Smoothed quantile regression for residual lifetime in censored data

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Abstract

We suggest a quantile regression method for residula lifetime given the prognostic factors and treatments at a ceratin followup time. To improve computational efficiency, we smooth an estimating function using induced smoothing approaches, and use mulipliер bootstrap and sandwich estimator method in finding a covariance matrix of regression estimator. We verify the performance of estimator in different combinationS of followup time point and censoring proportion of survival data in simulation settings, and apply proposed regression method to the longevity of dental restoration data from the Geriatric and Special Needs Dentistry Clinic at the University of Iowa Colloege of Dentistry (COD). We also establish asymptotic normality and consistency of proposed estimator.

Keywords: Induced smoothing, Quantile regression, Residual lifetime regression, Sandwich estimator, Survival analysis

1 Introduction

In general, most of medical or biological researches are interested in how long patient will survive or how long effect of medicine or treatment lasts given a certain circumstance. The ultimate goal of both fields are estimating a remaining lifetime, and only difference is residual duration of what. For this reason, plenty of studies have focused on estimating mean survival time by various methods. However, regressions based on mean are not appropriate to estimate residual lifetime due to heterogeneity in survival data. To makeup this disadvantage of mean-based regression, past statisticians found an alternative ways, and most popular approach was quantile-based regression.

First quantile regression model was introduced by Koenker and Bassett Jr (1978), and lots of studies have been derived from this idea. Jung et al. (2009) proposed a time-specific log-linear regression method that studies covariate effects on the conditional quantiles of residual lifetimes on fixed followup time point, and Kim et al. (2012) suggested advanced regression applied inverse probability of censoring weighted (IPCW) estimator of Van der Laan et al. (2003), and an alternative way to estimte covariance matrix of the estimator using a simple empirical likelihood inference method. Those two apporaches find their estimators to solve unsmoothed estimating equations using linear programming, however this approach is computationally inefficient. Furthermore, it also caused the difficulty in variance estimation for censored quantile regression. Thus, some statisticians dealt with this problem applying alternative methods, such as Kim et al. (2012) and Jung et al. (2009). However, they do not perfectly overcome disadvatage of unsmoothed estimating equation.

Fortunately, Brown and Wang (2007) introduced an induced smoothing method to solve unsmoothed issue. The induced smoothing idea smoothes unsmoothed estimating function by taking its expectation under the distribution of random perturbation. This concept makes solving estimating equation to computationally efficient, and also gives an opportunity to estimate variance. We suggest a variance estimation procedure derived from an approach of Chiou et al. (2015) based on multiplier boothstrapping and sandwich estimator. Because a sandwich estimator method is developed to estimate variance with computational efficiency, if an estimator satisfies asymptotic properties and consistency, it is very useful to estimate variance.

The rest of the article is organized as follows. In section 2, we introduce the process how to approach smoothed quantile regression estimator for residual lifetime. We discuss simulation results in section 3, and real data analysis in section 4. Discussion and some necessary proofs are at the end of this article.

2 Censored quantile regression for residual lifetime with induced smoothing

2.1 Semiparametric quantile model for residual lifetime of not censored data

We start from scenario where random sample subject to no censoring: $\{T_i, X_i\}_{i=1}^n$, where T_i, X_i denote the failure time, and covariate values of i^{th} subject respectively. $\theta_\tau(t_0)$ is τ-th quantile of residual lifetime at followup time t_0, and we simply express it as θ_τ. Corresponding to the AFT model and regression quantiles of residual lifetime, the τ-th quantile of residual lifetime based on a certain followup time t_0 given the covariate X_i is given by

$$\theta_\tau = log(T_i - t_0) = X_i'\beta(\tau, t_0) + \epsilon_i \tag{1}$$

where $\beta(\tau, t_0)$ represents τ-th quantile regression coefficient given covariate X_i and followup time t_0, and we simply use β instead of $\beta(\tau, t_0)$. ϵ_i is independent and have zero τ-th quantile.

Applying an idea of check function, introduced by Koenker and Bassett Jr (1978) and an estimating fucntion, suggested by Koenker and Regression (2005), τ-th quantile residual lifetime is a solution of derived estimating equation of no censroed data based on certain base time t_0 is:

$$0 = n^{-1} \sum_{i=0}^{n} I[T_i \geq t_0] X_i \Big(I\{\log(T_i - t_0) \leq X_i'\beta\} - \tau \Big) \tag{2}$$

2.2 Semiparametric quantile model for residual lifetime of censored data

Next we consider a concept of censoring in survival data. Suppose n iid $\{T_i, X_i\}_{i=1}^n$ are generated from model (1), and are right censored data: $\{Z_i, \delta_i, X_i\}_{i=1}^n$ where $Z_i = min(T_i, C_i), \delta_i = I[T_i \leq C_i]$ are independent. For dealing with censored data, we need to modify estimating equation adding weight w_i, which is the probability that Kaplan-Meier estimator based on $\{Z_i, \delta_i\}_{i=1}^n$ assigns on the case (Z_i, δ_i). Among many options, we use inverse probability of censoring weighting (IPCW) technique Robins and Rotnitzky (1992), as suggested at Kim et al. (2012) and Pang et al. (2012). Therefore, estimating equation of censored regression residual quantile estimator is:

$$U_n(\beta) = 0 = n^{-1} \sum_{i=1}^{n} I[Z_i \geq t_0] X_i \frac{\delta_i}{\hat{G}(Z_i)} \Big(I\{\log(Z_i - t_0) \leq X_i'\beta\} - \tau \Big) \tag{3}$$

where $\hat{G}(Z_i)$ is the Kaplan-Meier estimate of the survival unction of the censoring variable C_i. We prove consistency and asymptotic properties of unsmoothed estimator in appendix.

2.3 Induced smoothing approach

In Jung et al. (2009) and Kim et al. (2012), they find β by solving estimating equation using linear programming. Unfortunately, it has intensive computation issue, because their estimating functions are step function. To achive computational efficiency, we apply induced smoothing method, proposed by Brown and Wang (2007). By the asymptotic normality of $\hat{\beta}$, we can express $\hat{\beta} = \beta + \mathbf{H}^{1/2}V$, where $H = n^{-1}\Gamma$, $V \sim N(0, I_p)$, and I_p is the $p \times p$ identity matrix. After applying induced smoothing approach to equation (3), we define the residual quantile regression smoothed esimator at the followup time t_0 by a solution to

$$\begin{aligned} \tilde{U}_n(\beta, \mathbf{H}) &= E_v\{U_n(\beta + \mathbf{H}^{1/2}V)\} \\ &= n^{-1}\sum_{i=1}^{n} I[Z_i \geq t_0]X_i \frac{\delta_i}{\hat{G}(Z_i)}\left\{\Phi\Big(-\frac{log(Z_i - t_0) - X_i'\beta}{\sqrt{X_i'\mathbf{H}X_i}}\Big) - \tau\right\} \end{aligned} \tag{4}$$

or equivalently a solution to

$$\begin{aligned} \min_{\beta} n^{-1}\sum_{i=1}^{n} I[Z_i > t_0][log(Z_i - t_0) - X_i^T\beta]\frac{\delta_i}{\hat{G}(Z_i)}&\left\{\tau\Phi\Big(-\frac{log(Z_i - t_0) - X_i'\beta}{\sqrt{X_i'\mathbf{H}X_i}}\Big)\right\} \\ &+\phi\Big(-\frac{log(Z_i - t_0) - X_i'\beta}{\sqrt{X_i'\mathbf{H}X_i}}\Big)\sqrt{X_i'\mathbf{H}X_i} \end{aligned} \tag{5}$$

where $\Phi(\cdot)$ is the standard normal cumulative distribution function. As Pang et al. (2012) proposed, we use a positive definite $p \times p$ matrix $\tilde{\mathbf{H}} = O(n^{-1})$ as a smoothing matrix $\mathbf{H}$.

Estimated β from equation (4) is also consistent and asymptotically normal by Theorem 1 under below regularity conditions C1-C3:

C1 The conditional error distribution functions, $F_i(\cdot \mid X_i)_{i=1}^n$, are absolutely continuous with continuous densities $f_i(\cdot \mid X_i)$ uniformly bounded away from 0 and ∞ in a neighborhood of 0, and $f_i'(\cdot \mid X_i)$ exists and is uniformly bounded on the real line.

C2 For each $i = 1, \ldots, n, X_i$ satisfies the following conditions:

(a) $n^{-1}\Sigma_{i=1}^n X_i X_i' f_i(0 \mid X_i)$ converges to a positive definite matrix $\mathbf{A}$;
(b) $\sup_i \|X_i\| < \infty$, where $\|\cdot\|$ denotes the Euclidean norm.

C3 There exists $L > 0$ such that $P(C > L) = 0$ and $P(C = L) \geq \nu$, where ν is some positive constant.

Theorem 1. *Assume condition* $1-3$ *hold, and the smoothing matrix* H *is positive definite and* $O(n^{-1})$*, as* $n \to \infty$ *then we have*

$$n^{1/2}(\hat{\beta}_{IS} - \beta_0) \xrightarrow{d} \mathcal{N}(0, \Gamma)$$

We prove above Theorem 1 in appendix.

To estimate the variance-covariance matrix of $\hat{\beta}_{IS}$, we use sandwich estimator, which is computationally more efficient than full-multiplier bootstrap approach based on Chiou et al. (2015). For sandwich estimator, $\Sigma(\beta_0)$, two estimators, $A(\beta_0)$ and $V(\beta_0)$, are necessary. For estimating $A(\beta_0)$, we evaluate $A_n(\beta_0)$, the derivative of the smoothed estimating function:

$$A_n(\beta_0) = \frac{\partial \tilde{U}_n(\hat{\beta}_{IS}, \tilde{\mathbf{H}})}{\partial \beta}$$
$$= n^{-1} \sum_{i=1}^{n} I[Z_i > t_0] X_i \frac{\delta_i}{\hat{G}(Z_i)} \phi\left(\frac{X_i{}'\hat{\beta}_{IS} - \log(Z_i - t_0)}{\sqrt{X_i{}'\tilde{\mathbf{H}}X_i}}\right)\left(\frac{-X_i}{\sqrt{X_i{}'\tilde{\mathbf{H}}X_i}}\right) \tag{6}$$

at $\hat{\beta}_{IS}$, which is the solution of estimating equation (4), where $\phi\{\cdot\}$ is the density function of a standard normal distribution.

For estimating $V(\beta_0)$, we generate iid positive multiplier $\eta_i, i = 1, ..., n$, that are independent of the observed data from exp(1), and make perturbed estimating equation, $\tilde{U}_n^\star(\beta, \mathbf{H})$, with generated η_i:

$$\tilde{U}_n^\star(\beta, \tilde{\mathbf{H}}) = n^{-1} \sum_{i=1}^{n} I[Z_i \geq t_0] X_i \frac{\delta_i}{\hat{G}(Z_i)} \eta_i \left\{ \Phi\left(\frac{X_i'\beta - log(Z_i - t_0)}{\sqrt{X_i'\tilde{\mathbf{H}}X_i}}\right) - \tau \right\} \tag{7}$$

We evaluate $\tilde{U}_n^\star(\beta, \tilde{\mathbf{H}})$ at $\hat{\beta}_{IS}$. Repeating same process m times with new multipliers, and find sample variance of $\{\tilde{U}_n^{\star(1)}(\beta, \tilde{\mathbf{H}}), ..., \tilde{U}_n^{\star(m)}(\beta, \tilde{\mathbf{H}})\}$. $V(\beta_0)$ is approxiamated to sample variance of $\tilde{U}_n^\star(\beta, \tilde{\mathbf{H}})$.

Collating above $A(\beta_0)$, and $V(\beta_0)$, $\Sigma(\beta_0)$ is:

$$\Sigma(\beta_0) = A(\beta_0)^T V(\beta_0) A(\beta_0) \tag{8}$$

3 Simulation

To verify performance of smoothed estimator, we use a simple regression simulation setting that provided by Jung et al. (2009). Covariate X_i is a binary covariate, 0 for control and 1 for treatment group. $Z_i = min(T_i, C_i)$ where T_i generated from Weibull regression model with one binary covariate X_i and intercept, and where C_i generated from uniform distribution with range 0 and c, which is adjusted for censoring proportion. Size of single dataset was 200, and 2000 simulations were performed for every combination of t_0 and censoring proportion. In suggested estimating eqaution, equation (4), we need to decide how to calculate IPCW weight. Most popular and easiest ways is calculating $\hat{G}(Z_i)$ using survfit function of survival R package, and manually finding IPCW weight respectively. Alternative way is using jump weight, which is introduced by Stute (1996), as IPCW weight, and it is implemented by emplik R package. Equivalance of those two weight is proved by Zhou et al. (2012), and also we cannot find difference from simulation result using both weights. Furthermore, we changed δ_i of last subject to 1 manually. The reason is when we assume the last subject is uncensored, performance of simulation was slightly better than before, and also other article reports similar results.

In variance estimating process, we need to decide number of estimates of $\tilde{U}_n^\star(\beta, \tilde{\mathbf{H}})$ to find sample variance. Comparing sample variance of 100 estimates of $\tilde{U}_n^\star(\beta, \tilde{\mathbf{H}})$ and that of 500 estimates, we conclude that 100 estimates of $\tilde{U}_n^\star(\beta, \tilde{\mathbf{H}})$ are close enough to $V(\beta_0)$. For the computational efficiency, small number of estimates is more preferred.

From Table 1 to Table 3, we verify the performance of suggested estimator when covariate does not affect residual lifetime, which means identitcal survival distribution was assumed for both groups and $\beta_{t_0}^{(1)} = 0$ for all $t_0/geq0$. Based on Jung et al. (2009) simulation setting, true parameter $\beta_{t_0}^{(0)} = 1.61, 1.41, 1.22, 1.04$ at $t_0 = 0, 1, 2, 3$. $\beta^{(0)}$ and $\beta^{(1)}$ are sample means of true paramters, $\beta^{(0)}, \beta^{(1)}$, estimation using proposed method, and SE is mean of standard error of estimates using proposed variance estimation method. SD means sample standard deviation of estimates, and CV is a proportion that true parameter are included in 95% confidence interval of proposed estimates.

Table 1: Estimates of 25% quantile residual lifetime when $\beta^{(1)} = 0$

t_0	C%	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	CV	$\beta^{(1)}$	SE	SD	CV
0	0%	1.603	0.096	0.096	0.917	-0.004	0.137	0.138	0.935
	10%	1.604	0.099	0.099	0.911	-0.004	0.141	0.143	0.922
	30%	1.603	0.107	0.107	0.910	-0.003	0.155	0.157	0.922
	50%	1.604	0.138	0.128	0.911	-0.009	0.200	0.204	0.915
1	0%	1.406	0.117	0.118	0.914	-0.000	0.167	0.167	0.936
	10%	1.405	0.121	0.122	0.905	-0.004	0.174	0.168	0.939
	30%	1.404	0.135	0.128	0.915	-0.008	0.196	0.189	0.934
	50%	1.401	0.178	0.163	0.919	-0.003	0.261	0.264	0.915
2	0%	1.215	0.139	0.139	0.893	-0.013	0.200	0.189	0.934
	10%	1.214	0.143	0.144	0.892	-0.005	0.207	0.202	0.922
	30%	1.212	0.158	0.154	0.898	-0.006	0.233	0.227	0.920
	50%	1.205	0.218	0.208	0.883	-0.005	0.322	0.341	0.890
3	0%	1.035	0.154	0.161	0.890	-0.004	0.225	0.224	0.919
	10%	1.033	0.162	0.161	0.889	-0.004	0.239	0.232	0.926
	30%	1.036	0.187	0.184	0.884	-0.005	0.277	0.273	0.908
	50%	1.023	0.262	0.249	0.879	-0.015	0.392	0.411	0.888

Table 2: Estimates of 50% quantile residual lifetime when $\beta^{(1)} = 0$

t_0	C%	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	CV	$\beta^{(1)}$	SE	SD	CV
0	0%	1.606	0.069	0.068	0.929	-0.001	0.098	0.096	0.943
	10%	1.606	0.072	0.072	0.928	0.000	0.103	0.102	0.937
	30%	1.605	0.086	0.082	0.931	-0.000	0.123	0.120	0.943
	50%	1.606	0.134	0.118	0.913	-0.006	0.195	0.198	0.921
1	0%	1.408	0.084	0.086	0.916	-0.004	0.120	0.122	0.937
	10%	1.408	0.089	0.087	0.923	-0.004	0.128	0.126	0.940
	30%	1.410	0.110	0.105	0.922	-0.004	0.158	0.157	0.931
	50%	1.396	0.186	0.166	0.892	0.004	0.277	0.291	0.906
2	0%	1.219	0.100	0.103	0.904	-0.011	0.143	0.146	0.934
	10%	1.217	0.110	0.109	0.902	-0.003	0.157	0.157	0.929
	30%	1.213	0.141	0.130	0.924	0.003	0.202	0.198	0.930
	50%	1.212	0.247	0.221	0.874	-0.008	0.379	0.398	0.885
3	0%	1.035	0.120	0.121	0.906	-0.002	0.173	0.168	0.933
	10%	1.034	0.133	0.133	0.908	0.006	0.192	0.187	0.934
	30%	1.027	0.174	0.170	0.894	0.007	0.259	0.260	0.913
	50%	1.028	0.320	0.293	0.820	-0.018	0.487	0.523	0.820

Table 3: Estimates of 75% quantile residual lifetime when $\beta^{(1)} = 0$

t_0	C%	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	CV	$\beta^{(1)}$	SE	SD	CV
0	0%	1.609	0.059	0.058	0.930	0.004	0.083	0.082	0.936
	10%	1.609	0.064	0.062	0.926	0.004	0.091	0.089	0.941
	30%	1.610	0.084	0.078	0.927	0.002	0.120	0.117	0.933
	50%	1.612	0.150	0.135	0.855	-0.007	0.221	0.237	0.861
1	0%	1.409	0.073	0.074	0.925	0.001	0.104	0.103	0.932
	10%	1.407	0.082	0.082	0.916	0.004	0.116	0.116	0.934
	30%	1.411	0.116	0.109	0.917	0.001	0.168	0.170	0.920
	50%	1.392	0.176	0.193	0.718	-0.015	0.259	0.351	0.672
2	0%	1.216	0.090	0.091	0.914	0.002	0.129	0.127	0.935
	10%	1.216	0.107	0.102	0.912	0.001	0.153	0.146	0.937
	30%	1.210	0.165	0.153	0.886	0.001	0.253	0.246	0.902
	50%	1.145	0.169	0.238	0.611	-0.009	0.259	0.440	0.556
3	0%	1.033	0.114	0.119	0.893	0.003	0.166	0.165	0.917
	10%	1.031	0.139	0.138	0.888	0.000	0.203	0.201	0.920
	30%	1.035	0.232	0.218	0.843	-0.001	0.373	0.351	0.854
	50%	0.919	0.162	0.283	0.568	-0.039	0.270	0.514	0.504

In every combination of t_0, censoring proportion, and estimated quantile, proposed estimators show quite close value to their true parameter, and CV rate is realiable under

50% censoring proportion. In our median estimating simulation setting, we first generated dataset with 200 subjects, however it decreased datasize to $187, 168, 146$ depends on changing $t_0 = 1, 2, 3$. Furthermore, a dataset with high censoring proportion made number of perfect information data smaller, and it caused difficulty to estimate accurate value. In simuation setting to estimate 75% quantile from dataset with 70% censoring proportion, nearly 25% of datasets did not provide reasonable estimation. It comes from nleqslv function in nleqslv R package to solve proposed estimating equation (4), however other non-linear equation solvers also have similar issues. To support the conjecture that insufficient dataset size cause inaccurate estimation, we did same simulation with double up datasize from 200 to 400, and we got more accurate and stable result than before.

From Table 4 to Table 6, we verify that the smoothed estimator is able to estimate an effect of covariate. In this simulation scenario, we add one more assumption which is the difference in residual time between two groups is 5. In this scenario, true parameter $\beta^{(0)}_{t_0} = 1.61, 1.41, 1.22, 1.04$ at $t_0 = 0, 1, 2, 3$ and $\beta^{(1)}_{t_0} = 0.69, 0.80, 0.91, 1.02$ at $t_0 = 0, 1, 2, 3$.

Table 4: Estimates of 25% quantile residual lifetime when $\beta^{(1)} \neq 0$

t_0	C%	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	CV	$\beta^{(1)}$	SE	SD	CV
0	0%	1.603	0.097	0.096	0.917	0.690	0.138	0.138	0.935
	10%	1.604	0.098	0.098	0.912	0.689	0.142	0.142	0.934
	30%	1.603	0.103	0.104	0.906	0.691	0.158	0.153	0.938
	50%	1.602	0.113	0.113	0.911	0.693	0.231	0.173	0.969
1	0%	1.403	0.117	0.117	0.915	0.792	0.160	0.162	0.920
	10%	1.403	0.119	0.120	0.899	0.796	0.166	0.162	0.932
	30%	1.412	0.124	0.125	0.906	0.782	0.185	0.171	0.940
	50%	1.404	0.140	0.141	0.897	0.793	0.279	0.208	0.964
2	0%	1.221	0.136	0.135	0.899	0.879	0.183	0.176	0.927
	10%	1.210	0.141	0.139	0.903	0.888	0.190	0.185	0.927
	30%	1.217	0.149	0.146	0.895	0.872	0.219	0.203	0.929
	50%	1.212	0.169	0.173	0.888	0.880	0.329	0.249	0.962
3	0%	1.030	0.157	0.156	0.890	0.966	0.207	0.201	0.910
	10%	1.030	0.161	0.162	0.894	0.963	0.216	0.206	0.915
	30%	1.033	0.170	0.177	0.879	0.962	0.245	0.232	0.920
	50%	1.026	0.196	0.201	0.875	0.973	0.382	0.288	0.959

Table 5: Estimates of 50% quantile residual lifetime when $\beta^{(1)} \neq 0$

t_0	C%	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	CV	$\beta^{(1)}$	SE	SD	CV
0	0%	1.606	0.069	0.068	0.930	0.693	0.098	0.096	0.941
	10%	1.606	0.071	0.071	0.927	0.693	0.104	0.100	0.946
	30%	1.605	0.078	0.077	0.929	0.694	0.128	0.112	0.963
	50%	1.604	0.093	0.092	0.923	0.696	0.242	0.146	0.979
1	0%	1.407	0.084	0.083	0.926	0.787	0.114	0.114	0.930
	10%	1.409	0.087	0.087	0.920	0.785	0.121	0.121	0.937
	30%	1.407	0.096	0.096	0.919	0.789	0.155	0.135	0.959
	50%	1.404	0.121	0.123	0.913	0.793	0.284	0.193	0.974
2	0%	1.212	0.100	0.101	0.920	0.886	0.133	0.128	0.933
	10%	1.214	0.106	0.104	0.925	0.884	0.142	0.138	0.937
	30%	1.209	0.118	0.119	0.906	0.890	0.184	0.160	0.949
	50%	1.207	0.150	0.151	0.890	0.889	0.332	0.227	0.961
3	0%	1.037	0.120	0.119	0.901	0.968	0.153	0.147	0.923
	10%	1.031	0.127	0.125	0.911	0.968	0.166	0.158	0.914
	30%	1.041	0.141	0.139	0.899	0.957	0.219	0.184	0.945
	50%	1.023	0.186	0.192	0.891	0.970	0.377	0.280	0.955

Table 6: Estimates of 75% quantile residual lifetime when $\beta^{(1)} \neq 0$

t_0	C%	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	CV	$\beta^{(1)}$	SE	SD	CV
0	0%	1.609	0.059	0.058	0.925	0.697	0.083	0.082	0.938
	10%	1.609	0.062	0.061	0.927	0.698	0.091	0.087	0.949
	30%	1.609	0.072	0.069	0.925	0.697	0.129	0.103	0.965
	50%	1.610	0.096	0.104	0.898	0.629	0.155	0.138	0.967
1	0%	1.409	0.072	0.074	0.911	0.793	0.097	0.097	0.939
	10%	1.411	0.076	0.076	0.920	0.793	0.107	0.104	0.935
	30%	1.410	0.092	0.089	0.918	0.792	0.166	0.128	0.965
	50%	1.411	0.134	0.144	0.888	0.668	0.176	0.183	0.902
2	0%	1.216	0.091	0.091	0.914	0.888	0.118	0.117	0.929
	10%	1.216	0.097	0.101	0.898	0.889	0.131	0.128	0.918
	30%	1.218	0.116	0.117	0.891	0.886	0.212	0.158	0.953
	50%	1.206	0.158	0.197	0.828	0.727	0.191	0.239	0.825
3	0%	1.033	0.114	0.115	0.904	0.970	0.142	0.140	0.908
	10%	1.030	0.126	0.126	0.893	0.979	0.163	0.155	0.928
	30%	1.028	0.151	0.153	0.875	0.972	0.254	0.199	0.946
	50%	1.014	0.200	0.240	0.796	0.800	0.234	0.281	0.782

Table 7: Comparison of two estimators for median quantile: sample mean of 2,000 median estimates based on n=200 datasize when $\beta^{(1)} \neq 0$

t_0	C%	True beta	Proposed method		Quantreg package(rq function)	
			$\beta^{(0)}(SE)$	$\beta^{(1)}(SE)$	$\beta^{(0)}(SE)$	$\beta^{(1)}(SE)$
0	0%		1.606(0.069)	0.693(0.098)	1.611(0.074)	0.687(0.107)
	10%	1.61	1.606(0.071)	0.693(0.104)	1.607(0.077)	0.695(0.113)
	30%	0.69	1.605(0.078)	0.694(0.128)	1.606(0.084)	0.697(0.140)
	50%		1.604(0.093)	0.696(0.242)	1.605(0.102)	0.703(0.239)
1	0%		1.407(0.084)	0.787(0.114)	1.413(0.090)	0.780(0.124)
	10%	1.41	1.409(0.087)	0.785(0.121)	1.411(0.095)	0.786(0.132)
	30%	0.80	1.407(0.096)	0.789(0.155)	1.409(0.105)	0.790(0.168)
	50%		1.404(0.121)	0.793(0.284)	1.405(0.130)	0.805(0.284)
2	0%		1.212(0.100)	0.886(0.133)	1.219(0.108)	0.877(0.143)
	10%	1.22	1.214(0.106)	0.884(0.142)	1.215(0.114)	0.885(0.154)
	30%	0.91	1.209(0.118)	0.890(0.184)	1.211(0.128)	0.892(0.199)
	50%		1.207(0.150)	0.889(0.332)	1.208(0.162)	0.908(0.335)
3	0%		1.037(0.120)	0.968(0.153)	1.046(0.128)	0.957(0.165)
	10%	1.04	1.031(0.127)	0.968(0.166)	1.034(0.138)	0.968(0.179)
	30%	1.02	1.041(0.141)	0.957(0.219)	1.044(0.155)	0.958(0.236)
	50%		1.023(0.186)	0.970(0.377)	1.031(0.205)	0.992(0.387)

Next, we compare the performance of our proposed estimator to that of R software quantreg package, especially rq function. To compare the performance in similar setting, we applied same IPCW weight from jump estimates of WKM function (R software emplik) package to both simulation.

Table 7 verifies the proposed method provide similar estimation value, but lower standard error than existing R software quantreg package.

Table 8: Various quantile estimates of residual lifetime with high censoring (70%) when $\beta^{(1)} = 0$. When estimating quantile is high, results show big differences with true beta, or cannot estimate.

t_0	τ	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	CV	$\beta^{(1)}$	SE	SD	CV
0	25%	1.571	0.234	0.235	0.810	0.013	0.366	0.441	0.855
	50%	1.555	0.162	0.231	0.594	-0.005	0.243	0.433	0.521
	75%	1.412	0.075	0.157	0.514	0.001	0.117	0.299	0.371
1	25%	1.359	0.306	0.319	0.772	-0.001	0.471	0.599	0.791
	50%	1.293	0.159	0.301	0.480	-0.032	0.246	0.566	0.344
	75%	1.058	0.084	0.190	0.045	-0.021	0.142	0.367	0.357
2	25%	1.133	0.389	0.414	0.732	0.008	0.594	0.769	0.754
	50%	1.014	0.181	0.365	0.458	-0.005	0.289	0.682	0.345
	75%	0.697	0.103	0.240	0.032	0.041	0.165	0.423	0.357
3	25%	0.942	0.411	0.502	0.690	0.000	0.647	0.922	0.697
	50%	0.787	0.191	0.408	0.512	-0.055	0.336	0.761	0.366
	75%	0.421	0.101	0.251	0.031	-0.001	0.167	0.439	0.343

Table 9: Various quantile estimates of residual lifetime with high censoring (70%) when $\beta^{(1)} \neq 0$. When estimating quantile is high, results show big differences with true beta, or cannot estimate.

t_0	τ	$\beta^{(0)}$				$\beta^{(1)}$			
		$\beta^{(0)}$	SE	SD	CV	$\beta^{(1)}$	SE	SD	CV
0	25%	1.596	0.165	0.189	0.876	0.655	0.346	0.395	0.856
	50%	1.600	0.156	0.211	0.780	0.462	0.184	0.349	0.813
	75%	1.530	0.104	0.161	0.571	0.324	0.117	0.232	0.259
1	25%	1.380	0.207	0.247	0.861	0.753	0.379	0.496	0.810
	50%	1.345	0.179	0.262	0.685	0.530	0.204	0.413	0.764
	75%	1.234	0.127	0.194	0.459	0.355	0.139	0.268	0.270
2	25%	1.179	0.257	0.301	0.842	0.806	0.420	0.604	0.802
	50%	1.142	0.202	0.326	0.634	0.546	0.233	0.517	0.690
	75%	0.963	0.172	0.227	0.446	0.426	0.182	0.280	0.289
3	25%	0.972	0.305	0.361	0.806	0.904	0.482	0.684	0.803
	50%	0.908	0.247	0.376	0.601	0.642	0.282	0.554	0.680
	75%	0.722	0.201	0.256	0.473	0.507	0.210	0.294	0.326

Higher censoring proportions as well as larger followup times tends to inaccurate estimates, greater standard error, and lower CV rate. In 70% censoring proportion rate, our suggested method and R software quantreg package shows poor performance to estimate quantile regression coefficient. To enhance the performance, we tried to double-up the datasize as 400, and change other minor settings, however there was no major improvements. Table 8 and Table 9 provide 70% esimation in 3 different censoring proportions. This result show big difference in estimation performance with previous lower quantile (0%   50%) estimation.

4 Real data analysis

In this section, we apply the proposed estimator to analyze survival times of dental restoration longevity of older adults with on different circumstances. Dental restoration is a general term of dental cavity treatment. As older adult population grows, dental restoration becomes an important issue to health care system, and they focused on not only cost or side effect of treatment, but also the longevity of restoration. The Geriatric and Special Needs Dentistry Clinic at the University iof Iowa Collage of Dentistry (COD) has offered comprehensive dental care to 2,717 unique patients, and observed covariates of treatment and helath condition of patients. We refer criterion in the origianl paper Caplan et al. (2019), and drop out subjects included in cohort 1 - 3 and younger than 65. After first filitering, we randomly choose first restored tooth per one patient to remove the correlation effect from same subject and damage of multiple restoration. After filtered raw dataset, there are 1,551 patients/tooth data, and censoring proportion was 56.6%. Initial dataset has 23 kinds of covariates, among them, we consider 7 covariates: gender(male, female), age, cohort(cohort4, 5, 6), provider type (predoctoral student, graduate student/faculty), insurance type(private, Medicaid, self-pay), tooth type (molar, pre-molar, anterior), and restoration type (amalgam, composite, GIC, crown or bridge).

Table 10: 5% quantile estimates of dental restoration longevity with 7 covariates.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$	
	β_0	SE	β_0	SE	β_0	SE
Reference	-0.761	1.076	-0.458	1.420	-1.189	1.085
Male	-0.502	0.201	0.468	0.336	-0.640	0.803
Age	-0.016	0.015	-0.070	0.025	-0.077	0.066
Cohort4	0.804	0.324	1.417	1.385	1.589	0.693
Cohort5	0.425	0.329	0.374	1.273	-0.010	0.707
Predoc	-0.192	0.221	0.567	0.339	0.292	0.715
Private	-0.303	0.351	-0.640	0.521	-0.165	0.554
XIX	-0.371	0.688	0.301	0.434	-0.184	0.770
Molar	0.187	0.493	0.028	0.811	-0.438	0.496
Pre-molar	0.070	0.480	1.011	0.673	-0.150	0.480
Amalgam	-1.893	0.614	-2.021	0.612	0.063	0.803
Composite	-2.177	0.727	-2.412	0.377	0.226	0.651
GIC	-2.224	0.610	-2.559	0.515	-1.319	0.937

Table 11: 10% quantile estimates of dental restoration longevity with 7 covariates.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$	
	β_0	SE	β_0	SE	β_0	SE
Reference	-0.352	1.162	-0.156	0.910	-1.178	0.857
Male	-0.477	0.214	0.165	0.440	-0.658	0.551
Age	-0.018	0.016	-0.064	0.012	-0.075	0.020
Cohort4	0.916	0.376	1.210	0.566	1.576	0.406
Cohort5	0.491	0.281	0.274	0.678	0.334	1.078
Predoc	0.177	0.305	0.798	0.426	0.702	0.864
Private	0.019	0.582	0.023	0.451	-0.105	0.802
XIX	-0.288	0.439	0.124	0.469	-0.501	0.928
Molar	0.087	0.359	-0.008	0.450	-0.404	1.489
Pre-molar	0.072	0.384	0.914	0.656	-0.239	0.819
Amalgam	-2.076	0.934	-1.782	0.593	0.097	1.169
Composite	-2.307	0.907	-2.045	0.437	0.334	0.806
GIC	-2.079	0.866	-2.180	0.584	-1.121	1.290

Table 12: 15% quantile estimates of dental restoration longevity with 7 covariates.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$	
	β_0	SE	β_0	SE	β_0	SE
Reference	0.270	1.237	0.006	0.672	-1.164	2.738
Male	-0.353	0.449	-0.081	0.306	-0.642	0.436
Age	-0.027	0.019	-0.062	0.020	-0.067	0.021
Cohort4	0.978	0.483	1.228	0.432	1.634	0.534
Cohort5	0.369	0.371	0.305	0.673	0.695	0.430
Predoc	0.259	0.467	0.905	0.514	0.880	1.037
Private	0.158	0.594	0.026	0.316	-0.243	0.395
XIX	-0.203	0.441	-0.040	0.380	-0.617	1.044
Molar	0.099	0.443	-0.088	0.437	0.181	0.848
Pre-molar	0.003	0.608	0.651	0.609	-0.252	0.726
Amalgam	-2.221	1.138	-1.493	0.530	-0.013	2.091
Composite	-2.361	1.041	-1.515	0.647	0.326	2.375
GIC	-2.020	1.017	-1.886	0.483	-0.898	2.067

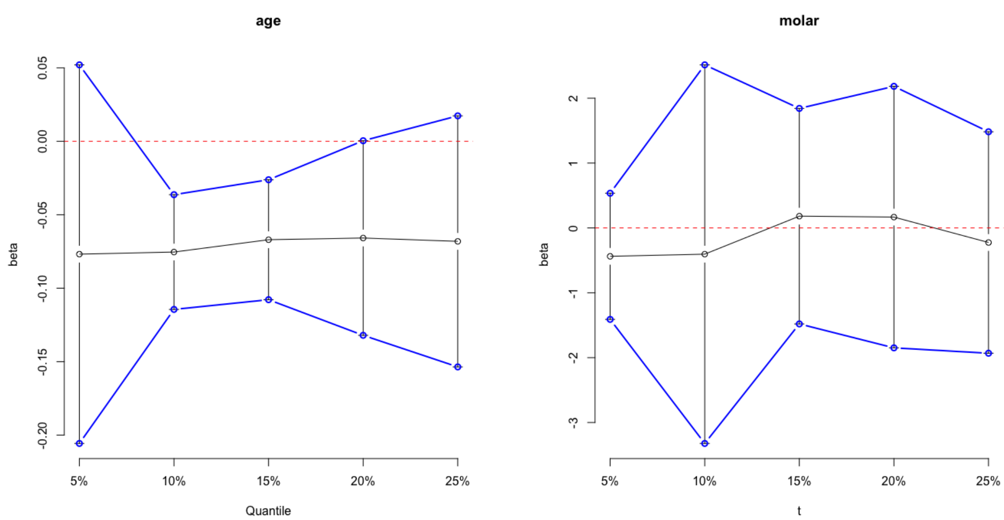
Table 13: 20% quantile estimates of dental restoration longevity with 7 covariates.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$	
	β_0	SE	β_0	SE	β_0	SE
Reference	0.599	0.443	0.098	0.943	-1.033	1.823
Male	-0.323	0.315	-0.179	0.348	-0.655	0.626
Age	-0.040	0.020	-0.059	0.020	-0.066	0.034
Cohort4	1.061	0.376	1.353	0.327	1.946	0.690
Cohort5	0.367	0.381	0.498	0.372	0.990	0.814
Predoc	0.385	0.259	0.823	0.508	0.938	1.466
Private	0.201	0.527	-0.010	0.368	-0.105	1.492
XIX	-0.189	0.371	-0.154	0.372	-0.088	0.532
Molar	0.051	0.318	-0.172	0.276	0.167	1.028
Pre-molar	0.007	0.580	0.469	0.482	-0.150	0.985
Amalgam	-1.994	0.487	-1.220	0.763	-0.275	0.876
Composite	-2.094	0.386	-1.170	0.762	0.054	1.523
GIC	-1.826	0.553	-1.673	0.659	-0.980	0.959

Table 14: 25% quantile estimates of dental restoration longevity with 7 covariates.

Covariate	$t_0 = 0$		$t_0 = 1$		$t_0 = 2$	
	β_0	SE	β_0	SE	β_0	SE
Reference	0.531	0.434	0.194	0.755	-0.362	1.368
Male	-0.307	0.317	-0.203	0.350	-0.504	0.595
Age	-0.043	0.014	-0.062	0.031	-0.068	0.044
Cohort4	1.153	0.395	1.401	0.518	2.325	0.848
Cohort5	0.461	0.291	0.550	0.390	1.454	0.893
Predoc	0.445	0.210	0.797	0.511	0.286	1.253
Private	0.292	0.483	-0.139	0.430	-0.434	0.840
XIX	-0.154	0.521	-0.253	0.708	0.009	0.644
Molar	0.007	0.379	-0.093	0.317	-0.225	0.871
Pre-molar	0.221	0.628	0.520	0.532	-0.092	1.268
Amalgam	-1.689	0.666	-0.977	0.612	-0.350	0.838
Composite	-1.717	0.510	-0.956	0.675	0.113	1.868
GIC	-1.582	0.534	-1.495	0.481	-0.921	0.635

Those tables summarizes quantile coefficient estimates and standard error of from 5% quantiles to 25% quantiles and various followup times. Among 7 covariates, we focused covariate effect of age and tooth type, especially molar, to longevity of dental restoration. Figure 1 are 95% confidence intervals of covariate effects of age of patient and pre-doctoral students for some quantiles coefficient of dental restoration longevity. Figure 1(a) shows that older age consistently affects negatively to lifetime of dental restoration for all quantiles, as most of survival data shows. It is an evidence that proposed smoothed estimator gives reasonable estimation result. Furthermore, figure 1(b) shows signs of coefficient are reversed based on estimation quantiles. At 5% and 10% quantile estimation, β_{predoc} is negative, however, it is changed to positive beyond 15% quantile. Finally, it goes back to negative at 25% quantile. It verifies that the proposed smoothed estimator can detect different covariate effect from changes of estimation quantiles.



(a) (b)

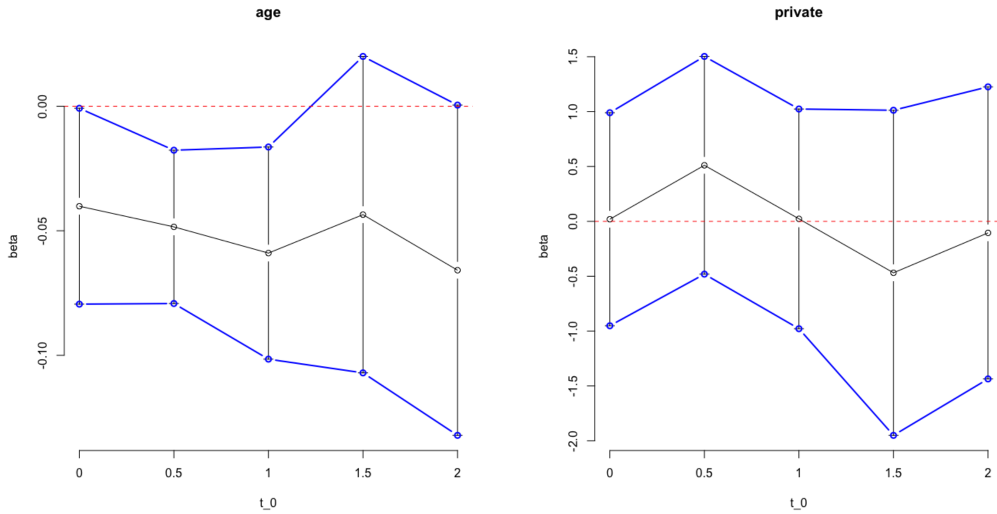
Figure 1: β is black line and 95% confidence interval is blue line. (a) 95% confidence interval of β_{age} are always less than 0. (b) β_{molar} varies with estimation quantiles.

Figure 2 are 95% confidence intervals of covariate effects of age of patient and pre-doctoral students for some quantiles coefficient of dental restoration longevity. Figure 1(a) shows that older age consistently affects negatively to lifetime of dental restoration for all quantiles, as most of survival data shows. It is an evidence that proposed smoothed estimator gives reasonable estimation result. Furthermore, figure 1(b) shows signs of coefficient are reversed based on estimation quantiles. At 5% and 10% quantile estimation, β_{predoc} is negative, however, it is changed to positive beyond 15% quantile. Finally, it goes back to negative at 25% quantile. It verifies that the proposed smoothed estimator can detect different covariate effect from changes of estimation quantiles.

Figure 4 are 95% confidence intervals of covariate effect of age and insurance type, especially private insurance, for 10% quantile of longevity of dental restoration at 4 different followup time. As result of 1(a), 2(a) shows that older age affect negatively on longevity of dental treatment at every followup time. 2(b) verifies $\beta_{private}$ is negative at $t_0 = 0$ to 1 and is changed to positive beyond $t_0 = 1$. Therefore, 2(b) proves proposed smoothed estimator can estimate different covariate effect from changes of basetime.

5 Discussion

In this article, we suggest computationally efficient inference method for censored quantile regression to accommodate change of followup time. Basis of our regression model starts



(a) (b)

Figure 2: β is black line and 95% confidence interval is blue line. (a) 95% confidence interval of β_{age} are always less than 0. (b) Different $\beta_{private}$ is estimated from different followup time.

from Jung et al. (2009), where the regression coefficient directly related to quantile residual lifetime, and we apply idea of time origin variation and inverse probability of censoring weighted (IPCW) estimator as suggested by Kim et al. (2012). However, inefficiency of solving estimating equation and lack of estimating variance-covariance matrix are still existed. Induced smoothing approach, developed by Brown and Wang (2007), is a perfect idea to overcome both problems. Smoothed estimating function provides computationally efficent way to find estimator using non-linear equation solver package, such as nleqslv or BB solver, and also makes possible to find approximated variance-covariance matrix of estimator based on multiplier bootstrapping and sandwich estimator, developed by Chiou et al. (2015). In simulation and real data anlaysis, we showed our method is computationally fast enough, and the result estimator is quite close to real parameter. Furthermore, we proved suggested estimator is theoretically valid in appendix.

Even though, proposed smoothed quantile regression estimator is valid in most cases, it needs further research in data with high censoring proportion. Especially, it is usually impossible to estimate covariate effect of high quantile residual lifetime in high censoring proprotion data. Because of ill-conditioned Jacobian matrix or local minimum issue of solution, non-linear equation solver cannot find an exact solution that we need. Thus, we expect the alternative way to solve non-linear estimating equation improves the performance of our method in data with high censoring proportion.

In section 2, we introduced two similar estimating equations, one suggested by Kim et al. (2012) and other derived from Peng and Fine (2009). Both estimating equations used IPCW estimator, and only difference is a position of weight applied. After conjugating induced smoothing approach to both estimating equation, and we compared the performance of them. In low censoring proportion data, performances of both estimating equation are good. However, as censoring proportion is getting higher, estimates from the estimating equation by Kim et al. (2012) shows significant difference to real parameter. On the other hands, estimates from that by Peng and Fine (2009) shows consistently similar result to real parameter. We checked both estimating equations are theoretically valid, and we cannot figure out the reason why those two estimating equationf shows performance difference in high censoring proportion data. We need to find out this point at future studies.

6 Appendix

6.1 Asymptotic properties of the unsmoothed estimator

In this part, we establish the consistency and asymptotic normality of β. We first impose the regularity conditions.

A1 There exists $\nu \geq 0$ such that $P(C = \nu) \geq 0$ and $P(C \geq \nu) = 0$.

A2 X is uniformly bounded, that is $\sup_i \|X_i\| \leq \infty$.

A3 (i) $\beta_0(\tau)$ is Lipschitz continuous for $\tau \in [\tau_L, \tau_U]$;
(ii) $f_i(t|X)$ is bounded above uniformly in t and X,where $f_1(t|X) = dF_1(t|X)/dt$.

A4 For some $\rho_0 \geq 0$ and $c_0 \geq 0, inf_{b\in\beta(\rho_0)} eigminA(b) \geq c_0$,
where $\beta(\rho) = b \in R^{P+1} : \inf_{\tau\in[\tau_L,\tau_U]} \|b - \beta_0(\tau)\| \leq \rho$ and $A(b) = E[Z^{\otimes 2} f_1\{\exp(X^T b)|X\}]$.
Here $\|\cdot\|$ is the Euclidean norm, and we define $u^{\otimes 2} = uu^T$ for a vector u.

Kim et al. (2012) shows an estimator from similar estimating equation

$$S_n(\beta, \tau) = n^{-1} \sum_{i=1}^{n} X_i \frac{\delta_i}{\hat{G}(Z_i)} \Big(I\{log(Z_i - t_0) \leq X_i'\beta\} - \tau \Big) \tag{9}$$

is consistent and satisfies asymptotic normality under above regularity conditions. If we change first X_i to $X_i^\star$ where $X_i^\star = X_i I[Z_i > t_0]$, we are simply able to prove our suggested estimator β_τ is also consistent and asymptotically normal: $n^{1/2}(\hat{\beta} - \beta_0) \xrightarrow{d} N(0, \Gamma)$ where $\Gamma = A^{-1}\Sigma A^{-1}$, $A = \lim_{n\to\infty} X_i X_i' f_i(0|X_i)$, and $\Sigma = \lim_{n\to\infty} Var\{U_n(\beta_0)\}$

6.2 Asymptotic properties of the smoothed estimator

For prove theorem 1, we need following lemma 2.

Lemma 2. *Let $W = O(n^{-1})$ any positive definite matrix, and define*

$$\tilde{S}_n(\beta, \boldsymbol{W}) = \tfrac{1}{n} \textstyle\sum_{i=1}^n I[Z_i \geq t_0] X_i \frac{\delta_i}{\hat{G}(Z_i)} \Phi\Big(\frac{X_i'\beta - log(Z_i - t_0)}{\sqrt{X_i' \boldsymbol{W} X_i}} \Big)$$

as the smoothed estimating function. Under condition 1-3, we have

$$\sup_{\|\beta-\beta_0\| \leq \epsilon_n} \|n^{1/2}\{\tilde{S}_n(\beta, \boldsymbol{W}) - S_n(\beta)\}\| \xrightarrow{p} 0, \; as \; n \to \infty,$$

where ϵ_n is a positive sequence that converges to 0.

Proof of lemma 2
Let $\sigma_i = (X_i' W X_i)^{1/2}$, $\epsilon_i^\beta = X_i\beta - log(Z_i - t_0)$, and $d_i(\beta) = sgn(\epsilon_i^\beta)\Phi(-|\epsilon_i^\beta/\sigma_i|)$

$$\begin{aligned} n^{1/2}\{\tilde{S}_n(\beta, \mathbf{W}) - S_n)\} &= n^{-1/2}\sum_{i=1}^n I[Z_i \geq t_0] X_i \frac{\delta_i}{\hat{G}(Z_i)}\left\{\Phi\left(\frac{-\epsilon_i^\beta}{\sigma_i}\right) - I(\epsilon_i^\beta < 0)\right\} \\ &= n^{-1/2}\sum_{i=1}^n \frac{\delta_i}{\hat{G}(Z_i)} X_i^\star d_i(\beta) \end{aligned}$$

Where $X_i^\star = X_i I(Z_i > t_0)$.

Denote $D_n(\beta) = n^{-1/2}\sum_{i=1}^n \frac{\delta_i}{\hat{G}(Z_i)} X_i^\star d_i(\beta)$ and $D_n^G(\beta) = n^{-1/2}\sum_{i=1}^n \frac{\delta_i}{G(Z_i)} X_i^\star d_i(\beta)$. It follows that

$$D_n(\beta) = D_n^G(\beta) - n^{-1/2}\sum_{i=1}^n \frac{X_i^\star \delta_i(\hat{G}(Z_i) - G(Z_i))}{\hat{G}^2(Z_i)} d_i(\beta) + o_p(1) \tag{10}$$

Expectation of $D_n^G(\beta)$ is $E\{D_n^G(\beta)\} = n^{-1/2}\sum_{i=1}^n X_i^\star E d_i(\beta)$, Where

$$\begin{aligned} E\{d_i(\beta)\} &= \int_{-\infty}^{\infty} sgn(\epsilon_i^\beta)\Phi(-|\epsilon_i^\beta/\sigma_i|) f_i\{\epsilon_i + X_i^{\star\prime}(\beta - \beta_0)\} d\epsilon_i^\beta \\ &= \sigma_i \int_{-\infty}^{\infty} \Phi(-|t|)\{2I(t>0) - 1\} f_i\{\sigma_i t + X_i^{\star\prime}(\beta - \beta_0)\} dt \\ &= \sigma_i \int_{-\infty}^{\infty} \Phi(-|t|)\{2I(t>0) - 1\}[f_i\{\sigma_i t + X_i^{\star\prime}(\beta - \beta_0)\} + f_i'\{\omega_i^\star(t)\}\sigma_i t] dt \end{aligned}$$

Where f_i is the density of $\epsilon_i = \epsilon_i^{\beta_0}$, and $\omega_i^\star(t)$ is between $x_i^{\star\prime}(\beta - \beta_0)$ and $X_i^{\star\prime}(\beta - \beta_0) + \sigma_i t$. Note that for β that satisfies $\|\beta - \beta_0\| \leq \epsilon_n$, where $\epsilon_n \to 0$, we have $\|X_i^{\star\prime}(\beta - \beta_0)\| \to 0$. It follows from assumption B1 that $sup_i f_i\{X_i^{\star\prime}(\beta - \beta_0)\} < \infty$ and since $\int_{-\infty}^{\infty}\Phi(-|t|)\{2I(t > 0) - 1\}dt = 0$, we have $\int_{-\infty}^{\infty}\Phi(-|t|)\{2I(t > 0) - 1\} f_i\{X_i^{\star\prime}(\beta - \beta_0)\}dt = 0$. In addition, by assumption B1, we can find $M > 0$ such that $sup_i|f_i'\{\omega_i^\star(t)\}| < M$. Thus, it follows that

$$|E\{d_i(\beta)\}| \leq \int_{-\infty}^{\infty}|t|\Phi(|t|)|f_\beta, i'\{\omega_i^\star(t)\}|dt \leq M\sigma_i^2/2$$

where the last equality holds becuase $\int_{-\infty}^{\infty}|t|\Phi(\{-|t|\})dt = 1/2$.

By assumption B2 and the fact that $W = O(n^{-1})$, $\sum_{i=1}^n \sigma_i^2 = tr(X^\star W X^{\star\prime}) = tr(W X^{\star\prime} X^\star)$ is bounded, and $\sum_{i=1}^n |E\{d_i(\beta)\}| \leq M\sum_{i=1}^n \sigma_i^2/2$ is also bounded. Therefore,

$$\|E\{D_n^G(\beta)\}\| \leq n^{-1/2}\sqrt{p}\sup_{i,j}|X_{ij}^\star \textstyle\sum_{i=1}^n |E\{d_i(\beta)\}| \to 0, \quad \text{as } n \to 0.$$

In addition, by assumption B3,

$$Var\{D_n^G(\beta)\} = \frac{1}{n}\sum_{i=1}^n X_i^\star X_i^{\star\prime} Var\Big\{\frac{\delta_i}{G(Z_i)}d_i(\beta)\Big\} \le \frac{1}{n}\sum_{i=1}^n \frac{X_i^\star X_i^{\star\prime}}{\nu}E\{d_i^2(\beta)\}$$

where

$$\begin{aligned}
E\{d_i^2(\beta)\} &= \int_{-\infty}^{\infty}\Phi^2(-|s|)f_i\{\sigma_i s + X_i^\star(\beta-\beta_0)\}d(\sigma_i s)\\
&= \int_{|s|>\Delta}\Phi^2(-|s|)f_i\{\sigma_i s + X_i^\star(\beta-\beta_0)\}d(\sigma_i s) + \int_{|s|\le\Delta}\Phi^2(-|s|)f_i\{\sigma_i s + X_i^\star(\beta-\beta_0)\}d(\sigma_i s)\\
&\le \Phi^2(-\Delta) + \sigma_i\int_{|s|\le\Delta} f_i\{\sigma_i s + X_i^\star(\beta-\beta_0)\}ds\\
&= \Phi^2(-\Delta) + 2\sigma_i\Delta f_i(\omega_i^\star).
\end{aligned}$$

Note that $\omega_i^\star \in (X_i^{\star\prime}(\beta-\beta_0)-\sigma_i\Delta, X_i^{\star\prime}(\beta-\beta_0)+\sigma_i\Delta)$. Let $\Delta = n^{1/4}$ and since $\sigma_i = O(n^{-1/2})$, both $\sigma_i\Delta$ and $\omega_i^\star$ go to 0 as n increases. As $f_i(\cdot)$ is uniformly bounded around zero, both $\Phi^2(-\Delta)$ and $\sigma_i\Delta f_i(\omega_i^\star)$ go to 0 as $n\to\infty$.

Thus, it follows that $\lim_{n\to\infty} E\{d_i^2(\beta)\} = 0$, and $\lim_{n\to\infty} Var\{D_n^G(\beta)\} = 0$. By the Weak Law of Large Numbers, for β that satisfies $\|\beta-\beta_0\| \le \epsilon_n$, we have

$$\|D_n^G(\beta)\| \xrightarrow{p} 0, \quad \text{as } n\to\infty. \tag{11}$$

The second term on the right side of (10) can be written as

$$\begin{aligned}
&n^{-1/2}\sum_{i=1}^n\Big\{\frac{\delta_i X_i^\star(\hat G(Z_i)-G(Z_i))}{G^2(Z_i)}\Big\}d_i(\beta) + o_p(1)\\
&= n^{-1/2}\sum_{j=1}^n\int_0^L\Big\{n^{-1}\sum_{i=1}^n\frac{\delta_i X_i^\star d_i(\beta Z_i(u))}{G(Z_i)}\Big\}\frac{dM_j^c(u)}{y(u)} + o_p(1)
\end{aligned}$$

where $M_i^c(u) = N_i^c(u) - \int_0^t I(Z_i \ge u)d\Lambda^c(s)$, $N_i^c(u) = (1-\delta_i)I(Z_i\le u)$, $\Lambda^c(u) = -log\{G(u)\}$ is the censoring cumulative hazard, $Z_i(u) = I(Z_i \ge u)$ is the ith at-risk process, and $y(u) = \lim_{n\to\infty} n^{-1}\sum_{i=1}^n Z_i(u)$ is bounded from below in $(0, L]$ by assumption B3.

Define $I_n(u,\beta) = n^{-1}\sum_{i=1}^n \frac{\delta_i X_i^\star d_i(\beta)Z_i(u)}{G(Z_i)}$, and $I(u,\beta) = E\{I_n(u,\beta)\}$. We have

$$I(u,\beta) = n^{-1}\sum_{i=1}^n X_i^\star E\{d_i(\beta)Z_i(u)\}$$

Where $|E\{d_i(\beta)Z_i(u)\}| \le E|d_i(\beta)|$, and

$$\begin{aligned}
E|d_i(\beta)| &= \int_{-\infty}^{\infty}\Phi(-|\epsilon_i^\beta/\sigma_i|)f_i\{\epsilon_i + X_i^{\star\prime}(\beta-\beta_0)\}d\epsilon_i^\beta\\
&= \sigma_i\int_{-\infty}^{\infty}\Phi(-|t|)f_i\{\sigma_i t + X_i^{\star\prime}(\beta-\beta_0)\}dt\\
&= \sigma_i f_i\{X_i^{\star\prime}(\beta-\beta_0)\}\int_{-\infty}^{\infty}\Phi(-|t|)dt + \sigma_i^2\int_{-\infty}^{\infty}t\Phi(-|t|)f_i'\{\omega_i^\star(t)\}dt.
\end{aligned}$$

By assumption B1, we have $f_i\{X_i^{\star\prime}(\beta-\beta_0)\}\int_{-\infty}^{\infty}\Phi(-|t|)dt \le \infty$, and $\int_{-\infty}^{\infty} t\Phi(-|t|)f_i'\{\omega_i^\star\}dt \le \infty$. Thus, it follows that $E|d_i(\beta)| =)(n^{-1/2})$, and

$$\|I(u,\beta)\| \le \sqrt{p}\sup_{i,j}|X_{ij}^\star|n^{-1}\textstyle\sum_{i=1}^n E|d_i(\beta)| = O(n^{-1/2}) \to 0.$$

Define $\mathcal{F} = \frac{\delta_i X_i^\star d_i(\beta) Z_i(u)}{G(Z_i)}$, $\|\beta-\beta_0\| \le \epsilon_n$ and $u \in (0,\infty)$. The function class $\mathcal{F}$ is Gilvenko-Cantelli Vaart and Wellner (1996) because the class of indicator functions is Gilvenko-Cantelli, and $X_i^\star, d_i(\beta), and 1/G(Z_i)$ are uniformly bounded. it follows that $sup_{\|\beta-\beta_0\|\le\epsilon_n, u\in(0,\infty)}\|I_n(u,\beta)-I(u,\beta)\| \xrightarrow{a.s.} 0$ and we have

$$n^{-1/2}\textstyle\sum_{j=1}^n\int_0^L I_n(u,\beta)\frac{dM_j^c(u)}{y(u)} = n^{-1/2}\sum_{j=1}^n\int_0^L I(u,\beta)\frac{dM_j^c(u)}{y(u)} + o_p(1)$$

By the Martingale Central Limit Theorem (Fleming and Harrington (2011)), $n^{-1/2}\sum_{j=1}^n\int_u^\beta\frac{dM_j^c(u)}{y(u)}$ is $o_p(1)$ as n goes to infinity. It follows that, for β that satisfies $\|\beta-\beta_0\| \le \epsilon_n$,

$$\left\|n^{-1/2}\sum_{j=1}^n\int_0^L I_n(u,\beta)\frac{dM_j^c(u)}{y(u)}\right\| \xrightarrow{p} 0 \tag{12}$$

Collating (11) and (12), we have

$$\|n^{1/2}\{\tilde{S}_n(\beta,\mathbf{W}) - S_n(\beta)\}\| \xrightarrow{p} 0.$$

for any β such that $\|\beta-\beta_0\| \le \epsilon_n$. Lemma 2 is thus proven by the fact that both $\tilde{S}_n(\beta,\mathbf{W})$ and $S_n(\beta)$ are monotone functions, thus the point-wise covergence could be strengthened to uniform convergence (Shorack and Wellner (2009)).

Proof of Theorem 1

After applying induced smoothing method, $\hat{\beta}_{IS} = \beta_0 + \mathbf{H}^{1/2}V$ where $\mathbf{H} = n^{-1}\Gamma$ and $V \sim \mathcal{N}(0, I_p)$. Since $\hat{\beta}_{IS}$ is a solution of

$$\tilde{U}_n(\hat{\beta}_{IS},\tilde{\mathbf{H}}) = n^{-1}\sum_{i=1}^n I[Z_i \ge t_0]X_i\frac{\delta_i}{\hat{G}(Z_i)}\left\{\Phi\left(\frac{X_i'\hat{\beta}_{IS} - log(Z_i-t_0)}{\sqrt{X_i'\tilde{\mathbf{H}}X_i}}\right) - \tau\right\} = 0 \tag{13}$$

Using Taylor expansion, we have

$$\sqrt{n}(\hat{\beta}_{IS}-\beta_0) = \frac{-\sqrt{n}\tilde{U}_n(\beta_0,\tilde{\mathbf{H}})}{\tilde{U}_n'(\beta_0,\tilde{\mathbf{H}})} \tag{14}$$

By Lemma (2), $-\sqrt{n}\tilde{U}_n(\beta_0,\tilde{\mathbf{H}}) \xrightarrow{p} -\sqrt{n}\tilde{U}(\beta_0)$, and Kim et al. (2012) shows $-\sqrt{n}\tilde{U}(\beta_0) \xrightarrow{d} \mathcal{N}(0,\Sigma)$. Since $\tilde{\mathbf{H}} = O(n^{-1})$, $\tilde{U}_n'(\hat{\beta}_0,\tilde{\mathbf{H}}) = \tilde{A}_n(\beta_0,\tilde{\mathbf{H}}) \xrightarrow{p} A$. In sum, $\sqrt{n}(\hat{\beta}_{IS}-\beta_0) \xrightarrow{d} \mathcal{N}(0,\Gamma)$, where $\Gamma = A^{-1}\Sigma A^{-1}$.

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