

## Problem Sequence - Solutions

This document will be filled up with the solutions from the problem sequence.

### **Solution 1.** (Greg)

From Axiom 1, there are three cases to consider:

1.  $p = 0$

$p$  is equal to the only element of  $M$ . From the definition of limit point, every segment containing  $p$  must contain a point of  $M$  different from  $p$ . However, there are no points in  $M$  different from  $p$ , so  $p$  must not be a limit point of  $M$ .

2.  $p > 0$

From Axiom 3, there exists a point  $a$  such that  $0 < a < p$ . There also exists a point  $b$  such that  $b > p$ . Since  $p > 0$  and  $b > p$ , Axiom 2 tells us that  $b > 0$ . We can then form the segment  $S = (a, b)$ . Since  $a > 0$  and  $b > 0$ , 0 is not between  $a$  and  $b$ , so  $S$  does not contain 0. However,  $a < p < b$ , so  $S$  contains  $p$ .  $S$  is a segment containing  $p$  that does not contain any element of  $M$ , so  $p$  is not a limit point of  $M$ .

3.  $p < 0$

(symmetric to the  $p > 0$  case)

Therefore, regardless of our choice of  $p$ , we can construct a segment that contradicts the requirements in the limit point definition, so  $p$  is not a limit point of  $M$ .

### **Solution 2.** (Jeff)

According to Definition 4, we can prove that  $p$  is not a limit point of  $M$  if we can construct a segment containing  $p$  but not a different point of  $M$ . Construct this segment as follows:

1. If all points of  $M$  are on the opposite side of  $p$ , choose any value as an endpoint.
2. If any point of  $M$  is on the same side of  $p$ , choose a point between the nearest point of  $M$  and  $p$ . (Axiom 3 confirms that there will be such a point.)

This segment contains  $p$  but will not contain any points of  $M$  (with the exception of  $p$ , if  $p$  is 0 or 1). Therefore, we have found a segment that does not fulfill the requirements of Definition 4, so  $p$  is not a limit point of  $M$ .

### **Solution 3.** (Erin, solved after Problem 4)

Let  $a = 0$  and  $b = 1$ . Then, according to Problem 4,  $b$  is a limit point of  $(a, b)$ . Therefore, 1 is a limit point of  $(0, 1)$ .

**Solution 4.** (Erin)

We will prove that  $b$  is a limit point of  $(a, b)$ ; The proof for  $a$  is similar.

Construct a segment  $(p, q)$  containing  $b$  ( $p < b < q$ ). According to Axiom 1, there are three cases, and we will deal with two of them simultaneously:

1.  $p < a$ 

In this case,  $p < a < b < q$ , so  $p < (a, b) < q$ . Since  $(p, q)$  contains every point of  $(a, b)$ , we have found points that satisfy Definition 4.

2.  $p \geq a$ 

In this case,  $a \leq p < b < q$ . According to Axiom 3, there is a point  $d$  between  $p$  and  $b$ . The inequality is then  $a \leq p < d < b < q$ . Then,

- $a < d < b$ , so  $d$  is in  $(a, b)$ .
- $p < d < q$ , so  $d$  is in  $(p, q)$ .

This means that  $(p, q)$  contains  $d$ , which is a point of  $(a, b)$ .

Therefore, every  $(p, q)$  containing  $b$  also contains a point of  $(a, b)$ , so  $b$  is a limit point of  $(a, b)$ .

**Solution 5.** (Greg)

Choose any point  $p$  from  $S$ . Construct a segment  $(x, y)$  that contains  $p$  (ie:  $x < p < y$ ). Put no other condition on  $y$ . According to Axiom 1, one of these three cases is true:

1.  $x > a$ 

Choose a point  $q$  between  $x$  and  $p$  ( $x < q < p$ ; this exists by Axiom 3). Apply Axiom 2 three times:

- $a < x$  and  $x < q$ , so  $a < q$
- $q < p$  and  $p < b$ , so  $q < b$
- $q < p$  and  $p < y$ , so  $q < y$

so  $a < q < b$  and  $x < q < y$ . Therefore,  $q$  is an element of  $S$  inside  $(a, b)$  that is different from  $p$ .

2.  $x = a$ 

Repeat the proof for  $x > a$  with one change:

- $a = x$  and  $x < q$ , so  $a < q$

3.  $x < a$ 

Choose  $q$  between  $a$  and  $p$ . Change:

- $x < a$  and  $a < q$ , so  $x < q$

(same conclusion as  $x > a$ )

In each of these three cases, every  $(x, y)$  contains a point  $q$  from  $(a, b)$ . Therefore, we have satisfied Definition 4, so  $p$  is a limit point of  $(a, b)$ .

**Solution 6.** (Greg)

There are two cases: if  $p$  is not a point of  $[a, b]$ , then either  $p < a$  or  $p > b$ . We will only consider the first case; the proof for the second follows identically.

Construct two points  $x$  and  $y$  such that  $x < p$  and  $p < y < a$  (Axiom 3 guarantees that  $y$  exists) and consider the segment  $(x, y)$ . Since  $x < p < y$ ,  $(x, y)$  contains  $p$ .

Then, take any point  $i$  from  $[a, b]$ . This requires that  $a \leq i$ . From Axiom 2, since  $y < a$  and  $a \leq i$ ,  $y < i$ . Then, since  $i \not< y$ ,  $(x, y)$  does not contain  $i$ .

Therefore,  $(x, y)$  is a segment containing  $p$  but no points of  $[a, b]$ . This segment contradicts the definition of a limit point, so  $p < a$  is not a limit point of  $[a, b]$ .

**Solution 7.** (Erin)

If  $p$  is a limit point of  $H$ , then any segment  $S$  which contains  $p$  also contains a different point of  $H$ . Since  $H \subseteq K$ , this point is also a point of  $K$ . Therefore,  $S$  also contains a different point of  $K$ , and  $p$  is a limit point of  $K$ .

**Solution 8.** (Greg)

Consider a point  $p$ . According to Axiom 4, there exists a largest integer  $M_x$  and a smallest integer  $M_y$  such that  $M_x < p < M_y$ . Then, choose points  $x$  and  $y$  from Axiom 3 such that  $M_x < x < p$  and  $p < y < M_y$ , and consider the segment  $S = (x, y)$ .

We will try to find an integer different from  $p$  inside  $S$ . Axiom 1 gives us three cases:

1.  $n = p$  (note: this is only possible if  $p$  is an integer)

Here,  $n$  is not different from  $p$ , so we have not found an integer different from  $p$ .

2.  $n < p$

Note that  $n$  must satisfy  $n \leq M_x$ ; if  $n > M_x$ , we have contradicted Axiom 4. Then, from Axiom 2,  $n \leq M_x$  and  $M_x < x$ , so  $n < x$ . This shows that  $n$  is not between  $x$  and  $y$ , so  $S$  does not contain  $n$ .

3.  $n > p$

This case is symmetric to the  $n < p$  case.

Therefore,  $S$  is a segment containing  $p$  that does not contain any integers different from  $p$ . Since every element of  $M$  is an integer, we have proven that there exists a segment  $S$  for every point  $p$  without any other points of  $M$ , so  $M$  has no limit points.

**Solution 9.** (Rayne)

Suppose that  $S$  does not contain 2 points of  $M$ . We will find a contradiction.

Let  $S$  be the segment  $(a, b)$  which contains  $p$  (ie:  $a < p < b$ ). According to Definition 4, it must also contain a point  $m$  of  $M$  that is different from  $p$ . We will assume that  $m < p$  (the case  $m > p$  is similar).

We have  $a < m < p < b$ . According to Axiom 3, we can create another segment that contains  $p$  but not  $m$ . Then, this new segment does not contain any points of  $M$ , so  $p$  must not be a limit point of  $M$ . However, we know that  $p$  is a limit point of  $M$ , so we have a contradiction, and  $S$  must contain 2 or more points of  $M$ .

**Solution 10.** (Greg)

If  $S$  is a segment containing  $p$ , then  $S$  contains a point of  $K$  different from  $p$ . Since every point of  $K$  is a limit point of  $H$ ,  $S$  contains a limit point of  $H$  different from  $p$ . According to Problem 9, if  $S$  contains a limit point of  $H$ ,  $S$  also contains 2 different points of  $H$ . If  $p$  is equal to one of these points, it will be different from the other one. Therefore,  $S$  contains a point of  $H$  different from  $p$ , satisfying Definition 4, and  $p$  is a limit point of  $H$ .

**Solution 11.** (Greg)

Construct a segment  $S = (x, y)$  that contains 0 ( $x < 0 < y$ ). Since the reciprocal of a positive number is positive, every element  $M_i$  of  $M$  is positive, so  $x < M_i$  from Axiom 2.

Then, we will attempt to find an element of  $M$  that is less than  $y$ . Since every element of  $M$  is of the form  $\frac{1}{n}$  for some integer  $n$ , we are looking for  $\frac{1}{n} < y$ . From the properties of reciprocals, this is equivalent to  $\frac{1}{y} < n$ . According to Axiom 4, the point  $\frac{1}{y}$  has an integer greater than it, so we can find an  $n$  that satisfies this inequality.

Therefore, every  $S$  containing 0 also contains a point of  $M$ , so 0 is a limit point of  $M$ .

**Solution 12.** (Erin)

We will attack this problem in several cases:

1.  $p < 0$ 

Construct a segment  $(a, b)$  that contains both  $p$  and 0. These points satisfy  $a < p < 0 < b$ . From Axiom 3, we can find another point  $q$  such that  $p < q < 0$ . Then, the segment  $(a, q)$  contains  $p$ , but no points of  $M$  (since every point of  $M$  is positive), so  $p$  must not be a limit point of  $M$ .

2.  $p = 1$ 

Construct an  $(a, b)$  containing  $p$  and  $\frac{1}{2}$ . The inequality is then  $a < \frac{1}{2} < p < b$ . Axiom 3: can find  $q$  such that  $\frac{1}{2} < q < p$ . Then,  $(q, b)$  contains  $p$  but no point of  $M$  other than  $p$  (since every other point of  $M$  is less than  $\frac{1}{2}$ ), so  $p$  must not be a limit point of  $M$ .

3.  $p > 1$

Construct  $(a, b)$  containing  $p$  and 1. ( $a < 1 < p < b$ ) Axiom 3: can find  $q$  such that  $1 < q < p$ . Then,  $(q, b)$  contains  $p$  but no points of  $M$ .

4.  $0 < p < 1$

First, note that if  $n$  is an integer, then  $n - 1 < n < n + 1$  (Axiom 5). Then,  $\frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1}$ . According to Axiom 3, we can find  $\frac{1}{a}, \frac{1}{b}$  such that  $\frac{1}{n+1} < \frac{1}{a} < \frac{1}{n} < \frac{1}{b} < \frac{1}{n-1}$ . We have two cases from here:

- $p = \frac{1}{n}$   
 $(\frac{1}{a}, \frac{1}{b})$  contains  $p$  but no other point of  $M$ , so  $p$  is not a limit point of  $M$ .
- $\frac{1}{n+1} < p < \frac{1}{n}$   
 From Axiom 3, we can find  $\frac{1}{c}$  such that  $p < \frac{1}{c} < \frac{1}{n}$ . Then,  $(\frac{1}{a}, \frac{1}{c})$  contains  $p$  but no other point of  $M$ , so  $p$  is not a limit point of  $M$ .

These cases cover every point  $p \neq 0$ . Therefore, if  $p$  is not zero,  $p$  is not a limit point of  $M$ .

**Solution 13.** (Erin)

First, note that the set  $H \cap K$  is a subset of  $H$  because every point of  $H \cap K$  must be a point of  $H$ . Similarly,  $H \cap K \subseteq K$ .

Then, from Problem 7, since  $p$  is a limit point of  $H \cap K$ ,  $p$  is also a limit point of  $H$ ; identically,  $p$  is a limit point of  $K$ .

**Solution 14.** (Jeff)

We will use a counterexample to prove that not every interval containing a limit point must contain another point of the set.

Consider the set  $M$  of the reciprocals of all positive integers. (This was the set discussed in problems 11 and 12.) We know from Problem 11 that 0 is a limit point of  $M$ , so pick the interval  $I = [i, 0]$  for any  $i < 0$  and let  $p = 0$ .

Since  $p = 0$ , which is an endpoint of  $I$ ,  $I$  contains  $p$ . However, every point  $m$  of  $M$  is positive ( $m > 0$ ), and is not contained by  $I$ . Therefore,  $I$  is an interval which contains a limit point of  $M$  and no other points of  $M$ . This shows that not every interval containing a limit point must contain a different point of the set.

**Solution 15.** (Greg)

Consider an arbitrary segment  $S = (S_a, S_b)$  that contains  $p$  (so  $S_a < p < S_b$ ). From Axiom 3, we can find two more points  $I_a, I_b$  such that  $S_a < I_a < p$  and  $p < I_b < S_b$ .

Then, we can construct the interval  $I = [I_a, I_b]$ . Note that every point of  $I$  is also in  $S$  - ie,  $I \subseteq S$ . Since  $I$  contains  $p$ , it must contain a point of  $M$  different from  $p$ . Therefore,  $S$  also contains a point of  $M$  different from  $p$ , and  $p$  is a limit point of  $M$ .

**Solution 16.** (Fernando)

Let  $S_1 = (a_1, b_1)$  and  $S_2 = (a_2, b_2)$ . Since  $p$  is in both segments,  $a_1 < p < b_1$  and  $a_2 < p < b_2$ . From Axiom 2, we know that:

- $a_1 < b_1$
- $a_1 < b_2$
- $a_2 < b_1$
- $a_2 < b_2$

Then, there are only 4 segments that could be the result of the intersection  $S_1 \cap S_2$ :

- $(a_1, b_1)$
- $(a_1, b_2)$
- $(a_2, b_1)$
- $(a_2, b_2)$

Since  $p$  is in all of these segments, we have shown that the resulting segment will always contain  $p$ .

**Solution 17.** (Greg)

$p$  is not a limit point of  $H$ , so there is a segment  $S_H = (H_a, H_b)$  that contains  $p$  but no point of  $H$  different from  $p$ . Likewise, there is a segment  $S_K = (K_a, K_b)$  that contains  $p$  but no different point of  $K$ . From these two segments, construct  $S = S_H \cap S_K$ .

Since  $p$  was in both of the original segments, Problem 16 tells us that  $S$  contains  $p$ . However, since there were no elements of  $H$  or  $K$  in these segments (except possibly  $p$ ),  $S$  contains no elements of  $H$  or  $K$ . Finally, because every element of  $H \cup K$  is an element of  $H$  or  $K$ , we know that  $S$  contains no elements of  $H \cup K$ .

Therefore,  $S$  is a segment that contains  $p$  but no other elements of  $H \cup K$ . We have found a counterexample to the definition of a limit point, so  $p$  is not a limit point of  $H \cup K$ .

**Solution 18.** (Greg)

We will use a proof by contrapositive. The contrapositive is:

*Show that if  $p$  is not a limit point of  $H$  and  $p$  is not a limit point of  $K$ , then  $p$  is not a limit point of  $H \cup K$ .*

This is problem 17 (which has been proved), so we are finished.

**Solution 19.** (Erin)

Consider a subset of  $M$  that contains  $n$  points. Since this subset has a finite number of points, we can order them  $x_0, x_1, x_2, \dots, x_n$  such that  $x_0 < x_1 < x_2 < \dots < x_n$ .

However, since there is no largest number in  $M$ , we can find another point  $x_{n+1}$  from  $M$  such that  $x_n < x_{n+1}$ . We can repeat this process indefinitely, so for any positive integer  $n$ , we can find at least  $n$  points in  $M$ , making  $M$  an infinite set.

**Solution 20.** (Fernando)

Since  $M$  is finite, there is a positive integer  $n$  such that  $M$  does not have  $n$  points. We will then say that the number of points in  $M$  is not greater than  $n - 1$ .

If  $M$  has only one point, then we can say that its one point is the largest point in  $M$ . This is our base case.

Then, if  $M$  has more than one point, we can use the following algorithm to find the largest point:

- Choose two different points  $a_1$  and  $a_2$  from  $M$ . Then, either  $a_1 > a_2$  or  $a_1 < a_2$ . Pick the larger of these two points and keep track of it. (We will assume that  $a_1$  was the larger of the two.)
- Choose another point  $a_3$  from  $M$  that is different from  $a_1$  and  $a_2$ . Then, either  $a_3 > a_1$  or  $a_3 < a_1$ . If  $a_3 > a_1$ , from Axiom 2,  $a_3 > a_2$ , so  $a_3$  is the new largest point. If  $a_3 < a_1$ , then  $a_1$  remains the largest.

Since  $M$  can have no more than  $n - 1$  points, we can then find the largest point using  $n - 2$  of these steps. Therefore, there must be a largest point of  $M$ .

**Solution 21.** (Erin)

According to Problem 9,  $S$  must contain at least two points of  $M$  different from  $p$ . Label these two points  $q$  and  $s$ . Assume that  $q$  is less than  $p$  - a similar argument applies if  $q > p$ .

If  $S$  is the segment  $(a, b)$  with  $a < p < b$ , then we can see that  $a < q < p$ . Then, we define a new segment  $(c, d)$  where  $q < c < p$  and  $p < d < b$ . This new segment does not contain  $q$ . However, it still contains  $p$ , so there must be another point of  $M$  different from  $p$  inside  $(c, d)$ .

Since every point of  $(c, d)$  is also in  $(a, b)$ , the new point that we have found is also in  $(a, b)$ . We can repeat this process indefinitely, so  $S$  must contain infinitely many points of  $M$ .

**Solution 22.** (no solution required)

**Solution 23.** (Erin)

First, let  $n$  be an even positive integer. Then,  $x_n = 0$  and  $x_{n+1} = 1$ . If the points  $a, b$  satisfy  $a < 0 < b < 1$ , then the segment  $(a, b)$  does not contain  $x_{n+1}$ . (Note that this does not depend on  $n$ .)

Then, let  $n$  be an odd positive integer, so  $x_n = x_{n+2} = 1$ . The segment  $(a, b)$  does not contain  $x_{n+2}$ .

Therefore, for every positive integer  $n$ , either  $x_{n+1}$  or  $x_{n+2}$  is not in  $(a, b)$ , so we cannot find an  $n$  that satisfies Definition 11, and the sequence does not converge to 0.

**Solution 24.** (no solution required)

**Solution 25.** (Erin)

First, note that ‘not infinite’ means the same thing as ‘finite’.

Since the sequence is convergent, there is a positive integer  $n$  such that  $x_n, x_{n+1}, x_{n+2}, \dots$  are all in  $(a, b)$ . This does not imply anything about the points  $x_1, x_2, x_3, \dots, x_{n-1}$ , so these points may or may not be in  $(a, b)$ . Counting these points tells us that there cannot be more than  $n - 1$  points of the sequence outside of  $(a, b)$ . Therefore, there are not  $n$  points of the sequence not in  $(a, b)$ , so this set of points is finite.

**Solution 26.** (Rayne)

Suppose that  $n$  is an odd positive integer - the proof for even  $n$  is similar.

Since the sequence is defined as

$$x_n = \begin{cases} \frac{1}{n-1}, & n \text{ odd} \\ \frac{1}{n+1}, & n \text{ even} \end{cases}$$

we can write out several terms of the sequence as

odd	even	odd	even
$x_n$	$x_{n+1}$	$x_{n+2}$	$x_{n+3}$
$\frac{1}{n+1}$	$\frac{1}{n}$	$\frac{1}{n+3}$	$\frac{1}{n+2}$

We can see from this sample that  $x_{n+2} < x_{n+3} < x_n < x_{n+1}$ . Then, if we make a segment  $(a, b)$  containing 0,  $x_n$  will be in this segment for large enough  $n$ . Since all consecutive terms of the sequence starting at  $n + 2$  are between 0 and  $b$ , then we can see that the definition of convergence is satisfied, and the sequence must converge to 0.

**Solution 27.** (Greg)

We can find three points  $a_1, a_2, a_3$  such that

$$a_1 < c < a_2 < d < a_3$$

(Axiom 3 guarantees that  $a_2$  exists). Construct the segments  $S_c = (a_1, a_2)$  and  $S_d = (a_2, a_3)$ .

Notice that any point  $p$  in  $S_c$  has  $p < a_2$  and any point  $q$  in  $S_d$  has  $q > a_2$ . If  $p = q$ , we contradict Axiom 1, so no point is in both  $S_c$  and  $S_d$ .

The sequence  $x_1, x_2, x_3, \dots$  converges to  $c$ , so there is a positive integer  $n$  such that  $x_m$  is in  $S_c$  for all  $m \geq n$ . From above,  $x_m$  is not in  $S_d$  for all  $m \geq n$ . Therefore, we cannot find an  $n$  that satisfies Definition 11 for  $S_d$ , and the sequence does not converge to  $d$ .

**Solution 28.**

**Solution 29.**

**Solution 30.**



**Solution 31.** (no solution required)

**Solution 32.**

**Solution 33.** (Greg)

We will assume that  $M$  has both a rightmost point and a smallest number largest points, and we will find a contradiction.

If the largest point of  $M$  is labelled  $x$ , then for any point  $p$  in  $M$ ,  $p \leq x$ . Also, if the smallest point greater than every point in  $M$  is  $y$ , then for every  $p$  in  $M$ ,  $p < y$ . Note that since  $x$  is in  $M$ ,  $x < y$ .

According to Axiom 3, there exists a point  $z$  such that  $x < z < y$ . Since  $x < z$ , Axiom 2 tells us that every point of  $M$  is less than  $z$ . However,  $z$  is smaller than  $y$ , so  $y$  is not the smallest point greater than every point of  $M$ , and we have a contradiction.

**Solution 34.** (Greg)

We will try to find a segment that contains  $p$  but no other point of  $M$ . Construct a segment  $S = (a, b)$  that contains  $p$ . This means that  $a < p$ .

If  $S$  does not contain another point of  $M$ , then every point  $q$  of  $M$  must satisfy  $q \leq a$  or  $q \geq b$ . No points of  $M$  are greater than  $b$ , since they are all less than  $p$ .

Now, Axiom 3 says that there is a point between  $a$  and  $p$ . This point is greater than every point of  $M$ , but is smaller than  $p$ . However,  $p$  was the smallest point with this property. We have a contradiction, so  $p$  must be a limit point of  $M$ .

**Solution 35.**

**Solution 36.**

**Solution 37.**

**Solution 38.** (Fernando)

**Solution 39.** (Rayne)