4.1 $\mathcal{T}_4$ mirror maps from coassociative fibrations

Let us begin by analyzing the generalized $\mathcal{T}_4$ mirror symmetry map in the context of TCS G_2 manifolds. As the Kovalev CYs $X_\pm$ admit mirrors by assumption, by the SYZ argument, these have two T^3 special lagrangian fibrations. Let us denote such fibrations by $L_\pm$. In the asymptotically cylindrical region of the manifold $X_\pm \sim \mathbb{R}^+ \times S^1 \times S_\pm$, the SYZ special lagrangians must asymptote to $L_\pm \sim S^1 \times \Lambda_\pm$, where $\Lambda_\pm$ are the SYZ special lagrangian T^2 fibrations within the asymptotic K3s (with respect to the K3 complex structure induced by the ambient CY). In particular, they do not extend along the $\mathbb{R}^+$ direction. Let us fix the phase of the holomorphic top form on $X_\pm$ such that the SYZ special lagrangians are calibrated as follows

$$\mathrm{vol}_{L_\pm} = -\mathrm{Im}\left(\Omega^{3,0}_\pm\right)|_{L_\pm}. \tag{4.4}$$

Notice that a special lagrangian L gives rise to a coassociative cycle $M_L \equiv S^1 \times L \subset S^1 \times X$ if and only if it satisfies (4.4). In particular,

$$\mathrm{Im}(\Omega^{3,0}_\pm) = d\theta_\pm \wedge I_{S_\pm} - dt_\pm \wedge R_{S_\pm} \tag{4.5}$$

therefore, for our special lagrangians $L_\pm$ we have

$$\mathrm{Im}(\Omega^{3,0}_\pm)|_{L_\pm} = -d\theta_\pm \wedge I_{S_\pm}. \tag{4.6}$$

Now, Equation (3.3) entails that the four-cycles $M_\pm \equiv (S^1)_\pm \times L_\pm \subset J_\pm$ are such that

$$\star\,\varphi_\pm|_{M_\pm} = d\xi_\pm \wedge d\theta_\pm \wedge \left(I_{S_\pm}\right)|_{\Lambda_\pm} = \mathrm{vol}_{M_\pm} \tag{4.7}$$

which entails these are coassociative submanifolds. Since

$$\begin{aligned}\Xi^*(\star\,\varphi_-|_{M_-}) &= \Xi^*(d\xi_- \wedge d\theta_- \wedge I_{S_-})\\ &= -d\theta_+ \wedge d\xi_+ \wedge -I_{S_+}\\ &= d\xi_+ \wedge d\theta_+ \wedge I_{S_+}\\ &= \star\,\varphi_+|_{M_+},\end{aligned} \tag{4.8}$$

the TCS glueing diffeomorphism Ξ is such that $M_\pm$ are glued into a coassociative submanifold $M \subset J$ which has the topology of a T^4, which may become singular along loci in J.

Performing three T-dualities along the $L_\pm$ SYZ fibres is mapping $X_\pm$ to their mirrors $X^\vee_\pm$ by construction. However, as the S^1 within the asymptotic CY cylinders are swapped with