1 Frame 29 – The Exponential Function

1.1 Definition

We define the **exponential function** e^z by writing

$$e^z = e^x e^{iy}$$

and we apply Euler's formula to get

$$e^z = e^x(\cos y + i \sin y)$$

Note that, when $y = 0$, e^z reduces to e^x.

Although we typically understand that $e^{1/n}$ would be the set of nth roots of e, here, we only use the real, positive root $\sqrt[n]{e}$.

1.2 Familar properties

First, in calculus, we know that

$$e^{x_1} e^{x_2} = e^{x_1 + x_2}$$

It is easy to verify that this holds true for complex numbers:

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

This also allows us to write

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

and, as a specific case,

$$\frac{1}{e^z} = e^{-z}$$

We showed earlier that e^z is differentiable everywhere in the complex plane, and that

$$\frac{d}{dz} e^z = e^z$$

We also know that e^z is never zero. This comes from the pair

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi$$

and since e^x is never zero, neither is e^z.

1.3 Unfamiliar properties

Since we can write

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

the exponential function is periodic with an imaginary period of $2\pi i$.

It is also possible for the complex exponential function to be negative. For an example, we know that Euler's identity states

$$e^{i\pi} = -1$$

In fact, e^z can be any given non-zero complex number.

Example: suppose we want solutions to the equation

$$e^z = 1 + i$$

The right side can be rewritten as

$$e^x e^{iy} = \sqrt{2} e^{i\pi/4}$$

and equating the parts of this equation gives

$$x = \ln\sqrt{2} = \frac{1}{2}\ln 2 \quad \textit{and} \quad y = \left(2n + \frac{1}{4}\right)\pi$$

so

$$z = \frac{1}{2}\ln 2 + \left(2n + \frac{1}{4}\right)\pi i$$

2 Frame 30 – The Logarithmic Function

2.1 Motivation

We said in the previous section that e^z can take on any non-zero complex value. To help us solve the equation

$$e^w = z$$

we will define a **logarithmic function**, such that

$$e^{\log z} = z \quad (z \neq 0)$$

We can solve for w by writing the two complex numbers in the form

$$z = re^{i\theta}$$
$$w = u + iv$$

Substituting these into the original equation gives

$$e^u e^{iv} = re^{i\theta}$$

so we get

$$w = \log z = \ln r + i(\theta + 2n\pi)$$

Note that this is a multi-valued function.

Example: if $z = -1 - i\sqrt{3}$*, then* $r = 2$ *and* $\theta = -2\pi/3$*, so*

$$\log(-1 - i\sqrt{3}) = \ln 2 + \left(n - \frac{1}{3}\right) 2\pi i$$

2.2 Precise definition

A more precise definition of the multi-valued logarithmic function is

$$\log z = \ln|z| + i \arg z$$

The **principal value** of $\log z$ is obtained by using the single-valued principal argument instead:

$$\text{Log } z = \ln|z| + i\theta$$

Note that

$$\log z = \text{Log } z + i2n\pi$$

2.3 Notes

The principal logarithmic function Log z reduces to the usual logarithm from calculus when z is positive and real – if $z = r$, then

$$\text{Log}\ r = \ln r$$

However, we are now able to find the logarithm of negative real numbers, which we were unable to do in calculus.

Example: the logarithm of -1 *is*

$$\log(-1) = \ln 1 + (1 + 2n)i\pi = (2n + 1)i\pi$$

and

$$\textit{Log}\ (-1) = i\pi$$