

# 1 Frame 37 – Derivatives with Real Variables

## 1.1 Definition

In the previous chapter, we looked at derivatives of complex functions of a complex variable  $z$ . Now, we look at the derivatives of a complex-valued function of a real variable  $t$ . If we write our function as

$$w(t) = u(t) + iv(t)$$

where  $u$  and  $v$  are real-valued, then we can define the derivative of  $w$  at a point  $t$  as

$$w'(t) = \frac{d}{dt}w(t) = u'(t) + iv'(t)$$

provided that  $u'$  and  $v'$  exist at  $t$ .

## 1.2 Properties

If  $z_0 = x_0 + iy_0$  is a complex constant, then we can show that

$$\begin{aligned}\frac{d}{dt}[z_0 w(t)] &= [(x_0 + iy_0)(u(t) + iv(t))]' \\ &= [x_0 u(t) - y_0 v(t)]' + i[y_0 u(t) + x_0 v(t)]' \\ &= [x_0 u'(t) - y_0 v'(t)] + i[y_0 u'(t) + x_0 v'(t)] \\ &= z_0 w'(t)\end{aligned}$$

as we expect.

Next, if  $z_0$  is still a complex constant, the derivative of  $e^{z_0 t}$  is

$$\begin{aligned}\frac{d}{dt}e^{z_0 t} &= \frac{d}{dt}e^{x_0 t}(\cos y_0 t + i \sin y_0 t) \\ &= \frac{d}{dt}e^{x_0 t} \cos y_0 t + i \frac{d}{dt}e^{x_0 t} \sin y_0 t \\ &= (x_0 + iy_0)(e^{x_0 t} \cos y_0 t + i e^{x_0 t} \sin y_0 t) \\ &= z_0 e^{z_0 t}\end{aligned}$$

Many other rules carry over from standard calculus. However, some rules no longer apply. For instance, in calculus, the mean value theorem for derivatives states that

$$w'(c) = \frac{w(b) - w(a)}{b - a}$$

for some  $c$  in the interval  $a \leq c \leq b$  as long as  $w$  is continuous. However, this is easily disproved by the function

$$w(t) = e^{it}$$

If  $a = 0$  and  $b = 2\pi$ , then  $w(a) = w(b) = 1$  and we expect to find a point  $c$  in  $[0, 2\pi]$  such that  $w'(c) = 0$ . However, no such points exist – the magnitude of the derivative is always 1.

## 2 Frame 38 – Definite Integrals of Complex Functions

### 2.1 Definitions

If  $w(t)$  is a complex-valued function of a real variable  $t$ , as in the previous section

$$w(t) = u(t) + iv(t)$$

then we define the **definite integral** of  $w(t)$  over the interval  $a \leq t \leq b$  as

$$\int_a^b w(t)dt = \int_a^b u(t)dt + i \int_a^b v(t)dt$$

provided the two right-side integrals exist. Then,

$$\begin{aligned}\Re \left[ \int_a^b w(t)dt \right] &= \int_a^b \Re[w(t)]dt \\ \Im \left[ \int_a^b w(t)dt \right] &= \int_a^b \Im[w(t)]dt\end{aligned}$$

Improper integrals over unbounded intervals are defined similarly.

The two real integrals will exist as long as  $u$  and  $v$  are **piecewise continuous** on the interval  $[a, b]$  – that is, continuous everywhere in the interval except possibly for a finite number of points where it has one-sided limits. When  $u$  and  $v$  are piecewise continuous, we say that  $w$  is also piecewise continuous.

### 2.2 Properties

The most common rules of integrals from calculus apply here as well:

- $\int z_0 w(t)dt = z_0 \int w(t)$
- $\int w_1(t) + w_2(t)dt = \int w_1(t)dt + \int w_2(t)dt$
- $\int_a^b w(t)dt = - \int_b^a w(t)dt$
- $\int_a^b w(t)dt = \int_a^c w(t)dt + \int_c^b w(t)dt$

We can also extend the fundamental theorem of calculus to complex integrals. Suppose that two functions

$$\begin{aligned}w(t) &= u(t) + iv(t) \\ W(t) &= U(t) + iV(t)\end{aligned}$$

are continuous on the interval  $[a, b]$  and  $W'(t) = w(t)$  when  $a \leq t \leq b$ . Then, we can write

$$\int_a^b w(t)dt = W(b) - W(a) = W(t)\Big|_a^b$$

*Example: noting that the derivative of  $\frac{1}{i}e^{it}$  is*

$$\frac{d}{dt} \left( \frac{1}{i}e^{it} \right) = \frac{1}{i}ie^{it} = e^{it}$$

*we can evaluate  $\int e^{it}dt$  as*

$$\begin{aligned} \int_0^{\pi/4} e^{it}dt &= \frac{e^{it}}{i} \Big|_0^{\pi/4} \\ &= \frac{1}{i} \left[ e^{\pi/4} - 1 \right] \\ &= \frac{1}{i} \left[ \frac{1}{\sqrt{2}} - 1 + \frac{i}{\sqrt{2}} \right] \\ &= \frac{1}{\sqrt{2}} + i \left( 1 - \frac{1}{\sqrt{2}} \right) \end{aligned}$$

As in the previous section, the mean value theorem for integrals does not apply. We can show this by finding the integral  $\int_0^{2\pi} e^{it}dt = 0$ , even though the function is never zero on this interval.

## 3 Frame 39 – Contours

### 3.1 Definitions

In calculus, integrals are defined on intervals of the real line. In complex analysis, we instead use curves in the complex plane.

An **arc** is a set of points  $z = (x, y)$  in the complex plane such that the functions

$$x = x(t), \quad y = y(t); \quad z = z(t) = x(t) + iy(t)$$

are continuous functions of the parameter  $t$ , where  $a \leq t \leq b$ . This definition is a continuous mapping of the interval  $a \leq t \leq b$  into the  $z$  plane.

We say that an arc is **simple** if it does not cross itself; ie:

$$z(t_1) \neq z(t_2) \quad \text{for all } t_1 \neq t_2$$

If a simple arc starts and ends at the same point ( $z(a) = z(b)$ ), it is called a **simple closed curve**. These curves are **positively oriented** when they are oriented in the counterclockwise direction.

*Example: the unit circle*

$$z = e^{i\theta}$$

where  $0 \leq \theta \leq 2\pi$  is a positively oriented simple closed curve centered at the origin with a radius of 1. A more general circle is

$$z = z_0 + Re^{i\theta}$$

which is centered at  $z_0$  and has a radius of  $R$ .

### 3.2 Uniqueness

Note that the parametric representation for any arc is not unique. If we know a function  $\phi$  such that

$$t = \phi(\tau)$$

maps the interval  $\alpha \leq \tau \leq \beta$  onto the interval  $a \leq t \leq b$ . Then, the two equations

$$z(t) \quad (a \leq t \leq b)$$

and

$$z(\phi(t)) \quad (\alpha \leq t \leq \beta)$$

represent the same arc.

### 3.3 Smoothness

Suppose that the real and imaginary components of  $z$  are differentiable, and their derivatives are continuous. Then, the arc  $z(t)$  is a **differentiable arc**, and

$$|z'(t)| = |x'(t) + iy'(t)| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

is integrable. This allows us to find the length of an arc as

$$L = \int_a^b |z'(t)| dt$$

If an arc is differentiable and  $z'(t)$  is never zero (except maybe at  $t = a$  or  $t = b$ ), then we call the arc a **smooth arc**. We can write the unit tangent vector

$$\mathbf{T} = \frac{z'(t)}{|z'(t)|}$$

which has an angle of inclination of  $\arg z'(t)$ .

A **contour** is an arc which consists of a finite number of smooth arcs joined together. Specifically, if  $z(t)$  represents a contour, then  $z(t)$  is continuous and  $z'(t)$  is piecewise continuous. If a contour is also a simple closed arc, we call it a **simple closed contour**.

The points on a simple closed arc are the boundary points of two different domains:

- The interior of the arc, which is bounded;
- The exterior of the arc, which is unbounded.

## 4 Frame 40 – Contour Integrals

### 4.1 Definitions and conditions

We can now integrate a complex function  $f$  along a contour  $C$ , which starts and ends at points  $z_1$  and  $z_2$ , respectively. This is effectively a line integral. These integrals can be written as

$$\int_C f(z)dz$$

or, if the integral does not depend on the path taken,

$$\int_{z_1}^{z_2} f(z)dz$$

This integral (along a complex path) represents an integral with respect to a real parameter  $t$ . If the contour  $C$  is written as  $z(t)$  on the interval  $a \leq t \leq b$ , then the integral represented is

$$\int_C f(z)dz = \int_a^b f[z(t)]z'(t)dt$$

Since  $z'(t)$  must be piecewise continuous, this integral exists as long as  $f[z(t)]$  is also piecewise continuous on this interval.

### 4.2 Basic properties

From the definition and the properties of integrals, we can write

$$\int_C z_0 f(z)dz = z_0 \int_C f(z)dz$$

and

$$\int_C [f(z) + g(z)]dz = \int_C f(z)dz + \int_C g(z)dz$$

We can also create a new contour  $-C$  that consists of the points in  $C$  in reversed order – this contour extends from  $z_2$  to  $z_1$ . Integrating along this reversed contour, we find that

$$\begin{aligned} \int_{-C} f(z)dz &= \int_{-b}^{-a} f[z(-t)] \frac{d}{dt} z(-t)dt \\ &= - \int_{-b}^{-a} f[z(-t)]z'(-t)dt \\ &= - \int_a^b f[z(t)]z'(t)dt \\ &= - \int_C f(z)dz \end{aligned}$$

We can also split up a contour  $C$  into multiple legs  $C_1, C_2, \dots$ . If we can write a contour this way, then we say that  $C = C_1 + C_2$ . The contour integral along  $C$  can then be written as

$$\int_C f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz$$



## 5 Frame 41 – Examples of Contour Integrals

This section will show several specific examples of contour integrals.

### 5.1 Example 1

Suppose that the contour  $C$  is the right hand half of the circle  $|z| = 2$ :

$$z = 2e^{i\theta}, \quad \left(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$$

Then,

$$\begin{aligned} \int_C \bar{z} dz &= \int_{-\pi/2}^{\pi/2} \overline{2e^{i\theta}} (2e^{i\theta})' d\theta \\ &= 4i \int_{-\pi/2}^{\pi/2} e^{-i\theta} e^{i\theta} d\theta \\ &= 4i \int_{-\pi/2}^{\pi/2} d\theta \\ &= 4\pi i \end{aligned}$$

Also, note that all of the points on this semicircle satisfy

$$z\bar{z} = |z|^2 = 4$$

so we can see from this result that

$$\int_C \frac{1}{z} dz = \pi i$$

### 5.2 Example 2

Suppose that the points  $O$ ,  $A$ , and  $B$  are  $0$ ,  $i$ , and  $1 + i$ , respectively. Then, if  $C_1$  is the polyline  $OAB$  and

$$f(z) = y - x - i3x^2$$

then the contour integral of  $f$  along  $C_1$  is

$$\begin{aligned}
\int_{C_1} f(z)dz &= \int_{OA} f(z)dz + \int_{AB} f(z)dz \\
&= \int_0^1 yidy + \int_0^1 (1-x-i3x^2)dx \\
&= \frac{i}{2} + \int_0^1 (1-x)dx - 3i \int_0^1 x^2 dx \\
&= \frac{i}{2} + \frac{1}{2} - i \\
&= \frac{1-i}{2}
\end{aligned}$$

Next, if  $C_2$  is the line  $OB$ , the contour integral along this curve is

$$\begin{aligned}
\int_{C_2} f(z)dz &= \int_0^1 -i3x^2(1+i)dx \\
&= 3(1-i) \int_0^1 x^2 \\
&= 1-i
\end{aligned}$$

Finally, the integral of  $f$  over the simple closed contour  $OABO$  is  $C_1 - C_2$ , which is

$$\int_{OABO} f(z)dz = \frac{-1+i}{2}$$

### 5.3 Example 3

Suppose that  $C$  is any arbitrary smooth arc from a fixed point  $z_1$  to another point  $z_2$ :

$$z = z(t) \quad (a \leq t \leq b)$$

The contour integral of  $f(z) = z$  along this curve is

$$\begin{aligned}
\int_C z dz &= \int_a^b z(t)z'(t)dt \\
&= \int_a^b \frac{d}{dt} \frac{[z(t)]^2}{2} dt \\
&= \left. \frac{[z(t)]^2}{2} \right|_a^b \\
&= \frac{z_2^2 - z_1^2}{2}
\end{aligned}$$

Note that this integral only depends on the endpoints of  $C$  and not the path. This lets us write

$$\int_{z_1}^{z_2} z dz = \frac{z_2^2 - z_1^2}{2}$$

This holds when  $C$  is not a smooth contour. Since all contours are sums of finite numbers of smooth arcs, this expression holds for each arc in  $C$ , leading to the same final expression.

Also, note that the integral of  $f(z) = z$  around any closed contour in the plane is zero.