Contents

1 Frame 12 – Functions of Complex Variables

1.1 Functions

If S is a set of complex numbers, then a **function** f is a rule that assigns a complex number w to each z in S. The number w is called the **value** of f at z. We denote it as

$$w = f(z)$$

The set S is called the **domain of definition** of f. Note that we need both a rule (f) and a domain (S) for a function to be well defined.

Suppose that $w = u + iv$ and $z = x + iy$. Then,

$$u + iv = f(x + iy)$$

Then, we can express $f(z)$ as a pair of real functions of x and y:

$$f(z) = u(x, y) + iv(x, y)$$

Alternatively, we could use polar coordinates to write

$$u + iv = f(re^{i\theta})$$

so

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Example: the function $f(z) = z^2$ can be written as

$$\begin{aligned} f(x + iy) &= (x + iy)^2 \\ &= (x^2 - y^2) + i2xy \end{aligned}$$

so

$$\begin{aligned} u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy \end{aligned}$$

In polar coordinates,

$$\begin{aligned} f(x + iy) &= (re^{i\theta})^2 \\ &= r^2 e^{i2\theta} \\ &= r^2 \cos 2\theta + ir^2 \sin 2\theta \end{aligned}$$

so

$$\begin{aligned} u(r, \theta) &= r^2 \cos 2\theta \\ v(r, \theta) &= r^2 \sin 2\theta \end{aligned}$$

1.2 Real-Valued Functions

We say that f is a **real-valued function** if v is zero everywhere.

Example: one real-valued function is

$$f(z) = |z|^2 = x^2 + y^2 + i0$$

1.3 Polynomials

If n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are complex numbers with $a_n \neq 0$, then the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a **polynomial** of degree n. Note that this sum has a finite number of terms and that the domain of definition is the entire z plane.

As in real numbers, a **rational function** is a quotient of two polynomials:

$$R(z) = \frac{P(z)}{Q(z)}$$

A rational function is defined everywhere that $Q(z) \neq 0$.

1.4 Multi-Valued Functions

A generalization of a function is a rule that assigns more than one value to a point z. These **multiple-valued functions** are usually studied by taking one of the possible values at each point and constructing a single-valued function.

Example: we know that we can write

$$z^{1/2} = \pm\sqrt{r}e^{i\theta/2}$$

where we denoted $-\pi < \theta \leq \pi$ as the ***principal value*** *of* $\arg z$*. To turn this into a single valued function, we can choose the positive value of r and write*

$$f(z) = \sqrt{r}e^{i\theta/2}$$

Then, f is well-defined on the entire plane.

2 Frame 13 – Mappings

2.1 Definitions

There is no convenient way to graph the function $w = f(z)$ – each of these complex numbers are located on a plane instead of a line. Instead, we can draw pairs of corresponding points on separate z and w planes. When we think of f this way, we call it a **mapping** or **transformation**.

If f is defined on the domain of definition S, then the **image** of a point $z \in S$ is the point $w = f(z)$. If T is a subset of S, then the set of the images of each point in T are called the image of T. In particular, the image of the entire domain, S, is called the **range** of f. The **inverse image** of a point w is the set of points z in S that map to w (possibly zero, one, or many points).

2.2 Basic transformations

Using this geometric interpretation, we can describe mappings using terms such as **translation**, **rotation**, and **reflection**. For instance, the mapping

$$w = z + 1 = (x + 1) + iy$$

can be thought of as a translation of each point z one unit to the right. Another example is the rotational mapping

$$w = iz$$

where, using $i = e^{i\pi/2}$ and $z = re^{i\theta}$, is

$$w = re^{i(\theta+\pi/2)}$$

or, in other words, a 90° rotation. Finally, the mapping

$$w = \overline{z} = x - iy$$

is a reflection across the real axis. Usually, it is more useful to sketch an image of a curve rather than a single point.

2.3 Mapping a curve

For an example, consider the mapping $w = z^2$. We showed earlier that this can be written as

$$u = x^2 - y^2, \quad v = 2xy$$

To sketch the image, we will first set $u = c_1$, which requires that

$$x^2 - y^2 = c_1, \quad c_1 > 0$$

which is the equation for a hyperbola. This equation can then be used to solve for the image points:

$$u = c_1, \quad v = \pm 2y\sqrt{y^2 + c_1}$$

where the plus-minus is resolved depending on which side the image point is on. Simply put, as z travels up the right-side hyperbola or down the left-side hyperbola, w travels up the vertical line $u = c_1$.

Next, we can set $v = c_2$, which requires

$$2xy = c_2, \quad c_2 > 0$$

This gives us the image set

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

As $x \to \pm\infty$, $u \to \infty$; as $x \to 0$, $u \to -\infty$. Thus, this hyperbola traces out the straight line $v = c_2$ towards the right as z travels towards the left.

2.4 Mapping a region

We can use some of the details from the previous example to find the image of a region, rather than a single curve.

Consider the domain $x > 0, y > 0, xy < 1$. This region consists of the upper branches of the hyperbolas

$$2xy = c, \quad 0 < c < 2$$

and we know from the previous example that these hyperbolas map to the straight lines

$$v = c$$

Thus, this region maps to the horizontal strip $0 < v < 2$.

We can also close the domain to contain the curves $x = 0$, $y = 0$, and $xy = 1$. From the function $w = z^2$, we know that the points $(0, y)$ and $(x, 0)$ map to the points $(-y^2, 0)$ and $(x^2, 0)$, so including the two straight lines simply extends the strip to include $v = 0$. Similarly, the hyperbola $xy = 1$ maps to the horizontal line $v = 2$.

Simply put, the image of the closed region $x \geq 0$, $y \geq 0$, $xy \leq 1$ is the closed region $0 \leq v \leq 2$.

2.5 Mapping with polar coordinates

Finally, we can use polar coordinates to simplify some mappings.

Again, consider the mapping $w = z^2$. If we write $z = re^{i\theta}$, then the image point can be written as

$$w = r^2 e^{2i\theta}$$

Looking at the magnitude of w, points on a circle $r = r_0$ are mapped onto a circle $r' = r_0^2$. Also, looking at the argument of w, the angle of the image is doubled. This means that the first quadrant, which is defined as

$$r \geq 0, \quad 0 \leq \theta \leq \pi/2$$

is in a one-to-one mapping with the top plane, $0 \leq \theta \leq \pi$. Simiarly, the top place is mapped onto the entire complex plane (although this is not one-to-one, since the inverse image of the positive real axis is both real axes).

Note that any mapping $w = z^n$ for positive integer n has a similar form, where each non-zero point in the w plane is the image of n distinct points in the z plane.

3 Frame 14 – Mappings by the Exponential Function

Now, we will look at the exponential function

$$e^z = e^{x+iy} = e^x e^{iy}$$

We can again look at straight lines and find their images in this mapping.

Consider the transformation

$$w = e^z = \rho e^{i\phi}$$

where

$$p = e^x \quad \phi = y$$

This means that the image of a vertical line $x = c_1$ *is a circle with radius* $p = e^{c_1}$. *Each point on the circle is the image of infinitely many points, each spaced* 2π *units apart on the vertical line. Similarly, the horizontal line* $y = c_2$ *is a ray with an angle of* $\phi = c_2$.

With these images in mind, we know that vertical and horizontal line segments are mapped onto arcs and rays, respectively. We can then use this information to map regions:

Now, consider the rectangular region

$$a \leq x \leq b \quad c \leq y \leq d$$

The image of this region under the mapping $w = e^z$ *is*

$$e^a \leq \rho \leq e^b \quad c \leq \phi \leq d$$

This is a one-to-one mapping if $d - c < 2\pi$. *In particular, the region with* $c = 0, d = \pi$ *is mapped onto half of a circular ring.*

4 Frame 15 – Limits

4.1 Definitions

Suppose that a function f is defined at all points z in some deleted neighborhood of z_0. The statement that the number w_0 is the **limit** of $f(z)$ as z approaches z_0 means that the point $w = f(z)$ can be made *arbitrarily close* to w_0 if we choose z close enough to z_0. We write this as

$$\lim_{z \to z_0} f(z) = w_0$$

To be more precise, if this limit exists, then for each positive number ϵ, there is a positive number δ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Geometrically, this definition says that each ϵ neighbourhood around w_0 has a corresponding deleted δ neighbourhood around z_0 such that the image of each point in the δ neighbourhood maps to a point in the ϵ neighbourhood.

Note that the deleted neighbourhood will always exist if z_0 is internal to the domain of definition of f. We can extend the definition of a limit to include boundary points by ignoring all of the neighbourhood's points that are outside the domain.

Also note that this definition only allows a given point to be tested as a limit – it does not provide a method for finding the limit. This will be covered in the next section.

4.2 Uniqueness

If the limit of a function $f(z)$ exists at z_0, it must be unique. To show this, consider two limits:

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} f(z) = w_1$$

This implies that we can find δ_0 and δ_1 such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_1$$

Now, suppose that δ is a positive number smaller than both δ_0 and δ_1. Then, for all $0 < |z - z_0| < \delta$, we find that the difference between the two limits is

$$\begin{aligned} |w_1 - w_0| &= ||f(z) - w_0| - |f(z) - w_1|| \\ &\leq |f(z) - w_0| + |f(z) - w_1| \\ &< \epsilon + \epsilon \\ &= 2\epsilon \end{aligned}$$

and since ϵ can be made arbitrarily small, we must have

$$w_1 = w_0$$

4.3 Example – basic limit

Consider the function $f(z) = \frac{i\overline{z}}{2}$. We can show that the limit of this function as $z \to 1$ is

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

To do this, we observe that

$$\begin{aligned} \left| f(z) - \frac{i}{2} \right| &= \left| \frac{i\overline{z}}{2} - \frac{i}{2} \right| \\ &= \frac{|z-1|}{2} \end{aligned}$$

Then, we can fulfill the limit definition by writing

$$\left| f(z) - \frac{i}{2} \right| < \epsilon \text{ whenever } |z - 1| < 2\epsilon$$

4.4 Example – direction dependence

In order for w_0 to be a limit of f at z_0, the limit conditions must hold if z approaches z_0 in any arbitrary manner.

Consider the function

$$f(z) = \frac{z}{\overline{z}}$$

Then, the limit

$$\lim_{z \to 0} f(z)$$

does not exist. To illustrate this, the function's value for any non-zero point $z = (x, 0)$ is

$$f(x, 0) = \frac{x + i0}{x - i0} = 1$$

but the value for any non-zero point $z = (0, y)$ *is*

$$f(0, y) = \frac{0 + iy}{0 - iy} = -1$$

so the limit would not be unique.

5 Frame 16 – Theorems on Limits

Next, it is helpful to connect limits of complex functions and real-valued functions, allowing us to use our knowledge of calculus to simplify the process of finding complex limits

5.1 Splitting into real functions

First, the following theorem is helpful:

Theorem 1. Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0$$

Then, the limit

$$\lim_{z \to z_0} = w_0$$

holds iff

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$$

The two implications of this theorem can be proved by considering the definitions of the neighbourhoods as open disks.

5.2 Combining simple limits

Theorem 2. Suppose that

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} F(z) = W_0$$

Then, we can write the following three limits:

$$\lim_{z \to z_0} f(z) + F(z) = w_0 + W_0$$

$$\lim_{z \to z_0} f(z)F(z) = w_0 W_0$$

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \text{ if } W_0 \neq 0$$

These can be proved easily by applying Theorem 1 to each limit.

5.3 Polynomials

Using the basic limit definition from the previous section, it is simple to show that

$$\lim_{z \to z_0} c = c$$

and

$$\lim_{z \to z_0} z = z_0$$

for any complex numbers c and z_0. Then, by the multiplication property,

$$\lim z \to z_0 z^n = z_0^n$$

for any positive integer z. These limits can be used to show that, for any polynomial

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

the limit as z approaches a point z_0 is the polynomial's value:

$$\lim_{z \to z_0} P(z) = P(z_0)$$

6 Frame 17 – Limits Involving Infinity

6.1 The point at infinity

Sometimes, it is useful to include the **point at infinity** with the complex plane. This point is denoted by ∞. In order to visualize it, the complex plane can be drawn with a unit sphere centered at the origin. Then, a line can be drawn from the top of the sphere (or the *north pole*, denoted by N) to any point on the plane; the line will pass through exactly one other point P on the sphere. This correspondence (between points on the plane, z, and the sphere, P) is called a **stereographic projection**, and the sphere is known as the **Riemann sphere**.

No point in the plane corresponds to the point N. We can let N correspond to the point at infinity, giving us a one-to-one mapping between points on the sphere and points in the extended complex plane.

We will make the distinction that a point z is a point in the finite plane unless we specifically describe the point at infinity – we will specifically mention ∞.

6.2 Neighbourhoods around infinity

Next, we can define neighbourhoods around the point at infinity. Looking at the Riemann sphere, we notice that all of the points P in the upper hemisphere project to points z outside of the unit disk.

Further, if ϵ is a small, positive number, then points in the plane such that

$$|z| > \frac{1}{\epsilon}$$

correspond to points on the sphere close to N. Thus, we call the set $|z| > 1/\epsilon$ an **(ϵ) neighbourhood** of ∞.

6.3 Limits with infinity

With this new point at infinity, we can give meaning to the statement

$$\lim_{z \to z_0} f(z) = w_0$$

when z_0 or w_0 are infinity. We can then use the following theorems:

$$\lim_{z\to z_0} f(z) = \infty \iff \lim_{z\to z_0} \frac{1}{f(z)} = 0$$
$$\lim_{z\to\infty} f(z) = w_0 \iff \lim_{z\to 0} f\left(\frac{1}{z}\right) = w_0$$
$$\lim_{z\to\infty} f(z) = \infty \iff \lim_{z\to 0} \frac{1}{f(1/z)} = 0$$

6.4 Examples

Three limits using these new properties follow.

- To find
$$\lim_{z\to -1} \frac{iz+3}{z+1}$$
we notice that
$$\lim_{z\to -1} \frac{z+1}{iz+3} = 0$$
so the limit is infinity.

- To find
$$\lim_{z\to\infty} \frac{2z+i}{z+1}$$
we evaluate
$$\begin{aligned}\lim_{z\to 0} \frac{(2/z)+i}{(1/z)+1} &= \lim_{z\to 0} \frac{2+iz}{1+z} \\ &= 2\end{aligned}$$

- To find
$$\lim_{z\to\infty} \frac{2z^3-1}{z^2+1}$$
we evaluate
$$\begin{aligned}\lim_{z\to 0} \frac{(1/z^2)+1}{(2/z^3)-1} &= \lim_{z\to 0} \frac{z+z^3}{2-z^3} \\ &= 0\end{aligned}$$
so the original limit is infinity.

7 Frame 18 – Continuity

7.1 Definitions

A function is **continuous** at a point z_0 if all of the three following conditions are true:

$$\lim_{z \to z_0} f(z) \text{ exists}$$
$$f(z_0) \text{ exists}$$
$$\lim_{z \to z_0} f(z) = f(z_0)$$

This final statement says that for each positive number ϵ there is a positive number δ such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } |z - z_0| < \delta$$

We say that a function is said to be continuous in a region R if it is continuous at each point in R.

7.2 Theorems

The basic limit identities allow us to find the continuity of more complex functions. If two functions are continuous at a point, their sum, product, and quotients are also continuous (in the last case, provided that the denominator is non-zero). A polynomial is continuous in the entire plane.

A complex function that can be split into its real and imaginary components

$$f(z) = u(x, y) + iv(x, y)$$

is continuous at $z_0 = (x_0, y_0)$ iff u and v are continuous at z_0.

We can state three more theorems about continuity:

- A composition of continuous functions is continuous.

  Suppose that $w = f(z)$ is defined in a neighbourhood of z_0 and $W = g(w)$ is defined in a neighbourhood of $f(z_0)$. Also, suppose that f is continuous at z_0 and g is continuous at $f(z_0)$. Then, the statement that the composition

  $$g[f(z)]$$

  is continuous is equivalent to the statement that

  $$|g[f(z)] - g[f(z_0)]| < \epsilon \textit{ whenever } |f(z) - f(z_0)| < \gamma$$

  Then, since f is continuous, we can find a δ such that the right side is satisfied, so $g \circ f$ is continuous.

- If f is continuous and non-zero at z_0, then there is some neighbourhood of z_0 where $f(z) \neq 0$.

  Suppose that we choose $\epsilon = |f(z_0)|/2$. Then, if there is a point where $f(z) = 0$ in a δ neighbourhood around z_0, the limit inequality is

  $$|f(z_0)| < \frac{|f(z_0)|}{2}$$

  so we have a contradiction, and there must be a neighbourhood where $f(z) = 0$.

- If f is continuous in a closed and bounded region R, then there exists a non-negative real number M such that

  $$|f(z)| \leq M$$

  where equality holds for one or more z.

8 Frame 19 – Derivatives

8.1 Definitions

Suppose that a function f is defined on a neighbourhood $|z - z_0| < \epsilon$. We define the **derivative** of f at z_0 as the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

or, if we write $\Delta z = z - z_0$, then the definition can be written as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

We say that f is **differentiable** at z_0 when $f'(z_0)$ exists.

As an augmented notation, we often write

$$\Delta w = f(z + \Delta z) - f(z)$$

and we use the derivative notation

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

8.2 Examples

Example: suppose $f(z) = z^2$. Then,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} 2z + \Delta z = 2z$$

Example: suppose $f(z) = \overline{z}$. Then,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

and we showed earlier that this limit does not exist – it has different values depending on the angle of approach.

Example: suppose $f(z) = |z|^2$. Then,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z + \Delta z} - z\overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \overline{z} + \overline{\Delta z} + z\frac{\overline{\Delta z}}{\Delta z}$$

We see, once again, that the $\frac{\overline{\Delta z}}{\Delta z}$ term stops us from obtaining a derivative. However, this term is not present if $z = 0$ – at this point, dw/dz is zero. Thus, the derivative exists only at $z = 0$, and $f'(0) = 0$.

8.3 Notes

We've seen that it is possible for a function to be differentiable at a point but nowhere else in a neighbourhood of that point. We can even note that our function, $f(z) = |z|^2$, can be written as

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0$$

so we have no trouble finding the partial derivatives, but the function is still not differentiable aside from $z = 0$.

We've also seen that our troublesome function, $f(z) = |z|^2$, was continuous but not differentiable. However, we can state that **differentiability implies continuity** at a point. We can see this easily in the numerator of the derivative limit: $f(z)$ must approach the point $f(z_0)$ as z approaches z_0, which implies continuity.

9 Frame 20 – Differentiation Formulas

The derivative definition in the previous section is extremely similar to the definition for real-valued functions. Similar basic differentiation formulas can be derived and used for complex functions. This section will cover the most useful formulas.

If c is a complex constant and f is a function that is differentiable at z, then:

$$\frac{d}{dz}c = 0$$
$$\frac{d}{dz}z = 1$$
$$\frac{d}{dz}[cf(z)] = cf'(z)$$

If n is a non-zero integer,

$$\frac{d}{dz}z^n = nz^{n-1}$$

If two functions f and g are differentiable at z, then

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$$
$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$$
$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - g'(z)f(z)}{g^2(z)} \text{ if } g(z) \neq 0$$

Finally, when composing functions (such as $F(z) = g[f(z)]$), the standard chain rule applies:

$$F'(z_0) = g'[f(z_0)]f'(z_0)$$

or, in a different form, if $w = f(z)$ and $W = F(z)$,

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}$$

10 Frame 21 – Cauchy-Riemann Equations

10.1 Basic proof

We will obtain a formula in this chapter that a function must satisfy at a point if its derivative exists there.

First, we can split up the following points into real and imaginary parts:

$$z_0 = x_0 + iy_0, \quad \Delta z = \Delta x + i\Delta y$$

Then, the term Δw becomes

$$\begin{aligned}\Delta w &= f(z_0 + \Delta z) - f(z_0) \\ &= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]\end{aligned}$$

Next, we know that we can split up the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

into its real and imaginary components

$$f'(z_0) = \lim_{\Delta z \to 0} \left(\Re \frac{\Delta w}{\Delta z} \right) + i \lim_{\Delta z \to 0} \left(\Im \frac{\Delta w}{\Delta z} \right)$$

We've seen several times that this expression must hold as Δz tends to zero in any direction. In particular, we can use the test points $(\Delta x, 0)$ and $(0, \Delta y)$ to test this derivative. In the first case, $\Delta y = 0$, so the derivative is

$$\begin{aligned}f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\ &= u_x(x_0, y_0) + iv_x(x_0, y_0)\end{aligned}$$

where u_x and v_x are the partial derivatives with respect to x. Similarly, we can set $\Delta x = 0$ and obtain

$$\begin{aligned}f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{i\Delta y} \\ &= \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta y} - i \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta y} \\ &= v_y(x_0, y_0) - iu_y(x_0, y_0)\end{aligned}$$

where u_y and v_y are the partial derivatives with respect to y.

Now, since these two values must be equal, we require two conditions for the derivative to exist

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$

and

$$u_y(x_0, y_0) = -v_x(x_0, y_0)$$

These are the **Cauchy-Riemann equations**. They are necessary conditions for a derivative to exist – if they are not satisfied at a point, the derivative does not exist there. Note, however, that the derivative is not guaranteed to exist – the Cauchy-Riemann equations are not a sufficient condition.

10.2 Formal theorem

We can summarize the above result.

Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then, the partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

there. Also, the derivative can be written as

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

10.3 Examples

Example: to illustrate the Cauchy-Riemann equations, consider the function

$$f(z) = z^2 = (x^2 - y^2) + i(2xy)$$

The real and imaginary functions are

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

and their partial derivatives are

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x$$

Finally, we can write

$$f'(z) = u_x + iv_x = 2x + i2y = 2z$$

as expected.

Example: for the function

$$f(z) = |z|^2$$

we have the real and imaginary components

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0$$

In order for the Cauchy-Riemann equations to hold at a point, we require that

$$\begin{aligned} u_x &= 2x = 0 \\ u_y &= 2y = 0 \end{aligned}$$

or, in other words, the derivative does not exist anywhere except possibly for the point $(0, 0)$.

11 Frame 22 – Sufficient Conditions for Differentiability

11.1 Extensions on Cauchy-Riemann

In the previous frame, we derived the Cauchy-Riemann equations, which are necessary for differentiability at a point. However, they are not sufficient. We can add some continuity conditions to improve the theorem as follows:

Suppose that a function

$$f(z) = u(x, y) + iv(x, y)$$

is defined in some ϵ neighbourhood of a point z_0. If the partial derivatives u_x, u_y, v_x, and v_y exist everywhere in the neighbourhood, are continuous at z_0, and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

then the derivative $f'(z_0)$ exists, and it has the value

$$f'(z_0) = u_x(z_0) + iv_x(z_0)$$

11.2 Examples

Example: suppose that

$$f(z) = e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

Then, the partial derivatives are

$$u_x = e^x \cos y = v_y, \quad u_y = -e^x \sin y = -v_x$$

These derivatives are continuous everywhere and the Cauchy-Riemann equations are satisfied, so $f'(z_0)$ exists everywhere, and it has the value

$$f'(z) = e^x \cos y + ie^x \sin y = e^z$$

Example: suppose that

$$f(z) = |z|^2 = (x^2 + y^2) + i0$$

We saw before that this function has a derivative at $z = 0$. However, at all other points, the Cauchy-RIemann equations are not satisfied, so the derivative does not exist anywhere else.

12 Frame 23 – Polar Coordinates

12.1 Basic concepts

We can restate the theorem from the previous section using the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

If we are using this choice of coordinates, then our goal is to express the first-order derivatives $u_r, u_\theta, \ldots$ in terms of $u_x, u_y, \ldots$ to allow us to rewrite the Cauchy-Riemann equations. We can do this using the chain rule:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

or, more simply,

$$u_r = u_x\cos\theta + u_y\sin\theta, \quad u_\theta = -u_x r\sin\theta + u_y r\cos\theta$$

The same applies to v. If the Cauchy-Riemann equations hold, then we require that

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

at z_0. The reverse is also true – if these hold, then the Cauchy-Riemann equations hold too – so these equations are equivalent.

12.2 Formal theorem

The above concepts can be stated as follows:

Suppose that the function

$$f(z) = u(r,\theta) + iv(r,\theta)$$

is defined in some ϵ neighbourhood of $z_0 = r_0e^{i\theta_0}$ ($r_0 \neq 0$). If the partial derivatives of u and v are continuous throughout the neighbourhood and they satisfy the polar Cauchy-Riemann equations

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

then $f'(z_0)$ exists and has the value

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

12.3 Examples

Example: consider the function

$$f(z) = \frac{1}{z} = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$$

Since the derivatives are

$$ru_r = -\frac{\cos\theta}{r} = v_\theta, \quad u_\theta = -\frac{\sin\theta}{r} = v_r$$

the derivative of f exists for all $z \neq 0$, with the value

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}$$

Example: suppose that $r > 0$ and α is a constant. Then, the function

$$f(z) = r^{1/3}e^{i\theta/3} = r^{1/3}\cos\frac{\theta}{3} + ir^1 3\sin\frac{\theta}{3} \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

has a derivative everywhere in the domain. Here,

$$ru_r = \frac{r^{1/3}}{3}\cos\frac{\theta}{3} = v_\theta, \quad u\theta = -\frac{r^{1/3}}{3}\sin\frac{\theta}{3} = -rv_r$$

so the derivative is

$$\begin{aligned} f'(z) &= e^{-i\theta}\left[\frac{1}{3r^{3/2}}\cos\frac{\theta}{3} + i\frac{1}{3r^{3/2}}\sin\frac{\theta}{3}\right] \\ &= \frac{1}{3(r^{1/3}e^{i\theta/3})^2} \\ &= \frac{1}{3f^2(z)} \end{aligned}$$

13 Frame 24 – Analytic Functions

13.1 Definitions

A function f is **analytic at a point** z_0 if it has a derivative at each point in some neighbourhood of z_0. By extension, if f is analytic at z_0, it must be analytic at each point in some neighbourhood of z_0. A function is **analytic in an open set** if it has a derivative everywhere in that set. For example, $f(z) = 1/z$ is analytic at each non-zero point, and $f(z) = |z|^2$ is analytic nowhere.

An **entire** function is a function that is analytic at each point in the finite complex plane. For example, a polynomial's derivative exists everywhere, so every polynomial is analytic.

If a function is not analytic at a point z_0 but is analytic at some point in every neighbourhood of z_0, then z_0 is called a **singular point** (or **singularity**) of f. For example, $f(z) = 1/z$ has a singular point at $z = 0$.

13.2 Extensions

We can obtain some useful rules about analytic functions by looking at the conditions for differentiability.

Here, suppose f and g are two analytic functions in a domain D. Since their sum, $f + g$, their product, fg, and their quotient, f/g, are all differentiable (provided the denominator does not vanish in the last case), all three of these new functions are also analytic in D.

We also know that the chain rule states that the composition $g[f(z)]$ is differentiable. Thus, the composition of two analytic functions is also also analytic.

We can derive another useful property about analytic functions:

If $f'(z) = 0$ everywhere in a domain D, then $f(z)$ must be constant through D.

This can be shown by writing f in the form

$$f(z) = u(x, y) + iv(x, y)$$

Since $f'(z) = u_x + iv_x = 0$, both of these partial derivatives must be zero. Then, from Cauchy-Riemann,

$$u_x = u_y = v_x = v_y = 0$$

at each point in D. Next, if we consider any line segment in D, we know that the derivative of u along its arc length s is

$$\frac{du}{ds} = (\text{grad } u) \cdot U$$

where U is a unit vector along the line segment. However, since $u_x = u_y = 0$, grad u is also zero, so u is constant on any line segment in D. Then, since we can connect any points in D with a finite number of line segments, u must have a constant value in D: ie,

$$u(x, y) = a$$

We can apply the same logic to get $v(x, y) = b$. Then, we conclude that $f(z) = a + bi$ at each point in D.

14 Frame 25 – Examples of Analyticity

This section will give some examples of analytic functions.

Example 1: the function

$$f(z) = \frac{z^3 + 4}{(z^2 - 3)(z^2 + 1)}$$

is analytic everywhere except for the points where the denominator vanishes due to the differentiation formulas available for polynomials, products, and quotients. There are four singular points at $z = \pm\sqrt{3}$ and $z = \pm i$.

Example 2: the function

$$f(z) = \cosh\, x \cos y + i\, \sinh\, x \sin y$$

has the component functions

$$u(x, y) = \cosh\, x \cos y, \quad v(x, y) = \sinh\, x \sin y$$

and the partial derivatives

$$u_x = \sinh\, x \cos y = v_y, \quad u_y = -\cosh\, x \sin y = -v_x$$

so f is an entire function.

Example 3: suppose that a function

$$f(z) = u(x, y) + iv(x, y)$$

and its conjugate

$$\overline{f(z)} = u(x, y) - iv(x, y)$$

are both analytic in a domain D. From applying the Cauchy-Riemann equations to f, we see that

$$u_x = v_y, \quad u_y = -v_x$$

and, applying them to the second function,

$$u_x = -v_y, \quad u_y = v_x$$

Solving these equations gives $u_x = v_x = 0$. Thus, the derivative of f is

$$f'(z) = 0 + i0 = 0$$

so f is constant throughout D.

Example 4: suppose that a function has a constant magnitude – ie,

$$|f(z)| = c$$

for all z in D. We can show that $f(z)$ is constant in D. First, if c is zero, then we are done – 0 is constant. Otherwise, we can write

$$f(z)\overline{f(z)} = c^2$$

Since $c \neq 0$, we know that $f(z)$ is never zero. Then, we can write

$$\overline{f(z)} = \frac{c^2}{f(z)}$$

and since the quotient of two analytic functions is also analytic, both f and its conjugate are analytic, so f must be constant (see previous example).

15 Frame 26 – Harmonic Functions

15.1 Definition

A **harmonic** function H of two real variables x and y is a function that has continuous first- and second-order derivatives that satisfy Laplace's equation:

$$H_{xx}(x, y) + H_{yy}(x, y) = 0$$

Example: it is easy to verify that $T(x, y) = e^{-y} \sin x$ *is harmonic over the entire* xy *plane.*

15.2 A source of harmonic functions

The following theorem provides a source of harmonic functions.

Theorem: if a function $f(z) = u(x, y) = iv(x, y)$ *is analytic in a domain* D*, then its component functions* u *and* v *are harmonic in* D*.*

This theorem can be proved by differentiating the Cauchy-Riemann equations with respect to x and y, giving

$$\begin{aligned} u_{xx} &= v_{yx}, \quad u_{yx} = -v_{xx} \\ u_{xy} &= v_{yy}, \quad u_{yy} = -v_{xy} \end{aligned}$$

Then, since $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$, it follows that

$$u_{xx} + u_{yy} = 0, \quad v_{xx} + v_{yy} = 0$$

so u and v are harmonic in D.

Example: the function

$$f(z) = e^{-y} \sin x - ie^{-y} \cos x$$

is entire, so its real component must be harmonic in the entire xy *plane. This matches the result from the previous example.*

Example: since the function

$$f(z) = \frac{i}{z^2}$$

is analytic for all $z \neq 0$*, we can write the function as*

$$f(z) = \frac{i}{z^2}\frac{\overline{z}^2}{\overline{z}^2} = \frac{i\overline{z}^2}{|z|^4} = \frac{2xy + i(x^2 - y^2)}{(x^2 + y^2)^2}$$

and we obtain the two harmonic functions

$$u(x, y) = \frac{2xy}{(x^2 + y^2)^2}$$
$$v(x, y) = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

15.3 Harmonic conjugates

If two functions u and v are harmonic in a domain D and their first-order derivatives satisfy Cauchy-Riemann in D, then we say that v is a **harmonic conjugate** of u. We can say that a function f is analytic in D iff v is a harmonic conjugate of u.

Example: if we have the functions

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

then we know that these functions are the components of $f(z) = z^2$, so v must be a harmonic conjugate of u. However, we verified earlier that $2xy + i(x^2 - y^2)$ is analytic nowhere, so u is not a harmonic conjugate of v.

Example: the component function

$$u(x, y) = y^3 - 3x^2 y$$

is harmonic because $u_{xx} + u_{yy} = 0$ in the entire xy plane. To find a harmonic conjugate, we can use the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

so we can write, using the first equation,

$$v_y(x, y) = -6xy$$

which integrates to give

$$v(x, y) = -3xy^2 + \phi(x)$$

Then, the other equation gives

$$3x^2 - 3y^2 = -3y^2 + \phi'(x)$$

so

$$v(x, y) = -3xy^2 + x^3 + C$$

These equations correspond to the form

$$f(z) = i(z^3 + C)$$