$$
\begin{aligned}
-\epsilon \iint_{\Omega\times\mathbb{S}^1} (\vec{w}\cdot\nabla_x\phi)(u-\bar{u}) &\leq C\epsilon \left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} \left\|\xi\right\|_{H^2(\Omega)} \\
&\leq C\epsilon \left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} \left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} .
\end{aligned} \tag{3.18}
$$

Using the trace theorem, we have

$$
\begin{aligned}
\epsilon \int_\Gamma u\phi \mathrm{d}\gamma = \epsilon \int_{\Gamma^+} u\phi \mathrm{d}\gamma + \epsilon \int_{\Gamma^-} u\phi \mathrm{d}\gamma &\leq C\epsilon \left\|\phi\right\|_{L^2(\Gamma)} \bigg(\left\|u\right\|_{L^2(\Gamma^+)} + \left\|h\right\|_{L^2(\Gamma^-)} \bigg) \\
\leq C\epsilon \left\|\phi\right\|_{H^1(\Omega\times\mathbb{S}^1)} \bigg(\left\|u\right\|_{L^2(\Gamma^+)} + \left\|h\right\|_{L^2(\Gamma^-)} \bigg) &\leq C\epsilon \left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} \bigg(\left\|u\right\|_{L^2(\Gamma^+)} + \left\|h\right\|_{L^2(\Gamma^-)} \bigg).
\end{aligned} \tag{3.19}
$$

Also, we obtain

$$
\iint_{\Omega\times\mathbb{S}^1} (u-\bar{u})\phi \leq C \left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} \left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} , \tag{3.20}
$$

$$
\iint_{\Omega\times\mathbb{S}^1} f\phi \leq C \left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} \left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)} . \tag{3.21}
$$

Collecting terms in (3.17), (3.18), (3.19), (3.20) and (3.21), we obtain

$$
\epsilon \left\|\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)} \leq C \left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} \bigg(\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} + \epsilon \left\|u\right\|_{L^2(\Gamma^+)} + \left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)} + \epsilon \left\|h\right\|_{L^2(\Gamma^-)} \bigg). \tag{3.22}
$$

Then this naturally implies that

$$
\epsilon \left\|\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} \leq C\bigg(\left\|u-\bar{u}\right\|_{L^2(\Omega\times\mathbb{S}^1)} + \epsilon \left\|u\right\|_{L^2(\Gamma^+)} + \left\|f\right\|_{L^2(\Omega\times\mathbb{S}^1)} + \epsilon \left\|h\right\|_{L^2(\Gamma^-)} \bigg). \tag{3.23}
$$

Step 2: Energy Estimate.
In the weak formulation (3.11), we may take the test function $\phi = u$ to get the energy estimate

$$
\frac{1}{2}\epsilon \int_\Gamma \left|u\right|^2 \mathrm{d}\gamma + \left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)} = \iint_{\Omega\times\mathbb{S}^1} fu. \tag{3.24}
$$

Then we have

$$
\frac{1}{2}\epsilon \left\|u\right\|^2_{L^2(\Gamma^+)} + \left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)} = \iint_{\Omega\times\mathbb{S}^1} fu + \epsilon \left\|h\right\|^2_{L^2(\Gamma^-)} . \tag{3.25}
$$

On the other hand, we can square on both sides of (3.36) to obtain

$$
\epsilon^2 \left\|\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)} \leq C\bigg(\left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)} + \epsilon^2 \left\|u\right\|^2_{L^2(\Gamma^+)} + \left\|f\right\|^2_{L^2(\Omega\times\mathbb{S}^1)} + \epsilon^2 \left\|h\right\|^2_{L^2(\Gamma^-)} \bigg). \tag{3.26}
$$

Multiplying a sufficiently small constant on both sides of (3.26) and adding it to (3.25) to absorb $\left\|u\right\|^2_{L^2(\Gamma^+)}$ and $\left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)}$, we deduce

$$
\epsilon \left\|u\right\|^2_{L^2(\Gamma^+)} + \epsilon^2 \left\|\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)} + \left\|u-\bar{u}\right\|^2_{L^2(\Omega\times\mathbb{S}^1)} \leq C\bigg(\left\|f\right\|^2_{L^2(\Omega\times\mathbb{S}^1)} + \iint_{\Omega\times\mathbb{S}^1} fu + \epsilon \left\|h\right\|^2_{L^2(\Gamma^-)} \bigg). \tag{3.27}
$$