

## 1 Frame 12 – Functions of Complex Variables

1(a) The function

$$f(z) = \frac{1}{z^2 + 1}$$

is defined everywhere except where  $z^2 + 1 = 0$ ; ie:

$$z \neq \pm i$$

1(b) The function

$$f(z) = \operatorname{Arg}\left(\frac{1}{z}\right)$$

is defined wherever  $\frac{1}{z}$  is defined:

$$z \neq 0$$

1(c) The function

$$f(z) = \frac{z}{z + \bar{z}}$$

can be written as

$$f(x, y) = \frac{x + iy}{(x + iy) + (x - iy)} = \frac{x + iy}{2x} = \frac{1}{2} + i\frac{y}{x}$$

so the domain is

$$\operatorname{Re}(z) \neq 0$$

1(d) The function

$$f(z) = \frac{1}{1 - |z|^2}$$

is equivalent to

$$f(r, \theta) = \frac{1}{1 - r^2}$$

so the domain is

$$r \neq 1$$

2 Substituting  $z = x + iy$  gives

$$\begin{aligned} f(x, y) &= (x + iy)^3 + (x + iy) + 1 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x + iy + 1 \\ &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) \end{aligned}$$

so

$$\begin{aligned} u(x, y) &= x^3 - 3xy^2 + x + 1 \\ v(x, y) &= 3x^2y - y^3 + y \end{aligned}$$

**3** Using the two expressions

$$x = \frac{z + \bar{z}}{2}$$
$$y = \frac{z - \bar{z}}{2i}$$

gives

$$\begin{aligned} f(z) &= \left(\frac{z + \bar{z}}{2}\right)^2 - \left(\frac{z - \bar{z}}{2i}\right)^2 - 2\frac{z - \bar{z}}{2i} \\ &\quad + i\left[2\frac{z + \bar{z}}{2}\left(1 - \frac{z - \bar{z}}{2i}\right)\right] \\ &= \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2 + z^2 - 2z\bar{z} + \bar{z}^2) + iz - i\bar{z} \\ &\quad + i\left[z + \bar{z} + \frac{iz^2}{2} - \frac{i\bar{z}^2}{2}\right] \\ &= \frac{1}{2}(z^2 + \bar{z}^2) + 2iz - \frac{iz^2}{2} + \frac{\bar{z}^2}{2} \\ &= \bar{z}^2 + 2iz \end{aligned}$$

**4** Using

$$z = re^{i\theta}$$

the function can be written as

$$\begin{aligned} f(z) &= re^{i\theta} + \frac{1}{re^{i\theta}} \\ &= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\ &= \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta \end{aligned}$$

## 2 Frame 14 – Mappings by the Exponential Function

1 We saw earlier that the hyperbolas

$$x^2 - y^2 = c_1$$

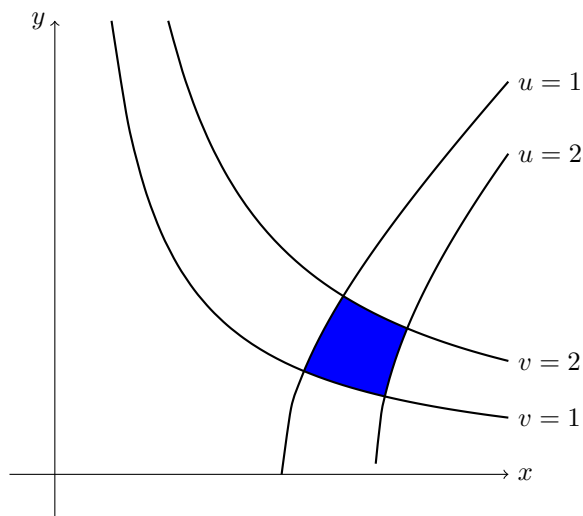
map onto the horizontal lines  $u = c_1$  and the hyperbolas

$$2xy = c_2$$

map onto the vertical lines  $v = c_2$ . Thus, a domain on the  $z$ -plane that maps onto  $1 \leq u \leq 2$  and  $1 \leq v \leq 2$  is

$$1 \leq x^2 - y^2 \leq 2 \quad 1 \leq 2xy \leq 2$$

A sketch of this region is:



2 The first hyperbola can be written as

$$y^2 - x^2 = |c_1|$$

Then, substitution into the  $v$  equation gives

$$u = c_1, \quad v = \begin{cases} 2x\sqrt{x^2 + |c_1|}, & y > 0 \\ -2x\sqrt{x^2 + |c_1|}, & y < 0 \end{cases}$$

This maps out the entire  $v$  line as  $x$  moves right (on the top branch) or left (on the bottom branch).

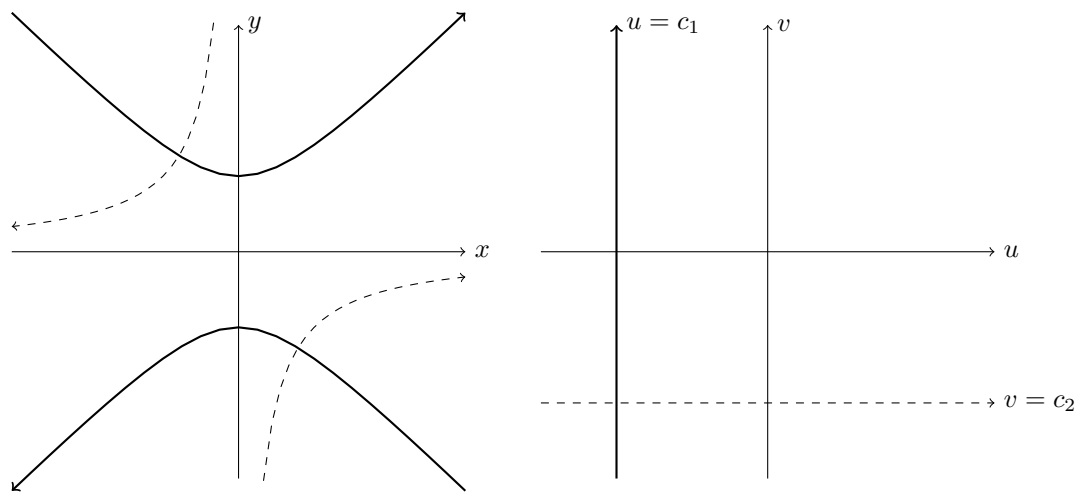
The second hyperbola can be written as

$$2xy = -|c_2|$$

and substituting this into the  $u$  equation gives

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

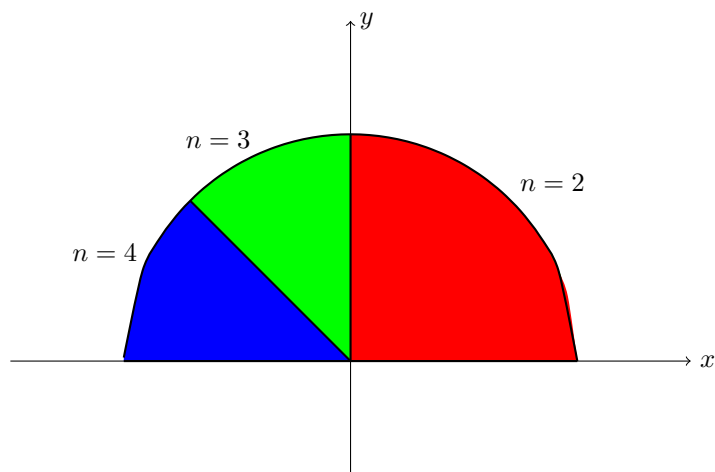
This maps out the entire  $u$  line: as  $x$  gets large in magnitude, so too does  $u$ . A sketch of these mappings is:



**3** The image of the sector  $r \leq 1, 0 \leq \theta \leq \pi/4$  under the mapping  $w = z^n$  is

$$\rho \leq 1, \quad 0 \leq \theta \leq n \frac{\pi}{4}$$

A sketch of these images for  $n = 2, 3, 4$  is:



**4** If  $z$  follows the straight line  $ay = x$ , then the mapping  $w = e^z$  is

$$\begin{aligned} w &= e^{x+iy} \\ &= e^{ay} e^{iy} \\ &= e^{a\phi} e^{i\phi} \\ &= \rho e^{i\phi} \end{aligned}$$

where  $\rho = a\phi$ .

**5** The rectangular region  $a \leq x \leq b$ ,  $c \leq y \leq d$  is made up of the horizontal line segments

$$x = t, \quad y = c_1$$

where  $t$  is a parameter running from  $a$  to  $b$  and  $c_1$  is a constant in the range  $[c, d]$ . These horizontal lines have the images

$$\rho = e^t, \quad \phi = c_1$$

Since  $t$  starts at  $a$  and ends at  $b$ , these images have a radius in the range  $[e^a, e^b]$ . Then, the entire image is the set of these lines, which range from  $\phi = c$  to  $\phi = d$ . Thus, the entire image is

$$e^a \leq \rho \leq e^b, \quad c \leq \phi \leq d$$

**6** Looking at the  $z$  plane, the initial set is the infinite strip

$$x \leq 0, \quad 0 \leq y \leq \pi$$

This maps to the image set

$$\lim_{a \rightarrow -\infty} e^a \leq \rho \leq e^0, \quad 0 \leq \phi \leq \pi$$

or

$$0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \pi$$

This is the upper half of the unit disk, as shown in the figure.

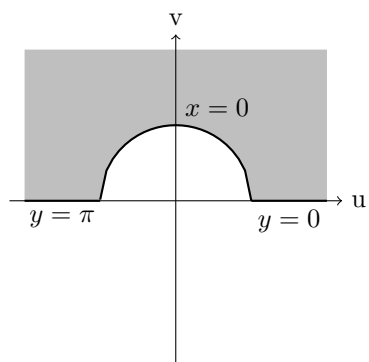
**7** In a similar manner to the previous problem, the image of the strip

$$x \geq 0, \quad 0 \leq y \leq \pi$$

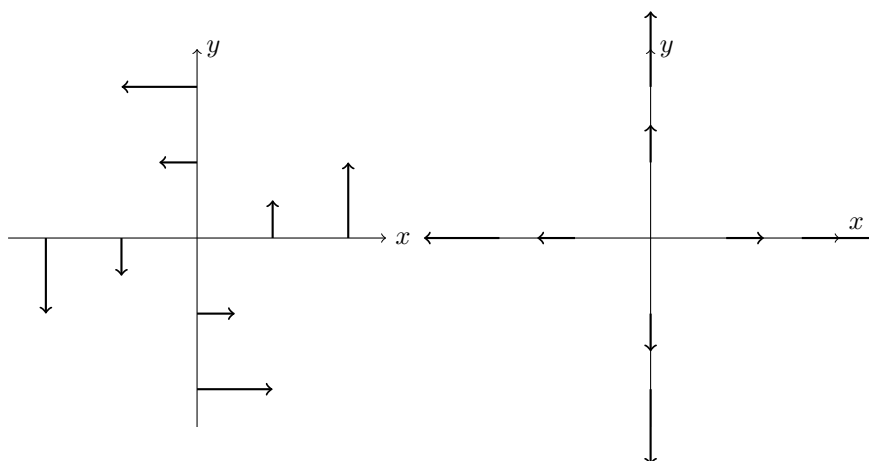
is the upper half-plane with a unit disk cut out:

$$\rho \geq 1, \quad 0 \leq \phi \leq \pi$$

A sketch of this region is:



8 Some sample vectors in these two fields are:



### 3 Frame 18 – Limits and Continuity

**1(a)** The left side of the limit is

$$|\Re(z) - \Re(z_0)| = |\Re(z - z_0)| < |z - z_0|$$

so

$$|\Re(z) - \Re(z_0)| < \delta \text{ whenever } |z - z_0| < \delta$$

**1(b)** The left side of the limit is

$$|\bar{z} - \bar{z}_0| = |\overline{z - z_0}| = |z - z_0|$$

so the limit holds.

**1(c)** The limit expression is

$$\left| \frac{\bar{z}^2}{z} \right| < \epsilon \text{ whenever } |z| < \delta$$

For  $z \neq 0$ , the left side expression is  $|z|$ , so the limit holds where  $\epsilon = \delta$ .

**2(a)** The left side of the limit is

$$|(az + b) - (az_0 + b)| = |a(z - z_0)| = |a||z - z_0|$$

so the limit holds for  $\delta = a\epsilon$ .

**2(b)** The left side is

$$|(z^2 + c) - (z_0^2 + c)| = |z^2 - z_0^2| = |z + z_0||z - z_0| \approx 2|z_0||z - z_0|$$

for  $\delta \ll |z_0|$ . Thus, the limit holds for  $\delta = 2z_0\epsilon$ . Note that if  $z_0 = 0$ , then this reduces to the limit of a constant value  $c$ , which is trivial.

**2(c)** The right side is

$$\begin{aligned} |z - (1 - i)| &= |[x - 1] + i[y + 1]| \\ &= \sqrt{(x - 1)^2 + (y + 1)^2} \end{aligned}$$

The left side is

$$\begin{aligned} |[x + i(2x + y)] - [1 + i]| &= |[x - 1] + i[2x + y - 1]| \\ &= |[x - 1] + i[2(x - 1) + (y + 1)]| \end{aligned}$$

Not sure how to prove limits – it appears as the right side goes to 0, the left side must too.

**3(a)** If  $z_0 \neq 0$ , the limit must be

$$\lim_{z \rightarrow z_0} \frac{1}{z^n} = \frac{\lim_{z \rightarrow z_0} 1}{\lim_{z \rightarrow z_0} z^n} = \frac{1}{z_0^n}$$

**3(b)**

$$\lim_{z \rightarrow i} \frac{iz^3 - 1}{z + i} = \frac{\lim_{z \rightarrow i} iz^3 - 1}{\lim_{z \rightarrow i} z + i} = \frac{i(i^3) - 1}{(i) + i} = 0$$

**3(c)**

$$\lim_{z \rightarrow z_0} \frac{P(z)}{Q(z)} = \frac{\lim_{z \rightarrow z_0} P(z)}{\lim_{z \rightarrow z_0} Q(z)} = \frac{P(z_0)}{Q(z_0)}$$

**4** The base case is

$$\lim_{z \rightarrow z_0} z = z_0$$

which was shown earlier.

If it is known that

$$\lim_{z \rightarrow z_0} z^k = z_0^k$$

then the following limit is

$$\lim_{z \rightarrow z_0} z^{k+1} = \left( \lim_{z \rightarrow z_0} z^k \right) \left( \lim_{z \rightarrow z_0} z \right) = (z_0^k)(z_0) = z_0^{k+1}$$

so, by induction, we are finished.

**5** First, for any point  $z = (x, 0)$ , the function is

$$f(x, 0) = \left( \frac{x}{x} \right)^2 = (1)^2 = 1$$

so the limit as  $x \rightarrow 0$  is 1.

Then, for any point  $z = (x, x)$ , the function is

$$f(x, x) = \left( \frac{x + ix}{x - ix} \right)^2 = (i)^2 = -1$$

so the limit as  $x \rightarrow 0$  is  $-1$ . This conflicts with the previous result, so the limit does not exist.

**6(a)** *The statement to be proved is*

$$\lim_{z \rightarrow z_0} [f(z) + F(z)] = w_0 + W_0$$



If  $f$  and  $F$  are split into their real and imaginary parts ( $u$ ,  $v$ ,  $U$ , and  $V$ ), then we know that

$$\begin{aligned}\lim_{z \rightarrow z_0} u(x, y) &= u_0 \\ \lim_{z \rightarrow z_0} v(x, y) &= v_0 \\ \lim_{z \rightarrow z_0} U(x, y) &= U_0 \\ \lim_{z \rightarrow z_0} V(x, y) &= V_0\end{aligned}$$

Then, the sum  $f(z) + F(z)$  has a real and imaginary part, which have the limits

$$\begin{aligned}\lim_{z \rightarrow z_0} u(x, y) + U(x, y) &= u_0 + U_0 \\ \lim_{z \rightarrow z_0} v(x, y) + V(x, y) &= v_0 + V_0\end{aligned}$$

so the result is

$$\lim_{z \rightarrow z_0} f(z) + F(z) = u_0 + v_0 + U_0 + V_0 = w_0 + W_0$$

**6(b)** The left side of the limit expression is

$$\begin{aligned}|f(z) + F(z) - w_0 - W_0| &= |f(z) - w_0 + F(z) - W_0| \\ &\leq |f(z) - w_0| + |F(z) - W_0| \\ &< \epsilon + \epsilon \\ &= 2\epsilon\end{aligned}$$

so the statement is true, and the limit is  $w_0 + W_0$ .

**7** The left hand side of the expression is

$$||f(z)| - |w_0|| \leq |f(z) - w_0|$$

which is the standard limit expression. Since  $\lim_{z \rightarrow z_0} f(z) = w_0$ , this expression is true, and the limit is  $|w_0|$ .

**10(a)** Making the replacement in the expression, this limit is

$$\begin{aligned}\lim_{z \rightarrow \infty} \frac{4z^2}{(z-1)^2} &= \lim_{z \rightarrow 0} \frac{4(1/z)^2}{(1 - (1/z))^2} \\ &= \lim_{z \rightarrow 0} \frac{4}{(z-1)^2} \\ &= \frac{4}{1} = 4\end{aligned}$$

**10(b)** The reciprocal of this expression has the limit

$$\lim_{z \rightarrow 1} (z - 1)^3 = 0$$

so the limit is infinity.

**10(c)** Taking both reciprocals,

$$\begin{aligned}\lim_{z \rightarrow \infty} \frac{z - 1}{z^2 + 1} &= \lim_{z \rightarrow 0} \frac{(1/z) - 1}{(1/z)^2 + 1} \\ &= \lim_{z \rightarrow 0} \frac{z - z^2}{1 + z^2} \\ &= \frac{0}{1} = 0\end{aligned}$$

so the limit is infinity.

## 4 Frame 20 – Differentiation

1 These four derivatives are:

- (a)

$$\frac{d}{dz}(3z^2 - 2z + 4) = 6z - 2$$

- (b)

$$\frac{d}{dz}(1 - 4z^2)^3 = 3(1 - 4z^2)^2(-8z) = -24z(1 - 4z^2)^2$$

- (c)

$$\frac{d}{dz} \frac{z-1}{2z+1} = \frac{(1)(2z+1) - (2)(z-1)}{(2z+1)^2} = \frac{-1}{(2z+1)^2}$$

- (d)

$$\begin{aligned} \frac{d}{dz} \frac{(1+z^2)^4}{z^2} &= \frac{4(1+z^2)^3(2z)(z^2) - 2z(1+z^2)^4}{z^4} \\ &= \frac{8z^3(1+z^2)^3 - (2z+2z^3)(1+z^2)^3}{z^4} \\ &= \frac{2z(3z^2-1)(1+z^2)^3}{z^4} \end{aligned}$$

2 First, each term of the polynomial  $P_k(z)$  is

$$P_k(z) = a_k z^k$$

All of these terms are differentiable everywhere, and the derivative is

$$P'_k(z) = k a_k z^{k-1}, \quad k \neq 0$$

where the derivative of  $a_0$  is simply zero. Then, we proved that the derivative of a sum is the sum of two derivatives, so we can apply this repeatedly to find

$$P'(z) = a_1 + 2a_2z + \cdots + na_nz^{n-1}$$

Notice that the function's value at zero is  $P(0) = a_0$ . The derivative's value at zero is similarly  $P'(0) = a_1$ . Applying the same process again, we find that

$$P''(0) = 2a_2$$

and

$$P^{(k)}(0) = k! \cdot a_k$$

Rearranging these terms, we can write

$$\begin{aligned} a_0 &= P(0) \\ a_1 &= \frac{P'(0)}{1!} \\ a_2 &= \frac{P''(0)}{2!} \\ &\dots \\ a_n &= \frac{P^{(n)}(0)}{n!} \end{aligned}$$

**3** We can find the derivative of  $f(z) = \frac{1}{z}$  by using the definition:

$$\frac{d}{dz} \frac{1}{z} = \lim_{\Delta z \rightarrow 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{(z) - (z + \Delta z)}{z\Delta z(z + \Delta z)} = \lim_{\Delta z \rightarrow 0} \frac{-1}{z(z + \Delta z)} = \frac{-1}{z^2}$$

**4** Applying the definition of a derivative, we can see that

$$\begin{aligned} \lim_{z \rightarrow z_0} \frac{f(z)}{g(z)} &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} \\ &= \lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \frac{z - z_0}{g(z) - g(z_0)} \\ &= \frac{f'(z_0)}{g'(z_0)} \end{aligned}$$

**5** Following the proof shown in the chapter, the derivative of a sum makes the term  $\Delta w$  into

$$\Delta w = f(z + \Delta z) - f(z) + g(z + \Delta z) - g(z)$$

Then, the derivative is

$$\begin{aligned} \frac{d}{dz}[f(z) + g(z)] &= \lim_{\Delta z \rightarrow 0} \frac{\Delta w}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z) + g(z + \Delta z) - g(z)}{\Delta z} \\ &= \lim_{\Delta z \rightarrow 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \rightarrow 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} \\ &= f'(z) + g'(z) \end{aligned}$$

**6** *Base case:* the derivative of  $z^1$  is 1.

$k + 1$  *case:* if the derivative of  $z^k$  is  $kz^{k-1}$ , then the derivative of  $z^{k+1}$  is

$$\frac{d}{dz} z^{k+1} = \frac{d}{dz} (z^k)(z) = (kz^{k-1})(z) + (z^k) = (k+1)z^k$$

Therefore, by induction, we are done.

**7** When  $n$  is a negative integer, we can write  $m = -n$  and rewrite the function as

$$f(z) = \frac{1}{z^m}$$

Then, using the quotient rule, the derivative of this function is

$$f'(z) = \frac{(0)(z^m) - (1)(mz^{m-1})}{z^{2m}} = \frac{nz^{-n-1}}{z^{-2n}} = nz^{n-1}$$

so the derivative formula is still valid.

**8(a)** The derivative of  $f(z) = \Re z$  is

$$\frac{d}{dz} \Re z = \lim_{\Delta z \rightarrow 0} \frac{\Re(z + \Delta z) - \Re z}{\Delta z} = \lim_{\Delta z \rightarrow 0} \frac{\Re(z)}{\Delta z}$$

To show that this limit doesn't exist, consider a point  $z_1 = (x, 0)$ . The limit from this direction then becomes

$$f'(x, 0) = \lim_{x \rightarrow 0} \frac{x}{x} = 1$$

Then, consider a different point  $z_2 = (0, y)$ . The limit this time is

$$f'(0, y) = \lim_{y \rightarrow 0} y \rightarrow 0 \frac{0}{y} = 0$$

Since the limit is different from these two directions, the derivative does not exist.

The proof for  $\Im(z)$  is nearly identical.