Problem 8

Consider any point p. According to Axiom 4, there exists a largest integer M_x and a smallest integer M_y such that $M_x < p < M_y$. Then, choose x and y such that $M_x < x < p$ and $p < y < M_y$ and consider the segment (x, y) (Axiom 3).

Then, we will try to find an integer different from p inside (x, y). Consider any integer n. There are three cases:

1. $n = p$

   Here, n is not different from p, so we have not found an integer different from p. (Note that this case is not possible if p is not an integer.)

2. $n < p$

   Note that n must satisfy $n \leq M_x$; if $n > M_x$, we have contradicted Axiom 4. Then, from $n \leq M_x$ and $M_x < x$, $n < x$; and from $x < y$, $n < y$ (Axiom 2). Thus, n is not between x and y, so (x, y) does not contain n.

3. $n > p$

   This case is symmetric to the $n < p$ case.

Therefore, we can construct a segment containing p that does not contain any integers (positive or otherwise) different from p. Since every element of M is an integer, we have proven that there is a segment containing any point without an element of M, so M has no limit points.

Problem 9

Show that if p is a limit point of the point set M and S is a segment containing p, then S contains 2 points of M.

Arbitrarily define S as the segment (a, b). From the definition of a limit point, S contains a point q_1 of M different from p. Construct a new segment S_2 as follows:

1. If $q_1 < p$, $S_2 = (q_1, b)$
2. If $q_1 > p$, $S_2 = (a, q_1)$

Note that S_2 contains p, and that every point of S_2 is between a and b. Since S_2 contains p, it must contain a point q_2 from M. q_2 cannot be equal to q_1 (because q_1 is not in S_2), and q_2 is an element of S. Therefore, we have found two points q_1 and q_2 in S that are in M.

Fun note: if we continue this procedure (shortening the segment), then we can find as many points from M as we like in S!

Problem 11

This proof assumes the following facts:

If $\frac{1}{a}$ denotes the *reciprocal* of a, and $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.

Positive means > 0.

Choose two points x, y such that $x < 0 < y$. Every point of M is positive, so Axiom 2 implies that every point is greater than x.

To show that the segment (x, y) contains a point i from M, we must find a point such that $i < y$. However, every point of M is a reciprocal of a positive integer n (ie: $i = \frac{1}{n}$). Therefore, we are trying to find a point such that $\frac{1}{n} < y$. From the fact given at the start of this proof, this means that $\frac{1}{y} < n$. According to Axiom 4, there is an integer greater than $\frac{1}{y}$, so we can find a point from M inside (x, y). Therefore, every segment containing 0 also contains a different point from M, so 0 is a limit point of M.

Problem 12

This proof assumes the following fact:

If $\frac{1}{a}$ denotes the *reciprocal* of a, and $0 < a < b$, then $0 < \frac{1}{b} < \frac{1}{a}$.

There are two cases to consider (Axiom 1):

1. $p < 0$

   Every element of M is the reciprocal of a positive integer. Since the reciprocal of a positive number is positive, every element in M is positive. We can construct $S = (x, y)$ such that $x < p$ and $p < y < 0$ (Axiom 3). Then, since $y < 0$ and $0 < q$ for every q in M, no point of M is less than y, so S contains no points of M, and p is not a limit point of M.

2. $p > 0$

   Consider the point $\frac{1}{p}$, which is also positive. Showing that p is a limit point of M is equivalent to showing that $\frac{1}{p}$ is a limit point of the set of positive integers. However, according to Problem 8, the set of positive integers has no limit points. Therefore, p is not a limit point of M.

Problem 13

Show that if H and K are two point sets with a common point and p is a limit point of $H \cap K$, then p is a limit point of H and p is a limit point of K.

If p is a limit point of $H \cap K$, then every segment S containing p contains a point q that is in $H \cap K$.

From the definition of an intersection, every point in $H \cap K$ is in H and K. Therefore, q is in H and q is in K.

Finally, since every S contains a q that's in H, p must be a limit point of H (and similar for K).

Problem 14

It is not required that every interval containing p contains a point of M.

We will construct a counterexample. Consider the point set $M = (0, 1)$, which has the limit point $p = 1$. Then consider the interval $I = [1, 2]$. From the definition of an interval, I contains 1, so I contains p. Then, we will show that no points of M are in I.

Consider an arbitrary point x in M. Since x is between 0 and 1, $x < 1$. In order to be in I, we would need $x = 1$ (bottom point) or $x > 1$ (segment and top point). However, either of these would contradict Axiom 1, so x cannot be in I. Thus, no points of M are in I, so we have found a counterexample.

Problem 17

Direct proof:

Since p is not a limit point of H, then we can find a segment that contains p with no other points of H.

Proof by contradiction:

(If p is a limit point of $H \cup K$, then we have a contradiction.)

Suppose that p is a limit point of $H \cup K$. Then, every segment containing p contains a point q of $H \cup K$. According to the definition of a union, either q is in H or q is in K. However, if q is in H, then S contains a point of H, so p is a limit point of H. Likewise, if q is in K, then S contains a point of K, so p is a limit point of K. Therefore, we have a contradiction, so p must not be a limit point of $H \cup K$.

(this looks faulty.)

Contrapositive:

(Show that if p is a limit point of $H \cup K$, then p is a limit point of H or p is a limit point of K.)

Every segment containing p also contains a point q of $H \cup K$. From the definition of a union, q is either a point of H or a point of K.

Problem 18

Show that if H and K are two point sets and p is a limit point of $H \cup K$, then p is a limit point of H or p is a limit point of K.

Let's prove the contrapositive:

Show that if p is not a limit point of H and p is not a limit point of K, then p is not a limit point of $H \cup K$.

...and this is true, as proven in the previous problem.