1 Frame 12 – Functions of Complex Variables

1(a) The function
$$f(z) = \frac{1}{z^2+1}$$
is defined everywhere except where $z^2 + 1 = 0$; ie:
$$z \neq \pm i$$

1(b) The function
$$f(z) = \mathrm{Arg}\left(\frac{1}{z}\right)$$
is defined wherever $\frac{1}{z}$ is defined:
$$z \neq 0$$

1(c) The function
$$f(z) = \frac{z}{z+\overline{z}}$$
can be written as
$$f(x, y) = \frac{x+iy}{(x+iy)+(x-iy)} = \frac{x+iy}{2x} = \frac{1}{2} + i\frac{y}{x}$$
so the domain is
$$\mathrm{Re}(z) \neq 0$$

1(d) The function
$$f(z) = \frac{1}{1-|z|^2}$$
is equivalent to
$$f(r, \theta) = \frac{1}{1-r^2}$$
so the domain is
$$r \neq 1$$

2 Substituting $z = x + iy$ gives
$$\begin{aligned} f(x, y) &= (x+iy)^3 + (x+iy) + 1 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x + iy + 1 \\ &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) \end{aligned}$$

so

$$\begin{aligned} u(x, y) &= x^3 - 3xy^2 + x + 1 \\ v(x, y) &= 3x^2y - y^3 + y \end{aligned}$$

3 Using the two expressions

$$x = \frac{z + \overline{z}}{2}$$
$$y = \frac{z - \overline{z}}{2i}$$

gives

$$\begin{aligned}
f(z) &= \Big(\frac{z+\overline{z}}{2}\Big)^2 - \Big(\frac{z-\overline{z}}{2i}\Big)^2 - 2\frac{z-\overline{z}}{2i} \\
&\quad + i\Big[2\frac{z+\overline{z}}{2}\Big(1 - \frac{z-\overline{z}}{2i}\Big)\Big] \\
&= \frac{1}{4}(z^2 + 2z\overline{z} + \overline{z}^2 + z^2 - 2z\overline{z} + \overline{z}^2) + iz - i\overline{z} \\
&\quad + i\Big[z + \overline{z} + \frac{iz^2}{2} - \frac{i\overline{z}^2}{2}\Big] \\
&= \frac{1}{2}(z^2 + \overline{z}^2) + 2iz - \frac{iz^2}{2} + \frac{\overline{z}^2}{2} \\
&= \overline{z}^2 + 2iz
\end{aligned}$$

4 Using

$$z = re^{i\theta}$$

the function can be written as

$$\begin{aligned}
f(z) &= re^{i\theta} + \frac{1}{re^{i\theta}} \\
&= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\
&= (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta
\end{aligned}$$

2 Frame 14 – Mappings by the Exponential Function

1 We saw earlier that the hyperbolas

$$x^2 - y^2 = c_1$$

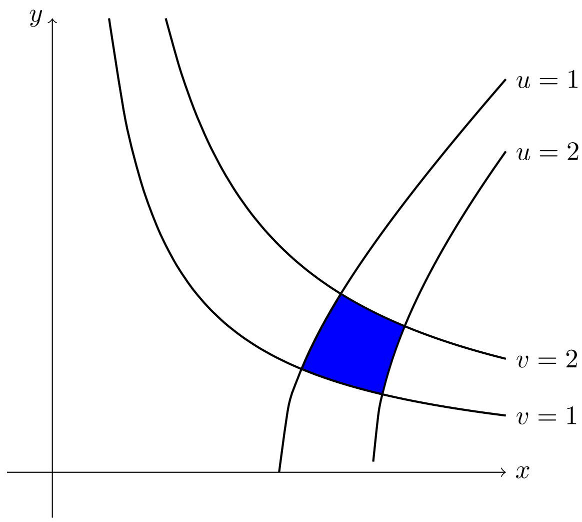
map onto the horizontal lines $u = c_1$ and the hyperbolas

$$2xy = c_2$$

map onto the vertical lines $v = c_2$. Thus, a domain on the z-plane that maps onto $1 \leq u \leq 2$ and $1 \leq v \leq 2$ is

$$1 \leq x^2 - y^2 \leq 2 \quad 1 \leq 2xy \leq 2$$

A sketch of this region is:



2 The first hyperbola can be written as

$$y^2 - x^2 = |c_1|$$

Then, substitution into the v equation gives

$$u = c_1, \quad v = \begin{cases} 2x\sqrt{x^2 + |c_1|}, & y > 0 \\ -2x\sqrt{x^2 + |c_1|}, & y < 0 \end{cases}$$

This maps out the entire v line as x moves right (on the top branch) or left (on the bottom branch).

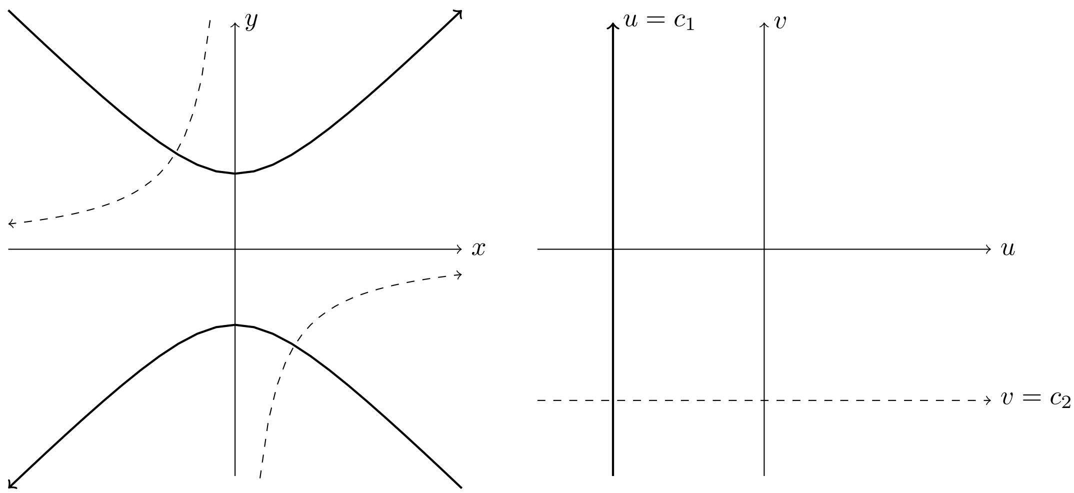
The second hyperbola can be written as

$$2xy = -|c_2|$$

and substituting this into the u equation gives

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

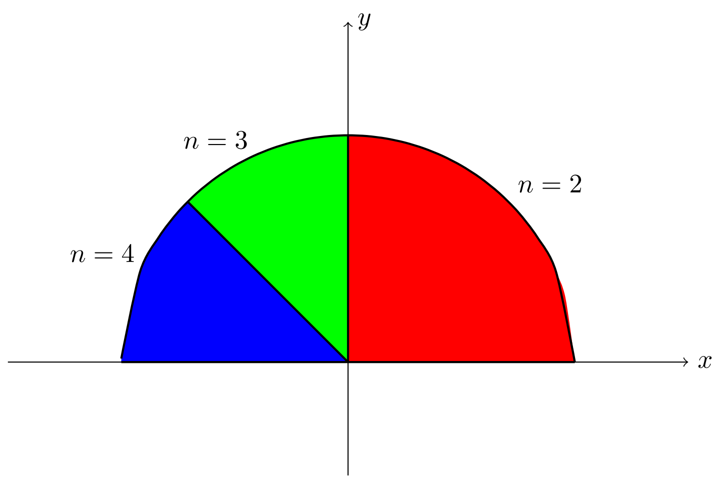
This maps out the entire u line: as x gets large in magnitude, so too does u. A sketch of these mappings is:



3 The image of the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ under the mapping $w = z^n$ is

$$\rho \leq 1, \quad 0 \leq \theta \leq n\frac{\pi}{4}$$

A sketch of these images for $n = 2, 3, 4$ is:



4 If z follows the straight line $ay = x$, then the mapping $w = e^z$ is

$$\begin{aligned} w &= e^{x+iy} \\ &= e^{ay}e^{iy} \\ &= e^{a\phi}e^{i\phi} \\ &= \rho e^{i\phi} \end{aligned}$$

where $\rho = a\phi$.

5 The rectangular region $a \le x \le b$, $c \le y \le d$ is made up of the horizontal line segments

$$x = t, \quad y = c_1$$

where t is a parameter running from a to b and c_1 is a constant in the range $[c, d]$. These horizontal lines have the images

$$\rho = e^t, \quad \phi = c_1$$

Since t starts at a and ends at b, these images have a radius in the range $[e^a, e^b]$. Then, the entire image is the set of these lines, which range from $\phi = c$ to $\phi = d$. Thus, the entire image is

$$e^a \le \rho \le e^b, \quad c \le \phi \le d$$

6 Looking at the z plane, the initial set is the infinite strip

$$x \le 0, \quad 0 \le y \le \pi$$

This maps to the image set

$$\lim_{a \to -\infty} e^a \le \rho \le e^0, \quad 0 \le \phi \le \pi$$

or

$$0 \le \rho \le 1, \quad 0 \le \phi \le \pi$$

This is the upper half of the unit disk, as shown in the figure.

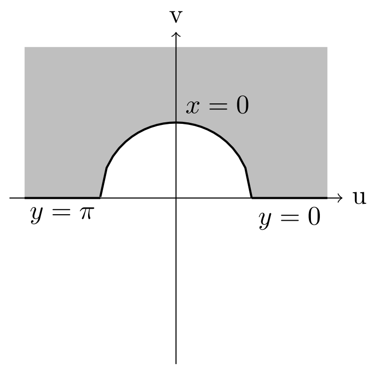
7 In a similar manner to the previous problem, the image of the strip

$$x \ge 0, \quad 0 \le y \le \pi$$

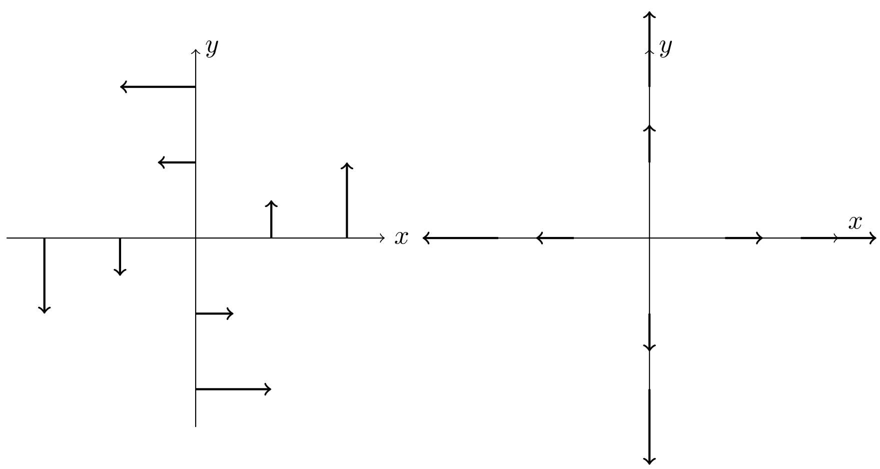
is the upper half-plane with a unit disk cut out:

$$\rho \ge 1, \quad 0 \le \phi \le \pi$$

A sketch of this region is:



8 Some sample vectors in these two fields are:



3 Frame 18 – Limits and Continuity

1(a) The left side of the limit is

$$|\Re(z) - \Re(z_0)| = |\Re(z - z_0)| < |z - z_0|$$

so

$$|\Re(z) - \Re(z_0)| < \delta \text{ whenever } |z - z_0| < \delta$$

1(b) The left side of the limit is

$$|\overline{z} - \overline{z_0}| = |\overline{z - z_0}| = |z - z_0|$$

so the limit holds.

1(c) The limit expression is

$$|\frac{\overline{z}^2}{z}| < \epsilon \text{ whenever } |z| < \delta$$

For $z \neq 0$, the left side expression is $|z|$, so the limit holds where $\epsilon = \delta$.

2(a) The left side of the limit is

$$|(az + b) - (az_0 + b)| = |a(z - z_0)| = |a||z - z_0|$$

so the limit holds for $\delta = a\epsilon$.

2(b) The left side is

$$|(z^2 + c) - (z_0^2 + c)| = |z^2 - z_0^2| = |z + z_0||z - z_0| \approx 2|z_0||z - z_0|$$

for $\delta << |z_0|$. Thus, the limit holds for $\delta = 2z_0\epsilon$. Note that if $z_0 = 0$, then this reduces to the limit of a constant value c, which is trivial.

2(c) The right side is

$$\begin{aligned} |z - (1 - i)| &= |[x - 1] + i[y + 1]| \\ &= \sqrt{(x - 1)^2 + (y + 1)^2} \end{aligned}$$

The left side is

$$\begin{aligned} |[x + i(2x + y)] - [1 + i]| &= |[x - 1] + i[2x + y - 1]| \\ &= |[x - 1] + i[2(x - 1) + (y + 1)]| \end{aligned}$$

Not sure how to prove limits – it appears as the right side goes to 0, the left side must too.

3(a) If $z_0 \neq 0$, the limit must be

$$\lim_{z \to z_0} \frac{1}{z^n} = \frac{\lim_{z \to z_0} 1}{\lim_{z \to z_0} z^n} = \frac{1}{z_0^n}$$

3(b)

$$\lim_{z \to i} \frac{iz^3 - 1}{z + i} = \frac{\lim z \to iiz^3 - 1}{\lim z \to iz + i} = \frac{i(i^3) - 1}{(i) + i} = 0$$

3(c)

$$\lim_{z \to z_0} \frac{P(z)}{Q(z)} = \frac{\lim_{z \to z_0} P(z)}{\lim_{z \to z_0} Q(z)} = \frac{P(z_0)}{Q(z_0)}$$

4 The base case is

$$\lim_{z \to z_0} z = z_0$$

which was shown earlier.

If it is known that

$$\lim_{z \to z_0} z^k = z_0^k$$

then the following limit is

$$\lim_{z \to z_0} z^{k+1} = \left(\lim_{z \to z_0} z^k \right) \left(\lim_{z \to z_0} z \right) = (z_0^k)(z_0) = z_0^{k+1}$$

so, by induction, we are finished.

5 First, for any point $z = (x, 0)$, the function is

$$f(x, 0) = \left(\frac{x}{x} \right)^2 = (1)^2 = 1$$

so the limit as $x \to 0$ is 1.

Then, for any point $z = (x, x)$, the function is

$$f(x, x) = \left(\frac{x + ix}{x - ix} \right)^2 = (i)^2 = -1$$

so the limit as $x \to 0$ is -1. This conflicts with the previous result, so the limit does not exist.

6(a) *The statement to be proved is*

$$\lim_{z \to z_0} [f(z) + F(z)] = w_0 + W_0$$

If f and F are split into their real and imaginary parts (u, v, U, and V), then we know that

$$\begin{aligned}
\lim_{z\to z_0} u(x,y) &= u_0 \\
\lim_{z\to z_0} v(x,y) &= v_0 \\
\lim_{z\to z_0} U(x,y) &= U_0 \\
\lim_{z\to z_0} V(x,y) &= V_0
\end{aligned}$$

Then, the sum $f(z)+F(z)$ has a real and imaginary part, which have the limits

$$\begin{aligned}
\lim_{z\to z_0} u(x,y) + U(x,y) &= u_0 + U_0 \\
\lim_{z\to z_0} v(x,y) + V(x,y) &= v_0 + V_0
\end{aligned}$$

so the result is

$$\lim_{z\to z_0} f(z) + F(z) = u_0 + v_0 + U_0 + V_0 = w_0 + W_0$$

6(b) The left side of the limit expression is

$$\begin{aligned}
|f(z) + F(z) - w_0 - W_0| &= |f(z) - w_0 + F(z) - W_0| \\
&\le |f(z) - w_0| + |F(z) - W_0| \\
&< \epsilon + \epsilon \\
&= 2\epsilon
\end{aligned}$$

so the statement is true, and the limit is $w_0 + W_0$.

7 The left hand side of the expression is

$$||f(z)| - |w_0|| \le |f(z) - w_0|$$

which is the standard limit expression. Since $\lim_{z\to z_0} f(z) = w_0$, this expression is true, and the limit is $|w_0|$.

10(a) Making the replacement in the expression, this limit is

$$\begin{aligned}
\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} &= \lim_{z\to 0} \frac{4(1/z)^2}{(1-(1/z))^2} \\
&= \lim_{z\to 0} \frac{4}{(z-1)^2} \\
&= \frac{4}{1} = 4
\end{aligned}$$

10(b) The reciprocal of this expression has the limit

$$\lim_{z \to 1} (z-1)^3 = 0$$

so the limit is infinity.

10(c) Taking both reciprocals,

$$\begin{aligned} \lim_{z \to \infty} \frac{z-1}{z^2+1} &= \lim_{z \to 0} \frac{(1/z)-1}{(1/z)^2+1} \\ &= \lim_{z \to 0} \frac{z-z^2}{1+z^2} \\ &= \frac{0}{1} = 0 \end{aligned}$$

so the limit is infinity.

4 Frame 20 – Differentiation

1 These four derivatives are:

- (a)
$$\frac{d}{dz}(3z^2 - 2z + 4) = 6z - 2$$

- (b)
$$\frac{d}{dz}(1 - 4z^2)^3 = 3(1 - 4z^2)^2(-8z) = -24z(1 - 4z^2)^2$$

- (c)
$$\frac{d}{dz}\frac{z - 1}{2z + 1} = \frac{(1)(2z + 1) - (2)(z - 1)}{(2z + 1)^2} = \frac{-1}{(2z + 1)^2}$$

- (d)
$$\begin{aligned}\frac{d}{dz}\frac{(1 + z^2)^4}{z^2} &= \frac{4(1 + z^2)^3(2z)(z^2) - 2z(1 + z^2)^4}{z^4}\\ &= \frac{8z^3(1 + z^2)^3 - (2z + 2z^3)(1 + z^2)^3}{z^4}\\ &= \frac{2z(3z^2 - 1)(1 + z^2)^3}{z^4}\end{aligned}$$

2 First, each term of the polynomial $P_k(z)$ is

$$P_k(z) = a_k z^k$$

All of these terms are differentiable everywhere, and the derivative is

$$P_k'(z) = k a_k z^{k-1}, \quad k \neq 0$$

where the derivative of a_0 is simply zero. Then, we proved that the derivative of a sum is the sum of two derivatives, so we can apply this repeatedly to find

$$P'(z) = a_1 + 2a_2 z + \cdots + n a_n z^{n-1}$$

Notice that the function's value at zero is $P(0) = a_0$. The derivative's value at zero is similarly $P'(0) = a_1$. Applying the same process again, we find that

$$P''(0) = 2a_2$$

and

$$P^{(k)}(0) = k! \cdot a_k$$

Rearranging these terms, we can write

$$\begin{aligned}
a_0 &= P(0) \\
a_1 &= \frac{P'(0)}{1!} \\
a_2 &= \frac{P''(0)}{2!} \\
&\dots \\
a_n &= \frac{P^{(n)}(0)}{n!}
\end{aligned}$$

3 We can find the derivative of $f(z) = \frac{1}{z}$ by using the definition:

$$\frac{d}{dz}\frac{1}{z} = \lim_{\Delta z \to 0} \frac{\frac{1}{z+\Delta z} - \frac{1}{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z) - (z + \Delta z)}{z\Delta z(z + \Delta z)} = \lim_{\Delta z \to 0} \frac{-1}{z(z + \Delta z)} = \frac{-1}{z^2}$$

4 Applying the definition of a derivative, we can see that

$$\begin{aligned}
\lim_{z \to z_0} \frac{f(z)}{g(z)} &= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{g(z) - g(z_0)} \\
&= \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} \frac{z - z_0}{g(z) - g(z_0)} \\
&= \frac{f'(z_0)}{g'(z_0)}
\end{aligned}$$

5 Following the proof shown in the chapter, the derivative of a sum makes the term Δw into

$$\Delta w = f(z + \Delta z) - f(z) + g(z + \Delta z) - g(z)$$

Then, the derivative is

$$\begin{aligned}
\frac{d}{dz}[f(z) + g(z)] &= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z) + g(z + \Delta z) - g(z)}{\Delta z} \\
&= \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} + \lim_{\Delta z \to 0} \frac{g(z + \Delta z) - g(z)}{\Delta z} \\
&= f'(z) + g'(z)
\end{aligned}$$

6 *Base case:* the derivative of z^1 is 1.

$k + 1$ *case:* if the derivative of z^k is kz^{k-1}, then the derivative of z^{k+1} is

$$\frac{d}{dz}z^{k+1} = \frac{d}{dz}(z^k)(z) = (kz^{k-1})(z) + (z^k) = (k + 1)z^k$$

Therefore, by induction, we are done.

7 When n is a negative integer, we can write $m = -n$ and rewrite the function as

$$f(z) = \frac{1}{z^m}$$

Then, using the quotient rule, the derivative of this function is

$$f'(z) = \frac{(0)(z^m) - (1)(mz^{m-1})}{z^{2m}} = \frac{nz^{-n-1}}{z^{-2n}} = nz^{n-1}$$

so the derivative formula is still valid.

8(a) The derivative of $f(z) = \Re z$ is

$$\frac{d}{dz}\Re z = \lim_{\Delta z \to 0} \frac{\Re(z + \Delta z) - \Re z}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Re(\Delta z)}{\Delta z}$$

To show that this limit doesn't exist, consider a point $z_1 = (x, 0)$. The limit from this direction then becomes

$$f'(x, 0) = \lim_{x \to 0} \frac{x}{x} = 1$$

Then, consider a different point $z_2 = (0, y)$. The limit this time is

$$f'(0, y) = \lim y \to 0 \frac{0}{y} = 0$$

Since the limit is different from these two directions, the derivative does not exist.

The proof for $\Im(z)$ is nearly identical.

5 Frame 23 – Differentiability

1(a) If

$$f(z) = \overline{z} = x - iy$$

then the derivatives are

$$u_x = 1, \quad u_y = 0, \quad v_x = 0, \quad v_y = -1$$

and since $1 \neq -1$, the Cauchy-Riemann equations do not hold, so f is not differentiable.

1(b) If

$$f(z) = z - \overline{z} = (x + iy) - (x - iy) = 2iy$$

then

$$u_x = u_y = v_x = 0, \quad v_y = 2$$

so f is not differentiable.

1(c) Since

$$u(x, y) = 2x, \quad v(x, y) = xy^2$$

we find

$$u_x = 2, v_y = 2xy, \quad u_y = 0, v_x = y^2$$

so f is not differentiable.

1(d) The function is

$$f(x, y) = e^x e^{-iy} = e^x \cos y - e^x \sin y$$

so

$$\begin{aligned} u_x &= e^x \cos y \\ v_y &= -e^x \cos y \\ u_y &= -e^x \sin y \\ v_x &= -e^x \sin y \end{aligned}$$

and f is not differentiable ($u_x \neq v_y$).

2(a) The function is

$$f(z) = iz + 2 = (-y + 2) + ix$$

so the partial derivatives are continuous everywhere. They are:

$$u_x = 0 = v_y, \quad u_y = -1 = -v_x$$

so the derivative is

$$f'(z) = u_x + iv_x = i$$

This function has

$$u_x = u_y = v_x = v_y = 0$$

so $f''(z)$ exists, and it is zero.

2(b) The function is

$$f(z) = e^{-x}e^{-iy} = e^{-x}\cos y - ie^{-x}\sin y$$

so the partial derivatives are

$$u_x = -e^{-x}\cos y = v_y, \quad u_y = -e^{-x}\sin y = -v_x$$

which are all continuous. Thus,

$$f'(z) = -e^{-x}\cos y + e^{-x}\sin y = -e^{-x}(\cos y - \sin y) = -e^{-x}e^{-iy} = -f(z)$$

Since $f'(z)$ only differs from f by a sign, the second derivative exists, and

$$f''(z) = f(z)$$

2(c) The function is

$$f(z) = z^3 = (x+iy)^3 = (x^3 - 3xy^2) + i(3x^2y - y^3)$$

The partial derivatives are

$$u_x = 3x^2 - 3y^2 = v_y, \quad u_y = -6xy = -v_x$$

so the derivative exists, and it is

$$f'(z) = (3x^2 - 3y^2) + i(6xy) = 3z^2$$

Then, the second partial derivatives exist, with the values

$$u_x = 6x = v_y \quad u_y = -6y = -v_x$$

Finally, $f''(z)$ exists, with the value

$$f''(z) = 6x + i6y = 6z$$

3(a) The function is

$$f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}$$

The partial derivatives are

$$\begin{aligned}
u_x &= \frac{(x^2+y^2)-(x)(2x)}{(x^2+y^2)^2} \\
&= \frac{-x^2+y^2}{(x^2+y^2)^2} \\
u_y &= \frac{-2xy}{(x^2+y^2)^2} \\
v_x &= \frac{2xy}{(x^2+y^2)^2} \\
v_y &= \frac{-(x^2+y^2)+2y^2}{(x^2+y^2)^2} \\
&= \frac{-x^2+y^2}{(x^2+y^2)^2}
\end{aligned}$$

so the Cauchy-Riemann equations hold for all x, y. The only point that they are not continuous at is $(0, 0)$, so the derivative exists for all $z \neq 0$. Its value is:

$$\begin{aligned}
f'(z) &= \frac{-x^2+y^2}{(x^2+y^2)^2} + i\frac{2xy}{(x^2+y^2)^2} \\
&= -\frac{(x-iy)^2}{(x+iy)^2(x-iy)^2} \\
&= \frac{-1}{z^2}
\end{aligned}$$

3(b) The function is

$$f(z) = x^2 + iy^2$$

so the partial derivatives are

$$u_x = 2xu_y = 0v_x = 0v_y = 2y$$

We see that $u_x = v_y$. However, for $u_y = -v_x$ to hold, we require

$$2x = -2y \iff x + y = 0$$

so the derivative exists only on the line $y = -x$. There, it has the value

$$f'(z) = 2x$$

3(c) The function is

$$f(z) = z\Im z = (x+iy)(y) = xy + iy^2$$

The partial derivatives are

$$u_x = yu_y = xv_x = 0v_y = 2y$$

Thus, the derivative is only defined where $y = 2y$ and $x = 0$. This only occurs at $z = 0$. The derivative's value is

$$f'(z) = y = 0$$

4(a) The function is

$$f(z) = \frac{1}{z^4} = \frac{1}{r^4}e^{-i4\theta} = \frac{1}{r^4}\cos 4\theta - \frac{1}{r^4}\sin 4\theta$$

The partial derivatives are

$$\begin{aligned}
u_r &= \frac{-4}{r^5}\cos 4\theta \\
u_\theta &= \frac{-4}{r^4}\sin 4\theta \\
v_r &= \frac{4}{r^5}\sin 4\theta \\
v_\theta &= \frac{-4}{r^4}\cos 4\theta
\end{aligned}$$

Since $ru_r = v_\theta$ and $u_\theta = -rv_r$, so the derivative exists and has the value

$$\begin{aligned}
f'(z) &= e^{-i\theta}\frac{4}{r^5}(-\cos 4\theta + \sin 4\theta) \\
&= -\frac{4}{r^5 e^{i5\theta}} \\
&= \frac{-4}{z^5}
\end{aligned}$$

for all $z \neq 0$.

4(b) The function is

$$f(z) = \sqrt{r}e^{i\theta/2} = \sqrt{r}\cos\frac{\theta}{2} + i\sqrt{r}\sin\frac{\theta}{2}$$

so the partial derivatives are

$$\begin{aligned}
u_r &= \frac{1}{2\sqrt{r}}\cos\frac{\theta}{2} \\
u_\theta &= -\frac{\sqrt{r}}{2}\sin\frac{\theta}{2} \\
v_r &= \frac{1}{2\sqrt{r}}\sin\frac{\theta}{2} \\
v_\theta &= \frac{\sqrt{r}}{2}\cos\frac{\theta}{2}
\end{aligned}$$

so the Cauchy-Riemann equations are satisfied, and the derivative has the value

$$f'(z) = e^{-i\theta}\frac{1}{2\sqrt{r}}\left(\cos\frac{\theta}{2} + \sin\frac{\theta}{2}\right) = \frac{1}{2\sqrt{2}e^{\theta/2}} = \frac{1}{2z^{1/2}}$$

5 Applying the Cauchy-Riemann equations, the partial derivatives are

$$\begin{aligned} u_x &= 3x^2 \\ u_y &= 0 \\ v_x &= 0 \\ v_y &= -3(1-y)^2 \end{aligned}$$

so the derivative only exists if

$$x^2 + (y-1)^2 = 0$$

The only solution to this is at $(x, y) = (0, 1)$. This corresponds to $z = i$, so that is the only point where the derivative exists.

6 Frame 25 – Analytic Functions

1(a) The function
$$f(z) = (3x + y) + i(3y - x)$$
has the partial derivatives

$$u_x = 3 = v_y, \quad u_y = 1 = -v_x$$

so it is differentiable everywhere and, thus, entire.

1(b) The components are

$$u(x, y) = \sin x \cosh y, \quad v(x, y) = \cos x \sinh y$$

so the partial derivatives are

$$u_x = \cos x \cosh y = v_y, \quad u_y = \sin x \sinh y = -v_x$$

so f is differentiable and analytic everywhere.

1(c) The components are

$$u(x, y) = e^{-y} \sin x, \quad v(x, y) = -e^{-y} \cos x$$

so the partial derivatives are

$$u_x = e^{-y} \cos x = v_y, \quad u_y = -e^{-y} \sin x = -v_x$$

and f is analytic.

2(a) The partial derivatives are

$$\begin{aligned} u_x &= y \\ u_y &= x \\ v_x &= 0 \\ v_y &= 1 \end{aligned}$$

so the function is only differentiable at $(x, y) = (0, 1)$ or $z = i$. Thus, it is not differentiable over any neighbourhood, so it is nowhere analytic.

2(b) The partial derivatives are

$$\begin{aligned} u_x &= 2y \\ u_y &= 2x \\ v_x &= 2x \\ v_y &= -2y \end{aligned}$$

so the function is only differentiable at $z = 0$, and is nowhere analytic.

2(c) The function is

$$f(z) = e^y e^{ix} = e^y \cos x + ie^y \sin x$$

so the partial derivatives are

$$\begin{aligned} u_x &= -e^y \sin x \\ u_y &= e^y \cos x \\ v_x &= e^y \cos x \\ v_y &= e^y \sin x \end{aligned}$$

The function is only differentiable when

$$2e^y \sin x = 0 \quad \text{and} \quad 2e^y \cos x = 0$$

Since e^y is never zero, this is only true if

$$\sin x = 0 \quad \text{and} \quad \cos x = 0$$

However, this is never true, so the function is nowhere differentiable and thus nowhere analytic.

4(a) The function has singular points where the denominator vanishes; ie:

$$z(z^2 + 1) = 0$$

or $z = 0, z = \pm i$.

4(b) The singular points are at

$$z^2 - 3z + 2 = 0$$

or $z = 1, 2$.

4(c) The singular points are at

$$(z + 2)(z^2 + 2z + 2)$$

or $z = -2, -1 \pm i$.

7 Suppose that f is a function of the form

$$f(z) = u(x, y) + iv(x, y)$$

If f is real-valued, then $v(x, y) = 0$ for all x, y. This means that

$$v_x = v_y = 0$$

Then, if f is analytic in a domain D, then the Cauchy-Riemann equations require that

$$u_x = v_y = 0, \quad u_y = -v_x = 0$$

Thus, all of the partial derivatives are zero, so $f'(z)$ is zero everywhere, and the function is constant throughout D.

7 Frame 26 – Harmonic Functions

1(a) The function $u(x,y) = 2x - 2xy$ has $u_{xx} = u_{yy} = 0$, so it is harmonic everywhere. Using Cauchy-Riemann,

$$v_y = u_x = 2 - 2y$$

so

$$v(x,y) = -y^2 + 2y + \phi(x)$$

Then,

$$v_x = \phi' x = -u_y = 2x$$

so finally

$$v(x,y) = x^2 - y^2 + 2y + C$$

1(b) The function $u(x,y) = 2x - x^3 + 3xy^2$ has $u_{xx} = -6x = -u_{yy}$, so it is harmonic everywhere. The harmonic conjugate has

$$v_y = u_x = 2 - 3x^2 + 3y^2$$

so

$$v(x,y) = 2y - 3x^2 y + y^3 + \phi(x)$$

Then,

$$v_x = -6xy + \phi'(x) = -6xy$$

so

$$v(x,y) = 2y - 3x^2 y + y^3 + C$$

1(c) The function $u(x,y) = \sinh x \sin y$ has $u_{xx} = u(x,y) = -u_{yy}$, so it is harmonic everywhere. From Cauchy-Riemann,

$$v_y = u_x = \cosh\, x \sin y$$

so

$$v(x,y) = -\cosh\, x \cos y + \phi(x)$$

and

$$v_x = -\sinh\, x \cos y + \phi'(x) = -u_y = -\sinh\, x \cos y$$

so

$$v(x,y) = -\cosh\, x \cos y + C$$