$$s = 4\pi L_1 L_2 r_h^{\theta-(1+\xi)} \sim T^{\frac{1+\xi-\theta}{z}} \,. \tag{3.22}$$

In (3.20) and (3.21) we replaced T with r_h by using (3.19) to simplify the expression. Thus, the diffusivities are

$$\begin{aligned} D_{T,x} &= \frac{\kappa_{xx}}{c_\rho} = \frac{z}{(2z-2)(\theta-1-\xi)} L_r L_1^{-2} r_h^{2-z} \,, \\ D_{T,y} &= \frac{\kappa_{yy}}{c_\rho} = \frac{z}{(2z-2(2-\xi))\,(\theta-1-\xi)} L_r L_2^{-2} r_h^{2\xi-z} \,, \end{aligned} \tag{3.23}$$

and the butterfly velocities (3.15) are

$$v_{B,x}^2 = \frac{2\pi T}{\theta-1-\xi} L_r L_1^{-2} r_h^{2-z} \,, \qquad v_{B,y}^2 = \frac{2\pi T}{\theta-1-\xi} L_r L_2^{-2} r_h^{2\xi-z} \,. \tag{3.24}$$

Finally, by noticing that $\tau_L = (2\pi T)^{-1}$ we have

$$\mathcal{E}_x = \frac{D_{T,x}}{v_{B,x}^2 \tau_L} = \frac{1}{2}\frac{z_x}{z_x-1} = \frac{1}{2}\frac{z}{z-1} \,, \tag{3.25}$$

$$\mathcal{E}_y = \frac{D_{T,y}}{v_{B,y}^2 \tau_L} = \frac{1}{2}\frac{z_y}{z_y-1} = \frac{1}{2}\frac{z}{z-\xi} \,. \tag{3.26}$$

Notice that the $\mathcal{E}_x$ and $\mathcal{E}_y$ depend only on z and ξ irrespective of θ and ζ. They are also independent of charge density ρ and momentum relaxations k_1 and k_2. This universality is nontrivial because the thermal conductivities, specific heat and butterfly velocity, all of them depend on $(\theta, \zeta, \rho, k_1, k_2)$ through (L_r, L_1, L_2, r_h). When it comes to the combinations $\mathcal{E}_x$ and $\mathcal{E}_y$, all L_r, L_1, L_2 and r_h are canceled out.

To investigate if there is any lower or upper bound of $\mathcal{E}_x$ and $\mathcal{E}_y$, we need to understand the parameter region of z and ξ. We will restrict ourselves to positive z_i. Based on the allowed parameter region obtained in section we find

- Class I and II

$$\frac{\lambda_2}{\lambda_1} \geq 1 \quad \Rightarrow \quad \frac{1}{2} \leq \mathcal{E}_x < \frac{1}{2}\left(\frac{1}{1-\xi^{-1}}\right) \,, \qquad \frac{1}{2} \leq \mathcal{E}_y \,, \tag{3.27}$$

$$\frac{\lambda_1}{\lambda_2} \geq 1 \quad \Rightarrow \quad \frac{1}{2} \leq \mathcal{E}_x \,, \qquad \frac{1}{2} \leq \mathcal{E}_y < \frac{1}{2}\left(\frac{1}{1-\xi}\right) \,, \tag{3.28}$$

  where $\xi = \frac{\lambda_2}{\lambda_1}$.

- Class I-i