where the operator $\mathcal{D}$ is defined as

$$\mathcal{D}[...] = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2 ...\right]. \tag{56}$$

The expressions we then obtain for the components $\hat{u}_r^\epsilon$, $\hat{u}_\theta^\epsilon$, $\hat{u}_\phi^\epsilon$ of the flow field $\hat{\boldsymbol{u}}^\epsilon$ are given in Appendix C.

B. First-order solution

We now consider the derivation for the first-order solution (in Reynolds) $\boldsymbol{u}^{(1)}$. That flow field contains terms of different frequencies, but we are here only interested in the steady part of the flow. For the sake of simplicity, we use $\boldsymbol{u}^{(1)}$ to denote to the steady component of this first-order flow. The latter sastifies the following set of equations

$$\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{(1)} = \frac{1}{4}[(\underline{\hat{\boldsymbol{u}}}^{(0)}\cdot\boldsymbol{\nabla})\,\hat{\boldsymbol{u}}^{(0)} + (\hat{\boldsymbol{u}}^{(0)}\cdot\boldsymbol{\nabla})\,\underline{\hat{\boldsymbol{u}}}^{(0)}], \tag{57}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u}^{(1)} = 0, \tag{58}$$

where complex conjugate quantities are underlined. In the first-order governing equations, the term $(\boldsymbol{v}^{\parallel}\cdot\boldsymbol{\nabla})\boldsymbol{u}^{(0)}$ has been dropped since this term is time-dependent (dimensionless frequency 1) and we are only interested in steady flows. Equations (57) and (58) have to be completed by the boundary conditions

$$\boldsymbol{u}^{(1)} = \boldsymbol{v}^{(1)} \quad \text{on } \mathcal{S}, \tag{59}$$

$$\boldsymbol{u}^{(1)} \to \boldsymbol{0} \quad \text{at infinity}. \tag{60}$$

where the unknown quantity $\boldsymbol{v}^{(1)}$ is linked to $\boldsymbol{v}^{\parallel}$ by the relationship

$$\boldsymbol{v}^{\parallel} = Re\,\boldsymbol{v}^{(1)}. \tag{61}$$

In order to obtain the first-order translation speed, we could try to derive the full velocity and stress fields $\boldsymbol{u}^{(1)}$ and $\boldsymbol{\sigma}^{(1)}$, and integrate the stress over the particle surface to obtain the propulsive force. However, it is more convenient to use a suitable version of the reciprocal theorem, as suggested by Ho & Leal [33] (the standard version of the Lorentz reciprocal theorem can be found in Ref. [35]).

C. Reciprocal theorem and propulsion speed

For the same geometry, we consider now an auxiliary Stokes velocity and stress fields $(\bar{\boldsymbol{u}}, \bar{\boldsymbol{\sigma}})$ satisfying

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\sigma}} = \boldsymbol{0}, \tag{62}$$

$$\boldsymbol{\nabla}\cdot\bar{\boldsymbol{u}} = 0, \tag{63}$$

with suitable boundary conditions to be specified below. Subtracting the inner product of equation (57) with $\bar{\boldsymbol{u}}$ and the inner product of equation (62) with $\boldsymbol{u}^{(1)}$, and integrating over the volume of fluid $\mathcal{V}$ leads to the equality of virtual powers as

$$\int_{\mathcal{V}}[\bar{\boldsymbol{u}}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{(1)}) - \boldsymbol{u}^{(1)}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\sigma}})]\,d\mathcal{V} = \frac{1}{4}\int_{\mathcal{V}}\bar{\boldsymbol{u}}\cdot[(\underline{\hat{\boldsymbol{u}}}^{(0)}\cdot\boldsymbol{\nabla})\,\hat{\boldsymbol{u}}^{(0)} + (\hat{\boldsymbol{u}}^{(0)}\cdot\boldsymbol{\nabla})\,\underline{\hat{\boldsymbol{u}}}^{(0)}]\,d\mathcal{V}. \tag{64}$$

Then, using the general vector identity

$$\bar{\boldsymbol{u}}\cdot(\boldsymbol{\nabla}\cdot\boldsymbol{\sigma}^{(1)}) - \boldsymbol{u}^{(1)}\cdot(\boldsymbol{\nabla}\cdot\bar{\boldsymbol{\sigma}}) = \boldsymbol{\nabla}\cdot(\bar{\boldsymbol{u}}\cdot\boldsymbol{\sigma}^{(1)} - \boldsymbol{u}^{(1)}\cdot\bar{\boldsymbol{\sigma}}) + (\boldsymbol{\nabla}\boldsymbol{u}^{(1)} : \bar{\boldsymbol{\sigma}} - \boldsymbol{\nabla}\bar{\boldsymbol{u}} : \boldsymbol{\sigma}^{(1)}), \tag{65}$$