(z)$ as z approaches z_0 means that the point $w = f(z)$ can be made *arbitrarily close* to w_0 if we choose z close enough to z_0. We write this as

$$\lim_{z \to z_0} f(z) = w_0$$

To be more precise, if this limit exists, then for each positive number ϵ, there is a positive number δ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Geometrically, this definition says that each ϵ neighbourhood around w_0 has a corresponding deleted δ neighbourhood around z_0 such that the image of each point in the δ neighbourhood maps to a point in the ϵ neighbourhood.

Note that the deleted neighbourhood will always exist if z_0 is internal to the domain of definition of f. We can extend the definition of a limit to include boundary points by ignoring all of the neighbourhood's points that are outside the domain.

Also note that this definition only allows a given point to be tested as a limit – it does not provide a method for finding the limit. This will be covered in the next section.

4.2 Uniqueness

If the limit of a function $f(z)$ exists at z_0, it must be unique. To show this, consider two limits:

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} f(z) = w_1$$

This implies that we can find δ_0 and δ_1 such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_1$$

Now, suppose that δ is a positive number smaller than both δ_0 and δ_1. Then, for all $0 < |z - z_0| < \delta$, we find that the difference between the two limits is

$$\begin{aligned} |w_1 - w_0| &= ||f(z) - w_0| - |f(z) - w_1|| \\ &\leq |f(z) - w_0| + |f(z) - w_1| \\ &< \epsilon + \epsilon \\ &= 2\epsilon \end{aligned}$$

and since ϵ can be made arbitrarily small, we must have

$$w_1 = w_0$$

4.3 Example – basic limit

Consider the function $f(z) = \frac{i\bar{z}}{2}$. We can show that the limit of this function as $z \to 1$ is

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

To do this, we observe that

$$\begin{aligned} \left| f(z) - \frac{i}{2} \right| &= \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| \\ &= \frac{|z - 1|}{2} \end{aligned}$$

Then, we can fulfill the limit definition by writing

$$\left| f(z) - \frac{i}{2} \right| < \epsilon \textit{ whenever } |z - 1| < 2\epsilon$$

4.4 Example – direction dependence

In order for w_0 to be a limit of f at z_0, the limit conditions must hold if z approaches z_0 in any arbitrary manner.

Consider the function

$$f(z) = \frac{z}{\bar{z}}$$

Then, the limit

$$\lim_{z \to 0} f(z)$$

does not exist. To illustrate this, the function's value for any non-zero point $z = (x, 0)$ is

$$f(x, 0) = \frac{x + i0}{x - i0} = 1$$

but the value for any non-zero point $z = (0, y)$ *is*

$$f(0, y) = \frac{0 + iy}{0 - iy} = -1$$

so the limit would not be unique.

5 Frame 16 – Theorems on Limits

Next, it is helpful to connect limits of complex functions and real-valued functions, allowing us to use our knowledge of calculus to simplify the process of finding complex limits

5.1 Splitting into real functions

First, the following theorem is helpful:

Theorem 1. Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0$$

Then, the limit

$$\lim_{z \to z_0} = w_0$$

holds iff

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$$

The two implications of this theorem can be proved by considering the definitions of the neighbourhoods as open disks.

5.2 Combining simple limits

Theorem 2. Suppose that

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} F(z) = W_0$$

Then, we can write the following three limits:

$$\lim_{z \to z_0} f(z) + F(z) = w_0 + W_0$$
$$\lim_{z \to z_0} f(z)F(z) = w_0 W_0$$
$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \text{ if } W_0 \neq 0$$

These can be proved easily by applying Theorem 1 to each limit.

5.3 Polynomials

Using the basic limit definition from the previous section, it is simple to show that

$$\lim_{z \to z_0} c = c$$

and

$$\lim_{z \to z_0} z = z_0$$

for any complex numbers c and z_0. Then, by the multiplication property,

$$\lim z \to z_0 z^n = z_0^n$$

for any positive integer z. These limits can be used to show that, for any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

the limit as z approaches a point z_0 is the polynomial's value:

$$\lim_{z \to z_0} P(z) = P(z_0)$$

6 Frame 17 – Limits Involving Infinity

6.1 The point at infinity

Sometimes, it is useful to include the **point at infinity** with the complex plane. This point is denoted by ∞. In order to visualize it, the complex plane can be drawn with a unit sphere centered at the origin. Then, a line can be drawn from the top of the sphere (or the *north pole*, denoted by N) to any point on the plane; the line will pass through exactly one other point P on the sphere. This correspondence (between points on the plane, z, and the sphere, P) is called a **stereographic projection**, and the sphere is known as the **Riemann sphere**.

No point in the plane corresponds to the point N. We can let N correspond to the point at infinity, giving us a one-to-one mapping between points on the sphere and points in the extended complex plane.

We will make the distinction that a point z is a point in the finite plane unless we specifically describe the point at infinity – we will specifically mention ∞.

6.2 Neighbourhoods around infinity

Next, we can define neighbourhoods around the point at infinity. Looking at the Riemann sphere, we notice that all of the points P in the upper hemisphere project to points z outside of the unit disk.

Further, if ϵ is a small, positive number, then points in the plane such that

$$|z| > \frac{1}{\epsilon}$$

correspond to points on the sphere close to N. Thus, we call the set $|z| > 1/\epsilon$ an **(ϵ) neighbourhood** of ∞.