1 Frame 12 – Functions of Complex Variables

1(a) The function
$$f(z) = \frac{1}{z^2+1}$$
is defined everywhere except where $z^2+1=0$; ie:
$$z \neq \pm i$$

1(b) The function
$$f(z) = \mathrm{Arg}\left(\frac{1}{z}\right)$$
is defined wherever $\frac{1}{z}$ is defined:
$$z \neq 0$$

1(c) The function
$$f(z) = \frac{z}{z+\bar{z}}$$
can be written as
$$f(x,y) = \frac{x+iy}{(x+iy)+(x-iy)} = \frac{x+iy}{2x} = \frac{1}{2} + i\frac{y}{x}$$
so the domain is
$$\mathrm{Re}(z) \neq 0$$

1(d) The function
$$f(z) = \frac{1}{1-|z|^2}$$
is equivalent to
$$f(r,\theta) = \frac{1}{1-r^2}$$
so the domain is
$$r \neq 1$$

2 Substituting $z = x+iy$ gives
$$\begin{aligned} f(x,y) &= (x+iy)^3 + (x+iy) + 1 \\ &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 + x + iy + 1 \\ &= (x^3 - 3xy^2 + x + 1) + i(3x^2y - y^3 + y) \end{aligned}$$

so

$$\begin{aligned} u(x,y) &= x^3 - 3xy^2 + x + 1 \\ v(x,y) &= 3x^2y - y^3 + y \end{aligned}$$

3 Using the two expressions

$$x = \frac{z+\bar{z}}{2}$$
$$y = \frac{z-\bar{z}}{2i}$$

gives

$$\begin{aligned} f(z) &= \Big(\frac{z+\bar{z}}{2}\Big)^2 - \Big(\frac{z-\bar{z}}{2i}\Big)^2 - 2\frac{z-\bar{z}}{2i} \\ &\quad + i\Big[2\frac{z+\bar{z}}{2}\Big(1 - \frac{z-\bar{z}}{2i}\Big)\Big] \\ &= \frac{1}{4}(z^2 + 2z\bar{z} + \bar{z}^2 + z^2 - 2z\bar{z} + \bar{z}^2) + iz - i\bar{z} \\ &\quad + i\Big[z + \bar{z} + \frac{iz^2}{2} - \frac{i\bar{z}^2}{2}\Big] \\ &= \frac{1}{2}(z^2 + \bar{z}^2) + 2iz - \frac{iz^2}{2} + \frac{\bar{z}^2}{2} \\ &= \bar{z}^2 + 2iz \end{aligned}$$

4 Using

$$z = re^{i\theta}$$

the function can be written as

$$\begin{aligned} f(z) &= re^{i\theta} + \frac{1}{re^{i\theta}} \\ &= re^{i\theta} + \frac{1}{r}e^{-i\theta} \\ &= (r + \frac{1}{r})\cos\theta + i(r - \frac{1}{r})\sin\theta \end{aligned}$$

2 Frame 14 – Mappings by the Exponential Function

1 We saw earlier that the hyperbolas

$$x^2 - y^2 = c_1$$

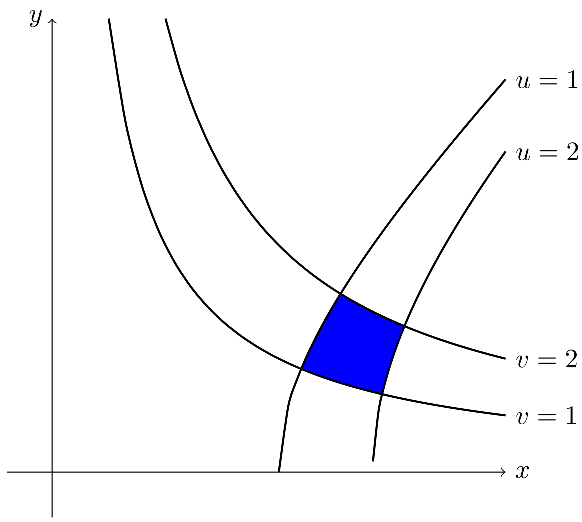
map onto the horizontal lines $u = c_1$ and the hyperbolas

$$2xy = c_2$$

map onto the vertical lines $v = c_2$. Thus, a domain on the z-plane that maps onto $1 \le u \le 2$ and $1 \le v \le 2$ is

$$1 \le x^2 - y^2 \le 2 \quad 1 \le 2xy \le 2$$

A sketch of this region is:



2 The first hyperbola can be written as

$$y^2 - x^2 = |c_1|$$

Then, substitution into the v equation gives

$$u = c_1, \quad v = \begin{cases} 2x\sqrt{x^2 + |c_1|}, & y > 0 \\ -2x\sqrt{x^2 + |c_1|}, & y < 0 \end{cases}$$

This maps out the entire v line as x moves right (on the top branch) or left (on the bottom branch).

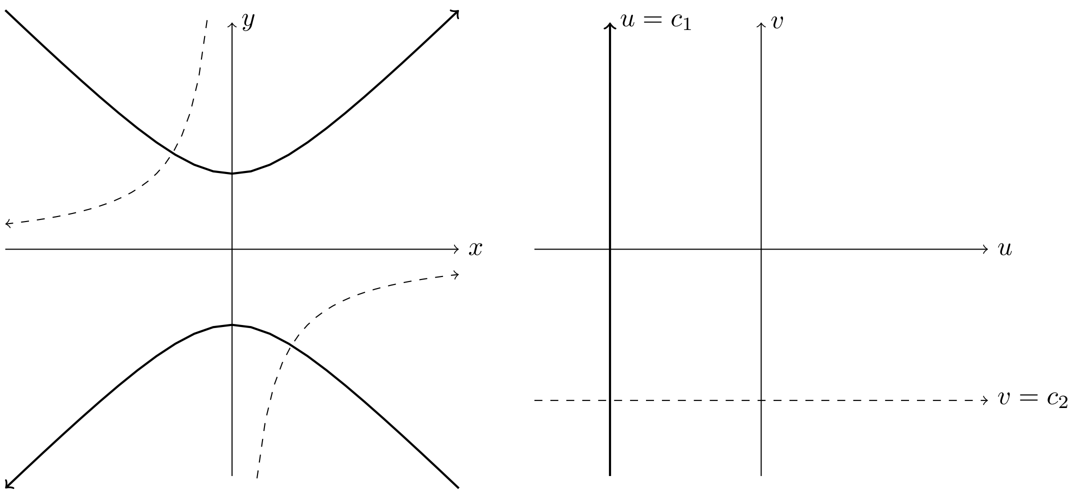
The second hyperbola can be written as

$$2xy = -|c_2|$$

and substituting this into the u equation gives

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

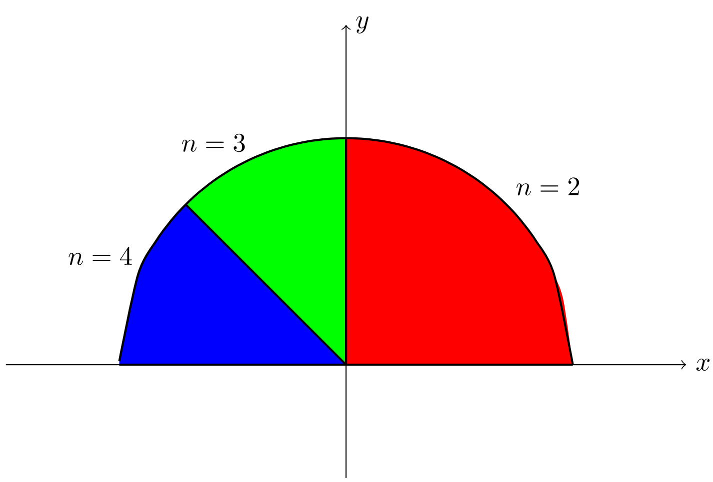
This maps out the entire u line: as x gets large in magnitude, so too does u. A sketch of these mappings is:



3 The image of the sector $r \leq 1, 0 \leq \theta \leq \pi/4$ under the mapping $w = z^n$ is

$$\rho \leq 1, \quad 0 \leq \theta \leq n\frac{\pi}{4}$$

A sketch of these images for $n = 2, 3, 4$ is:



4 If z follows the straight line $ay = x$, then the mapping $w = e^z$ is

$$\begin{aligned} w &= e^{x+iy} \\ &= e^{ay}e^{iy} \\ &= e^{a\phi}e^{i\phi} \\ &= \rho e^{i\phi} \end{aligned}$$

where $\rho = a\phi$.

5 The rectangular region $a \leq x \leq b$, $c \leq y \leq d$ is made up of the horizontal line segments

$$x = t, \quad y = c_1$$

where t is a parameter running from a to b and c_1 is a constant in the range $[c, d]$. These horizontal lines have the images

$$\rho = e^t, \quad \phi = c_1$$

Since t starts at a and ends at b, these images have a radius in the range $[e^a, e^b]$. Then, the entire image is the set of these lines, which range from $\phi = c$ to $\phi = d$. Thus, the entire image is

$$e^a \leq \rho \leq e^b, \quad c \leq \phi \leq d$$

6 Looking at the z plane, the initial set is the infinite strip

$$x \leq 0, \quad 0 \leq y \leq \pi$$

This maps to the image set

$$\lim_{a \to -\infty} e^a \leq \rho \leq e^0, \quad 0 \leq \phi \leq \pi$$

or

$$0 \leq \rho \leq 1, \quad 0 \leq \phi \leq \pi$$

This is the upper half of the unit disk, as shown in the figure.

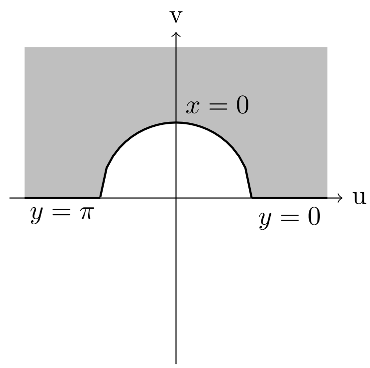
7 In a similar manner to the previous problem, the image of the strip

$$x \geq 0, \quad 0 \leq y \leq \pi$$

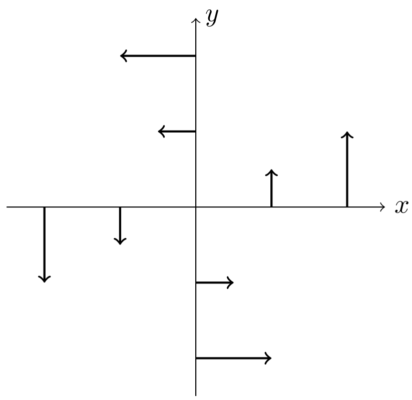
is the upper half-plane with a unit disk cut out:

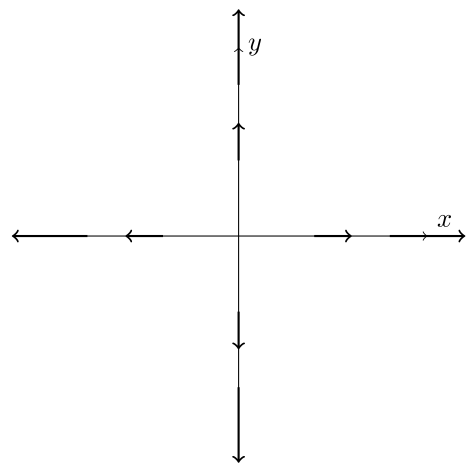
$$\rho \geq 1, \quad 0 \leq \phi \leq \pi$$

A sketch of this region is:



8 Some sample vectors in these two fields are:





3 Frame 18 – Limits and Continuity

1(a) The left side of the limit is

$$|\Re(z) - \Re(z_0)| = |\Re(z - z_0)| < |z - z_0|$$

so

$$|\Re(z) - \Re(z_0)| < \delta \text{ whenever } |z - z_0| < \delta$$