

1 Frame 78 – Evaluating Improper Integrals

1.1 Improper Integrals

The **improper integral** of a continuous function $f(x)$ over the interval $0 \leq x < \infty$ is defined as

$$\int_0^\infty f(x) dx = \lim_{R \rightarrow \infty} \int_0^R f(x) dx$$

If this limit exists, we say that the improper integral **converges** to this limit. The improper integral of f over the infinite interval $-\infty < x < \infty$ is

$$\int_{-\infty}^\infty f(x) dx = \lim_{R_1 \rightarrow \infty} \int_{-R_1}^0 f(x) dx + \lim_{R_2 \rightarrow \infty} \int_0^{R_2} f(x) dx$$

If both of these limits exist, we say that the integral converges to their sum.

1.2 The Cauchy Principal Value

We say that the **Cauchy Principal Value** of an indefinite integral is

$$\text{P.V.} \int_{-\infty}^\infty f(x) dx = \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx$$

as long as this limit exists.

If the regular improper integral converges, then

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx &= \lim_{R \rightarrow \infty} \left[\int_{-R}^0 f(x) dx + \int_0^R f(x) dx \right] \\ &= \lim_{R \rightarrow \infty} \int_{-R}^0 f(x) dx + \lim_{R \rightarrow \infty} \int_0^R f(x) dx \end{aligned}$$

so the principal value also exists. However, the converse is not true – the existence of the principal value does not imply the existence of the improper integral.

Next, suppose that $f(x)$ is an even function; ie:

$$f(-x) = f(x)$$

and assume that the principal value exists. Then, the evenness of f allows us to write

$$\begin{aligned} \int_{-R_1}^0 f(x) dx &= \frac{1}{2} \int_{-R_1}^{R_1} f(x) dx \\ \int_0^{R_2} f(x) dx &= \frac{1}{2} \int_{-R_2}^{R_2} f(x) dx \end{aligned}$$

so we can convert both single-sided limits into double-sided limits, and

$$\int_{-\infty}^{\infty} f(x) dx = \text{P.V.} \int_{-\infty}^{\infty} f(x) dx$$

Also, extending this formula,

$$\int_0^{\infty} f(x) dx = \frac{1}{2} \text{P.V.} \int_{-\infty}^{\infty} f(x) dx$$

1.3 Using Residues

Now, we apply our knowledge of residues to integrate $f(z) = p(z)/q(z)$ along the real axis when p and q are polynomials. In this discussion, suppose that q has at least one zero above the real axis and no zeroes on the real axis.

From our knowledge of polynomials, we know that q has a finite number of distinct zeroes, which we can label as z_1, z_2, \dots, z_n . Then, we can integrate the function $f(z)$ along the contour:

- Along the real axis from $-R$ to R ;
- Along the upper semicircle with a radius of R from $(R, 0)$ to $(-R, 0)$, which we call C_R .

This contour allows us to write

$$\int_{-R}^R f(x) dx + \int_{C_R} f(z) dz = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z)$$

or

$$\int_{-R}^R f(x) dx = 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z) - \int_{C_R} f(z) dz$$

Using this expression, we can say that if

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$$

then the following three equations hold (the latter two if f is even):

$$\begin{aligned} \text{P.V.} \int_{-\infty}^{\infty} f(x) dx &= 2\pi i \sum_{k=1}^n \text{Res}_{z=z_k} f(z) \\ \int_{-\infty}^{\infty} f(x) dx &= 2\pi i \sum_{z=z_k} f(z) \\ \int_0^{\infty} f(x) dx &= \pi i \sum_{z=z_k} f(z) \end{aligned}$$

2 Frame 79 – Example

This section will show a sample integral that can be calculated using the method from the previous section.

The goal of this example is to calculate

$$\int_0^\infty \frac{x^2}{x^6 + 1}$$

To do this, we can define

$$f(z) = \frac{z^2}{z^6 + 1}$$

and note that this has isolated singularities at the sixth roots of -1 , or

$$c_k = e^{i(1+2k)\pi/6}$$

Three of these roots lie on the upper half-plane, at

$$c_0 = e^{i\pi/6}$$

$$c_1 = e^{i\pi/2} = i$$

$$c_2 = e^{i5\pi/6}$$

We can find the residue at these three points through the formula

$$B_k = \text{Res}_{z=c_k} \frac{z^2}{z^6 + 1} = \frac{z^2}{6z^5} \Big|_{z=c_k} = \frac{1}{6z^3} \Big|_{z=c_k} = \frac{1}{6c_k^3}$$

so we can write

$$2\pi i \sum_{k=1}^n B_k = 2\pi i \left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i} \right) = \frac{\pi}{3}$$

and, as long as $R > 1$,

$$\int_{-R}^R f(x) dx = \frac{\pi}{3} - \int_{C_R} f(z) dz$$

Next, when $|z| = R$, we can write

$$\frac{|z^2|}{|z^6 + 1|} < \frac{R^2}{R^6 - 1}$$

so

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{R^2}{R^6 - 1} \pi R = \frac{R^3}{R^6 - 1}$$

and, as $R \rightarrow \infty$, this approaches 0. Thus,

$$\text{P.V.} \int_{-\infty}^{\infty} f(z) dz = \frac{\pi}{3}$$

so

$$\int_0^\infty \frac{x^2}{x^6 + 1} = \frac{\pi}{6}$$

3 Frame 80 – Improper Integrals from Fourier Analysis

In this section, we will look at integrals of the form

$$\int_{-\infty}^{\infty} f(x) \sin ax \, dx$$

and

$$\int_{-\infty}^{\infty} f(x) \cos ax \, dx$$

where $f(x) = p(x)/q(x)$ is a rational function with no poles on the real axis, but at least one above it.

Note that we can't apply the process from the previous section directly, since

$$|\sin az|^2 = \sin^2 ax + \sinh^2 ay$$

is unbounded as $y \rightarrow \infty$.

3.1 Example

Consider the integral

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2 + 1)^2} dx = \frac{2\pi}{e^3}$$

We can begin by defining the function

$$f(z) = \frac{1}{(z^2 + 1)^2}$$

and using the regular semicircular contour from $-R$ to $+R$. Since $f(z)$ only has a single pole in the upper half-plane (at $z = i$), as long as $R > 1$, we can write

$$\int_{-R}^R \frac{e^{i3x}}{(x^2 + 1)^2} dx = 2\pi i \operatorname{Res}_{z=i} [f(z)e^{i3z}] - \int_{C_R} f(z)e^{i3z} dz$$

Now, this pole at $z = i$ is a second-order pole, so

$$\begin{aligned} B &= \frac{d}{dz} \frac{e^{i3z}}{(z+i)^2} \Big|_{z=i} \\ &= \frac{3ie^{3iz}(z+i)^2 - 2e^{3iz}(z+i)}{(z+i)^4} \Big|_{z=i} \\ &= \frac{e^{3iz}[3iz - 5]}{(z+i)^3} \Big|_{z=i} \\ &= \frac{-8e^{-3}}{(2i)^3} \\ &= \frac{1}{ie^3} \end{aligned}$$

Next, the integral on the semicircle satisfies

$$\begin{aligned}\left|\int_{C_R} f(z)e^{i3z}\right| &\leq \pi R \cdot \frac{|e^{-3y}|}{(R^2-1)^2} \\ &\leq \pi R \cdot \frac{1}{(R^2-1)^2}\end{aligned}$$

and this vanishes as R tends to infinity. Thus, equating real parts of the integral formula,

$$\lim_{R \rightarrow \infty} \int_{-R}^R \frac{\cos 3x}{(x^2+1)^2} = 2\pi i \frac{1}{ie^3} = \frac{2\pi}{e^3}$$

4 Frame 81 – Jordan’s Lemma

4.1 Jordan’s Lemma

The following lemma is helpful for evaluating the integrals from the previous section.

Theorem. Suppose that:

- A function f is analytic at all points above the real axis outside some circle $|z| = R_0$
- C_R is a semicircle $z = Re^{i\theta}$ ($0 \leq \theta \leq \pi$) where $R > R_0$
- On C_R , there is a constant M_R such that

$$|f(z)| \leq M_R$$

and

$$\lim_{R \rightarrow \infty} M_R = 0$$

Then, for any positive a ,

$$\lim_{R \rightarrow \infty} \int_{C_R} f(z) e^{iaz} dz = 0$$

To prove this, we can write

$$\int_{C_R} f(z) e^{iaz} dz = \int_0^\pi f(Re^{i\theta}) \exp(iaRe^{i\theta}) Rie^{i\theta} d\theta$$

Then, we know that

$$|f(Re^{i\theta})| \leq M_R$$

and

$$|\exp(iaRe^{i\theta})| \leq e^{-aR \sin \theta}$$

Next, Jordan’s inequality tells us that

$$\int_0^\pi e^{-R \sin \theta} d\theta < \frac{\pi}{R}$$

so we can write the contour integral as

$$\begin{aligned} \left| \int_{C_R} f(z) e^{iaz} dz \right| &\leq M_R R \int_0^\pi e^{-aR \sin \theta} d\theta \\ &< M_R R \frac{\pi}{aR} \\ &= \frac{M_R \pi}{a} \end{aligned}$$

and this tends to zero as $R \rightarrow \infty$.

4.2 Example

We can use this theorem to help us find the principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2} dx$$

If we write

$$f(z) = \frac{z}{z^2 + 2z + 2}$$

then we can find the residue at the point $z_1 = -1 + i$ as

$$B = \frac{z_1 e^{iz_1}}{z_1 - \bar{z}_1}$$

Next, we can use the standard contour integral to write

$$\int_{-R}^R \frac{x e^{ix}}{x^2 + 2x + 2} = 2\pi i B - \int_{C_R} f(z) e^{iz} dz$$

and, taking the imaginary components of both sides,

$$\int_{-R}^R \frac{x \sin x}{x^2 + 2x + 2} dx = \Im(2\pi i B) - \Im \int_{C_R} f(z) e^{iz} dz$$

Then, if $|z| = R$, we can write

$$|f(z)| \leq \frac{R}{(R - \sqrt{2})^2}$$

and this tends to zero as $R \rightarrow \infty$, so the integral vanishes, leaving

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2} dx = \Im(2\pi i B) = \frac{\pi}{e} (\sin 1 + \cos 1)$$