1 Frame 37 – Derivatives with Real Variables

1.1 Definition

In the previous chapter, we looked at derivatives of complex functions of a complex variable z. Now, we look at the derivatives of a complex-valued function of a real variable t. If we write our function as

$$w(t) = u(t) + iv(t)$$

where u and v are real-valued, then we can define the derivative of w at a point t as

$$w'(t) = \frac{d}{dt}w(t) = u'(t) + iv'(t)$$

provided that u' and v' exist at t.

1.2 Properties

If $z_0 = x_0 + iy_0$ is a complex constant, then we can show that

$$\begin{aligned}
\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u(t) + iv(t)]' \\
&= [x_0 u(t) - y_0 v(t)]' + i[y_0 u(t) + x_0 v(t)]' \\
&= [x_0 u'(t) - y_0 v'(t)] + i[y_0 u'(t) + x_0 v'(t)] \\
&= z_0 w'(t)
\end{aligned}$$

as we expect.

Next, if z_0 is still a complex constant, the derivative of $e^{z_0 t}$ is

$$\begin{aligned}
\frac{d}{dt}e^{z_0 t} &= \frac{d}{dt}e^{x_0 t}(\cos y_0 t + isiny_0 t) \\
&= \frac{d}{dt}e^{x_0 t}\cos y_0 t + i\frac{d}{dt}e^{x_0 t}\sin y_0 t \\
&= (x_0 + iy_0)(e^{x_0 t}\cos y_0 t + ie^{x_0 t}\sin y_0 t) \\
&= z_0 e^{z_0 t}
\end{aligned}$$

Many other rules carry over from standard calculus. However, some rules no longer apply. For instance, in calculus, the mean value theorem for derivatives states that

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

for some c in the interval $a \leq c \leq b$ as long as w is continuous. However, this is easily disproved by the function

$$w(t) = e^{it}$$

If $a = 0$ and $b = 2\pi$, then $w(a) = w(b) = 1$ and we expect to find a point c in $[0, 2\pi]$ such that $w'(c) = 0$. However, no such points exist – the magnitude of the derivative is always 1.

2 Frame 38 – Definite Integrals of Complex Functions

2.1 Definitions

If $w(t)$ is a complex-valued function of a real variable t, as in the previous section

$$w(t) = u(t) + iv(t)$$

then we define the **definite integral** of $w(t)$ over the interval $a \le t \le b$ as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

provided the two right-side integrals exist. Then,

$$\Re\left[\int_a^b w(t)dt\right] = \int_a^b \Re[w(t)]dt$$
$$\Im\left[\int_a^b w(t)dt\right] = \int_a^b \Im[w(t)]dt$$

Improper integrals over unbounded intervals are defined similarly.

The two real integrals will exist as long as u and v are **piecewise continuous** on the interval $[a, b]$ – that is, continuous everywhere in the interval except possibly for a finite number of points where it has one-sided limits. When u and v are piecewise continuous, we say that w is also piecewise continuous.

2.2 Properties

The most common rules of integrals from calculus apply here as well:

- $\int z_0 w(t)dt = z_0 \int w(t)$
- $\int w_1(t) + w_2(t)dt = \int w_1(t)dt + \int w_2(t)dt$
- $\int_a^b w(t)dt = -\int_b^a w(t)dt$
- $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$

We can also extend the fundamental theorem of calculus to complex integrals. Suppose that two functions

$$w(t) = u(t) + iv(t)$$
$$W(t) = U(t) + iV(t)$$

are continuous on the interval $[a, b]$ and $W'(t) = w(t)$ when $a \leq t \leq b$. Then, we can write

$$\int_a^b w(t)dt = W(b) - W(a) = W(t)\Big|_a^b$$

Example: noting that the derivative of $\frac{1}{i}e^{it}$ *is*

$$\frac{d}{dt}\left(\frac{1}{i}e^{it}\right) = \frac{1}{i}ie^{it} = e^{it}$$

we can evaluate $\int e^{it}dt$ *as*

$$\begin{aligned}
\int_0^{\pi/4} e^{it}dt &= \frac{e^{it}}{i}\Big|_0^{\pi/4} \\
&= \frac{1}{i}\left[e^{\pi/4} - 1\right] \\
&= \frac{1}{i}\left[\frac{1}{\sqrt{2}} - 1 + \frac{i}{\sqrt{2}}\right] \\
&= \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)
\end{aligned}$$

As in the previous section, the mean value theorem for integrals does not apply. We can show this by finding the integral $\int_0^{2\pi} e^{it}dt = 0$, even though the function is never zero on this interval.