1 Frame 55 – Sequences and Convergence

1.1 Definitions

An **infinite sequence** of complex numbers,

$$z_1, z_2, \ldots, z_n, \ldots$$

has a **limit** if, for each positive number ϵ, there exists a positive integer n_0 such that

$$|z_n - z| < \epsilon \quad \text{whenever} \quad n > n_0$$

Geometrically, this limit implies that for all $n > n_0$, each number z_n in the sequence will be inside an ϵ neighbourhood of z.

A sequence can only have one limit, at most. When this limit exists, we say that the sequence **converges** to z, and we write

$$\lim_{n\to\infty} z_n = z$$

If a sequence has no limit, it **diverges**.

1.2 Components

Theorem: If we write $z_n = x_n + iy_n$ and $z = x + iy$, then

$$\lim_{n\to\infty} z_n = z \iff \lim_{n\to\infty} x_n = x \text{ and } \lim_{n\to\infty} y_n = y$$

This theorem allows us to write

$$\lim_{n\to\infty} (x_n + iy_n) = \lim_{n\to\infty} x_n + i \lim_{n\to\infty} y_n$$

as long as the limits on either side of this equation exist.

1.3 Examples

Example 1: we can evaluate the following limit easily:

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{n^3 + i} &= \lim_{n\to\infty} \frac{1}{n^3} + i \lim_{n\to\infty} 1 \\
&= 0 + i \cdot 1 \\
&= i
\end{aligned}$$

Example 2: Polar coordinates require some extra care. Looking at the sequence

$$z_n = -2 + i\frac{(-1)^n}{n^2}$$

we can see that

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty}(-2) + i\lim_{n\to\infty}\frac{(-1)^n}{n^2} = -2$$

However, we can find that the principal polar representation of these numbers is

$$r_n = \sqrt{4 + \frac{1}{n^2}}$$
$$\Theta_n = \mathrm{Arg}\, z_n = \tan^{-1}\left(\frac{(-1)^n}{-2n^2}\right)$$

Evaluating the first limit, we find that

$$\lim_{n\to\infty} r_n = \sqrt{4} = 2$$

which is fine. However, the second sequence does not converge. Looking at every second term, we see that

$$\lim_{n\to\infty} \Theta_{2n} = \pi$$

and

$$\lim_{n\to\infty} \Theta_{2n-1} = -\pi$$

so Θ_n *diverges.*