$$\sum_{i,\tilde{h},j}(-1)^{i+j}q^{\frac{i}{2}+\tilde{h}}\check{c}^{i,\tilde{h},j}(y^{i+1}-y^{-(i+1)})u^{-j}=\left(\frac{-i\theta_1(\tau,2z)\eta(\tau)^6}{q^{3/8}\theta_4(\tau,z)^2L^{\mathrm{NS}}(\tau,z)}\right)\prod_{m,\ell,\ell'}\frac{1}{(1-q^my^\ell u^{\ell'})^{c^{(2,2)}(m,\ell,\ell')}},\tag{B.31}$$

where again we have dropped the subscript of N on $\check{c}^{i,\tilde{h},j}$ to indicate we have taken the limit of $N\to\infty$, and we define

$$\sum_{m,\ell,\ell'}c^{(2,2)}(m,\ell,\ell')q^my^\ell u^{\ell'}=f^{(2,2)}(u^{-1},q,y,u)-(1-2u^{-1}+u^{-2})\equiv g^{(2,2)}(q,y,u).\tag{B.32}$$

Then we take the large i limit of $\check{c}^{i,\tilde{h},3i+5\tilde{h}-k'}$. Redefining $\tilde{q}\equiv qy^2u^{-6}$, removing $(1-\tilde{q}^{1/2})^{-1}$, and using (B.2), we obtain

$$\lim_{i\to\infty}\sum_{\tilde{h},k'}(-1)^{\tilde{h}+k'}\check{c}^{i,\tilde{h},3i+5\tilde{h}-k'}y^{-2\tilde{h}}u^{\tilde{h}+k'}=\left(\frac{-i\theta_1(6\nu-2z,2z)\eta(6\nu-2z)^6}{y^{1/4}u^{9/4}\theta_4(6\nu-2z,z)^2L^{\mathrm{NS}}(6\nu-2z,z)}\right)\prod_{\ell,\ell'}\frac{1}{(1-y^\ell u^{\ell'})^{\tilde{d}^{(2,2)}(\ell,\ell')}}\tag{B.33}$$

where we define

$$\sum_{\ell,\ell'}\tilde{d}^{(2,2)}(\ell,\ell')y^\ell u^{\ell'}=g^{(2,2)}(y^{-2}u^6,y,u)-1.\tag{B.34}$$

Finally we rewrite (B.33) by defining $\alpha\equiv uy^{-2}$ to get

$$\lim_{i\to\infty}\sum_{\tilde{h},k'}(-1)^{\tilde{h}+k'}\check{c}^{i,\tilde{h},3i+5\tilde{h}-k'}\alpha^hu^{k'}=-\frac{\alpha}{u}\left(\prod_{n=1}^{\infty}\frac{(1-\alpha^nu^{5n})^6(1-\alpha^{n-2}u^{5n-4})(1-\alpha^{n+1}u^{5n-1})}{(1+\alpha^{n-1}u^{5n-2})^4(1+\alpha^nu^{5n-3})^4}\right)\left(\prod_{\ell,\ell'}\frac{1}{(1-(-1)^{\ell+\ell'}\alpha^\ell u^{\ell'})^{d^{(2,2)}(\ell,\ell')}}\right),\tag{B.35}$$

where we define

$$\sum_{\ell,\ell'}d^{(2,2)}(\ell,\ell')\alpha^\ell u^{\ell'}=g^{(2,2)}(\alpha u^5,\alpha^{-1/2}u^{1/2},u)-1\equiv F^{(2,2)}(\alpha,u).\tag{B.36}$$