$$(1-\widehat{\delta}')\phi^{(t+1)} \geq \phi^{(t)} - \mu\sqrt{(\phi^{(t)})^2 + (\theta^{(t)})^2}, \tag{157}$$

$$(1-\widehat{\delta}')\theta^{(t+1)} \leq \theta^{(t)} + \mu\sqrt{(\phi^{(t)})^2 + (\theta^{(t)})^2}, \tag{158}$$

where μ is the upper bound of term $\eta(1.5\|\widetilde{\Delta}_l'^{(t)}\| + \|\widetilde{\Delta}_u'^{(t)}\|)\|\mathbf{T}'^{-1}\|$ and can be obtained by

$$\mu \triangleq \eta\rho\mathcal{S}\mathcal{P}(4+62\widehat{c}). \tag{159}$$

Quantifying the Norm of $\mathbf{v}^{(t)}$ Projected at Different Subspaces: Then, we will use mathematical induction to prove

$$\theta^{(t)} \leq 4\mu t\phi^{(t)}. \tag{160}$$

It is true when $t=0$ since $\|\theta^{(0)}\| \overset{(115)}{=} 0$.

Assuming that equation (160) is true at the tth iteration, we need to prove

$$\theta^{(t+1)} \leq 4\mu(t+1)\phi^{(t+1)}. \tag{161}$$

Applying (157) into RHS of (161), we have

$$4\mu(t+1)\phi^{(t+1)} \geq \frac{4\mu(t+1)}{1-\widehat{\delta}'}\left(\phi^{(t)} - \mu\sqrt{(\phi^{(t)})^2 + (\theta^{(t)})^2}\right), \tag{162}$$

and substituting (158) into LHS of (161), we have

$$\theta^{(t+1)} \leq \frac{(4\mu t\phi^{(t)}) + \mu\sqrt{(\phi^{(t)})^2 + (\theta^{(t)})^2}}{1-\widehat{\delta}'}. \tag{163}$$

Then, our goal is to prove RHS of (162) is greater than RHS of (163). After some manipulations, it is sufficient to show

$$(1+4\mu(t+1))\left(\sqrt{(\phi^{(t)})^2 + (\theta^{(t)})^2}\right) \leq 4\phi^{(t)}. \tag{164}$$

In the following, we will show that the above relation is true.

First step : We know that

$$4\mu(t+1) \leq 4\mu T \overset{(159)}{\leq} 4\eta\rho\mathcal{S}\mathcal{P}(4+62\widehat{c})\widehat{c}\mathcal{T} \overset{(107d)(159)}{\leq} \frac{4\widehat{c}\eta^2 L_{\max}^2(4+62\widehat{c})}{\kappa\log(\frac{d\kappa}{\delta})} \overset{(a)}{\leq} 1 \tag{165}$$

where (a) is true because we choose $c'_{\max} = 1/(2\widehat{c}(4+62\widehat{c}))$ and $\eta \leq c'_{\max}/L_{\max}$.

Second step : Also, we know that

$$4\phi^{(t)} \geq 2\sqrt{2(\phi^{(t)})^2} \overset{(160),(165)}{\geq} (1+4\mu(t+1))\sqrt{(\phi^{(t)})^2 + (\theta^{(t)})^2}.$$

With the above two steps, we have $\theta^{(t+1)} \leq 4\mu(t+1)\phi^{(t+1)}$, which completes the induction.