1 Evaluating Improper Integrals

1.1 Improper Integrals

The **improper integral** of a continuous function $f(x)$ over the interval $0 \leq x < \infty$ is defined as

$$\int_0^\infty f(x)\ dx = \lim_{R\to\infty} \int_0^R f(x)\ dx$$

If this limit exists, we say that the improper integral **converges** to this limit. The improper integral of f over the infinite interval $-\infty < x < \infty$ is

$$\int_{-\infty}^\infty = \lim_{R_1\to\infty} \int_{-R_1}^0 f(x)\ dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\ dx$$

If both of these limits exist, we say that the integral converges to their sum.

1.2 The Cauchy Principal Value

We say that the **Cauchy Principal Value** of an indefinite integral is

$$\text{P.V.} \int_{-\infty}^\infty f(x)\ dx = \lim_{R\to\infty} \int_{-R}^R f(x)\ dx$$

as long as this limit exists.

If the regular improper integral converges, then

$$\begin{aligned} \lim_{R\to\infty} \int_{-R}^R f(x)\ dx &= \lim_{R\to\infty} \left[\int_{-R}^0 f(x)\ dx + \int_0^R f(x)\ dx \right] \\ &= \lim_{R\to\infty} \int_{-R}^0 f(x)\ dx + \lim_{R\to\infty} \int_0^R f(x)\ dx \end{aligned}$$

so the principal value also exists. However, the converse is not true – the existence of the principal value does not imply the existence of the improper integral.

Next, suppose that $f(x)$ is an even function; ie:

$$f(-x) = f(x)$$

and assume that the principal value exists. Then, the evenness of f allows us to write

$$\begin{aligned} \int_{-R_1}^0 f(x)\ dx &= \frac{1}{2} \int_{-R_1}^{R_1} f(x)\ dx \\ \int_0^{R_2} f(x)\ dx &= \frac{1}{2} \int_{-R_2}^{R_2} f(x)\ dx \end{aligned}$$

so we can convert both single-sided limits into double-sided limits, and

$$\int_{-\infty}^{\infty} f(x)\ dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx$$

Also, extending this formula,

$$\int_{0}^{\infty} f(x)\ dx = \frac{1}{2}\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx$$

1.3 Using Residues

Now, we apply our knowledge of residues to integrate $f(z) = p(z)/q(z)$ along the real axis when p and q are polynomials. In this discussion, suppose that q has at least one zero above the real axis and no zeroes on the real axis.

From our knowledge of polynomials, we know that q has a finite number of distinct zeroes, which we can label as $z_1, z_2, \ldots, z_n$. Then, we can integrate the function $f(z)$ along the contour:

- Along the real axis from $-R$ to R;
- Along the upper semicircle with a radius of R from $(R, 0$ to $(-R, 0)$, which we call C_R.

This contour allows us to write

$$\int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$$

or

$$\int_{-R}^{R} f(x)\ dx = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z) - \int_{C_R} f(z)\ dz$$

Using this expression, we can say that if

$$\lim_{R\to\infty} \int_{C_R} f(z)\ dz = 0$$

then the following three equations hold (the latter two if f is even):

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum k = 1^n \text{Res}_{z=z_k} f(z)$$

$$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum_{z=z_k} f(z)$$

$$\int_{0}^{\infty} f(x)\ dx = \pi i \sum_{z=z_k} f(z)$$