$\mathcal{M}$	$\dim(V_{\text{off}})$	$L_2^a(c_1)$	$H_1^a(c_1)$	$L_2^a(c_2)$	$H_1^a(c_2)$	$H^Q(c_1,c_2)$
Standard GmsFEM (un-coupled)						
6	1452	0.837	1.925	0.837	30.542	13.869
8	1936	0.293	0.358	0.293	26.427	11.912
12	2904	0.148	0.193	0.148	23.165	10.440
16	3872	0.108	0.089	0.108	17.945	8.086
Simplified basis functions (un-coupled)						
all	536	2.343	3.870	2.343	36.649	16.872
Standard GmsFEM (coupled)						
6	1452	1.944	2.584	1.944	6.942	3.934
8	1936	1.070	1.200	1.070	2.197	1.452
12	2904	0.359	0.544	0.359	0.788	0.606
16	3872	0.129	0.105	0.129	0.375	0.193
Simplified basis functions (coupled)						
all	830	2.105	3.399	2.105	4.122	3.557

Table 3: Dual-continuum background. Numerical results of relative errors (%) at the final simulation time. $DOF_f = 17834$. $Q_1 = 250 \cdot \kappa_m$ and $Q_2 = 920 \cdot \kappa_m$.

using RVE computations. This is based on a localization assumption, which we introduce next.

We consider $\mathcal{H}^\omega$, which is the harmonic expansion in ω, which is defined by solving local problems in each K. We can use

$$\int_D \kappa \nabla \mathcal{H}^{\omega_j}(\phi_{i,fine}^{\omega_j}) \cdot \nabla \mathcal{H}^{\omega_l}(\phi_{m,fine}^{\omega_l})\, dx$$

to approximate the elements of the stiffness matrix. Our localization assumption uses the local snapshots computed in the RVE for each ω_i, which we denote by RVE_i. We denote these RVE snapshots by $\psi_{j,fine}^{RVE_i}$. Then, we propose the following localization assumption

$$\int_D \kappa \nabla \mathcal{H}^{\omega_j}(\phi_{i,fine}^{\omega_j}) \cdot \nabla \mathcal{H}^{\omega_l}(\phi_{m,fine}^{\omega_l})\, dx \approx \int_D \kappa \nabla \mathcal{H}^{RVE_j}(\chi_{fine}^{\omega_j}\psi_{i,fine}^{RVE_j}) \cdot \nabla \mathcal{H}^{RVE_l}(\chi_{fine}^{\omega_l}\psi_{m,fine}^{RVE_l})\, dx.$$

3.2 Numerical simulation of the shale gas transport

In this section, we add a case study for our method. We follow the example considered in [2], where a shale gas transport with dual-continuum (organic and inorganic pores) (see also [3]) is studied. In inorganic matter, we have

$$\varphi_i \frac{\partial c}{\partial t} = \operatorname{div}((\varphi_i D_i + cZRT\frac{\kappa_i}{\mu})\nabla c) + Q_{ki}.$$