1 Frame 37 – Derivatives with Real Variables

1.1 Definition

In the previous chapter, we looked at derivatives of complex functions of a complex variable z. Now, we look at the derivatives of a complex-valued function of a real variable t. If we write our function as

$$w(t) = u(t) + iv(t)$$

where u and v are real-valued, then we can define the derivative of w at a point t as

$$w'(t) = \frac{d}{dt}w(t) = u'(t) + iv'(t)$$

provided that u' and v' exist at t.

1.2 Properties

If $z_0 = x_0 + iy_0$ is a complex constant, then we can show that

$$\begin{aligned}
\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u(t) + iv(t)]' \\
&= [x_0 u(t) - y_0 v(t)]' + i[y_0 u(t) + x_0 v(t)]' \\
&= [x_0 u'(t) - y_0 v'(t)] + i[y_0 u'(t) + x_0 v'(t)] \\
&= z_0 w'(t)
\end{aligned}$$

as we expect.

Next, if z_0 is still a complex constant, the derivative of $e^{z_0 t}$ is

$$\begin{aligned}
\frac{d}{dt}e^{z_0 t} &= \frac{d}{dt}e^{x_0 t}(\cos y_0 t + isiny_0 t) \\
&= \frac{d}{dt}e^{x_0 t}\cos y_0 t + i\frac{d}{dt}e^{x_0 t}\sin y_0 t \\
&= (x_0 + iy_0)(e^{x_0 t}\cos y_0 t + ie^{x_0 t}\sin y_0 t) \\
&= z_0 e^{z_0 t}
\end{aligned}$$

Many other rules carry over from standard calculus. However, some rules no longer apply. For instance, in calculus, the mean value theorem for derivatives states that

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

for some c in the interval $a \leq c \leq b$ as long as w is continuous. However, this is easily disproved by the function

$$w(t) = e^{it}$$

If $a = 0$ and $b = 2\pi$, then $w(a) = w(b) = 1$ and we expect to find a point c in $[0, 2\pi]$ such that $w'(c) = 0$. However, no such points exist – the magnitude of the derivative is always 1.

2 Frame 38 – Definite Integrals of Complex Functions

2.1 Definitions

If $w(t)$ is a complex-valued function of a real variable t, as in the previous section

$$w(t) = u(t) + iv(t)$$

then we define the **definite integral** of $w(t)$ over the interval $a \leq t \leq b$ as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

provided the two right-side integrals exist. Then,

$$\Re\left[\int_a^b w(t)dt\right] = \int_a^b \Re[w(t)]dt$$
$$\Im\left[\int_a^b w(t)dt\right] = \int_a^b \Im[w(t)]dt$$

Improper integrals over unbounded intervals are defined similarly.

The two real integrals will exist as long as u and v are **piecewise continuous** on the interval $[a, b]$ – that is, continuous everywhere in the interval except possibly for a finite number of points where it has one-sided limits. When u and v are piecewise continuous, we say that w is also piecewise continuous.

2.2 Properties

The most common rules of integrals from calculus apply here as well:

- $\int z_0 w(t)dt = z_0 \int w(t)$
- $\int w_1(t) + w_2(t)dt = \int w_1(t)dt + \int w_2(t)dt$
- $\int_a^b w(t)dt = -\int_b^a w(t)dt$
- $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$

We can also extend the fundamental theorem of calculus to complex integrals. Suppose that two functions

$$w(t) = u(t) + iv(t)$$
$$W(t) = U(t) + iV(t)$$

are continuous on the interval $[a, b]$ and $W'(t) = w(t)$ when $a \leq t \leq b$. Then, we can write

$$\int_a^b w(t)dt = W(b) - W(a) = W(t)\Big|_a^b$$

Example: noting that the derivative of $\frac{1}{i}e^{it}$ *is*

$$\frac{d}{dt}\left(\frac{1}{i}e^{it}\right) = \frac{1}{i}ie^{it} = e^{it}$$

we can evaluate $\int e^{it}dt$ *as*

$$\begin{aligned}
\int_0^{\pi/4} e^{it}dt &= \frac{e^{it}}{i}\Big|_0^{\pi/4} \\
&= \frac{1}{i}\left[e^{\pi/4} - 1\right] \\
&= \frac{1}{i}\left[\frac{1}{\sqrt{2}} - 1 + \frac{i}{\sqrt{2}}\right] \\
&= \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)
\end{aligned}$$

As in the previous section, the mean value theorem for integrals does not apply. We can show this by finding the integral $\int_0^{2\pi} e^{it}dt = 0$, even though the function is never zero on this interval.

3 Frame 39 – Contours

3.1 Definitions

In calculus, integrals are defined on intervals of the real line. In complex analysis, we instead use curves in the complex plane.

An **arc** is a set of points $z = (x, y)$ in the complex plane such that the functions

$$x = x(t), \quad y = y(t); \quad z = z(t) = x(t) + iy(t)$$

are continuous functions of the parameter t, where $a \leq t \leq b$. This definition is a continuous mapping of the interval $a \leq t \leq b$ into the z plane.

We say that an arc is **simple** if it does not cross itself; ie:

$$z(t_1) \neq z(t_2) \quad \text{for all } t_1 \neq t_2$$

If a simple arc starts and ends at the same point ($z(a) = z(b)$), it is called a **simple closed curve**. These curves are **positively oriented** when they are oriented in the counterclockwise direction.

Example: the unit circle

$$z = e^{i\theta}$$

where $0 \leq \theta \leq 2\pi$ *is a positively oriented simple closed curve centered at the origin with a radius of* 1. *A more general circle is*

$$z = z_0 + Re^{i\theta}$$

which is centered at z_0 *and has a radius of* R.

3.2 Uniqueness

Note that the parametric representation for any arc is not unique. If we know a function ϕ such that

$$t = \phi(\tau)$$

maps the interval $\alpha \leq \tau \leq \beta$ onto the interval $a \leq t \leq b$. Then, the two equations

$$z(t) \quad (a \leq t \leq b)$$

and

$$z(\phi(t)) \quad (\alpha \leq t \leq \beta)$$

represent the same arc.

3.3 Smoothness

Suppose that the real and imaginary components of z are differentiable, and their derivatives are continuous. Then, the arc $z(t)$ is a **differentiable arc**, and

$$|z'(t)| = |x'(t) + iy'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable. This allows us to find the length of an arc as

$$L = \int_a^b |z'(t)| dt$$

If an arc is differentiable and $z'(t)$ is never zero (except maybe at $t = a$ or $t = b$), then we call the arc a **smooth arc**. We can write the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

which has an angle of inclination of $\arg z'(t)$.

A **contour** is an arc which consists of a finite number of smooth arcs joined together. Specifically, if $z(t)$ represents a contour, then $z(t)$ is continuous and $z'(t)$ is piecewise continuous. If a contour is also a simple closed arc, we call it a **simple closed contour**.

The points on a simple closed arc are the boundary points of two different domains:

- The interior of the arc, which is bounded;
- The exterior of the arc, which is unbounded.

4 Frame 40 – Contour Integrals

4.1 Definitions and conditions

We can now integrate a complex function f along a contour C, which starts and ends at points z_1 and z_2, respectively. This is effectively a line integral. These integrals can be written as

$$\int_C f(z)dz$$

or, if the integral does not depend on the path taken,

$$\int_{z_1}^{z_2} f(z)dz$$

This integral (along a complex path) represents an integral with respect to a real parameter t. If the contour C is written as $z(t)$ on the interval $a \leq t \leq b$, then the integral represented is

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$$

Since $z'(t)$ must be piecewise continuous, this integral exists as long as $f[z(t)]$ is also piecewise continuous on this interval.

4.2 Basic properties

From the definition and the properties of integrals, we can write

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$$

and

$$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

We can also create a new contour $-C$ that consists of the points in C in reversed order – this contour extends from z_2 to z_1. Integrating along this reversed contour, we find that

$$\begin{aligned}
\int_- Cf(z)dz &= \int_{-b}^{-a} f[z(-t)]\frac{d}{dt}z(-t)dt \\
&= -\int_{-b}^{-a} f[z(-t)]z'(-t)dt \\
&= -\int_a^b f[z(t)]z'(t)dt \\
&= -\int_C f(z)dz
\end{aligned}$$

We can also split up a contour C into multiple legs $C_1, C_2, \ldots$. If we can write a contour this way, then we say that $C = C_1 + C_2$. The contour integral along C can then be written as

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$