Proof The identity (2.15), for $d = r = k+1$, is by definition true in the initial seed. Moreover, the exchange polynomials $\theta_1^0, \ldots, \theta_{n-1}^0$ are exactly the ones that appear in a standard cluster algebra of type C, and they are unaffected by mutation : indeed, only the monomials $u_k^{\pm}$ change, in accordance with the mutations of the exchange matrix. Therefore, cluster variables that do not correspond to diameters behave the same way as in a standard cluster algebra of type C, as described in [13]. The proof for the first case is thus similar to those found in [3] and [13].

The second equation (2.16), for $k = n-1$, is also by definition true in the initial seed. Since every cluster contains exactly one variable of the form $x_{a,\overline{a}}$, and any mutation of a variable $x_{a,\overline{a}}$ yields a variable corresponding to another diameter, we can deduce from the initial cluster that all variables $x_{a,\overline{a}}$ are linked by a mutation in direction n. In the initial cluster $(x_{\overline{2n},2k},\ k \in [\![1,n]\!])$, we have

$$x_{\overline{2n},2n}x_{\overline{2n-2},2n-2} = x_{\overline{2n},2n-2}^2 + \lambda x_{\overline{2n},2n-2} + 1. \tag{2.17}$$

The general relation (2.16) can be obtained directly in the following cluster (see Figure 6) :

$$\begin{aligned} \mu_{n-1}\mu_{n-2}\ldots\mu_{k+1}&(x_{\overline{2n},2k},\ k \in [\![1,n]\!]) \\ &= (x_{\overline{2n},2}, x_{\overline{2n},4}, \ldots, x_{\overline{2n},2k}, x_{2k,2k+4}, x_{2k,2k+6}, \ldots, x_{2k,2n}, x_{\overline{2n},2n}), \end{aligned} \tag{2.18}$$

where performing the mutation μ_n maps $x_{\overline{2n},2n}$ to $x_{\overline{2k},2k}$, and θ_n^0 gives (2.16).

Indeed, recall that θ_n^0 is unaffected by mutation, so that in order to understand μ_n, it is enough to know how the matrix B mutates, namely in the standard way (Definition 2). This determines the variables x_{ab} appearing in the monomials u_n^+ and u_n^- in the mutated cluster above, thus yielding (2.16). $\square$

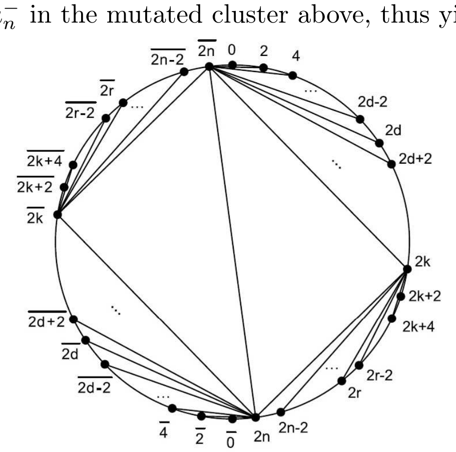

FIGURE 6 – The mutated cluster $\mu_{n-1}\mu_{n-2}\ldots\mu_{k+1}(x_{\overline{2n},2k},\ k \in [\![1,n]\!])$