Main Course Notes

There are 6 axioms that we will use in this class:

Axiom 1. If each of a and b is a number, then exactly one of the following is true:

1. $a = b$
2. $a < b$
3. $a > b$

Axiom 2. If each of a, b, c is a number, $a < b$, and $b < c$, then $a < c$.

Axiom 3. If a and b are two points on the number line, then there is a point between them.

Axiom 4. If a is a point, then there is a smallest integer b such that $b > a$ and a largest integer c such that $c < a$.

Axiom 5. If n is an integer, then $n + 1$ and $n - 1$ are integers, and n is the only integer between $n - 1$ and $n + 1$.

Axiom 6. If M is a point set which is bounded above (below) then there is a point p such that either

1. $p \in M$ and p is the largest (smallest) number in M, or
2. $p \notin M$ and p is the smallest (largest) number such that each point of M is to the left (right) of p.

Then, as the course continues, we will make various definitions and notation for these definitions.

Definition 1. If a and b are two points and $a < b$, the statement that the point p is **between** the points a and b means that $a < p$ and $p < b$.

Definition 2. A **point set** is a set of one or more points.

Definition 3. The statement that the point set S is a **segment** means that there are two points a and b, called the *endpoints* of S, such that S is the set of all points between a and b.

Notation 1. If a and b are two points and $a < b$ then (a, b) denotes the segment consisting of all points between a and b.

Definition 4. If M is a point set and p is a point, the statement that p is a **limit point** of the point set M means that every segment that contains p contains a point of M different from p.

Definition 5. The statement that the point set I is an **interval** means that there are two points a and b, called the *endpoints* of I, such that I is the set containing a, b, and (a, b). I is denoted by $[a, b]$.

Notation 2. If a and b are two points and $a < b$ then $[a, b]$ denotes the interval with endpoints a and b.

Definition 6. The statement that the point set H is a **subset** of the point set K means that if p is a point of H, then p is a point of K.

Notation 3. If H is a point set and K is a point set then $H \subseteq K$ means that H is a subset of K.

Definition 7. If each of H and K is a point set and there is a point that is in both of them, then the **intersection** of H and K is the set to which a point p belongs if and only if p is in both H and K.

Notation 4. If each of H and K is a point set and there is a point that is in both of them, then $H \cap K$ denotes the intersection of H and K.

Definition 8. If each of H and K is a point set, the **union** of H and K is the set to which the point p belongs if and only if p is in H or p is in K.

Notation 5. If each of H and K is a point set, then $H \cup K$ denotes the union of H and K. Thus $H \cup K$ is the set of all points in H together with the points in K.

Notation 6. If each of a, b, and c is a number, we will use the notation $M = \{a, b, c\}$ to mean the set containing the points a, b, and c and no other point. Similarly, we will denote infinite sets, when the pattern is clear, by $M = \{a_1, a_2, a_3, \ldots\}$. For example, the set of all positive integers is denoted by $M = \{1, 2, 3, \ldots\}$.

Definition 9. The statement that the point set M is **infinite** means that for every positive integer n, M contains at least n points.

Definition 10. The statement that the point set M is **finite** means that it is not infinite. That is, there is a positive integer n such that M does not contains n points.

By a **function** we mean a set of ordered number pairs, no two of which have the same first term. Or, if you prefer, a function is a set of points in the number plane with no two on the same vertical line. If f is a function, then the **domain** of f is the set of all first terms of ordered pairs of f and the **range** of f is the set of all second terms of ordered pairs of f. By a **sequence** we mean a function whose domain is the set of positive integers and whose range is a point set.

Usually if f is a function and (x, y) is one of the ordered pairs in f, then we denote y by $f(x)$. When f is a sequence and (n, y) is one of the ordered pairs in f, then we usually denote y by the short hand f_n. Thus we might refer to a sequence by the name of the function as f for example. Or we might refer to a sequence as $f_1, f_2, f_3, \ldots$. By a term of a sequence f, or a point of (or in) the sequence, we mean f_n for some positive integer n.

Definition 11. The statement that the sequence $x_1, x_2, x_3, \ldots$ **converges** to the number c or has c as a **limit** means that for every segment S containing c, there is a positive integer n such that each of $x_n, x_{n+1}, x_{n+2}, \ldots$ is in S. (In other words, for every positive integer $m \geq n$, x_m is in S.)

Note that a sequence is not a point set, but a set of ordered pairs. As such, it does not have a limit point. However, the range of a sequence is a point set and thus might or might not have a limit point.

Definition 12. The statement that the point set M is **bounded above** means that there is a number c such that each point of M is to the left of c.

Definition 13. The statement that the point set M is **bounded below** means that there is a number c such that each point of M is to the right of c.

(Notation) If p is a point and M is a point set, then $p \in M$ means that p is a point in M. Similarly, $p \notin M$ means that p is not in M.

Definition 14. The statement that the sequence $x_1, x_2, x_3, \ldots$ is an **increasing** (**decreasing**) sequence means that for each positive integer n, $x_n < x_{n+1}$ ($x_n < x_{n+1}$).

Definition 15. The statement that the sequence $x_1, x_2, x_3, \ldots$ is **non decreasing** (**non increasing**) means that for each positive integer n, $x_n \leq x_{n+1}$ ($x_n \geq x_{n+1}$).

Definition 16. The statement that the point set M is **bounded** means that M is bounded above and M is bounded below. Equivalently, there is a segment that contains M.

Definition 17. If M is a point set that is bounded above, then the **least upper bound** of M is the number that is the largest number in M or the smallest number that is larger than each number in M. It is denoted by $\mathrm{lub}(M)$.

Definition 18. If M is a point set that is bounded below, then the **greatest lower bound** of M is the number that is the smallest number in M or the largest number that is smaller than each number in M. It is denoted by $\mathrm{glb}(M)$.

From here on, we will be talking about points in a plane. We will denote them by their coordinates as (x, y). You will have to determine from the context whether (x, y) denotes a segment from x to y or the point with coordinates x and y. We will also have to be careful to distinguish between points and points on the x-axis, which are really points of the form $(x, 0)$ but which we often think of as numbers.

Definition 19. The statement that f is a **simple graph** (or a **function**) means that f is a set of points in a plane such that no vertical line contains two of them.

The following two definitions are equivalent:

Definition 20. The statement that the simple graph f is **continuous** at the point $(x, f(x))$ means that if S is a segment containing $f(x)$ then there is a segment T containing x such that if $t \in T$ and t is in the domain of f, then $f(t) \in S$.

Definition 21. The statement that the simple graph f is **continuous** at the point $(x, f(x))$ means that if A and B are any two horizontal lines with the point $(x, f(x))$ between them, then there are two vertical lines a and b with the point $(x, f(x))$ between them such that every point of the graph f that is between a and b is also between A and B.