1 Frame 29 – The Exponential Function

1.1 Definition

We define the **exponential function** e^z by writing

$$e^z = e^x e^{iy}$$

and we apply Euler's formula to get

$$e^z = e^x(\cos y + i \sin y)$$

Note that, when $y = 0$, e^z reduces to e^x.

Although we typically understand that $e^{1/n}$ would be the set of nth roots of e, here, we only use the real, positive root $\sqrt[n]{e}$.

1.2 Familar properties

First, in calculus, we know that

$$e^{x_1} e^{x_2} = e^{x_1 + x_2}$$

It is easy to verify that this holds true for complex numbers:

$$e^{z_1} e^{z_2} = e^{z_1 + z_2}$$

This also allows us to write

$$\frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$$

and, as a specific case,

$$\frac{1}{e^z} = e^{-z}$$

We showed earlier that e^z is differentiable everywhere in the complex plane, and that

$$\frac{d}{dz} e^z = e^z$$

We also know that e^z is never zero. This comes from the pair

$$|e^z| = e^x \quad \text{and} \quad \arg(e^z) = y + 2n\pi$$

and since e^x is never zero, neither is e^z.

1.3 Unfamiliar properties

Since we can write

$$e^{z+2\pi i} = e^z e^{2\pi i} = e^z$$

the exponential function is periodic with an imaginary period of $2\pi i$.

It is also possible for the complex exponential function to be negative. For an example, we know that Euler's identity states

$$e^{i\pi} = -1$$

In fact, e^z can be any given non-zero complex number.

Example: suppose we want solutions to the equation

$$e^z = 1 + i$$

The right side can be rewritten as

$$e^x e^{iy} = \sqrt{2} e^{i\pi/4}$$

and equating the parts of this equation gives

$$x = \ln\sqrt{2} = \frac{1}{2}\ln 2 \quad \textit{and} \quad y = \left(2n + \frac{1}{4}\right)\pi$$

so

$$z = \frac{1}{2}\ln 2 + \left(2n + \frac{1}{4}\right)\pi i$$