1 Frame 37 – Derivatives with Real Variables

1.1 Definition

In the previous chapter, we looked at derivatives of complex functions of a complex variable z. Now, we look at the derivatives of a complex-valued function of a real variable t. If we write our function as

$$w(t) = u(t) + iv(t)$$

where u and v are real-valued, then we can define the derivative of w at a point t as

$$w'(t) = \frac{d}{dt}w(t) = u'(t) + iv'(t)$$

provided that u' and v' exist at t.

1.2 Properties

If $z_0 = x_0 + iy_0$ is a complex constant, then we can show that

$$\begin{aligned}\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u(t) + iv(t)]' \\ &= [x_0 u(t) - y_0 v(t)]' + i[y_0 u(t) + x_0 v(t)]' \\ &= [x_0 u'(t) - y_0 v'(t)] + i[y_0 u'(t) + x_0 v'(t)] \\ &= z_0 w'(t)\end{aligned}$$

as we expect.

Next, if z_0 is still a complex constant, the derivative of $e^{z_0 t}$ is

$$\begin{aligned}\frac{d}{dt}e^{z_0 t} &= \frac{d}{dt}e^{x_0 t}(\cos y_0 t + isiny_0 t) \\ &= \frac{d}{dt}e^{x_0 t}\cos y_0 t + i\frac{d}{dt}e^{x_0 t}\sin y_0 t \\ &= (x_0 + iy_0)(e^{x_0 t}\cos y_0 t + ie^{x_0 t}\sin y_0 t) \\ &= z_0 e^{z_0 t}\end{aligned}$$

Many other rules carry over from standard calculus. However, some rules no longer apply. For instance, in calculus, the mean value theorem for derivatives states that

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

for some c in the interval $a \le c \le b$ as long as w is continuous. However, this is easily disproved by the function

$$w(t) = e^{it}$$

If $a = 0$ and $b = 2\pi$, then $w(a) = w(b) = 1$ and we expect to find a point c in $[0, 2\pi]$ such that $w'(c) = 0$. However, no such points exist – the magnitude of the derivative is always 1.

2 Frame 38 – Definite Integrals of Complex Functions

2.1 Definitions

If $w(t)$ is a complex-valued function of a real variable t, as in the previous section

$$w(t) = u(t) + iv(t)$$

then we define the **definite integral** of $w(t)$ over the interval $a \leq t \leq b$ as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

provided the two right-side integrals exist. Then,

$$\begin{aligned}
\Re\left[\int_a^b w(t)dt\right] &= \int_a^b \Re[w(t)]dt \\
\Im\left[\int_a^b w(t)dt\right] &= \int_a^b \Im[w(t)]dt
\end{aligned}$$

Improper integrals over unbounded intervals are defined similarly.

The two real integrals will exist as long as u and v are **piecewise continuous** on the interval $[a, b]$ – that is, continuous everywhere in the interval except possibly for a finite number of points where it has one-sided limits. When u and v are piecewise continuous, we say that w is also piecewise continuous.

2.2 Properties

The most common rules of integrals from calculus apply here as well:

- $\int z_0 w(t)dt = z_0 \int w(t)$
- $\int w_1(t) + w_2(t)dt = \int w_1(t)dt + \int w_2(t)dt$
- $\int_a^b w(t)dt = -\int_b^a w(t)dt$
- $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$

We can also extend the fundamental theorem of calculus to complex integrals. Suppose that two functions

$$\begin{aligned}
w(t) &= u(t) + iv(t) \\
W(t) &= U(t) + iV(t)
\end{aligned}$$

are continuous on the interval $[a, b]$ and $W'(t) = w(t)$ when $a \le t \le b$. Then, we can write

$$\int_a^b w(t)dt = W(b) - W(a) = W(t)\Big|_a^b$$

Example: noting that the derivative of $\frac{1}{i}e^{it}$ *is*

$$\frac{d}{dt}\left(\frac{1}{i}e^{it}\right) = \frac{1}{i}ie^{it} = e^{it}$$

we can evaluate $\int e^{it}dt$ *as*

$$\begin{aligned}
\int_0^{\pi/4} e^{it}dt &= \frac{e^{it}}{i}\Big|_0^{\pi/4} \\
&= \frac{1}{i}\left[e^{\pi/4} - 1\right] \\
&= \frac{1}{i}\left[\frac{1}{\sqrt{2}} - 1 + \frac{i}{\sqrt{2}}\right] \\
&= \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)
\end{aligned}$$

As in the previous section, the mean value theorem for integrals does not apply. We can show this by finding the integral $\int_0^{2\pi} e^{it}dt = 0$, even though the function is never zero on this interval.

3 Frame 39 – Contours

3.1 Definitions

In calculus, integrals are defined on intervals of the real line. In complex analysis, we instead use curves in the complex plane.

An **arc** is a set of points $z = (x, y)$ in the complex plane such that the functions

$$x = x(t), \quad y = y(t); \quad z = z(t) = x(t) + iy(t)$$

are continuous functions of the parameter t, where $a \leq t \leq b$. This definition is a continuous mapping of the interval $a \leq t \leq b$ into the z plane.

We say that an arc is **simple** if it does not cross itself; ie:

$$z(t_1) \neq z(t_2) \quad \text{for all } t_1 \neq t_2$$

If a simple arc starts and ends at the same point ($z(a) = z(b)$), it is called a **simple closed curve**. These curves are **positively oriented** when they are oriented in the counterclockwise direction.

Example: the unit circle

$$z = e^{i\theta}$$

where $0 \leq \theta \leq 2\pi$ *is a positively oriented simple closed curve centered at the origin with a radius of* 1*. A more general circle is*

$$z = z_0 + Re^{i\theta}$$

which is centered at z_0 *and has a radius of* R*.*

3.2 Uniqueness

Note that the parametric representation for any arc is not unique. If we know a function ϕ such that

$$t = \phi(\tau)$$

maps the interval $\alpha \leq \tau \leq \beta$ onto the interval $a \leq t \leq b$. Then, the two equations

$$z(t) \quad (a \leq t \leq b)$$

and

$$z(\phi(t)) \quad (\alpha \leq t \leq \beta)$$

represent the same arc.

3.3 Smoothness

Suppose that the real and imaginary components of z are differentiable, and their derivatives are continuous. Then, the arc $z(t)$ is a **differentiable arc**, and

$$|z'(t)| = |x'(t) + iy'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable. This allows us to find the length of an arc as

$$L = \int_a^b |z'(t)|dt$$

If an arc is differentiable and $z'(t)$ is never zero (except maybe at $t = a$ or $t = b$), then we call the arc a **smooth arc**. We can write the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

which has an angle of inclination of $\arg z'(t)$.

A **contour** is an arc which consists of a finite number of smooth arcs joined together. Specifically, if $z(t)$ represents a contour, then $z(t)$ is continuous and $z'(t)$ is piecewise continuous. If a contour is also a simple closed arc, we call it a **simple closed contour**.

The points on a simple closed arc are the boundary points of two different domains:

- The interior of the arc, which is bounded;
- The exterior of the arc, which is unbounded.

4 Frame 40 – Contour Integrals

4.1 Definitions and conditions

We can now integrate a complex function f along a contour C, which starts and ends at points z_1 and z_2, respectively. This is effectively a line integral. These integrals can be written as

$$\int_C f(z)dz$$

or, if the integral does not depend on the path taken,

$$\int_{z_1}^{z_2} f(z)dz$$

This integral (along a complex path) represents an integral with respect to a real parameter t. If the contour C is written as $z(t)$ on the interval $a \leq t \leq b$, then the integral represented is

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$$

Since $z'(t)$ must be piecewise continuous, this integral exists as long as $f[z(t)]$ is also piecewise continuous on this interval.

4.2 Basic properties

From the definition and the properties of integrals, we can write

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$$

and

$$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

We can also create a new contour $-C$ that consists of the points in C in reversed order – this contour extends from z_2 to z_1. Integrating along this reversed contour, we find that

$$\begin{aligned}
\int_- Cf(z)dz &= \int_{-b}^{-a} f[z(-t)]\frac{d}{dt}z(-t)dt \\
&= -\int_{-b}^{-a} f[z(-t)]z'(-t)dt \\
&= -\int_a^b f[z(t)]z'(t)dt \\
&= -\int_C f(z)dz
\end{aligned}$$

We can also split up a contour C into multiple legs $C_1, C_2, \ldots$. If we can write a contour this way, then we say that $C = C_1 + C_2$. The contour integral along C can then be written as

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

5 Frame 41 – Examples of Contour Integrals

This section will show several specific examples of contour integrals.

5.1 Example 1

Suppose that the contour C is the right hand half of the circle $|z| = 2$:

$$z = 2e^{i\theta}, \quad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$$

Then,

$$\begin{aligned}
\int_C \overline{z} dz &= \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}} (2e^{i\theta})' d\theta \\
&= 4i \int_{-\pi/2}^{\pi/2} e^{-i\theta} e^{i\theta} d\theta \\
&= 4i \int_{-\pi/2}^{\pi/2} d\theta \\
&= 4\pi i
\end{aligned}$$

Also, note that all of the points on this semicircle satisfy

$$z\overline{z} = |z|^2 = 4$$

so we can see from this result that

$$\int_C \frac{1}{z} dz = \pi i$$

5.2 Example 2

Suppose that the points O, A, and B are 0, i, and $1 + i$, respectively. Then, if C_1 is the polyline OAB and

$$f(z) = y - x - i3x^2$$

then the contour integral of f along C_1 is

$$\begin{aligned}\int_{C_1} f(z)dz &= \int_{OA} f(z)dz + \int_{AB} f(z)dz \\ &= \int_0^1 yidy + \int_0^1 (1 - x - i3x^2)dx \\ &= \frac{i}{2} + \int_0^1 (1-x)dx - 3i\int_0^1 x^2dx \\ &= \frac{i}{2} + \frac{1}{2} - i \\ &= \frac{1-i}{2}\end{aligned}$$

Next, if C_2 is the line OB, the contour integral along this curve is

$$\begin{aligned}\int_{C_2} f(z)dz &= \int_0^1 -i3x^2(1+i)dx \\ &= 3(1-i)\int_0^1 x^2 \\ &= 1 - i\end{aligned}$$

Finally, the integral of f over the simple closed contour $OABO$ is $C_1 - C_2$, which is

$$\int_{OABO} f(z)dz = \frac{-1+i}{2}$$

5.3 Example 3

Suppose that C is any arbitrary smooth arc from a fixed point z_1 to another point z_2:

$$z = z(t) \quad (a \le t \le b)$$

The contour integral of $f(z) = z$ along this curve is

$$\begin{aligned}\int_C zdz &= \int_a^b z(t)z'(t)dt \\ &= \int_a^b \frac{d}{dt}\frac{[z(t)]^2}{2}dt \\ &= \frac{[z(t)]^2}{2}\Big|_a^b \\ &= \frac{z_2^2 - z_1^2}{2}\end{aligned}$$

Note that this integral only depends on the endpoints of C and not the path. This lets us write

$$\int_{z_1}^{z_2} z dz = \frac{z_2^2 - z_1^2}{2}$$

This holds when C is not a smooth contour. Since all contours are sums of finite numbers of smooth arcs, this expression holds for each arc in C, leading to the same final expression.

Also, note that the integral of $f(z) = z$ around any closed contour in the plane is zero.

6 Frame 42 – Examples with Branch Cuts

A contour integral's path can include a point on a branch cut. The following two examples show this.

6.1 Example 1

Suppose we want to integrate the function

$$f(z) = z^{1/2} = e^{\frac{1}{2}\log z} \quad (0 < \arg z < 2\pi)$$

on the semicircle

$$z = 3e^{i\theta} \quad (0 \le \theta \le \pi)$$

Although the function is not defined at $\theta = 0$, we can still write

$$f[z(\theta)] = e^{\frac{1}{2}(\ln 3 + i\theta)} = \sqrt{3}e^{i\theta/2}$$

and the right hand limit of this function exists at $\theta = 0$. Thus, the integrand exists as long as we define the missing point as

$$f[z(0)]z'(0) = i3\sqrt{3}$$

Then,

$$\begin{aligned}\int_C f(z)dz &= 3\sqrt{3}\int_0^{\pi} e^{i3\theta/2} \\ &= 3\sqrt{3}\frac{2}{3i}e^{i3\theta/2}\Big|_0^{\pi} \\ &= -\frac{2}{3i}(1+i) \\ &= -2\sqrt{3}(1+i)\end{aligned}$$

6.2 Example 2

Suppose that we want to integrate the function

$$f(z) = z^{a-1} = e^{(a-1)\operatorname{Log} z} \quad (-\pi < \operatorname{Arg} z < \pi)$$

on the positively oriented circle

$$z = Re^{i\theta} \quad (-\pi \le \theta \le \pi)$$

The contour integral is

$$\begin{aligned}
\int_C z^{a-1} dz &= \int_{-\pi}^{\pi} iR^a e^{ia\theta} d\theta \\
&= iR^a \int_{-\pi}^{\pi} e^{ia\theta} d\theta \\
&= iR^a \left(\frac{e^{ia\theta}}{ia} \right)_{-\pi}^{\pi} \\
&= i\frac{2R^a}{a} \frac{e^{ia\pi} - e^{-ia\pi}}{2i} \\
&= i\frac{2R^a}{a} \sin a\pi
\end{aligned}$$

Note that if a is a non-zero integer, this integral is zero; if $a = 0$, this integral reduces to

$$\int_C \frac{dz}{z} = 2\pi i$$

7 Frame 43 – Upper Bounds for Contour Integrals

We can put a bound on the modulus of a contour integral by observing that

$$\left| \int_a^b w(t)dt \right| \leq \int_a^b |w(t)|\, dt$$

7.1 Theorem

Suppose that C is a contour with a length of L and that $f(z)$ is a function that is piecewise continuous on C. If $M \geq 0$ is a real constant such that

$$|f(z)| \leq M$$

for all points on C, then

$$\left| \int_C f(z)dz \right| \leq ML$$

Note that such a number M will always exist because f is continuous on C.

7.2 Examples

Example: Suppose that

$$f(z) = \frac{z+4}{z^3-1}$$

and the contour C is a quarter circle with a radius of 2 in the first quadrant (running from $z = 2$ to $z = 2i$). Since $|z| = 2$ at all points on this contour, we can write that

$$|z+4| \leq |z| + 4 = 6$$

and

$$|z^3 - 1| \geq |z|^3 - 1 = 7$$

Since the length of the contour is $L = \pi$, we can write the upper bound

$$\left| \int_C \frac{z+4}{z^3-1} dz \right| \leq \frac{6\pi}{7}$$

Example: suppose that C_R is the semicircular contour

$$z = Re^{i\theta} \quad (0 \leq \theta \leq \pi)$$

and f is the function

$$f(z) = \frac{z^{1/2}}{z^2+1}$$

where $z^{1/2}$ *denotes the branch* $-\pi/2 < \theta < 3\pi/2$*. Anywhere on this semicircle,*

$$|z^{1/2}| = \sqrt{R}$$

and

$$|z^2 + 1| \geq ||z^2| - 1| = R^2 - 1$$

Since the contour has a length of πR*, the contour integral of* f *along* C *can be limited by*

$$\begin{aligned}\int_C f(z)dz &\leq \frac{\sqrt{R}}{R^2 - 1} \cdot \pi R \\ &= \frac{\pi R^{3/2}}{R^2 - 1} \\ &= \frac{\pi/\sqrt{R}}{1 - (1/R^2)}\end{aligned}$$

As R *approaches infinity, this bound approaches zero, so*

$$\lim_{R\to\infty} \int_C f(z)dz = 0$$

8 Frame 44 – Antiderivatives

We saw earlier that some functions have integrals from z_1 to z_2 that are independent of path. This section will look more closely at such functions to extend the fundamental theorem of calculus.

8.1 Antiderivatives and their theorem

Recall that an antiderivative of a continuous function $f(z)$ on a domain D is an analytic function $F(z)$ such that $F'(z) = f(z)$ for all z in D. Also, note that antiderivatives are unique to an additive constant: if $F(z)$ and $G(z)$ are two antiderivatives of the same function, then $F'(z) - G'(z) = 0$, so $F(z) - G(z)$ must be constant everywhere.

The following theorem makes several relationships between antiderivatives and their properties.

Theorem: If $f(z)$ is a continuous function on a domain D, then any one of these statements implies the other two:

1. *$f(z)$ has an antiderivative $F(z)$ throughout D;*
2. *Any integral of $f(z)$ along a contour in D is path independent, and*

$$\int_{z_1}^{z_2} f(z)dz = F(z_2) - F(z_1)$$

3. *The integral of $f(z)$ around any closed contour in D is zero.*

8.2 Examples

Example 1: the function $f(z) = z^2$ has an antiderivative $F(z) = z^3/3$ throughout the entire plane. This allows us to write, for any contour extending from $z = 0$ to $1 + i$,

$$\begin{aligned}
\int_0^{1+i} z^2 dz &= \frac{z^3}{3}\Big|_0^{1+i} \\
&= \frac{1}{3}(1+i)^3 \\
&= \frac{1}{3}(\sqrt{2}e^{i\pi/4} \\
&= \frac{2}{3}\sqrt{2}e^{i3\pi/4} \\
&= \frac{2}{3}(-1+i)
\end{aligned}$$

Example 2: the function $f(z) = \frac{1}{z^2}$ is continuous everywhere except for the point $z = 0$. In this domain, it has an antiderivative $F(z) = \frac{-1}{z}$. Thus, if C is a positively oriented circle centered at the origin,

$$\int_C \frac{dz}{z^2} = 0$$

Example 3: the function $f(z) = \frac{1}{z}$ does not have a simple antiderivative on the entire plane. Although $\log z$ is an antiderivative where it is defined, it requires a branch cut to be single valued.

Suppose we want to evaluate the integral

$$\int_C \frac{dz}{z}$$

where C is the full circle

$$z = 2e^{i\theta}, \quad -\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$$

We can do this by splitting the contour into two legs: C_1 is the right semicircle $(-\pi/2 \le \theta \le \pi/2)$ and C_2 is the left semicircle $(\pi/2 \le \theta \le 3\pi/2)$. Then, the principal branch of the logarithm is a suitable antiderivative for C_1, so

$$\int_{C_1} \frac{dz}{z} = \mathrm{Log}(2i) - \mathrm{Log}(-2i) = (\ln 2 + i\pi/2) - (\ln 2 - i\pi/2) = i\pi$$

To evaluate the integral along C_2, we switch to the branch $0 < \theta < 2\pi$. Here,

$$\int_{C_2} \frac{dz}{z} = \log(-2i) - \log(2i) = (\ln 2 + i3\pi/2) - (\ln 2 + i\pi/2) = i\pi$$

so, adding these up,

$$\int_C \frac{dz}{z} = \int_{C_1} \frac{dz}{z} + \int_{C_2} \frac{dz}{z} = i\pi + i\pi = 2\pi i$$

Example 4: suppose we want to evaluate the integral

$$\int_{-3}^{3} z^{1/2} dz$$

where $z^{1/2}$ denotes the branch for $0 < \theta < 2\pi$ and we use any contour that is always above the real axis (except at the endpoints). Since this branch is not defined at the endpoints, we can replace the integrand with the branch

$$z^{1/2} = \sqrt{r}e^{i\theta/2} \quad (-\frac{\pi}{2} < \theta < \frac{3\pi}{2})$$

This new function has an antiderivative of $\frac{2}{3}z^{3/2}$*, so we can write*

$$\int_{-3}^{3} z^{1/2}dz = R^{3/2}e^{i3\theta/2}\Big|_{-3}^{3} = 2\sqrt{3}(e^0 - e^{i3\pi/2}) = 2\sqrt{3}(1+i)$$

We could evaluate the same integral for any contour below the real axis in a similar manner.

9 Frame 46 – Cauchy-Goursat Theorem

9.1 The theorem

We will show some simple conditions under which a contour integral is guaranteed to be zero.

Suppose that C is a simple, closed, positively-oriented contour $z(t)$ (where $a \leq t \leq b$) and f is a function that analytic at each point interior to and on C. Then, we can write

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$$

Then, if

$$f(z) = u(x, y) + iv(x, y)$$

and

$$z(t) = x(t) + iy(t)$$

we can write this integral as

$$\int_C f(z)dz = \int_C udx - vdy + i \int_C vdx + udy$$

We can then use Green's theorem to write these line integrals as double integrals over the region R bounded by C:

$$\int_C f(z)dz = \int \int_R (-v_x - u_y)dA + i \int \int_R (u_x - v_y)dA$$

Finally, due to the Cauchy Riemann equations, both of these integrands are zero, so

$$\int_C f(z)dz = 0$$

whenever f is analytic and f' is continuous in R. Note that, due to Goursat, the continuity condition is actually unnecessary.

9.2 Example

Example: if C is a simple closed contour, then

$$\int_C e^{z^3} dz = 0$$

since e^{z^3} is analytic everywhere.

10 Frame 48 – Simply Connected Domains

The conditions on the contour in the previous section can be relaxed when the domain of interest is simply connected. The following theorem is an adapted version:

Theorem: if a function f is analytic throughout a simply connected domain D, then

$$\int_C f(z)dz = 0$$

for every closed contour C (simple or not) in D.

This is a simple extension of the theorem – if a contour intersects itself a finite number of times, k, then the contour can be viewed as k different simple and closed legs, which must all have an integral of zero. Thus,

$$\int_C f(z)dz = \sum_{i=0}^{k} \int_{C_i} f(z)dz = 0$$

Example: if C is any closed contour in the disk $|z| < 2$, then

$$\int_C \frac{ze^z}{(z^2+9)^5} dz = 0$$

because the only singularities of this function are outside the disk (at $z = \pm 3i$).

Note that this implies, from the theorem earlier in the chapter, that *every entire function has an antiderivative everywhere.*

11 Frame 49 – Multiply Connected Domains

We can further adapt the Cauchy-Goursat theorem to be useful on multiply connected domains.

11.1 Extended theorem

Theorem: Suppose that

- *C is a simple, closed, positively oriented contour*
- *C_k are simple, closed, negatively oriented contours interior to C that do not overlap (interiors and contours are disjoint)*

If a function f is analytic on each contour and the multiply connected domain inside C but outside C_k, then

$$\int_C f(z) + \sum_{k=1}^{n} \int_{C_k} f(z)dz = 0$$

Note that the multiply connected domain is to the left of each contour.

11.2 Applications and examples

A corollary of this extended theorem follows.

Corollary: if C_1 and C_2 are two positively oriented contours such that C_1 is contained inside C_2, then for any function f analytic inside C_2,

$$\int_{C_2} f(z)dz + \int_{-C_1} f(z)dz = 0$$

so

$$\int_{C_2} f(z)dz = \int_{C_1} f(z)dz$$

Example: if C is any positively-oriented, simple, closed contour that encloses the origin, then we can make positively oriented circular contour C_0 that is entirely inside C and contains the origin. We've shown before that

$$\int_{C_0} \frac{dz}{z} = 2\pi i$$

so

$$\int_C \frac{dz}{z} = \int_{C_0} \frac{dz}{z} = 2\pi i$$