## Problem Sequence - Solutions

This document will be filled up with the solutions from the problem sequence.

**Solution 1.** (Greg)
From Axiom 1, there are three cases to consider:

1. $p = 0$

   $p$ is equal to the only element of $M$. From the definition of limit point, every segment containing $p$ must contain a point of $M$ different from $p$. However, there are no points in $M$ different from $p$, so $p$ must not be a limit point of $M$.

2. $p > 0$

   From Axiom 3, there exists a point $a$ such that $0 < a < p$. There also exists a point $b$ such that $b > p$. Since $p > 0$ and $b > p$, Axiom 2 tells us that $b > 0$. We can then form the segment $S = (a, b)$. Since $a > 0$ and $b > 0$, 0 is not between $a$ and $b$, so $S$ does not contain 0. However, $a < p < b$, so $S$ contains $p$. $S$ is a segment containing $p$ that does not contain any element of $M$, so $p$ is not a limit point of $M$.

3. $p < 0$

   (symmetric to the $p > 0$ case)

Therefore, regardless of our choice of $p$, we can construct a segment that contradicts the requirements in the limit point definition, so $p$ is not a limit point of $M$.

**Solution 2.** (Jeff)
According to Definition 4, we can prove that $p$ is not a limit point of $M$ if we can construct a segment containing $p$ but not a different point of $M$. Construct this segment as follows:

1. If all points of $M$ are on the opposite side of $p$, choose any value as an endpoint.

2. If any point of $M$ is on the same side of $p$, choose a point between the nearest point of $M$ and $p$. (Axiom 3 confirms that there will be such a point.)

This segment contains $p$ but will not contain any points of $M$ (with the exception of $p$, if $p$ is 0 or 1). Therefore, we have found a segment that does not fulfill the requirements of Definition 4, so $p$ is not a limit point of $M$.

**Solution 3.** (Erin, solved after Problem 4)
Let $a = 0$ and $b = 1$. Then, according to Problem 4, $b$ is a limit point of $(a, b)$. Therefore, 1 is a limit point of $(0, 1)$.

**Solution 4.** (Erin)

We will prove that $b$ is a limit point of $(a, b)$; The proof for $a$ is similar.

Construct a segment $(p, q)$ containing $b$ ($p < b < q$). According to Axiom 1, there are three cases, and we will deal with two of them simultaneously:

1. $p < a$

   In this case, $p < a < b < q$, so $p < (a, b) < q$. Since $(p, q)$ contains every point of $(a, b)$, we have found points that satisfy Defintion 4.

2. $p \geq a$

   In this case, $a \leq p < b < q$. According to Axiom 3, there is a point $d$ between $p$ and $b$. The inequality is then $a \leq p < d < b < q$. Then,

   - $a < d < b$, so $d$ is in $(a, b)$.
   - $p < d < q$, so $d$ is in $(p, q)$.

   This means that $(p, q)$ contains $d$, which is a point of $(a, b)$.

Therefore, every $(p, q)$ containing $b$ also contains a point of $(a, b)$, so $b$ is a limit point of $(a, b)$.

**Solution 5.** (Greg)

Choose any point $p$ from $S$. Construct a segment $(x, y)$ that contains $p$ (ie: $x < p < y$). Put no other condition on $y$. According to Axiom 1, one of these three cases is true:

1. $x > a$

   Choose a point $q$ between $x$ and $p$ ($x < q < p$; this exists by Axiom 3). Apply Axiom 2 three times:

   - $a < x$ and $x < q$, so $a < q$
   - $q < p$ and $p < b$, so $q < b$
   - $q < p$ and $p < y$, so $q < y$

   so $a < q < b$ and $x < q < y$. Therefore, $q$ is an element of $S$ inside $(a, b)$ that is different from $p$.

2. $x = a$

   Repeat the proof for $x > a$ with one change:

   - $a = x$ and $x < q$, so $a < q$

3. $x < a$

   Choose $q$ betweem $a$ and $p$. Change:

   - $x < a$ and $a < q$, so $x < q$

(same conclusion as $x > a$)

In each of these three cases, every $(x, y)$ contains a point $q$ from $(a, b)$. Therefore, we have satisfied Definition 4, so $p$ is a limit point of $(a, b)$.

**Solution 6.** (Greg)
There are two cases: if $p$ is not a point of $[a, b]$, then either $p < a$ or $p > b$. We will only consider the first case; the proof for the second follows identically.

Construct two points $x$ and $y$ such that $x < p$ and $p < y < a$ (Axiom 3 guarantees that $y$ exists) and consider the segment $(x, y)$. Since $x < p < y$, $(x, y)$ contains $p$.

Then, take any point $i$ from $[a, b]$. This requires that $a \leq i$. From Axiom 2, since $y < a$ and $a \leq i$, $y < i$. Then, since $i \not< y$, $(x, y)$ does not contain $i$.

Therefore, $(x, y)$ is a segment containing $p$ but no points of $[a, b]$. This segment contradicts the definition of a limit point, so $p < a$ is not a limit point of $[a, b]$.

**Solution 7.** (Amber)

**Solution 8.** (Greg)
Consider a point $p$. According to Axiom 4, there exists a largest integer $M_x$ and a smallest integer $M_y$ such that $M_x < p < M_y$. Then, choose points $x$ and $y$ from Axiom 3 such that $M_x < x < p$ and $p < y < M_y$, and consider the segment $S = (x, y)$.

We will try to find an integer different from $p$ inside $S$. Axiom 1 gives us three cases:

1. $n = p$ (note: this is only possible if $p$ is an integer)

   Here, $n$ is not different from $p$, so we have not found an integer different from $p$.

2. $n < p$

   Note that $n$ must satisfy $n \leq M_x$; if $n > M_x$, we have contradicted Axiom 4. Then, from Axiom 2, $n \leq M_x$ and $M_x < x$, so $n < x$. This shows that $n$ is not between $x$ and $y$, so $S$ does not contain $n$.

3. $n > p$

   This case is symmetric to the $n < p$ case.

Therefore, $S$ is a segment containing $p$ that does not contain any integers different from $p$. Since every element of $M$ is an integer, we have proven that there exists a segment $S$ for every point $p$ without any other points of $M$, so $M$ has no limit points.

**Solution 9.** (Rayne)
Suppose that $S$ does not contain 2 points of $M$. We will find a contradiction.

Let $S$ be the segment $(a, b)$ which contains $p$ (ie: $a < p < b$). According to Definition 4, it must also contain a point $m$ of $M$ that is different from $p$. We will assume that $m < p$ (the case $m > p$ is similar).

We have $a < m < p < b$. According to Axiom 3, we can create another segment that contains $p$ but not $m$. Then, this new segment does not contain any points of $M$, so $p$ must not be a limit point of $M$. However, we know that $p$ is a limit point of $M$, so we have a contradiction, and $S$ must contain 2 or more points of $M$.

**Solution 10.**

**Solution 11.** (Greg)

Construct a segment $S = (x, y)$ that contains 0 ($x < 0 < y$). Since the reciprocal of a positive number is positive, every element $M_i$ of $M$ is positive, so $x < M_i$ from Axiom 2.

Then, we will attempt to find an element of $M$ that is less than $y$. Since every element of $M$ is of the form $\frac{1}{n}$ for some integer $n$, we are looking for $\frac{1}{n} < y$. From the properties of reciprocals, this is equivalent to $\frac{1}{y} < n$. According to Axiom 4, the point $\frac{1}{y}$ has an integer greater than it, so we can find an $n$ that satisfies this inequality.

Therefore, every $S$ containing 0 also contains a point of $M$, so 0 is a limit point of $M$.

**Solution 12.** (Erin)

We will attack this problem in several cases:

1. $p < 0$

   Construct a segment $(a, b)$ that contains both $p$ and 0. These points satisfy $a < p < 0 < b$. From Axiom 3, we can find another point $q$ such that $p < q < 0$. Then, the segment $(a, q)$ contains $p$, but no points of $M$ (since every point of $M$ is positive), so $p$ must not be a limit point of $M$.

2. $p = 1$

   Construct an $(a, b)$ containing $p$ and $\frac{1}{2}$. The inequality is then $a < \frac{1}{2} < p < b$. Axiom 3: can find $q$ such that $\frac{1}{2} < q < p$ Then, $(q, b)$ contains $p$ but no point of $M$ other than $p$ (since every other point of $M$ is less than $\frac{1}{2}$, so $p$ must not be a limit point of $M$.

3. $p > 1$

   Construct $(a, b)$ containing $p$ and 1. ($a < 1 < p < b$) Axiom 3: can find $q$ such that $1 < q < p$. Then, $(q, b)$ contains $p$ but no points of $M$.

4. $0 < p < 1$

   First, note that if $n$ is an integer, then $n - 1 < n < n + 1$ (Axiom 5). Then, $\frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1}$. According to Axiom 3, we can find $\frac{1}{a}, \frac{1}{b}$ such that $\frac{1}{n+1} < \frac{1}{a} < \frac{1}{n} < \frac{1}{b} < \frac{1}{n-1}$. We have two cases from here:

- $p = \frac{1}{n}$
  $(\frac{1}{a}, \frac{1}{b})$ contains $p$ but no other point of $M$, so $p$ is not a limit point of $M$.
- $\frac{1}{n+1} < p < \frac{1}{n}$
  From Axiom 3, we can find $\frac{1}{c}$ such that $p < \frac{1}{c} < \frac{1}{n}$. Then, $(\frac{1}{a}, \frac{1}{c})$ contains $p$ but no other point of $M$, so $p$ is not a limit point of $M$.

These cases cover every point $p \neq 0$. Therefore, if $p$ is not zero, $p$ is not a limit point of $M$.

**Solution 13.** (Erin)
First, note that the set $H \cap K$ is a subset of $H$ because every point of $H \cap K$ must be a point of $H$. Similarly, $H \cap K \subseteq K$.
Then, from Problem 7, since $p$ is a limit point of $H \cap K$, $p$ is also a limit point of $H$; identically, $p$ is a limit point of $K$.

**Solution 14.** (Jeff)
We will use a counterexample to prove that not every interval containing a limit point must contain another point of the set.
Consider the set $M$ of the reciprocals of all positive integers. (This was the set discussed in problems 11 and 12.) We know from Problem 11 that 0 is a limit point of $M$, so pick the interval $I = [i, 0]$ for any $i < 0$ and let $p = 0$.
Since $p = 0$, which is an endpoint of $I$, $I$ contains $p$. However, every point $m$ of $M$ is positive ($m > 0$), and is not contained by $I$. Therefore, $I$ is an interval which contains a limit point of $M$ and no other points of $M$. This shows that not every interval containing a limit point must contain a different point of the set.

**Solution 15.**

**Solution 16.** (Fernando)
Let $S_1 = (a_1, b_1)$ and $S_2 = (a_2, b_2)$. Since $p$ is in both segments, $a_1 < p < b_1$ and $a_2 < p < b_2$. From Axiom 2, we know that:

- $a_1 < b_1$
- $a_1 < b_2$
- $a_2 < b_1$
- $a_2 < b_2$

Then, there are only 4 segments that could be the result of the intersection $S_1 \cap S_2$:

- $(a_1, b_1)$
- $(a_1, b_2)$

- $(a_2, b_1)$
- $(a_2, b_2)$

Since $p$ is in all of these segments, we have shown that the resulting segment will always contain $p$.

**Solution 17.** (Greg)

$p$ is not a limit point of $H$, so there is a segment $S_H = (H_a, H_b)$ that contains $p$ but no point of $H$ different from $p$. Likewise, there is a segment $S_K(K_a, K_b)$ that contains $p$ but no different point of $K$. From these two segments, construct $S = S_H \cap S_K$.

Since $p$ was in both of the original segments, Problem 16 tells us that $S$ contains $p$. However, since there were no elements of $H$ or $K$ in these segments (except possibly $p$), $S$ contains no elements of $H$ or $K$. Finally, because every element of $H \cup K$ is an element of $H$ or $K$, we know that $S$ contains no elements of $H \cup K$.

Therefore, $S$ is a segment that contains $p$ but no other elements of $H \cup K$. We have found a counterexample to the definition of a limit point, so $p$ is not a limit point of $H \cup K$.

**Solution 18.** (Greg)

We will use a proof by contrapositive. The contrapositive is:

*Show that if $p$ is not a limit point of $H$ and $p$ is not a limit point of $K$, then $p$ is not a limit point of $H \cup K$.*

This is problem 17 (which has been proved), so we are finished.

**Solution 19.**

**Solution 20.** (Fernando)

Since $M$ is finite, there is a positive integer $n$ such that $M$ does not have $n$ points. We will then say that the number of points in $M$ is not greater than $n - 1$.

If $M$ has only one point, then we can say that its one point is the largest point in $M$. This is our base case.

Then, if $M$ has more than one point, we can use the following algorithm to find the largest point:

- Choose two different points $a_1$ and $a_2$ from $M$. Then, either $a_1 > a_2$ or $a_1 < a_2$. Pick the larger of these two points and keep track of it. (We will assume that $a_1$ was the larger of the two.)
- Choose another point $a_3$ from $M$ that is different from $a_1$ and $a_2$. Then, either $a_3 > a_1$ or $a_3 < a_1$. If $a_3 > a_1$, from Axiom 2, $a_3 > a_2$, so $a_3$ is the new largest point. If $a_3 < a_1$, then $a_1$ remains the largest.

Since $M$ can have no more than $n - 1$ points, we can then find the largest point using $n - 2$ of these steps. Therefore, there must be a largest point of $M$.

**Solution 21.**

**Solution 22.**

**Solution 23.** (Erin)

First, let $n$ be an even positive integer. Then, $x_n = 0$ and $x_{n+1} = 1$. If the points $a, b$ satisfy $a < 0 < b < 1$, then the segment $(a, b)$ does not contain $x_{n+1}$. (Note that this does not depend on $n$.)

Then, let $n$ be an odd positive integer, so $x_n = x_{n+2} = 1$. The segment $(a, b)$ does not contain $x_{n+2}$.

Therefore, for every positive integer $n$, either $x_{n+1}$ or $x_{n+2}$ is not in $(a, b)$, so we cannot find an $n$ that satisfies Definition 11, and the sequence does not converge to 0.

**Solution 24.** (no solution required)

**Solution 25.** (Erin)

First, note that 'not infinite' means the same thing as 'finite'.

Since the sequence is convergent, there is a positive integer $n$ such that $x_n, x_{n+1}, x_{n+2}, \ldots$ are all in $(a, b)$. This does not imply anything about the points $x_1, x_2, x_3, \ldots, x_{n-1}$, so these points may or may not be in $(a, b)$. Counting these points tells us that there cannot be more than $n-1$ points of the sequence outside of $(a, b)$. Therefore, there are not $n$ points of the sequence not in $(a, b)$, so this set of points is finite.

**Solution 26.**

**Solution 27.** (Greg)

We can find three points $a_1, a_2, a_3$ such that

$$a_1 < c < a_2 < d < a_3$$

(Axiom 3 guarantees that $a_2$ exists). Construct the segments $S_c = (a_1, a_2)$ and $S_d = (a_2, a_3)$.

Notice that any point $p$ in $S_c$ has $p < a_2$ and any point $q$ in $S_d$ has $q > a_2$. If $p = q$, we contradict Axiom 1, so no point is in both $S_c$ and $S_d$.

The sequence $x_1, x_2, x_3, \ldots$ converges to $c$, so there is a positive integer $n$ such that $x_m$ is in $S_c$ for all $m \geq n$. From above, $x_m$ is not in $S_d$ for all $m \geq n$. Therefore, we cannot find an $n$ that satisfies Definition 11 for $S_d$, and the sequence does not converge to $d$.

**Solution 28.**

**Solution 29.**

**Solution 30.**

**Solution 31.** (no solution required)

**Solution 32.**

**Solution 33.** (Greg)

We will assume that $M$ has both a rightmost point and a smallest number largest points, and we will find a contradiction.

If the largest point of $M$ is labelled $x$, then for any point $p$ in $M$, $p \leq x$. Also, if the smallest point greater than every point in $M$ is $y$, then for every $p$ in $M$, $p < y$. Note that since $x$ is in $M$, $x < y$.

According to Axiom 3, there exists a point $z$ such that $x < z < y$. Since $x < z$, Axiom 2 tells us that every point of $M$ is less than $z$. However, $z$ is smaller than $y$, so $y$ is not the smallest point greater than every point of $M$, and we have a contradiction.

**Solution 34.**