Boosting Data Reduction for the Maximum Weight Independent Set Problem Using Increasing Transformations

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Article summary

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Maximum Weight Independent Set Problem

- Given a graph $G = (V, E)$ and a weight function $w : V \to \mathbb{R}^+$, the goal of the *Maximum Weight Independent Set Problem* is to compute a set of pair-wise non-adjacent vertices $I \subseteq V$, whose total weight is maximum.

Related work

- **Exact methods**: the most used methods are based on branch-and-bound with special pruning rules so that the explored space it's reduced significantly. Also, the most promising one is the branch-and-reduce algorithm for MWIS, presented by Lamm et al. [2].
- **Heuristic methods**: The most used heuristic approach is the Local Search (or variants of this method), which usually computes an initial solution and then tries to improve it by simple operations (insertions, removals or swaps).

Structions

- Originally, the **struction** (STability number RedUCTION) was introduced by Ebenegger et al. [3] and was later improved by Alexe et al. [4].
- This method is a graph transformation that can be applied to an arbitrary vertex and reduces the stability number by exactly one. Thus, after successive applications of the struction, the stability number of the graph can be determined.

Structions

- In general, a struction is applied on a center vertex v, which has the neighborhood $N(v)$. The struction removes the vertex v from the graph G, producing a new graph G', and reduces the weighted independence number of the graph G by its weight, i.e. $\alpha_w(G) = \alpha_w(G') + w(v)$.

- Layering is a method that partitions a set M that contains vertices $v_{x,y}$. These vertices are indexed by two parameters $x \in X, y \in Y$, where the sets X and Y either contain vertices or vertex sets. A layer L_k is defined as the set of all vertices having k as first parameter ($k \in X$). Conversely, the layer of a vertex $v_{x,y}$ is $L(v_{x,y}) = k$.

Original Struction

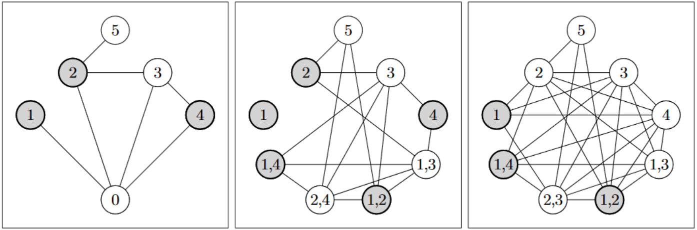
- Let $v \in V$ be a vertex with minimum weight $w(v)$ among its neighbors. The graph it's transformed as follows:
    1. Remove v and lower the weight of each neighbor by $w(v)$;
    2. For each pair of non-adjacent neighbors $x < y$, create a new vertex $v_{x,y}$ with weight $w(v_{x,y}) = w(v)$;
    3. Insert edges between $v_{q,x}$ and $v_{r,y}$ if either x and y are adjacent or $L_q \neq L_r$;
    4. Each vertex $v_{x,y}$ is also connected to vertex $w \in V \setminus \{v\}$ adjacent to either x or y.

- After an independent set I' (of graph G') is constructed, the original independent set I (of graph G) can be reconstructed using $I = I' \cup \{v\}$, if $I' \cap N(v) = \emptyset$, or $I = I' \cap V$, otherwise.

Modified Weighted Struction

- When using this struction, the newly created vertex for each pair of non-adjacent neighbors $x, y \in N(v)$ with $x < y$ is now assigned weight $w(v_{x,y}) = w(y)$. Furthermore, in addition to the edges created in the original struction, each neighbor $k \in N(v)$ is connected to each vertex $v_{x,y}$ belonging to a different layer than k and $N(v)$ is extended to a clique by adding edges between any vertices $x, y \in N(v)$.
- To get the original I from the resulted I', we use $I = I' \cup \{v\}$, if $I' \cap N(v) = \emptyset$, or $I = (I' \cap V) \cup \{v_y | v_{x,y} \in I' \setminus V\}$, otherwise.

Example



(a) Original Graph (b) Original Struction (c) Modified Struction

Figure 1: Application of original struction and modified struction on center vertex $v = 0$. Vertices representing the same independent set in the different graphs are highlighted in gray.

Extended Weighted Struction

- Let $v \in V$ be an arbitrary vertex and C the set of all independent sets c in $G[N(v)]$ (the subgraph induced by $N(v)$) with $w(c) > w(v)$. The transformed graph G' is derived using the following steps:
  1. Remove v together with its neighbors;
  2. Create a new vertex v_c with weight $w(v_c) = w(c) - w(v)$ for each independent set $c \in C$;
  3. Each vertex v_c is connected to each non-neighbor (of v) w adjacent to at least one vertex in c;
  4. The vertices v_c are connected with each other to form a clique.

- For an MWIS I' (of G') we obtain I (of G) as follows: if $I' \setminus V = \{v_c\}$, then $I = (I' \cap V) \cup c$, otherwise $I = I' \cup \{v\}$.

Extended Reduced Weighted Struction

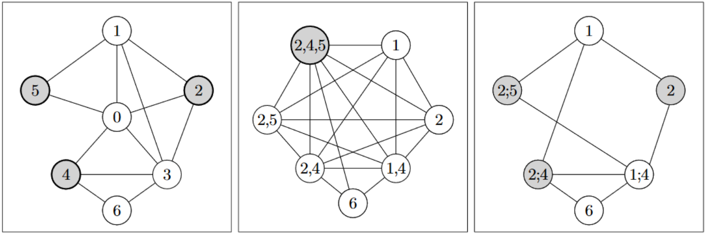
- To use the third struction, a subset of C is defined as follows: $C' = \{c \in C | w(c) - w(M(c)) \leq w(v)\}$, where $M(c)$ is the vertex from $N(v)$ with the highest index in an arbitrary (but fixed) ordering. Then, the same construction as for the extended struction is used, but only create vertices for the subset C'. The resulting set is denoted by V_C.

- Since this construction might not be valid anymore, additional vertices, that are connected to each other with layering, are added.

Extended Reduced Weighted Struction

- For each pair of an independent set $c \in C'$ and a vertex $y \in N(v)$, a vertex $v_{c,y}$ with weight $w(v_{c,y}) = w(y)$ is created, if y is not adjacent to any vertex in c. The resulting set is denoted by V_E. Then new edges are inserted between two vertices $v_{c,y}, v_{c',y'}$ if they either belong to different layers or y and y' are adjacent. Moreover, each vertex $v_{c,y}$ is connected to each non-neighbour w (of v), if w has been connected to either y or a vertex $x \in c$. Also, each vertex v_c to each vertex $v_{c',y}$ belonging to a different layer than c.
- For an MWIS I' the authors obtain an MWIS I as follows: if $I' \cap V_C = \emptyset$, then $I = I' \cup \{v\}$, otherwise, $I = (I' \cap V) \cup c \cup \{v_y | v_{c,y} \in I' \cap V_E\}$.

Example



(a) Original Graph (b) Extended Struction (c) Extended Reduced Struction

Figure 2: Application of extended struction and extended reduced struction on center vertex $v = 0$. Vertices representing the same independent set in the different graphs are highlighted in gray. We assume some weight constraints in the original graph for the construction in (b) and (c): $w(1) > w(0)$, $w(2) > w(0)$ and $w(3) + w(4) + w(5) \leq w(0)$.

Transformations

- The structions described in the previous sections are used in three different forms, called transformations:
  1. **Decreasing Transformations** : transformations where graph G' has less vertices than G;
  2. **Plateau Transformations** : transformations where graph G' has the same number of vertices as G (but the size and weight of the MWIS are reduced);
  3. **Increasing Transformations** : transformations where graph G' has more vertices than G.

Non-Increasing Reduction Algorithm

- The authors use the standard branch-and-reduce framework and apply transformations instead of simple reductions.
- When applying a struction variant, in the form of a transformation, the new method generally keep track of the number of vertices that will be created.
- If this number exceeds a given maximum value n_{max}, the corresponding struction is discarded to ensure that not too many vertices are created.

Non-Increasing Reduction Algorithm

Algorithm 1 Branch-and-Reduce Algorithm for MWIS

input graph $G = (V, E)$, current solution weight $curr$ (initially zero), best solution weight $\mathcal{W}$ (initially zero)
procedure Solve(G, $curr$, $\mathcal{W}$)
  $(G, curr) \leftarrow$ Reduce$(G, curr)$
  if $\mathcal{W} = 0$ **then** $\mathcal{W} \leftarrow curr + \text{ILS}(G)$
  if $curr$ + UpperBound$(G) \leq \mathcal{W}$ **then return** $\mathcal{W}$
  if G is empty **then return** $\max\{\mathcal{W}, curr\}$
  if G is not connected **then**
    for all $G_i \in$ Components(G) **do**
      $curr \leftarrow curr +$ Solve$(G_i, 0, 0)$
    return $\max(\mathcal{W}, curr)$
  $(G_1, curr_1), (G_2, curr_2) \leftarrow$ Branch$(G, curr)$
  {Run 1st case, update currently best solution}
  $\mathcal{W} \leftarrow$ Solve$(G_1, curr_1, \mathcal{W})$
  {Use updated $\mathcal{W}$ to shrink the search space}
  $\mathcal{W} \leftarrow$ Solve$(G_2, curr_2, \mathcal{W})$
return $\mathcal{W}$

Cyclic Blow-Up Algorithm

- Another method used in the article is an extension of the previous one.
- The main idea is to alternate between computing an irreducible graph using the non-increasing algorithm (*the Reduce Phase*) and then apply increasing transformations (structions) while ensuring that the graph size does not increase too much (*the Blow-Up Phase*).

Cyclic Blow-Up Algorithm

Algorithm 2 Cyclic Blow-Up Algorithm

input graph $G = (V, E)$, unsuccessful iteration threshold $X \in [1, \infty)$, maximum blowup $\alpha \in [1, \infty)$

```
procedure CyclicBlowUp(G, X)
    K ← Reduce(G)
    K* ← K
    count ← 0
    while |V(K)| ≤ α · |V(K*)| and count < X do
        K' ← BlowUp(K)
        if K' = K then
            return K*
        K'' ← Reduce(K')
        K ← Accept(K'', K)
        if K < K* then
            K* ← K
return K*
```

Experimental results

- The experimental configuration included the standard branch-and-reduce algorithm, where the 2 new preprocessing algorithms are applied.
- Cyclic Blow-Up Algorithm is used in two forms: C_{strong}, with max number of unsuccessful blow-up phases $X = 64$ and $n_{max} = 2048$, and C_{fast}, with $X = 25$ and $n_{max} = 512$. Also, instead of using ILS, the authors use hybrid iterated local search (HILS).

Experimental results

Graph	n	t_r	n	t_r	n	t_r	n	t_r	n	t_r
OSM instances	Basic-Dense		Basic-Sparse		NonIncreasing		Cyclic-Fast		Cyclic-Strong	
alabama-AM2	173	0.06	173	0.07	0	0.01	0	0.01	0	0.01
district-of-columbia-AM2	6 360	11.86	6 360	14.39	5 606	0.85	1 855	2.51	1 484	84.91
florida-AM3	1 069	31.52	1 069	35.20	814	0.13	661	0.44	267	42.26
georgia-AM3	861	8.99	861	10.14	796	0.08	587	0.69	425	12.84
greenland-AM3	3 942	3.81	3 942	24.77	3 953	3.94	3 339	10.27	3 339	54.44
new-hampshire-AM3	247	4.99	247	5.69	164	0.02	0	0.07	0	0.09
rhode-island-AM2	1 103	0.55	1 103	0.68	845	0.17	0	0.53	0	4.57
utah-AM3	568	8.21	568	8.97	396	0.03	0	0.09	0	0.40
Empty graphs	0% (0/34)		0% (0/34)		11.8% (4/34)		41.2% (14/34)		50% (17/34)	
SNAP instances	Basic-Dense		Basic-Sparse		NonIncreasing		Cyclic-Fast		Cyclic-Strong	
as-skitter	26 584	25.82	8 585	36.69	3 426	4.75	2 782	5.50	2 343	6.80
ca-AstroPh	0	0.02	0	0.02	0	0.02	0	0.03	0	0.03
email-EuAll	0	0.08	0	0.09	0	0.06	0	0.09	0	0.07
p2p-Gnutella06	0	0.01	0	0.01	0	0.01	0	0.01	0	0.01
roadNet-PA	133 814	2.43	35 442	7.73	300	1.05	0	1.19	0	1.14
soc-LiveJournal1	60 041	236.88	29 508	213.74	4 319	22.27	3 530	24.13	1 314	37.77
web-Google	2 810	1.57	1 254	2.42	361	1.75	46	1.88	46	7.97
wiki-Vote	477	0.03	0	0.02	0	0.02	0	0.02	0	0.02
Empty graphs	58.1% (18/31)		67.7% (21/31)		67.7% (21/34)		80.6% (25/31)		80.6% (25/31)	
mesh instances	Basic-Dense		Basic-Sparse		NonIncreasing		Cyclic-Fast		Cyclic-Strong	
buddha	380 315	5.56	107 265	26.19	86	1.83	0	1.87	0	1.91
dragon	51 885	0.89	12 893	1.34	0	0.18	0	0.19	0	0.21
ecat	239 787	4.07	26 270	10.09	274	2.12	0	2.12	0	2.14
Empty graphs	0% (0/15)		0% (0/15)		66.7% (10/15)		100% (15/15)		100% (15/15)	
FE instances	Basic-Dense		Basic-Sparse		NonIncreasing		Cyclic-Fast		Cyclic-Strong	
fe_ocean	141 283	1.05	0	5.94	138 338	8.90	138 134	9.61	138 049	10.78
fe_sphere	15 269	0.21	15 269	1.47	2 961	0.34	147	0.62	0	0.75
Empty graphs	0% (0/7)		14.3% (1/7)		0% (0/7)		28.6% (2/7)		42.9% (3/7)	

Table 1: Smallest irreducible graph found by each algorithm and time (in seconds) required to compute it. Rows are highlighted in gray if one of our algorithms is able to obtain an empty graph.

Experimental results

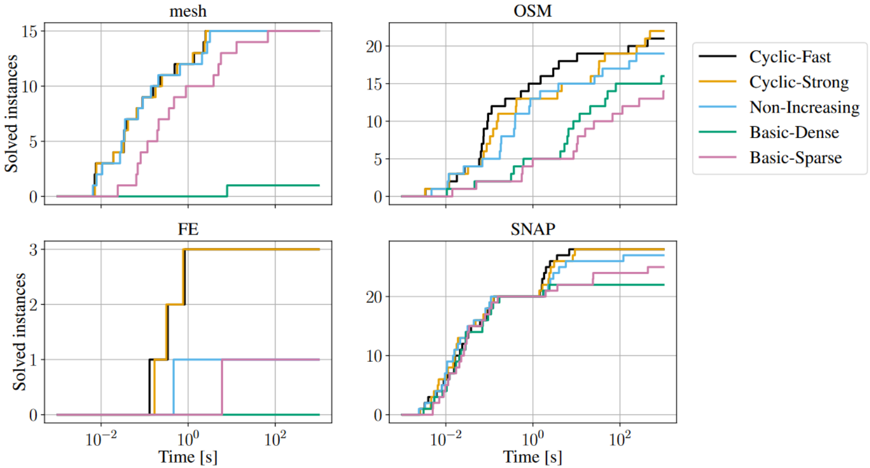


Figure 3: Cactus plots for the different instance families and evaluated solvers.

Experimental results

Graph	t_{max}	w_{max}	t_{max}	w_{max}	t_{max}	w_{max}	t_{max}	w_{max}
OSM instances	DynWVC2		HILS		Cyclic-Fast		Cyclic-Strong	
alabama-AM2	0.24	174 269	0.03	**174 309**	0.01	**174 309**	0.01	**174 309**
district-of-columbia-AM2	915.18	208 977	400.69	**209 132**	4.21	**209 132**	84.21	209 131
florida-AM3	862.04	237 120	3.98	**237 333**	1.57	**237 333**	40.97	**237 333**
georgia-AM3	1.31	**222 652**	0.04	**222 652**	0.98	**222 652**	12.97	**222 652**
greenland-AM3	640.46	14 010	1.18	**14 011**	10.95	**14 011**	58.24	14 008
new-hampshire-AM3	1.63	**116 060**	0.03	**116 060**	0.05	**116 060**	0.08	**116 060**
rhode-island-AM2	13.90	184 576	0.24	**184 596**	0.41	**184 596**	4.37	**184 596**
utah-AM3	136.90	**98 847**	0.07	**98 847**	0.09	**98 847**	0.27	**98 847**
Solved instances						61.8% (21/34)		64.7% (22/34)
Optimal weight		68.2% (15/22)		100.0% (22/22)				
SNAP instances	DynWVC2		HILS		Cyclic-Fast		Cyclic-Strong	
as-skitter	383.97	123 273 938	999.32	122 658 804	346.69	124 137 148	354.71	**124 137 365**
ca-AstroPh	125.05	797 480	13.47	**797 510**	0.02	**797 510**	0.02	**797 510**
email-EuAll	132.62	**25 286 322**	338.14	**25 286 322**	0.07	**25 286 322**	0.07	**25 286 322**
p2p-Gnutella06	186.97	548 611	1.29	**548 612**	0.01	**548 612**	0.01	**548 612**
roadNet-PA	469.18	60 990 177	999.94	60 037 011	0.96	**61 731 589**	1.04	**61 731 589**
soc-LiveJournal1	999.99	279 231 875	1 000.00	255 079 926	51.33	284 036 222	44.19	**284 036 239**
web-Google	324.65	56 206 250	995.92	56 008 278	1.72	**56 326 504**	6.44	**56 326 504**
wiki-Vote	0.32	**500 079**	10.34	**500 079**	0.02	**500 079**	0.02	**500 079**
Solved instances						90.3% (28/31)		90.3% (28/31)
Optimal weight		28.6% (8/28)		57.1% (16/28)				
mesh instances	DynWVC2		HILS		Cyclic-Fast		Cyclic-Strong	
buddha	797.35	56 757 052	999.94	55 490 134	1.75	**57 555 880**	1.77	**57 555 880**
dragon	981.51	7 944 042	996.01	7 940 422	0.21	**7 956 530**	0.22	**7 956 530**
ecat	542.87	36 129 804	999.91	35 512 644	2.19	**36 650 298**	2.29	**36 650 298**
Solved instances						100.0% (15/15)		100.0% (15/15)
Optimal weight		0.0% (0/15)		0.0% (0/15)				
FE instances	DynWVC1		HILS		Cyclic-Fast		Cyclic-Strong	
fe_ocean	983.53	**7 222 521**	999.57	7 069 279	18.85	6 591 832	19.04	6 591 537
fe_sphere	875.87	616 978	843.67	616 528	0.63	**617 816**	0.67	**617 816**
Solved instances						42.9% (3/7)		42.9% (3/7)
Optimal weight		0.0% (0/3)		0.0% (0/3)				

Table 2: Best solution found by each algorithm and time (in seconds) required to compute it. The global best solution is highlighted in bold. Rows are highlighted in gray if one of our exact solvers is able to solve the corresponding instances.

References

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