2-round Ramsey games for cycles in random graphs

Kyriakos Katsamaktsis

August 6, 2025

1 Introduction

The starting point of Ramsey theory is the theorem due Ramsey [18] that for any graph H, if n is sufficiently large, any red-blue colouring of K_n contains a monochromatic copy of H. For a graph G we write $G \to H$ if any red-blue colouring of G has a monochromatic copy of H, and say that G is *H-Ramsey*. Ramsey theory is a central area of combinatorics. One of the most famous long-standing open problems in combinatorics is to determine the rate of growth of the diagonal Ramsey number i.e. the smallest n as a function of r so that $K_n \to K_r$. Recently, there was major breakthrough [3] by Campos, Griffiths, Morris and Sahasrabudhe on this problem, who obtained the first exponential improvement to the upper bound, as well as its off-diagonal variant by Campos, Jenssen, Michelen and Sahasrabudhe [4].

One important variant of the Ramsey problem, and the topic of this paper, is when the 'host' graph is the binomial random graph $\mathbf{G}(n,p)$ instead of the complete graph. The original motivation for studying the Ramsey property of $\mathbf{G}(n,p)$ was for finding sparse H-Ramsey graphs. This line of work started in the 1980's with the work of Frankl and Rödl [8], who proved that there are K_4-free graphs that are K_3-Ramsey; and Łuczak, Ruciński and Voigt [15], who studied more general Ramsey properties of the binomial random graph. The graph property $\mathbf{G}(n,p) \to H$ is increasing, and hence, by a well-known theorem of Bollobás and Thomason [2], it has a threshold. This was determined in an sequence of breakthroughs [19–21] by Rödl and Ruciński for a wide class of graphs H, proving what is now known as the 'random Ramsey theorem'.

Theorem 1.1 (Rödl–Ruciński [21]). *Let H be a graph which is not a forest of stars and P_3's. Then there exist constants c, C such that*

$$\mathbf{P}[\mathbf{G}(n,p) \to H] = \begin{cases} 0 & p \le cn^{-1/m_2(H)} \\ 1 & p \ge Cn^{-1/m_2(H)}, \end{cases}$$

where

$$m_2(H) = \max_{H' \subseteq H : e(H') \geq 2} \frac{e_{H'} - 1}{v_{H'} - 2}.$$

It is worth pausing for a moment to motivate the appearance of the density function $m_2(H)$ in the above theorem. Clearly, we can avoid monochromatic copies of H if there are no monochromatic copies of some subgraph $H' \subseteq H$ with $e(H') \geq 2$. Intuitively, it is easier to avoid monochromatic copies of H' when, on average, there are few copies of H' per edge. That is, when for some small constant $c > 0$ we have $n^{v_{H'}} p^{e_{H'}} \leq cpn^2$. Solving for p, and minimising over $H' \subseteq H$ gives a value for the Rödl–Ruciński threshold. Random Ramsey theory studies variants of this question in the binomial random graph, and in other random graph models [6, 7] and more general random discrete structures such as groups [9, 22].

The topic of the present paper is a two-round variant of the theorem of Rödl and Ruciński. Let $\mathbf{G}_1 \sim \mathbf{G}(n, p)$ with $p \leq cn^{-1/m_2(H)}$ so that, with high probability, $\mathbf{G}_1$ has a red-blue colouring avoiding monochromatic copies of H. Fix such a colouring ϕ of $\mathbf{G}_1$. Once this colouring is fixed, a second independent copy $\mathbf{G}_2 \sim \mathbf{G}(n, q)$ on the same vertex set is revealed. Is there a colouring ψ of $\mathbf{G}_2 \setminus \mathbf{G}_1$ (i.e. the edges of $\mathbf{G}_2$ not in $\mathbf{G}_1$) so that, with high probability, $\mathbf{G}_1 \cup \mathbf{G}_2$ has no monochromatic copy of H under ϕ and ψ? We say that the combined colouring *extends* ϕ. The crux is that we must first colour $\mathbf{G}_1$, without any knowledge of $\mathbf{G}_2$ whatsoever besides its distribution, and yet ensure that the colouring of $\mathbf{G}_1$ can be extended with high probability to $\mathbf{G}_1 \cup \mathbf{G}_2$.

The motivation for this problem comes from the central role it played in two well-studied questions of random Ramsey theory. Firstly, in determining the *sharpness* of the threshold in the Rödl–Ruciński theorem for triangles [10] i.e. showing that for all n the thresholds for the 0 and 1-statements are within a $o(1)$ factor of each other (as opposed to a constant factor apart as in Theorem 1.1). Secondly, in determining the maximum duration of the *online random Ramsey game*. In this game a player colours the random graph process online i.e. as soon as each edge of the random graph process is reavealed, they must colour it irrevocably either red or blue. The game finishes when the player is forced to create a monochromatic H.

For triangles, Friedgut, Kohayakawa, Rödl, Ruciński and Tetali [11], and for a wide class of graphs H (which does not include all graphs in the Rödl – Ruciński theorem) Marciniszyn, Spöhel and Steger [17] determined a function $m^*(H)$ so that, with high probability, the game lasts $o(m^*(H))$ rounds (the corresponding 0-statement, with the same threshold $m^*(H)$, is also known [11, 16]). To prove this, the authors [11, 17] considered the two-round game where both random graphs have $m^*(H)$ edges. (The 0-statement does however use the random graph process directly). As one might expect, the online threshold is well below the Rödl-Ruciński threshold i.e. $m^*(H) = o(n^{2-1/m_2(H)})$.

Following [1], we say that $\hat{q} = \hat{q}(p, n, H)$ is a *Ramsey completion threshold for H* if

- when $q = o(\hat{q})$, with high probability there exists a 2-colouring of $\mathbf{G}_1$ that extends to a colouring of $\mathbf{G}_1 \cup \mathbf{G}_2$ so that it has no monochromatic copy of H (the 0-statement);
- when $q = \omega(\hat{q})$, with high probability no 2-colouring of $\mathbf{G}_1$ extends to a colouring of $\mathbf{G}_1 \cup \mathbf{G}_2$ without a monochromatic copy of H (the 1-statement).

The two-round Ramsey game was first studied in its own right by Conlon, Das, Lee and Mészáros [5], who considered the regime $p \geq \varepsilon n^{-1/m_2(H)}$, for any fixed $\varepsilon < c$, where c is as in the Rödl–Ruciński theorem. They proved that $\hat{q} = n^{-2}$ is a Ramsey completion threshold for a large class of graphs H. In other words, when $q = \omega(n^{-2})$, with high probability no colouring of $\mathbf{G}_1$ extends to a colouring of $\mathbf{G}_1 \cup \mathbf{G}_2$ avoiding monochromatic copies of H. When $q = o(n^{-2})$, with high probability $\mathbf{G}_2$ has no edges at all, and so trivially any two-colouring of $\mathbf{G}_1$ which avoids monochromatic copies of H will do (and such a colouring exists with high probability by the Rödl – Ruciński theorem).

What if $p = o(n^{-1/m_2(H)})$? Observe that if $p = o(m^*(H)/n^2)$, then we can colour the edges of $\mathbf{G}_1 \cup \mathbf{G}_2$ online; the interesting case is when p is between these two extremes. The two-round Ramsey game was first studied in this range by Alon, Morris and Samotij when H is the triangle [1]. Remarkably, they discovered that there are two distinct regimes. They showed that, when p is in this range, there is a $\hat{p}$ and $\hat{q}_1, \hat{q}_2$ so that, for $p = o(\hat{p})$, the Ramsey completion threshold for the triangle is $\hat{q}_1$; and for $p = \omega(\hat{p})$ the Ramsey completion threshold for the triangle is $\hat{q}_2$.

Theorem 1.2 (Alon, Morris, Samotij [1])**.** *Suppose $p = o(n^{-1/2})$ and $p = \omega(n^{-2/3})$. Then*

$$\hat{q} = \begin{cases} p^{-7/2} n^{-3}, & p = o(n^{-3/5}) \\ p^{-6} n^{-8}, & p = \omega(n^{-3/5}) \end{cases}$$

In the present paper I prove two 0-statements for all cycles, thus lower-bounding the Ramsey completion thresholds, and providing evidence that there are two distinct regimes for all cycles. The values of these thresholds, as well as the point where they change, are a direct generalisation of the results in [1]. For the theorem below, note that $m_2(C_\ell) = -1 + 1/(\ell - 1)$; and that the online Ramsey game for C_ℓ lasts with high probability at most $n^{1+1/\ell}$ rounds, so $\mathbf{G}(n, p)$ with $p = \omega(n^{-1+1/\ell})$ cannot be coloured online to avoid monochromatic copies of C_ℓ. Before explaining the values of the thresholds and the point at which they change, let us state the main theorem of this paper, Theorem 1.3.

Theorem 1.3. *Let $\ell \geq 3$ and suppose $p = o(n^{-1+1/(\ell-1)})$ and $p = \omega(n^{-1+1/\ell})$. Then there exists* *a* $c > 0$ *such that the following holds.*

$$\hat{q}_{C_\ell} \geq \begin{cases} n^{-\ell}\, p^{-\ell-1/(\ell-1)}, \text{ if } p \leq cn^{-\frac{\ell-2}{\ell-1-1/\ell}} \\ n^{-\ell(\ell-1)}\, p^{-\ell^2+1}, \text{ if } p \geq n^{-\frac{\ell-2}{\ell-1-1/\ell}} \end{cases}$$

For the remainder of the paper we set

$$\hat{m}(C_\ell) = \frac{\ell - 1 - 1/\ell}{\ell - 2}$$

so that the completion threshold changes when $p = n^{-1/\hat{m}(C_\ell)}$; and we let

$$\hat{q}_{\text{up}} = n^{-\ell(\ell-1)}\, p^{-\ell^2+1}, \hat{q}_{\text{lo}} = n^{-\ell} p^{-\ell-1/(\ell-1)}$$

denote the completion thresholds in the upper and lower range respectively. It is straightforward to check that $\hat{q}_{\text{up}} \ll \hat{q}_{\text{lo}}$, as we may expect since $\mathbf{G}_1$ has many more edges in the upper than in the lower range and hence intuitively $\mathbf{G}_2$ can afford to have fewer edges before a monochromatic C_ℓ is forced. Setting $\ell = 3$, we see the bounds coincide with the values of the thredholds in the Alon – Morris – Samotij theorem. The value of each threshold is determined by the appearance of a coloured subgraph. Showing that in any 2-colouring of $\mathbf{G}_1$ the number of these coloured subgraphs is up to a constant factor the same as the copies of the underlying uncoloured subgraphs would prove the 1-statements missing from Theorem 1.3. I have work in progress towards proving the 1-statement in the lower range.

The coloured graph for the threshold in the upper range is as follows. It consists of one red and one blue $P_{\ell-1}$ that have the same ends and share no other vertex; and a copy of C_ℓ on each edge of the red (say) $P_{\ell-1}$, so that these C_ℓ's share no other vertex with one another or the $P_{\ell-1}$'s, and their edges not on the $P_{\ell-1}$ are coloured blue. Clearly, once this colouring is fixed, if $\mathbf{G}_2$ contains the 'diagonal' edge connecting the ends of the two monochromatic $P_{\ell-1}$'s a monochromatic C_ℓ is forced.

The coloured graph for the threshold in the lower range is similar, but here $\mathbf{G}_2$ needs to hit $\ell - 1$ missing edges for a monochromatic C_ℓ to be forced. It consists of a blue $P_{\ell(\ell-1)}$ with one red edge connecting two vertices $\ell - 1$ edges apart. This graph has $\ell - 1$ potential edges whose addition would create a C_ℓ, and any of these edges must be coloured red to avoid a blue C_ℓ. If all are hit by $\mathbf{G}_2$, a red C_ℓ is forced, using the red edge from $\mathbf{G}_1$. The number of copies of this (uncoloured) graph in $\mathbf{G}_1$ is, with high probability, $n^{\ell(\ell-1)}p^{\ell(\ell-1)+1}$. Hence the expected number of copies with all $\ell - 1$ dangerous potential edges present in $\mathbf{G}_2$ is $n^{\ell(\ell-1)}p^{\ell(\ell-1)+1}q^{\ell-1}$. Setting this equal to 1 and solving for q gives the threshold in the lower range.

A heuristic similar to the one behind the Rödl–Ruciński theorem explains the value of p where the thresholds change, $n^{-\frac{\ell-2}{\ell-1-1/\ell}}$. There are two constraints that we want to satisfy in order to avoid creating a monochromatic C_ℓ: $\mathbf{G}_1$ must have no monochromatic C_ℓ and not too many subgraphs with dangerous potential edges. Setting the expected number of C_ℓ's and the subgraphs that determine the threshold in the upper range to be equal and solving for p yields $p = n^{-\frac{\ell-2}{\ell-1-1/\ell}}$.

Notation We will use $a \ll b$ and $c \gg d$ to mean $a = o(b)$ and $c = \omega(d)$ respectively. Given two graphs G and H we write $G \cup H$ for the graph with vertices $V(G) \cup V(H)$ and edges $E(G) \cup E(H)$. We write $G \setminus H$ for the graph with vertex set $V(G)$ and edges $E(G) \setminus E(H)$. Let X_1, X_2 be two copies of C_ℓ where $E(X_1) \cap E(X_2)$ consists of exactly one edge e, and $V(X_1) \cap V(X_2)$ consists of the two vertices of e. We denote the graph $X_1 \cup X_2$ by $2C_\ell$.

2 Proof of Theorem 1.3

We call a red-blue colouring of a graph G *good* if

1. it has no monochromatic C_ℓ;
2. every red edge lies in a copy of C_ℓ.

We call a coloured copy of $C_{2\ell-2}$ *dangerous* if it consists of a red $P_{\ell-1}$ and a blue $P_{\ell-1}$. We call the potential edge between the ends of the red and blue path a *dangerous potential edge*. Clearly, if our colouring of $\mathbf{G}_1$ were to contain a dangerous $C_{2\ell-2}$, and if $\mathbf{G}_2$ were to hit the potential dangerous edge, then we would be forced to have a monochromatic C_ℓ in $\mathbf{G}_1 \cup \mathbf{G}_2$. To prove Theorem 1.3, we will find a good colouring of $\mathbf{G}_1$ with so few dangerous $C_{2\ell-2}$'s so that, with high probability, $\mathbf{G}_2$ hits no dangerous potential edge. For this we will need to understand the graphs that may give rise to a dangerous $C_{2\ell-2}$'s, thus motivating the next definition. In particular, as Proposition 2.3 shows, in a good colouring every dangerous $C_{2\ell-2}$ is contained in one of the graphs in Definition 2.1.

Definition 2.1 ($\mathcal{G}_{C_\ell}$, $\mathcal{G}^+_{C_\ell}$, G_{C_ℓ}, $G^+_{C_\ell}$, attached C_ℓ's)**.** *We will define two collections of graphs $\mathcal{G}_{C_\ell}, \mathcal{G}^+_{C_\ell}$ obtained by the following procedure. Let H be a copy of $2C_\ell$, let xy be the shared edge and F_0, F_{-1} the two edge-disjoint paths of length $\ell - 1$ with ends x, y.*

For every edge $e \in E(F_0)$, let F_e be a copy of C_ℓ such that $e \in E(F_e) \cap E(F_0)$ and $xy \notin E(F_e)$. We say F_e is attached *to e, and F_e is an* attached copy *of C_ℓ or an* attached C_ℓ. *Define $G^+ := H \cup \bigcup_{e \in E(F_0)} F_e$ and $G := G^+ \setminus \{xy\}$. We call F_0 the* central P_ℓ *and F_{-1} the* attached P_ℓ *of G and G^+, and $F_0 \cup \{xy\}$ the* central C_ℓ *of G^+.*

We define $\mathcal{G}_{C_\ell}$ (resp. $\mathcal{G}^+_{C_\ell}$) to be the collection of graphs that can be obtained as a graph G (resp. G^+) by the above procedure, and which are minimal with respect to the property that for every $e \in E(F_0)$ there is a C_ℓ attached to e.

We denote by $G^+_{C_\ell} \in \mathcal{G}^+_{C_\ell}$ the unique graph consisting of $\ell-1$ attached copies of C_ℓ which share no edge among themselves, each one shares exactly two vertices with the central C_ℓ, and none has a vertex in $V(F_{-1}) \setminus \{x, y\}$. We set $G_{C_\ell} := G^+_{C_\ell} \setminus \{xy\}$.

Remark 2.2. Any $G \in \mathcal{G}^+_{C_\ell}$ has at least 3 distinct copies of C_ℓ apart from the central copy, and any $G \in \mathcal{G}_{C_\ell}$ has at least 2 attached C_ℓ's

The motivation behind Definition 2.1 is that F_0, the central P_ℓ, plays the role of the red P_ℓ in a dangerous $C_{2\ell-2}$ and F_{-1}, the attached P_ℓ, plays the role of the blue P_ℓ. This is the essence of the next proposition and its proof.

Proposition 2.3. *In any colouring where every red edge lies in a C_ℓ, every dangerous $C_{2\ell-2}$ is a subgraph of some $G \in \mathcal{G}_{C_\ell}$.*

Proof. Let H be a dangerous $C_{2\ell-2}$ and let P_0 and P_{-1} be the red and blue paths of length ℓ respectively. For every $e \in E(P_0)$, there exists a copy F_e of C_ℓ. Let $\mathcal{F} = \{F_e : e \in E(P_0)\}$ be a collection of such copies such that for any $\mathcal{F}' \subsetneq \mathcal{F}$, there exists some $e \in E(P_0)$ which does not lie in any C_ℓ in $\mathcal{F}'$. Then the graph with vertices $V(H) \cup \bigcup_{F\in\mathcal{F}} V(F)$ and edges $E(H) \cup \bigcup_{F\in\mathcal{F}} E(F)$ is in $\mathcal{G}_{C_\ell}$, with $\mathcal{F}$ being the collection of attached C_ℓ's and P_0, P_1 being the central and attached P_ℓ respectively. $\square$

Most of the technical content of this paper is in the next lemma, which shows that in the lower range, with high probability we can colour $\mathbf{G}_1$ so that it has few dangerous copies of $C_{2\ell-2}$.

Lemma 2.4. *There exists $c > 0$ such that for $p \leq cn^{-1/\hat{m}(C_\ell)}$ and $\mathbf{G}_1 \sim \mathbf{G}(n,p)$ the following holds. With high probability, $\mathbf{G}_1$ has a good red-blue colouring with at most $\hat{q}_{\,lo}^{-1}$ dangerous copies of $C_{2\ell-2}$.*

The proof of Lemma 2.4 is split into a 'probabilistic lemma', Lemma 4.3 which is proven in Section 5; and a deterministic lemma, Lemma 4.4, which is proven in Section 6. In Section 4 we put these together to prove Lemma 2.4.

The next lemma is the corresponding statement in the upper range, which we prove at the end of this section.

Lemma 2.5. *Suppose $p \geq n^{-1/\hat{m}(C_\ell)}$ and let $\mathbf{G}_1 \sim \mathbf{G}(n,p)$. Then with high probability, $\mathbf{G}_1$ has a good red-blue colouring with at most $\hat{q}_{\,up}^{-1}$ dangerous copies of $C_{2\ell-2}$.*

To prove Lemma 2.5, as well as Lemma 2.7 below we will need the following result which says that there are many more copies of G_{C_ℓ} in $\mathbf{G}_1$ than of any other graph in $\mathcal{G}_{C_\ell}$.

Lemma 2.6. *For a graph H let X_H denote the number of copies of H in $\mathbf{G}_1$. Then for any $G \in \mathcal{G}_{C_\ell} \setminus G_{C_\ell}$ we have*

$$\mathbb{E}[X_G] \ll \mathbb{E}\Big[X_{G_{C_\ell}}\Big].$$

To prove Lemma 2.6 we need some technical lemmas about how the attached C_ℓ's interact; we postpone its proof until the end of Section 3.2.

The next lemma, which is crucial for extending a colouring of $\mathbf{G}_1$ to $\mathbf{G}_1 \cup \mathbf{G}_2$, says that any $G \in \mathcal{G}^+_{C_\ell}$ has at most one edge in $\mathbf{G}_2$. In other words, we only need to worry about the dangerous potential edge that connects the ends of the central and attached copy of P_ℓ in G.

Lemma 2.7. *With high probability over $\mathbf{G}_1 \sim \mathbf{G}(n,p), \mathbf{G}_2 \sim \mathbf{G}(n,q)$ with $n^{-1+1/\ell} \ll p \ll n^{-1+1/(\ell-1)}$ and $q \ll \hat{q}_{\,lo}$ the following holds. For every $G \in \mathcal{G}^+_{C_\ell}$, every copy of G in $\mathbf{G}_1 \cup \mathbf{G}_2$ has at most one edge in $\mathbf{G}_2$.*

Proof of Lemma 2.7. For a graph $G \in \mathcal{G}_{C_\ell}$ let X_G denote the expected number of copies of G in $\mathbf{G}_1$ and let Y_G denote the number of copies of G^+ in $\mathbf{G}_1 \cup \mathbf{G}_2$ with at least two edges in $\mathbf{G}_2$. Since $q \ll p$, for any $G \in \mathcal{G}_{C_\ell}$, $\mathbb{E}[Y_G] = \Theta\left(q^2 p^{-1}\,\mathbb{E}[X_G]\right)$. Hence Lemma 2.6 implies that for any $G \in \mathcal{G}_{C_\ell}$, $\mathbb{E}[Y_G] = o\Big(\mathbb{E}\Big[Y_{G_{C_\ell}}\Big]\Big)$. Therefore it suffices to show that $\mathbb{E}\Big[Y_{G_{C_\ell}}\Big] \ll 1$, and the required conclusion follows from Markov's inequality and a union bound over the $O(1)$ choices for $G \in \mathcal{G}_{C_\ell}$.

Recalling the definition of $G^+_{C_\ell}$ it is easy to see that $v(G^+_{C_\ell}) = \ell(\ell-1)$ and $e(G^+_{C_\ell}) = \ell^2$. Since $q \ll p$, the expected number of $G^+_{C_\ell}$ with at least two edges in $\mathbf{G}_2$ is at most $O\left(n^{\ell(\ell-1)}\,p^{\ell^2-2}q^2\right)$ which is $o(1)$ for $q \ll \hat{q}_{\,\mathrm{lo}}$. □

The next lemma shows how, given a colouring of the first random graph such as the ones in Lemmas 2.4 and 2.5 one can extend it to the second random graph.

Lemma 2.8. *Suppose $n^{-1+1/\ell} \ll p \ll n^{-1+1/\ell}$ and $q \ll Q^{-1}$, for some Q with $Q \geq \hat{q}_{\,lo}^{\,-1}$. Let $\mathbf{G}_1 \sim \mathbf{G}(n,p), \mathbf{G}_2 \sim \mathbf{G}(n,q)$ be independent random graphs on the same vertex set. Suppose that, with high probability, $\mathbf{G}_1$ has a good colouring ϕ with at most Q dangerous $C_{2\ell-2}$'s. Then with high probablity, there exists a colouring ϕ' of $\mathbf{G}_1 \cup \mathbf{G}_2$ that agrees with ϕ on $E(\mathbf{G}_1)$ that has no monochromatic C_ℓ.*

Proof. The next claim contains the properties we need the 'realisation' of the first random graph to satisfy.

Claim 2.9. *With high probability, the random graph $\mathbf{G}_1$ is a graph G_1 satisfying the following.*

- *There is a good colouring ϕ of G_1 with at most Q dangerous copies of $C_{2\ell-2}$.*

- *With high probability over $\mathbf{G}_2 \sim \mathbf{G}(n,q)$, for every $G \in \mathcal{G}^+_{C_\ell}$, every copy of G in $G_1 \cup \mathbf{G}_2$ has at most one edge in $\mathbf{G}_2$.*
- *With high probability over $\mathbf{G}_2 \sim \mathbf{G}(n,q)$, $\mathbf{G}_2$ contains no edges which are dangerous potential edges with respect to ϕ.*

Proof. The first bullet-point follows clearly from the assumptions of the lemma. The second bullet-point follows from Fubini's theorem and Lemma 2.7.

The last bullet-point follows from Markov's inequality. Indeed, the number of dangerous potential edges is at most the number of dangerous copies of $C_{2\ell-2}$. Since each edge is present with probability q, the expected number of such edges in $\mathbf{G}_2$ is at most $Q \cdot q \ll 1$. □

Fix a colouring ϕ of G_1 satisfying the first bullet point of Claim 2.9. By Claim 2.9, with high probability, $\mathbf{G}_2$ is a graph G_2 such that for every $G \in \mathcal{G}^+_{C_\ell}$, every copy of G in $G_1 \cup G_2$ has at most one edge in G_2; and G_2 contains no edges which are dangerous potential edges with respect to ϕ.

We extend the colouring ϕ to a colouring ϕ' of $G_1 \cup G_2$ as follows. We go through $E(G_2) \setminus E(G_1)$ in an arbitrary order, and for each $e \in E(G_2) \setminus E(G_1)$ we colour e blue, unless this creates a blue C_ℓ along with $E(G_1)$ and the already coloured edges of $E(G_2)$, in which case we colour e red.

Suppose for the sake of contradiction that this colouring fails to be C_ℓ-free. Then there is an edge $xy \in E(G_2) \setminus E(G_1)$ such that there is a dangerous $C_{2\ell-2}$, K, contained in the union of $E(G_1)$ and the edges of G_2 coloured thus far, with x, y being the ends of the red and blue P_ℓ of K. Since the colouring of $E(G_2) \setminus E(G_1)$ maintains the property that every red edge lies in a copy of C_ℓ, by Proposition 2.3 K is a subgraph of a $G \in \mathcal{G}_{C_\ell}$. Hence $K \cup \{xy\}$ lies in a copy of some $G^+ \in \mathcal{G}^+_{C_\ell}$. By the second bullet point of Claim 2.9, at most one edge of G^+ lies in G_2, so xy is the only edge of G^+ that lies in G_2 and K is a dangerous $C_{2\ell-2}$ in ϕ and a subgraph of G_1. Therefore the pair x, y is a dangerous potential edge with respect to ϕ. Hence the third bullet point of Claim 2.9 implies that $xy \notin E(G_2)$, which is a contradiction. □

Proof of Theorem 1.3. Theorem 1.3 is a direct consequence of Lemma 2.8 and Lemma 2.5, for the upper range; and Lemma 2.8 and Lemma 2.4 in the lower range. □

We end this section by completing the proof of the 0-statement for the upper range. For this we will need the following well-known bound on the number of copies of small graphs in $\mathbf{G}(n,p)$.

Proposition 2.10 (Chapter 3 [13])**.** *Let H be a strictly balanced graph on a bounded number of vertices and suppose $p \gg n^{-1/m(H)}$. Then with high probability, the number of copies of H in $\mathbf{G}(n,p)$ is $\Theta(n^{v_H} p^{e_H})$.*

Proof of Lemma 2.5. By Theorem 1.1, with high probability $\mathbf{G}_1$ has a 2-colouring with no monochromatic C_ℓ. Fix such a colouring. Then, change the colour of every red edge that is not on a C_ℓ to blue. The resulting colouring is good, so by Proposition 2.3 every dangerous $C_{2\ell-2}$ lies in a copy of some $G \in \mathcal{G}_{C_\ell}$. Hence the total number of all dangerous $C_{2\ell-2}$ is at most the sum over $G \in \mathcal{G}_{C_\ell}$ of the number of copies of $G \in \mathcal{G}_{C_\ell}$ in $\mathbf{G}_1$. By Lemma 2.6 and Proposition 2.10, the number each such copy is, with high probability, at most $O\left(n^{\ell(\ell-1)} p^{\ell^2-1}\right)$. This is $O\left(\hat{q}_{\text{up}}^{-1}\right)$, as required. $\square$

3 Preliminary lemmas

3.1 Graph densities and graph counts

At several places we will use the following observation.

Observation 3.1. *Let $a, b, x, y, C > 0$ with $a \geq x$ and $b \geq y$.*

- *Suppose $a > x$ and $b > y$. Then $\frac{a-x}{b-y} \geq \frac{a}{b} \Leftrightarrow \frac{a}{b} \geq \frac{x}{y}$, with equality if and only if $a/b = x/y$.*
- *If $\frac{a}{b}, \frac{x}{y} \geq C$, then $\frac{a+x}{b+y} \geq C$, with equality if and only if $\frac{a}{b} = \frac{x}{y} = C$*
- *If $x > y$ then $\frac{x+C}{y+C} < \frac{x}{y}$ and $\frac{x-C}{y-C} > \frac{x}{y}$.*

The density of a non-empty graph G is $d(G) = e_G/v_G$ and we define $m(G) = \max_{G' \subseteq G} d(G')$, where G' has at least one vertex. The 1-density for G with $v_G \geq 2$ is $d_1(G) = \frac{e_G}{v_G - 1}$ and the 2-density if $v_G \geq 3$ is $d_2(G) = \frac{e_G - 1}{v_G - 2}$. The maximum 1- and 2-densities are defined to be

$$m_1(G) = \max_{H \subseteq G} d_1(H) \text{ and } m_2(G) = \max_{H \subseteq G} d_2(H).$$

For $k \in \{1, 2\}$, we say G is *strictly k-balanced* if $m_k(G) = d_k(G)$ and $d_k(H) < d_k(G)$ for every proper subgraph H of G.

Observation 3.2. *Every cycle is strictly 1- and 2-balanced.*

Observation 3.3. *We have $\hat{m}(C_\ell) < m_2(C_\ell)$.*

Proposition 3.4. *For every strictly 2-balanced graph H and every subgraph $F \subseteq H$ with $2 \leq v_F \leq v_H - 1$,*

$$\frac{e_H - e_F}{v_H - v_F} \geq m_2(H),$$

with equality if and only if F is a single edge.

Proof. If $v_F \geq 3$ then
$$\frac{e_H - e_F}{v_H - v_F} = \frac{e_H - 1 - (e_F - 1)}{(v_H - 2) - (v_F - 2)} > \frac{e_H - 1}{v_H - 2},$$
using $d_2(F) < d_2(H)$ and Observation 3.1. For $v_F = 2$ it is easy to check the inequality holds, with equality if and only if F is an edge. □

3.2 Attached C_ℓ's

Throughout the paper, when considering a graph $G \in \mathcal{G}_{C_\ell} \cup \mathcal{G}^+_{C_\ell}$, F_0 and F_{-1} will always denote the central and the attached P_ℓ of G. It will be useful to consider different orderings on the attached C_ℓ's for the proof of the 0-statement in the lower range.

Definition 3.5 (Order of attached C_ℓ's, vertices and edges.)**.** *Let $G \in \mathcal{G}_{C_\ell}$, let F_0 be the central P_ℓ of G with ends x, y.*

We define the following linear order on $V(F_0)$: for $u \neq v \in V(F_0)$, $u < v$ if we encounter u before v as we traverse F_0 from x to y.

We define the following linear order on $E(F_0)$: for $e \neq e' \in E(F_0)$, $e < e'$ if we encounter e before e' as we traverse F_0 from x to y.

Finally, we define the following linear order on the attached C_ℓ's: for two attached C_ℓ's $F \neq F'$, we have $F < F'$ if $E(F) \cap E(F_0)$ is less than $E(F') \cap E(F_0)$ in the lexicographic order of subsets of $E(F_0)$ induced by the linear order of $E(F_0)$. If G has k attached C_ℓ's, $F_1, \ldots, F_k$ will always denote an enumeration of the attached C_ℓ's in this order i.e. $F_1 < \cdots < F_k$, with $x \in V(F_1)$ and $y \in V(F_k)$.

Observation 3.6. *Let $G \in \mathcal{G}_{C_\ell}$ with k attached C_ℓ's.*

- *For every $i \in [2, k]$, F_i and $\bigcup_{j<i} F_j$ share the first vertex in $V(F_0) \cap V(F_i)$;*
- *For every $i \in [2, k-1]$, F_i shares the first and last vertex in $V(F_i) \cap V(F_0)$ with $\bigcup_{j \neq i} F_j$.*

Proposition 3.7. *Let $G \in \mathcal{G}_{C_\ell}$ with k attached C_ℓ's. For $i \geq 1$ set $F'_i = F_i \cap \bigcup_{0 \leq j < i} F_i$ and $F'_{-1} = F_{-1} \cap \bigcup_{i=1}^k F_i$. For $i \in [k] \cup \{-1\}$ let $v_i = v(F'_i)$ and $e_i = e(F'_i)$. Then the following hold.*

1. *$\sum_{i=1}^k v_i \geq k + \ell - 1$ and $e_i \leq v_i - 1$.*
2. *$v_{-1} \geq 2$ and either F_{-1} has at least two connected components, or $F'_{-1} = F_{-1}$ and $\sum_{i=1}^k v_i \geq k + \ell$.*
3. *v_G and e_G satisfy $v_G = (k+2)\ell - \sum_{i=1}^k v_i - v_{-1}$ and*

$$e_G = (k+2)\ell - 2 - \sum_{i=1}^k e_i - e_{-1} \geq (k+2)\ell - 2 + k - \sum_{i=1}^k v_i - e_{-1}$$

Proof. Let $i \geq 1$. Since every edge of F_0 lies in a copy of C_ℓ and G is minimal with respect to this property, F_i' is a proper subgraph of F_i, and thus it is a linear forest, so $e(F_i') \leq v(F_i') - 1$. Let $v_i = v(F_i')$ and for $i \geq 1$ set

$$v_i^1 = \left| V\left(F_i' \cap F_0\right) \setminus \bigcup_{0<j<i} V\left(F_j\right) \right|,$$

$$v_i^2 = \left| \left(V(F_i') \cap \bigcup_{0<j<i} V(F_j) \right) \setminus V(F_0) \right|,$$

$$v_i^3 = \left| V(F_i') \cap \left(\bigcup_{0<j<i} V(F_j) \right) \cap V(F_0) \right|,$$

so that $v_i = v_i^1 + v_i^2 + v_i^3$. Because $\bigcup_{i\geq 1} F_i' = \bigcup_{i \geq 1} F_i$ covers $V(F_0)$, we have $\sum_{i=1}^k v_i^1 = \ell$. Observation 3.6 implies that for every $i \geq 2$, $v_i^3 \geq 1$. Hence $\sum_{i=1}^k v_i^3 \geq k-1$. We can thus conclude $\sum_{i=1}^k v_i \geq \sum_{i=1}^k (v_i^1 + v_i^3) \geq k + \ell - 1$.

For the second part of the proposition, note that the two vertices in $V(F_{-1}) \cap V(F_0)$ are the ends of F_{-1}. Hence they lie in distinct connected components of F_{-1}' unless $F_{-1}' = F_{-1}$. If $F_{-1}' = F_{-1}$, we have $F_{-1} \subseteq \bigcup_{j \geq 1} F_j$. Cleary F_{-1} cannot be a subgraph of a single C_ℓ, since the ends of F_{-1} are non-adjacent. Hence there are edges $uv, vw \in E(F_{-1})$ and $1 \leq j < i \leq k$ with $uv \in E(F_i), vw \in E(F_j)$. Hence $v \in V(F_i) \cap V(F_j)$, and observe that $v \notin V(F_0)$, since it is an internal vertex of F_{-1}. Hence $v_i^2 \geq 1$, which gives $\sum_{i=1}^k v_i \geq k + \ell$.

For the third part of the proposition we used the bound $e_i \leq v_i - 1$ for each $i \in [k]$. $\square$

The last item of this section is the proof of Lemma 2.7.

Proof of Lemma 2.6. We will show that for any $G \in \mathcal{G}_{C_\ell} \setminus \{G_{C_\ell}\}$ and $p \gg n^{-1+1/\ell}$,

$$n^{v(G)} p^{e(G)} \ll n^{v(G_{C_\ell})} p^{e(G_{C_\ell})}. \tag{1}$$

Observe that for all $G \in \mathcal{G}_{C_\ell} \setminus \{G_{C_\ell}\}$ with $e(G) = e(G_{C_\ell})$, $v(G) \leq v(G_{C_\ell}) - 1$. Hence (1) is satisfies for such G and for the remainder of the proof we will consider the case $e(G) < e(G_{C_\ell})$. Then (1) can be rewritten as $p \gg n^{\frac{v(G)-v(G_{C_\ell})}{e(G_{C_\ell})-e(G)}}$. Since $p \gg n^{-1+1/\ell}$, (1) follows from

$$\frac{e(G_{C_\ell}) - e(G)}{v(G_{C_\ell}) - v(G)} \leq \frac{\ell}{\ell - 1}, \tag{2}$$

which we will show to hold in the remainder of the proof. Since $e(G_{C_\ell}) = \ell^2 - 1$ and $v(G_{C_\ell}) = \ell^2 - \ell$,(2) can be rewritten as

$$\frac{\ell^2 - (1 + e_G)}{\ell^2 - \ell - v_G} \leq \frac{\ell^2}{\ell^2 - \ell},$$

which, using Observation 3.1, is equivalent to

$$\frac{1+e_G}{\ell^2-\ell-v_G} \geq \frac{\ell}{\ell-1}.$$

Let F_0, F_{-1} be the central and attached copy of C_ℓ respectively and let $F_1, \ldots, F_k$ be the attached C_ℓ's of G in the linear order of Definition 3.5. Let $F_i' = F_i \cap \bigcup_{0\leq j<i} F_i$ and $F_{-1}' = F_{-1} \cap \bigcup_{i=1}^{k} F_i$. For $i \in [k] \cup \{-1\}$ let $v_i = v(F_i')$ and $e_i = e(F_i')$. Using the expressions for e_G, v_G from Proposition 3.7 the left hand side of the last inequality is at least

$$\frac{-1+(k+2)\ell+k-\sum_{i=1}^{k} v_i - e_{-1}}{(k+2)\ell - \sum_{i=1}^{k} v_i - v_{-1}} = 1 + \frac{k-1+v_{-1}-e_{-1}}{(k+2)\ell - \sum_{i=1}^{k} v_i - v_{-1}}.$$

If $F_{-1} = F_{-1}'$, by Proposition 3.7 we have $\sum_{i=1}^{k} v_i \geq k+\ell$ and hence the above expresion is at least

$$1+\frac{k}{k\ell-k} = \frac{\ell}{\ell-1},$$

as required. Otherwise by Proposition 3.7 F_{-1}' has at least two components, and using the bounds $\sum_{i=1}^{k} v_i \geq k+\ell-1$ and $v_{-1} \geq 2$ from Proposition 3.7, the previous expression is at least

$$1+\frac{k+1}{(k+2)\ell-k-\ell+1-2} = \frac{\ell}{\ell-1},$$

as required. □

4 The 0-statement in the lower range

In this section we prove the key lemma for the 0-statement in the lower range, Lemma 2.4, subject to Lemmas 4.3 and 4.4 which are the main results of Sections 4 and 5 respectively.

To find a good colouring of $\mathbf{G}_1$ with few dangerous $C_{2\ell-2}$'s we need to understand how copies of C_ℓ and $G \in \mathcal{G}_{C_\ell}$ interact. Borrowing a concept common in random Ramsey theory in general, and its incarnation in the work of Alon, Morris and Samotij [1] in particular, we will do this by studying the following hypergraph. The definitions in this section, as well as the techniques in the remainder of the paper, build on [1].

Definition 4.1 (Collage)**.** *For an integer $\ell \geq 4$ let $\mathcal{H}$ be the hypergraph with vertex set $E(K_n)$ where a subset $S \subseteq E(K_n)$ is an edge of $\mathcal{H}$ if S spans a copy of a graph in $\{C_\ell\} \cup \mathcal{G}_{C_\ell}$. We call a graph $C \subseteq K_n$ a C_ℓ-*collage *or a* collage *if $\mathcal{H}[E(C)]$ is connected. We denote the collection of all C_ℓ-collages by $\mathcal{C}_\ell$ or $\mathcal{C}$.*

Definition 4.2 (Good collage)**.** *We say a C_ℓ-collage $C \in \mathcal{C}_\ell$ is* good *if*

1. $v(C) \le \log n$;

2. *for every* $C' \subseteq C$ *with* $C' \in \mathcal{C}_\ell$ *we have* $e(C')/v(C') < \hat{m}(C_\ell)$.

We say that C *is* very good *if it contains no subgraph* H *with* $\ell+1 \le v_H \le \ell^3$ *and* $e_H \ge \ell+1$ *satisfying*

$$\frac{e_H - \ell - 1/(\ell-1)}{v_H - \ell} > \hat{m}(C_\ell) \tag{3}$$

where recall

$$\hat{m}(C_\ell) = \frac{\ell^2-\ell-1}{\ell(\ell-2)} = \frac{\ell-1-1/\ell}{\ell-2} = 1 + \frac{1-1/\ell}{\ell-2}$$

The proof of the key lemma for the 0-statement in the lower range, Lemma 2.4, splits into a 'probabilistic lemma', Lemma 4.3 below; and a 'deterministic lemma', Lemma 4.4 below. The former says that, with high probability, every collage that $\mathbf{G}_1$ contains is good.

Lemma 4.3. *There exists* $c > 0$ *such that the following holds. Suppose* $p \le cn^{-1/\hat{m}(C_\ell)}$ *and let* $\mathbf{G}_1 \sim \mathbf{G}(n,p)$. *Then with high probability every* $C \in \mathcal{C}_\ell$ *with* $C \subseteq \mathbf{G}_1$ *is good.*

The deterministic lemma is a little more subtle. Here we show that a very good collage, i.e. one without any 'dense' small subgraph (i.e. without any subgraph satisfying (3)) can be coloured so that there is no dangerous $C_{2\ell-2}$ at all. Our final colouring may have dangerous $C_{2\ell-2}$'s in collages which are good but not very good; we can obtain a good upper bound on the dangerous $C_{2\ell-2}$'s in such collages by using that they have order at most $\log n$ and contain subgraphs satisfying (3).

Lemma 4.4. *Every very good* C_ℓ*-collage admits a good colouring that has no dangerous* $C_{2\ell-2}$.

Proof of Lemma 2.4. The property of good but not very good collages that we need is captured in the following claim, whose proof we defer until the end.

Claim 4.5. *With high probability, there are at most* $O(n^{-\varepsilon}\,\hat{q}_{lo}^{\,-1})$ *collages which are not very good, for some constant* $\varepsilon > 0$.

With high probability, $\mathbf{G}_1$ is a graph G_1 satisfying the conclusions of Theorem 1.1, Proposition 2.10, Lemma 4.3 and Claim 4.5.

By Theorem 1.1, G_1 has a red-blue colouring ϕ_0 that avoids monochromatic C_ℓ's. Let $(C_i)_{i\in I\cup J}$ be a maximal collection of edge-disjoint collages of G with $(C_i)_{i\in J}$ being the very good collages, and let $E_0 = E(G) \setminus \bigcup_{i\in I\cup J} C_i$ be the edges not on any graph in $\{C_\ell\} \cup \mathcal{G}_{C_\ell}$. By Lemma 4.3, for every $i \in I$, C_i is a good collage. For every $j \in J$, by Lemma 4.4, every C_j has a very good colouring ϕ_j.

Let ϕ be the following colouring:

- for every $j \in J$, ϕ agrees with ϕ_j on C_j;
- ϕ colours every $e \in E_0$ blue;
- for every $i \in I$, ϕ agrees with ϕ_0 on C_i on all edges lying in a C_ℓ, and all edges not in a C_ℓ are coloured blue.

Since each copy of C_ℓ lies in a collage, ϕ avoids monochromatic C_ℓ's; and since every red edge lies in a C_ℓ, ϕ is a good colouring.

It remains to upper bound the number of dangerous $C_{2\ell-2}$'s. Because the colouring is good, by Proposition 2.3 every dangerous $C_{2\ell-2}$ lies in some $G \in \mathcal{G}_{C_\ell}$, and hence in some collage. The colouring ϕ avoid dangerous $C_{2\ell-2}$ in every $C_j, j \in J$. Therefore it suffices to show that there are $o\left(\hat{q}_{\,\mathrm{lo}}^{-1}\right)$ copies of each $G \in \mathcal{G}_{C_\ell}$ in $\bigcup_{i\in I} C_i$.

By Claim 4.5, $|I| \le n^{-\varepsilon} \cdot \hat{q}_{\,\mathrm{lo}}^{-1}$. By Lemma 4.3 for every $i \in I$, C_i has order at most $\log n$ and hence has at most $(\log n)^{O(1)}$ copies of graphs in $\mathcal{G}_{C_\ell}$. Hence the number of copies of graphs $G \in \mathcal{G}_{C_\ell}$ in $\bigcup_{i\in I} C_i$ is at most $|I| \cdot (\log n)^{O(1)} \le n^{-\varepsilon/2} \cdot \hat{q}_{\,\mathrm{lo}}^{-1} \ll \hat{q}_{\,\mathrm{lo}}^{-1}$, as required.

Proof of Claim 4.5. Since every collage which is not very good contains a copy of a graph satisfying (3), the number of collages which are not very good is at most the number of copies of such graphs in G_1. Let $\delta > 0$ be the minimum of $\frac{e_H-\ell-1/(\ell-1)}{v_H-\ell} - \hat{m}(C_\ell)$ over all H with $\ell+1 \le v_H \le \ell^3$ and $e_H \ge \ell+1$ satisfying (3). Then, using $v_H \ge \ell+1$, we have $e_H \ge \hat{m}(C_\ell)(v_H-\ell)+\delta+\ell+1/(\ell-1)$, for every such H. By Proposition 2.10 there are at most $O(n^{v_H}p^{e_H})$ copies of every such H in G_1. Using $\hat{q}_{\,\mathrm{lo}} = n^{-\ell}\,p^{-\ell-1/(\ell-1)}$, the lower bound for e_H, and that $p \le cn^{-1/\hat{m}(C_\ell)}$ we have

$$\hat{q}_{\,\mathrm{lo}} \cdot n^{v_H}p^{e_H} \le n^{v_H-\ell}p^{\hat{m}(C_\ell)\,(v_H-\ell)+\delta} \le n^{-\delta/\hat{m}(C_\ell)}c^{\hat{m}(C_\ell)(v_H-\ell)+\delta} \le n^{-\varepsilon},$$

for some $\varepsilon > 0$, using $v_H \le \ell^3$. Rearranging gives $n^{v_H}\,p^{e_H} \le n^{-\varepsilon}\,\hat{q}_{\,\mathrm{lo}}^{-1}$. Since there are at most $2^{2\ell^6}$ choices for H, we deduce there are at most $O(n^{-\varepsilon}\,\hat{q}_{\,\mathrm{lo}}^{-1})$ copies of such H, and hence at most this many collages which are not very good. $\square$

This completes the proof of Lemma 2.4. $\square$

5 The probabilistic lemma: all collages are good

For a graph G and a subgraph I of G with $v(I) < v(G)$ define

$$\hat{d}(G,I) = \frac{e(G)-e(I)}{v(G)-v(I)} \tag{4}$$

so $\hat{m}(C_\ell) = \hat{d}(G_{C_\ell}, C_\ell)$.

To prove Lemma 4.3 we will analyse an exploration algorithm on a collage. The crux of the analysis will be the next lemma. It essentiall says that at each step of the exploration, the density of the collage cannot increase too much.

Lemma 5.1. *Let $G \in \mathcal{G}_{C_\ell}$ and let I be a subgraph of G with $v(I) < v(G)$ that satisfies the following:*

- *$E(I) \neq \emptyset$;*
- *every copy of C_ℓ in G is either contained in I or shares no edge with I.*

Then

$$\hat{d}(G, I) \geq \hat{d}(G_{C_\ell}, C_\ell),$$

with equality if and only if $G = G_{C_\ell}$ and $I = C_\ell$.

Before proving Lemma 5.1, which is rather technical, we show how it is used to prove the probabilistic lemma, Lemma 4.3. The proof is partly based on the proof of the probabilistic statement in [14].

Proof of Lemma 4.3. Let $\mathcal{C}_{\text{bad}} = \{C \in \mathcal{C}_\ell : v(C) > \log n \text{ or } e(C)/v(C) \geq \hat{m}(C_\ell)\}$, so that $\mathcal{C}_{\text{bad}}$ contains every collage which is not good. Let ε, L and Γ be parameters to be determined later, whose value depends only on ℓ. We will describe an exploration algorithm that given a $C \in \mathcal{C}_{\text{bad}}$, it outputs a $S \subseteq C$ such that both conditions a) and b) below hold.

a) Either $e(S) \geq \hat{m}(C_\ell) \cdot v(S) + \varepsilon$; or $v(S) \geq \log n$ and $e(S) \geq \hat{m}(C_\ell)\,(v(S) - \ell)$.

b) For each $k \leq n$, there are at most Ln^k possible outputs S of the algorithm with $v(S) = k$.

Let $\mathcal{S}$ be the collection of all outputs of our exploration algorithm. Before describing the algorithm, we show how the existence of such a collection implies the lemma. We have

$$\mathbf{P}[C \subseteq \mathbf{G}_1 \text{ for some } C \in \mathcal{C}_{\text{bad}}] \leq \mathbf{P}[S \subseteq \mathbf{G}_1 \text{ for some } S \in \mathcal{S}] \leq \sum_{S \in \mathcal{S}} p^{e_S}.$$

Using a), b) and choosing c sufficiently small, this is at most

$$\sum_{k \leq \log n} (Ln)^k p^{\hat{m}(C_\ell)k+\varepsilon} + \sum_{k \geq \log n} (Ln)^k p^{\hat{m}(C_\ell)(k-\ell)} \leq 2p^\varepsilon + n^\ell \cdot n^{-\ell-1} \ll 1$$

We now proceed to define the exploration algorithm on input $C \in \mathcal{C}_{\text{bad}}$. Fix a labelling of $V(K_n)$ and order $E(K_n)$ according to the lexicographic order. This induces a lexicographic order of the subgraphs of K_n. Let C_0 be the copy of C_ℓ in C that is first in this order. While $C_i \neq C$, since C is a collage there exists either a copy of C_ℓ or a graph in $\mathcal{G}_{C_\ell}$ that intersects C_i on an edge. We call iteration $i+1$ of the algorithm *regular* if there is a copy G of G_{C_ℓ} such that its intersection with C_i is exactly a copy of C_ℓ, and call G a *regular copy* of G_{C_ℓ}. We call iteration $i+1$ *degenerate* otherwise. The *root* of a regular copy is the unique edge that lies both in the central copy P_ℓ and the copy of C_ℓ in C_i.

At each step of the algorithm we keep track of five first-in-first-out queues $L_V, L_E, L_R, , L'_R, L_D$ that will log information about the execution of the algorithm. We will ultimately show that we can reconstruct the output of the algorithm from these logs, and thus we can upper-bound the number of possible outputs by the number of possible values for the logs. Throughout the exploration, L_V will be a sequence of vertices in $V(C)$ and L_E a sequence of edges in $E(C)$, so that at the end of the i-th iteration L_V and L_E are the vertex and edge sets of C_i. L_R will be an increasing (but *not* necessarily *strictly* increasing!) sequence of positive integers, each one corrsponding to an entry in L_E. Entries of L'_R will be integers in $[\ell-1]$. Each entry of L_D will be a step i of the execution of the algorithm and a collection of edges. We first set L_V to be $V(C)$ and L_E to be $E(C)$. For each $i \geq 0$, we update C_i and the logs to obtain C_{i+1} as follows.

1) Suppose there is a regular copy of G_{C_ℓ}. Let G be the regular copy whose root was added first in L_E. Let $j \leq i$ be the position of the root and insert j (at the end of) L_R. Since j is the smallest possible 'birth time' for a root, this maintains the property that L_R is a increasing sequence (L_R will only record the roots of regular steps). Let x, y be the vertices that are both in the central and the attached copy of P_ℓ and suppose $x < y$ in the lexicographic order. Let $H_1, \dots, H_{\ell-1}$ be the ordering of the copies of C_ℓ on G_{C_ℓ} as we move from x to y along the central P_ℓ. Insert into L'_R the position $s \in [\ell-1]$ of $G \cap C_i$. Insert $V(G) \setminus V(C_i)$ into L_V in the following order. We go through the cycles $H_1, \dots, H_{s-1}, H_{s+1}, \dots, H_{\ell-1}$ in order. For H_i, if its vertices are $v_1, \dots, v_\ell$ with $v_t v_{t+1} \in E(H_i)$ and $v_1 v_\ell$ is the edge on the central copy of P_ℓ, then we add $V(H_i)$ in this order in L_V. Finally, we add the vertices of the attached copy of P_ℓ as we traverse it from x to y. It is not too hard to see that we can reconstruct $E(G) \setminus E(C_i)$ given the ℓ-cycle $G \cap C_i$, its position among the other cycles in G, and the ordered sequence of $V(G) \setminus V(C_i)$ in L_V. Insert $E(G) \setminus E(C_i)$ into L_E and set $C_{i+1} := C_i \cup E(G)$.

2) Suppose condition 1) fails. If there is a copy of C_ℓ that intersects C_i on at least one edge, let H be such a copy chosen arbitrarilly. Insert $V(H) \setminus V(C_i)$ into L_V, $E(H) \setminus E(C_i)$ into L_E, and $(i, E(H) \setminus E(C_i))$ into L_D. Set $C_{i+1} := C_i \cup H$.

3) Suppose conditions 1) and 2) fail. Then there exists a copy of some $G \in \mathcal{G}_{C_\ell}$ such that its intersection with C_i is a subgraph I with $(G, I) \neq (G_{C_\ell}, C_\ell)$ and moreover for every copy of C_ℓ in G, I either contains it or it is edge-disjoint from it. In particular, I satisfies the assumptions of Lemma 5.1. Insert $V(G) \setminus V(C_i)$ into L_V, $E(G) \setminus E(C_i)$ into L_E, and $(i, E(G) \setminus E(C_i))$ into L_D. Set $C_{i+1} := C_i \cup H$.

Let $\tau(C)$ be the first iteration $i \geq 1$ such that one of the following holds.

- The i-th iteration is the Γ-th degenerate iteration.
- $v_{C_i} \geq \log n$.
- $C_i = C$

The exploration process stops at iteration $\tau(C)$ and we output $C_{\tau(C)}$. Let $\mathcal{S} = \{C_{\tau(C)} : C \in \mathcal{C}\}$ be the possible outputs of the process.

Claim 5.2. *There exist $\varepsilon, \Gamma > 0$ depending only on ℓ such that condition a) holds. Moreover, there exists $A > 0$ depending only on ℓ such that $e(C_i) \leq Ai$ for all i.*

Proof. Let $S = C_{\tau(C)}$ for some $C \in \mathcal{C}_{\text{bad}}$.

We first show that when $S \in \mathcal{C}_{\text{bad}}$, there exists $\varepsilon > 0$ (depending only on ℓ) so that $e_S \geq \hat{m}(C_\ell)\, v_S + \varepsilon$. Let $a = (\ell-1)^2 - 1$ and note that $\hat{m}(C_\ell) \cdot a$ is an integer. Hence $a \cdot e_S > a \cdot \hat{m}(C_\ell)\, v_S$ implies $a \cdot e_S \geq a \cdot \hat{m}(C_\ell)\, v_S + 1$, since both sides are integers. Then setting $\varepsilon := 1/a$ yields $e_S \geq \hat{m}(C_\ell)\, v_S + \varepsilon$.

Suppose now $S \in \mathcal{S} \setminus \mathcal{C}_{\text{bad}}$ is the output of the algorithm when executed on a collage $C \in \mathcal{C}_{\text{bad}}$ and let C_i be the subgraph of C at the i-th iteration of the exploration algorithm, so $S = C_{\tau(C)}$ and $S \neq C$.

Let α_0 be the minimum value of $d(G, I)$ over all pairs $G \in \mathcal{G}_{C_\ell}$ and $I \subsetneq G$ where I contains an edge of G, $(G, I) \neq (G_{C_\ell}, C_\ell)$ and for every copy of C_ℓ in G, either I contains it or it is edge disjoint from I. By Lemma 5.1, $\alpha_0 > \hat{m}(C_\ell)$. Let $\alpha = \min\{m_2(C_\ell), \alpha_0\}$, and note that $\alpha > \hat{m}(C_\ell)$. Let $\eta := \alpha - \hat{m}(C_\ell)$. Let d_i be the number of denerate steps the algorithm makes up to iteration i. We will now show that

$$(e_{C_i} - \ell) - \hat{m}(C_\ell)(v_{C_i} - \ell) \geq \eta d_i, \text{ for all } i. \tag{5}$$

For $i = 0$ both sides of (5) are 0. Suppose that (5) holds for i. If the $(i+1)$-th step is regular, then $d_{i+1} = d_i$, $e_{C_{i+1}} = (e(G_{C_\ell}) - \ell) + e_{C_i}$ and $v_{C_{i+1}} = (v(G_{C_\ell}) - \ell) + v_{C_i}$, so both sides of (5) remain unchanged, since $e(G_{C_\ell}) - \ell = \hat{m}(C_\ell) \cdot (v(G_{C_\ell}) - \ell)$. Suppose the $(i+1)$-th step is degenerate, so we are either in case 2) or in case 3). Let G be the copy of the graph in $\{C_\ell\} \cup \mathcal{G}_{C_\ell}$ for this step and $I = G \cap C_i$ its intersection with the collage. Then the right-hand-side of (5) increases by η and left-hand-side increases by

$$(e(G) - e(I)) - \hat{m}(C_\ell)(v(G) - v(I)) \geq (\alpha - \hat{m}(C_\ell))(v(G) - v(I)) \geq \eta.$$

Here for case 2) we used that for any $I \subsetneq C_\ell$ that contains at least one edge, $\ell - e(I) \geq m_2(C_\ell)(\ell - v(I))$ by Proposition 3.4, which is at least $\alpha(\ell - v(I))$ by definition of α. We thus deduce that (5) holds for all $i \geq 1$. Hence, if the exploration continues for $i \geq \log n$ steps, S satisfies condition a).

Set $\Gamma := \lceil \ell \cdot \eta^{-1} \cdot \hat{m}(C_\ell) \rceil$, and suppose the algorithm terminates at the Γ-th degenerate step. Then (5) gives

$$e_S \geq \ell + \eta \cdot \Gamma + \hat{m}(C_\ell)(v_S - \ell) \geq \ell + \hat{m}(C_\ell)\, v_S$$

and condition a) holds.

Finally, for the second part of the claim, at each iteration we add $e(G) - e(I)$ edges to the collage, where $G \in \mathcal{G}_{C_\ell} \cup C_\ell$ is the graph we extend the collage by and I is its intersection with the collage. It is not hard to see that the maximum of $e(G) - e(I)$ over all valid choices of G and I is $e(G_{C_\ell}) - \ell$, which is a function of ℓ. □

Claim 5.3. *There exists a constant $L > 0$ depending only on ℓ such that condition b) holds.*

Proof. Let $S \in \mathcal{S}$ with k vertices and $C \in \mathcal{C}_{\text{bad}}$ such that S is the output of the algorithm when ran on C. We first claim that we can reconstruct L_E given the logs L_V, L_R, L'_R, L_D, and thus we can reconstruct S. We will show that given the logs L_V, L_R, L'_R, L_D and the entries of L_E up to iteration $i-1$, we can deduce the entries of L_E up to iteration i. We can tell if the i-th step is degenerate by inspecting whether $(i, Z) \in L_D$, for some collection of edges Z. If this is the case, then $Z = E(C_i) \setminus E(C_{i-1})$, and we are done. Suppose then that the i-th step is regular, and that it is the j-th regular step. Then the j-the entry of L_R contains the position in L_E of the root of the regular copy, and the j-th entry of L'_R contains the position of the ℓ-cycle $C_{i-1} \cap G$ among the other ℓ-cycles in G_{C_ℓ}. Examining the next $v(G_{C_\ell}) - \ell$ entries in L_V allows us then to reconstruct the edges of the regular copy of G_{C_ℓ} uniquely, as explained in the description of the algorithm. Thus we can reconstruct L_E at the end of the i-th iteration.

It remains to show there are at most $L^k n^k$ possibilites for the logs L_V, L_R, L'_R, L_D, for some constant L.

There are at most n^k possibilities for L_V. Each step adds at least one vertex to the explored graph, so there are at most k regular steps, and L_R, L'_R have length at most k, while L_D has length at most Γ.

Each entry of L'_R is in $[\ell - 1]$, so there are at most ℓ^k possibilities for L'_R. By Claim 5.2 $e(C_i) \leq Ai \leq Ak$ for all i and hence L_E has length at most Ak. Hence L_R is an increasing sequence of length at most k of integers in $[Ak]$. Hence the number of possibilities for L_R is at most $\binom{Ak+k-1}{k} \leq 2^{(A+1)k}$. Finally, for each entry of L_D there are at most $k \cdot (Ak)^{\ell^3}$ possibilities, so L_D can take at most $\left(k \cdot (Ak)^{\ell^3}\right)^{\Gamma} \leq k^B$ values, for some constant B depending only on ℓ (using that Γ is a function of ℓ by Claim 5.2).

Hence, the total number of possibilities for the logs L_V, L_R, L'_R, L_D is at most $L^k \cdot n^k$ for some constant L depending only on ℓ, as claimed. □

This completes the proof of Lemma 4.3. □

5.1 Proof of Lemma 5.1

Lemma 5.4. *Let $G \in \mathcal{G}_{C_\ell}$ and let F_{-1} be the attached C_ℓ of G. Let I be a subgraph of G with $v(I) < v(G)$ and $E(I) \neq \emptyset$ that satisfies the following.*

- *I contains no edge that lies in an attached C_ℓ;*
- *$E(I) \subseteq E(F_{-1})$.*

Then $\hat{d}(G, I) > \hat{d}(G_{C_\ell}, C_\ell)$.

Proof. Let F_0 be the central copy of P_ℓ. Let $F_1, \dots, F_k$ be the copies of C_ℓ in G according to the order in Definition 3.5. For $i \geq 1$, let $F_i' = F_i \cap \bigcup_{j=0}^{i-1} F_i$ and $F_{-1}' = F_{-1} \cap \bigcup_{i=0}^{k} F_i$. For each $i \in [k] \cup \{-1\}$ let $e_i = e(F_i')$, $v_i = v(F_i')$. Since the ends of F_{-1} are in F_0, we have $v_{-1}' \geq 2$. By Proposition 3.7, $\sum_{i=1}^{k} v_i \geq \ell + k - 1$ and $e_i \leq v_i - 1$, for all $i \in [k]$. Since I contains no edge that lies in an attached C_ℓ (but may have isolated vertices in attached C_ℓ's) we have $E(I) \subseteq E(F_{-1}) \setminus \left(\bigcup_{i=1}^{k} E(F_i)\right)$ and thus by Proposition 3.7 $F_{-1} \cap \bigcup_{i \geq 1} F_i$ has at least two components.

Let $e = e(I \cap F_{-1})$ and $v = v(I \cap F_{-1})$, and note that $1 \leq e \leq v - 1$ and $v \geq 2$. We can write $v(G) = \ell + \sum_{i=1}^{k}(\ell - v_i) + (\ell - v_{-1})$ and $e(G) = \ell - 1 + \sum_{i=1}^{k}(\ell - e_i) + (\ell - 1 - e_{-1})$.

First suppose that $F_{-1} \subseteq I \cup \bigcup_{i \geq 1} F_i$. Because $F_{-1} \cap \bigcup_{i \geq 1} F_i$ has at least two components, I has a vertex from each component. Therefore I contains at least two vertices in $\bigcup_{i \geq 1} F_i$, i.e.

$$v(G) - v(I) \leq \ell + \sum_{i=1}^{k}(\ell - v_i) - 2 = (k+1)\ell - 2 - \sum_{i=1}^{k} v_i.$$

We also have

$$e(G) - e(I) = \ell - 1 + \sum_{i=1}^{k}(\ell - e_i) \geq \ell - 1 + \sum_{i=1}^{k}(\ell - v_i + 1) = (k+1)\ell + k - 1 - \sum_{i=1}^{k} v_i.$$

Hence, using $\sum_{i=1}^{k} v_i \geq k + \ell - 1$ and Observation 3.1 we obtain

$$\frac{e(G) - e(I)}{v(G) - v(I)} \geq \frac{(k+1)\ell + k - 1 - \sum_{i=1}^{k} v_i}{(k+1)\ell - 2 - \sum_{i=1}^{k} v_i} \geq \frac{k\ell}{k\ell - k - 1} = \frac{\ell}{\ell - 1 - 1/k}$$

which is at least

$$\frac{\ell}{\ell - 1 - 1/(\ell - 1)},$$

using $k \leq \ell - 1$ for the last inequality. This is striclty larger than $\hat{m}(C_\ell) = \frac{\ell - 1/(\ell-1)}{\ell - 1 - 1/(\ell-1)}$.

Suppose now that the path F_{-1} is not a subgraph of $I \cup \bigcup_{i \geq} F_i$. Since $F_{-1} \cap (\bigcup_{i \geq 1} F_i)$ has at least two components, so does $F_{-1} \cap (I \cup \bigcup_{i \geq 1} F_i)$. Let $\hat{v} = v\left(F_{-1} \cap (I \cup \bigcup_{i \geq 1} F_i)\right)$ and

$\hat{e} = e\left(F_{-1} \cap (I \cup \bigcup_{i\geq 1} F_i)\right)$. Observe that $\hat{e} \geq 1$ and $\hat{v} \geq 3$ since $F_{-1} \cap (I \cup \bigcup_{i\geq 1} F_i)$ has at least one edge and it contains the two ends of F_{-1}. We can write

$$\begin{aligned} v(G) - v(I) &= \ell + \sum_{i=1}^{k}(\ell - v_i) + (\ell - \hat{v}) - v\left(I \cap \bigcup_{i\geq 1} F_i\right) \\ &= (k+2)\ell - \sum_{i=1}^{k} v_i - \hat{v} - v\left(I \cap \bigcup_{i\geq 1} F_i\right) \end{aligned}$$

and, using $e_i \leq v_i - 1$ for each $i \in [k]$,

$$e(G) - e(I) = \ell - 1 + \sum_{i=1}^{k}(\ell - e_i) + (\ell - 1 - \hat{e}) \geq (k+2)\ell + k - 2 - \sum_{i=1}^{k} v_i - \hat{e}$$

Using Observation 3.1 and $\sum_{i=1}^{k} v_i \geq k + \ell - 1$ we have

$$\frac{e(G) - e(I)}{v(G) - v(I)} \geq \frac{(k+1)\ell - 1 - \hat{e}}{(k+1)\ell - k + 1 - v\left(I \cap \bigcup_{i\geq 1} F_i\right) - \hat{v}} \tag{6}$$

If $F_{-1} \cap (I \cup \bigcup_{i\geq 1} F_i)$ has exactly two connected components, then one of the edges of I must contain a vertex in $F_{-1} \cap \bigcup_{i\geq 1} F_i$. Hence $v\left(I \cap \bigcup_{i\geq 1} F_i\right) \geq 1$. Substituting this and $\hat{e} = \hat{v} - 2$ in (6) yields

$$\frac{e(G) - e(I)}{v(G) - v(I)} \geq \frac{(k+1)\ell - \hat{v} + 1}{(k+1)\ell - k - \hat{v}} \geq \frac{(k+1)\ell - 2}{(k+1)\ell - k - 3} = \frac{\ell - 2/(k+1)}{\ell - 1 - 2/(k+1)} > \hat{m}(C_\ell),$$

using $\hat{v} \geq 3$ and Observation 3.1 for the penultimate inequality, and $k \leq \ell - 1$ and $\ell > 3$ for the last ienquality.

If $F_{-1} \cap (I \cup \bigcup_{i\geq 1} F_i)$ has at least 3 connected components, notice that it must also have at least four vertices, since $\hat{e} \geq 1$. Substituting $\hat{e} \leq \hat{v} - 3$ in (6) yields

$$\frac{e(G) - e(I)}{v(G) - v(I)} \geq \frac{(k+1)\ell - \hat{v} + 2}{(k+1)\ell - k + 1 - \hat{v}} \geq \frac{(k+1)\ell - 2}{(k+1)\ell - k - 3} > \hat{m}(C_\ell)$$

using $\hat{v} \geq 4$ for the penultimate inequality and the same caclulation as above for the last inequality. □

Lemma 5.5. *Let $G \in \mathcal{G}_{C_\ell}$ and let I be a subgraph of G with $v(I) < v(G)$ that satisfies the following:*

- *I contains at least one copy of C_ℓ in G.*
- *for every copy F of C_ℓ in G, either $E(F) \subseteq E(I)$ or $E(F) \cap E(I) = \emptyset$.*

Then

$$\hat{d}(G,I) \geq \hat{d}(G_{C_\ell}, C_\ell),$$

with equality if and only if $G = G_{C_\ell}$ *and* $I = C_\ell$.

Proof. Let F_0, F_{-1} be the central and attached copies of P_ℓ in G and let x, y be their ends. First suppose that I contains all copies of C_ℓ in G. Let $F'_{-1} = F_{-1} \cap I$, and notice that $x, y \in I$, since they both lie in a copy of C_ℓ; and $v(F'_{-1}) < v(F_{-1})$, since $v(I) < v(G)$. Thus if we let $V_0 := V(F'_{-1})$ and $E_0 := E(F'_{-1}) \cup \{xy\}$, this defines a subgraph $H_0 = (V_0, E_0)$ of C_ℓ with $2 \leq v(H_0) < \ell$. Then

$$\frac{e(G)-e(I)}{v(G)-v(I)} = \frac{\ell - 1 - e(F'_{-1})}{v(F_{-1}) - v(F'_{-1})} = \frac{\ell - e(H_0)}{\ell - v(H_0)} > m_2(C_\ell) > \hat{m}(C_\ell),$$

using Proposition 3.4 for the penultimate inequality.

Suppose now that I contains at least one but not all copies of C_ℓ in G, and let H be a copy of C_ℓ in I. Order the path F_0 from left to right such that x is the leftmost vertex. We will consider a slightly different order on the attached C_ℓ's than the one in Definition 3.5. Let e be the leftmost edge of $I \cap F_0$, and suppose there are $\ell_0 \geq 0$ edges on the left of e in F_0. Let $e_1, \ldots, e_{\ell_0}$ be an enumeration of the edges on F_0 in the order that we encounter them as we move left along F_0 from e, with $e_1 \cap e \neq 0$ and $e_{\ell_0} \ni x$. Let $e_{\ell_0+1}, \ldots, e_{\ell-2}$ be an enumeration of the edges of F_0 in the order that we encounter them as we move to the right along F_0 from e, with $e \cap e_{\ell_0+1} \neq \emptyset$ and $e_{\ell-2} \ni y$. Let $\phi : E(F_0) \to [\ell-1]$ be the map with $\phi(e) = \ell - 1$ and $\phi(e_i) = i$. Consider the linear order $<_\phi$ on the attached C_ℓ's as follows: for two attached C_ℓ's $F \neq F'$, we have $F <_\phi F'$ if $\phi\left(E(F) \cap E(F_0)\right)$ is less than $\phi\left(E(F') \cap E(F_0)\right)$ in the lexicographic ordering. Suppose there are $k \leq \ell - 2$ copies of C_ℓ in G that are not contained in I, and let $H_1 <_\phi \cdots <_\phi H_k$ be an enumeration according to this order. We define an order $\prec_\phi$ on $V(F_0) \cap \bigcup_{j \geq 1} V(H_j)$ as follows. Let $u \neq v \in V(F_0) \cap \bigcup_{j\geq 1} V(H_j)$ and let E_u, E_v be the edges in $F_0 \cap \bigcup_{j \geq 1} V(H_j)$ containing u and v respectively. Then $u \prec_\phi v$ if $\phi(E_u)$ is less than $\phi(E_v)$ in the lexicographic ordering. Let $H'_i = H_i \cap \left(I \cup \bigcup_{1 \leq j < i} H_j\right)$ and $H'_{-1} = F_{-1} \cap \left(I \cup \bigcup_j H_j\right)$. Observe that for every $i \in [k]$, $v\left(H'_i\right) \geq 1$, since the first vertex of $V(H_i) \cap V(F_0)$ in the order $\prec_\phi$ lies in $V(I) \cup \bigcup_{1 \leq j < i} V(H_j)$. Also, H'_i is a proper subgraph of H_i: indeed, otherwise, since H_i shares no edges with I, $E(H_i) \subseteq \bigcup_{1 \leq j < i} H_i$, which contradicts the minimality of G with respect to every edge of F_0 having an attached C_ℓ. Moreover, H'_{-1} contains $\{x, y\}$ and hence consists of at least two components unless $F_{-1} \subseteq I \cup \bigcup_j H_j$.

Then we may rewrite $\hat{d}(G, I)$ as follows.

$$\hat{d}(G,I) = \frac{\sum_{i=1}^{k} \left(\ell - e(H'_i)\right) + \ell - 1 - e(H'_{-1})}{\sum_{i=1}^{k} \left(\ell - v(H'_i)\right) + \ell - v(H'_{-1})}.$$

Let $J = \{i \in [k] : v(H'_i) > 1\}$ and $s = k - |J|$. Then we can rewrite the right hand side of the last inequality as

$$\frac{\sum_{i \in J} (\ell - e(H'_i)) + s\ell + \ell - 1 - e(H'_{-1})}{\sum_{i \in J} (\ell - v(H'_i)) + s(\ell - 1) + \ell - v(H'_{-1})}.$$

By Proposition 3.4, for every $i \in [J]$

$$\frac{\ell - e(H'_i)}{\ell - v(H'_i)} > m_2(C_\ell) > \hat{m}(C_\ell). \tag{7}$$

We now want to lower bound the ratio of the remaining terms, and use Observation 3.1 to deduce $\hat{d}(G, I) \geq \hat{m}(C_\ell)$. First suppose that F_{-1} is not equal to H'_{-1}. Then H'_{-1} has at least two connected components i.e. $e(H'_{-1}) \leq v(H'_{-1}) - 2$. This gives

$$\frac{s\ell + \ell - 1 - e(H'_{-1})}{s(\ell - 1) + \ell - v(H'_{-1})} \geq \frac{(s+1)\ell + 1 - v(H'_{-1})}{(s+1)(\ell - 1) + 1 - v(H'_{-1})}$$

which is equal to

$$\frac{\ell - (v(H'_{-1}) - 1)/(s+1)}{\ell - 1 - (v(H'_{-1}) - 1)/(s+1)} \geq \hat{m}(C_\ell).$$

The last inequality follows from Observation 3.1 and $\frac{v(H'_{-1})-1}{s+1} \geq \frac{1}{\ell-1}$, which is a consequence of $s \leq k \leq \ell - 2$ and $v(H'_{-1}) \geq 2$. This holds with equality if and only if $V(H'_{-1}) = \{x, y\}$ and $s = \ell - 2$. This implies $J = \emptyset$ and thus, under the assumption F_{-1} is not fully contained in I, we have that $\hat{d}(G, I) = \hat{m}(C_\ell)$ if and only if $G = G_{C_\ell}$ and $I = C_\ell$.

Claim 5.6. *If $F_{-1} = H'_{-1}$, then $J \neq \emptyset$.*

Before proving Claim 5.6, we show how it implies the lemma. Suppose $F_{-1} = H'_{-1}$ and let $j^* \in J$. Note that in this case $s = k - |J| < \ell - 2$. Then, using $e(H_{j^*}) \leq v(H_{j^*}) - 1$, we have that $\frac{\ell - e(H_{j^*}) + s\ell}{\ell - v(H'_{j^*}) + s(\ell - 1)}$ is at least

$$\frac{\ell - v(H'_{j^*}) + 1 + s\ell}{\ell - v(H'_{j^*}) + s(\ell - 1)} = \frac{(s+1)\ell + 1 - v(H'_{j^*})}{(s+1)(\ell - 1) + 1 - v(H'_{j^*})} = \frac{\ell - (v(H'_{j^*}) - 1)/(s+1)}{\ell - 1 - (v(H'_{j^*}) - 1)/(s+1)}.$$

By the same calculation as for the preceding inequality we see that this is strictly larger than $\hat{m}(C_\ell)$. Finally, if $|J| \geq 2$, using (7) and the last item of Observation 3.1, we have

$$\frac{\sum_{i \in J \setminus \{j^*\}} (\ell - e(H'_i))}{\sum_{i \in J \setminus \{j^*\}} (\ell - v(H'_i))} > \hat{m}(C_\ell)$$

and hence $\hat{d}(G, I) > \hat{m}(C_\ell)$ follows using one more time the last item of Observation 3.1. We now prove Claim 5.6, which completes the proof of the lemma.

Proof of Claim 5.6. Suppose first that $F_{-1} \subseteq \bigcup_{j=1}^{k} H_j$. Clearly a single attached C_ℓ cannot contain F_{-1} as a subgraph, so for some pair of incident edges uv, vw of F_{-1} there are $i \prec_\phi j$ with $uv \in E(H_i)$ and $vw \in E(H_j)$. In particular, v is an internal vertex of F_{-1} i.e. $v \notin \{x, y\}$ and so $v \notin V(F_0)$. It is not hard to see that H_j shares the first vertex (in $\prec_\phi$) in $E(H_j) \cap E(F_0)$ with $I \cup \bigcup_{s<j} H_s$. Therefore $v(H'_j) \geq 2$, as required.

Next suppose that $F_{-1} \subseteq I$. Let u, v be the first and last vertices of $V(H_k) \cap V(F_0)$ in $\prec_\phi$, and note that $u \neq v$, since H_k contains at least one edge of F_0. We will show that $u, v \in V(H'_k)$. It is not hard to see that $u \in V(I) \cup \bigcup_{1 \leq j<k} V(H_j)$. Note that v is the last vertex that has an atttached C_ℓ not in I. So either v lies in another attached C_ℓ which is in I; or it is one of $\{x, y\}$. In either case, $v \in V(I)$.

Finally suppose that neither of the above holds. Then, similarly to the first case, there is a pair of incident edges uv, vw of F_{-1} with $uv \in I$, $vw \in \bigcup_{j \geq 1} H_j$. Then $v \notin V(F_0)$, and for some $i \in [k]$, $v \in V(H_i)$, so $v \in V(H'_i)$. Since H'_i contains also a vertex in F_0, we have $v(H'_i) \geq 2$. □

□

Proof of Lemma 5.1. This is a direct consequence of Lemmas 5.4 and 5.5. □

6 The deterministic lemma: colouring very good collages

Even when not mentioned explicitly, every collage considered in this section will be a very good collage. In this section we will prove Lemma 4.4, which says that every very good collage C admits a good colouring (i.e. one with no monochromatic C_ℓ and with every red edge on a C_ℓ) with no dangerous $C_{2\ell-2}$. Recall that a dangerous $C_{2\ell-2}$ is a coloured $C_{2\ell-2}$ consisting of a red and a blue $P_{\ell-1}$. We call an uncoloured copy of $C_{2\ell-2}$ *potentially dangerous* if it is the central $C_{2\ell-2}$ of some $G \in \mathcal{G}_{C_\ell}$. By Proposition 2.3, in a graph H coloured with a good colouring every dangerous $C_{2\ell-2}$ is a potentially dangerous $C_{2\ell-2}$ in H. *We will use this fact throughout this section without mentioning Proposition 2.3.* From now on, we say a colouring of a very good collage is *very good* if i) it has no monochromatic C_ℓ, ii) every red edge lies in a C_ℓ and iii) it has no dangerous $C_{2\ell-2}$. In other words, we will show that every very good collage admits a very good colouring. We will do so by removing a carefully selected subset of edges from the collage so that we can extend a very good colouring of the rest of the collage to these edges. We will use a 'discharging' method to find these edges. The discharging method will distribute a weight assigned initially to vertices and edges of the collage to a collection of edge disjoint subgraphs of the collage, which we call blocks. A *block* in the collage is a subgraph X that satisfies one of the following

i) $X \cong 2C_\ell$ and X shares no edge with a copy of C_ℓ which is not a subgraph of X;

ii) $X \cong C_\ell$ and X shares no edge with another copy of C_ℓ.

As the next lemma implies, the collection of all blocks in a very good collage consists of all the copies of C_ℓ in the collage. *We will use this throughout this section without mentioning Lemma 6.1.*

Lemma 6.1. *Let $X, Y \cong C_\ell$ be subgraphs of a very good collage and suppose that $E(X) \cap E(Y) \neq \emptyset$. Then $X \cap Y$ is exactly one edge, i.e. $X \cup Y \cong 2C_\ell$.*

We prove this lemma, along with others that exclude other graphs as subgraphs of very good collages, in Section 6.2. The carefully selected edges that we will remove from a very good collage, to extend a very good colouring of the rest of the collage, is given in the next lemma.

Lemma 6.2. *Let C be a very good C_ℓ-collage. Then C contains a block X that satisfies one of the following.*

a) $X \cong 2C_\ell$ and denoting the C_ℓ's of X by X_1, X_2, for some $i \in \{1,2\}$ X_i has two edges other than $X_1 \cap X_2$ such that neither of them is on a copy of $C_{2\ell-2}$ not contained in X.

b) $X \cong C_\ell$ and there is an edge $xy \in E(X)$ such that neither x nor y lies in any other block.

c) $\ell = 4$, $X \cong C_4$, and every potentially dangerous C_6 containing an edge of X shares at most one other edge with a block other than X.

We are now ready to prove the main result of this section, Lemma 4.4, before proving Lemma 6.2 in the remainder of this section.

Proof of Lemma 4.4. Suppose for the sake of contradiction that there exists a very good collage C_0 which does not admit a very good colouring. Let $C \subseteq C_0$ be a collage that does not admit a very good colouring but for any $e \in E(C)$, $C \setminus \{e\}$ does; such C exists since, for example, a sinlge C_ℓ admits a very good colouring. We wil show that C admits a very good colouring, thus deriving a contradiction. Observe that being a very good collage is defined by excluding subgraphs and hence C and every collage that is a subgraph of C is very good. By Lemma 6.2, C contains a block X that satisfies one of a), b), c).

Suppose that C contains a block $X \cong 2C_\ell$ satisfying a) and let $xy, zw \in E(X)$ be the edges as in a), where we may have $\{x,y\} \cap \{z,w\} \neq \emptyset$. Let $C_1, \ldots, C_k$ be the (edge-disjoint) collection of maximal collages of $C \setminus \{xy, zw\}$ and let $E_0 = C \setminus \left(\{xy, zw\} \cup \bigcup_{i=1}^k E(C_i)\right)$ be the remaining edges of $C \setminus \{xy, zw\}$ that lie neither on a C_ℓ nor on some $G \in \mathcal{G}_{C_\ell}$ fully contained in $C \setminus \{xy, zw\}$. By the minimality of C, every C_i has a very good colouring ϕ_i. Let ϕ be the colouring on $E_0 \cup \bigcup_{i=1}^k E(C_i)$ that agrees with ϕ_i on $E(C_i)$, for every i, and colours every edge of E_0 blue. We will extend ϕ to a colouring $\tilde{\phi}$ that assigns a colour also to xy and zw. Observe that $\tilde{\phi}$ can only fail to be very good due to either a C_ℓ containing one of xy and zw being monochromatic; or a

$C_{2\ell-2}$ containing one of xy, zw being dangerous; any other C_ℓ or $C_{2\ell-2}$ is guaranteed to satisfy the properties of a very good colouring because ϕ_i is a very good colouring of the collage C_i, for every $i \in [k]$, and the edges E_0 can only lie in a C_ℓ or $C_{2\ell-2}$ that also contains one of xy, zw. We first try extending ϕ by colouring xy blue and zw red. The single copy of C_ℓ containing both of them is not monochromatic, and the only dangerous $C_{2\ell-2}$ that this colour assignment creates is a subgraph of X (by assumption a) on xy and zw). If this colouring does not create a dangerous $C_{2\ell-2}$, we extend ϕ to a colouring $\tilde{\phi}$ assigning blue xy and red to zw; then $\tilde{\phi}$ is a very good colouring of C, and this completes the proof if C contains a block X satisfying a).

Suppose now that colouring xy blue and zw red results in a dangerous $C_{2\ell-2}$. Observe that in every dangerous $C_{2\ell-2}$, every blue edge is incident to at least one blue edge and every red edge is incident to at least one red edge. Then at least one end of xy, say x, is incident to a blue edge in ϕ and at least one end of zw, say z, is incident to a red edge in ϕ. Suppose $\{x, y\} \cap \{z, w\} \neq \emptyset$. Then in the colouring ϕ extended by colouring xy blue and zw red, the pair $\{xy, zw\}$ is one of the two possible pairs of incident edges of different colours in a dangerous $C_{2\ell-2}$.INSERT PICTURE HERE FOR THIS CASE Then extending ϕ by colouring instead xy red and zw blue ensures that the $C_{2\ell-2}$ in X is not dangerous, and hence ϕ is extended to a very good colouring. Suppose then $\{x, y\} \cap \{z, w\} = \emptyset$. In this case, we again extend ϕ to a colouring ϕ by colouring xy red and zw blue, and claim that this results in a colouring with no dangerous $C_{2\ell-2}$ in X. Suppose for the sake of contradiction that this is not the case. Then in $\tilde{\phi}$, y is incident to a red edge and w is incident to a blue edge (and both of these edges are different from xy, zw, since $\{x, y\} \cap \{z, w\} \neq \emptyset$), so $\{xy, zw\}$ is one of the two possible pairs of different-coloured, non-adjacent edges in a dangerous $C_{2\ell-2}$ such that each edge is incident to both colours. Then there must be $\ell - 2$ red edges between y and z and $\ell - 2$ blue edges between x and w, along the $C_{2\ell-2}$ in X. But xy, zw are in the same C_ℓ and neither of them is the intersection of the two C_ℓ's, so the shortest path between them in the $C_{2\ell-2}$ has at most $\ell-3$ edges between them, yielding a contradiction. We conclude that $\tilde{\phi}$ has no dangerous $C_{2\ell-2}$ and thus is a very good colouring.

Suppose there is a block $X \cong C_\ell$ satisfying b), and let $xy \in E(X)$ such that neither x nor y lie in any other block. Let $C_1, \dots, C_k$ be the edge-disjoint collection of maximal collages of $C \setminus \{xy\}$ and let $E_0 = C \setminus \left(\{xy\} \cup \bigcup_{i=1}^k E(C_i)\right)$ be the remaining edges of C that do not lie on any C_ℓ or copy of a $G \in \mathcal{G}_{C_\ell}$. By the minimality of C, every C_i has a very good colouring ϕ_i. Let ϕ be the colouring on $E_0 \cup \bigcup_{i=1}^k E(C_i)$ that agrees with ϕ_i on $E(C_i)$, for every i, and colours every edge of E_0 blue. We extend ϕ to xy by colouring xy red, and claim the resulting colouring is very good. First, this does not create any monochromatic C_ℓ: the only copy of C_ℓ that may be monochromatic is one containing xy, and X is the unique such C_ℓ, since blocks are edge disjoint. Moreover $E(X) \setminus \{xy\} \subseteq E_0$ (again because blocks are edge disjoint), so $E(X) \setminus \{xy\}$ is coloured blue. Second, there is no dangerous $C_{2\ell-2}$. Every potentially dangerous $C_{2\ell-2}$ in the collages $C_1, \ldots, C_k$ is contained in a single collage which has one of the very good colourings $\phi_1, \ldots, \phi_k$. Hence any dangerous $C_{2\ell-2}$ when extending ϕ to xy must contain xy. Observe that in a dangerous $C_{2\ell-2}$, every red edge has at least one incident red edge. However, all edges incident to x, y other than xy do not lie in a block, so they are in E_0 and are coloured blue. We conclude that C has a very good colouring.

Suppose $\ell = 4$ and there is a block $X \cong C_4$ satisfying c). Let $xy \in E(X)$ and define C_i and ϕ_i, $i \in [k]$, and E_0 and ϕ as above. We claim that we can extend ϕ to a very good colouring of C by colouring xy red. First notice that $E(X) \setminus \{xy\} \setminus E_0$, since the only block that any edge in $E(X)$ lies in is X. Hence ϕ assigns blue to every edge in $E(X) \setminus \{xy\}$, and colouring xy red ensures that X is not monochromatic. It remains to show that colouring xy red does not create any dangerous C_6. Because this colouring is good, any dangerous C_6 will be a subgraph of some potentially dangerous C_6. By condition c), every potentially C_6 that contains xy shares at most one other of its edges with another block. Hence any potentially dangerous C_6 containing xy can have at most two red edges (since we maintain the property that red edges are in blocks), and therefore cannot be dangerous. Therefore there is no dangerous C_6 containing xy. By the definition of ϕ, there is no dangerous C_6 in $E(C) \setminus \{xy\}$, so we conclude that the colouring is very good. $\square$

6.1 Discharging

As mentioned earlier, we will use a 'discharging' method to find the block in Lemma 6.2. This approach to proving the deterministic lemma in a 0-statement in random Ramsey theory was pioneered in the recent works [1, 12, 14]. Our proof builds on the corresponding lemma in the work of Alon, Morris and Samotij [1].

We will give two distinct discharging procedures, depending on whether the collage is a C_4-collage or a C_ℓ-collage with $\ell \geq 5$. Recall that $\hat{m}(C_\ell) = \frac{\ell^2-\ell-1}{\ell(\ell-2)}$, so $\hat{m}(C_4) = 11/8$, and that every good C_ℓ-collage C has $e_C/v_C < \hat{m}(C_\ell)$.

Discharging for very good C_4-collages. Assign weight -8 to each edge and 11 to every vertex of the collage.

i) Every edge which lies in a block sends its weight to its own block.

ii) Every vertex in a block splits its weight equally among all the blocks it lies in.

   After the end of stage ii) the only vertices and edges with non-zero weight are those not in any block.

iii) Every edge with both ends in different blocks splits its weight equally among all the blocks that its ends lie in.

At this stage edges with at most one end in a block and vertices not lying in any block are the only vertices and edges with non-zero weight. Since such edges and vertices lie on an attached P_3 of some $G \in \mathcal{G}_{C_6}$, every such vertex and every end of such an edge has a neighbour in a block. We will first move all weight to edges with exactly one end in a block and then to blocks.

iv) For every edge which has no end in a block, both ends have a neighbour in a block. Every such edge splits its weight equally between its ends.

v) Every vertex splits its weight equally among all its incident edges whose other end is on a block.

At this stage, every vertex has weight zero, and only edges with exactly one end in a block have non-zero weight.

vi) Split the weight of every edge with exactly one end in a block equally among all the blocks that this end lies in.

Since C is a very good C_4-collage, $e_C/v_C < \hat{m}(C_4) = 11/8$. Hence, the total weight of the collage at the beginning of the discharging procedure is positive. Since all weight is distributed to the blocks, at the end of the discharging procedure there is a block with positive weight. The next lemma allows us to argue locally about whether a block has positive weight. Given a subgraph C' of a collage C that contains all blocks of C and a block X we write $w_{C'}(X)$ for the weight of X at the end of the discharging procedure when it is executed on input C'. We write $w(X)$ for $w_C(X)$.

Lemma 6.3. *Let C be a good C_4-collage and let C' be a subgraph of C that contains every block of C. Then for every block X, $w_{C'}(X) \geq w_C(X)$.*

Proof. Because C', C have the same blocks, they only differ at edges and vertices which do not lie in any block. Hence, at the end of stage ii) blocks have the same weight in both C, C'. Any edge considered in stage iii) that is in C and not in C' will decrease the weight of its incident blocks (since edges have negative weight at the beginning of the discharging procedure).

Observe that the weight of blocks changes again in stage vi). Moreover, at the end of stage v), every vertex has weight zero and only edges with exactly one end in a block have non-zero weight. Then these transmit their weight to a block. Therefore, to complete the proof of the lemma, it suffices to show that for every edge e with exactly one end in a block, at the end of stage v)

a) $w_{C'}(e) \geq w_C(e)$, if e lies in both C, C';

b) $w_C(e) \leq 0$, if e lies in C and not in C'.

The lemma is then an immediate consequence of a) and b). From now one, we write $w_C(e), w_{C'}(e)$ to denote the weight of e at the end of stage v) when the discharging procedure is executed on C and C' respectively.

Let $e = xy$ be an edge in C' and suppose x is in a block and y is not. Let $b_{C'}(y)$ be the number of neighbours of y in C' which lie in a block and $n_{C'}(y)$ be the number of neighbours not lying in any block. Define $b_C(y), n_C(y)$ in the same manner for C. Clearly, $n_C(y) + b_C(y) \geq 2$, since any collage has minimum degree at least 2, and $b_C(y) \geq 1$, since x is in a block. We have

$$w_C(xy) = -8 + 11/b_C(y) - 4 \cdot n_C(y),$$

where the term $-4 \cdot n_C(y)$ comes from stage iv) (and that this weight is passed on to xy in the next stage). If $n_C(y) \geq 1$, then $w_C(xy) \leq -8 + 11 - 4 \cdot 1 = -1$; otherwise, $b_C(y) \geq 2$, and then we have $w_C(y) \leq -8 + 11/2 = -5/2$, thus proving b). For a), we have

$$w_{C'}(xy) = -8 + 11/b_{C'}(y) - 4 \cdot n_{C'}(y) \geq -8 + 11/b_C(y) - 4 \cdot n_C(y) = w_C(xy),$$

using $n_{C'}(y) \leq n_C(y)$ and $b_{C'}(y) \leq b_C(y)$. □

Discharging for very good C_ℓ-collages, $\ell \geq 5$. Assign weight $\ell^2 - \ell - 1$ to each vertex in a block and $-\ell(\ell - 2)$ to each edge in a block. *We do not assign any weight to vertices and edges not in any block.*

1) For each edge that is on a block, move its weight to this block.
2) For each vertex that is on at least one block, split its weight equally among all the blocks it lies in.

As the next lemma shows, the subgraph of the collage consisting of the union of all C_ℓ's has density less than $\hat{m}(C_\ell)$, and hence the total weight of the collage at the beginning is positive. Since all the weight is reassigned to the blocks at the end of the discharging procedure there is a block X with $w(X) > 0$.

Lemma 6.4. *Let C be a very good C_ℓ-collage with $\ell \geq 5$ and let $C' \subseteq C$ be the union of all blocks. Then $e(C')/v(C') < \hat{m}(C_\ell)$.*

Proof. By the definition of a collage, every edge which is not contained in a C_ℓ i.e. any edge not contained in a block, is contained in an attached P_ℓ of some $G \in \mathcal{G}_{C_\ell}$. Let $F_1, \dots, F_k$ be an enumeration of all the attached P_ℓ's in C. Let $H_i := F_i \cap (C' \cup \bigcup_{j<i} F_j)$, and note that $v(H_i) \geq 2$, since the ends of F_i lie in some central P_ℓ and hence in some C_ℓ. Let $I \subseteq [k]$ be the indices i with $V(H_i) \subsetneq V(F_i)$ i.e. those for which F_i is not a subgraph of $C' \cup \bigcup_{j<i} F_j$.

We have $e_C/v_C < \hat{m}(C_\ell)$. We can write $e(C) = e(C') + \sum_{i\in I}(\ell - 1 - e(H_i))$ and $v(C) = v(C') + \sum_{i\in I}(\ell - v(H_i))$. Then by Observation 3.1 it suffices to show that for every $i \in I$,

$$\frac{\ell - 1 - e(H_i)}{\ell - v(H_i)} > \hat{m}(C_\ell).$$

Let $\hat{F}_i$ be a copy of C_ℓ on $V(F_i)$ with edges $E(F_i)$ and the edge between the ends of F_i. Let $\hat{H}_i$ be the subgraph of $\hat{F}_i$ on $V(H_i)$ with edges $E(H_i)$ and the edge between the ends of F_i. Then the left hand side of the above inequality equals $\frac{\ell - e(\hat{H}_i)}{\ell - v(\hat{H}_i)}$, and by Proposition 3.4 it is strictly greater than $m_2(C_\ell)$. Since $m_2(C_\ell) > \hat{m}(C_\ell)$ this completes the proof. □

The next three lemmas give the properties of a positive weight block we need in order to extend a very good colouring of a collage; Lemma 6.2 is a direct consequence of these three lemmas, and the fact that every very good collage has a block whose weight is positive at the end of the discharging procedure. Recall from above that given a subgraph C' of a collage C that contains all blocks of C and a block X we write $w_{C'}(X)$ for the weight of X at the end of the discharging procedure when it is executed on input C'. We write $w(X)$ for $w_C(X)$.

Lemma 6.5. *Let C be a very good C_ℓ-collage, where $\ell \geq 4$, and let $X \cong C_\ell$ be a subgraph of C. If X shares at least three vertices with another block, then $w(X) \leq 0$.*

Lemma 6.6. *Let C be a very good C_4-collage and let $X \cong C_4$ be a subgraph of C. Suppose that for every edge of X at least one end is in another C_4 and $w(X) > 0$. Then any potentially dangerous C_6 that contains at least one edge of X, shares at most one other edge with a block other than X.*

Lemma 6.7. *Let C be a very good C_ℓ-collage, where $\ell \geq 4$, and let $X \cong 2C_\ell$ be a subgraph of C. Let X_1, X_2 be the two C_ℓ's of X. If $w(X) > 0$ at the end of either discharging procedure for C_ℓ-collages, then for some $i \in \{1, 2\}$, X_i has two edges other than $X_1 \cap X_2$ such that neither of them is in a potentially dangerous $C_{2\ell-2}$ not contained in X.*

Proof of Lemma 6.2. For $\ell \geq 5$ it follows from Lemma 6.4 that the weight of the collage at the beginning of the discharging procedure is positive; for $\ell = 4$ it follows immediately from the definition of a very good collage. Hence at the end of either discharging procedure, since the weight assigned initially to edges and vertices is distributed to blocks, there exists a block X with positive weight. If $X \cong 2C_\ell$ then Lemma 6.7 gives the required conlcusion. If instead $X \cong C_\ell$, by Lemma 6.5 X shares at most two of its vertices with another block. Since the vertex cover of C_ℓ for $\ell \geq 5$ is at least 3, it follows that if $\ell \geq 5$ then for one edge of X neither end is shared with another block. Finally, if $X \cong C_4$ and every edge of X shares one of its ends with another block, the lemma follows from Lemma 6.6. □

Proof of Lemma 6.5. Recall that at the beginning of either discharging procedure each vertex is assigned weight $\ell^2 - \ell - 1$ and each edge $-(\ell^2 - 2\ell)$. Let C' be the subgraph of C that is the union of all blocks of C. If $s \geq 3$ vertices of X are shared with another block they contribute at most half of their weight to X, yielding

$$\begin{aligned} w_{C'}(X) &\leq ((\ell - s) + s/2)\,(\ell^2 - \ell - 1) - \ell(\ell^2 - 2\ell) \\ &\leq (\ell - 3/2)\,(\ell^2 - \ell - 1) - \ell(\ell^2 - 2\ell) \\ &= \left(\ell^3 - 5\ell^2/2 + \ell/2 + 3/2\right) - \left(\ell^3 - 2\ell^2\right) \\ &= -\ell^2/2 + \ell/2 + 3/2 \\ &\leq -16/2 + 4/2 + 3/2 = -9/2 < 0, \end{aligned}$$

where we used for the last inequality that $-\ell^2/2 + \ell/2 + 3/2$ is decreasing for $\ell \geq 4$. For $\ell \geq 5$, this immediately implies that $w(X) < 0$ at the end of the discharging procedure, and for $\ell = 4$ it follows from Lemma 6.3. □

For the proof of Lemma 6.6 we will need the following two corollaries of lemmas which are stated and proven in the next subsection. The proofs of Corollaries 6.8 and 6.9 are given after the proof of Lemma 6.12.

Corollary 6.8. *Let $G \in \mathcal{G}_{C_4}$ be a subgraph of a very good C_4-collage. Then G is a copy of $2C_4, G_{C_4}$ or the graph G_0 in Figure 2.*

Corollary 6.9. *Let $H \cong C_4$ and $G \in \mathcal{G}_{C_4}$ be subgraphs of a very good C_4-collage. Suppose that H shares an edge with the attached P_3 of G. Then $G \cong G_{C_4}$, and $G \cap H$ is exactly one edge.*

Proof of Lemma 6.6. We will use several times without mentioning Lemma 6.3, that the weight of a block in the collage is at most the weight by examining only a subgraph of the collage that contains all blocks. Our aim is to see how subgraphs containing X pass weight on to X and show that, if X does not satisfy the conclusion of the lemma, then $w(X) < 0$.

By Lemma 6.5, at most 2 vertices of X are shared with other blocks; and if only one vertex is shared, clearly one edge of X shares neither of its ends with another block, which contradicts the assumption on X. Hence X has exactly two vertices that it shares with another block, and these are non-adjacent: otherwise again for one edge of X neither end is shared with another block. If one of these vertices is shared by two other blocks, then

$$w(X) \leq 4 \cdot (-8) + 2 \cdot 11 + 11/2 + 11/3 < 0,$$

so we can conclude that each of the two vertices is shared with exactly one other block. We thus have $w(X) \leq 4 \cdot (-8) + 2 \cdot 11 + 2 \cdot 11/2 = 1$, so if some edge (that we have not considered yet) with only one end in X sends weight -1 or less to X we can deduce $w(X) \leq 0$.

Consider a potentially dangerous C_6 that shares an edge with X and let G be a copy of a graph in $\mathcal{G}_{C_4}$ containing this C_6. By Corollary 6.8, either $G \cong G_{C_4}$ or $G \cong G_0$, where G_0 is the graph in Figure 2.

First suppose that X is not one of the attached C_4's of G. Then it shares an edge with the attached P_3 of G and by Corollary 6.9 $G \cong G_{C_4}$, and $G \cap X$ is a single edge. That is, $G \cup X$ is one of the graphs in Figure 1. We may assume that $G \cup X$ is isomorphic to the left graph in Figure 1 since otherwise we can view X as an attached C_4 of G_{C_ℓ}, a case that we will consider next. Let y be the unique vertex in the edge of $X \cap G$ which is shared with another block, and let x be the other vertex on this edge which does not lie in another block (since no two neighbouring vertices of X lie in another block). Then the other edge in the attached P_3 incident to x, xu, is not in a block (since x is not in a block) and hence sends weight at most -4 to X, yielding $w(X) < 0$. Therefore we may assume for the remainder of the proof that there is no $G \in \mathcal{G}_{C_4}$ such that X shares an edge with G and X is not an attached C_4 of G. Hence X is an attached C_4 of either a G_{C_4} or a G_0.

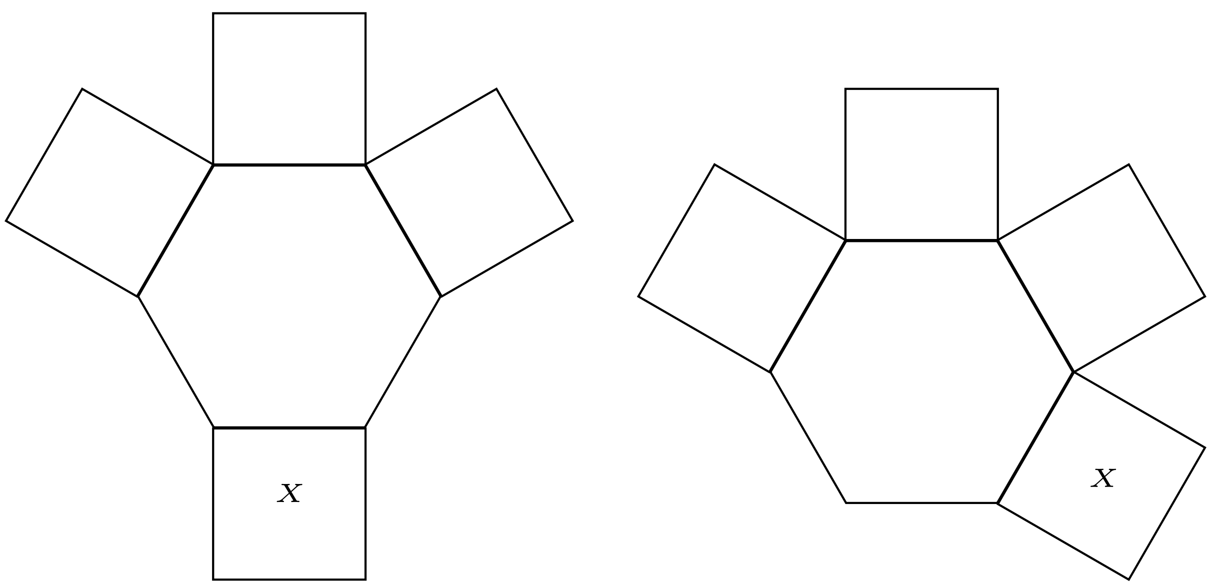

Figure 1: X is not an attached C_4.

X is not the 'middle' attached C_4 of a G_{C_4} since otherwise it shares two adjacent vertices with other blocks. Hence $G \cup X$ is one of the graphs in Figure 2. ADD FIGURE OF X BEING THE C4 WITH SINGLE EDGE ON CENTRAL C6 OF G0 Following all three graphs in Figure 2, let ab, bc, cd be the edges of the C_6 without an attached C_4, with $a \in V(X)$. We will consider different cases based on whether b, c and a are on other blocks, treating all three cases simultaneously at the beginning until we exclude the second and third graph of Figure 2.

Observe though that a can only be in another block in the first graph of Figure 2, since in the other two graphs X would then have two neighbouring vertices shared with other blocks.

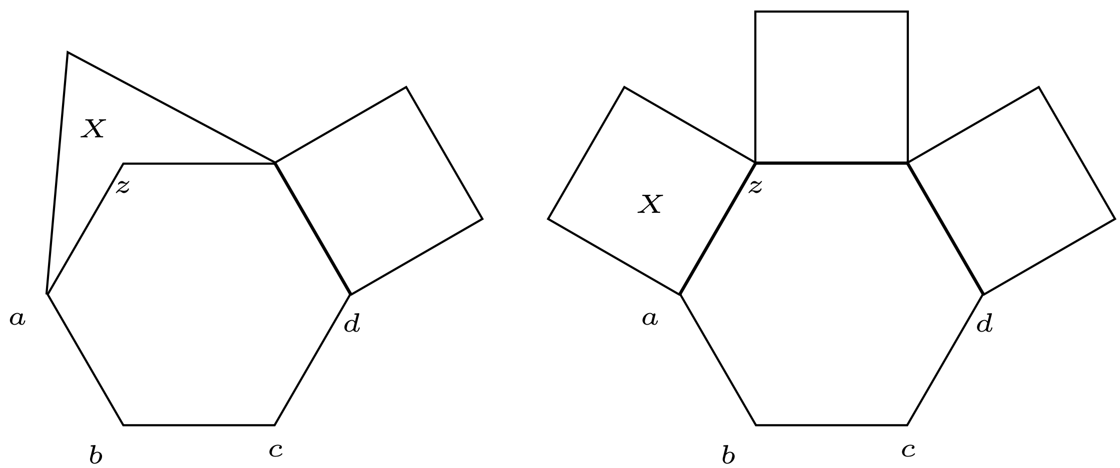

Figure 2: G_0 on the left and G_{C_4} on the right. These and $2C_4$ are the only graphs in $\mathcal{G}_{C_4}$ that can be subgraphs of a very good C_4-collage.

If b is on another block, then X receives weight at most $\frac{1}{2} \cdot w(ab)/2 = -2$ via ab. If b is not on another block and c is, then X received via ab weight $\frac{1}{2}(w(b)/2 + w(ab)) = \frac{1}{2}(11/2 - 8) = -5/4$. Suppose that neither b nor c is on another block, and also that a is not on another block. The weight of bc then is split equally between b and c. Then at the end of stage vi) X receives via ab a further weight

$$w(ab) + w(b) + w(bc)/2 = -8 + 11 - 4 = -1$$

and thus $w(X) \leq 0$. Since a cannot be on another block when $G \cong G_{C_\ell}$ or $G \cong G_0$ and X is the C_4 sharing only one edge with the central C_6, we may assume for the raminder that $G \cong G_0$ and X is the C_4 sharing two edges with the central C_6 i.e. G, X are as in the first graph of Figure 2. If the only potentially dangerous C_6 sharing an edge with X is the one in G the lemma holds. Suppose then that there exists another potentially dangerous C_6 that X shares an edge with, and let $G' \neq G$ be a copy of either G_0 or G_{C_ℓ} containing this C_6 as a central C_6 and having X as an attached C_ℓ (by the argument at the beginning of the proof we have excluded other possibilities for G', X).

We may assume that X is not the attached C_4 of G' sharing only one edge with the central C_6 (i.e. G' and X cannot be as in the second or third graph in Figure 2), since otherwise the above argument implies $w(X) \leq 0$. Hence G' and X are as in the first graph in Figure 2.

We claim that if there is no G' whose C_4 other than X contains a, then X satisfies the lemma. To see this, let X' be the unique block other than X containing y. Then the C_4 of G' other than

X must be X'. By Corollary 6.9, there is no other C_ℓ sharing an edge with G', since $G' \cong G_0$. Since G' is arbitrary, it follows that every potentially dangerous C_6, which shares at least one edge with X, shares exactly one edge with X', and shares no edge with any other copy of C_4. Hence X satisfies the lemma.

To complete the proof, suppose that there is such a G', with attached P_3 $ab'c'd'$ (where potentially $\{b', c', d'\} \cap \{b, c, d\} \neq \emptyset$) and let X' be the attached C_4 of G' on edge ab'. We claim that $d' \neq d$. Indeed: otherwise, Y is a C_ℓ that shares an edge with G' and is not an attached C_ℓ of G', which is excluded by Corollary 6.9 (since $G' \cong G_0$). Then the weight of the edges $yd', c'd'$ has not been accounted so far. X receives via yd' from these edges weight at most

$$\frac{1}{2}\left(w(yd') + w(d') + w(c'd')/2\right) = -1/2,$$

where the $1/2$ in front is because the weight going through yd' is shared with block Y. Hence $w(X) \leq 0$ which completes the proof.

$\square$

For the proof of Lemma 6.7 we will need the following lemma. Essentially it says that, inside a very good collage, an $X \cong 2C_\ell$ can share an edge with a $C_{2\ell-2}$ which is not contained in X only in one way: they share one edge only, the $C_{2\ell-2}$ is in a copy G of G_{C_ℓ}, and the C_ℓ in X sharing an edge with the $C_{2\ell-2}$ is one of the attached C_ℓ's of G. We will prove Lemma 6.10 in the next subsection.

Lemma 6.10. *Let C be a very good collage. Let X, Y be subgraphs of C with $Y \cong C_{2\ell-2}$ contained in some graph in $\mathcal{G}_{C_\ell}$ and let $X \cong 2C_\ell$. Suppose that Y is not a subgraph of X and that it contains an edge of X other than the intersection of the two C_ℓ's in X. Then*

- *$X \cap Y$ is one edge;*
- *Y is contained in a copy G of G_{C_ℓ};*
- *the only copies of C_ℓ sharing an edge with Y are those in G, and X contains one of them.*

Proof of Lemma 6.7. Suppose for the sake of contradiction that neither X_i has such a pair of edges. By Lemma 6.10 for both X_1, X_2 among all edges apart from $X_1 \cap X_2$, all but one lies on the $C_{2\ell-2}$ of a G_{C_ℓ}, and X_1, X_2 play the role of an attached C_ℓ. Observe that in a G_{C_ℓ}, for every attached C_ℓ, one of the vertices on the edge of the C_ℓ that is also on the $C_{2\ell-2}$ is shared with another C_ℓ. Hence there are $V_1 \subseteq V(X_1), V_2 \subseteq V(X_2)$ with V_i covering all but one edge of $X_i \setminus (X_1 \cap X_2)$, with every $v \in V_i$ being shared with another block. Let $s = |V_1 \cup V_2|$, and notice that $s \geq 2$: if $s = 1$ some X_i has two edges that do not intersect $V_1 \cup V_2$. Let $C' \subseteq C$ be the union of all blocks in C. Since s vertices contribute weight at most $(\ell^2-\ell-1)/2$ to X, we have

$$\begin{aligned} w_{C'}(X) &\leq (2\ell-2-s/2)(\ell^2-\ell-1)-(2\ell-1)(\ell^2-2\ell) \\ &\leq (2\ell-3)(\ell^2-\ell-1)-(2\ell-1)(\ell^2-2\ell) \\ &= -\ell+3 \leq -1. \end{aligned}$$

For $\ell \geq 5$, this immidiately implies that $w(X) < 0$ at the end of the discharging procedure, and for $\ell = 4$ it follows from Lemma 6.3. This gives the required contradiction. □

6.2 Small graphs excluded from very good collages

First we prove Lemma 6.1, that shows that any C_ℓ can share an edge with at most one other copy of C_ℓ, and that their intersection consists of at most one edge.

Proof of Lemma 6.1. Let $Y \neq X$ be another copy of C_ℓ. We will first show that Y can share at most two vertices with X. Suppose to the contrary that $v(Y \cap X) \geq 3$. We will show that the graph $X \cup Y$ satisfies (3), and thus cannot be contained in a very good collage. Since $X \cap Y$ is a path, we have

$$e(X \cup Y) = 2\ell - e(X \cap Y) \geq 2\ell + 1 - v(X \cap Y).$$

Moreover $v(X \cup Y) = 2\ell - v(X \cap Y)$, so

$$\frac{e(X \cup Y) - \ell - 1/(\ell-1)}{v(X \cup Y) - \ell} \geq \frac{\ell + 1 - v(X \cap Y) - 1/(\ell-1)}{\ell - v(X \cap Y)} = 1 + \frac{1 - 1/(\ell-1)}{\ell - v(X \cap Y)},$$

which is at least $1 + \frac{1-1/(\ell-1)}{\ell-3}$. This is strictly larger than $\hat{m}(C_\ell) = 1 + \frac{1-1/\ell}{\ell-2}$ if

$$(\ell-2)(1-1/(\ell-1)) > (\ell-3)(1-1/\ell).$$

This can be rewriten as $(\ell-2)(\ell-1) > 1$, which hold for every $\ell \geq 3$.

Let Y_1, Y_2 be different copies of C_ℓ which share at least one edge with X. By the above argument, each shares exactly one edge with X. We will again show that $X \cup Y_1 \cup Y_2$ satisfies (3) and hence cannot be a subgraph of a very good collage. Let $v_1 = v(Y_1 \cap X)$, $v_2 = v(Y_2 \cap (X \cup Y_1))$, and note that both $v_1, v_2 \geq 2$. Then, using that $Y_1 \cap X$, $Y_2 \cap (Y_1 \cup X)$ are linear forests on v_1 and v_2 vertices respectively, we have $e(X \cup Y_1 \cup Y_2) \geq 3\ell + 2 - v_1 - v_2$. Then

$$\frac{e(X \cup Y_1 \cup Y_2) - \ell - 1/(\ell-1)}{v(X \cup Y_1 \cup Y_2) - \ell} \geq 1 + \frac{2 - 1/(\ell-1)}{2\ell - v_1 - v_2} \geq 1 + \frac{2 - 1/(\ell-1)}{2\ell - 4}.$$

This is stricly greater than $\hat{m}(C_\ell)$ if $(2 - 1/(\ell-1))(\ell-2) > (2\ell-4)(1-1/\ell)$, which holds for every $\ell \geq 3$. $\square$

In the remainder of this section we work towards proving Lemma 6.10. We will need two preparatory lemmas.

Lemma 6.11. *Suppose G is obtained from a $G' \in \mathcal{G}_{C_\ell}$, which has at most $\ell - 2$ copies of C_ℓ, by adding one more copy of C_ℓ to G' so that the intersection of G' and this copy of C_ℓ contains an edge. Then G is not a subgraph of any very good collage.*

Proof. Let F_0 and F_{-1} be the central and attached copies of P_ℓ of G'. Suppose G' has $k \leq \ell - 2$ copies of C_ℓ, $F_1, \dots, F_k$, enumerated in the linear order in Definition 3.5. For $i \geq 1$ set $F_i' = F_i \cap \bigcup_{0 \leq j < i} F_i$ and let $F_{-1}' = F_{-1} \cap \bigcup_{i=1}^k F_i$. For $i \in [k] \cup \{-1\}$ let $v_i = v(F_i'), e_i = e(F_i')$. By Proposition 3.7, $e_i \leq v_i - 1$. Let H be the additional copy of C_ℓ that we attach to G' and let $e_{k+1} = e(H \cap G')$, $v_{k+1} = v(H \cap G')$, so $v_{k+1} \geq 2$ and $e_{k+1} \leq v_{k+1} - 1$, since $H \cap G'$ contains an edge and is a linear forest. Then

$$e_G \geq \ell - 1 + \sum_{i=1}^{k+1} (\ell - v_i + 1) + (\ell - 1 - e_{-1}) = (k+3)\ell - 1 + k - \sum_{i=1}^{k+1} v_i - e_{-1}$$

and

$$v_G \geq \ell + \sum_{i=1}^{k+1} (\ell - v_i) + (\ell - v_{-1}) = (k+3)\ell - \sum_{i=1}^{k+1} v_i - v_{-1}.$$

First suppose that $F_{-1}' \neq F_{-1}$. Then by Proposition 3.7 $e_{-1} \leq v_{-1} - 2$ and $\sum_{i=1}^k v_i \geq \ell + k - 1$, so $\sum_{i=1}^{k+1} v_i \geq \ell + k + 1$, and we have

$$\begin{aligned}
\frac{e_G - \ell - 1/(\ell-1)}{v_G - \ell} &\geq \frac{(k+2)\ell + k + 1 - \sum_{i=1}^{k+1} v_i - v_{-1} - 1/(\ell-1)}{(k+2)\ell - \sum_{i=1}^{k+1} v_i - v_{-1}} \\
&\geq \frac{(k+1)\ell - v_{-1} - 1/(\ell-1)}{(k+1)\ell - k - 1 - v_{-1}} \\
&\geq \frac{(k+1)\ell - 2 - 1/(\ell-1)}{(k+1)\ell - k - 3} \\
&= 1 + \frac{k + 1 - 1/(\ell-1)}{(k+1)(\ell-1) - 2} \\
&= 1 + \frac{(k+1)(\ell-1) - 1}{(k+1)(\ell-1)^2 - 2(\ell-1)},
\end{aligned}$$

where we used $\sum_{i=1}^{k+1} v_i \geq \ell + k + 1$ and Observation 3.1 for the second inequality; and $v_{-1} \geq 2$ and Observation 3.1 for the third inequality. Hence (3) is equivalent to

$$\frac{(k+1)(\ell-1) - 1}{(k+1)(\ell-1)^2 - 2(\ell-1)} > \frac{\ell - 1}{\ell(\ell-2)} .$$

After rearranging this can be rewritten as

$$(\ell-1)^2+1>(k+1)(\ell-1)$$

which holds since $k\le \ell-2$.

Now suppose that $F'_{-1}=F_{-1}$, so $e_{-1}=\ell-1$ and $v_{-1}=\ell$. By Proposition 3.7, $\sum_{i=1}^{k} v_i\ge \ell+k$, so $\sum_{i=1}^{k+1} v_i\ge \ell+k+2$, and we have

$$\begin{aligned}\frac{e_G-\ell-1/(\ell-1)}{v_G-\ell}&=\frac{(k+1)\ell+k-\sum_{i=1}^{k+1} v_i-1/(\ell-1)}{(k+1)\ell-\sum_{i=1}^{k+1} v_i}\\&\ge\frac{k\ell-2-1/(\ell-1)}{k\ell-k-2}\\&=1+\frac{k-1/(\ell-1)}{k(\ell-1)-2}\\&=1+\frac{k(\ell-1)-1}{k(\ell-1)^2-2(\ell-1)},\end{aligned}$$

and (3) is equivalent to

$$\frac{k(\ell-1)-1}{k(\ell-1)^2-2(\ell-1)}>\frac{\ell-1}{\ell(\ell-2)}\ .$$

After rearranging this can be rewritten as $(\ell-1)^2+1>k(\ell-1)$ which holds for $k\le\ell-2$. $\square$

Lemma 6.12. *Let $G\in\mathcal{G}_{C_\ell}$ be a subgraph of a very good collage with $k\in[2,\ell-1]$ attached C_ℓ's. Let F_0,F_{-1} be the central and attached P_ℓ of G and let $F_1,\dots,F_k$ enumerate the attached C_ℓ's in the linear order of Definition 3.5. Then either*

- *for every $i\in[k]$, $F_i\cap\bigcup_{0\le j<i}F_j$ is a path contained in F_0, and $F_i\cap\bigcup_{0<j<i}F_j$ consists of the ends of this path and has no edges;*
- *or $G\cong 2C_\ell$.*

Proof. If G fails to satisfy the first bullet-point of the lemma, then it satisfies one of the following conditions:

a) for some $i\in[k]$, F_i shares at least one vertex in $V(F_i)\setminus V(F_0)$ with $F_i\cap\bigcup_{0<j<i}F_j$;

b) for some $i\in[k]$, $F_i\cap F_0\cap\bigcup_{0<j<i}F_j$ contains an edge.

c) for some $i\in[k]$, $F_i\cap F_0$ has at least two connected components;

d) F_{-1} shares a vertex with $\bigcup_{i=1}^{k}F_i$ which is not in $F_0\cap F_{-1}$.

We will show that if G satisfies any of the above, then it satisfies (3) and hence cannot be a subgraph of a very good collage. For $i\ge 1$ let $F'_i=F_i\cap\bigcup_{0\le j<i}F_j$ and $v_i=v(F'_i)$, $e_i=e(F'_i)$.

Let $F'_{-1} = F_{-1} \cap \bigcup_{i=1}^{k} F_i$ and $v_{-1} = v(F'_{-1})$, $e_{-1} = e(F'_{-1})$. From Proposition 3.7 we have $v_{-1} \geq 2$. Moreover, following the proof of Proposition 3.7, let

$$v_i^1 = v\left((F'_i \cap F_0) \setminus \bigcup_{0<j<i} F_j \right),$$
$$v_i^2 = v\left(\left(F'_i \cap \bigcup_{0<j<i} F_j \right) \setminus F_0 \right),$$
$$v_i^3 = v\left(F'_i \cap \left(\bigcup_{0<j<i} F_j \right) \cap F_0 \right),$$

so that $v_i = v_i^1 + v_i^2 + v_i^3$. From the proof of Proposition 3.7, we have that $\sum_{i=1}^{k} v_i^1 = \ell$ and $v_i^3 \geq 1$ for every $i \in [2, k]$.

Claim 6.13. *If G satisfies one of a), b), c), then $\sum_{i=1}^{k} v_i \geq k + \ell$.*

Proof. If G satisfies a), then $\sum_{i=1}^{k} v_i^2 \geq 1$ and the Claim follows, using $\sum_{i=1}^{k} v_i^1 = \ell$ and $v_i^3 \geq 1$, for every $i \in [2, k]$. If G satisfies b) for some $i \in [k]$, then $v_i^3 \geq 2$ and hence $\sum_{i=1}^{k} v_i^3 \geq k$, which implies the Claim using $\sum_{i=1}^{k} v_i^1 \geq \ell$.

Suppose G satisfies c) i.e. suppose for some $i \in [k]$, $F_i \cap F_0$ has at least two connected components. Recall the order on the vertices of F_0 given at the end of Definition 3.5, i.e. we consider $V(F_0)$ in the order that we traverse the path from edge e_1 to $e_{\ell-1}$, with the end of F_0 in e_1 being the first vertex and the end of F_0 in $e_{\ell-1}$ being the last vertex. Note we have $v_j^3 \geq 1$ for each $j \geq 2$ because F_j shares the first vertex on the first edge of $F_j \cap F_0$ with $\bigcup_{0<j'<j} F_{j'}$. Let v be the first vertex of the last component of $F_i \cap F_0$, let uv the edge of F_0 not in F_i, and suppose F_j contains the edge uv. If $j < i$ then $v_i^3 \geq 2$, since then F_i shares v, which is not the first vertex of its first edge in F_0; and if $j > i$, then F_j shares the second vertex of one of its edges, so $v_j^3 \geq 2$. In either case, we deduce $\sum_{i=1}^{k} v_i^3 \geq k$ which implies the Claim using $\sum_{i=1}^{\ell} v_i^1 = \ell$ and $v_{-1} \geq 2$. □

If G satisfies d), then $v_{-1} \geq 3$ and using $\sum_{i=1}^{k} v_i \geq k + \ell - 1$ from Proposition 3.7 we have $v_{-1} + \sum_{i=1}^{k} v_i \geq k + \ell + 2$. Using Claim 6.13 and the lower bound $v_{-1} \geq 2$ we have $v_{-1} + \sum_{i=1}^{k} v_i \geq k + \ell + 2$ if G satisfies one of a), b), c).

First suppose that F_{-1} is not a subgraph of $\bigcup_{i=1}^{k} F_i$, and that it satisfies one of a), b), c), d). We will show that it then satisfies the first bullet point of the lemma. Because $F'_{-1} \neq F_{-1}$ and it contains both ends of F_{-1}, it has two connected components. Substituting the bounds for v_G, e_G

from Proposition 3.7, using $e_{-1} \le v_{-1} - 2$ and the bound $v_{-1} + \sum_{i=1}^k v_i \ge k + \ell + 2$ we have

$$\begin{aligned}
\frac{e_G - \ell - 1/(\ell-1)}{v_G - \ell} &\ge \frac{(k+1)\ell - 2 + k - \sum_{i=1}^k v_i - v_{-1} + 2 - 1/(\ell-1)}{(k+1)\ell - \sum_{i=1}^k v_i - v_{-1}} \\
&\ge \frac{k\ell - 2 - 1/(\ell-1)}{k\ell - k - 2} \\
&= 1 + \frac{k - 1/(\ell-1)}{k(\ell-1) - 2} \\
&= 1 + \frac{k(\ell-1) - 1}{k(\ell-1)^2 - 2(\ell-1)},
\end{aligned}$$

where we used for the second inequality Observation 3.1 and $v_{-1} + \sum_{i=1}^k v_i \ge k + \ell + 2$. Hence (3) is equivalent to

$$\frac{k(\ell-1) - 1}{k(\ell-1)^2 - 2(\ell-1)} > \frac{\ell-1}{\ell(\ell-2)} .$$

This can be rewritten as $(\ell-1)^2 + 1 > k(\ell-1)$ which holds for $k \le \ell - 1$. Hence if F'_{-1} is not a subgraph of F_{-1}, then G does not satisfy any of a), b), c), d) and hence G satisfies the first bullet-point of the lemma.

Suppose now that $F_{-1} \subseteq \bigcup_{j\ge1} F_j$. We will show that in this case $G \cong 2C_\ell$. Substituting $e_{-1} = \ell - 1$, $v_{-1} = \ell$ in the expression for e_G, v_G in Proposition 3.7, we have $e_G \ge (k+1)\ell - 1 + k - \sum_{i=1}^k v_i$ and $v_G = (k+1)\ell - \sum_{i=1}^k v_i$.

Claim 6.14. *If $\sum_{i=1}^k v_i \ge k + \ell + 1$, then G is not a subgraph of a very good collage.*

Proof. We have

$$\begin{aligned}
\frac{e_G - \ell - 1/(\ell-1)}{v_G - \ell} &= \frac{k\ell - 1 + k - \sum_{i=1}^k v_i - 1/(\ell-1)}{k\ell - \sum_{i=1}^k v_i} \\
&\ge \frac{(k-1)\ell - 2 - 1/(\ell-1)}{(k-1)\ell - 1 - k} \\
&= 1 + \frac{(k-1)(\ell-1) - 1}{(k-1)(\ell-1)^2 - 2(\ell-1)},
\end{aligned}$$

and (3) is equivalent to

$$\frac{(k-1)(\ell-1) - 1}{(k-1)(\ell-1)^2 - 2(\ell-1)} > \frac{\ell-1}{\ell(\ell-2)}.$$

This can be rewritten as $(\ell-1)^2 + 1 > (k-1)(\ell-1)$, which holds since $k \le \ell - 1$. □

Since the ends of F_{-1} are not connected by an edge, F_{-1} cannot be a subgraph of a single C_ℓ. Hence there are edges $uv, vw \in E(F_{-1})$ and $F_i, F_{i^*}, i < i^*$, with $uv \in E(F_i), vw \in E(F_{i^*})$. In particular, v is an internal vertex of F_{-1} i.e. $v \in V(F_{-1}) \setminus V(F_0)$.

Claim 6.15. *For every $i \in [k] \setminus \{i^*\}$, $F_i \cap \bigcup_{0 \le j < i} F_j$ is a path contained in F_0, and $F_i \cap \bigcup_{0<j<i} F_j$ consists of the ends of this path and has no edges. $F_{i^*} \cap \bigcup_{0 \le j < i^*} F_j$ consists of a vertex in $V(F_{-1}) \setminus V(F_0)$ and a path in F_0; and $F_{i^*} \cap \bigcup_{0<j<i^*} F_j$ consists of the ends of this path and a vertex in $V(F_{-1}) \setminus V(F_0)$, and has no edges.*

Proof. By Claim 6.14 it suffices to show that if G fails to satisfy the conditions of the Claim then $\sum_{i=1}^k v_i \ge k + \ell + 1$.

By the discussion before the Claim, we have $v_{i^*}^2 \ge 1$. If F_i^* shares two vertices in $V(F_{-1}) \setminus V(F_0)$ or one vertex outside $V(F_{-1}) \cup V(F_0)$ with $\bigcup_{j<i^*} F_j$ then $v_{i^*}^2 \ge 2$ and hence $\sum_{i=1}^k v_i \ge k + \ell + 1$, using $\sum_{i=1}^k v_i^1 = \ell$ and that $v_i^3 \ge 1$ for all $i \in [2, k]$. If G fails to satisfy any other conditions of the claim, arguing as in Claim 6.13, we either have $v_i^2 \ge 1$ for some $i \ne i^*$ or $v_i^3 \ge 2$ for some $i \in [k]$; along with $v_{i^*}^2 \ge 1$, both imply $\sum_{i=1}^k v_i \ge k + \ell + 1$. □

Recall that F_1, F_k contain the ends of F_{-1}, since $e_1 \in E(F_1)$ and $e_k \in E(F_k)$. We claim that $F_{-1} \subseteq F_1 \cup F_k$. Suppose otherwise, and let $Q_1 \subseteq F_1 \cap F_{-1}, Q_k \subseteq F_k \cap F_{-1}$ be the maximal subpaths of F_{-1} containing the end of F_{-1} in e_1 and the end of F_{-1} in $e_{\ell-1}$ respectively. If there is an edge of F_{-1} not in $Q_1 \cup Q_k$, then Q_1, Q_k are vertex disjoint, and for some $1 < i, j < k$ (with potentially $i = j$) F_i shares a vertex with Q_1 and F_j shares a vertex with Q_k. Suppose $z \in \{0, 1, 2\}$ of these vertices are the ends of F_{-1}. Recall that for each $i \ge 2$, F_i shares the first vertex on the first edge of $F_0 \cap F_i$ with $\bigcup_{0<j<i} F_j$, and note that this vertex cannot be an end of F_{-1}: one end of F_{-1} is the last vertex in the ordering, and the other end is in e_1 which is not in any $F_i, i \ge 2$ since the attached C_ℓ's are edge-disjoint, by Claim 6.15. Thus $\sum_{i=1}^k v_i^3 \ge k - 1 + z$. We also have $\sum_{i=1}^k v_i^2 \ge 2 - z$, since F_i and F_j together have $2 - z$ internal vertices of F_{-1}, which are not in $V(F_0)$. Hence $\sum_{i=1}^k v_i \ge k + \ell + 1$, which contradicts G being a subgraph of a very good collage by Claim 6.14.

Claim 6.16. *If $k \ge 3$, then G is not a subgraph of a very good collage.*

Proof. Suppose $k \ge 3$. We will show that G satisfies (3), and hence cannot be a subgraph of a very good collage. By Claim 6.15 the attached C_ℓ's are edge-disjoint and contain F_{-1}, so $e_G = k\ell$. By Claim 6.15, for every $i \in [2, k-1]$, F_i shares exactly two vertices with $\bigcup_{j \ge 1 : j \ne i} F_j$ and these vertices are the first and last vertex of $V(F_0) \cap V(F_i)$; and F_1, F_k share exactly one vertex, which is contained in $V(F_{-1})$. Hence, by counting first the vertices in $F_1 \cup F_k$ and then in $F_2, \dots, F_{k-1}$, we have $v_G = 2\ell - 1 + (k-3)(\ell-1) + (\ell-2) = k\ell - k$. Then

$$\begin{aligned}\frac{e_G - \ell - 1/(\ell-1)}{v_G - \ell} &= \frac{(k-1)\ell - 1/(\ell-1)}{(k-1)\ell - k} \\ &= 1 + \frac{k - 1/(\ell-1)}{(k-1)(\ell-1) - 1} \\ &= 1 + \frac{k(\ell-1) - 1}{(k-1)(\ell-1)^2 - (\ell-1)}.\end{aligned}$$

and (3) is equivalent to

$$\frac{k(\ell-1) - 1}{(k-1)(\ell-1)^2 - (\ell-1)} > \frac{\ell-1}{\ell(\ell-2)}.$$

This can be rewritten as $k(\ell-1) < (\ell-1)^2 + 1$, which holds, since $k \leq \ell - 1$. □

Hence we conclude that the edges of both F_0 and F_{-1} are contained in exactly two C_ℓ's, F_1, F_2, which share exactly one vertex in F_0, by Claim 6.15, and exactly one vertex in F_{-1}, by the discussion preceding Claim 6.15. Hence $F_1 \cup F_2$ spans $2\ell - 2$ vertices i.e. $V(F_1) \cup V(F_2) = V(F_{-1}) \cup V(F_0)$. Let v_1, v_2 be the vertices in $V(F_1) \cap V(F_2)$. It remains to show that v_1, v_2 span an edge in both F_1, F_2. Suppose otherwise. Then, since the longest path in C_ℓ between non-adjacent vertices is $\ell - 2$, the longest cycle in $F_1 \cup F_2$ has length at most $\ell - 2 + \ell - 1 = 2\ell - 3$, which contradicts that $F_1 \cup F_2$ contains a copy of $C_{2\ell-2}$. □

Proof of Corollary 6.8. This is a direct consequence of Lemma 6.12. □

Proof of Corollary 6.9. We will in fact prove the lemma for any $\ell \geq 4$ i.e. we will show the following. If $H \cong C_\ell$, $G \in \mathcal{G}_{C_4}$ are subgraphs of a very good collage so that $H \cap G$ contains an edge, then $G \cong G_{C_\ell}$, and $G \cap H$ is exactly one edge.

The fact that $G \cong G_{C_\ell}$ is a direct consequence of Lemma 6.11, which says that in a very good collage no $G \in \mathcal{G}_{C_\ell}$ with at most $\ell - 1$ attached C_ℓ's can share an edge with a C_ℓ which is not one of the attached ones; and Lemma 6.12, which says that the only $G \in \mathcal{G}_{C_\ell}$ with ℓ attached C_ℓ's contained in very good collages is G_{C_ℓ}.

To show that $G \cap H$ is exactly one edge, we will show that otherwise $G \cup H$ satisfies (3) and hence it cannot be a subgraph of a very good collage. Let $e = e(G \cap H) \geq 1$ and $v = v(G \cap H)$ and suppose that $v \geq 3$. Then $e(G \cup H) = \ell^2 - 1 + \ell - e(G \cap H) \geq \ell^2 + \ell - v$ and $v(G \cup H) = \ell^2 - \ell - v$. Then

$$\frac{e(G \cup H) - \ell - 1/(\ell-1)}{v(G \cup H) - \ell} \geq \frac{\ell^2 - v - 1/(\ell-1)}{\ell^2 - \ell - v} \geq \frac{\ell^2 - 1/(\ell-1) - 3}{\ell^2 - \ell - 3},$$

using Observation 3.1 for the last inequality and $v \geq 3$. After some calculations, we see this is strictly greater than $\hat{m}(C_\ell) = \frac{\ell^2 - \ell - 1}{\ell^2 - 2\ell}$ if and only if $\ell^2 - 3\ell + 3 > 0$ which holds for $\ell \geq 4$. Hence $G \cup H$ satisfies (3), as required. □

Finally we are ready to prove Lemma 6.10.

Proof of Lemma 6.10. Let G be a copy in C of some graph in $\mathcal{G}_{C_\ell}$ which contains Y. Let X_1, X_2 be the copies of C_ℓ in X. If $G \cong 2C_\ell$, since X is a block and blocks are edge-disjoint, we have $G = X$. But then Y is a subgraph of X which is a contradiction. Thus G is not isomorphic to $2C_\ell$ and the first setting of Lemma 6.12 applies. Hence G cannot have two attached C_ℓ's that share an edge, so either one or none of X_1, X_2 is an attached C_ℓ of G. In either case, if G has at most $\ell - 2$ attached C_ℓ's, then $G \cup X$ satisfies the assumption of Lemma 6.11 and thus cannot be contained in a very good collage. We deduce that G has $\ell - 1$ attached C_ℓ's. The only such member of $\mathcal{G}_{C_\ell}$ which also satisfies the properties in Lemma 6.12 is G_{C_ℓ}. This proves the first bullet-point of the Lemma.

Suppose that there are two copies of C_ℓ, Z_1, Z_2 which are not among the attached C_ℓ's in G, such that i) Z_1 shares an edge with G and ii) Z_2 shares an edge with $G \cup Z_1$ (these may be equal to or different from X_1, X_2) Let $e_1 = e(Z_1 \cap G)$, $v_1 = v(Z_1 \cap G)$, $e_2 = e(Z_2 \cap (G \cup Z_1))$ and $v_2(Z_2 \cap (G \cup Z_1))$. Note that $e_i \leq v_i - 1$ and $e(G) = \ell^2 - 1$ and $v(G) = \ell^2 - \ell$. We have

$$e(Z \cup G) = \ell^2 - 1 + \ell - e_1 + \ell - e_2 \geq \ell^2 + 2\ell + 1 - v_1 - v_2$$

and

$$v(Z \cup G) = \ell^2 - \ell + \ell - v_1 + \ell - v_2 = \ell^2 + \ell - v_1 - v_2.$$

Then the left-hand-side of (3) with $H_{(3)} = G \cup Z_1 \cup Z_2$ is at least

$$\frac{\ell^2 + \ell + 1 - v_1 - v_2 - 1/(\ell - 1)}{\ell^2 - v_1 - v_2} \geq \frac{\ell^2 + \ell - 3 - 1/(\ell - 1)}{\ell^2 - 4},$$

using that $v_i \geq 2$ and Observation 3.1. Then (3) reduces to

$$\ell^2 + \ell - 3 - 1/(\ell - 1) > (\ell + 2)(\ell - 1 - 1/\ell)$$

which after rearranging is equivalent to $\ell > 2$. Hence there are no such copies of Z_1, Z_2. This implies that X_1 (say) is one of the attached C_ℓ's in G and that there is no other C_ℓ sharing an edge with $G \cup X$ (since the argument above shows at most one C_ℓ, X_2 in this case, which is not an attached C_ℓ can share an edge with G). This proves the third bullet-point of the Lemma.

Finally, to prove the second bullet-point of the lemma, we show that X_2 shares no vertex with G other than those in $X_1 \cap X_2$. Let $e = e(X_2 \cap G)$ and $v = v(X_2 \cap G)$ and suppose for contradiction that $v \geq 3$. We have $e(G \cup X_2) = \ell^2 - 1 + \ell - e \geq \ell^2 + \ell - v$ and $v(G \cup X_2) = \ell^2 - \ell + \ell - v = \ell^2 - v$ Then the left hand side of (3) is at least

$$\frac{\ell^2 - v - 1/(\ell - 1)}{\ell^2 - \ell - v} \geq 1 + \frac{\ell - 1/(\ell - 1)}{\ell^2 - \ell - 3}$$

and (3) reduces to $(\ell-2)(\ell-1/(\ell-1)) > (1-1/\ell)(\ell^2-\ell-3)$, which holds for all $\ell \geq 3$. $\square$

References

[1] Y. Alon, P. Morris, and W. Samotij, *Two-round ramsey games on random graphs*, arXiv:2305.02725 (2023). 2, 3, 12, 26

[2] B. Bollobás and A. G. Thomason, *Threshold functions*, Combinatorica **7** (1987), no. 1, 35–38. 1

[3] M. Campos, S. Griffiths, R. Morris, and J. Sahasrabudhe, *An exponential improvement for diagonal Ramsey*, Annals of Mathematics (to appear) (2025). 1

[4] M. Campos, M. Jenssen, M. Michelen, and J. Sahasrabudhe, *A new lower bound for the Ramsey numbers $R(3, k)$*, arXiv:2505.13371 (2025). 1

[5] D. Conlon, S. Das, J. Lee, and T. Mészáros, *Ramsey games near the critical threshold*, Random Structures & Algorithms **57** (2020), no. 4, 940–957. 3

[6] S. Das, P. Morris, and A. Treglown, *Vertex Ramsey properties of randomly perturbed graphs*, Random Structures & Algorithms **57** (2020), no. 4, 983–1006. 2

[7] S. Das and A. Treglown, *Ramsey properties of randomly perturbed graphs: cliques and cycles*, Combinatorics, Probability and Computing **29** (2020), no. 6, 830–867. 2

[8] P. Frankl and V. Rödl, *Large triangle-free subgraphs in graphs without K_4*, Graphs and Combinatorics **2** (1986), no. 1, 135–144. 1

[9] A. Freschi, R. Hancock, and A. Treglown, *Typical Ramsey properties of the primes, abelian groups and other discrete structures*, arXiv:2405.19113 (2024). 2

[10] E. Friedgut, *A sharp threshold for random graphs with a monochromatic triangle in every edge coloring*, Memoirs of the American Mathematical Society **179** (2006), no. 845, vi+66. 2

[11] E. Friedgut, Y. Kohayakawa, V. Rodl, A. Ruciński, and P. Tetali, *Ramsey games against a one-armed bandit*, Combinatorics, Probability and Computing **12** (2003), no. 5-6, 515–545. 2

[12] E. Friedgut, E. Kuperwasser, W. Samotij, and M. Schacht, *Sharp thresholds for Ramsey properties*, arXiv:2207.13982 (2022). 26

[13] S. Janson, A. Ruciński, and T. Łuczak, *Random Graphs*, John Wiley & Sons, 2011. 8

[14] E. Kuperwasser and W. Samotij, *The list-Ramsey threshold for families of graphs*, Combinatorics, Probability and Computing **33** (2024), no. 6, 829–851. 15, 26

[15] T. Łuczak, A. Ruciński, and B. Voigt, *Ramsey properties of random graphs*, Journal of Combinatorial Theory, Series B **56** (1992), no. 1, 55–68. 1

[16] M. Marciniszyn, R. Spöhel, and A. Steger, *Online Ramsey games in random graphs*, Combinatorics, Probability and Computing **18** (2009), no. 1-2, 271–300. 2

[17] ———, *Upper bounds for online Ramsey games in random graphs*, Combinatorics, Probability and Computing **18** (2009), no. 1-2, 259–270. 2

[18] F. P. Ramsey, *On a Problem of Formal Logic*, Proceedings of the London Mathematical Society **s2-30** (1930), no. 1, 264–286. 1

[19] V. Rödl and A. Rucinski, *Lower bounds on probability thresholds for Ramsey properties*, Combinatorics, Paul Erdős is eighty **1** (1993), no. 317-346, 8. 1

[20] V. Rödl and A. Ruciński, *Random graphs with monochromatic triangles in every edge coloring*, Random Structures & Algorithms **5** (1994), no. 2, 253–270. 1

[21] ———, *Threshold functions for Ramsey properties*, Journal of the American Mathematical Society **8** (1995), no. 4, 917–942. 1

[22] V. Rödl and A. Ruciński, *Rado Partition Theorem for Random Subsets of Integers*, Proceedings of the London Mathematical Society **74** (1997), no. 3, 481–502. 2