1 Frame 71 – Residues

1(a) This function is

$$\begin{aligned}\frac{1}{z(1+z)} &= \frac{1}{z}\frac{1}{1+z} \\ &= \frac{1}{z}(1 - z + z^2 - \dots) \\ &= \frac{1}{z} - 1 + z - \dots\end{aligned}$$

so the residue at 0 is 1.

1(b) This function is

$$\begin{aligned}z\cos\left(\frac{1}{z}\right) &= z\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{-2n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{1-2n} \\ &= z - \frac{1}{2z} + \frac{1}{24z^3} - \dots\end{aligned}$$

so the residue at zero is $-1/2$.

1(c) This function is

$$\begin{aligned}\frac{z - \sin z}{z} &= 1 - \frac{\sin z}{z} \\ &= 1 - \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1} \\ &= 1 - \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}\end{aligned}$$

This series has no $1/z$ term, so the residue at zero is 0.

1(d) The Laurent series expansion for this function is

$$\begin{aligned}\frac{1}{z^4}\cot z &= \frac{1}{z^4}\left(\frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \dots\right) \\ &= \frac{1}{z^5} - \frac{1}{3z^3} - \frac{1}{45z} - \frac{2z}{945} - \dots\end{aligned}$$

so the residue at $z = 0$ is $-1/45$.

1(e) A series expansion for this function is

$$\begin{aligned}
\frac{\sinh z}{z^4(1-z^2)} &= \frac{1}{z^4}\left(\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}z^{2n+1}\right)\left(\sum_{n=0}^{\infty}z^{2n}\right)\\
&= \frac{1}{z^4}\left(z+\frac{z^3}{6}+\frac{z^5}{120}\right)\left(1+z^2+z^4+\dots\right)\\
&= \frac{1}{z^4}\left(z+\frac{7z^3}{6}+\frac{141z^5}{120}+\dots\right)\\
&= \frac{1}{z^3}+\frac{7}{6z}+\frac{141z}{120}+\dots
\end{aligned}$$

so the residue at zero is $7/6$.

2(a) This function only has a singularity at $z=0$. Finding the Laurent series here, the expansion is

$$\begin{aligned}
\frac{1}{z^2}e^{-z} &= \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}z^n\\
&= \frac{1}{z^2}\left(1-z+\frac{z^2}{2}-\frac{z^3}{6}+\dots\right)\\
&= \frac{1}{z^2}-\frac{1}{z}+\frac{1}{2}-\frac{z}{6}+\dots
\end{aligned}$$

so the residue at zero is -1, and

$$\int_C \frac{e^{-z}}{z^2}\,dz = 2\pi i(-1) = -2\pi i$$

2(b) This function now has a singular point at $z=1$. The series expansion here is

$$\begin{aligned}
\frac{1}{(z-1)^2}e^{-z} &= \frac{1}{(z-1)^2}e^{-(z-1)}\frac{1}{e}\\
&= \frac{1}{e(z-1)^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(z-1)^n\\
&= \frac{1}{e}\left(\frac{1}{(z-1)^2}-\frac{1}{z-1}+\frac{1}{2}-\frac{z-1}{6}+\dots\right)
\end{aligned}$$

so the residue at $z=1$ is $-1/e$, and

$$\int_C f(z)\,dz = 2\pi i(-1/e) = -\frac{2\pi}{e}i$$

2(c) This function only has a singular point at $z = 0$, with the series expansion

$$\begin{aligned} z^2 e^{1/z} &= z^2 \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \\ &= z^2 \left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots\right) \\ &= z^2 + z + \frac{1}{2} + \frac{1}{6z} + \dots \end{aligned}$$

so the residue here is 1/6, and

$$\int_C z^2 e^{1/z} dz = 2\pi i \frac{1}{6} = \frac{\pi i}{3}$$

2(d) This function has singular points at $z = 0$ and $z = 2$. Expanding the function at $z = 0$ gives

$$\begin{aligned} \frac{z+1}{z} \frac{1}{z-2} &= \left(1 + \frac{1}{z}\right) \frac{-1}{2(1 - z/2)} \\ &= \left(1 + \frac{1}{z}\right) \frac{-1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} + \dots\right) \\ &= -\frac{1}{2z} - \frac{3}{2} - \frac{3z}{4} - \dots \end{aligned}$$

so the residue at $z = 0$ is $-1/2$. Then,

$$\begin{aligned} \frac{z+1}{z-2} \frac{1}{z} &= \frac{(z-2)+3}{z-2} \frac{1}{2 + (z-2)} \\ &= \frac{1}{2} \left(1 + \frac{3}{z-2}\right) \frac{1}{1 + (z-2)/2} \\ &= \frac{1}{2} \left(1 + \frac{3}{z-2}\right) \left(1 - \frac{z-2}{2} + \frac{(z-2)^2}{4} - \dots\right) \\ &= \frac{3}{2(z-2)} - \frac{1}{4} + \dots \end{aligned}$$

so the residue at $z = 2$ is $3/2$. Thus,

$$\int_C \frac{z+1}{z^2 - 2z} dz = 2\pi i(-1/2 + 3/2) = 2\pi i$$

3(a) The residue at infinity can be found by writing the function

$$\begin{aligned} \frac{1}{z^2} \frac{(1/z)^5}{1 - (1/z)^3} &= \frac{-1}{z^4} \frac{1}{1 - z^3} \\ &= \frac{-1}{z^4} \left(1 + z^3 + z^6 + \dots\right) \\ &= -\frac{1}{z^4} - \frac{1}{z} - z^2 - \dots \end{aligned}$$

so the residue at infinity is $-(-1)$, and

$$\int_C f(z)\ dz = 2\pi i \cdot (-1) = -2\pi i$$

3(b) The residue at infinity can be found via

$$\begin{aligned}\frac{1}{z^2}\frac{1}{1+(1/z)^2} &= \frac{1}{1+z^2} \\ &= 1 - z^2 + z^4 - \dots\end{aligned}$$

so the residue at infinity is zero, and

$$\int_C f(z)\ dz = 0$$

3(c) The residue at infinity, from

$$\frac{1}{z^2}\frac{1}{1/z} = \frac{1}{z}$$

is -1, so

$$\int_C f(z)\ dz = 2\pi i$$

2 Frame 72 – Singular Points

1(a) This function is

$$ze^{1/z} = z\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \dots\right) = z + 1 + \frac{1}{2z} + \dots$$

so it has an essential singular point at the origin.

1(b) This function is

$$\frac{z^2}{z+1} = \frac{(z+1)^2 - 2(z+1) + 1}{z+1} = (z+1) - 2 + \frac{1}{z+1}$$

so it has a simple pole at $z = -1$.

1(c) This function is

$$\frac{\sin z}{z} = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

so it has a removable singular point at the origin.

1(d) This function is

$$\frac{\cos z}{z} = \frac{1}{z}\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \dots\right) = \frac{1}{z} - \frac{z}{2!} + \frac{z^3}{4!} - \dots$$

so it has a simple pole at the origin.

1(e) This function is already in principal form. It has a third order pole at $z = 2$.

2(a) This function is

$$\begin{aligned}\frac{1 - \cosh z}{z^3} &= \frac{1}{z^3}\left[1 - \left(1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \dots\right)\right] \\ &= \frac{1}{z^3}\left[-\frac{z^2}{2!} - \frac{z^4}{4!} - \dots\right] \\ &= -\frac{1}{2! \cdot z} - \frac{z}{4!} - \dots\end{aligned}$$

so it has a first-order pole at the origin with a residue of $B = -1/2$.

2(b) This function is

$$\begin{aligned}\frac{1 - e^{2z}}{z^4} &= \frac{1}{z^4}\left[-\frac{2z}{1!} - \frac{4z^2}{2!} - \frac{8z^3}{3!} - \frac{16z^4}{4!} \dots\right] \\ &= -\frac{2}{z^3} - \frac{2}{z^2} - \frac{4}{3z} - \frac{2}{3} - \dots\end{aligned}$$

so it has a third-order pole at the origin with a residue of $B = -4/3$.

4 To solve the equation

$$e^{1/z} = -1$$

we note that this occurs when

$$\frac{1}{z} = (2n+1)\pi i$$

or

$$z = \frac{1}{(2n+1)\pi i} = -\frac{i}{(2n+1)\pi}$$

5 If we write the function

$$f(z) = \frac{8a^3z^2}{(z^2+a^2)^3}$$

as

$$f(z) = \frac{\phi(z)}{(z-ai)^3} \quad \text{where} \quad \phi(z) = \frac{8a^3z^2}{(z+ai)^3}$$

then, since $\phi(z)$ has no singular points at $z = ai$, we can write its Taylor series as

$$\begin{aligned}\phi(z) &= \frac{8a^3z^2}{(z+ai)^3} \\ &= \phi(ai) + \phi'(ai)(z-ai) + \frac{\phi''(ai)}{2}(z-ai)^2 + \dots\end{aligned}$$

To find these coefficients, the derivative of $\phi(z)$ is

$$\begin{aligned}\phi'(z) &= \frac{d}{dz}\frac{8a^3z^2}{(z+ai)^3} \\ &= \frac{16a^3z(z+ai)^3 - 24a^3z^2(z+ai)^2}{(z+ai)^6} \\ &= \frac{16a^3z(z+ai) - 24a^3z^2}{(z+ai)^4} \\ &= \frac{8a^3z(-z+2ai)}{(z+ai)^4}\end{aligned}$$

and the second derivative is

$$\begin{aligned}\phi''(z) &= \frac{d}{dz}\frac{8a^3z(-z+2ai)}{(z+ai)^4} \\ &= \frac{d}{dz}\frac{-8a^3z^2+16a^4zi}{(z+ai)^4} \\ &= \frac{(-16a^3z+16a^4i)(z+ai)^4 - 4(z+ai)^3(-8a^3z^2+16a^4zi)}{(z+ai)^8} \\ &= \frac{(-16a^3z+16a^4i)(z+ai) - 4(-8a^3z^2+16a^4zi)}{(z+ai)^5}\end{aligned}$$

Evaluating these at $z = ai$,

$$\begin{aligned}
\phi(ai) &= \frac{8a^3(ai)^2}{(2ai)^3} \\
&= -i\frac{8a^5}{8a^3} \\
&= -a^2 i
\end{aligned}$$

$$\begin{aligned}
\phi'(ai) &= \frac{8a^3(ai)^2}{(2ai)^4} \\
&= -\frac{8a^5}{16a^4} \\
&= -\frac{a}{2}
\end{aligned}$$

$$\begin{aligned}
\phi''(ai) &= \frac{(-16a^3(ai) + 16a^4 i)(2ai) - 4(8a^3(ai)^2)}{(2ai)^5} \\
&= \frac{(0) + 4(8a^5)}{32a^5 i} \\
&= -i
\end{aligned}$$

so we find that

$$\phi(z) = -a^2 i - \frac{a}{2}(z - ai) - \frac{i}{2}(z - ai)^2$$

and

$$f(z) = \frac{\phi(z)}{(z - ai)^3} = \frac{-i/2}{z - ai} - \frac{a/2}{(z - ai)^2} - \frac{a^2 i}{(z - ai)^3}$$

3 Frame 74 – Residues and Poles

1(a) This function has a simple pole at $z = 1$, with

$$B = \phi(1) = (1)^2 + 2 = 3$$

1(b) This function has a third-order pole at $z = -1/2$, with

$$B = \frac{(-1/2)^3}{2^3} = -1/64$$

1(c) This function has simple poles at $z = \pm\pi i$. At $z = \pi$,

$$B_1 = \left.\frac{e^z}{z + i\pi}\right|_{z=i\pi} = \frac{e^{i\pi}}{2i\pi} = \frac{i}{2\pi}$$

and at $z = -\pi$,

$$B_2 = \left.\frac{e^z}{z - i\pi}\right|_{z=-i\pi} = \frac{e^{-i\pi}}{-2i\pi} = \frac{-i}{2\pi}$$

2(a) This residue is

$$\mathrm{Res}_{z=-1}\,\frac{z^{1/4}}{z+1} = \left.z^{1/4}\right|_{z=-1} = 1e^{\pi/4} = \frac{1+i}{\sqrt{2}}$$

2(b) This residue is

$$\begin{aligned}
\mathrm{Res}_{z=i}\,\frac{\operatorname{Log} z}{(z^2+1)^2} &= \left.\frac{d}{dz}\frac{\operatorname{Log} z}{(z+i)^2}\right|_{z=i} \\
&= \left.\frac{\frac{1}{z}(z+i)^2 - 2(z+i)\operatorname{Log} z}{(z+i)^4}\right|_{z=i} \\
&= \left.\frac{1 + i/z - 2\operatorname{Log} z}{(z+i)^3}\right|_{z=i} \\
&= \frac{1 + 1 - 2(0 + i\pi/2)}{-8i} \\
&= \frac{2 - i\pi}{-8i} \\
&= \frac{\pi + 2i}{8}
\end{aligned}$$

3(a) This circle only contains the singular point $z = 1$, so

$$\int_C f(z)\,dz = 2\pi i \cdot \mathrm{Res}_{z=1}\, f(z) = 2\pi i \cdot \left.\frac{3z^3+2}{z^2+9}\right|_{z=1} = 2\pi i \cdot \frac{3+2}{1+9} = \pi i$$

3(b) Now, the circle contains every singular point of f. The residue at infinity is then

$$\begin{aligned}
\mathrm{Res}_{z=\infty} f(z) &= -\,\mathrm{Res}_{z=0} \frac{1}{z^2} f(1/z) \\
&= -\,\mathrm{Res}_{z=0} \frac{1}{z^2} \frac{3+2z^3}{(1-z)(1+9z^2)} \\
&= -\frac{d}{dz} \frac{3+2z^3}{-9z^3+9z^2-z+1}\Big|_{z=0} \\
&= -\frac{(6z^2)(-9z^3+9z^2-z+1)-(-27z^2+18z-1)(3+2z^3)}{(1-z+9z^2-9z^3)^2}\Big|_{z=0} \\
&= -\frac{0-(-1)(3)}{1} \\
&= -3
\end{aligned}$$

so

$$\int_C f(z)\ dz = -2\pi i(-3) = 6\pi i$$

4(a) This contour only contains the singular point at $z=0$. The residue here is

$$\begin{aligned}
\mathrm{Res}_{z=0} f(z) &= \frac{1}{2} \frac{d^2}{dz^2} \frac{1}{z+4}\Big|_{z=0} \\
&= \frac{1}{2} \frac{d}{dz} \frac{-1}{(z+4)^2}\Big|_{z=0} \\
&= \frac{1}{2} \frac{2}{(z+4)^3}\Big|_{z=0} \\
&= \frac{1}{64}
\end{aligned}$$

so

$$\int_C f(z)\ dz = 2\pi i \frac{1}{64} = \frac{\pi i}{32}$$

4(b) This contour now contains both singular points. The second residue (at $z=-4$) is

$$\mathrm{Res}_{z=-4} f(z) = \frac{1}{z^3}\Big|_{z=-4} = -\frac{1}{64}$$

so the sum of the two residues is zero, and

$$\int_C f(z)\ dz = 0$$

4 Frame 76 – Poles and Zeroes

1 The residue at $z = 0$ is

$$B = \frac{1}{\cos 0} = 1$$

2(a) The derivative of the denominator is

$$\frac{d}{dz} z^2 \sinh z = z^2 \cosh z + 2z \sinh z$$

At $z = i\pi$, then, the residue is

$$B = \frac{i\pi - \sinh(i\pi)}{(i\pi)^2 \cosh(i\pi) + (i\pi)\sinh(i\pi)} = \frac{i\pi}{-\pi^2 \cdot (-1)} = \frac{i}{\pi}$$

3(a) Since this function is $\frac{z}{\cos z}$, the residue is

$$B = \frac{z_n}{-\sin(z_n)} = \frac{z_n}{-(-1)^n} = (-1)^{n+1} z_n$$

4(a) Since $\tan z = \sin z / \cos z$, this contour contains the two poles at $z = \pm\pi/2$. Here, the residues are

$$B_n = \frac{\sin(\pm\pi/2)}{-\sin(\pm\pi/2)} = -1$$

so the sum of the residues is -2 and the contour integral evaluates to

$$\int_C \tan z \; dz = 2\pi i(-2) = -4\pi i$$