1. induced smoothing method to Quantile residual life model fitting estimating equation
   Estimaitng Equation

$$S(\beta) = \textstyle\sum_{i=1}^{n} w_i I[Z_i > t_0] X_i \psi_\tau\{\log(Z_i - t_0) - X_i^T \beta\} = 0$$

   Using induced smoothing

$$\begin{aligned}
\bar{S}(\beta) &= E_D[S(\beta + \Gamma^{\frac{1}{2}} D)] \\
&= E_D[\sum_{i=1}^{n} w_i I[Z_i > t_0] X_i \psi_\tau\{\log(Z_i - t_0) - X_i^T(\beta + \Gamma^{\frac{1}{2}} D)\}] \\
&= E_D[\sum_{i=1}^{n} w_i I[Z_i > t_0] X_i (\tau - I\{\log(Z_i - t_0) - X_i^T(\beta + \Gamma^{\frac{1}{2}} D) \leq 0\})] \\
&= \sum_{i=1}^{n} w_i I[Z_i > t_0] X_i (\tau - E_D[I\{\log(Z_i - t_0) - X_i^T(\beta + \Gamma^{\frac{1}{2}} D) \leq 0\}]) \\
&= \sum_{i=1}^{n} w_i I[Z_i > t_0] X_i (\tau - P[X_i^T \Gamma^{\frac{1}{2}} D \geq \log(Z_i - t_0) - X_i^T \beta]) \\
&= \sum_{i=1}^{n} w_i I[Z_i > t_0] X_i (\tau - P[D \geq \frac{\log(Z_i - t_0) - X_i^T \beta}{\sqrt{X_i^T \Gamma X_i}}]) \\
&= \sum_{i=1}^{n} w_i I[Z_i > t_0] X_i (\tau - 1 + \Phi(\frac{\log(Z_i - t_0) - X_i^T \beta}{\sqrt{X_i^T \Gamma X_i}}))
\end{aligned}$$

   where $D \sim N_P(0, I_P)$, $Z_i = min(T_i, C_i)$ and X_i is covariate.

2. Sandwich estimator $A(\beta)$

$$
\begin{aligned}
A(\beta) &= \frac{\partial U(\beta)}{\partial \beta} \\
&= \frac{\partial \sum w_i I[Z_i > T_0] X_i \left(\tau - 1 + \Phi\left(\frac{\log(Z_i - t_0) - {X_i}^T \beta}{\sqrt{{X_i}^T \Gamma X_i}} \right) \right)}{\partial \beta} \\
&= \sum w_i I[Z_i > T_0] X_i \phi\left(\frac{\log(Z_i - t_0) - {X_i}^T \beta}{\sqrt{{X_i}^T \Gamma X_i}} \right) \left(\frac{-X_i}{\sqrt{{X_i}^T \Gamma {X_i}^T}} \right)
\end{aligned}
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