Censored quantile regression for residual lifetime with induced smoothing method

Kyu Hyun Kim

December 4, 2019

Abstract

Keywords

Induced smoothing, Quantile regression, Residual lifetime regression, Sandwich estimator, Survival analysis

1. Introduction

In general, most of medical researches are interested in how long patients will survive or how long effect of medicine or treatment lasts given a certain circumstances. Both of them relates to estimate survival time of remaining lifetime, and only difference is lifetime of what. For this reason, many studies has focused on estimating mean survival time by various regression method. However, a regression method based on mean is not appropriate to estimate survival time due to heterogeneity in the survival data. To makeup disadvantage of mean-based regression, some statistician find alternative way, which is quantile regression.

First quantile regression models were introduced by Koenker and Bassett (1978). From this research, lots of studies have been progressed. Jung et al. (2009) proposed a time-specific log-linear regression method on quantile residual lifetime. Kim et al. (2012) suggested regression method that studies covariate effects on the conditional quantiles of residual lifetimes at a certain followup time point. Because estimating equations of both studies include indicator function, which is unsmoothed, finding estimator from solving those equations usually use linear programming, and this method is computationally inefficient. Furthermore, it caused the difficulty in variance estimation for censored quantile regression. Although Kim et al. (2012) solves this problem using empirical likelihood inference method that does not require estimating the covariance matrix of the estimator or resampling, it does not perfectly overcome unsmoothed estimating equation.

The unsmoothness problem can be solved by induced smoothing method, introduced by Brown and Wang (2005). The induced smoothing idea smoothes the estimating function by taking its expectation under the distribution of random perturbation. This idea makes solving estimating equation very efficient, and also gives opportunity to estimate variance. We derived a variance estimation procedure from sandwich estimator Chiou et al. (2015). Because sandwich estimator is developed to estimate variance with computational efficiency, it is very useful if asymptotically normal and consistency is guaranteed. We will prove this part end of this paper.

The rest of the article is organized as follows. In section 2, we introduced the process how to approach smoothed quantile regression estimator fore residual lifetime. We discussed simulation results in section 3 and real data analysis in section 4. Discussion and some necessary proof are at the end of this article.

2. Censored quantile regression for residual lifetime with induced smoothing

2.1 AFT model with no censored data

We start from scenario where random sample subject to no censoring: $\{T_i, X_i\}_{i=1}^n$, where T_i, X_i denote the failure time, and covariate values of i^{th} subject respectively.

$\theta_\tau(t_0)$ is τ-th quantile of residual lifetime at followup time t_0, and we simply express it as θ_τ. Corresponding to the AFT model and regression quantiles of residual lifetime, the τ-th quantile of residual lifetime given the covariate X_i is given by

$$\theta_\tau = log(T_i - t_0) = X_i'\beta(\tau, t_0) + \epsilon_i \tag{1}$$

where $\beta(\tau, t_0)$ represents τ-th quantile regression coefficient given covariate X_i and followup time t_0, and we simply use β instead of $\beta(\tau, t_0)$. ϵ_i is independent and have zero τ-th quantile.

In this case, the τ-th quantile residual lifetime quantile is solution of estimating equation:

$$0 = n^{-1} \sum_{i=1}^{n} I[(T_i - t_0) \leq \theta_\tau] - (1-\tau)I[T_i \leq t_0] - \tau \tag{2}$$

After some manipulations, estimating equation for τ-th quantile regression coefficient β is

$$0 = n^{-1} \sum_{i=0}^{n} I[T_i \geq t_0] X_i I\Big(I\{\log(T_i - t_0) \leq X_i'\beta\} - \tau\Big) \tag{3}$$

2.2 AFT model with censored data

And we consider concept of censoring of subject. Suppose n iid $\{T_i, X_i\}_{i=1}^n$ are generated from model (1), and we have right censored data $\{Z_i, \delta_i, X_i\}_{i=1}^n$ where $Z_i = min(T_i, C_i), \delta_i = I[T_i \leq C_i]$ are independent. In this case, we need to consider weight w_i, which are the probability that Kaplan-Meier estimator based on $\{Z_i, \delta_i\}_{i=1}^n$ assigns on the case (Z_i, δ_i). For this weight, we use inverse probability of censoring weighting (IPCW) technique (Robins and Ronitzky, 1992).

Therefore, estimating equation of censored regression residual quantile estimator is:

$$U_n(\beta) = 0 = n^{-1}\sum_{i=1}^{n} I[Z_i \geq t_0]X_i\Big(I\{\log(Z_i - t_0) \leq X_i'\beta\}\frac{\delta_i}{\hat{G}(Z_i)} - \tau\Big) \tag{4}$$

where $\hat{G}(Z_i)$ is the Kaplan-Meier estimate of the survival unction of the censoring variable C_i. We prove consistancy and asymptotic properties of unsmoothed estimator in appendix.

There is similar estimating equation of censored regression residual quantile estimator that suggested at Pang et al.(2010) and Kim et al.(2012). Both estimating equations used IPCW as weight to apply censored data, however, only difference is the position of weight. Both equations are theoretically correct equations, and Peng and Fine (2012) also similar estimating equation.

2.3 Induced smoothing approach

Even though solving equation (4) is possible by using linear programming approach, it has much intensive computation issue. To minimize computation burden, we apply induced smoothing method, proposed by Brown and Wang (2005). By the asymptotic normality of $\hat{\beta}$, we can express $\hat{\beta} = \beta + \mathbf{H}^{1/2}V$, where $H = n^{-1}\Gamma$, $V \sim N(0, I_p)$, and I_p is the $p \times p$ identity matrix. After applying induced smoothing approach to equation (4)

$$\tilde{U}_n(\beta, \mathbf{H}) = E_v\{U_n(\beta + \mathbf{H}^{1/2}V)\} = n^{-1}\sum_{i=1}^{n} I[Z_i \geq t_0]X_i\Big\{\tau - \frac{\delta_i}{\hat{G}(Z_i)}\Phi\Big(\frac{X_i'\beta - log(Z_i - t_0)}{\sqrt{X_i'\mathbf{H}X_i}}\Big)\Big\} \tag{5}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function. As Pang et al.(2010) proposed, we use a positive definite $p \times p$ matrix $\tilde{\mathbf{H}} = O(n^{-1})$ as a smoothing matrix $\mathbf{H}$. Estimated β from equation 5 is also consistent and asymptotically normal by 1 under below regularity conditions C1-C3:

- C1 The conditional error distribution functions, $F_i(\cdot \mid X_i)_{i=1}^n$, are absolutely continuous with continuous densities $f_i(\cdot \mid X_i)$ uniformly bounded away from 0 and ∞ in a neighborhood of 0, and $f_i'(\cdot \mid X_i)$ exists and is uniformly bounded on the real line.
- C2 For each $i = 1, \ldots, n, X_i$ satisfies the following conditions:

  (a) $n^{-1}\Sigma_{i=1}^n X_i X_i' f_i(0 \mid X_i)$ converges to a positive definite matrix $\mathbf{A}$;
  (b) $\sup_i \|X_i\| < \infty$, where $\|\cdot\|$ denotes the Euclidean norm.
- C3 There exists $L > 0$ such that $P(C > L) = 0$ and $P(C = L) \geq \nu$, where ν is some positive constant.

Theorem 1. *Assume condition* $1-3$ *hold, and the smoothing matrix* H *is positive definite and* $O(n^{-1})$*, as* $n \to \infty$ *then we have*

$$n^{1/2}(\hat{\beta}_{IS} - \beta_0) \xrightarrow{d} \mathcal{N}(0, \Gamma)$$

We prove above theorem 1 in appendix.

To estimate the asymptotic variance of $\hat{\beta}_{IS}$, we use sandwich estimators, which is computationally more efficient than full-multiplier bootstrap approach based on Chiou et al.(2015). For sandwich estimator, $\Sigma(\beta_0)$, two estimators, $A(\beta_0)$ and $V(\beta_0)$, are necessary. For estimating $A(\beta_0)$, we evaluate $A_n(\beta_0)$, the derivative of the smoothed estimating function:

$$A_n(\beta_0) = \frac{\partial \tilde{U}_n(\hat{\beta}_{IS}, \tilde{\mathbf{H}})}{\partial \beta} = n^{-1} \sum_{i=1}^{n} I[Z_i > t_0] X_i \frac{\delta_i}{\hat{G}(Z_i)} \phi\Big(\frac{X_i{}'\hat{\beta}_{IS} - \log(Z_i - t_0)}{\sqrt{X_i{}'\tilde{\mathbf{H}}X_i}}\Big)\Big(\frac{-X_i}{\sqrt{X_i{}'\tilde{\mathbf{H}}X_i}}\Big) \tag{6}$$

at $\hat{\beta}_{IS}$, which is the solution of estimating equation (5), where $\phi\{\cdot\}$ is the density function of a standard normal distribution.

For estimating $V(\beta_0)$, we generate iid positive multiplier $\eta_i, i = 1, ..., n$, that are independent of the observed data from exp(1), and make perturbed estimating equation, $\tilde{U}_n^{\star}(\beta, \mathbf{H})$, with generated η_i:

$$\tilde{U}_n^{\star}(\beta, \tilde{\mathbf{H}}) = n^{-1} \sum_{i=1}^{n} I[Z_i \geq t_0] X_i \eta_i \Big\{\tau - \frac{\delta_i}{\hat{G}(Z_i)} \Phi\Big(\frac{X_i'\beta - log(Z_i - t_0)}{\sqrt{X_i'\tilde{\mathbf{H}}X_i}}\Big)\Big\} \tag{7}$$

We evaluate $\tilde{U}_n^{\star}(\beta, \tilde{\mathbf{H}})$ at $\hat{\beta}_{IS}$. Repeating same process m times with new multipliers, and find sample variance of $\{\tilde{U}_n^{\star(1)}(\beta, \tilde{\mathbf{H}}), ..., \tilde{U}_n^{\star(m)}(\beta, \tilde{\mathbf{H}})\}$. $V(\beta_0)$ is approxiamated to sample variance of $\tilde{U}_n^{\star}(\beta, \tilde{\mathbf{H}})$.
Using above $A(\beta_0)$, and $V(\beta_0)$, $\Sigma(\beta_0)$ is:

$$\Sigma(\beta_0) = A(\beta_0)^T V(\beta_0) A(\beta_0) \tag{8}$$

3. Simulation

To verify performance of smoothed estimator, we used same simple regression simulation setting that provided by Jung et al. (2009). Covariate X_i is a binary covariate, 0 for control and 1 for treatment group. $Z_i = min(T_i, C_i)$ where T_i generated from Weibull regression model with one binary covariate X_i and intercept and where C_i generated from uniform distribution with range 0 and c, which is adjusted for censoring proportion. One dataset size was 200, and 2000 simulations were performed for every combination of t_0 and censoring proportion. In variance estimating process, we need to fix how many $\tilde{U}_n^{\star}(\beta, \tilde{\mathbf{H}})$ we will use to find sample variance. Comparing sample variance of 100 $V(\beta_0)$s and sample variance of 500 $V(\beta_0)$s, we conclude that 100 $\tilde{U}_n^{\star}(\beta, \tilde{\mathbf{H}})$s are close enough to $V(\beta_0)$.

From table 1 to table 12, we verify the performance of suggested estimator when covariate does not affect residual lifetime, which means $\beta_{t_0}^{(1)} = 0$. Based on Jung et al.(2009) simulation setting, true parameter $\beta_{t_0}^{(0)} = 1.61, 1.41, 1.22, 1.04$ at $t_0 = 0, 1, 2, 3$. $\beta^(0)$ and $\beta^(1)$ are mean of empirical estimates of true paramters $\beta^(0), \beta^(1)$, and SE is mean of standard error of empirical estimates of each true parameter. SD is standard deviation of empirical estimates, and Coverage is proportion that true parameter are included in 95% confidence interval of proposed estimates.

Table 1: Estimates of 25% quantile residual lifetime when $\beta^{(1)} = 0$

t_0	censor	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	Coverage	$\beta^{(1)}$	SE	SD	Coverage
$t_0 = 0$	0%	1.607	0.069	0.069	0.931	0.000	0.098	0.069	0.946
	10%	1.608	0.073	0.069	0.937	0.000	0.104	0.069	0.944
	30%	1.608	0.083	0.081	0.926	0.001	0.119	0.081	0.941
	50%	1.606	0.096	0.091	0.924	0.003	0.137	0.091	0.933
$t_0 = 1$	0%	1.406	0.084	0.084	0.927	-0.001	0.120	0.084	0.940
	10%	1.408	0.089	0.090	0.916	-0.003	0.127	0.090	0.934
	30%	1.403	0.103	0.098	0.928	0.002	0.147	0.098	0.932
	50%	1.408	0.120	0.115	0.908	-0.001	0.174	0.115	0.938
$t_0 = 2$	0%	1.214	0.100	0.099	0.924	0.002	0.143	0.099	0.943
	10%	1.214	0.108	0.106	0.918	0.000	0.154	0.106	0.942
	30%	1.215	0.126	0.121	0.923	-0.002	0.181	0.121	0.938
	50%	1.220	0.151	0.144	0.912	0.000	0.222	0.144	0.925
$t_0 = 3$	0%	1.038	0.120	0.121	0.913	-0.004	0.171	0.121	0.923
	10%	1.038	0.129	0.133	0.904	-0.004	0.186	0.133	0.934
	30%	1.037	0.157	0.153	0.914	0.005	0.225	0.153	0.935
	50%	1.040	0.186	0.177	0.906	0.005	0.284	0.177	0.937

Table 2: Estimates of 50% quantile residual lifetime when $\beta^{(1)} = 0$

t_0	censor	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	Coverage	$\beta^{(1)}$	SE	SD	Coverage
$t_0 = 0$	0%	1.607	0.069	0.069	0.931	0.000	0.098	0.069	0.946
	10%	1.607	0.073	0.069	0.937	-0.001	0.104	0.069	0.944
	30%	1.608	0.083	0.081	0.926	-0.002	0.119	0.081	0.941
	50%	1.606	0.096	0.091	0.924	0.003	0.137	0.091	0.933
$t_0 = 1$	0%	1.406	0.084	0.084	0.927	-0.001	0.120	0.084	0.940
	10%	1.408	0.090	0.090	0.916	-0.003	0.127	0.090	0.934
	30%	1.403	0.103	0.098	0.928	0.002	0.147	0.098	0.932
	50%	1.408	0.120	0.115	0.908	-0.001	0.174	0.115	0.938
$t_0 = 2$	0%	1.214	0.100	0.099	0.924	0.002	0.143	0.099	0.943
	10%	1.214	0.108	0.106	0.918	0.000	0.154	0.106	0.942
	30%	1.215	0.126	0.121	0.922	-0.002	0.181	0.121	0.938
	50%	1.220	0.151	0.144	0.911	0.000	0.222	0.144	0.925
$t_0 = 3$	0%	1.038	0.120	0.121	0.913	-0.004	0.171	0.121	0.923
	10%	1.038	0.129	0.133	0.904	-0.003	0.186	0.133	0.934
	30%	1.036	0.157	0.153	0.914	0.005	0.225	0.153	0.935
	50%	1.040	0.186	0.177	0.906	0.005	0.284	0.177	0.937

Table 3: Estimates of 75% quantile residual lifetime when $\beta^{(1)} = 0$

t_0	censor	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	Coverage	$\beta^{(1)}$	SE	SD	Coverage
$t_0 = 0$	0%	1.609	0.058	0.058	0.926	0.003	0.082	0.058	0.938
	10%	1.610	0.069	0.066	0.951	0.001	0.098	0.066	0.941
	30%	1.610	0.092	0.080	0.955	0.008	0.133	0.080	0.948
	50%	1.615	0.133	0.101	0.948	0.002	0.200	0.101	0.961
$t_0 = 1$	0%	1.410	0.072	0.075	0.917	-0.001	0.103	0.075	0.931
	10%	1.410	0.086	0.081	0.936	0.002	0.122	0.081	0.942
	30%	1.410	0.117	0.104	0.940	0.010	0.171	0.104	0.948
	50%	1.396	0.207	0.120	0.944	0.014	0.327	0.120	0.969
$t_0 = 2$	0%	1.215	0.090	0.092	0.906	0.005	0.129	0.092	0.930
	10%	1.219	0.109	0.103	0.935	0.000	0.156	0.103	0.934
	30%	1.222	0.154	0.140	0.926	0.006	0.228	0.140	0.929
	50%	1.175	0.264	0.136	0.941	0.018	0.530	0.136	0.987
$t_0 = 3$	0%	1.036	0.116	0.121	0.905	0.000	0.166	0.121	0.929
	10%	1.037	0.140	0.138	0.907	0.006	0.205	0.138	0.925
	30%	1.040	0.215	0.191	0.904	0.016	0.330	0.191	0.945
	50%	0.946	0.651	0.177	0.937	0.036	0.905	0.177	0.991

In every combination of t_0, censoring proportion, and estimated quantile, proposed estimator is quite close to true paramter except for estimating high qunatile 75% from data with high censoring proportion, 70%. Furthermore, small data size due to large t_0 makes estimating true parameter more difficult.

From table 13 to table 24, we verify that smoothed estimator is able to estimate an effect of covariate. In this simulation scenario, we add one more assumption that the difference in residual time between two groups are 5. Then, true parameter $\beta_{t_0}^{(0)} = 1.61, 1.41, 1.22, 1.04$ at $t_0 = 0, 1, 2, 3$ and $\beta_{t_0}^{(1)} = 0.69, 0.80, 0.91, 1.02$ at $t_0 = 0, 1, 2, 3$.

Table 4: Estimates of 25% quantile residual lifetime when $\beta^{(1)} \neq 0$

t_0	censor	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	Coverage	$\beta^{(1)}$	SE	SD	Coverage
$t_0 = 0$	0%	1.604	0.096	0.098	0.910	0.692	0.137	0.098	0.929
	10%	1.604	0.098	0.099	0.908	0.693	0.141	0.099	0.934
	30%	1.605	0.102	0.102	0.904	0.693	0.150	0.102	0.933
	50%	1.606	0.106	0.107	0.905	0.692	0.162	0.107	0.937
$t_0 = 1$	0%	1.403	0.116	0.116	0.909	0.789	0.159	0.116	0.933
	10%	1.410	0.119	0.119	0.907	0.787	0.164	0.119	0.937
	30%	1.399	0.124	0.124	0.898	0.797	0.176	0.124	0.925
	50%	1.407	0.129	0.128	0.905	0.793	0.188	0.128	0.930
$t_0 = 2$	0%	1.210	0.137	0.140	0.894	0.884	0.182	0.140	0.923
	10%	1.209	0.142	0.140	0.908	0.886	0.191	0.140	0.927
	30%	1.216	0.144	0.146	0.900	0.878	0.201	0.146	0.927
	50%	1.209	0.152	0.152	0.899	0.893	0.218	0.152	0.925
$t_0 = 3$	0%	1.031	0.154	0.154	0.895	0.965	0.204	0.154	0.917
	10%	1.032	0.158	0.162	0.886	0.972	0.211	0.162	0.923
	30%	1.040	0.166	0.168	0.889	0.957	0.226	0.168	0.909
	50%	1.037	0.182	0.174	0.891	0.963	0.254	0.174	0.909

Table 5: Estimates of 50% quantile residual lifetime when $\beta^{(1)} \neq 0$

t_0	censor	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	Coverage	$\beta^{(1)}$	SE	SD	Coverage
$t_0 = 0$	0%	1.607	0.068	0.068	0.921	0.692	0.097	0.068	0.949
	10%	1.607	0.072	0.070	0.931	0.691	0.104	0.070	0.944
	30%	1.606	0.078	0.075	0.932	0.695	0.119	0.075	0.943
	50%	1.606	0.086	0.081	0.933	0.695	0.149	0.081	0.945
$t_0 = 1$	0%	1.408	0.084	0.086	0.912	0.790	0.114	0.086	0.932
	10%	1.406	0.087	0.088	0.930	0.790	0.121	0.088	0.944
	30%	1.408	0.095	0.095	0.922	0.791	0.139	0.095	0.938
	50%	1.409	0.105	0.103	0.925	0.789	0.190	0.103	0.943
$t_0 = 2$	0%	1.216	0.100	0.100	0.918	0.884	0.132	0.100	0.937
	10%	1.217	0.106	0.103	0.922	0.880	0.142	0.103	0.946
	30%	1.215	0.116	0.114	0.916	0.879	0.164	0.114	0.931
	50%	1.216	0.131	0.123	0.915	0.872	0.239	0.123	0.941
$t_0 = 3$	0%	1.030	0.121	0.122	0.915	0.972	0.154	0.122	0.913
	10%	1.037	0.125	0.127	0.896	0.965	0.163	0.127	0.902
	30%	1.037	0.137	0.137	0.902	0.964	0.187	0.137	0.927
	50%	1.037	0.158	0.154	0.904	0.943	0.312	0.154	0.925

Table 6: Estimates of 75% quantile residual lifetime when $\beta^{(1)} \neq 0$

t_0	censor	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	Coverage	$\beta^{(1)}$	SE	SD	Coverage
$t_0 = 0$	0%	1.610	0.058	0.0593	0.922	0.695	0.082	0.059	0.944
	10%	1.610	0.066	0.063	0.938	0.695	0.098	0.063	0.940
	30%	1.611	0.080	0.074	0.939	0.698	0.134	0.074	0.955
	50%	1.627	0.105	0.093	0.961	0.614	0.430	0.093	0.955
$t_0 = 1$	0%	1.408	0.072	0.073	0.929	0.795	0.098	0.073	0.935
	10%	1.407	0.081	0.079	0.933	0.796	0.115	0.079	0.947
	30%	1.410	0.097	0.092	0.935	0.799	0.159	0.092	0.951
	50%	1.435	0.132	0.118	0.935	0.654	0.618	0.118	0.923
$t_0 = 2$	0%	1.215	0.091	0.092	0.9212	0.889	0.117	0.092	0.927
	10%	1.218	0.101	0.101	0.912	0.888	0.138	0.101	0.934
	30%	1.221	0.125	0.119	0.917	0.887	0.200	0.119	0.946
	50%	1.251	0.175	0.150	0.931	0.689	0.858	0.150	0.881
$t_0 = 3$	0%	1.033	0.117	0.115	0.919	0.974	0.144	0.115	0.916
	10%	1.037	0.130	0.126	0.907	0.968	0.169	0.126	0.922
	30%	1.046	0.158	0.150	0.919	0.970	0.241	0.150	0.943
	50%	1.079	0.240	0.209	0.906	0.741	0.604	0.209	0.893

In data with high censoring proportion, estimating β is unstable. When we find a solution for proposed estimating equation (5), we use nleqslv function in nleqslv R package. However, solutions from high censored proportion data are not unique or unsolvable, other nonlinear equation solvers also have similar issues. If both estimating quantile and censored proportion are high, we are hard to find solution. Futhermore, size of dataset affects performance of proposed estimator. Smaller dataset caused more error in solving the estimating equation, and also shows poor performance.

Table 7: Various quantile estimates of residual lifetime with high censoring (70%) when $\beta^{(1)} = 0$. When estimating quantile is high, results show big differences with true beta, or cannot estimate.

t_0	Quantile	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	Coverage	$\beta^{(1)}$	SE	SD	Coverage
$t_0 = 0$	25%	1.600	0.596	0.102	0.940	-0.002	0.662	0.102	0.956
	50%	1.600	0.596	0.102	0.939	-0.002	0.662	0.102	0.956
	75%	1.481	0.231	0.075	0.818	0.016	0.538	0.075	1.000
$t_0 = 1$	25%	1.378	0.179	0.115	0.938	0.003	0.339	0.115	0.968
	50%	1.378	0.179	0.115	0.938	0.003	0.339	0.115	0.968
	75%	1.218	0.269	NA	1.000	-0.104	0.290	NA	1.000
$t_0 = 2$	25%	1.145	0.258	0.141	0.922	0.038	0.425	0.141	0.981
	50%	1.145	0.258	0.141	0.921	0.038	0.425	0.141	0.981
	75%	NA	NA	NA	NA	NA	NA	NA	NA
$t_0 = 3$	25%	0.900	1.216	0.166	0.908	0.088	1.520	0.166	0.986
	50%	0.900	1.216	0.166	0.908	0.088	1.520	0.166	0.986
	75%	NA	NA	NA	NA	NA	NA	NA	NA

Table 8: Various quantile estimates of residual lifetime with high censoring (70%) when $\beta^{(1)} \neq 0$. When estimating quantile is high, results show big differences with true beta, or cannot estimate.

t_0	Quantile	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	Coverage	$\beta^{(1)}$	SE	SD	Coverage
$t_0 = 0$	25%	1.605	0.113	0.112	0.909	0.683	0.212	0.112	0.944
	50%	1.615	0.104	0.099	0.915	0.517	0.361	0.099	0.827
	75%	1.602	0.156	0.081	1.000	0.294	0.603	0.081	0.653
$t_0 = 1$	25%	1.406	0.140	0.140	0.896	0.767	0.273	0.140	0.943
	50%	1.418	0.140	0.130	0.909	0.550	0.499	0.130	0.788
	75%	1.376	0.170	0.114	1.000	0.283	1.124	0.114	0.571
$t_0 = 2$	25%	1.215	0.165	0.168	0.893	0.837	0.281	0.168	0.918
	50%	1.246	0.160	0.157	0.886	0.553	0.730	0.157	0.783
	75%	1.128	0.228	0.112	1.000	0.301	0.617	0.112	0.467
$t_0 = 3$	25%	1.035	0.192	0.199	0.862	0.908	0.346	0.199	0.909
	50%	1.048	0.234	0.198	0.85	0.596	0.667	0.198	0.743
	75%	0.917	1.482	0.196	0.808	0.380	2.369	0.196	0.731

4. Real data analysis

In this section, we apply the proposed estimator to analyze survival times of dental restoration longevity of older adults with on different circumstances. Dental restoration is a general term of treatments of dental cavity. As older adult population grows, dental restoration becomes an important issue to health care system, and they focused on not only cost or side effect of treatment, but also the longevity of restoration. The Geriatric and Special Needs Dentistry Clinic at the University iof Iowa Collage of Dentistry (COD) has offered comprehensive dental care to 2,717 unique patients, and observed covariates of treatment and helath condition of patients. Among this data, we randomly choose first restored tooth per one patient to remove the correlation effect from same patient and damage of multiple restoration. Data satisfied previous conditions includes 2,717 patients/tooth data, and censoring proportion was 62.8%. Initial dataset has 23 kinds of covariates, among them, we consider 5 covariates: gender, age, cohort, provider type, and payment method.

Table 9: 5% quantile estimates of dental restoration longevity with 5 covariates.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$		$t_0 = 3$	
	β_0	SE	β_0	SE	β_0	SE	β_0	SE
Base	-1.7816	0.3761	0.1954	0.5686	0.5682	0.3555	0.8895	0.4390
Male	-0.0150	0.1068	-0.4253	0.1964	-0.2774	0.1868	-0.1627	0.2781
Age	-0.0307	0.0040	-0.0323	0.0066	-0.0300	0.0048	-0.0201	0.0079
Cohort2	0.2593	0.1951	-0.0187	0.4145	-0.1983	0.3864	-0.4695	0.4925
Cohort3	0.1798	0.1929	0.1192	0.3842	-0.2912	0.3230	-1.0804	0.4027
Cohort4	0.4304	0.2165	0.2403	0.4252	0.2289	0.3190	-1.0257	0.5401
Cohort5	0.6769	0.2007	0.4928	0.4374	0.0170	0.3345	-1.0390	0.6020
Cohort6	0.7019	0.2522	0.9969	0.4983	1.1625	0.4971	-0.4324	0.8048
Grad	0.1687	0.2880	-1.3587	2.0702	-0.7369	0.6424	-0.0542	0.4970
Predoc	0.0232	0.1500	0.0980	0.2282	-0.0426	0.2157	-0.0485	0.3366
Private	-0.1819	0.2162	-0.2369	0.3309	0.1029	0.2716	0.4399	0.4568
XIX	-0.5256	0.2217	-0.8442	0.4206	-0.2417	0.3346	-0.1602	0.6649

Table 10: 10% quantile estimates of dental restoration longevity with 5 covariates.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$		$t_0 = 3$	
	β_0	SE	β_0	SE	β_0	SE	β_0	SE
Base	-0.9836	0.2759	0.9749	0.3142	1.0960	0.3291	1.5036	0.3744
Male	-0.0617	0.1032	-0.3862	0.1410	-0.1863	0.1673	0.0278	0.2026
Age	-0.0323	0.0032	-0.0305	0.0043	-0.0249	0.0049	-0.0194	0.0061
Cohort2	0.2055	0.1724	-0.0762	0.2526	-0.0495	0.3541	-0.2198	0.3179
Cohort3	0.0572	0.1549	-0.0835	0.2271	-0.2976	0.2786	-0.9242	0.2837
Cohort4	0.3406	0.1719	0.1498	0.2645	0.0802	0.2975	-0.7619	0.3679
Cohort5	0.5727	0.1805	0.2940	0.2648	-0.0690	0.3250	-0.6667	0.4357
Cohort6	0.6851	0.2050	1.0851	0.4776	1.4173	0.9233	0.2818	1.4891
Grad	0.1441	0.2759	-0.4511	0.6600	-0.6532	0.6341	-0.5009	0.5144
Predoc	0.0124	0.1198	0.0480	0.1573	-0.1232	0.1813	-0.2720	0.2416
Private	-0.1755	0.1695	-0.2294	0.2298	0.0256	0.2425	0.3932	0.3217
XIX	-0.4939	0.1798	-0.6143	0.2721	-0.2407	0.2983	-0.0962	0.3776

Table 11: 15% quantile estimates of dental restoration longevity with 5 covariates. When estimation quantile is 15% and t_0 is greater than 1, proposed estimating equation is not solvable.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$		$t_0 = 3$	
	β_0	SE	β_0	SE	β_0	SE	β_0	SE
Base	-0.5916	0.2172	1.3364	0.2767	NA	NA	NA	NA
Male	-0.0857	0.0852	-0.3272	0.1198	NA	NA	NA	NA
Age	-0.0322	0.0028	-0.0274	0.0037	NA	NA	NA	NA
Cohort2	0.1749	0.1547	-0.1074	0.2361	NA	NA	NA	NA
Cohort3	-0.0078	0.1334	-0.1831	0.2002	NA	NA	NA	NA
Cohort4	0.2685	0.1653	0.0697	0.2237	NA	NA	NA	NA
Cohort5	0.5137	0.1673	0.1587	0.2299	NA	NA	NA	NA
Cohort6	0.6380	0.1965	1.3439	0.5394	NA	NA	NA	NA
Grad	0.1451	0.2448	-0.3032	0.4599	NA	NA	NA	NA
Predoc	0.0384	0.0987	-0.0152	0.1476	NA	NA	NA	NA
Private	-0.1488	0.1459	-0.1810	0.2062	NA	NA	NA	NA
XIX	-0.4362	0.1572	-0.5396	0.2319	NA	NA	NA	NA

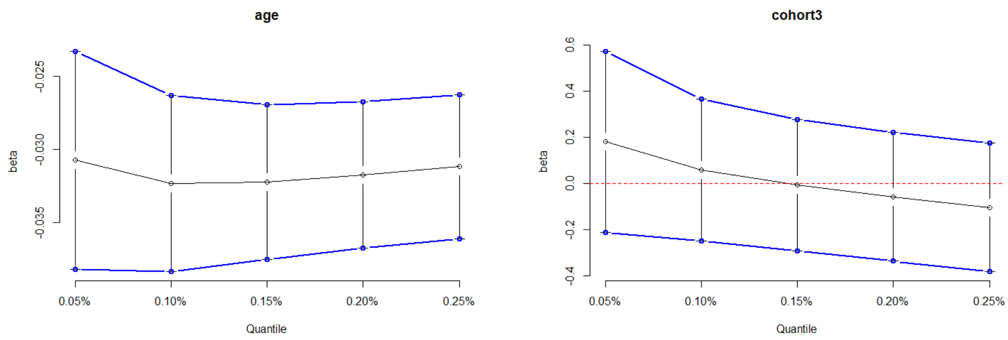
Table 12: 20% quantile estimates of dental restoration longevity with 5 covariates. When estimation quantile is 20% and t_0 greater than 1, proposed estimating equation is not solvable.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$		$t_0 = 3$	
	β_0	SE	β_0	SE	β_0	SE	β_0	SE
Base	-0.3329	0.1979	1.5523	0.2281	NA	NA	NA	NA
Male	-0.0955	0.0788	-0.2771	0.1153	NA	NA	NA	NA
Age	-0.0317	0.0025	-0.0244	0.0032	NA	NA	NA	NA
Cohort2	0.1470	0.1501	-0.1154	0.2189	NA	NA	NA	NA
Cohort3	-0.0591	0.1341	-0.2440	0.1870	NA	NA	NA	NA
Cohort4	0.2220	0.1616	-0.0005	0.2249	NA	NA	NA	NA
Cohort5	0.4696	0.1587	0.0778	0.2318	NA	NA	NA	NA
Cohort6	0.6001	0.1890	1.8729	1.0286	NA	NA	NA	NA
Grad	0.1393	0.2409	-0.2459	0.3607	NA	NA	NA	NA
Predoc	0.0589	0.0963	-0.0632	0.1302	NA	NA	NA	NA
Private	-0.1304	0.1425	-0.1504	0.2134	NA	NA	NA	NA
XIX	-0.3947	0.1476	-0.5048	0.2309	NA	NA	NA	NA

Table 13: 25% quantile estimates of dental restoration longevity with 5 covariates. When estimation quantile is 25% and t_0 is greater than 0, proposed estimating equation is not solvable.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$		$t_0 = 3$	
	β_0	SE	β_0	SE	β_0	SE	β_0	SE
Base	-0.1330	0.1894	NA	NA	NA	NA	NA	NA
Male	-0.1006	0.0830	NA	NA	NA	NA	NA	NA
Age	-0.0312	0.0024	NA	NA	NA	NA	NA	NA
Cohort2	0.1193	0.1530	NA	NA	NA	NA	NA	NA
Cohort3	-0.1045	0.1404	NA	NA	NA	NA	NA	NA
Cohort4	0.1892	0.1615	NA	NA	NA	NA	NA	NA
Cohort5	0.4319	0.1696	NA	NA	NA	NA	NA	NA
Cohort6	0.5746	0.2046	NA	NA	NA	NA	NA	NA
Grad	0.1224	0.2147	NA	NA	NA	NA	NA	NA
Predoc	0.0725	0.1025	NA	NA	NA	NA	NA	NA
Private	-0.1197	0.1443	NA	NA	NA	NA	NA	NA
XIX	-0.3671	0.1514	NA	NA	NA	NA	NA	NA

Those tables summarizes the quantile coefficient estimates and standard error from 5% quantiles to 25% quantiles. Among 5 covariates, we focused on age and cohort3 covariates. Figure 1(a) and 1(b) show that older age affects negatively to longevity of dental restoration for all quantiles, as most of survival data shows. It is an evidence that proposed smoothed estimator gives reasonable estimation result. Furthermore, coefficients of cohort3 shows signs of coefficient are reversed based on estimation quantiles. For 5% and 10% estimation quantiles, β_{cohort} are positive, on the other hand, beyond 15% qunatiles β_{cohort} are negative. It gives an evidence that the proposed smoothed estimator can shows the diffenrence of covariate effect depending on estimation quantiles.



(a) (b)

Figure 1: β is black line and 95% confidence interval is blue line. (a) 95% confidence interval of β_{age} are always less than 0. (b) Sign of $\beta_{cohort3}$ is changed depending on quantiles.

5. Discussion

6. Appendix

6.1 Asymptotic properties of the unsmoothed estimator

In this part, we establish the consistency and asymptotic normality of β. We first impose the regularity conditions.

A1 There exists $\nu \geq 0$ such that $P(C = \nu) \geq 0$ and $P(C \geq \nu) = 0$.

A2 X is uniformly bounded, that is $\sup_i \|X_i\| \leq \infty$.

A3 (i) $\beta_0(\tau)$ is Lipschitz continuous for $\tau \in [\tau_L, \tau_U]$;
(ii) $f_i(t|X)$ is bounded above uniformly in t and X,where $f_1(t|X) = dF_1(t|X)/dt$.

A4 For some $\rho_0 \geq 0$ and $c_0 \geq 0, inf_{b \in \beta(\rho_0)} eigmin A(b) \geq c_0$,
where $\beta(\rho) = b \in R^{P+1} : \inf_{\tau \in [\tau_L, \tau_U]} \|b - \beta_0(\tau)\| \leq \rho$ and $A(b) = E[Z^{\otimes 2} f_1\{\exp(X^T b)|X\}]$.
Here $\|\cdot\|$ is the Euclidean norm, and we define $u^{\otimes 2} = uu^T$ for a vector u.

Peng and Fine (2012) shows an estimator from similar estimating equation

$$S_n(\beta, \tau) = n^{-1} \sum_{i=1}^{n} X_i \Big(I\{log(Z_i - t_0) \leq X_i'\beta\} \frac{\delta_i}{\hat{G}(Z_i)} - \tau \Big) \tag{9}$$

is consistent and satisfies asymptotic normality under above regularity conditions. If we change first X_i to $X_i^\star$ where $X_i^\star = X_i I[Z_i > t_0]$, we are simply able to prove our suggested estimator β_τ is also consistent and asymptotically normal: $n^{1/2}(\hat{\beta} - \beta_0) \xrightarrow{d} N(0, \Gamma)$ where $\Gamma = A^{-1} \Sigma A^{-1}$, $A = \lim_{n \to \infty} X_i X_i' f_i(0|X_i)$, and $\Sigma = \lim_{n \to \infty} Var\{U_n(\beta_0)\}$

6.2 Asymptotic properties of the smoothed estimator

For prove theorem 1, we need following lemma 2.

Lemma 2. *Let $W = O(n^{-1})$ any positive definite matrix, and define*

$$\tilde{S}_n(\beta, \boldsymbol{W}) = \frac{1}{n}\sum_{i=1}^n I[Z_i \geq t_0] X_i \frac{\delta_i}{\hat{G}(Z_i)} \Phi\left(\frac{X_i'\beta - log(Z_i - t_0)}{\sqrt{X_i'\boldsymbol{W}X_i}}\right)$$

as the smoothed estimating function. Under condition 1-3, we have

$$\sup_{\|\beta-\beta_0\|\leq\epsilon_n} \|n^{1/2}\{\tilde{S}_n(\beta, \boldsymbol{W}) - S_n(\beta)\}\| \xrightarrow{p} 0, \text{ as } n \to \infty,$$

where ϵ_n is a positive sequence that converges to 0.

Proof of lemma 2
Let $\sigma_i = (X_i'WX_i)^{1/2}, \epsilon_i^\beta = X_i\beta - log(Z_i - t_0)$, and $d_i(\beta) = sgn(\epsilon_i^\beta)\Phi(-|\epsilon_i^\beta/\sigma_i|)$

$$\begin{aligned} n^{1/2}\{\tilde{S}_n(\beta, \mathbf{W}) - S_n)\} &= n^{-1/2}\sum_{i=1}^n I[Z_i \geq t_0] X_i \frac{\delta_i}{\hat{G}(Z_i)}\left\{\Phi\left(\frac{-\epsilon_i^\beta}{\sigma_i}\right) - I(\epsilon_i^\beta < 0)\right\} \\ &= n^{-1/2}\sum_{i=1}^n \frac{\delta_i}{\hat{G}(Z_i)} X_i^\star d_i(\beta) \end{aligned}$$

Where $X_i^\star = X_i I(Z_i > t_0)$.

Denote $D_n(\beta) = n^{-1/2}\sum_{i=1}^n \frac{\delta_i}{\hat{G}(Z_i)} X_i^\star d_i(\beta)$ and $D_n^G(\beta) = n^{-1/2}\sum_{i=1}^n \frac{\delta_i}{G(Z_i)} X_i^\star d_i(\beta)$. It follows that

$$D_n(\beta) = D_n^G(\beta) - n^{-1/2}\sum_{i=1}^n \frac{X_i^\star \delta_i(\hat{G}(Z_i) - G(Z_i))}{\hat{G}^2(Z_i)} d_i(\beta) + o_p(1) \tag{10}$$

Expectation of $D_n^G(\beta)$ is $E\{D_n^G(\beta)\} = n^{-1/2}\sum_{i=1}^n X_i^\star E d_i(\beta)$, Where

$$\begin{aligned} E\{d_i(\beta)\} &= \int_{-\infty}^{\infty} sgn(\epsilon_i^\beta)\Phi(-|\epsilon_i^\beta/\sigma_i|) f_i\{\epsilon_i + X_i^{\star\prime}(\beta - \beta_0)\} d\epsilon_i^\beta \\ &= \sigma_i \int_{-\infty}^{\infty} \Phi(-|t|)\{2I(t > 0) - 1\} f_i\{\sigma_i t + X_i^{\star\prime}(\beta - \beta_0)\} dt \\ &= \sigma_i \int_{-\infty}^{\infty} \Phi(-|t|)\{2I(t > 0) - 1\}[f_i\{\sigma_i t + X_i^{\star\prime}(\beta - \beta_0)\} + f_i'\{\omega_i^\star(t)\}\sigma_i t] dt \end{aligned}$$

Where f_i is the density of $\epsilon_i = \epsilon_i^{\beta_0}$, and $\omega_i^\star(t)$ is between $x_i^{\star\prime}(\beta - \beta_0)$ and $X_i^{\star\prime}(\beta - \beta_0) + \sigma_i t$. Note that for β that satisfies $\|\beta - \beta_0\| \leq \epsilon_n$, where $\epsilon_n \to 0$, we have $\|X_i^{\star\prime}(\beta - \beta_0)\| \to 0$. It follows from assumption B1 that $sup_i f_i\{X_i^{\star\prime}(\beta - \beta_0)\} < \infty$ and since $\int_{-\infty}^{\infty} \Phi(-|t|)\{2I(t > 0) - 1\} dt = 0$, we have $\int_{-\infty}^{\infty} \Phi(-|t|)\{2I(t > 0) - 1\} f_i\{X_i^{\star\prime}(\beta - \beta_0)\} dt = 0$. In addition, by assumption B1, we can find $M > 0$ such that $sup_i|f_i'\{\omega_i^\star(t)\}| < M$. Thus, it follows that

$$|E\{d_i(\beta)\}| \leq \int_{-\infty}^{\infty}|t|\Phi(|t|)|f_\beta, i'\{\omega_i^\star(t)\}|dt \leq M\sigma_i^2/2$$

where the last equality holds becuase $\int_{-\infty}^{\infty}|t|\Phi(\{-|t|\})dt = 1/2$.

By assumption B2 and the fact that $W = O(n^{-1})$, $\sum_{i=1}^n \sigma_i^2 = tr(X^\star W X^{\star\prime}) = tr(W X^{\star\prime} X^\star)$ is bounded, and $\sum_{i=1}^n|E\{d_i(\beta)\}| \leq M\sum_{i=1}^n \sigma_i^2/2$ is also bounded. Therefore,

$$\|E\{D_n^G(\beta)\}\| \leq n^{-1/2}\sqrt{p}\sup_{i,j}|X_{ij}^\star \sum_{i=1}^n|E\{d_i(\beta)\}| \to 0, \quad \text{as } n \to 0.$$

In addition, by assumption B3,

$$Var\{D_n^G(\beta)\} = \tfrac{1}{n}\sum_{i=1}^n X_i^\star X_i^{\star\prime} Var\left\{\tfrac{\delta_i}{G(Z_i)}d_i(\beta)\right\} \leq \tfrac{1}{n}\sum_{i=1}^n \tfrac{X_i^\star X_i^{\star\prime}}{\nu}E\{d_i^2(\beta)\}$$

where

$$\begin{aligned}
E\{d_i^2(\beta)\} &= \int_{-\infty}^{\infty}\Phi^2(-|s|)f_i\{\sigma_i s + X_i^\star(\beta-\beta_0)\}d(\sigma_i s)\\
&= \int_{|s|>\Delta}\Phi^2(-|s|)f_i\{\sigma_i s + X_i^\star(\beta-\beta_0)\}d(\sigma_i s) + \int_{|s|\leq\Delta}\Phi^2(-|s|)f_i\{\sigma_i s + X_i^\star(\beta-\beta_0)\}d(\sigma_i s)\\
&\leq \Phi^2(-\Delta) + \sigma_i\int_{|s|\leq\Delta} f_i\{\sigma_i s + X_i^\star(\beta-\beta_0)\}ds\\
&= \Phi^2(-\Delta) + 2\sigma_i\Delta f_i(\omega_i^\star).
\end{aligned}$$

Note that $\omega_i^\star \in (X_i^{\star\prime}(\beta-\beta_0)-\sigma_i\Delta, X_i^{\star\prime}(\beta-\beta_0)+\sigma_i\Delta)$. Let $\Delta = n^{1/4}$ and since $\sigma_i = O(n^{-1/2})$, both $\sigma_i\Delta$ and $\omega_i^\star$ go to 0 as n increases. As $f_i(\cdot)$ is uniformly bounded around zero, both $\Phi^2(-\Delta)$ and $\sigma_i\Delta f_i(\omega_i^\star)$ go to 0 as $n \to \infty$.

Thus, it follows that $\lim_{n\to\infty} E\{d_i^2(\beta)\} = 0$, and $\lim_{n\to\infty} Var\{D_n^G(\beta)\} = 0$. By the Weak Law of Large Numbers, for β that satisfies $\|\beta-\beta_0\| \leq \epsilon_n$, we have

$$\|D_n^G(\beta)\| \xrightarrow{p} 0, \quad \text{as } n \to \infty. \tag{11}$$

The second term on the right side of (10) can be written as

$$\begin{aligned}
&n^{-1/2}\sum_{i=1}^n\left\{\frac{\delta_i X_i^\star(\hat{G}(Z_i)-G(Z_i))}{G^2(Z_i)}\right\}d_i(\beta) + o_p(1)\\
&= n^{-1/2}\sum_{j=1}^n\int_0^L\left\{n^{-1}\sum_{i=1}^n\frac{\delta_i X_i^\star d_i(\beta Z_i(u))}{G(Z_i)}\right\}\frac{dM_j^c(u)}{y(u)} + o_p(1)
\end{aligned}$$

where $M_i^c(u) = N_i^c(u) - \int_0^t I(Z_i \geq u)d\Lambda^c(s)$, $N_i^c(u) = (1-\delta_i)I(Z_i \leq u)$, $\Lambda^c(u) = -log\{G(u)\}$ is the censoring cumulative hazard, $Z_i(u) = I(Z_i \geq u)$ is the ith at-risk process, and $y(u) = \lim_{n\to\infty} n^{-1}\sum_{i=1}^n Z_i(u)$ is bounded from below in $(0, L]$ by assumption B3.

Define $I_n(u,\beta) = n^{-1}\sum_{i=1}^n \frac{\delta_i X_i^\star d_i(\beta)Z_i(u)}{G(Z_i)}$, and $I(u,\beta) = E\{I_n(u,\beta)\}$. We have

$$I(u,\beta) = n^{-1}\sum_{i=1}^{n} X_i^\star E\{d_i(\beta)Z_i(u)\}$$

Where $|E\{d_i(\beta)Z_i(u)\}| \le E|d_i(\beta)|$, and

$$\begin{aligned}
E|d_i(\beta)| &= \int_{-\infty}^{\infty} \Phi(-|\epsilon_i^\beta/\sigma_i|) f_i\{\epsilon_i + X_i^{\star\prime}(\beta-\beta_0)\} d\epsilon_i^\beta \\
&= \sigma_i \int_{-\infty}^{\infty} \Phi(-|t|) f_i\{\sigma_i t + X_i^{\star\prime}(\beta-\beta_0)\} dt \\
&= \sigma_i f_i\{X_i^{\star\prime}(\beta-\beta_0)\} \int_{-\infty}^{\infty} \Phi(-|t|)dt + \sigma_i^2 \int_{-\infty}^{\infty} t\Phi(-|t|) f_i'\{\omega_i^\star(t)\}dt.
\end{aligned}$$

By assumption B1, we have $f_i\{X_i^{\star\prime}(\beta-\beta_0)\}\int_{-\infty}^{\infty}\Phi(-|t|)dt \le \infty$, and $\int_{-\infty}^{\infty} t\Phi(-|t|)f_i'\{\omega_i^\star\}dt \le \infty$. Thus, it follows that $E|d_i(\beta)| =)(n^{-1/2})$, and

$$\|I(u,\beta)\| \le \sqrt{p}\sup_{i,j}|X_{ij}^\star| n^{-1}\sum_{i=1}^{n} E|d_i(\beta)| = O(n^{-1/2}) \to 0.$$

Define $\mathcal{F} = \frac{\delta_i X_i^\star d_i(\beta) Z_i(u)}{G(Z_i)}$, $\|\beta-\beta_0\| \le \epsilon_n$ and $u \in (0,\infty)$. The function class $\mathcal{F}$ is Gilvenko-Cantelli (van der Vaart and Wellner, 1996) because the class of indicator functions is Gilvenko-Cantelli, and $X_i^\star, d_i(\beta), and 1/G(Z_i)$ are uniformly bounded. it follows that $sup_{\|\beta-\beta_0\|\le\epsilon_n, u\in(0,\infty)}\|I_n$ $I(u,\beta)\| \xrightarrow{a.s.} 0$ and we have

$$n^{-1/2}\sum_{j=1}^{n}\int_0^L I_n(u,\beta)\frac{dM_j^c(u)}{y(u)} = n^{-1/2}\sum_{j=1}^{n}\int_0^L I(u,\beta)\frac{dM_j^c(u)}{y(u)} + o_p(1)$$

By the Martingale Central Limit Theorem (Fleming and Harrington, 1991), $n^{-1/2}\sum_{j=1}^{n}\int_u^\beta \frac{dM_j^c(u)}{y(u)}$ is $o_p(1)$ as n goes to infinity. It follows that, for β that satisfies $\|\beta-\beta_0\| \le \epsilon_n$,

$$\left\| n^{-1/2}\sum_{j=1}^{n}\int_0^L I_n(u,\beta)\frac{dM_j^c(u)}{y(u)} \right\| \xrightarrow{p} 0 \tag{12}$$

Collating (11) and (12), we have

$$\sup_{\|\beta-\beta_0\|\le\epsilon_n} \|n^{1/2}\{\tilde{S}_n(\beta,\mathbf{W}) - S_n(\beta)\}\| \xrightarrow{p} 0.$$

for any β such that $\|\beta-\beta_0\| \le \epsilon_n$. Lemma 2 is thus proven by the fact that both $\tilde{S}_n(\beta,\mathbf{W})$ and $S_n(\beta)$ are monotone functions, thus the point-wise covergence could be strengthened to uniform convergence (Shorack and Wellner, 1986).

Proof of Theorem 1

After applying induced smoothing method, $\hat{\beta}_{IS} = \beta_0 + \mathbf{H}^{1/2}V$ where $\mathbf{H} = n^{-1}\Gamma$ and $V \sim \mathcal{N}(0, I_p)$. Since $\hat{\beta}_{IS}$ is a solution of

$$\tilde{U}_n(\hat{\beta}_{IS},\tilde{\mathbf{H}}) = n^{-1}\sum_{i=1}^{n} I[Z_i \ge t_0]X_i\left\{\tau - \frac{\delta_i}{\hat{G}(Z_i)}\Phi\left(\frac{X_i'\hat{\beta}_{IS} - log(Z_i - t_0)}{\sqrt{X_i'\tilde{\mathbf{H}}X_i}}\right)\right\} = 0 \tag{13}$$

Using Taylor expansion, we have

$$\sqrt{n}(\hat{\beta}_{IS} - \beta_0) = \frac{-\sqrt{n}\tilde{U}_n(\beta_0, \tilde{\mathbf{H}})}{\tilde{U}_n'(\beta_0, \tilde{\mathbf{H}})} \tag{14}$$

By Lemma (2), $-\sqrt{n}\tilde{U}_n(\beta_0, \tilde{\mathbf{H}}) \xrightarrow{p} -\sqrt{n}\tilde{U}(\beta_0)$, and Kim et al (2012) shows $-\sqrt{n}\tilde{U}(\beta_0) \xrightarrow{d} \mathcal{N}(0, \Sigma)$. Since $\tilde{\mathbf{H}} = O(n^{-1})$, $\tilde{U}_n'(\hat{\beta}_0, \tilde{\mathbf{H}}) = \tilde{A}_n(\beta_0, \tilde{\mathbf{H}}) \xrightarrow{p} A$. In sum, $\sqrt{n}(\hat{\beta}_{IS} - \beta_0) \xrightarrow{d} \mathcal{N}(0, \Gamma)$, where $\Gamma = A^{-1}\Sigma A^{-1}$.