1 Frame 29 – The Exponential Function

1(a) The value is
$$e^{2\pm 3\pi i} = e^2 e^{\pm 3\pi i} = e^2(-1) = e^2$$

1(b) The value is

$$e^{1/2+i\pi/4} = e^{1/2}e^{i\pi 4} = \sqrt{e}\frac{1+i}{\sqrt{2}} = \sqrt{\frac{e}{2}}(1+i)$$

1(c) This expression can be split up as

$$e^{z+\pi i} = e^z e^{i\pi} = e^z \cdot -1 = -e^z$$

3 The function can be written as

$$e^{\overline{z}} = e^x e^{-iy} = e^x \cos y - ie^x \sin y$$

so the partial derivatives are

$$\begin{aligned} u_x &= e^x \cos y \\ u_y &= -e^x \sin y \\ v_x &= -e^x \sin y \\ v_y &= -e^x \cos y \end{aligned}$$

These don't satisfy Cauchy-Riemann, so the function is differentiable nowhere and thus analytic nowhere.

8(a) The equation $e^z = -2$ can be written as

$$e^z = 2e^{i\pi}$$

so

$$\begin{gathered} e^x = 2 \to x = \ln 2 \\ e^{iy} = e^{i\pi} \to y = (2n+1)\pi \end{gathered}$$

and
$$z = \ln 2 + (2n+1)i\pi$$

8(b) The equation $e^z = 1 + i\sqrt{3}$ can be written as
$$e^z = 2e^{i\pi/3}$$

so

$$\begin{aligned} x &= \ln 2 \\ y &= \left(\frac{1}{3} + 2n\right)\pi \end{aligned}$$

and

$$z = \ln 2 + \left(\frac{1}{3} + 2n\right) i\pi$$

8(c) If

$$e^{2z-1} = 1$$

then

$$2z - 1 = i2n\pi$$

or, solving for z,

$$z = \frac{1}{2} + in\pi$$

2 Frame 31 – Logarithms

1(a) Evaluating the logarithm,

$$\mathrm{Log}(-ei) = \ln(e) + \mathrm{Arg}(-ei) = 1 - i\frac{\pi}{2}$$

1(b) As above,

$$\mathrm{Log}(1-i) = \ln(\sqrt{2}) + \mathrm{Arg}(1-i) = \frac{1}{2}\ln(2) - i\frac{\pi}{4}$$

2(a) The set of values is

$$\log(e) = \ln(e) + \arg(e) = 1 + i2n\pi$$

2(b) As above,

$$\log(i) = \ln(1) + \arg(i) = 0 + i\left(2n + \frac{1}{2}\right)\pi$$

2(c) As above,

$$\log(-1+i\sqrt{3}) = \ln(\sqrt{4}) + \arg(-1+i\sqrt{3}) = \ln 2 + i\left(2n + \frac{2}{3}\right)\pi$$

3(a) The left side is

$$\mathrm{Log}(1+i)^2 = \ln(2) + i\frac{\pi}{2}$$

and the right side is

$$2\,\mathrm{Log}(1+i) = 2\left[\frac{1}{2}\ln(2) + i\frac{\pi}{4}\right] = \ln(2) + i\frac{\pi}{2} = \mathrm{Log}(1+i)^2$$

3(b) The left side is

$$\mathrm{Log}(-1+i)^2 = \ln(2) - i\frac{\pi}{2}$$

and the right side is

$$2\,\mathrm{Log}(-1+i) = 2\left[\frac{1}{2}\ln(2) + i\frac{3\pi}{4}\right] = \ln(2) + i\frac{3\pi}{2}$$

5(a) Since $i^{1/2}$ can be written as the set

$$i^{1/2} = e^{\pi/4+\pi k} \quad (k = 0, \pm 1, \pm 2, \dots)$$

it has the logarithm

$$\log(i^{1/2}) = \ln(1) + i\arg(i^{1/2}) = i\left(n + \frac{1}{4}\right)\pi$$

Then,

$$\frac{1}{2}\log i = \frac{1}{2}\left[\ln(1) + i\left(2n + \frac{1}{2}\right)\pi\right] = i\left(n + \frac{1}{4}\right)\pi = \log(i^{1/2})m$$

6 Differentiating and using the chain rule, the identity

$$z = e^{\log z}$$

becomes

$$1 = e^{\log z} \cdot \frac{d}{dz}\log z$$

or

$$\frac{d}{dz}\log z = \frac{1}{e^{\log z}} = \frac{1}{z}$$

7 To solve the equation

$$\log z = i\frac{\pi}{2}$$

we write

$$z = e^{i\pi/2} = i$$

3 Frame 33 – Complex Exponents

1(a) The values of $(1+i)^i$ are

$$(1+i)^i = e^{i\cdot(\ln\sqrt{2}+i(2n+1/4)\pi)} = e^{i\ln(2)/2}e^{-\pi/4+2n\pi}$$

1(b) The values of $(-1)^{1/\pi}$ are

$$(-1)^{1/pi} = e^{\frac{1}{\pi}\cdot i(2n+1)\pi} = e^{(2n+1)i}$$

2(a) The principal value of i^i is

$$i^i = e^{i\,\mathrm{Log}\,i} = e^{i(i\pi/2)} = e^{-\pi/2}$$

2(b) The principal value of $[\frac{e}{2}(-1-i\sqrt{3})]^{3\pi i}$ is

$$[\frac{e}{2}(-1-i\sqrt{3})]^{3\pi i} = \exp\left[3\pi i\left(1-i\frac{2\pi}{3}\right)\right] = e^{3\pi i}e^{-2\pi^2} = -e^{-2\pi^2}$$

2(c) The principal value of $(1-i)^{4i}$ is

$$(1-i)^{4i} = e^{4i(\frac{1}{2}\ln 2-i\pi/4)} = e^{i2\ln(2)}e^{\pi} = e^{\pi}[\cos(2\ln 2)+i\sin(2\ln 2)]$$

3 The values of $(-1+i\sqrt{3})^{3/2}$ are

$$(-1+i\sqrt{3})^{3/2} = e^{3/2\cdot(\ln 2+i\pi(2n+2/3))} = 2^{3/2}e^{i(\pi+3n\pi)} = \pm 2\sqrt{2}$$

4 Frame 34 – Trigonometric Functions

15 To find all roots of the equation

$$\sin z = \cosh 4$$

we can write

$$\sin z = \sin x \cosh y + i \cos x \sinh y$$

Then, equating real and imaginary parts, we find that

$$\begin{aligned} \sin x \cosh y &= \cosh 4 \\ \cos x \sinh y &= 0 \end{aligned}$$

The second equation says that $x = (n + 1/2)\pi$. Since $\sin x$ here is one of ± 1, the first equation becomes

$$\cosh y = \pm \cosh 4$$

However, $\cosh y > 0$ for all y snd $\cosh y = \cosh(-y)$, so the only solutions are

$$z = \left(2n + \frac{1}{2}\right)\pi \pm 4i$$

16 To find all roots of the equation

$$\cos z = 2$$

we can write

$$\cos z = \cos x \cosh y - i \sin x \sinh y$$

so

$$\begin{aligned} \cos x \cosh y &= 2 \\ \sin x \sinh y &= 0 \end{aligned}$$

The second equation states that $x = n\pi$ or $y = 0$. The second case has no solutions, so we must have $x = n\pi$. Then, $\cos x = \pm 1$. As above, we require that $\cos x = 1$ for any solutions to be found. This gives

$$x = 2n\pi$$

so

$$y = \cosh^{-1}(2)$$

or, putting these together,

$$z = 2n\pi + i \cosh^{-1}(2)$$

To simplify this, we can try to find a simpler expression for y. If $\cosh y = 2$, then

$$\frac{e^y + e^{-y}}{2} = 2$$

or

$$(e^y)^2 - 4(e^y) + 1 = 0$$

This has solutions when

$$e^y = 2 \pm \sqrt{3}$$

or

$$y = \ln(2 \pm \sqrt{3})$$

However,

$$\frac{1}{2 - \sqrt{3}} = 2 + \sqrt{3}$$

so

$$\ln(2 + \sqrt{3}) = -\ln(2 - \sqrt{3})$$

and the solutions are

$$z = 2n\pi \pm i \ln(2 + \sqrt{3})$$

5 Frame 35 – Hyperbolic Trigonometry

1 Since $\sinh z$ is defined as

$$\sinh z = \frac{e^z - e^{-z}}{2}$$

its derivative is

$$\frac{d}{dz}\sinh z = \frac{d}{dz}\frac{e^z - e^{-z}}{2} = \frac{e^z + e^{-z}}{2} = \cosh z$$

2 Using the definitions,

$$\sinh 2z = \frac{e^{2z} - e^{-2z}}{2} = 2\frac{e^z + e^{-z}}{2}\frac{e^z - e^{-z}}{2} = 2\cosh z \sinh z$$

4 Since $z = x + iy$, we can write

$$\sinh z = \sinh(x+iy) = \sinh(x)\cosh(iy)+\sinh(iy)\cosh(x) = \sinh(x)\cos(y)+i\cosh(x)\sin(y)$$

8 The zeroes of sinh are at

$$\sinh x \cos y = 0$$
$$\cosh x \sin y = 0$$

The second equation only holds when $y = n\pi$. At all of these points, $\cos y \neq 0$, so the first equation only holds when $x = 0$. The zeroes, then, occur at

$$z = 0 + in\pi$$

Since $\cosh z = \sinh(z + \pi/2)$, the zeroes of cosh are at

$$z = 0 + i(n + 1/2)\pi$$

15(a) Splitting sinh into its real and imaginary components, the equation is

$$\sinh x \cos y = 0$$
$$\cosh x \sin y = 1$$

The first equation says that $x = 0$ or $y = (n+1/2)\pi$. In the first case, $\cosh(0) = 1$, so the second equation simply becomes

$$\sin y = 1$$

which has solutions at $y = (2n + 1/2)\pi$. In the second case, $\sin(y) = \pm 1$; examining these solutions, they result in the same set of numbers. Finally, putting the parts together, the solutions are at

$$z = 0 + i(2n + 1/2)\pi$$

15(b) Splitting cosh into its real and imaginary parts, the equation is

$$\cosh x \cos y = \frac{1}{2}$$
$$\sinh x \sin y = 0$$

The second equation has two solutions:

- $x = 0$: Here, $\cosh(0) = 1$, so the top equation becomes

  $$\cos y = \frac{1}{2}$$

  which has solutions at

  $$y = \begin{cases} \frac{\pi}{3} + 2n\pi, \\ \frac{-\pi}{3} + 2n\pi \end{cases}$$

- $y = n\pi$: Here, $\cos y = \pm 1$, so

  $$\cosh x = \pm \frac{1}{2}$$

  which has no solutions.

Overall, the solutions are the set

$$z = 0 + i\left(2n \pm \frac{1}{3}\right)\pi$$

16 Equating the real and imaginary parts,

$$\cosh x \cos y = -2$$
$$\sinh x \sin y = 0$$

As in the previous problem, there are two solutions to the second equation:

- $x = 0$: The first equation reduces to

  $$\cos y = -2$$

  which has no solutions.

- $y = n\pi$: Depending on y, this reduces the first equation to

  $$\cosh x = \pm 2$$

  This only has solutions in the positive case, where

  $$x = \cosh^{-1} 2$$

The solution set is simply

$$z = \cosh^{-1} 2 + i(2n + 1)\pi$$

Thinking back to the previous section, this can be simplified to

$$z = \pm \ln(2 + \sqrt{3}) + i(2n + 1)\pi$$

6 Frame 36 – Inverse Trigonometry

1(a) The values of $\tan^{-1}(2i)$ are

$$\begin{aligned}\arctan(2i) &= \frac{i}{2}\log\frac{3i}{-i}\\ &= \frac{i}{2}\log -3\\ &= \frac{i}{2}[\ln 3 + i(2n+1)\pi]\\ &= (n+1/2)\pi + i\frac{\ln 3}{2}\end{aligned}$$

1(b) The values of $\tan^{-1}(1+i)$ are

$$\begin{aligned}\arctan(1+i) &= \frac{i}{2}\log\frac{1+2i}{-1}\\ &= \frac{i}{2}\log -(1+2i)\\ &= \frac{i}{2}[\ln\sqrt{5} + i(2n + \arctan(2) + 1)\pi]\\ &= [n - (1+\arctan 2)]\pi + \frac{i}{4}\ln 5\end{aligned}$$

1(c) Since $(-1)^2 - 1 = 0$, the values of $\cosh^{-1}(-1)$ are

$$\cosh^{-1}(-1) = \log(-1) = i(2n+1)\pi$$

1(d) The values of $\tanh^{-1}(0)$ are

$$\begin{aligned}\tanh^{-1}(0) &= \frac{1}{2}\log 1\\ &= \frac{1}{2}(2n\pi i)\\ &= in\pi\end{aligned}$$

2 The solutions to the equation $\sin z = 2$ are

$$\begin{aligned}z &= \sin^{-1} 2\\ &= -i\log[2i + (-3)^{1/2}]\\ &= -i\log[i(2\pm\sqrt{3})]\end{aligned}$$

The values of this logarithm are

$$\log[i(2+\sqrt{3})] = \ln(2+\sqrt{3}) + i(2n+1/2)\pi$$

and

$$\log[i(2-\sqrt{3})] = \ln(2-\sqrt{3}) + i(2n+1/2)\pi = -\ln(2+\sqrt{3}) + i(2n+1/2)\pi$$

or, combining these two sets,

$$\log[i(2\pm\sqrt{3})] = \pm\ln(2+\sqrt{3}) + i(2n+1/2)\pi$$

Then,

$$\sin^{-1} 2 = \left(2n+\frac{1}{2}\right)\pi \pm i\ln(2+\sqrt{3})$$

3 The values of $\cos^{-1}\sqrt{2}$ are

$$\cos^{-1}\sqrt{2} = -i\log[\sqrt{2}+i(-1)^{1/2}] = -i\log[\sqrt{2}\pm 1]$$

This logarithm has two values:

$$\log[\sqrt{2}+1] = \ln(\sqrt{2}+1) + i2n\pi$$

and

$$\log[\sqrt{2}-1] = \ln(\sqrt{2}-1) + i2n\pi = -\ln(\sqrt{2}+1) + i2n\pi$$

so the solutions are

$$\cos^{-1}\sqrt{2} = 2n\pi \pm i\ln(1+\sqrt{2})$$