1 Frame 38 – Derivatives and Integrals

1(b) Breaking the derivative into its complex components,

$$\begin{aligned}\frac{d}{dt}[w(t)]^2 &= \frac{d}{dt}[u(t)+iv(t)]^2 \\ &= 2[u(t)+iv(t)][u(t)+iv(t)]' \\ &= 2w(t)w'(t)\end{aligned}$$

2(a) Evaluating the integral,

$$\begin{aligned}\int_1^2 \left(\frac{1}{t}-i\right)^2 dt &= \int_1^2 \frac{1}{t^2} - 1 - i\frac{2}{t}dt \\ &= -\frac{1}{t} - t - 2i\ln t\Big|_1^2 \\ &= -(\frac{1}{2}-1) - (2-1) - 2i(\ln 2 - 0) \\ &= -\frac{1}{2} - i\ln 4\end{aligned}$$

2(b)

$$\begin{aligned}\int_0^{\pi/6} e^{i2t} &= \frac{1}{2i}e^{i2t}\Big|_0^{\pi/6} \\ &= \frac{1}{2i}(e^{i\pi/3}-1) \\ &= \frac{1}{2i}\left(i\frac{\sqrt{3}}{2}-\frac{1}{2}\right) \\ &= \frac{\sqrt{3}}{4}+i\frac{1}{4}\end{aligned}$$

2(c) Converting this improper integral into a limit,

$$\begin{aligned}\int_0^{\infty} e^{-zt} &= \lim_{L\to\infty}\int_0^L e^{-zt} \\ &= \lim_{L\to\infty} -\frac{1}{z}e^{-zt}\Big|_0^L \\ &= -\frac{1}{z}\lim_{L\to\infty} e^{-zL} - 1 \\ &= \frac{1}{z}\end{aligned}$$

4 Evaluating the left-side integral,

$$\begin{aligned}\int_0^\pi e^{1+i} x dx &= \frac{1}{1+i} e^{1+i} x \Big|_0^\pi \\ &= \frac{1}{1+i}(e^{\pi+i\pi} - 1) \\ &= \frac{1-i}{2}(-e^\pi - 1) \\ &= -\frac{1}{2}(e^\pi + 1) + \frac{i}{2}(e^\pi + 1)\end{aligned}$$

2 Frame 39 – Contours

2 First, the original parametrization can be written as

$$z(\theta) = 2e^{i\theta} = 2\cos\theta + 2i\sin\theta$$

If $\theta = \arctan \frac{y}{\sqrt{4-y^2}}$, then this becomes

$$z(y) = 2\cos\arctan\frac{y}{\sqrt{4-y^2}} + 2i\sin\arctan\frac{y}{\sqrt{4-y^2}}$$

Next, these terms can be simplified using basic geometry. The expression $\arctan \frac{y}{\sqrt{4-y^2}}$ represents a right-angled triangle with legs of lengths $\sqrt{4-y^2}$ and y, so the hypotenuse must have a length of 2. Then,

$$\begin{aligned}\cos\arctan\frac{y}{\sqrt{4-y^2}} &= \frac{\sqrt{4-y^2}}{2} \\ \sin\arctan\frac{y}{\sqrt{4-y^2}} &= \frac{y}{2}\end{aligned}$$

so the arc is

$$z(y) = \sqrt{4-y^2} + iy$$

6 (a) First, the function

$$z(t) = t + iy(t) = t + it^3\sin(\pi/t)$$

intersects the real axis whenever $y(t) = 0$. If $t = 1/n$, then this expression becomes

$$y(1/n) = \frac{\sin\left(\frac{\pi}{1/n}\right)}{n^3} = \frac{\sin(n\pi)}{n^3} = 0$$

as predicted.

(b) An arc is smooth if the function $z(t)$ is continuous and its derivative is piecewise continuous.

First, $z(t)$ is continuous for $0 < x \leq 1$ because $x(t) = x$ and $y(t) = y(x)$ are both continuous on this interval. To show continuity at $t = 0$, we must show that

$$\lim_{t\to 0+} y(t) = 0$$

However, the magnitude of $y(t)$ must be in the range

$$0 \leq \left| t^3 \sin\left(\frac{\pi}{t}\right)\right| \leq t^3$$

and the left- and right-hand limits are

$$\lim_{t \to 0+} 0 = 0$$
$$\lim_{t \to 0+} t^3 = 0$$

so, by the squeeze theorem, the original limit holds, and $y(t)$ is continuous at $t = 0$.

Finally, the derivative of $z(t)$ is

$$z'(t) = 1 + i\left[3t^2 \sin(\pi/t) - \pi t \cos(\pi/t)\right]$$

Using the same process as above, the limit as t goes to zero is

$$\lim_{t \to 0+} z'(t) = 1 + i0$$

How can I tell whether this is continuous? The derivative isn't defined at zero.