Problem Sequence - Solutions

This document will be filled up with the solutions from the problem sequence.

Solution 1. (Greg)

From Axiom 1, there are three cases to consider:

1. $p = 0$

   p is equal to the only element of M. From the definition of limit point, every segment containing p must contain a point of M different from p. However, there are no points in M different from p, so p must not be a limit point of M.

2. $p > 0$

   From Axiom 3, there exists a point a such that $0 < a < p$. There also exists a point b such that $b > p$. Since $p > 0$ and $b > p$, Axiom 2 tells us that $b > 0$. We can then form the segment $S = (a, b)$. Since $a > 0$ and $b > 0$, 0 is not between a and b, so S does not contain 0. However, $a < p < b$, so S contains p. S is a segment containing p that does not contain any element of M, so p is not a limit point of M.

3. $p < 0$

   (symmetric to the $p > 0$ case)

Therefore, regardless of our choice of p, we can construct a segment that contradicts the requirements in the limit point definition, so p is not a limit point of M.

Solution 2. (Jeff)

According to Definition 4, we can prove that p is not a limit point of M if we can construct a segment containing p but not a different point of M. Construct this segment as follows:

1. If all points of M are on the opposite side of p, choose any value as an endpoint.

2. If any point of M is on the same side of p, choose a point between the nearest point of M and p. (Axiom 3 confirms that there will be such a point.)

This segment contains p but will not contain any points of M (with the exception of p, if p is 0 or 1). Therefore, we have found a segment that does not fulfill the requirements of Definition 4, so p is not a limit point of M.

Solution 3. (Erin, solved after Problem 4)

Let $a = 0$ and $b = 1$. Then, according to Problem 4, b is a limit point of (a, b). Therefore, 1 is a limit point of $(0, 1)$.

Solution 4. (Erin)

We will prove that b is a limit point of (a, b); The proof for a is similar.

Construct a segment (p, q) containing b ($p < b < q$). According to Axiom 1, there are three cases, and we will deal with two of them simultaneously:

1. $p < a$

   In this case, $p < a < b < q$, so $p < (a, b) < q$. Since (p, q) contains every point of (a, b), we have found points that satisfy Defintion 4.

2. $p \geq a$

   In this case, $a \leq p < b < q$. According to Axiom 3, there is a point d between p and b. The inequality is then $a \leq p < d < b < q$. Then,

   - $a < d < b$, so d is in (a, b).
   - $p < d < q$, so d is in (p, q).

   This means that (p, q) contains d, which is a point of (a, b).

Therefore, every (p, q) containing b also contains a point of (a, b), so b is a limit point of (a, b).

Solution 5. (Greg)

Choose any point p from S. Construct a segment (x, y) that contains p (ie: $x < p < y$). Put no other condition on y. According to Axiom 1, one of these three cases is true:

1. $x > a$

   Choose a point q between x and p ($x < q < p$; this exists by Axiom 3). Apply Axiom 2 three times:

   - $a < x$ and $x < q$, so $a < q$
   - $q < p$ and $p < b$, so $q < b$
   - $q < p$ and $p < y$, so $q < y$

   so $a < q < b$ and $x < q < y$. Therefore, q is an element of S inside (a, b) that is different from p.

2. $x = a$

   Repeat the proof for $x > a$ with one change:

   - $a = x$ and $x < q$, so $a < q$

3. $x < a$

   Choose q betweem a and p. Change:

   - $x < a$ and $a < q$, so $x < q$

(same conclusion as $x > a$)

In each of these three cases, every (x, y) contains a point q from (a, b). Therefore, we have satisfied Definition 4, so p is a limit point of (a, b).

Solution 6. (Greg)

There are two cases: if p is not a point of $[a, b]$, then either $p < a$ or $p > b$. We will only consider the first case; the proof for the second follows identically.

Construct two points x and y such that $x < p$ and $p < y < a$ (Axiom 3 guarantees that y exists) and consider the segment (x, y). Since $x < p < y$, (x, y) contains p.

Then, take any point i from $[a, b]$. This requires that $a \leq i$. From Axiom 2, since $y < a$ and $a \leq i$, $y < i$. Then, since $i \not< y$, (x, y) does not contain i.

Therefore, (x, y) is a segment containing p but no points of $[a, b]$. This segment contradicts the definition of a limit point, so $p < a$ is not a limit point of $[a, b]$.

Solution 7. (Amber)

Solution 8. (Greg)

Consider a point p. According to Axiom 4, there exists a largest integer M_x and a smallest integer M_y such that $M_x < p < M_y$. Then, choose points x and y from Axiom 3 such that $M_x < x < p$ and $p < y < M_y$, and consider the segment $S = (x, y)$.

We will try to find an integer different from p inside S. Axiom 1 gives us three cases:

1. $n = p$ (note: this is only possible if p is an integer)

   Here, n is not different from p, so we have not found an integer different from p.

2. $n < p$

   Note that n must satisfy $n \leq M_x$; if $n > M_x$, we have contradicted Axiom 4. Then, from Axiom 2, $n \leq M_x$ and $M_x < x$, so $n < x$. This shows that n is not between x and y, so S does not contain n.

3. $n > p$

   This case is symmetric to the $n < p$ case.

Therefore, S is a segment containing p that does not contain any integers different from p. Since every element of M is an integer, we have proven that there exists a segment S for every point p without any other points of M, so M has no limit points.

Solution 9. (Rayne)

Suppose that S does not contain 2 points of M. We will find a contradiction.

Let S be the segment (a, b) which contains p (ie: $a < p < b$). According to Definition 4, it must also contain a point m of M that is different from p. We will assume that $m < p$ (the case $m > p$ is similar).

We have $a < m < p < b$. According to Axiom 3, we can create another segment that contains p but not m. Then, this new segment does not contain any points of M, so p must not be a limit point of M. However, we know that p is a limit point of M, so we have a contradiction, and S must contain 2 or more points of M.

Solution 10.

Solution 11. (Greg)

Construct a segment $S = (x, y)$ that contains 0 ($x < 0 < y$). Since the reciprocal of a positive number is positive, every element M_i of M is positive, so $x < M_i$ from Axiom 2.

Then, we will attempt to find an element of M that is less than y. Since every element of M is of the form $\frac{1}{n}$ for some integer n, we are looking for $\frac{1}{n} < y$. From the properties of reciprocals, this is equivalent to $\frac{1}{y} < n$. According to Axiom 4, the point $\frac{1}{y}$ has an integer greater than it, so we can find an n that satisfies this inequality.

Therefore, every S containing 0 also contains a point of M, so 0 is a limit point of M.

Solution 12. (Erin)

We will attack this problem in several cases:

1. $p < 0$

   Construct a segment (a, b) that contains both p and 0. These points satisfy $a < p < 0 < b$. From Axiom 3, we can find another point q such that $p < q < 0$. Then, the segment (a, q) contains p, but no points of M (since every point of M is positive), so p must not be a limit point of M.

2. $p = 1$

   Construct an (a, b) containing p and $\frac{1}{2}$. The inequality is then $a < \frac{1}{2} < p < b$. Axiom 3: can find q such that $\frac{1}{2} < q < p$ Then, (q, b) contains p but no point of M other than p (since every other point of M is less than $\frac{1}{2}$, so p must not be a limit point of M.

3. $p > 1$

   Construct (a, b) containing p and 1. ($a < 1 < p < b$) Axiom 3: can find q such that $1 < q < p$. Then, (q, b) contains p but no points of M.

4. $0 < p < 1$

   First, note that if n is an integer, then $n - 1 < n < n + 1$ (Axiom 5). Then, $\frac{1}{n+1} < \frac{1}{n} < \frac{1}{n-1}$. According to Axiom 3, we can find $\frac{1}{a}, \frac{1}{b}$ such that $\frac{1}{n+1} < \frac{1}{a} < \frac{1}{n} < \frac{1}{b} < \frac{1}{n-1}$. We have two cases from here:

- $p = \frac{1}{n}$
  $(\frac{1}{a}, \frac{1}{b})$ contains p but no other point of M, so p is not a limit point of M.
- $\frac{1}{n+1} < p < \frac{1}{n}$
  From Axiom 3, we can find $\frac{1}{c}$ such that $p < \frac{1}{c} < \frac{1}{n}$. Then, $(\frac{1}{a}, \frac{1}{c})$ contains p but no other point of M, so p is not a limit point of M.

These cases cover every point $p \neq 0$. Therefore, if p is not zero, p is not a limit point of M.

Solution 13. (Erin)
First, note that the set $H \cap K$ is a subset of H because every point of $H \cap K$ must be a point of H. Similarly, $H \cap K \subseteq K$.

Then, from Problem 7, since p is a limit point of $H \cap K$, p is also a limit point of H; identically, p is a limit point of K.

Solution 14. (Jeff)
We will use a counterexample to prove that not every interval containing a limit point must contain another point of the set.

Consider the set M of the reciprocals of all positive integers. (This was the set discussed in problems 11 and 12.) We know from Problem 11 that 0 is a limit point of M, so pick the interval $I = [i, 0]$ for any $i < 0$ and let $p = 0$.

Since $p = 0$, which is an endpoint of I, I contains p. However, every point m of M is positive ($m > 0$), and is not contained by I. Therefore, I is an interval which contains a limit point of M and no other points of M. This shows that not every interval containing a limit point must contain a different point of the set.

Solution 15. (Jeff)

Solution 16. (Fernando)
Let $S_1 = (a_1, b_1)$ and $S_2 = (a_2, b_2)$. Since p is in both segments, $a_1 < p < b_1$ and $a_2 < p < b_2$. From Axiom 2, we know that:

- $a_1 < b_1$
- $a_1 < b_2$
- $a_2 < b_1$
- $a_2 < b_2$

Then, there are only 4 segments that could be the result of the intersection $S_1 \cap S_2$:

- (a_1, b_1)
- (a_1, b_2)

- (a_2, b_1)
- (a_2, b_2)

Since p is in all of these segments, we have shown that the resulting segment will always contain p.

Solution 17. (Greg)

p is not a limit point of H, so there is a segment $S_H = (H_a, H_b)$ that contains p but no point of H different from p. Likewise, there is a segment $S_K(K_a, K_b)$ that contains p but no different point of K. From these two segments, construct $S = S_H \cap S_K$.

Since p was in both of the original segments, Problem 16 tells us that S contains p. However, since there were no elements of H or K in these segments (except possibly p), S contains no elements of H or K. Finally, because every element of $H \cup K$ is an element of H or K, we know that S contains no elements of $H \cup K$.

Therefore, S is a segment that contains p but no other elements of $H \cup K$. We have found a counterexample to the definition of a limit point, so p is not a limit point of $H \cup K$.

Solution 18. (Greg)

We will use a proof by contrapositive. The contrapositive is:

Show that if p is not a limit point of H and p is not a limit point of K, then p is not a limit point of $H \cup K$.

This is problem 17 (which has been proved), so we are finished.