1 Frame 78 – Evaluating Improper Integrals

1.1 Improper Integrals

The **improper integral** of a continuous function $f(x)$ over the interval $0 \leq x < \infty$ is defined as

$$\int_0^\infty f(x)\ dx = \lim_{R\to\infty} \int_0^R f(x)\ dx$$

If this limit exists, we say that the improper integral **converges** to this limit. The improper integral of f over the infinite interval $-\infty < x < \infty$ is

$$\int_{-\infty}^{\infty} = \lim_{R_1\to\infty} \int_{-R_1}^{0} f(x)\ dx + \lim_{R_2\to\infty} \int_0^{R_2} f(x)\ dx$$

If both of these limits exist, we say that the integral converges to their sum.

1.2 The Cauchy Principal Value

We say that the **Cauchy Principal Value** of an indefinite integral is

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx$$

as long as this limit exists.

If the regular improper integral converges, then

$$\begin{aligned} \lim_{R\to\infty} \int_{-R}^{R} f(x)\ dx &= \lim_{R\to\infty} \left[\int_{-R}^{0} f(x)\ dx + \int_0^R f(x)\ dx \right] \\ &= \lim_{R\to\infty} \int_{-R}^{0} f(x)\ dx + \lim_{R\to\infty} \int_0^R f(x)\ dx \end{aligned}$$

so the principal value also exists. However, the converse is not true – the existence of the principal value does not imply the existence of the improper integral.

Next, suppose that $f(x)$ is an even function; ie:

$$f(-x) = f(x)$$

and assume that the principal value exists. Then, the evenness of f allows us to write

$$\begin{aligned} \int_{-R_1}^{0} f(x)\ dx &= \frac{1}{2} \int_{-R_1}^{R_1} f(x)\ dx \\ \int_0^{R_2} f(x)\ dx &= \frac{1}{2} \int_{-R_2}^{R_2} f(x)\ dx \end{aligned}$$

so we can convert both single-sided limits into double-sided limits, and

$$\int_{-\infty}^{\infty} f(x)\ dx = \text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx$$

Also, extending this formula,

$$\int_{0}^{\infty} f(x)\ dx = \frac{1}{2}\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx$$

1.3 Using Residues

Now, we apply our knowledge of residues to integrate $f(z) = p(z)/q(z)$ along the real axis when p and q are polynomials. In this discussion, suppose that q has at least one zero above the real axis and no zeroes on the real axis.

From our knowledge of polynomials, we know that q has a finite number of distinct zeroes, which we can label as $z_1,\ z_2,\ \ldots,\ z_n$. Then, we can integrate the function $f(z)$ along the contour:

- Along the real axis from $-R$ to R;
- Along the upper semicircle with a radius of R from $(R, 0$ to $(-R, 0)$, which we call C_R.

This contour allows us to write

$$\int_{-R}^{R} f(x)\ dx + \int_{C_R} f(z)\ dz = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z)$$

or

$$\int_{-R}^{R} f(x)\ dx = 2\pi i \sum_{k=1}^{n} \text{Res}_{z=z_k} f(z) - \int_{C_R} f(z)\ dz$$

Using this expression, we can say that if

$$\lim_{R\to\infty} \int_{C_R} f(z)\ dz = 0$$

then the following three equations hold (the latter two if f is even):

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum k = 1^n \text{Res}_{z=z_k} f(z)$$
$$\int_{-\infty}^{\infty} f(x)\ dx = 2\pi i \sum_{z=z_k} f(z)$$
$$\int_{0}^{\infty} f(x)\ dx = \pi i \sum_{z=z_k} f(z)$$

2 Frame 79 – Example

This section will show a sample integral that can be calculated using the method from the previous section.

The goal of this example is to calculate

$$\int_0^\infty \frac{x^2}{x^6+1}$$

To do this, we can define

$$f(z) = \frac{z^2}{z^6+1}$$

and note that this has isolated singularities at the sixth roots of -1, or

$$c_k = e^{i(1+2k)\pi/6}$$

Three of these roots lie on the upper half-plane, at

$$\begin{aligned} c_0 &= e^{i\pi/6} \\ c_1 &= e^{i\pi/2} = i \\ c_2 &= e^{i5\pi/6} \end{aligned}$$

We can find the residue at these three points through the formula

$$B_k = \mathrm{Res}_{z=c_k} \frac{z^2}{z^6+1} = \left.\frac{z^2}{6z^5}\right|_{z=c_k} = \left.\frac{1}{6z^3}\right|_{z=c_k} = \frac{1}{6c_k^3}$$

so we can write

$$2\pi i \sum_{k=1}^{n} B_k = 2\pi i \left(\frac{1}{6i} - \frac{1}{6i} + \frac{1}{6i}\right) = \frac{\pi}{3}$$

and, as long as $R > 1$,

$$\int_{-R}^{R} f(x)\ dx = \frac{\pi}{3} - \int_{C_R} f(z)\ dz$$

Next, when $|z| = R$, we can write

$$\frac{|z^2|}{|z^6+1|} < \frac{R^2}{R^6-1}$$

so

$$\left|\int_{C_R} f(z)\ dz\right| \leq \frac{R^2}{R^6-1}\pi R = \frac{R^3}{R^6-1}$$

and, as $R \to \infty$, this approaches 0. Thus,

$$\text{P.V.} \int_{-\infty}^{\infty} f(z)\ dz = \frac{\pi}{3}$$

so

$$\int_0^\infty \frac{x^2}{x^6+1} = \frac{\pi}{6}$$

3 Frame 80 – Improper Integrals from Fourier Analysis

In this section, we will look at integrals of the form

$$\int_{-\infty}^{\infty} f(x)\sin ax\ dx$$

and

$$\int_{-\infty}^{\infty} f(x)\cos ax\ dx$$

where $f(x) = p(x)/q(x)$ is a rational function with no poles on the real axis, but at least one above it.

Note that we can't apply the process from the previous section directly, since

$$|\sin az|^2 = \sin^2 ax + \sinh^2 ay$$

is unbounded as $y \to \infty$.

3.1 Example

Consider the integral

$$\int_{-\infty}^{\infty} \frac{\cos 3x}{(x^2+1)^2} dx = \frac{2\pi}{e^3}$$

We can begin by defining the function

$$f(z) = \frac{1}{(z^2+1)^2}$$

and using the regular semicircular contour from $-R$ to $+R$. Since $f(z)$ only has a single pole in the upper half-plane (at $z = i$), as long as $R > 1$, we can write

$$\int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2} dx = 2\pi i \operatorname{Res}_{z=i}[f(z)e^{i3z}] - \int_{C_R} f(z)e^{i3z} dz$$

Now, this pole at $z = i$ is a second-order pole, so

$$\begin{aligned}
B &= \frac{d}{dz}\frac{e^{i3z}}{(z+i)^2}\Big|_{z=i} \\
&= \frac{3ie^{3iz}(z+i)^2 - 2e^{3iz}(z+i)}{(z+i)^4}\Big|_{z=i} \\
&= \frac{e^{3iz}[3iz-5]}{(z+i)^3}\Big|_{z=i} \\
&= \frac{-8e^{-3}}{(2i)^3} \\
&= \frac{1}{ie^3}
\end{aligned}$$

Next, the integral on the semicircle satisfies

$$\begin{aligned}\left|\int_{C_R} f(z)e^{i3z}\right| &\leq \pi R \cdot \frac{|e^{-3y}|}{(R^2-1)^2} \\ &\leq \pi R \cdot \frac{1}{(R^2-1)^2}\end{aligned}$$

and this vanishes as R tends to infinity. Thus, equating real parts of the integral formula,

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos 3x}{(x^2+1)^2} = 2\pi i \frac{1}{ie^3} = \frac{2\pi}{e^3}$$

4 Frame 81 – Jordan's Lemma

4.1 Jordan's Lemma

The following lemma is helpful for evaluating the integrals from the previous section.

Theorem: Suppose that:

- *A function f is analytic at all points above the real axis outside some circle $|z| = R_0$*
- *C_R is a semicircle $z = Re^{i\theta}$ $(0 \leq \theta \leq \pi)$ where $R > R_0$*
- *On C_R, there is a constant M_R such that*

  $$|f(z)| \leq M_R$$

  and

  $$\lim_{R\to\infty} M_R = 0$$

Then, for any positive a,

$$\lim_{R\to\infty} \int_{C_R} f(z)e^{iaz} dz = 0$$

To prove this, we can write

$$\int_{C_R} f(z)e^{iaz} dz = \int_0^\pi f(Re^{i\theta}) \exp(iaRe^{i\theta}) Rie^{i\theta} d\theta$$

Then, we know that

$$|f(Re^{i\theta})| \leq M_R$$

and

$$|\exp(iaRe^{i\theta})| \leq e^{-aR\sin\theta}$$

Next, Jordan's inequality tells us that

$$\int_0^\pi e^{-R\sin\theta} d\theta < \frac{\pi}{R}$$

so we can write the contour integral as

$$\begin{aligned} \left| \int_{C_R} f(z)e^{iaz} dz \right| &\leq M_R R \int_0^\pi e^{-aR\sin\theta} d\theta \\ &< M_R R \frac{\pi}{aR} \\ &= \frac{M_R \pi}{a} \end{aligned}$$

and this tends to zero as $R \to \infty$.

4.2 Example

We can use this theorem to help us find the principal value of the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2} dx$$

If we write

$$f(z) = \frac{z}{z^2 + 2z + 2}$$

then we can find the residue at the point $z_1 = -1 + i$ as

$$B = \frac{z_1 e^{iz_1}}{z_1 - \overline{z_1}}$$

Next, we can use the standard contour integral to write

$$\int_{-R}^{R} \frac{x e^{ix}}{x^2 + 2x + 2} = 2\pi i B - \int_{C_R} f(z) e^{iz} \, dz$$

and, taking the imaginary components of both sides,

$$\int_{-R}^{R} \frac{x \sin x}{x^2 + 2x + 2} \, dx = \Im(2\pi i B) - \Im \int_{C_R} f(z) e^{iz} \, dz$$

Then, if $|z| = R$, we can write

$$|f(z)| \leq \frac{R}{(R - \sqrt{2})^2}$$

and this tends to zero as $R \to \infty$, so the integral vanishes, leaving

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 2} \, dx = \Im(2\pi i B) = \frac{\pi}{e}(\sin 1 + \cos 1)$$

5 Frame 82 – Indented Paths

5.1 Main Theorem

This section will examine the use of indented paths to carry out some more integrals. The following theorem will be useful here:

Theorem: Suppose that f is a function with a simple pole at $z = x_0$ on the real axis with a Laurent series that is valid for $0 < |z - x_0| < R_2$ and with a residue of B_0. Then, if C_ρ is the clockwise upper semicircle

$$|z - x_0| = \rho \quad (\rho < R_2)$$

then

$$\lim_{\rho \to 0} \int_{C_\rho} f(z)\ dz = -B_0 \pi i$$

To prove this, we can write f as

$$f(z) = g(z) + \frac{B_0}{z - x_0}$$

where

$$g(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n$$

Then, we can write the integral as

$$\int_{C_\rho} f(z) = \int_{C_\rho} g(z) + B_0 \int_{C_\rho} \frac{1}{z - x_0}$$

and look at these two terms separately. First, $g(z)$ is continuous, so it must be bounded on some disk centered at ρ_0 (so that $|g(z)| \leq M$). This allows us to write

$$\left| \int_{C_\rho} g(z)\ dz \right| \leq ML = M\pi\rho$$

which vanishes as $\rho \to 0$. For the second term, we can write out the contour as the equation

$$z = x_0 + \rho e^{i\theta}$$

so that

$$\int_{C_\rho} \frac{dz}{z - x_0} = -\int_0^\pi \frac{1}{\rho e^{i\theta}} \rho i e^{i\theta}\ d\theta = -i \int_0^\pi d\theta = -i\pi$$

so

$$\int_{C_\rho} f(z)\ dz = -B_0 \pi i$$

5.2 Example

We can evaluate the integral

$$\int_0^\infty \frac{\sin x}{x}\ dx$$

by using the following contour:

- L_1, the real axis from ρ to R.
- C_R, a positively-oriented semicircle from R to $-R$;
- L_2, the real axis from $-R$ to $-\rho$;
- C_ρ, a negatively-oriented semicircle from $-\rho$ to ρ;

We do this to avoid the singularity at $z = 0$ in e^{iz}/z. Using the Cauchy-Goursat theorem, we can write

$$\int_{C_R} \frac{e^{iz}}{z}\ dz + \int_{L_1} \frac{e^{iz}}{z}\ dz + \int_{C_\rho} \frac{e^{iz}}{z}\ dz + \int_{L_2} \frac{e^{iz}}{z}\ dz$$

Now, since we can write L_1 as

$$z = re^{i0} \quad (\rho \leq r \leq R)$$

and $-L_2$ as

$$z = re^{i\pi} = -r \quad (\rho \leq r \leq R)$$

we can combine the integrals along the line segments into

$$\int_\rho^R \frac{e^{ir}}{r}\ dr - \int_\rho^R \frac{e^{-ir}}{r}\ dr = 2i \int_\rho^R \frac{\sin r}{r}\ dr$$

ie:

$$2i \int_\rho^R \frac{\sin r}{r}\ dr = -\int_{C_\rho} \frac{e^{iz}}{z}\ dz - \int_{C_R} \frac{e^{iz}}{z}\ dz$$

Now, according to the theorem above, since this function has a residue of 1 at the origin, we can write

$$\lim_{\rho \to 0} \int_{C_\rho} \frac{e^{iz}}{z}\ dz = -\pi i$$

and, since $|1/z| = 1/R$, we know from Jordan's lemma that

$$\lim_{R \to \infty} \int_{C_R} \frac{e^{iz}}{z}\ dz = 0$$

so, applying these limits, we find that

$$\int_0^\infty \frac{\sin r}{r}\ dr = \frac{1}{2i} - (-\pi i) = \frac{\pi}{2}$$