1 Frame 56 – Sequences and Series

3 If

$$\lim_{n\to\infty} z_n = z$$

then, for some integer n_0, all of the terms z_k $(k > n_0)$ will be in some ϵ neighbourhood of z; ie:

$$|z_k - z| < \epsilon$$

However,

$$||z_k| - |z| \le \epsilon$$

so all of the terms $|z_k|$ must be inside the same ϵ neighbourhood of $|z|$, and we can say that

$$\lim_{n\to\infty} |z_n| = |z|$$

4 Starting from the series

$$\sum_{n=1}^{\infty} z^n = \frac{1}{1-z} - 1 = \frac{z}{1-z}$$

the components of this expression can be written as

$$\begin{aligned}
\frac{z}{1-z} &= \frac{r\cos\theta + ir\sin\theta}{1 - r\cos\theta - ir\sin\theta} \\
&= \frac{r\cos\theta - r^2\cos^2\theta - r^2\sin^2\theta}{(1-r\cos\theta)^2 + r^2\sin^2\theta} + i\frac{r\sin\theta - r^2\sin\theta\cos\theta + r^2\sin\theta\cos\theta}{(1-r\cos\theta)^2 + r^2\sin^2\theta} \\
&= \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2} + i\frac{r\sin\theta}{1-2r\cos\theta + r^2}
\end{aligned}$$

so, equating the real and imaginary parts of the sum,

$$\sum_{n=1}^{\infty} r^n \cos n\theta = \frac{r\cos\theta - r^2}{1 - 2r\cos\theta + r^2}$$

and

$$\sum_{n=1}^{\infty} r^n \sin n\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$