1 Frame 56 – Sequences and Series

3 If

$$\lim_{n\to\infty} z_n = z$$

then, for some integer n_0, all of the terms z_k $(k > n_0)$ will be in some ϵ neighbourhood of z; ie:

$$|z_k - z| < \epsilon$$

However,

$$||z_k| - |z| \le \epsilon$$

so all of the terms $|z_k|$ must be inside the same ϵ neighbourhood of $|z|$, and we can say that

$$\lim_{n\to\infty} |z_n| = |z|$$

4 Starting from the series

$$\sum_{n=1}^{\infty} z^n = \frac{1}{1-z} - 1 = \frac{z}{1-z}$$

the components of this expression can be written as

$$\begin{aligned}
\frac{z}{1-z} &= \frac{r\cos\theta + ir\sin\theta}{1 - r\cos\theta - ir\sin\theta} \\
&= \frac{r\cos\theta - r^2\cos^2\theta - r^2\sin^2\theta}{(1-r\cos\theta)^2 + r^2\sin^2\theta} + i\frac{r\sin\theta - r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta}{(1-r\cos\theta)^2 + r^2\sin^2\theta} \\
&= \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} + i\frac{r\sin\theta}{1 - 2r\cos\theta + r^2}
\end{aligned}$$

so, equating the real and imaginary parts of the sum,

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2}$$

and

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

2 Frame 59 – Taylor Series

1 The Maclaurin series for $z\cosh(z^2)$ is

$$z\cosh(z^2) = z \cdot \sum_{n=0}^{\infty} \frac{(z^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{z^{4n+1}}{(2n)!}$$

3 The Maclaurin series for this function is

$$\frac{z}{9}\frac{1}{1+(z^4/9)} = \frac{z}{9}\sum_{n=0}^{\infty}(-1)^n (z^4/9)^n = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+1}}{9^{n+1}}$$

4 Starting with the Maclaurin series for $\sin z$, this function's expansion is

$$\sin(z^2) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}(z^2)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{4n+2}$$

Thus, a_k is only non-zero for $k = 2, 6, 10, 14, \ldots$.

10 The function $\tanh z$ has singularities wherever $\cosh z = 0$, which occurs at $z = (k + 1/2)\pi i$. Thus, the closest singularity has a radius of $\pi/2$, and this is the radius of convergence.

We can find some of the terms of the Taylor series. The constant is

$$\tanh(0) = 0$$

The first derivative is

$$\frac{d}{dz}\tanh(z)\Big|_{z=0} = \operatorname{sech}^2(0) = 1$$

The second derivative is

$$\frac{d}{dz}\operatorname{sech}^2(z)\Big|_{z=0} = 2\operatorname{sech}(z)\cdot(-\operatorname{sech} z\tanh z)\Big|_{z=0} = -2\operatorname{sech}^2 z\tanh z\Big|_{z=0} = 0$$

The third derivative is

$$\frac{d}{dz} - 2\operatorname{sech}^2 z\tanh z\Big|_{z=0} = -2\left[-2\operatorname{sech}^2 z\tanh^2 z + \operatorname{sech}^4 z\right]\Big|_{z=0} = -2$$

so the first few terms of the Taylor series are

$$\tanh z \approx z - \frac{z^3}{3}$$

11(a) The series for this function is

$$\frac{e^z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!} = \sum_{n=0}^{\infty}\frac{z^{n-2}}{n!} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=0}^{\infty}\frac{z^n}{(n+2)!}$$

11(b) The series for this function is

$$\begin{aligned}\frac{\sin(z^2)}{z^4} &= \frac{1}{z^4}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(z^2)^{2n+1} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{4n-2} \\ &= \frac{1}{z^2}+\sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+3)!}z^{4n+2}\end{aligned}$$

3 Frame 62 – Laurent Series

1 The Laurent series for this function is

$$\begin{aligned} z^2 \sin\left(\frac{1}{z^2}\right) &= z^2 \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \left(\frac{1}{z^2}\right)^{2n+1} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{4n}} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{4n}} \end{aligned}$$

2 The Laurent series for this function is

$$\begin{aligned} \frac{e^z}{(z+1)^2} &= \frac{1}{(z+1)^2} \frac{e^{z+1}}{e} \\ &= \frac{1}{e} \frac{1}{(z+1)^2} \sum_{n=0}^{\infty} \frac{(z+1)^n}{n!} \\ &= \frac{1}{e} \sum_{n=0}^{\infty} \frac{(z+1)^{n-2}}{n!} \\ &= \frac{1}{e} \left[\sum_{n=0}^{\infty} \frac{(z+1)^n}{(n+2)!} + \frac{1}{z+1} + \frac{1}{(z+1)^2} \right] \end{aligned}$$

3 The Laurent series for this function is

$$\begin{aligned} \frac{1}{1+z} &= \frac{1}{z} \frac{1}{1+1/z} \\ &= \frac{1}{z} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^n} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{n+1}} \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{z^n} \end{aligned}$$

6 This function's Laurent series is

$$
\begin{aligned}
\frac{z}{(z-1)(z-3)} &= \frac{(z-1)+1}{(z-1)\cdot(z-1-2)} \\
&= -\frac{1}{2}\frac{1}{1-(z-1)/2} - \frac{1}{2}\frac{1}{(z-1)\cdot(1-(z-1)/2} \\
&= -\frac{1}{2}\left[\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^n} + \sum_{n=0}^{\infty}\frac{(z-1)^{n-1}}{2^n}\right] \\
&= -\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+1}} - \sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2}\frac{1}{z-1} \\
&= -3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}} - \frac{1}{2(z-1)}
\end{aligned}
$$

7 In the domain $1 < |z| < \infty$, this function is

$$
\begin{aligned}
\frac{1}{z(1+z^2)} &= \frac{1}{z^3}\frac{1}{1+1/z^2} \\
&= \frac{1}{z^3}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n}} \\
&= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{z^{2n+1}}
\end{aligned}
$$

4 Frame 66 – Integrating and Differentiating Power Series

1 Since

$$\frac{d}{dz}\frac{1}{1-z} = \frac{1}{(1-z)^2}$$

we can write

$$\begin{aligned}
\frac{1}{(1-z)^2} &= \sum_{n=0}^{\infty} \frac{d}{dz} z^n \\
&= \sum_{n=-1}^{\infty} n z^{n-1} \\
&= \sum_{n=0}^{\infty} (n+1) z^n
\end{aligned}$$

3 The Taylor series for $1/z$ is

$$\begin{aligned}
\frac{1}{z} &= \frac{1}{2+(z-2)} \\
&= \frac{1}{2}\frac{1}{1+(z-2)/2} \\
&= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n \frac{(z-2)^n}{2^n} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{1}{2^{n+1}}(z-2)^n
\end{aligned}$$

Then, since $-\frac{d}{dz}\frac{1}{z} = \frac{1}{z^2}$, the series for $\frac{1}{z^2}$ is

$$\begin{aligned}
\frac{1}{z^2} &= -\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} \frac{d}{dz}(z-2)^n \\
&= -\sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+1}} n(z-2)^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{n+2}} (n+1)(z-2)^n \\
&= \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n (n+1)\left(\frac{z-2}{2}\right)^n
\end{aligned}$$

4 The Maclaurin series for $\sin z/z$ is

$$\begin{aligned}\frac{\sin z}{z} &= \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n+1}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}z^{2n}\end{aligned}$$

Evaluating this at $z = 0$ gives

$$\lim_{z\to 0}\frac{\sin z}{z} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(0)^{2n} = 1 = f(0)$$

so this function is entire.

5 A series for

$$f(z) = \frac{\cos z}{(z+\pi/2)(z-\pi/2)}$$

centered at $z_0 = -\pi/2$ is

$$\begin{aligned}f(z) &= \frac{1}{z+\pi/2}z - \pi/2\sin(z+\pi/2)\\ &= \frac{1}{z+\pi/2}z - \pi/2\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(z+\pi/2)^{2n+1}\\ &= \frac{1}{z-\pi/2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}(z+\pi/2)^{2n}\end{aligned}$$

At $z = -\pi/2$, this is

$$\begin{aligned}f(\pi/2) &= \frac{1}{-\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)}!(0)^{2n}\\ &= -\frac{1}{\pi}\end{aligned}$$

so the function is continuous at $z = -\pi/2$. A similar proof would work for $z = +\pi/2$.

6 On this limited domain,

$$\int_1^z \frac{1}{w}dw = \operatorname{Log} w\Big|_1^z = \operatorname{Log} z - 0 = \operatorname{Log} z$$

so

$$\begin{aligned}
\operatorname{Log} z &= \sum_{n=0}^{\infty} \int (-1)^n (z-1)^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (z-1)^{n+1} \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n
\end{aligned}$$

7 By dividing through by a factor of $z-1$, we find that

$$\begin{aligned}
\frac{\operatorname{Log} z}{z-1} &= \frac{1}{1-z} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^n \\
&= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (z-1)^{n-1} \\
&= \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (z-1)^n
\end{aligned}$$

Then, at $z = 1$, this sum is 1, so the function is analytic at this point and throughout the specified domain.