1 Frame 29 – The Exponential Function

1(a) The value is
$$e^{2\pm 3\pi i} = e^2 e^{\pm 3\pi i} = e^2(-1) = e^2$$

1(b) The value is
$$e^{1/2+i\pi/4} = e^{1/2}e^{i\pi 4} = \sqrt{e}\frac{1+i}{\sqrt{2}} = \sqrt{\frac{e}{2}}(1+i)$$

1(c) This expression can be split up as
$$e^{z+\pi i} = e^z e^{i\pi} = e^z \cdot -1 = -e^z$$

3 The function can be written as
$$e^{\overline{z}} = e^x e^{-iy} = e^x \cos y - ie^x \sin y$$
so the partial derivatives are
$$\begin{aligned} u_x &= e^x \cos y \\ u_y &= -e^x \sin y \\ v_x &= -e^x \sin y \\ v_y &= -e^x \cos y \end{aligned}$$
These don't satisfy Cauchy-Riemann, so the function is differentiable nowhere and thus analytic nowhere.

8(a) The equation $e^z = -2$ can be written as
$$e^z = 2e^{i\pi}$$
so
$$\begin{aligned} e^x = 2 &\to x = \ln 2 \\ e^{iy} = e^{i\pi} &\to y = (2n+1)\pi \end{aligned}$$
and
$$z = \ln 2 + (2n+1)i\pi$$

8(b) The equation $e^z = 1 + i\sqrt{3}$ can be written as
$$e^z = 2e^{i\pi/3}$$
so
$$\begin{aligned} x &= \ln 2 \\ y &= \left(\frac{1}{3} + 2n\right)\pi \end{aligned}$$

and

$$z = \ln 2 + \left(\frac{1}{3} + 2n\right) i\pi$$

8(c) If

$$e^{2z-1} = 1$$

then

$$2z - 1 = i2n\pi$$

or, solving for z,

$$z = \frac{1}{2} + in\pi$$

2 Frame 31 – Logarithms

1(a) Evaluating the logarithm,

$$\mathrm{Log}(-ei) = \ln(e) + \mathrm{Arg}(-ei) = 1 - i\frac{\pi}{2}$$

1(b) As above,

$$\mathrm{Log}(1-i) = \ln(\sqrt{2}) + \mathrm{Arg}(1-i) = \frac{1}{2}\ln(2) - i\frac{\pi}{4}$$

2(a) The set of values is

$$\log(e) = \ln(e) + \arg(e) = 1 + i2n\pi$$

2(b) As above,

$$\log(i) = \ln(1) + \arg(i) = 0 + i\left(2n + \frac{1}{2}\right)\pi$$

2(c) As above,

$$\log(-1+i\sqrt{3}) = \ln(\sqrt{4}) + \arg(-1+i\sqrt{3}) = \ln 2 + i\left(2n + \frac{2}{3}\right)\pi$$

3(a) The left side is

$$\mathrm{Log}(1+i)^2 = \ln(2) + i\frac{\pi}{2}$$

and the right side is

$$2\,\mathrm{Log}(1+i) = 2\left[\frac{1}{2}\ln(2) + i\frac{\pi}{4}\right] = \ln(2) + i\frac{\pi}{2} = \mathrm{Log}(1+i)^2$$

3(b) The left side is

$$\mathrm{Log}(-1+i)^2 = \ln(2) - i\frac{\pi}{2}$$

and the right side is

$$2\,\mathrm{Log}(-1+i) = 2\left[\frac{1}{2}\ln(2) + i\frac{3\pi}{4}\right] = \ln(2) + i\frac{3\pi}{2}$$

5(a) Since $i^{1/2}$ can be written as the set

$$i^{1/2} = e^{\pi/4+\pi k} \quad (k = 0, \pm1, \pm2, \dots)$$

it has the logarithm

$$\log(i^{1/2}) = \ln(1) + i\arg(i^{1/2}) = i\left(n + \frac{1}{4}\right)\pi$$

Then,

$$\frac{1}{2}\log i = \frac{1}{2}\left[\ln(1) + i\left(2n + \frac{1}{2}\right)\pi\right] = i\left(n + \frac{1}{4}\right)\pi = \log(i^{1/2})m$$

6 Differentiating and using the chain rule, the identity

$$z = e^{\log z}$$

becomes

$$1 = e^{\log z} \cdot \frac{d}{dz}\log z$$

or

$$\frac{d}{dz}\log z = \frac{1}{e^{\log z}} = \frac{1}{z}$$

7 To solve the equation

$$\log z = i\frac{\pi}{2}$$

we write

$$z = e^{i\pi/2} = i$$