Further Graphics Exercise Set I

1. (a) Briefly compare parametric surfaces to implicit surfaces and name at least three aspects where one offers advantages over the other. When and why would you use a parametric or implicit surface in practice?

   How do point set surfaces differ from parametric and implicit surfaces?

   (b) Which of these three representations (parametric, implicit, point set) do you use for texture mapping?

2. What geometric shape does the following planar curve describe?

$$\big(x(t), y(t)\big) = \big(e^t \cos(t), e^t \sin(t)\big)$$

3. (a) How do you compute the surface normal $\mathbf{n}$ for a parametric surface?

   (b) Below you find the parametrisation of a *torus*. Compute the surface normal $\mathbf{n}$ for $u = \frac{1}{3}\pi, v = \frac{3}{4}\pi$.

$$s(u, v) = \Big(\big(3 + \sqrt{2}\cos(v)\big)\cos(u), \big(3 + \sqrt{2}\cos(v)\big)\sin(u), \sqrt{2}\sin(v)\Big)$$

4. The following implicit equation describes an ellipsoid in $\mathbb{R}^3$.

$$\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{4} - 1 = 0$$

   (a) Find a parametric equation that describes the same ellipsis.

   Hint: start off with the parametric equation of a sphere and then modify it as appropriate.

   (b) Prove that $P = (2, 3, -2)$ lies on the ellipsoid's surface.

   (c) Determine the surface normal $\mathbf{n}$ at P.

5. If we define a curve in $\mathbb{R}^2$ with the implicit function $f(x, y) = x^2y^2$, what problem could there arise when computing the surface normal?

6. (a) What data structure is used to represent triangle meshes? That is: how is the data that describes a triangle mesh stored?

   (b) Which parts of this data structure pertain to the *topology* and which to the *geometry* of a surface?

7. Compared to a triangle mesh, a point set lacks a vital piece information in that it no longer identifies the neighbouring vertices of any given vertex. Why is such information necessary or useful in the first place and how do point set surfaces work without it?

8. What is a *manifold* and how does it differ from a *manifold with boundaries*?

9. For a surface S in $\mathbb{R}^3$ and any point P on S, you can choose your coordinate system so that P lies at the centre $\mathcal{O}(0,0,0)$ and the xy-plane is tangent to the surface. You can then express the surface S locally as a function $z = f(x,y)$.

   In this situation, you can calculate the curvatures as follows (where II is called the "second fundamental form"):

$$\mathrm{II} = \begin{bmatrix} \frac{\partial^2}{\partial x^2} f & \frac{\partial}{\partial x}\frac{\partial}{\partial y} f \\ \frac{\partial}{\partial y}\frac{\partial}{\partial x} f & \frac{\partial^2}{\partial y^2} f \end{bmatrix} \qquad H = \frac{\mathrm{II}_{11} + \mathrm{II}_{22}}{2} \qquad G = \det(\mathrm{II})$$

   (a) Compute the mean and Gaussian curvature for: $f(x,y) = 5x^2 - 4y^2$
   (b) Compute the mean and Gaussian curvature for: $f(x,y) = 66x^2 - 24xy + 59y^2$
   (c) Considering the second fundamental form II above, what could you say about the principal curvatures κ_1 and κ_2 if $\frac{\partial}{\partial x}\frac{\partial}{\partial y} f = 0$?
   (d) A cylinder with radius $r = 1$ can locally be parametrised as $f(x,y) = 1 - \sqrt{1-(x-y)^2}$. Show that it has mean curvature $H = 1$ and zero Gaussian curvature, i.e. $G = 0$.

10. Compute the Laplacian of the given functions $f(\mathbf{x})$ (assume that everything except for $\mathbf{x}$ is constant).

   (a) $f(\mathbf{x}) = e^{-\|\mathbf{x}-\mathbf{c}\|^2/s^2}$
   (b) $f(\mathbf{x}) = \sin(\mathbf{x}^T\mathbf{d})$

11. On a triangulated surface, the vertex $v_0 = (4,5,6)$ is surrounded by four neighbouring vertices, so that $v_0v_1v_2$, $v_0v_2v_3$, $v_0v_3v_4$ and $v_0v_4v_1$ build triangles.

$$v_1 = (5,-1,4), \quad v_2 = (0,1,2), \quad v_3 = (5,3,0), \quad v_4 = (6,5,2)$$

   (a) Compute the discrete Laplacian at the vertex v_0 and derive the mean curvature H at this point.
   (b) How would you compute the Gaussian curvature?

12. The sphere can be represented by the parametric form:

$$s(u,v) = \big(r\cos(u)\cos(v), r\sin(u)\cos(v), r\sin(v)\big)$$

   Determine if this parametric form of the sphere is a *conformal* map form the plane to the sphere or not.

13. A rotation by θ around the z-axis is typically represented by the 3×3-matrix, which is then applied to a vector $\vec{v} = (v_0, v_1, v_2)^T$:

$$R_\theta = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \vec{v}' = R_\theta \cdot \vec{v}$$

When using quaternions, the same rotation can be written using $r_\theta = \cos(\theta/2) + k\sin(\theta/2)$ (where $i^2 = j^2 = k^2 = -1$) as:

$$\mathbf{v}' = \mathbf{r}_\theta \mathbf{v} \mathbf{r}_\theta^{-1}$$

The vector $\vec{v}$ is written as the quaternion $v_o i + v_1 j + v_2 k$ and $r_\theta^{-1} = \cos(\theta/2) - k\sin(\theta/2)$.

Demonstrate that the quaternion rotation $\mathbf{r}_\theta \mathbf{v} \mathbf{r}_\theta^{-1}$ yields indeed the same rotation as the classic 3×3-matrix above.

14. (a) What is rigging? Briefly describe what kind of structure and data are added to the geometry/surface model.

    (b) Assume we have a cylinder with an axis aligned with the x-axis. The cylinder extends from -1 to 1 on the x-axis and has a base radius of 1. We embed two bones inside the cylinder along the x-axis: one extends from -1 to 0, the other from 0 to 1. We define the influence of each bone on each point on the cylinder as the inverse distance of the point to the bone. Assume we transform the second bone with the transformation T.

    Determine the concrete weights and give a formula of how the new position x' would be computed from the weights $w_1(x)$, $w_2(x)$, x and T for a point on the surface (a) in the middle and (b) on either end.

15. *(Past exam question)* **Transformation Blending (I) Thinking about fundamental properties.** First, please answer the following questions in maximum two sentences for each.

    (a) What is the advantage of representing rigid transformations with dual quaternions for blending?

    (b) Briefly explain one fundamental disadvantage of using quaternions based shortest path blending for rotations as compared to linear blend skinning (i.e. averaging rotation matrices)?

    Hint: think about the continuity of both blending methods for 2D rotations.

16. *(Past exam question)* **Transformation Blending (II) Derivations and deeper understanding.** Below we list all required properties of dual quaternions for the rest of the exercise.

    A dual quaternion $\hat{\mathbf{q}}$ can be written in the form $\hat{\mathbf{q}} = \mathbf{q}_0 + \epsilon \mathbf{q}_\epsilon$, where $\mathbf{q}_0$ and $\mathbf{q}_\epsilon$ are quaternions and ϵ is the dual unit with the property $\epsilon^2 = 0$. The norm of $\hat{\mathbf{q}}$ is then given by:

$$\|\hat{q}\| = \|q_0\| + \epsilon \frac{\langle \mathbf{q}_0, \mathbf{q}_\epsilon \rangle}{\|\mathbf{q}_0\|}$$

Dual quaternions representing rigid transformations can be written in the following form:

$$\hat{\mathbf{q}} = \cos\left(\hat{\theta}/2\right) + \hat{\mathbf{s}} \sin\left(\hat{\theta}/2\right)$$

where $\hat{\theta} = \theta_0 + \epsilon\,\theta_\epsilon$ and $\hat{\mathbf{s}} = \mathbf{s}_0 + \epsilon\,\mathbf{s}_\epsilon$. Here, $\mathbf{s}_0$ is the axis of rotation, θ_0 is the rotation angle, and θ_ϵ is the amount of translation along $\mathbf{s}_0$. Since this is a unit dual quaterion it can be shown that $\langle \mathbf{s}_0, \mathbf{s}_\epsilon \rangle = 0$ and $\langle \mathbf{s}_0, \mathbf{s}_0 \rangle = 1$.

The power of a dual quaternion is defined by

$$\hat{\mathbf{q}}^t = e^{t \log(\hat{\mathbf{q}})}$$

where:

$$e^{\hat{\mathbf{q}}} = \cos(\|\hat{\mathbf{q}}\|) + \frac{\hat{\mathbf{q}}}{\|\hat{\mathbf{q}}\|}\sin(\|\hat{\mathbf{q}}\|) \qquad \log\left(\cos\left(\hat{\theta}/2\right) + \hat{\mathbf{s}}\sin\left(\hat{\theta}/2\right)\right) = \hat{\mathbf{s}}\frac{\hat{\theta}}{2}$$

(c) Utilizing the properties above, for a dual quaternion $\hat{\mathbf{q}} = \cos\left(\hat{\theta}/2\right) + \hat{\mathbf{s}}\sin\left(\hat{\theta}/2\right)$, prove that:

$$\hat{\mathbf{q}}^t = \cos\left(t\hat{\theta}/2\right) + \hat{\mathbf{s}}\sin\left(t\hat{\theta}/2\right)$$

Hint: you do not need to know the expression for $\cos(\hat{\theta})$ or $\sin(\hat{\theta})$.

(d) Now, consider rigid transformations in the 2D xy-plane. For these transformations, the rotation is always around the z (or $-z$)-axis, i.e. $\mathbf{s}_0$ is fixed to the z-axis. On the other hand, a dual quaternion encodes translations only along $\mathbf{s}_0$, which are in this case always zero, since we can only translate in the xy-plane. Then, how can a dual quaterion represent a rotation and translation in the xy-plane, such as the one depicted in Figure 1? Please answer in maximum two sentences without any equations.

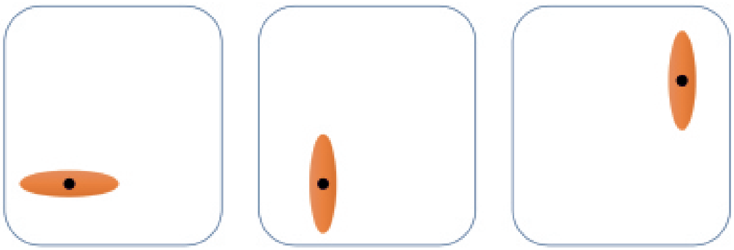

FIGURE 1. An object (left) is first rotated around its center of mass (middle) and then translated (right).