1 Frame 12 – Functions of Complex Variables

1.1 Functions

If S is a set of complex numbers, then a **function** f is a rule that assigns a complex number w to each z in S. The number w is called the **value** of f at z. We denote it as

$$w = f(z)$$

The set S is called the **domain of definition** of f. Note that we need both a rule (f) and a domain (S) for a function to be well defined.

Suppose that $w = u + iv$ and $z = x + iy$. Then,

$$u + iv = f(x + iy)$$

Then, we can express $f(z)$ as a pair of real functions of x and y:

$$f(z) = u(x, y) + iv(x, y)$$

Alternatively, we could use polar coordinates to write

$$u + iv = f(re^{i\theta})$$

so

$$f(z) = u(r, \theta) + iv(r, \theta)$$

Example: the function $f(z) = z^2$ can be written as

$$f(x + iy) = (x + iy)^2$$
$$= (x^2 - y^2) + i2xy$$

so

$$u(x, y) = x^2 - y^2$$
$$v(x, y) = 2xy$$

In polar coordinates,

$$f(x + iy) = (re^{i\theta})^2$$
$$= r^2 e^{i2\theta}$$
$$= r^2 \cos 2\theta + ir^2 \sin 2\theta$$

so

$$u(r, \theta) = r^2 \cos 2\theta$$
$$v(r, \theta) = r^2 \sin 2\theta$$

1.2 Real-Valued Functions

We say that f is a **real-valued function** if v is zero everywhere.

Example: one real-valued function is

$$f(z) = |z|^2 = x^2 + y^2 + i0$$

1.3 Polynomials

If n is a non-negative integer and $a_0, a_1, a_2, \ldots, a_n$ are complex numbers with $a_n \neq 0$, then the function

$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

is a **polynomial** of degree n. Note that this sum has a finite number of terms and that the domain of definition is the entire z plane.

As in real numbers, a **rational function** is a quotient of two polynomials:

$$R(z) = \frac{P(z)}{Q(z)}$$

A rational function is defined everywhere that $Q(z) \neq 0$.

1.4 Multi-Valued Functions

A generalization of a function is a rule that assigns more than one value to a point z. These **multiple-valued functions** are usually studied by taking one of the possible values at each point and constructing a single-valued function.

Example: we know that we can write

$$z^{1/2} = \pm\sqrt{r}e^{i\theta/2}$$

*where we denoted $-\pi < \theta \leq \pi$ as the **principal value** of $\arg z$. To turn this into a single valued function, we can choose the positive value of r and write*

$$f(z) = \sqrt{r}e^{i\theta/2}$$

Then, f is well-defined on the entire plane.

2 Frame 13 – Mappings

2.1 Definitions

There is no convenient way to graph the function $w = f(z)$ – each of these complex numbers are located on a plane instead of a line. Instead, we can draw pairs of corresponding points on separate z and w planes. When we think of f this way, we call it a **mapping** or **transformation**.

If f is defined on the domain of definition S, then the **image** of a point $z \in S$ is the point $w = f(z)$. If T is a subset of S, then the set of the images of each point in T are called the image of T. In particular, the image of the entire domain, S, is called the **range** of f. The **inverse image** of a point w is the set of points z in S that map to w (possibly zero, one, or many points).

2.2 Basic transformations

Using this geometric interpretation, we can describe mappings using terms such as **translation**, **rotation**, and **reflection**. For instance, the mapping

$$w = z + 1 = (x + 1) + iy$$

can be thought of as a translation of each point z one unit to the right. Another example is the rotational mapping

$$w = iz$$

where, using $i = e^{i\pi/2}$ and $z = re^{i\theta}$, is

$$w = re^{i(\theta + \pi/2)}$$

or, in other words, a 90° rotation. Finally, the mapping

$$w = \bar{z} = x - iy$$

is a reflection across the real axis. Usually, it is more useful to sketch an image of a curve rather than a single point.

2.3 Mapping a curve

For an example, consider the mapping $w = z^2$. We showed earlier that this can be written as

$$u = x^2 - y^2, \quad v = 2xy$$

To sketch the image, we will first set $u = c_1$, which requires that

$$x^2 - y^2 = c_1, \quad c_1 > 0$$

3

which is the equation for a hyperbola. This equation can then be used to solve for the image points:

$$u = c_1, \quad v = \pm 2y\sqrt{y^2 + c_1}$$

where the plus-minus is resolved depending on which side the image point is on. Simply put, as z travels up the right-side hyperbola or down the left-side hyperbola, w travels up the vertical line $u = c_1$.

Next, we can set $v = c_2$, which requires

$$2xy = c_2, \quad c_2 > 0$$

This gives us the image set

$$u = x^2 - \frac{c_2^2}{4x^2}, \quad v = c_2$$

As $x \to \pm\infty$, $u \to \infty$; as $x \to 0$, $u \to -\infty$. Thus, this hyperbola traces out the straight line $v = c_2$ towards the right as z travels towards the left.

2.4   Mapping a region

We can use some of the details from the previous example to find the image of a region, rather than a single curve.

Consider the domain $x > 0, y > 0, xy < 1$. This region consists of the upper branches of the hyperbolas

$$2xy = c, \quad 0 < c < 2$$

and we know from the previous example that these hyperbolas map to the straight lines

$$v = c$$

Thus, this region maps to the horizontal strip $0 < v < 2$.

We can also close the domain to contain the curves $x = 0$, $y = 0$, and $xy = 1$. From the function $w = z^2$, we know that the points $(0, y)$ and $(x, 0)$ map to the points $(-y^2, 0)$ and $(x^2, 0)$, so including the two straight lines simply extends the strip to include $v = 0$. Similarly, the hyperbola $xy = 1$ maps to the horizontal line $v = 2$.

Simply put, the image of the closed region $x \geq 0$, $y \geq 0$, $xy \leq 1$ is the closed region $0 \leq v \leq 2$.

2.5   Mapping with polar coordinates

Finally, we can use polar coordinates to simplify some mappings.

Again, consider the mapping $w = z^2$. If we write $z = re^{i\theta}$, then the image point can be written as

$$w = r^2 e^{2i\theta}$$

Looking at the magnitude of w, points on a circle $r = r_0$ are mapped onto a circle $r' = r_0^2$. Also, looking at the argument of w, the angle of the image is doubled. This means that the first quadrant, which is defined as

$$r \geq 0, \quad 0 \leq \theta \leq \pi/2$$

is in a one-to-one mapping with the top plane, $0 \leq \theta \leq \pi$. Simiarly, the top place is mapped onto the entire complex plane (although this is not one-to-one, since the inverse image of the positive real axis is both real axes).

Note that any mapping $w = z^n$ for positive integer n has a similar form, where each non-zero point in the w plane is the image of n distinct points in the z plane.

3 Frame 14 – Mappings by the Exponential Function

Now, we will look at the exponential function

$$e^z = e^{x+iy} = e^x e^{iy}$$

We can again look at straight lines and find their images in this mapping.

Consider the transformation

$$w = e^z = \rho e^{i\phi}$$

where

$$p = e^x \quad \phi = y$$

This means that the image of a vertical line $x = c_1$ is a circle with radius $p = e^{c_1}$. Each point on the circle is the image of infinitely many points, each spaced 2π units apart on the vertical line. Similarly, the horizontal line $y = c_2$ is a ray with an angle of $\phi = c_2$.

With these images in mind, we know that vertical and horizontal line segments are mapped onto arcs and rays, respectively. We can then use this information to map regions:

Now, consider the rectangular region

$$a \le x \le b \quad c \le y \le d$$

The image of this region under the mapping $w = e^z$ is

$$e^a \le \rho \le e^b \quad c \le \phi \le d$$

This is a one-to-one mapping if $d - c < 2\pi$. In particular, the region with $c = 0, d = \pi$ is mapped onto half of a circular ring.

4 Frame 15 – Limits

4.1 Definitions

Suppose that a function f is defined at all points z in some deleted neighborhood of z_0. The statement that the number w_0 is the **limit** of $f(z)$ as z approaches z_0 means that the point $w = f(z)$ can be made *arbitrarily close* to w_0 if we choose z close enough to z_0. We write this as

$$\lim_{z \to z_0} f(z) = w_0$$

To be more precise, if this limit exists, then for each positive number ϵ, there is a positive number δ such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta$$

Geometrically, this definition says that each ϵ neighbourhood around w_0 has a corresponding deleted δ neighbourhood around z_0 such that the image of each point in the δ neighbourhood maps to a point in the ϵ neighbourhood.

Note that the deleted neighbourhood will always exist if z_0 is internal to the domain of definition of f. We can extend the definition of a limit to include boundary points by ignoring all of the neighbourhood's points that are outside the domain.

Also note that this definition only allows a given point to be tested as a limit – it does not provide a method for finding the limit. This will be covered in the next section.

4.2 Uniqueness

If the limit of a function $f(z)$ exists at z_0, it must be unique. To show this, consider two limits:

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} f(z) = w_1$$

This implies that we can find δ_0 and δ_1 such that

$$|f(z) - w_0| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_0$$

and

$$|f(z) - w_1| < \epsilon \text{ whenever } 0 < |z - z_0| < \delta_1$$

Now, suppose that δ is a positive number smaller than both δ_0 and δ_1. Then, for all $0 < |z - z_0| < \delta$, we find that the difference between the two limits is

$$\begin{aligned} |w_1 - w_0| &= ||f(z) - w_0| - |f(z) - w_1|| \\ &\leq |f(z) - w_0| + |f(z) - w_1| \\ &< \epsilon + \epsilon \\ &= 2\epsilon \end{aligned}$$

and since ϵ can be made arbitrarily small, we must have

$$w_1 = w_0$$

4.3   Example – basic limit

Consider the function $f(z) = \frac{i\bar{z}}{2}$. We can show that the limit of this function as $z \to 1$ is

$$\lim_{z \to 1} f(z) = \frac{i}{2}$$

To do this, we observe that

$$\begin{aligned} \left| f(z) - \frac{i}{2} \right| &= \left| \frac{i\bar{z}}{2} - \frac{i}{2} \right| \\ &= \frac{|z - 1|}{2} \end{aligned}$$

Then, we can fulfill the limit definition by writing

$$\left| f(z) - \frac{i}{2} \right| < \epsilon \text{ whenever } |z - 1| < 2\epsilon$$

4.4   Example – direction dependence

In order for w_0 to be a limit of f at z_0, the limit conditions must hold if z approaches z_0 in any arbitrary manner.

Consider the function

$$f(z) = \frac{z}{\bar{z}}$$

Then, the limit

$$\lim_{z \to 0} f(z)$$

does not exist. To illustrate this, the function's value for any non-zero point $z = (x, 0)$ is

$$f(x, 0) = \frac{x + i0}{x - i0} = 1$$

but the value for any non-zero point $z = (0, y)$ is

$$f(0, y) = \frac{0 + iy}{0 - iy} = -1$$

so the limit would not be unique.

5 Frame 16 – Theorems on Limits

Next, it is helpful to connect limits of complex functions and real-valued functions, allowing us to use our knowledge of calculus to simplify the process of finding complex limits

5.1 Splitting into real functions

First, the following theorem is helpful:

Theorem 1. Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and

$$z_0 = x_0 + iy_0, \quad w_0 = u_0 + iv_0$$

Then, the limit

$$\lim_{z \to z_0} = w_0$$

holds iff

$$\lim_{(x,y) \to (x_0,y_0)} u(x, y) = u_0 \text{ and } \lim_{(x,y) \to (x_0,y_0)} v(x, y) = v_0$$

The two implications of this theorem can be proved by considering the definitions of the neighbourhoods as open disks.

5.2 Combining simple limits

Theorem 2. Suppose that

$$\lim_{z \to z_0} f(z) = w_0 \text{ and } \lim_{z \to z_0} F(z) = W_0$$

Then, we can write the following three limits:

$$\lim_{z \to z_0} f(z) + F(z) = w_0 + W_0$$

$$\lim_{z \to z_0} f(z)F(z) = w_0 W_0$$

$$\lim_{z \to z_0} \frac{f(z)}{F(z)} = \frac{w_0}{W_0} \text{ if } W_0 \neq 0$$

These can be proved easily by applying Theorem 1 to each limit.

5.3   Polynomials

Using the basic limit definition from the previous section, it is simple to show that
$$\lim_{z \to z_0} c = c$$
and
$$\lim_{z \to z_0} z = z_0$$
for any complex numbers c and z_0. Then, by the multiplication property,

$$\lim z \to z_0 z^n = z_0^n$$

for any positive integer z. These limits can be used to show that, for any polynomial
$$P(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n$$

the limit as z approaches a point z_0 is the polynomial's value:

$$\lim_{z \to z_0} P(z) = P(z_0)$$

6 Frame 17 – Limits Involving Infinity

6.1 The point at infinity

Sometimes, it is useful to include the **point at infinity** with the complex plane. This point is denoted by ∞. In order to visualize it, the complex plane can be drawn with a unit sphere centered at the origin. Then, a line can be drawn from the top of the sphere (or the *north pole*, denoted by N) to any point on the plane; the line will pass through exactly one other point P on the sphere. This correspondence (between points on the plane, z, and the sphere, P) is called a **stereographic projection**, and the sphere is known as the **Riemann sphere**.

No point in the plane corresponds to the point N. We can let N correspond to the point at infinity, giving us a one-to-one mapping between points on the sphere and points in the extended complex plane.

We will make the distinction that a point z is a point in the finite plane unless we specifically describe the point at infinity – we will specifically mention ∞.

6.2 Neighbourhoods around infinity

Next, we can define neighbourhoods around the point at infinity. Looking at the Riemann sphere, we notice that all of the points P in the upper hemisphere project to points z outside of the unit disk.

Further, if ϵ is a small, positive number, then points in the plane such that

$$|z| > \frac{1}{\epsilon}$$

correspond to points on the sphere close to N. Thus, we call the set $|z| > 1/\epsilon$ an (ϵ) **neighbourhood** of ∞.

6.3 Limits with infinity

With this new point at infinity, we can give meaning to the statement

$$\lim_{z \to z_0} f(z) = w_0$$

when z_0 or w_0 are infinity. We can then use the following theorems:

$$\lim_{z \to z_0} f(z) = \infty \iff \lim_{z \to z_0} \frac{1}{f(z)} = 0$$

$$\lim_{z \to \infty} f(z) = w_0 \iff \lim_{z \to 0} f\left(\frac{1}{z}\right) = w_0$$

$$\lim_{z \to \infty} f(z) = \infty \iff \lim_{z \to 0} \frac{1}{f(1/z)} = 0$$

6.4   Examples

Three limits using these new properties follow.

- To find
$$\lim_{z \to -1} \frac{iz + 3}{z + 1}$$
we notice that
$$\lim_{z \to -1} \frac{z + 1}{iz + 3} = 0$$
so the limit is infinity.

- To find
$$\lim_{z \to \infty} \frac{2z + i}{z + 1}$$
we evaluate
$$\lim_{z \to 0} \frac{(2/z) + i}{(1/z) + 1} = \lim_{z \to 0} \frac{2 + iz}{1 + z}$$
$$= 2$$

- To find
$$\lim_{z \to \infty} \frac{2z^3 - 1}{z^2 + 1}$$
we evaluate
$$\lim_{z \to 0} \frac{(1/z^2) + 1}{(2/z^3) - 1} = \lim_{z \to 0} \frac{z + z^3}{2 - z^3}$$
$$= 0$$
so the original limit is infinity.

13

7 Continuity

7.1 Definitions

A function is **continuous** at a point z_0 if all of the three following conditions are true:

$$\lim_{z \to z_0} f(z) \text{ exists}$$

$$f(z_0) \text{ exists}$$

$$\lim_{z \to z_0} f(z) = f(z_0)$$

This final statement says that for each positive number ϵ there is a positive number δ such that

$$|f(z) - f(z_0)| < \epsilon \text{ whenever } |z - z_0| < \delta$$

We say that a function is said to be continuous in a region R if it is continuous at each point in R.

7.2 Theorems

The basic limit identities allow us to find the continuity of more complex functions. If two functions are continuous at a point, their sum, product, and quotients are also continuous (in the last case, provided that the denominator is non-zero). A polynomial is continuous in the entire plane.

A complex function that can be split into its real and imaginary components

$$f(z) = u(x, y) + iv(x, y)$$

is continuous at $z_0 = (x_0, y_0)$ iff u and v are continuous at z_0.

We can state three more theorems about continuity:

- A composition of continuous functions is continuous.

  Suppose that $w = f(z)$ is defined in a neighbourhood of z_0 and $W = g(w)$ is defined in a neighbourhood of $f(z_0)$. Also, suppose that f is continuous at z_0 and g is continuous at $f(z_0)$. Then, the statement that the composition

  $$g[f(z)]$$

  is continuous is equivalent to the statement that

  $$|g[f(z)] - g[f(z_0)]| < \epsilon \text{ whenever } |f(z) - f(z_0)| < \gamma$$

  Then, since f is continuous, we can find a δ such that the right side is satisfied, so $g \circ f$ is continuous.

- If f is continuous and non-zero at z_0, then there is some neighbourhood of z_0 where $f(z) \neq 0$.

  Suppose that we choose $\epsilon = |f(z_0)|/2$. Then, if there is a point where $f(z) = 0$ in a δ neighbourhood around z_0, the limit inequality is

  $$|f(z_0)| < \frac{|f(z_0)|}{2}$$

  so we have a contradiction, and there must be a neighbourhood where $f(z) = 0$.

- If f is continuous in a closed and bounded region R, then there exists a non-negative real number M such that

  $$|f(z)| \leq M$$

  where equality holds for one or more z.

8 Frame 19 – Derivatives

8.1 Definitions

Suppose that a function f is defined on a neighbourhood $|z - z_0| < \epsilon$. We define the **derivative** of f at z_0 as the limit

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

or, if we write $\Delta z = z - z_0$, then the definition can be written as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

We say that f is **differentiable** at z_0 when $f'(z_0)$ exists.

As an augmented notation, we often write

$$\Delta w = f(z + \Delta z) - f(z)$$

and we use the derivative notation

$$\frac{dw}{dz} = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

8.2 Examples

Example: suppose $f(z) = z^2$. Then,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} 2z + \Delta z = 2z$$

Example: suppose $f(z) = \overline{z}$. Then,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{z} + \overline{\Delta z} - \overline{z}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\overline{\Delta z}}{\Delta z}$$

and we showed earlier that this limit does not exist – it has different values depending on the angle of approach.

Example: suppose $f(z) = |z|^2$. Then,

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = \lim_{\Delta z \to 0} \frac{|z + \Delta z|^2 - |z|^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{(z + \Delta z)(\overline{z + \Delta z} - z\overline{z})}{\Delta z} = \lim_{\Delta z \to 0} \overline{z} + \overline{\Delta z} + z \frac{\overline{\Delta z}}{\Delta z}$$

We see, once again, that the $\frac{\overline{\Delta z}}{\Delta z}$ term stops us from obtaining a derivative. However, this term is not present if $z = 0$ – at this point, dw/dz is zero. Thus, the derivative exists only at $z = 0$, and $f'(0) = 0$.

8.3 Notes

We've seen that it is possible for a function to be differentiable at a point but nowhere else in a neighbourhood of that point. We can even note that our function, $f(z) = |z|^2$, can be written as

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0$$

so we have no trouble finding the partial derivatives, but the function is still not differentiable aside from $z = 0$.

We've also seen that our troublesome function, $f(z) = |z|^2$, was continuous but not differentiable. However, we can state that **differentiability implies continuity** at a point. We can see this easily in the numerator of the derivative limit: $f(z)$ must approach the point $f(z_0)$ as z approaches z_0, which implies continuity.

9 Frame 20: Differentiation Formulas

The derivative definition in the previous section is extremely similar to the definition for real-valued functions. Similar basic differentiation formulas can be derived and used for complex functions. This section will cover the most useful formulas.

If c is a complex constant and f is a function that is differentiable at z, then:

$$\frac{d}{dz}c = 0$$

$$\frac{d}{dz}z = 1$$

$$\frac{d}{dz}[cf(z)] = cf'(z)$$

If n is a non-zero integer,

$$\frac{d}{dz}z^n = nz^{n-1}$$

If two functions f and g are differentiable at z, then

$$\frac{d}{dz}[f(z) + g(z)] = f'(z) + g'(z)$$

$$\frac{d}{dz}[f(z)g(z)] = f(z)g'(z) + f'(z)g(z)$$

$$\frac{d}{dz}\left[\frac{f(z)}{g(z)}\right] = \frac{f'(z)g(z) - g'(z)f(z)}{g^2(z)} \text{ if } g(z) \neq 0$$

Finally, when composing functions (such as $F(z) = g[f(z)]$), the standard chain rule applies:

$$F'(z_0) = g'[f(z_0)]f'(z_0)$$

or, in a different form, if $w = f(z)$ and $W = F(z)$,

$$\frac{dW}{dz} = \frac{dW}{dw}\frac{dw}{dz}$$

10 Frame 21: Cauchy-Riemann Equations

10.1 Basic proof

We will obtain a formula in this chapter that a function must satisfy at a point if its derivative exists there.

First, we can split up the following points into real and imaginary parts:

$$z_0 = x_0 + iy_0, \quad \Delta z = \Delta x + i\Delta y$$

Then, the term Δw becomes

$$\begin{aligned}
\Delta w &= f(z_0 + \Delta z) - f(z_0) \\
&= [u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)] + i[v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0)]
\end{aligned}$$

Next, we know that we can split up the derivative

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

into its real and imaginary components

$$f'(z_0) = \lim_{\Delta z \to 0} \left(\Re \frac{\Delta w}{\Delta z} \right) + i \lim_{\Delta z \to 0} \left(\Im \frac{\Delta w}{\Delta z} \right)$$

We've seen several times that this expression must hold as Δz tends to zero in any direction. In particular, we can use the test points $(\Delta x, 0)$ and $(0, \Delta y)$ to test this derivative. In the first case, $\Delta y = 0$, so the derivative is

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\
&= u_x(x_0, y_0) + iv_x(x_0, y_0)
\end{aligned}$$

where u_x and v_x are the partial derivatives with respect to x. Similarly, we can set $\Delta x = 0$ and obtain

$$\begin{aligned}
f'(z_0) &= \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{i\Delta y} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{i\Delta y} \\
&= \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta y} - i \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta y} \\
&= v_y(x_0, y_0) - iu_y(x_0, y_0)
\end{aligned}$$

where u_y and v_y are the partial derivatives with respect to y.

Now, since these two values must be equal, we require two conditions for the derivative to exist

$$u_x(x_0, y_0) = v_y(x_0, y_0)$$

and

$$u_y(x_0, y_0) = -v_x(x_0, y_0)$$

These are the **Cauchy-Riemann equations**. They are necessary conditions for a derivative to exist – if they are not satisfied at a point, the derivative does not exist there. Note, however, that the derivative is not guaranteed to exist – the Cauchy-Riemann equations are not a sufficient condition.

10.2   Formal theorem

We can summarize the above result.

Suppose that

$$f(z) = u(x, y) + iv(x, y)$$

and that $f'(z)$ exists at a point $z_0 = x_0 + iy_0$. Then, the partial derivatives of u and v must exist at (x_0, y_0), and they must satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

there. Also, the derivative can be written as

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0).$$

10.3   Examples

Example: to illustrate the Cauchy-Riemann equations, consider the function

$$f(z) = z^2 = (x^2 - y^2) + i(2xy)$$

The real and imaginary functions are

$$u(x, y) = x^2 - y^2, \quad v(x, y) = 2xy$$

and their partial derivatives are

$$u_x = 2x = v_y, \quad u_y = -2y = -v_x$$

Finally, we can write

$$f'(z) = u_x + iv_x = 2x + i2y = 2z$$

as expected.

Example: for the function

$$f(z) = |z|^2$$

we have the real and imaginary components

$$u(x, y) = x^2 + y^2, \quad v(x, y) = 0$$

In order for the Cauchy-Riemann equations to hold at a point, we require that

$$u_x = 2x = 0$$
$$u_y = 2y = 0$$

or, in other words, the derivative does not exist anywhere except possibly for the point $(0, 0)$.

11 Frame 22 – Sufficient Conditions for Differentiability

11.1 Extensions on Cauchy-Riemann

In the previous frame, we derived the Cauchy-Riemann equations, which are necessary for differentiability at a point. However, they are not sufficient. We can add some continuity conditions to improve the theorem as follows:

Suppose that a function

$$f(z) = u(x, y) + iv(x, y)$$

is defined in some ϵ neighbourhood of a point z_0. If the partial derivatives u_x, u_y, v_x, and v_y exist everywhere in the neighbourhood, are continuous at z_0, and satisfy the Cauchy-Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

then the derivative $f'(z_0)$ exists, and it has the value

$$f'(z_0) = u_x(z_0) + iv_x(z_0)$$

11.2 Examples

Example: suppose that

$$f(z) = e^z = e^x e^{iy} = e^x \cos y + ie^x \sin y$$

Then, the partial derivatives are

$$u_x = e^x \cos y = v_y, \quad u_y = -e^x \sin y = -v_x$$

These derivatives are continuous everywhere and the Cauchy-Riemann equations are satisfied, so $f'(z_0)$ exists everywhere, and it has the value

$$f'(z) = e^x \cos y + ie^x \sin y = e^z$$

Example: suppose that

$$f(z) = |z|^2 = (x^2 + y^2) + i0$$

We saw before that this function has a derivative at $z = 0$. However, at all other points, the Cauchy-RIemann equations are not satisfied, so the derivative does not exist anywhere else.

12 Frame 23 – Polar Coordinates

12.1 Basic concepts

We can restate the theorem from the previous section using the polar coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$

If we are using this choice of coordinates, then our goal is to express the first-order derivatives $u_r, u_\theta, \ldots$ in terms of $u_x, u_y, \ldots$ to allow us to rewrite the Cauchy-Riemann equations. We can do this using the chain rule:

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}, \quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

or, more simply,

$$u_r = u_x \cos\theta + u_y \sin\theta, \quad u_\theta = -u_x r \sin\theta + u_y r \cos\theta$$

The same applies to v. If the Cauchy-Riemann equations hold, then we require that

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

at z_0. The reverse is also true – if these hold, then the Cauchy-Riemann equations hold too – so these equations are equivalent.

12.2 Formal theorem

The above concepts can be stated as follows:

Suppose that the function

$$f(z) = u(r, \theta) + iv(r, \theta)$$

is defined in some ϵ neighbourhood of $z_0 = r_0 e^{i\theta_0}$ ($r_0 \neq 0$). If the partial derivatives of u and v are continuous throughout the neighbourhood and they satisfy the polar Cauchy-Riemann equations

$$ru_r = v_\theta, \quad u_\theta = -rv_r$$

then $f'(z_0)$ exists and has the value

$$f'(z_0) = e^{-i\theta}(u_r + iv_r)$$

12.3 Examples

Example: consider the function

$$f(z) = \frac{1}{z} = \frac{\cos\theta}{r} - i\frac{\sin\theta}{r}$$

Since the derivatives are

$$ru_r = -\frac{\cos\theta}{r} = v_\theta, \quad u_\theta = -\frac{\sin\theta}{r} = v_r$$

the derivative of f exists for all $z \neq 0$, with the value

$$f'(z) = e^{-i\theta}\left(-\frac{\cos\theta}{r^2} + i\frac{\sin\theta}{r^2}\right) = -\frac{1}{(re^{i\theta})^2} = -\frac{1}{z^2}$$

Example: suppose that $r > 0$ and α is a constant. Then, the function

$$f(z) = r^{1/3}e^{i\theta/3} = r^{1/3}\cos\frac{\theta}{3} + ir^13\sin\frac{\theta}{3} \quad (r > 0, \alpha < \theta < \alpha + 2\pi)$$

has a derivative everywhere in the domain. Here,

$$ru_r = \frac{r^{1/3}}{3}\cos\frac{\theta}{3} = v_\theta, \quad u\theta = -\frac{r^{1/3}}{3}\sin\frac{\theta}{3} = -rv_r$$

so the derivative is

$$f'(z) = e^{-i\theta}\left[\frac{1}{3r^{3/2}}\cos\frac{\theta}{3} + i\frac{1}{3r^{3/2}}\sin\frac{\theta}{3}\right]$$
$$= \frac{1}{3(r^{1/3}e^{i\theta/3})^2}$$
$$= \frac{1}{3f^2(z)}$$