We claim that q is adjacent to p in C_k. Indeed, conditions C1 and C3 hold trivially. To verify C2, consider a curve $e \in S_2$ touching d at a point in the arc of d from q to q'. This touching point (as the whole arc of d from q to q') lies inside T_i. However, e must intersect b, so it must leave T_i. It touches a, so it must leave through either b_i or c'_i. Here c'_i has fewer than $2\alpha^2 k$ points in X, by our choice of i, while b_i contains fewer than $\alpha^2 k$ points in X, since Case 2 does not hold. Hence, e has fewer than $3\alpha^2 k$ points where it can leave T_i, and there are at most $3\alpha^2 k$ possible choices for the curve e. This means that condition C2 is satisfied and q is adjacent to p in C_k.

By our choice of i, we can select the curve d in at least k/α different ways, each giving rise to a different edge in C_k incident to p. The total weight of these edges is α. This completes the analysis of the last case, showing that the total weight of all edges in A_k, A'_k, A''_k, B_k and C_k incident to p is at least α.

Summing over all l possible values of k and over all n^2 touching points in T, we conclude that the total weight of G is at least $\alpha\lceil\log n\rceil n^2$. Comparing this lower bound with the upper bound proved in the preceding subsection and substituting $\alpha = \sqrt{\log n / \log\log n}$, Theorem 7 follows.

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