# 1 Frame 37 – Derivatives with Real Variables

## 1.1 Definition

In the previous chapter, we looked at derivatives of complex functions of a complex variable $z$. Now, we look at the derivatives of a complex-valued function of a real variable $t$. If we write our function as

$$w(t) = u(t) + iv(t)$$

where $u$ and $v$ are real-valued, then we can define the derivative of $w$ at a point $t$ as

$$w'(t) = \frac{d}{dt}w(t) = u'(t) + iv'(t)$$

provided that $u'$ and $v'$ exist at $t$.

## 1.2 Properties

If $z_0 = x_0 + iy_0$ is a complex constant, then we can show that

$$\begin{aligned}
\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u(t) + iv(t)]' \\
&= [x_0 u(t) - y_0 v(t)]' + i[y_0 u(t) + x_0 v(t)]' \\
&= [x_0 u'(t) - y_0 v'(t)] + i[y_0 u'(t) + x_0 v'(t)] \\
&= z_0 w'(t)
\end{aligned}$$

as we expect.

Next, if $z_0$ is still a complex constant, the derivative of $e^{z_0 t}$ is

$$\begin{aligned}
\frac{d}{dt}e^{z_0 t} &= \frac{d}{dt}e^{x_0 t}(\cos y_0 t + isiny_0 t) \\
&= \frac{d}{dt}e^{x_0 t}\cos y_0 t + i\frac{d}{dt}e^{x_0 t}\sin y_0 t \\
&= (x_0 + iy_0)(e^{x_0 t}\cos y_0 t + ie^{x_0 t}\sin y_0 t) \\
&= z_0 e^{z_0 t}
\end{aligned}$$

Many other rules carry over from standard calculus. However, some rules no longer apply. For instance, in calculus, the mean value theorem for derivatives states that

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

for some $c$ in the interval $a \leq c \leq b$ as long as $w$ is continuous. However, this is easily disproved by the function

$$w(t) = e^{it}$$

If $a = 0$ and $b = 2\pi$, then $w(a) = w(b) = 1$ and we expect to find a point $c$ in $[0, 2\pi]$ such that $w'(c) = 0$. However, no such points exist – the magnitude of the derivative is always 1.

# 2 Frame 38 – Definite Integrals of Complex Functions

## 2.1 Definitions

If $w(t)$ is a complex-valued function of a real variable $t$, as in the previous section

$$w(t) = u(t) + iv(t)$$

then we define the **definite integral** of $w(t)$ over the interval $a \leq t \leq b$ as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i\int_a^b v(t)dt$$

provided the two right-side integrals exist. Then,

$$\begin{aligned} \Re\left[\int_a^b w(t)dt\right] &= \int_a^b \Re[w(t)]dt \\ \Im\left[\int_a^b w(t)dt\right] &= \int_a^b \Im[w(t)]dt \end{aligned}$$

Improper integrals over unbounded intervals are defined similarly.

The two real integrals will exist as long as $u$ and $v$ are **piecewise continuous** on the interval $[a, b]$ – that is, continuous everywhere in the interval except possibly for a finite number of points where it has one-sided limits. When $u$ and $v$ are piecewise continuous, we say that $w$ is also piecewise continuous.

## 2.2 Properties

The most common rules of integrals from calculus apply here as well:

- $\int z_0 w(t)dt = z_0 \int w(t)$
- $\int w_1(t) + w_2(t)dt = \int w_1(t)dt + \int w_2(t)dt$
- $\int_a^b w(t)dt = -\int_b^a w(t)dt$
- $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$

We can also extend the fundamental theorem of calculus to complex integrals. Suppose that two functions

$$\begin{aligned} w(t) &= u(t) + iv(t) \\ W(t) &= U(t) + iV(t) \end{aligned}$$

are continuous on the interval $[a, b]$ and $W'(t) = w(t)$ when $a \leq t \leq b$. Then, we can write

$$\int_a^b w(t)dt = W(b) - W(a) = W(t)\Big|_a^b$$

*Example: noting that the derivative of* $\frac{1}{i}e^{it}$ *is*

$$\frac{d}{dt}\left(\frac{1}{i}e^{it}\right) = \frac{1}{i}ie^{it} = e^{it}$$

*we can evaluate* $\int e^{it}dt$ *as*

$$\begin{aligned}\int_0^{\pi/4} e^{it}dt &= \frac{e^{it}}{i}\Big|_0^{\pi/4} \\ &= \frac{1}{i}\left[e^{\pi/4} - 1\right] \\ &= \frac{1}{i}\left[\frac{1}{\sqrt{2}} - 1 + \frac{i}{\sqrt{2}}\right] \\ &= \frac{1}{\sqrt{2}} + i\left(1 - \frac{1}{\sqrt{2}}\right)\end{aligned}$$

As in the previous section, the mean value theorem for integrals does not apply. We can show this by finding the integral $\int_0^{2\pi} e^{it}dt = 0$, even though the function is never zero on this interval.

# 3 Frame 39 – Contours

## 3.1 Definitions

In calculus, integrals are defined on intervals of the real line. In complex analysis, we instead use curves in the complex plane.

An **arc** is a set of points $z = (x, y)$ in the complex plane such that the functions

$$x = x(t), \quad y = y(t); \quad z = z(t) = x(t) + iy(t)$$

are continuous functions of the parameter $t$, where $a \leq t \leq b$. This definition is a continuous mapping of the interval $a \leq t \leq b$ into the $z$ plane.

We say that an arc is **simple** if it does not cross itself; ie:

$$z(t_1) \neq z(t_2) \quad \text{for all } t_1 \neq t_2$$

If a simple arc starts and ends at the same point ($z(a) = z(b)$), it is called a **simple closed curve**. These curves are **positively oriented** when they are oriented in the counterclockwise direction.

*Example: the unit circle*

$$z = e^{i\theta}$$

*where* $0 \leq \theta \leq 2\pi$ *is a positively oriented simple closed curve centered at the origin with a radius of* $1$. *A more general circle is*

$$z = z_0 + Re^{i\theta}$$

*which is centered at* $z_0$ *and has a radius of* $R$.

## 3.2 Uniqueness

Note that the parametric representation for any arc is not unique. If we know a function $\phi$ such that

$$t = \phi(\tau)$$

maps the interval $\alpha \leq \tau \leq \beta$ onto the interval $a \leq t \leq b$. Then, the two equations

$$z(t) \quad (a \leq t \leq b)$$

and

$$z(\phi(t)) \quad (\alpha \leq t \leq \beta)$$

represent the same arc.

## 3.3 Smoothness

Suppose that the real and imaginary components of $z$ are differentiable, and their derivatives are continuous. Then, the arc $z(t)$ is a **differentiable arc**, and

$$|z'(t)| = |x'(t) + iy'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable. This allows us to find the length of an arc as

$$L = \int_a^b |z'(t)|dt$$

If an arc is differentiable and $z'(t)$ is never zero (except maybe at $t = a$ or $t = b$), then we call the arc a **smooth arc**. We can write the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

which has an angle of inclination of $\arg z'(t)$.

A **contour** is an arc which consists of a finite number of smooth arcs joined together. Specifically, if $z(t)$ represents a contour, then $z(t)$ is continuous and $z'(t)$ is piecewise continuous. If a contour is also a simple closed arc, we call it a **simple closed contour**.

The points on a simple closed arc are the boundary points of two different domains:

- The interior of the arc, which is bounded;
- The exterior of the arc, which is unbounded.

# 4 Frame 40 – Contour Integrals

## 4.1 Definitions and conditions

We can now integrate a complex function $f$ along a contour $C$, which starts and ends at points $z_1$ and $z_2$, respectively. This is effectively a line integral. These integrals can be written as

$$\int_C f(z)dz$$

or, if the integral does not depend on the path taken,

$$\int_{z_1}^{z_2} f(z)dz$$

This integral (along a complex path) represents an integral with respect to a real parameter $t$. If the contour $C$ is written as $z(t)$ on the interval $a \leq t \leq b$, then the integral represented is

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$$

Since $z'(t)$ must be piecewise continuous, this integral exists as long as $f[z(t)]$ is also piecewise continuous on this interval.

## 4.2 Basic properties

From the definition and the properties of integrals, we can write

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$$

and

$$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

We can also create a new contour $-C$ that consists of the points in $C$ in reversed order – this contour extends from $z_2$ to $z_1$. Integrating along this reversed contour, we find that

$$\begin{aligned}
\int_{-} Cf(z)dz &= \int_{-b}^{-a} f[z(-t)]\frac{d}{dt}z(-t)dt \\
&= -\int_{-b}^{-a} f[z(-t)]z'(-t)dt \\
&= -\int_a^b f[z(t)]z'(t)dt \\
&= -\int_C f(z)dz
\end{aligned}$$

We can also split up a contour $C$ into multiple legs $C_1, C_2, \dots$. If we can write a contour this way, then we say that $C = C_1 + C_2$. The contour integral along $C$ can then be written as

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$

# 5 Frame 41 – Examples of Contour Integrals

This section will show several specific examples of contour integrals.

## 5.1 Example 1

Suppose that the contour $C$ is the right hand half of the circle $|z| = 2$:

$$z = 2e^{i\theta}, \quad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$$

Then,

$$\begin{aligned}
\int_C \overline{z} dz &= \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}} (2e^{i\theta})' d\theta \\
&= 4i \int_{-\pi/2}^{\pi/2} e^{-i\theta} e^{i\theta} d\theta \\
&= 4i \int_{-\pi/2}^{\pi/2} d\theta \\
&= 4\pi i
\end{aligned}$$

Also, note that all of the points on this semicircle satisfy

$$z\overline{z} = |z|^2 = 4$$

so we can see from this result that

$$\int_C \frac{1}{z} dz = \pi i$$

## 5.2 Example 2

Suppose that the points $O$, $A$, and $B$ are $0$, $i$, and $1 + i$, respectively. Then, if $C_1$ is the polyline $OAB$ and

$$f(z) = y - x - i3x^2$$

then the contour integral of $f$ along $C_1$ is

$$\begin{aligned}\int_{C_1} f(z)dz &= \int_{OA} f(z)dz + \int_{AB} f(z)dz \\ &= \int_0^1 yidy + \int_0^1 (1 - x - i3x^2)dx \\ &= \frac{i}{2} + \int_0^1 (1-x)dx - 3i\int_0^1 x^2dx \\ &= \frac{i}{2} + \frac{1}{2} - i \\ &= \frac{1-i}{2}\end{aligned}$$

Next, if $C_2$ is the line $OB$, the contour integral along this curve is

$$\begin{aligned}\int_{C_2} f(z)dz &= \int_0^1 -i3x^2(1+i)dx \\ &= 3(1-i)\int_0^1 x^2 \\ &= 1 - i\end{aligned}$$

Finally, the integral of $f$ over the simple closed contour $OABO$ is $C_1 - C_2$, which is

$$\int_{OABO} f(z)dz = \frac{-1+i}{2}$$

## 5.3 Example 3

Suppose that $C$ is any arbitrary smooth arc from a fixed point $z_1$ to another point $z_2$:

$$z = z(t) \quad (a \leq t \leq b)$$

The contour integral of $f(z) = z$ along this curve is

$$\begin{aligned}\int_C zdz &= \int_a^b z(t)z'(t)dt \\ &= \int_a^b \frac{d}{dt}\frac{[z(t)]^2}{2}dt \\ &= \frac{[z(t)]^2}{2}\bigg|_a^b \\ &= \frac{z_2^2 - z_1^2}{2}\end{aligned}$$

Note that this integral only depends on the endpoints of $C$ and not the path. This lets us write

$$\int_{z_1}^{z_2} z dz = \frac{z_2^2 - z_1^2}{2}$$

This holds when $C$ is not a smooth contour. Since all contours are sums of finite numbers of smooth arcs, this expression holds for each arc in $C$, leading to the same final expression.

Also, note that the integral of $f(z) = z$ around any closed contour in the plane is zero.

## 6 Frame 42 – Examples with Branch Cuts

A contour integral's path can include a point on a branch cut. The following two examples show this.

### 6.1 Example 1

Suppose we want to integrate the function

$$f(z) = z^{1/2} = e^{\frac{1}{2}\log z} \quad (0 < \arg z < 2\pi)$$

on the semicircle

$$z = 3e^{i\theta} \quad (0 \leq \theta \leq \pi)$$

Although the function is not defined at $\theta = 0$, we can still write

$$f[z(\theta)] = e^{\frac{1}{2}(\ln 3 + i\theta)} = \sqrt{3}e^{i\theta/2}$$

and the right hand limit of this function exists at $\theta = 0$. Thus, the integrand exists as long as we define the missing point as

$$f[z(0)]z'(0) = i3\sqrt{3}$$

Then,

$$\begin{aligned}
\int_C f(z)dz &= 3\sqrt{3}\int_0^{\pi} e^{i3\theta/2} \\
&= 3\sqrt{3}\frac{2}{3i}e^{i3\theta/2}\Big|_0^{\pi} \\
&= -\frac{2}{3i}(1+i) \\
&= -2\sqrt{3}(1+i)
\end{aligned}$$

### 6.2 Example 2

Suppose that we want to integrate the function

$$f(z) = z^{a-1} = e^{(a-1)\,\mathrm{Log}\,z} \quad (-\pi < \mathrm{Arg}\,z < \pi)$$

on the positively oriented circle

$$z = Re^{i\theta} \quad (-\pi \leq \theta \leq \pi)$$

The contour integral is

$$
\begin{aligned}
\int_C z^{a-1} dz &= \int_{-\pi}^{\pi} iR^a e^{ia\theta} d\theta \\
&= iR^a \int_{-\pi}^{\pi} e^{ia\theta} d\theta \\
&= iR^a \left( \frac{e^{ia\theta}}{ia} \right)_{-\pi}^{\pi} \\
&= i\frac{2R^a}{a} \frac{e^{ia\pi} - e^{-ia\pi}}{2i} \\
&= i\frac{2R^a}{a} \sin a\pi
\end{aligned}
$$

Note that if $a$ is a non-zero integer, this integral is zero; if $a = 0$, this integral reduces to

$$\int_C \frac{dz}{z} = 2\pi i$$

# 7 Frame 43 – Upper Bounds for Contour Integrals

We can put a bound on the modulus of a contour integral by observing that

$$\left|\int_a^b w(t)dt\right| \leq \int_a^b |w(t)|\,dt$$

## 7.1 Theorem

Suppose that $C$ is a contour with a length of $L$ and that $f(z)$ is a function that is piecewise continuous on $C$. If $M \geq 0$ is a real constant such that

$$|f(z)| \leq M$$

for all points on $C$, then

$$\left|\int_C f(z)dz\right| \leq ML$$

Note that such a number $M$ will always exist because $f$ is continuous on $C$.

## 7.2 Examples

*Example: Suppose that*

$$f(z) = \frac{z+4}{z^3-1}$$

*and the contour $C$ is a quarter circle with a radius of 2 in the first quadrant (running from $z = 2$ to $z = 2i$). Since $|z| = 2$ at all points on this contour, we can write that*

$$|z+4| \leq |z| + 4 = 6$$

*and*

$$|z^3 - 1| \geq |z|^3 - 1 = 7$$

*Since the length of the contour is $L = \pi$, we can write the upper bound*

$$\left|\int_C \frac{z+4}{z^3-1}dz\right| \leq \frac{6\pi}{7}$$

*Example: suppose that $C_R$ is the semicircular contour*

$$z = Re^{i\theta} \quad (0 \leq \theta \leq \pi)$$

*and $f$ is the function*

$$f(z) = \frac{z^{1/2}}{z^2+1}$$

*where* $z^{1/2}$ *denotes the branch* $-\pi/2 < \theta < 3\pi/2$. *Anywhere on this semicircle,*

$$|z^{1/2}| = \sqrt{R}$$

*and*

$$|z^2 + 1| \geq ||z^2| - 1| = R^2 - 1$$

*Since the contour has a length of* $\pi R$, *the contour integral of* $f$ *along* $C$ *can be limited by*

$$\begin{aligned} \int_C f(z)dz &\leq \frac{\sqrt{R}}{R^2 - 1} \cdot \pi R \\ &= \frac{\pi R^{3/2}}{R^2 - 1} \\ &= \frac{\pi/\sqrt{R}}{1 - (1/R^2)} \end{aligned}$$

*As* $R$ *approaches infinity, this bound approaches zero, so*

$$\lim_{R \to \infty} \int_C f(z)dz = 0$$