yz709-FG-sup2

Question 13

13. A rotation by θ around the z-axis is typically represented by the 3×3-matrix, which is then applied to a vector $\vec{v} = (v_0, v_1, v_2)^T$:

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{v}' = R_\theta \cdot \vec{v}$$

When using quaternions, the same rotation can be written using $r_\theta = \cos(\theta/2) + k\sin(\theta/2)$ (where $i^2 = j^2 = k^2 = -1$) as:

$$\mathbf{v}' = \mathbf{r}_\theta \mathbf{v} \mathbf{r}_\theta^{-1}$$

The vector $\vec{v}$ is written as the quaternion $v_o i + v_1 j + v_2 k$ and $r_\theta^{-1} = \cos(\theta/2) - k\sin(\theta/2)$. Demonstrate that the quaternion rotation $\mathbf{r}_\theta \mathbf{v} \mathbf{r}_\theta^{-1}$ yields indeed the same rotation as the classic 3×3-matrix above.

Q13

$$R_\theta \cdot \vec{v} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} V_0 \\ V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_0 \cos\theta - V_1 \sin\theta \\ V_0 \sin\theta + V_1 \cos\theta \\ V_2 \end{bmatrix}$$

$$\vec{v}' = \vec{r}_\theta \vec{v} \vec{r}_\theta^{-1} = (\cos\tfrac{\theta}{2} + k\sin\tfrac{\theta}{2})(V_0 i + V_1 j + V_2 k)(\cos\tfrac{\theta}{2} - k\sin\tfrac{\theta}{2})$$

$$= (\cos\tfrac{\theta}{2} + k\sin\tfrac{\theta}{2})(V_0\cos\tfrac{\theta}{2} i - V_0\sin\tfrac{\theta}{2} i\cdot k + V_1\cos\tfrac{\theta}{2} j - V_1\sin\tfrac{\theta}{2} j\cdot k + V_2\cos\tfrac{\theta}{2}\cdot k + V_2\sin\tfrac{\theta}{2})$$

since i, j, k are perpendicular to each other, $i\cdot j\cdot k = -1 \Rightarrow i\cdot j = k,\ j\cdot i = -k$

$k\cdot i = j,\ i\cdot k = -j$

$$\vec{v}' = (\cos\tfrac{\theta}{2} + k\sin\tfrac{\theta}{2})(V_0\cos\tfrac{\theta}{2} i + V_0\sin\tfrac{\theta}{2} j + V_1\cos\tfrac{\theta}{2} j - V_1\sin\tfrac{\theta}{2} i + V_2\cos\tfrac{\theta}{2}\cdot k + V_2\sin\tfrac{\theta}{2})$$

$$= V_0(\cos\tfrac{\theta}{2})^2 i + \underline{\underline{\frac{V_0\sin\theta}{2} j}} + V_1(\cos\tfrac{\theta}{2})^2 j - \underline{\frac{V_1\sin\theta}{2} i} + V_2(\cos\tfrac{\theta}{2})^2\cdot k + \cancel{V_2\frac{\sin\theta}{2}}$$

$$+ \underline{\underline{\frac{V_0\sin\theta}{2} j}} - V_0(\sin\tfrac{\theta}{2})^2 i - \underline{\frac{V_1\sin\theta}{2} i} - V_1(\sin\tfrac{\theta}{2})^2 j - \cancel{V_2\frac{\sin\theta}{2}} + V_2(\sin\tfrac{\theta}{2})^2 k$$

$$= \left[(\cos\tfrac{\theta}{2})^2 - (\sin\tfrac{\theta}{2})^2\right](V_0 i + V_1 j) - V_1\sin\theta\, i + V_0\sin\theta\, j + V_2 k$$

$$= \cos\theta(V_0 i + V_1 j) - V_1\sin\theta\, i + V_0\sin\theta\, j + V_2 k$$

$$= (V_0\cos\theta - V_1\sin\theta) i + (V_1\cos\theta + V_0\sin\theta) j + V_2 k$$

$$\equiv R_\theta \cdot \vec{v}$$

💡 Comments:

Question 15

15. *(Past exam question)* **Transformation Blending (I) Thinking about fundamental properties.** First, please answer the following questions in maximum two sentences for each.

    (a) What is the advantage of representing rigid transformations with dual quaternions for blending?

    (b) Briefly explain one fundamental disadvantage of using quaternions based shortest path blending for rotations as compared to linear blend skinning (i.e. averaging rotation matrices)?
    Hint: think about the continuity of both blending methods for 2D rotations.

- (a):
  - Easy to interpolate between two quaternions and less memory required for storage.
  - No matrix is involved so much easier and convenient for blending computation; we would suffer fewer rounding defects when normalising floating-point values.
- (b):
  - If there is no shortest path on the manifold for two points (i.e., the discontinuity between two points), quaternions could not work as they need to find the shortest path between two points on the manifold. But the linear blending skinning would work as it considers simply a point on the straight line connecting the two points.

💡 Comments:

Question 16

16. *(Past exam question)* **Transformation Blending (II) Derivations and deeper understanding.** Below we list all required properties of dual quaternions for the rest of the exercise.

    A dual quaternion $\hat{\mathbf{q}}$ can be written in the form $\hat{\mathbf{q}} = \mathbf{q}_0 + \epsilon \mathbf{q}_\epsilon$, where $\mathbf{q}_0$ and $\mathbf{q}_\epsilon$ are quaternions and ϵ is the dual unit with the property $\epsilon^2 = 0$. The norm of $\hat{\mathbf{q}}$ is then given by:

$$||\hat{q}|| = ||q_0|| + \epsilon \frac{\langle \mathbf{q}_0, \mathbf{q}_\epsilon \rangle}{||\mathbf{q}_0||}$$

    Dual quaternions representing rigid transformations can be written in the following form:

$$\hat{\mathbf{q}} = \cos\left(\hat{\theta}/2\right) + \hat{\mathbf{s}} \sin\left(\hat{\theta}/2\right)$$

where $\hat{\theta} = \theta_0 + \epsilon\, \theta_\epsilon$ and $\hat{\mathbf{s}} = \mathbf{s}_0 + \epsilon\, \mathbf{s}_\epsilon$. Here, $\mathbf{s}_0$ is the axis of rotation, θ_0 is the rotation angle, and θ_ϵ is the amount of translation along $\mathbf{s}_0$. Since this is a unit dual quaterion it can be shown that $\langle \mathbf{s}_0, \mathbf{s}_\epsilon \rangle = 0$ and $\langle \mathbf{s}_0, \mathbf{s}_0 \rangle = 1$.

The power of a dual quaternion is defined by

$$\hat{\mathbf{q}}^t = e^{t \log(\hat{\mathbf{q}})}$$

where:

$$e^{\hat{\mathbf{q}}} = \cos(||\hat{\mathbf{q}}||) + \frac{\hat{\mathbf{q}}}{||\hat{\mathbf{q}}||} \sin(||\hat{\mathbf{q}}||) \qquad \log\left(\cos\left(\hat{\theta}/2\right) + \hat{\mathbf{s}} \sin\left(\hat{\theta}/2\right)\right) = \hat{\mathbf{s}} \frac{\hat{\theta}}{2}$$

(c) Utilizing the properties above, for a dual quaternion $\hat{\mathbf{q}} = \cos\left(\hat{\theta}/2\right) + \hat{\mathbf{s}}\sin\left(\hat{\theta}/2\right)$, prove that:

$$\hat{\mathbf{q}}^t = \cos\left(t\hat{\theta}/2\right) + \hat{\mathbf{s}}\sin\left(t\hat{\theta}/2\right)$$

Hint: you do not need to know the expression for $\cos(\hat{\theta})$ or $\sin(\hat{\theta})$.

Q1b

$\hat{q} = q_0 + \epsilon q_\epsilon \qquad \epsilon^2 = 0$

$\|\hat{q}\| = \|q_0\| + \epsilon \dfrac{\langle q_0, q_\epsilon \rangle}{\|q_0\|}$

$\hat{q} = \cos(\frac{\hat{\theta}}{2}) + \hat{s}\sin(\frac{\hat{\theta}}{2})$

(c)

Assume $\hat{q} = \cos(\frac{\hat{\theta}}{2}) + \hat{s}\sin(\frac{\hat{\theta}}{2})$

$(\hat{q})^t = (\cos\frac{\hat{\theta}}{2} + \hat{s}\sin\frac{\hat{\theta}}{2})^t = e^{t \log(\cos\frac{\hat{\theta}}{2} + \hat{s}\sin\frac{\hat{\theta}}{2})}$

$= e^{t\hat{s}\frac{\hat{\theta}}{2}}$

$= \cos(\|t\hat{s}\frac{\hat{\theta}}{2}\|) + \dfrac{t\hat{s}\frac{\hat{\theta}}{2}}{\|t\hat{s}\frac{\hat{\theta}}{2}\|}\sin(\|t\hat{s}\frac{\hat{\theta}}{2}\|)$

$= \cos(t\|\hat{s}\frac{\hat{\theta}}{2}\|) + \dfrac{\hat{s}\frac{\hat{\theta}}{2}}{\|\hat{s}\frac{\hat{\theta}}{2}\|}\sin(t\|\hat{s}\frac{\hat{\theta}}{2}\|)$

$= \cos(t\frac{\hat{\theta}}{2}) + \hat{s}\sin(t\frac{\hat{\theta}}{2})$

Since $\|\hat{s}\frac{\hat{\theta}}{2}\| = \|\hat{s}\|\|\frac{\hat{\theta}}{2}\| = (\|S_0\| + \varepsilon\dfrac{\langle S_0, S_\varepsilon\rangle}{\|S_0\|}) \cdot \frac{1}{2} \cdot \|\hat{\theta}\|$

$= \|S_0\| \frac{1}{2} \cdot \|\hat{\theta}\|$

$= \dfrac{\|\hat{\theta}\|}{2}$

$\overset{?}{=} \dfrac{\hat{\theta}}{2}$ I didn't get this transition

(d) Now, consider rigid transformations in the 2D xy-plane. For these transformations, the rotation is always around the z (or $-z$)-axis, i.e. $\mathbf{s}_0$ is fixed to the z-axis. On the other hand, a dual quaternion encodes translations only along $\mathbf{s}_0$, which are in this case always zero, since we can only translate in the xy-plane. Then, how can a dual quaterion represent a rotation and translation in the xy-plane, such as the one depicted in Figure 1? Please answer in maximum two sentences without any equations.

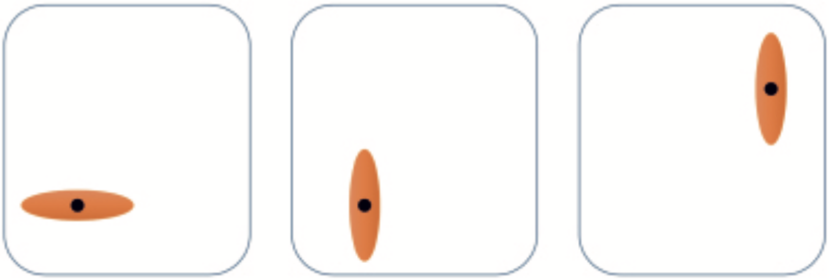


FIGURE 1. An object (left) is first rotated around its center of mass (middle) and then translated (right).

- Dual quaternions can represent rotation and translation by representing each as a component and multiplying them together.
- Assume we rotate at the origin by 90 degrees clockwise $(R_{rot} = cos(\theta/2) + isin(\theta/2) = \sqrt{2}/2 + i\sqrt{2}/2)$ and then translate up-right $(R_{trans} = 1 + x/2i\varepsilon + y/2j\varepsilon + z/2k\varepsilon)$ where x, y, z are the transformed distance and $z = 0$, then we multiple these two together to form the dual quaternions.

> 💡 Comments:
> Good resource explaining quaternions: [Maths - Dual Quaternions - Martin Baker]

Question 1

1. Simple 3D engines often approximate lighting as the superposition/combination of diffuse ambient light and direct illumination from point sources. This is a rather crude (yet efficient) approximation to the full rendering equation. What kind of effects are not captured by this simple approximation, i.e. what kind of visual effects cannot be rendered?

- Indirect illumination is not considered, reflected lights from other objects are omitted; hence we cannot simulate mirrored effects, reflected surfaces can be not rendered correctly.
- All objects under shadow would be pretty dim, but we could minimise the effects with indirect illumination.

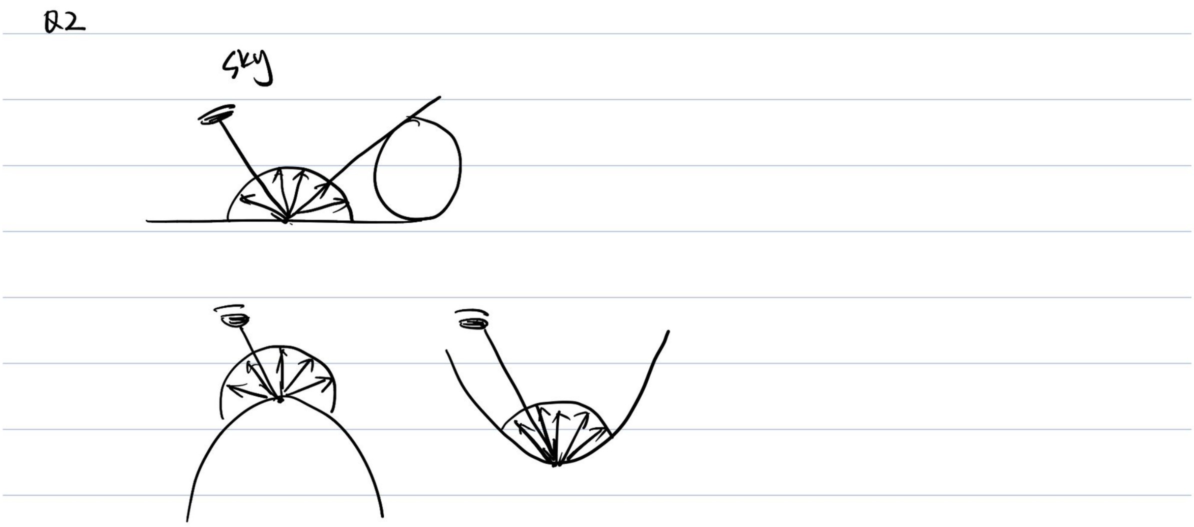
💡 Comments:

Question 2

2. If there is only diffuse ambient light, we can easily determine the radiosity of a point on the surface through ambient occlusion, i.e. by measuring how much of the 'sky' is visible from that specific point.

   Briefly describe how information about the curvature of a surface can help in this situation.

- We want to determine the radiosity of a point on the surface. If the curvature of that surface is 0, then the surface is a plane; we could find the radiosity by integrating occlusion over the hemisphere around the normal at the point.
- If the curvature is not 0, then it helps determine the volume we need to integrate on.

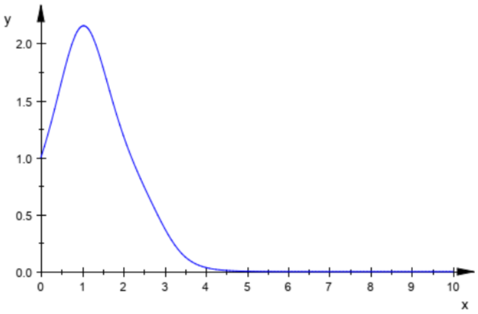


💡 Comments: I am pretty confused about this question...

Question 3

3. Consider the function $f(x)$ with the graph shown below. In the range 0 to 10, it covers an area of $I = \int_0^{10} f(x)\mathrm{d}x \approx 4.329268516$.

$$f(x) = \frac{40}{(1+e^x)\cdot\left((x-2)^4+4\right)}$$



(a) Approximate the integral I through uniform sampling at the five points $x_k = 1, 3, 5, 7, 9$, i.e.:

$$\langle I\rangle = \sum_k \frac{f(x_k)}{p(x_k)}$$

Since there are five uniformly distributed points, the density for all of them is $p = \frac{5}{10} = \frac{1}{2}$. What is the relative error of the number you get?

Q3

(a) $\langle I\rangle = \sum_K \frac{f(x_k)}{p(x_k)} = \frac{1}{2}\sum_k f(x_k) \qquad k \in \{1,3,5,7,9\}$

$= \frac{1}{2}\times 40\times\left(\frac{1}{(1+e^1)\times 5} + \frac{1}{(1+e^3)\times 5} + \frac{1}{(1+e^5)\times(3^4+4)} + \frac{1}{(1+e^7)\times(5^4+4)} + \frac{1}{(1+e^9)\times(7^4+4)}\right)$

≈ 1.2671

$\varepsilon = |I - \langle I\rangle| \approx 3.0622$

(b) According to the idea of importance sampling, you concentrate the evaluation of function values on the region where the values are largest, i.e. between 0 and 4 in our case. Sample the function at the points $x = 0.5, 1.5, 2.5, 3.5$ as well as $x = 7$ and compute again the estimator $\langle I\rangle$ for the integral I. The density for the first four points (each covering an interval of length 1) is $p = \frac{1}{1} = 1$ whereas the density for $x = 7$ (which covers an interval of length 6) is $p = \frac{1}{6}$.

(b) $\langle I\rangle = 1\times 40\times\left(\frac{1}{(1+e^{0.5})\times(-1.5^4+4)} + \frac{1}{(1+e^{1.5})\times(-0.5^4+4)} + \frac{1}{(1+e^{2.5})\times(0.5^4+4)}\right.$

$\left. + \frac{1}{(1+e^{3.5})(1.5^4+4)}\right) + \frac{1}{6}\times 40\times\frac{1}{(1+e^7)\times(5^4+4)}$

≈ 4.3389

$\varepsilon = |I - \langle I\rangle| \approx -9.6067\times 10^{-3}$

Comments:

Question 4

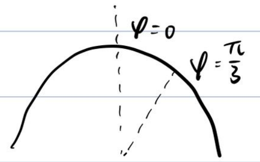
4. You can calculate the surface integral over a hemisphere $\mathcal{H}^2$ with the following formula:

$$\int_{\mathcal{H}^2} f(\vec{\omega})\mathrm{d}\vec{\omega} = \int_0^{\pi/2}\int_0^{2\pi} f(\varphi,\theta)\sin(\varphi)\mathrm{d}\theta\mathrm{d}\varphi$$

Now consider the 'top' part of the hemisphere ranging from $\varphi = 0$ to some arbitrary $\varphi = \Phi$, which we will denote as $\mathbf{H}_\Phi f$:

$$\mathbf{H}_\Phi f = \int_0^{\Phi}\int_0^{2\pi} f(\varphi,\theta)\sin(\varphi)\mathrm{d}\theta\mathrm{d}\varphi$$

(a) Show that $\mathbf{H}_{\pi/3}$ covers exactly half of the entire surface area $\mathbf{H}_{\pi/2}$, i.e. if you have φ range from 0 to $\frac{\pi}{3}$ (instead of 0 to $\frac{\pi}{2}$), you cover $50\,\%$.



(a)

$$\frac{H_{\frac{\pi}{3}}}{H_{\frac{\pi}{2}}} = \frac{\int_0^{\frac{\pi}{3}}\int_0^{2\pi} \sin(\varphi)\, d\theta\, d\varphi}{\int_0^{\frac{\pi}{2}}\int_0^{2\pi} \sin(\varphi)\, d\theta d\varphi}$$

$$= \frac{\int_0^{\frac{\pi}{3}} \sin(\varphi)\, d\varphi}{\int_0^{\frac{\pi}{2}} \sin(\varphi)\, d\varphi} = \frac{[-\cos(\varphi)]_0^{\frac{\pi}{3}}}{[-\cos(\varphi)]_0^{\frac{\pi}{2}}} = \frac{1}{2}$$

$$= 50\%$$

(b) Let $f(\varphi,\theta) = \cos(\varphi)$ and compute $I = \mathbf{H}_{\pi/2} f$ as well as $I' = \mathbf{H}_{\pi/3} f$. What fraction of I is 'covered' by I'? That is, if we were to approximate I by I', how good would that approximation (relatively) be?

(b) Let $f(\varphi,\theta) = \cos(\varphi)$

$$\frac{I'}{I} = \frac{H_{\frac{\pi}{3}} f}{H_{\frac{\pi}{2}} f} = \frac{\int_0^{\frac{\pi}{3}} \int_0^{2\pi} \cos\varphi \sin\varphi \, d\theta \, d\varphi}{\int_0^{\frac{\pi}{2}} \int_0^{2\pi} \cos\varphi \sin\varphi \, d\theta \, d\varphi}$$

$$= \frac{\frac{1}{2}\int_0^{\frac{\pi}{3}} \sin 2\varphi \, d\varphi}{\frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2\varphi \, d\varphi}$$

$$= \frac{\left[-\cancel{\frac{1}{2}} \cos 2\varphi\right]_0^{\frac{\pi}{3}}}{\left[-\cancel{\frac{1}{2}} \cos 2\varphi\right]_0^{\frac{\pi}{2}}}$$

$$= \frac{-\frac{1}{2} - 1}{0 - 1} = \frac{3}{2}$$

(c) Briefly explain the significance of (b) with respect to incident light at a point on a surface. In particular, why is *importance sampling Monte Carlo integration* in the context of the rendering equation a good idea?

- The incident light makes an angle of φ with the normal at that point; thus, $f(\vec{\omega})$ changes with respect to the incident angle.
- Using importance sampling, where the probability of sampling an f value matches with the distribution of f, we can find a better approximation of the expected value of f to that point.

(d) Using the cosine as weight for importance sampling MC integration of the rendering equation is based on some basic assumptions. What are these assumptions and under what circumstances might they fail? That is, under what circumstances would cosine-weighted importance sampling *not* deliver better results?

- It assumes the surface is diffusely reflected, where the diffuse terms are the same at any view angle (a.k.a Lambertian diffusion); hence the brightness is the same.
- However, if the surface is a mirror or some glossy surfaces, the assumption breaks, and cosine importance sampling does not work.

💡 Comments:

Question 5

5. *(Past exam question)* Write down the directional form of the rendering equation. Briefly explain each of the terms and the integration domain.

$$L_0(\mathbf{x}, \vec{\omega}) = \dots$$

- Because of energy conservation, no light has been absorbed by the point on a surface, so all outgoing light $L_o(\vec{x}, \vec{\omega})$ from this point is the emitted lights from light sources $L_e(\vec{x}, \vec{\omega})$ and the reflected light through that point $L_r(\vec{x}, \vec{\omega})$.

$$L_o(\vec{x}, \vec{\omega_o}) = L_e(\vec{x}, \vec{\omega_o}) + L_r(\vec{x}, \vec{\omega_o})$$
$$L_o(\vec{x}, \vec{\omega_o}) = L_e(\vec{x}, \vec{\omega_o}) + \int_{H^2} f_r(\vec{x}, \vec{\omega_i}, \vec{\omega_r}) L_i(\vec{x}, \vec{\omega_i}) \cos\theta_i \, d\vec{\omega_i}$$

- The integration integrates across the whole hemisphere, with the reflectance radiance per solid angle in that hemisphere. $L_i(\vec{x}, \vec{\omega})$ is the incidence radiance, $\cos\theta_i$ where θ_i is the angle between $\vec{x}$ and the surface normal at that point.
- The reflected light $L_r(\vec{x}, \vec{\omega})$ can be calculated using the BRDF $f_r(\vec{x}, \vec{\omega_i}, \vec{\omega_r})$ which provides a relation between radiance and differential reflected radiance. Hence the reflected radiance comes from the illumination of all directions. $E_i(\vec{\omega_i})$ is the irradiance which is the power received by a surface per unit area: $E_e = \frac{d\Phi_e}{dA}$

$$f_r(\vec{x}, \vec{\omega_i}, \vec{\omega_r}) = \frac{dL_r(\vec{x}, \vec{\omega_r})}{dE_i(\vec{\omega_i})} = \frac{dL_r(\vec{x}, \vec{\omega_r})}{L_i(\vec{x}, \vec{\omega_i}) \cos\theta_i \, d\vec{\omega_i}}$$
$$\text{Unit of function}: \frac{1}{steradians}$$
$$f_r(\vec{x}, \vec{\omega_i}, \vec{\omega_r}) L_i(\vec{x}, \vec{\omega_i}) \cos\theta_i = \frac{dL_r(\vec{x}, \vec{\omega_r})}{d\vec{\omega_i}}$$
$$\text{Reflection Equation} = L_r(\vec{x}, \vec{\omega_r}) = \int_{H^2} f_r(\vec{x}, \vec{\omega_i}, \vec{\omega_r}) L_i(\vec{x}, \vec{\omega_i}) \cos\theta_i \, d\vec{\omega_i}$$

💡 Comments:

Question 6

6. *(Past exam question)* Assume a scene containing only diffuse surfaces with diffuse reflectance (i.e. albedo) $\rho(\mathbf{x})$. The scene is surrounded by a single *distant* light source with constant (i.e. directionally-invariant) emission $\bar{L}$. Let $V(\mathbf{x}, \vec{\omega})$ be a function returning the visibility of the light source from point $\mathbf{x}$ along direction $\vec{\omega}$.

   Change/simplify the rendering equation you provided in (a) as much as possible to estimate *only direct illumination* due to the light source.

- For the diffuse reflectance term, BRDF is a constant: $L_r(\vec{x}) = \rho(\vec{x})$
- For the direct illumination term, we only consider the single distant light source with directionally invariant emission $\overline{L}$, so $L_r(\vec{x}) = \int_{H^2} f_r(\vec{x}, \vec{\omega})\overline{L}V(\vec{x}, \vec{\omega})d(\vec{\omega})$
- Hence the rendering equation becomes:

$$L_o(\vec{x}, \vec{\omega_o}) = L_e(\vec{x}, \vec{\omega_o}) + \rho(\vec{x}) + \int_{H^2} f_r(\vec{x}, \vec{\omega})\overline{L}V(\vec{x}, \vec{\omega})d(\vec{\omega})$$

💡 Comments:

Question 7

7. *(Past exam question)* Recall the surface area form of the rendering equation:

$$L(\mathbf{x}, \mathbf{z}) = L_e(\mathbf{x}, \mathbf{z}) + \int_A f_r(\mathbf{x}, \mathbf{y}, \mathbf{z}) \cdot L(\mathbf{x}, \mathbf{y}) \cdot G(\mathbf{x}, \mathbf{y})\, \mathrm{d}A(\mathbf{y})$$

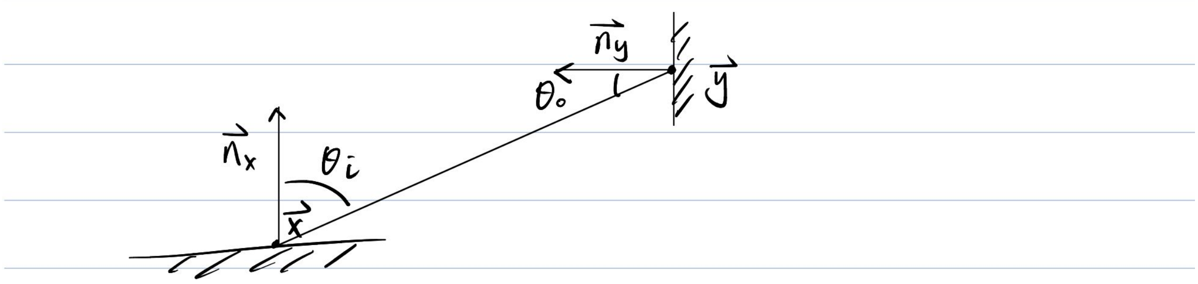
   (a) Provide the mathematical formulation of $G(\mathbf{x}, \mathbf{y})$ and explain its terms.

- $V(\vec{x}, \vec{y})$ is the visibility term which is a function returning the visibility of a light source from point $\vec{x}$ along direction $\vec{y}$.

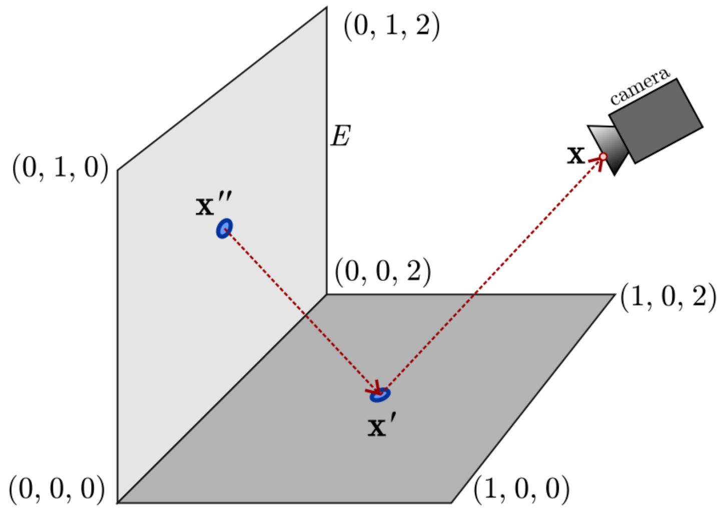
- The term $\frac{|\cos\theta_i||\cos\theta_o|}{||\vec{x}-\vec{y}||^2}$ represents the chance that a photon emitted from a differential patch will hit another differential patch. θ_i is the angle between normal vector at $\vec{x}$ and the line between $\vec{x}$ and $\vec{y}$, while θ_o is the angle between normal vector at $\vec{y}$ and the same line between $\vec{x}$ and $\vec{y}$.
  - The chance decreases when the patches face away from each other, so $|\cos\theta_i||\cos\theta_o|$ decreases.
  - The change decreases when the patch move away from each other $||\vec{x}-\vec{y}||^2$

$$G(\vec{x},\vec{y}) = V(\vec{x},\vec{y})\frac{|\cos\theta_i||\cos\theta_o|}{||\vec{x}-\vec{y}||^2}$$

Q7



(b) Consider the following scene configuration. The rectangle E on the left is an area emitter with $L_e = 1$ and a black BSDF, i.e. $f_r = 0$. The coordinates of its corners are labeled. The ground plane is a non-emissive surface with $f_r = \frac{1}{\pi}$. Provide pseudo-code for a Monte-Carlo estimator of L, given as input a point on the ground plane $\mathbf{x}'$, a point on the camera $\mathbf{x}$, and the desired amount of samples N. You can obtain random numbers uniformly in $[0, 1)$ by calling `RAND()`.



```
def monteEst(x', x, N): ''' Apply uniformly sampling to sample N points, p(w,k) = 1/2*pi Get the cosine term using RAND() ''' refRadSum = 0 for i in range(N): refRadSum += ((1/pi * 1 * RAND()) / (1/2*pi)) refRadSum / N return refRadSum * 2 / N
```

$$L_r(\vec{x}, \vec{\omega_r}) = \int_{H_2} f_r(\vec{x}, \vec{\omega_i}, \vec{\omega_r}) L_i(\vec{x}, \vec{\omega_i}) \cos\theta_i d_{\omega_i}$$

$$\text{Ambient occlusion}: L_r(\vec{x}) = \frac{\rho}{\pi} \int_{H^2} V(\vec{x}, \vec{\omega_i}) \cos\theta_i d\vec{\omega_i}$$

$$\approx \frac{\rho}{\pi N} \sum_{k=1}^{N} \frac{V(\vec{x}, \vec{\omega_i}) \cos\theta_{i,k}}{p(\vec{\omega_{i,k}})}$$

$$\text{When doing uniform sampling, we have:}$$

$$p(\vec{\omega_{i,k}}) = \frac{1}{2}\pi$$

$$L_r(\vec{x}) \approx \frac{2\rho}{N} \sum_{k=1}^{N} V(\vec{x}, \vec{\omega_{i,k}}) \cos\theta_{i,k}$$

💡 Comments:

Question 8

8. *(Past exam question)* Consider the following *single-sample* Monte Carlo (MC) estimator $\langle F \rangle$ of an integral F of an arbitrary non-negative integrand $f(x)$ over an arbitrary domain D with volume 1, driven by a random variable $X \sim p(x)$.

$$F = \int_D f(x) \mathrm{d}x \approx \frac{f(X)}{p(X)} = \langle F \rangle, \qquad p(x) = \frac{1}{2}\left(p_{\text{good}}(x) + p_{\text{bad}}(x)\right)$$

The "good" probability density function (PDF) $p_{\text{good}}(x)$ is proportional to $f(x)$ (i.e. $p_{\text{good}}(x) = cf(x)$ for some constant c) and the "bad" PDF $p_{\text{bad}}(x)$ is zero wherever $f(x)$ is *non*-zero (i.e. $\forall x. f(x) \neq 0 \Rightarrow p_{\text{bad}}(x) = 0$).

Derive *step-by-step* the variance of the estimator $\langle F \rangle$ as a function of F.

Hint: Recall that the variance is defined as:

$$\text{Var}[Y] = E\left[(Y - E[Y])^2\right] = E\left[Y^2\right] - E[Y]^2$$

Q8 $X \sim p(x)$

$$\text{Variance}[\langle F \rangle] = E[\langle F^2 \rangle] - (E[\langle F \rangle])^2$$

$$\begin{cases} f(x) = 0 & p(x) = \frac{1}{2} P_{bad}(x) \\ f(x) \neq 0 & p(x) = \frac{1}{2} P_{good}(x) = \frac{c}{2} f(x) \end{cases}$$

$$E[\langle F \rangle] = E[\frac{f(x)}{p(x)}] = \int_0^1 \frac{f(x)}{p(x)} P_X(X=x)\, dx = f(x)$$

$$E[\langle F^2 \rangle] = E[(\frac{f(x)}{p(x)})^2] = \int_0^1 \frac{f^2(x)}{p^2(x)} P_X(X=x)\, dx = \lim_{a \to 0} \int_a^1 \frac{f^2(x)}{\frac{c}{2} f(x)}\, dx$$

$$= \frac{2}{c} f(x)$$

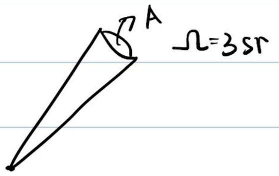
$$\text{Variance}[\langle F \rangle] = \frac{2}{c} f(x) - (f(x))^2$$

Comments:

Question 9

9. *(Past exam question)* Consider a spotlight emitting radiant intensity $I = 20\,\frac{\text{W}}{\text{sr}}$ confined in a cone of directions with solid angle $\omega = 3\,\text{sr}$ (recall that sr stands for steradian). Compute the total emitted radiant flux from this light.

Q9



Radiant flux $\Phi(A) = \frac{E}{T} = \int_{S^2} I(\vec{\omega})\, d\vec{\omega}$

since $I(\vec{\omega}) = 20\,\frac{W}{sr}$, $\Omega = \int_{S^2} d\vec{\omega} = 3sr$

$\Rightarrow \Phi(A) = 20\,\frac{W}{sr} \cdot 3sr = 60\ W$

Comments: