Problem Sequence - Solutions

This document will be filled up with the solutions from the problem sequence.

Solution 1. (Greg)
From Axiom 1, there are three cases to consider:

1. $p = 0$

   p is equal to the only element of M. From the definition of limit point, every segment containing p must contain a point of M different from p. However, there are no points in M different from p, so p must not be a limit point of M.

2. $p > 0$

   From Axiom 3, there exists a point a such that $0 < a < p$. There also exists a point b such that $b > p$. Since $p > 0$ and $b > p$, Axiom 2 tells us that $b > 0$. We can then form the segment $S = (a, b)$. Since $a > 0$ and $b > 0$, 0 is not between a and b, so S does not contain 0. However, $a < p < b$, so S contains p. S is a segment containing p that does not contain any element of M, so p is not a limit point of M.

3. $p < 0$

   (symmetric to the $p > 0$ case)

Therefore, regardless of our choice of p, we can construct a segment that contradicts the requirements in the limit point definition, so p is not a limit point of M.

Solution 2. (Jeff)
According to Definition 4, we can prove that p is not a limit point of M if we can construct a segment containing p but not a different point of M. Construct this segment as follows:

1. If all points of M are on the opposite side of p, choose any value as an endpoint.

2. If any point of M is on the same side of p, choose a point between the nearest point of M and p. (Axiom 3 confirms that there will be such a point.)

This segment contains p but will not contain any points of M (with the exception of p, if p is 0 or 1). Therefore, we have found a segment that does not fulfill the requirements of Definition 4, so p is not a limit point of M.

Solution 3. (Erin, solved after Problem 4)
Let $a = 0$ and $b = 1$. Then, according to Problem 4, b is a limit point of (a, b). Therefore, 1 is a limit point of $(0, 1)$.

Solution 4. (Erin)

We will prove that b is a limit point of (a, b); The proof for a is similar.

Construct a segment (p, q) containing b ($p < b < q$). According to Axiom 1, there are three cases, and we will deal with two of them simultaneously:

1. $p < a$

   In this case, $p < a < b < q$, so $p < (a, b) < q$. Since (p, q) contains every point of (a, b), we have found points that satisfy Defintion 4.

2. $p \geq a$

   In this case, $a \leq p < b < q$. According to Axiom 3, there is a point d between p and b. The inequality is then $a \leq p < d < b < q$. Then,

   - $a < d < b$, so d is in (a, b).
   - $p < d < q$, so d is in (p, q).

   This means that (p, q) contains d, which is a point of (a, b).

Therefore, every (p, q) containing b also contains a point of (a, b), so b is a limit point of (a, b).

Solution 5. (Greg)

Choose any point p from S. Construct a segment (x, y) that contains p (ie: $x < p < y$). Put no other condition on y. According to Axiom 1, one of these three cases is true:

1. $x > a$

   Choose a point q between x and p ($x < q < p$; this exists by Axiom 3). Apply Axiom 2 three times:

   - $a < x$ and $x < q$, so $a < q$
   - $q < p$ and $p < b$, so $q < b$
   - $q < p$ and $p < y$, so $q < y$

   so $a < q < b$ and $x < q < y$. Therefore, q is an element of S inside (a, b) that is different from p.

2. $x = a$

   Repeat the proof for $x > a$ with one change:

   - $a = x$ and $x < q$, so $a < q$

3. $x < a$

   Choose q betweem a and p. Change:

   - $x < a$ and $a < q$, so $x < q$

(same conclusion as $x > a$)

In each of these three cases, every (x, y) contains a point q from (a, b). Therefore, we have satisfied Definition 4, so p is a limit point of (a, b).

Solution 6. (Zack)

Solution 7. (Amber)

Solution 8. (Greg)

Solution 9.

Solution 10.

Solution 11.

Solution 12.

Solution 13.

Solution 14.

Solution 15.

Solution 16.

Solution 17.

Solution 18.