Problem Sequence - Solutions

This document will be filled up with the solutions from the problem sequence.

Solution 1. (Greg)
From Axiom 1, there are three cases to consider:

1. $p = 0$

   p is equal to the only element of M. From the definition of limit point, every segment containing p must contain a point of M different from p. However, there are no points in M different from p, so p must not be a limit point of M.

2. $p > 0$

   From Axiom 3, there exists a point a such that $0 < a < p$. There also exists a point b such that $b > p$. Since $p > 0$ and $b > p$, Axiom 2 tells us that $b > 0$. We can then form the segment $S = (a, b)$. Since $a > 0$ and $b > 0$, 0 is not between a and b, so S does not contain 0. However, $a < p < b$, so S contains p. S is a segment containing p that does not contain any element of M, so p is not a limit point of M.

3. $p < 0$

   (symmetric to the $p > 0$ case)

Therefore, regardless of our choice of p, we can construct a segment that contradicts the requirements in the limit point definition, so p is not a limit point of M.

Solution 2. (Jeff)
According to Definition 4, we can prove that p is not a limit point of M if we can construct a segment containing p but not a different point of M. Construct this segment as follows:

1. If all points of M are on the opposite side of p, choose any value as an endpoint.

2. If any point of M is on the same side of p, choose a point between the nearest point of M and p. (Axiom 3 confirms that there will be such a point.)

This segment contains p but will not contain any points of M (with the exception of p, if p is 0 or 1). Therefore, we have found a segment that does not fulfill the requirements of Definition 4, so p is not a limit point of M.

Solution 3. (Erin, solved after Problem 4)
Let $a = 0$ and $b = 1$. Then, according to Problem 4, b is a limit point of (a, b). Therefore, 1 is a limit point of $(0, 1)$.

Solution 4. (Erin)

We will prove that b is a limit point of (a, b); The proof for a is similar.

Construct a segment (p, q) containing b ($p < b < q$). According to Axiom 1, there are three cases, and we will deal with two of them simultaneously:

1. $p < a$

   In this case, $p < a < b < q$, so $p < (a, b) < q$. Since (p, q) contains every point of (a, b), we have found points that satisfy Defintion 4.

2. $p \geq a$

   In this case, $a \leq p < b < q$. According to Axiom 3, there is a point d between p and b. The inequality is then $a \leq p < d < b < q$. Then,

   - $a < d < b$, so d is in (a, b).
   - $p < d < q$, so d is in (p, q).

   This means that (p, q) contains d, which is a point of (a, b).

Therefore, every (p, q) containing b also contains a point of (a, b), so b is a limit point of (a, b).

Solution 5. (Greg)

Choose any point p from S. Construct a segment (x, y) that contains p (ie: $x < p < y$). Put no other condition on y. According to Axiom 1, one of these three cases is true:

1. $x > a$

   Choose a point q between x and p ($x < q < p$; this exists by Axiom 3). Apply Axiom 2 three times:

   - $a < x$ and $x < q$, so $a < q$
   - $q < p$ and $p < b$, so $q < b$
   - $q < p$ and $p < y$, so $q < y$

   so $a < q < b$ and $x < q < y$. Therefore, q is an element of S inside (a, b) that is different from p.

2. $x = a$

   Repeat the proof for $x > a$ with one change:

   - $a = x$ and $x < q$, so $a < q$

3. $x < a$

   Choose q betweem a and p. Change:

   - $x < a$ and $a < q$, so $x < q$

(same conclusion as $x > a$)

In each of these three cases, every (x, y) contains a point q from (a, b). Therefore, we have satisfied Definition 4, so p is a limit point of (a, b).

Solution 6. (Greg)

There are two cases: if p is not a point of $[a, b]$, then either $p < a$ or $p > b$. We will only consider the first case; the proof for the second follows identically.

Construct two points x and y such that $x < p$ and $p < y < a$ (Axiom 3 guarantees that y exists) and consider the segment (x, y). Since $x < p < y$, (x, y) contains p.

Then, take any point i from $[a, b]$. This requires that $a \leq i$. From Axiom 2, since $y < a$ and $a \leq i$, $y < i$. Then, since $i \not< y$, (x, y) does not contain i.

Therefore, (x, y) is a segment containing p but no points of $[a, b]$. This segment contradicts the definition of a limit point, so $p < a$ is not a limit point of $[a, b]$.

Solution 7. (Erin)

If p is a limit point of H, then any segment S which contains p also contains a different point of H. Since $H \subseteq K$, this point is also a point of K. Therefore, S also contains a different point of K, and p is a limit point of K.

Solution 8. (Greg)

Consider a point p. According to Axiom 4, there exists a largest integer M_x and a smallest integer M_y such that $M_x < p < M_y$. Then, choose points x and y from Axiom 3 such that $M_x < x < p$ and $p < y < M_y$, and consider the segment $S = (x, y)$.

We will try to find an integer different from p inside S. Axiom 1 gives us three cases:

1. $n = p$ (note: this is only possible if p is an integer)

   Here, n is not different from p, so we have not found an integer different from p.

2. $n < p$

   Note that n must satisfy $n \leq M_x$; if $n > M_x$, we have contradicted Axiom 4. Then, from Axiom 2, $n \leq M_x$ and $M_x < x$, so $n < x$. This shows that n is not between x and y, so S does not contain n.

3. $n > p$

   This case is symmetric to the $n < p$ case.

Therefore, S is a segment containing p that does not contain any integers different from p. Since every element of M is an integer, we have proven that there exists a segment S for every point p without any other points of M, so M has no limit points.

Solution 9. (Rayne)

Suppose that S does not contain 2 points of M. We will find a contradiction.

Let S be the segment (a, b) which contains p (ie: $a < p < b$). According to Definition 4, it must also contain a point m of M that is different from p. We will assume that $m < p$ (the case $m > p$ is similar).

We have $a < m < p < b$. According to Axiom 3, we can create another segment that contains p but not m. Then, this new segment does not contain any points of M, so p must not be a limit point of M. However, we know that p is a limit point of M, so we have a contradiction, and S must contain 2 or more points of M.

Solution 10. (Greg)

If S is a segment containing p, then S contains a point of K different from p. Since every point of K is a limit point of H, S contains a limit point of H different from p. According to Problem 9, if S contains a limit point of H, S also contains 2 different points of H. If p is equal to one of these points, it will be different from the other one. Therefore, S contains a point of H different from p, satisfying Definition 4, and p is a limit point of H.

Solution 11. (Greg)

Construct a segment $S = (x, y)$ that contains 0 ($x < 0 < y$). Since the reciprocal of a positive number is positive, every element M_i of M is positive, so $x < M_i$ from Axiom 2.

Then, we will attempt to find an element of M that is less than y. Since every element of M is of the form $\frac{1}{n}$ for some integer n, we are looking for $\frac{1}{n} < y$. From the properties of reciprocals, this is equivalent to $\frac{1}{y} < n$. According to Axiom 4, the point $\frac{1}{y}$ has an integer greater than it, so we can find an n that satisfies this inequality.

Therefore, every S containing 0 also contains a point of M, so 0 is a limit point of M.

Solution 12. (Erin)

We will attack this problem in several cases:

1. $p < 0$

   Construct a segment (a, b) that contains both p and 0. These points satisfy $a < p < 0 < b$. From Axiom 3, we can find another point q such that $p < q < 0$. Then, the segment (a, q) contains p, but no points of M (since every point of M is positive), so p must not be a limit point of M.

2. $p = 1$

   Construct an (a, b) containing p and $\frac{1}{2}$. The inequality is then $a < \frac{1}{2} < p < b$. Axiom 3: can find q such that $\frac{1}{2} < q < p$ Then, (q, b) contains p but no point of M other than p (since every other point of M is less than $\frac{1}{2}$, so p must not be a limit point of M.

3. $p > 1$

   Construct (a, b) containing p and 1. ($a < 1 < p < b$) Axiom 3: can find q such that $1 < q < p$. Then, (q, b) contains p but no points of M.

4. $0 < p < 1$

   First, note that if n is an integer, then $n - 1 < n < n + 1$ (Axiom 5). Then, $\frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1}$. According to Axiom 3, we can find $\frac{1}{a}, \frac{1}{b}$ such that $\frac{1}{n+1} < \frac{1}{a} < \frac{1}{n} < \frac{1}{b} < \frac{1}{n-1}$. We have two cases from here:

   - $p = \frac{1}{n}$
     $(\frac{1}{a}, \frac{1}{b})$ contains p but no other point of M, so p is not a limit point of M.
   - $\frac{1}{n+1} < p < \frac{1}{n}$
     From Axiom 3, we can find $\frac{1}{c}$ such that $p < \frac{1}{c} < \frac{1}{n}$. Then, $(\frac{1}{a}, \frac{1}{c})$ contains p but no other point of M, so p is not a limit point of M.

These cases cover every point $p \neq 0$. Therefore, if p is not zero, p is not a limit point of M.

Solution 13. (Erin)
First, note that the set $H \cap K$ is a subset of H because every point of $H \cap K$ must be a point of H. Similarly, $H \cap K \subseteq K$.

Then, from Problem 7, since p is a limit point of $H \cap K$, p is also a limit point of H; identically, p is a limit point of K.

Solution 14. (Jeff)
We will use a counterexample to prove that not every interval containing a limit point must contain another point of the set.

Consider the set M of the reciprocals of all positive integers. (This was the set discussed in problems 11 and 12.) We know from Problem 11 that 0 is a limit point of M, so pick the interval $I = [i, 0]$ for any $i < 0$ and let $p = 0$.

Since $p = 0$, which is an endpoint of I, I contains p. However, every point m of M is positive ($m > 0$), and is not contained by I. Therefore, I is an interval which contains a limit point of M and no other points of M. This shows that not every interval containing a limit point must contain a different point of the set.

Solution 15. (Greg)
Consider an arbitrary segment $S = (S_a, S_b)$ that contains p (so $S_a < p < S_b$). From Axiom 3, we can find two more points I_a, I_b such that $S_a < I_a < p$ and $p < I_b < S_b$.

Then, we can construct the interval $I = [I_a, I_b]$. Note that every point of I is also in S - ie, $I \subseteq S$. Since I contains p, it must contain a point of M different from p. Therefore, S also contains a point of M different from p, and p is a limit point of M.

Solution 16. (Fernando)

Let $S_1 = (a_1, b_1)$ and $S_2 = (a_2, b_2)$. Since p is in both segments, $a_1 < p < b_1$ and $a_2 < p < b_2$. From Axiom 2, we know that:

- $a_1 < b_1$
- $a_1 < b_2$
- $a_2 < b_1$
- $a_2 < b_2$

Then, there are only 4 segments that could be the result of the intersection $S_1 \cap S_2$:

- (a_1, b_1)
- (a_1, b_2)
- (a_2, b_1)
- (a_2, b_2)

Since p is in all of these segments, we have shown that the resulting segment will always contain p.

Solution 17. (Greg)

p is not a limit point of H, so there is a segment $S_H = (H_a, H_b)$ that contains p but no point of H different from p. Likewise, there is a segment $S_K(K_a, K_b)$ that contains p but no different point of K. From these two segments, construct $S = S_H \cap S_K$.

Since p was in both of the original segments, Problem 16 tells us that S contains p. However, since there were no elements of H or K in these segments (except possibly p), S contains no elements of H or K. Finally, because every element of $H \cup K$ is an element of H or K, we know that S contains no elements of $H \cup K$.

Therefore, S is a segment that contains p but no other elements of $H \cup K$. We have found a counterexample to the definition of a limit point, so p is not a limit point of $H \cup K$.

Solution 18. (Greg)

We will use a proof by contrapositive. The contrapositive is:

Show that if p is not a limit point of H and p is not a limit point of K, then p is not a limit point of $H \cup K$.

This is problem 17 (which has been proved), so we are finished.

Solution 19. (Erin)

Consider a subset of M that contains n points. Since this subset has a finite number of points, we can order them $x_0, x_1, x_2, \ldots, x_n$ such that $x_0 < x_1 < x_2 < \cdots < x_n$.

However, since there is no largest number in M, we can find another point x_{n+1} from M such that $x_n < x_{n+1}$. We can repeat this process indefinitely, so for any positive integer n, we can find at least n points in M, making M an infinite set.

Solution 20. (Fernando)

Since M is finite, there is a positive integer n such that M does not have n points. We will then say that the number of points in M is not greater than $n-1$.

If M has only one point, then we can say that its one point is the largest point in M. This is our base case.

Then, if M has more than one point, we can use the following algorithm to find the largest point:

- Choose two different points a_1 and a_2 from M. Then, either $a_1 > a_2$ or $a_1 < a_2$. Pick the larger of these two points and keep track of it. (We will assume that a_1 was the larger of the two.)
- Choose another point a_3 from M that is different from a_1 and a_2. Then, either $a_3 > a_1$ or $a_3 < a_1$. If $a_3 > a_1$, from Axiom 2, $a_3 > a_2$, so a_3 is the new largest point. If $a_3 < a_1$, then a_1 remains the largest.

Since M can have no more than $n-1$ points, we can then find the largest point using $n-2$ of these steps. Therefore, there must be a largest point of M.

Solution 21. (Erin)

According to Problem 9, S must contain at least two points of M different from p. Label these two points q and s. Assume that q is less than p - a similar argument applies if $q > p$.

If S is the segment (a, b) with $a < p < b$, then we can see that $a < q < p$. Then, we define a new segment (c, d) where $q < c < p$ and $p < d < b$. This new segment does not contain q. However, it still contains p, so there must be another point of M different from p inside (c, d).

Since every point of (c, d) is also in (a, b), the new point that we have found is also in (a, b). We can repeat this process indefinitely, so S must contain infinitely many points of M.

Solution 22. (no solution required)

Solution 23. (Erin)

First, let n be an even positive integer. Then, $x_n = 0$ and $x_{n+1} = 1$. If the points a, b satisfy $a < 0 < b < 1$, then the segment (a, b) does not contain x_{n+1}. (Note that this does not depend on n.)

Then, let n be an odd positive integer, so $x_n = x_{n+2} = 1$. The segment (a, b) does not contain x_{n+2}.

Therefore, for every positive integer n, either x_{n+1} or x_{n+2} is not in (a, b), so we cannot find an n that satisfies Definition 11, and the sequence does not converge to 0.

Solution 24. (no solution required)

Solution 25. (Erin)

First, note that 'not infinite' means the same thing as 'finite'.

Since the sequence is convergent, there is a positive integer n such that $x_n, x_{n+1}, x_{n+2}, \ldots$ are all in (a, b). This does not imply anything about the points $x_1, x_2, x_3, \ldots, x_{n-1}$, so these points may or may not be in (a, b). Counting these points tells us that there cannot be more than $n-1$ points of the sequence outside of (a, b). Therefore, there are not n points of the sequence not in (a, b), so this set of points is finite.

Solution 26. (Rayne)

Suppose that n is an odd positive integer - the proof for even n is similar.

Since the sequence is defined as

$$x_n = \begin{cases} \frac{1}{n-1}, & n \text{ odd} \\ \frac{1}{n+1}, & n \text{ even} \end{cases}$$

we can write out several terms of the sequence as

odd	even	odd	even
x_n	x_{n+1}	x_{n+2}	x_{n+3}
$\frac{1}{n+1}$	$\frac{1}{n}$	$\frac{1}{n+3}$	$\frac{1}{n+2}$

We can see from this sample that $x_{n+2} < x_{n+3} < x_n < x_{n+1}$. Then, if we make a segment (a, b) containing 0, x_n will be in this segment for large enough n. Since all consecutive terms of the sequence starting at $n+2$ are between 0 and b, then we can see that the definition of convergence is satisfied, and the sequence must converge to 0.

Solution 27. (Greg)

We can find three points a_1, a_2, a_3 such that

$$a_1 < c < a_2 < d < a_3$$

(Axiom 3 guarantees that a_2 exists). Construct the segments $S_c = (a_1, a_2)$ and $S_d = (a_2, a_3)$.

Notice that any point p in S_c has $p < a_2$ and any point q in S_d has $q > a_2$. If $p = q$, we contradict Axiom 1, so no point is in both S_c and S_d.

The sequence $x_1, x_2, x_3, \ldots$ converges to c, so there is a positive integer n such that x_m is in S_c for all $m \geq n$. From above, x_m is not in S_d for all $m \geq n$. Therefore, we cannot find an n that satisfies Definition 11 for S_d, and the sequence does not converge to d.

Solution 28.

Solution 29.

Solution 30.

Solution 31. (no solution required)

Solution 32.

Solution 33. (Greg)

We will assume that M has both a rightmost point and a smallest number largest points, and we will find a contradiction.

If the largest point of M is labelled x, then for any point p in M, $p \leq x$. Also, if the smallest point greater than every point in M is y, then for every p in M, $p < y$. Note that since x is in M, $x < y$.

According to Axiom 3, there exists a point z such that $x < z < y$. Since $x < z$, Axiom 2 tells us that every point of M is less than z. However, z is smaller than y, so y is not the smallest point greater than every point of M, and we have a contradiction.

Solution 34.