

INSTITUTE OF MATHEMATICS FOR INDUSTRY,
KYUSHU UNIVERSITY

LOGIC AND COMPUTATION PROJECT

Coq Modules for Relational Calculus

(Ver.0.1)

Hisaharu TANAKA
Saga University

Toshiaki MATSUSHIMA
Kyushu University

Shuichi INOKUCHI
Fukuoka Institute of Technology

Yoshihiro MIZOGUCHI
Kyushu University

October 20, 2015

Repository: `svn://sakura.imi.kyushu-u.ac.jp/Coq/RelationalCalculus/branches/20150716/`

Contents

1	Library <code>Basic_Notations</code>	4
1.1	このライブラリについて	4
2	Library <code>Basic_Notations_Rel</code>	5
2.1	定義	5
2.2	関数の定義	8
2.3	関係の公理	8
2.3.1	Dedekind 圏の公理	8
2.3.2	排中律	13
2.3.3	単域	13
2.3.4	選択公理	13
2.3.5	関係の有理性	14
2.3.6	直和と直積	14
3	Library <code>Basic_Notations_Set</code>	15
3.1	定義	15
3.2	関数の定義	16
3.3	関係の公理	17
3.3.1	Dedekind 圏の公理	17
3.3.2	排中律	25
3.3.3	単域	26
3.3.4	選択公理	27
3.3.5	関係の有理性	28
3.3.6	直和と直積	31
4	Library <code>Basic_Lemmas</code>	36
4.1	束論に関する補題	36
4.1.1	和関係, 共通関係	36
4.1.2	分配法則	45
4.1.3	原子性	47
4.2	Heyting 代数に関する補題	49
4.3	補関係に関する補題	56
4.4	Bool 代数に関する補題	60

5	Library <code>Relation_Properties</code>	63
5.1	関係計算の基本的な性質	63
5.2	<code>comp_inc_compat</code> と派生補題	74
5.3	逆関係に関する補題	76
5.4	合成に関する補題	81
5.5	単域と Tarski の定理	86
6	Library <code>Functions_Mappings</code>	91
6.1	全域性, 一価性, 写像に関する補題	91
6.2	全射, 単射に関する補題	101
7	Library <code>Dedekind</code>	110
7.1	Dedekind formula に関する補題	110
7.2	Dedekind formula と全関係	111
7.3	Dedekind formula と恒等関係	114
8	Library <code>Rationality</code>	117
8.1	有理性から導かれる系	117
9	Library <code>Conjugate</code>	119
9.1	共役性の定義	119
9.2	共役の例	120
10	Library <code>Domain</code>	129
10.1	定義域の定義	129
10.2	定義域の性質	129
10.2.1	基本的な性質	129
10.2.2	合成と定義域	132
10.2.3	その他の性質	136
10.2.4	矩形関係	141
11	Library <code>Residual</code>	145
11.1	剰余合成関係の性質	145
11.1.1	基本的な性質	145
11.1.2	単調性と分配法則	148
11.1.3	剰余合成と関数	151
11.2	Galois 同値とその系	154
11.3	その他の性質	155
11.4	順序の関係と左剰余合成	165
11.4.1	\max, \sup, \min, \inf	165
11.4.2	左剰余合成	170
12	Library <code>Schroder</code>	172
12.1	Schröder 圏の性質	172

13 Library	Sum_Product	180
13.1	関係の直和	180
13.1.1	入射対, 関係直和の定義	180
13.1.2	関係直和の性質	182
13.1.3	分配法則	186
13.2	関係の直積	187
13.2.1	射影対, 関係直積の定義	187
13.2.2	関係直積の性質	189
13.2.3	鋭敏性	198
13.2.4	分配法則	204
14 Library	Point_Axiom	208
14.1	I-点	208
14.1.1	I-点の定義	208
14.1.2	I-点の性質	208
14.2	I-点に関する諸公理	211
14.2.1	点公理	211
14.2.2	全域性公理	215
14.2.3	その他の公理	218

Chapter 1

Library `Basic_Notations`

1.1 このライブラリについて

- このライブラリは河原康雄先生の“関係の理論 - Dedekind 圏概説 -”をもとに制作されている.
- 現状サポートしているのは,
 - 1.4 節大半, 1.5 - 1.6 節全部
 - 2.1 - 2.3 節全部, 2.4 - 2.5 節大半, 2.6 節全部, 2.7 節大半, 2.8 節有理性
 - 4.2 - 4.3 節全部, 4.4 - 4.5 節大半, 4.6 節命題 4.6.1, 4.7 節大半, 4.9 節全部
 - 4.8 節は部分域公理を用いるので, そちらが終わり次第といったところである.
- 関係論で話を進めたい場合は, 下の行に `Require Export Basic_Notations_Rel.` を, 集合論で話を進めたい場合は, `Require Export Basic_Notations_Set.` を記述する.

`Require Export Basic_Notations_Rel.`

なお, 証明の書き方が悪いと, まれに“関係論では証明が通ったのに, 集合論では通らない”といったことも起こるようなので, ある程度注意しておく必要がある.

Chapter 2

Library `Basic_Notations_Rel`

`Require Export ssreflect eqtype bigop.`

`Require Export Logic.ClassicalFacts.`

`Axiom prop_extensionality_ok : prop_extensionality.`

2.1 定義

- A, B を `eqType` として, A から B への関係の型を $(\text{Rel } A B)$ と書き, $A \rightarrow B \rightarrow \text{Prop}$ として定義する. 本文中では型 $(\text{Rel } A B)$ を $A \rightarrow B$ と書く.
- 関係 $\alpha : A \rightarrow B$ の逆関係 $\alpha^\# : B \rightarrow A$ は $(\text{inverse } \alpha)$ で, Coq では $(\alpha \#)$ と記述する.
- 2つの関係 $\alpha : A \rightarrow B, \beta : B \rightarrow C$ の合成関係 $\alpha \cdot \beta$ (or $\alpha\beta$) : $A \rightarrow C$ は $(\text{composite } \alpha \beta)$ で, $(\alpha \cdot \beta)$ と記述する.
- 剰余合成関係 $\alpha \triangleright \beta : A \rightarrow C$ は $(\text{residual } \alpha \beta)$ で, $(\alpha \triangleright \beta)$ と記述する.
- 恒等関係 $\text{id}_A : A \rightarrow A$ は $(\text{identity } A)$ で, $(\text{Id } A)$ と記述する.
- 空関係 $\phi_{AB} : A \rightarrow B$ は $(\text{empty } A B)$ で, (ϕ_{AB}) と記述する.
- 全関係 $\nabla_{AB} : A \rightarrow B$ は $(\text{universal } A B)$ で, (∇_{AB}) と記述する.
- 2つの関係 $\alpha : A \rightarrow B, \beta : A \rightarrow B$ の和関係 $\alpha \sqcup \beta : A \rightarrow B$ は $(\text{cup } \alpha \beta)$ で, $(\alpha \sqcup \beta)$ と記述する.
- 共通関係 $\alpha \sqcap \beta : A \rightarrow B$ は $(\text{cap } \alpha \beta)$ で, $(\alpha \sqcap \beta)$ と記述する.
- 相対擬補関係 $\alpha \Rightarrow \beta : A \rightarrow B$ は $(\text{rpc } \alpha \beta)$ で, $(\alpha \Rightarrow \beta)$ と記述する.
- 関係 $\alpha : A \rightarrow B$ の補関係 $\alpha^- : A \rightarrow B$ は $(\text{complement } \alpha)$ で, Coq では $(\alpha \sim)$ と記述する.

	数式	Coq	Notation
逆関係	$\alpha^\#$	(inverse α)	($\alpha \#$)
合成関係	$\alpha \cdot \beta$	(composite $\alpha \beta$)	($\alpha \cdot \beta$)
剰余合成関係	$\alpha \triangleright \beta$	(residual $\alpha \beta$)	($\alpha \triangleright \beta$)
恒等関係	id_A	(identity A)	(Id A)
空関係	ϕ_{AB}	(empty $A B$)	($\emptyset A B$)
全関係	∇_{AB}	(universal $A B$)	($\forall A B$)
和関係	$\alpha \sqcup \beta$	(cup $\alpha \beta$)	($\alpha \sqcup \beta$)
共通関係	$\alpha \sqcap \beta$	(cap $\alpha \beta$)	($\alpha \sqcap \beta$)
相対擬補関係	$\alpha \Rightarrow \beta$	(rpc $\alpha \beta$)	($\alpha \Rightarrow \beta$)
補関係	α^-	(complement α)	(α^-)
差関係	$\alpha - \beta$	(difference $\alpha \beta$)	($\alpha - \beta$)
添字付和関係	$\sqcup_{P(\alpha)} f(\alpha)$	(cupP $P f$)	($\sqcup_{\{P\}} f$)
添字付共通関係	$\sqcap_{P(\alpha)} f(\alpha)$	(capP $P f$)	($\sqcap_{\{P\}} f$)

Table 2.1: 関係の表記について

- 2 つの関係 $\alpha : A \rightarrow B$, $\beta : A \rightarrow B$ の差関係 $\alpha - \beta : A \rightarrow B$ は (difference $\alpha \beta$) で, ($\alpha - \beta$) と記述する.
- (cupP) と (capP) は添字付の和関係と共通関係であり, 述語 P に対し, $\{f(\alpha) \mid P(\alpha)\}$ の和関係, 共通関係を表す.
- また, 1 点集合 $I = \{*\}$ は i と表記する.
- なお, 通常の共通関係, 和関係も添字付のもので表現することができるため, ここではそれを用いて表記する.
- 後で述べるように, 剰余合成 $\alpha \triangleright \beta$ も $(\alpha \cdot \beta^-)^-$ のように表現することは可能だが, “剰余合成が存在すれば, それは $(\alpha \cdot \beta^-)^-$ に等しい” というレベルのものであるため, 剰余合成に関する公理はやはり必要となる.

表 2.1 に関係の表記についてまとめる.

Definition $Rel (A B : eqType) := A \rightarrow B \rightarrow Prop$.

Parameter $inverse : (\forall A B : eqType, Rel A B \rightarrow Rel B A)$.

Notation " $a \#$ " $:= (inverse _ a)$ (at level 20).

Parameter $composite : (\forall A B C : eqType, Rel A B \rightarrow Rel B C \rightarrow Rel A C)$.

Notation " $a \cdot \cdot b$ " $:= (composite _ _ a b)$ (at level 50).

Parameter $residual : (\forall A B C : eqType, Rel A B \rightarrow Rel B C \rightarrow Rel A C)$.

Notation " $a \triangleright b$ " $:= (residual _ _ a b)$ (at level 50).

Parameter $identity : (\forall A : eqType, Rel A A)$.

Notation " Id " $:= identity$.

CHAPTER 2. LIBRARY BASIC_NOTATIONS_REL

Parameter *empty* : $(\forall A B : eqType, Rel A B)$.

Notation "'''" := *empty*.

Parameter *universal* : $(\forall A B : eqType, Rel A B)$.

Notation "'''" := *universal*.

Parameter *include* : $(\forall A B : eqType, Rel A B \rightarrow Rel A B \rightarrow Prop)$.

Notation "a' ' b" := (*include* _ _ a b) (at level 50).

Parameter *cupP* : $(\forall A B C D : eqType, (Rel C D \rightarrow Prop) \rightarrow (Rel C D \rightarrow Rel A B) \rightarrow Rel A B)$.

Notation "'' _{ ' p ' }' f" := (*cupP* _ _ _ p f) (at level 50).

Parameter *capP* : $(\forall A B C D : eqType, (Rel C D \rightarrow Prop) \rightarrow (Rel C D \rightarrow Rel A B) \rightarrow Rel A B)$.

Notation "'' _{ ' p ' }' f" := (*capP* _ _ _ p f) (at level 50).

Definition *cup* {A B : eqType} (alpha beta : Rel A B)

:= _{fun gamma : Rel A B \Rightarrow gamma = alpha \vee gamma = beta} id.

Notation "a' ' b" := (*cup* a b) (at level 50).

Definition *cap* {A B : eqType} (alpha beta : Rel A B)

:= _{fun gamma : Rel A B \Rightarrow gamma = alpha \vee gamma = beta} id.

Notation "a' ' b" := (*cap* a b) (at level 50).

Parameter *rpc* : $(\forall A B : eqType, Rel A B \rightarrow Rel A B \rightarrow Rel A B)$.

Notation "a' »' b" := (*rpc* _ _ a b) (at level 50).

Definition *complement* {A B : eqType} (alpha : Rel A B) := alpha » A B.

Notation "a' ^'" := (*complement* a) (at level 20).

Definition *difference* {A B : eqType} (alpha beta : Rel A B) := alpha beta ^.

Notation "a - b" := (*difference* a b) (at level 50).

Notation "i'" := *unit_eqType*.

2.2 関数の定義

$\alpha : A \rightarrow B$ に対し, 全域性 `total_r`, 一価性 `univalent_r`, 関数 `function_r`, 全射 `surjective_r`, 単射 `injective_r`, 全単射 `bijection_r` を以下のように定義する.

- `total_r` : $id_A \sqsubseteq \alpha \cdot \alpha^\#$
- `univalent_r` : $\alpha^\# \cdot \alpha \sqsubseteq id_B$
- `function_r` : $id_A \sqsubseteq \alpha \cdot \alpha^\# \wedge \alpha^\# \cdot \alpha \sqsubseteq id_B$
- `surjection_r` : $id_A \sqsubseteq \alpha \cdot \alpha^\# \wedge \alpha^\# \cdot \alpha = id_B$
- `injection_r` : $id_A = \alpha \cdot \alpha^\# \wedge \alpha^\# \cdot \alpha \sqsubseteq id_B$
- `bijection_r` : $id_A = \alpha \cdot \alpha^\# \wedge \alpha^\# \cdot \alpha = id_B$

Definition `total_r` $\{A\ B : eqType\} (\alpha : Rel\ A\ B) := (Id\ A) \quad (\alpha \cdot \alpha^\#)$.

Definition `univalent_r` $\{A\ B : eqType\} (\alpha : Rel\ A\ B) := (\alpha^\# \cdot \alpha) \quad (Id\ B)$.

Definition `function_r` $\{A\ B : eqType\} (\alpha : Rel\ A\ B)$
 $:= (total_r\ \alpha) \wedge (univalent_r\ \alpha)$.

Definition `surjection_r` $\{A\ B : eqType\} (\alpha : Rel\ A\ B)$
 $:= (function_r\ \alpha) \wedge (total_r\ (\alpha^\#))$.

Definition `injection_r` $\{A\ B : eqType\} (\alpha : Rel\ A\ B)$
 $:= (function_r\ \alpha) \wedge (univalent_r\ (\alpha^\#))$.

Definition `bijection_r` $\{A\ B : eqType\} (\alpha : Rel\ A\ B)$
 $:= (function_r\ \alpha) \wedge (total_r\ (\alpha^\#)) \wedge (univalent_r\ (\alpha^\#))$.

2.3 関係の公理

今後の諸定理の証明は, 原則以下の公理群, およびそれらから導かれる補題のみを用いて行っていくことにする.

2.3.1 Dedekind 圏の公理

Axiom 1 (`comp_id_l`, `comp_id_r`) *Let $\alpha : A \rightarrow B$. Then,*

$$id_A \cdot \alpha = \alpha \cdot id_B = \alpha.$$

Definition `axiom1a` $:= \forall (A\ B : eqType) (\alpha : Rel\ A\ B), Id\ A \cdot \alpha = \alpha$.

Axiom `comp_id_l` : `axiom1a`.

Definition `axiom1b` $:= \forall (A\ B : eqType) (\alpha : Rel\ A\ B), \alpha \cdot Id\ B = \alpha$.

Axiom `comp_id_r` : `axiom1b`.

Axiom 2 (comp_assoc) Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$, and $\gamma : C \rightarrow D$. Then,

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

Definition *axiom2* :=

$\forall (A\ B\ C\ D : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ C\ D),$
 $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$

Axiom *comp_assoc* : *axiom2*.

Axiom 3 (inc_refl) Let $\alpha : A \rightarrow B$. Then,

$$\alpha \sqsubseteq \alpha.$$

Definition *axiom3* := $\forall (A\ B : eqType)(\alpha : Rel\ A\ B), \alpha \sqsubseteq \alpha.$

Axiom *inc_refl* : *axiom3*.

Axiom 4 (inc_trans) Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma \Rightarrow \alpha \sqsubseteq \gamma.$$

Definition *axiom4* := $\forall (A\ B : eqType)(\alpha\ \beta\ \gamma : Rel\ A\ B),$
 $\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma \Rightarrow \alpha \sqsubseteq \gamma.$

Axiom *inc_trans* : *axiom4*.

Axiom 5 (inc_antisym) Let $\alpha, \beta : A \rightarrow B$. Then,

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \alpha \Rightarrow \alpha = \beta.$$

Definition *axiom5* := $\forall (A\ B : eqType)(\alpha\ \beta : Rel\ A\ B),$
 $\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \alpha \Rightarrow \alpha = \beta.$

Axiom *inc_antisym* : *axiom5*.

Axiom 6 (inc_empty_alpha) Let $\alpha : A \rightarrow B$. Then,

$$\phi_{AB} \sqsubseteq \alpha.$$

Definition *axiom6* := $\forall (A\ B : eqType)(\alpha : Rel\ A\ B), \phi_{AB} \sqsubseteq \alpha.$

Axiom *inc_empty_alpha* : *axiom6*.

Axiom 7 (inc_alpha_universal) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \nabla_{AB}.$$

Definition *axiom7* := $\forall (A B : eqType)(\alpha : Rel A B), \alpha \sqsubseteq \nabla_{AB}$.

Axiom *inc_alpha_universal* : *axiom7*.

Axiom 8 (inc_capP, inc_cap)

1. **inc_capP** : *Let $\alpha : A \rightarrow B$, $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and P : predicate. Then,*

$$\alpha \sqsubseteq (\cap_{P(\beta)} f(\beta)) \Leftrightarrow \forall \beta : C \rightarrow D, P(\beta) \Rightarrow \alpha \sqsubseteq f(\beta).$$

2. **inc_cap** : *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq (\beta \cap \gamma) \Leftrightarrow \alpha \sqsubseteq \beta \wedge \alpha \sqsubseteq \gamma.$$

Definition *axiom8a* :=

$\forall (A B C D : eqType)$
 $(\alpha : Rel A B)(f : Rel C D \rightarrow Rel A B)(P : Rel C D \rightarrow Prop),$
 $\alpha \sqsubseteq (\cap_{P} f) \Leftrightarrow \forall \beta : Rel C D, P \beta \rightarrow \alpha \sqsubseteq f \beta.$

Axiom *inc_capP* : *axiom8a*.

Definition *axiom8b* := $\forall (A B : eqType)(\alpha \beta \gamma : Rel A B),$
 $\alpha \sqsubseteq (\beta \cap \gamma) \Leftrightarrow (\alpha \sqsubseteq \beta) \wedge (\alpha \sqsubseteq \gamma).$

Lemma *inc_cap* : *axiom8b*.

Proof.

move $\Rightarrow A B \alpha \beta \gamma$.

rewrite *inc_capP*.

split; move $\Rightarrow H$.

split; apply *H*.

by [left].

by [right].

move $\Rightarrow \Delta H0$.

case *H0* $\Rightarrow H1$; rewrite *H1*; apply *H*.

Qed.

Axiom 9 (inc_cupP, inc_cup)

1. **inc_cupP** : Let $\alpha : A \rightarrow B$, $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,

$$(\sqcup_{P(\beta)} f(\beta)) \sqsubseteq \alpha \Leftrightarrow \forall \beta : C \rightarrow D, P(\beta) \Rightarrow f(\beta) \sqsubseteq \alpha.$$

2. **inc_cup** : Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,

$$(\beta \sqcup \gamma) \sqsubseteq \alpha \Leftrightarrow \beta \sqsubseteq \alpha \wedge \gamma \sqsubseteq \alpha.$$

Definition *axiom9a* :=

$\forall (A\ B\ C\ D : \text{eqType})$
 $(\alpha : \text{Rel } A\ B)(f : \text{Rel } C\ D \rightarrow \text{Rel } A\ B)(P : \text{Rel } C\ D \rightarrow \text{Prop}),$
 $(_ \{P\} f) \quad \alpha \leftrightarrow \forall \text{beta} : \text{Rel } C\ D, P\ \text{beta} \rightarrow f\ \text{beta} \quad \alpha.$

Axiom *inc_cupP* : *axiom9a*.

Definition *axiom9b* := $\forall (A\ B : \text{eqType})(\alpha\ \text{beta}\ \gamma : \text{Rel } A\ B),$
 $(\text{beta}\ \gamma) \quad \alpha \leftrightarrow (\text{beta}\ \alpha) \wedge (\gamma\ \alpha).$

Lemma *inc_cup* : *axiom9b*.

Proof.

move $\Rightarrow A\ B\ \alpha\ \text{beta}\ \gamma$.

rewrite *inc_cupP*.

split; move $\Rightarrow H$.

split; apply *H*.

by [left].

by [right].

move $\Rightarrow \text{delta } H0$.

case *H0* $\Rightarrow H1$; rewrite *H1*; apply *H*.

Qed.

Axiom 10 (inc_rpc) Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,

$$\alpha \sqsubseteq (\beta \Rightarrow \gamma) \Leftrightarrow (\alpha \sqcap \beta) \sqsubseteq \gamma.$$

Definition *axiom10* := $\forall (A\ B : \text{eqType})(\alpha\ \text{beta}\ \gamma : \text{Rel } A\ B),$
 $\alpha \sqsubseteq (\text{beta} \gg \gamma) \leftrightarrow (\alpha\ \text{beta}) \sqsubseteq \gamma.$

Axiom *inc_rpc* : *axiom10*.

Axiom 11 (inv_invol) Let $\alpha : A \rightarrow B$. Then,

$$(\alpha^\#)^\# = \alpha.$$

Definition *axiom11* := $\forall (A\ B : \text{eqType})(\alpha : \text{Rel } A\ B), (\alpha^\#)^\# = \alpha.$

Axiom *inv_invol* : *axiom11*.

CHAPTER 2. LIBRARY BASIC_NOTATIONS_REL

Axiom 12 (comp_inv) Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then,

$$(\alpha \cdot \beta)^\# = \beta^\# \cdot \alpha^\#.$$

Definition *axiom12* := $\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C),$
 $(\alpha \cdot \beta)^\# = (\beta^\# \cdot \alpha^\#).$

Axiom *comp_inv* : *axiom12*.

Axiom 13 (inc_inv) Let $\alpha, \beta : A \rightarrow B$. Then,

$$\alpha \sqsubseteq \beta \Rightarrow \alpha^\# \sqsubseteq \beta^\#.$$

Definition *axiom13* :=

$\forall (A\ B : eqType)(\alpha\ \beta : Rel\ A\ B), \alpha \rightarrow \beta \Rightarrow \alpha^\# \rightarrow \beta^\#.$

Axiom *inc_inv* : *axiom13*.

Axiom 14 (dedekind) Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$, and $\gamma : A \rightarrow C$. Then,

$$(\alpha \cdot \beta) \sqcap \gamma \sqsubseteq (\alpha \sqcap (\gamma \cdot \beta^\#)) \cdot (\beta \sqcap (\alpha^\# \cdot \gamma)).$$

Definition *axiom14* :=

$\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ A\ C),$
 $((\alpha \cdot \beta) \sqcap \gamma) \sqsubseteq ((\alpha \sqcap (\gamma \cdot \beta^\#)) \cdot (\beta \sqcap (\alpha^\# \cdot \gamma))).$

Axiom *dedekind* : *axiom14*.

Axiom 15 (inc_residual) Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$, and $\gamma : A \rightarrow C$. Then,

$$\gamma \sqsubseteq (\alpha \triangleright \beta) \Leftrightarrow \alpha^\# \cdot \gamma \sqsubseteq \beta.$$

Definition *axiom15* :=

$\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ A\ C),$
 $\gamma \sqsubseteq (\alpha \triangleright \beta) \Leftrightarrow (\alpha^\# \cdot \gamma) \sqsubseteq \beta.$

Axiom *inc_residual* : *axiom15*.

2.3.2 排中律

Dedekind 圏の公理のほかに, 以下の“排中律”を仮定すれば, 与えられる圏は Schröder 圏となり, Bool 代数の性質も満たされる. ちなみに剰余合成は補関係から定義可能なので, 本来 Schröder 圏には剰余合成に関する公理は存在しない.

Axiom 16 (complement_classic) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqcup \alpha^- = \nabla_{AB}$$

Definition *axiom16* := $\forall (A B : eqType)(\alpha : Rel A B),$
 $\alpha \sqcup \alpha^- = \nabla_{AB}.$

Axiom *complement_classic* : *axiom16.*

2.3.3 単域

1 点集合 I が単域となるための条件は

$$\phi_{II} \neq id_I \wedge id_I = \nabla_{II} \wedge \forall X, \nabla_{XI} \cdot \nabla_{IX} = \nabla_{XX}$$

だが, Rel の定義から左 2 つは証明できるため, 右の式だけ仮定する.

Axiom 17 (unit_universal)

$$\nabla_{AI} \cdot \nabla_{IA} = \nabla_{AA}$$

Definition *axiom17* := $\forall (A : eqType), \nabla_{AI} \cdot \nabla_{IA} = \nabla_{AA}.$

Axiom *unit_universal* : *axiom17.*

2.3.4 選択公理

この“選択公理”を仮定すれば, 排中律と単域の存在 (厳密には全域性公理) を利用して点公理を導出できる.

Axiom 18 (axiom_of_choice) *Let $\alpha : A \rightarrow B$ be a total relation. Then,*

$$\exists \beta : A \rightarrow B, \beta \sqsubseteq \alpha.$$

Definition *axiom18* := $\forall (A B : eqType)(\alpha : Rel A B),$
 $total_r \alpha \rightarrow \exists \beta : Rel A B, function_r \beta \wedge \beta \sqsubseteq \alpha.$

Axiom *axiom_of_choice* : *axiom18.*

2.3.5 関係の有理性

集合論では色々インポートしながら頑張って証明したので、できればそちらもご覧ください。

Axiom 19 (rationality) *Let $\alpha : A \rightarrow B$. Then,*

$$\exists R, \exists f : R \rightarrow A, \exists g : R \rightarrow B, \alpha = f^\# \cdot g \wedge f \cdot f^\# \sqcap g \cdot g^\# = id_R.$$

Definition *axiom19* := $\forall (A B : eqType)(alpha : Rel A B),$

$\exists (R : eqType)(f : Rel R A)(g : Rel R B),$

$function_r f \wedge function_r g \wedge alpha = f^\# \cdot g \wedge ((f \cdot f^\#) \sqcap (g \cdot g^\#)) = Id R.$

Axiom *rationality* : *axiom19*.

2.3.6 直和と直積

任意の直和に対して、入射対が存在することを仮定する。

Axiom 20 (pair_of_inclusions) $\exists j : A \rightarrow A + B, \exists k : B \rightarrow A + B,$

$$j \cdot j^\# = id_A \wedge k \cdot k^\# = id_B \wedge j \cdot k^\# = \phi_{AB} \wedge j^\# \cdot j \sqcup k^\# \cdot k = id_{A+B}.$$

Definition *axiom20* :=

$\forall (A B : eqType), \exists (j : Rel A (sum_eqType A B))(k : Rel B (sum_eqType A B)),$

$j \cdot j^\# = Id A \wedge k \cdot k^\# = Id B \wedge j \cdot k^\# = \phi_{AB} \wedge$

$(j^\# \cdot j) \sqcup (k^\# \cdot k) = Id (sum_eqType A B).$

Axiom *pair_of_inclusions* : *axiom20*.

任意の直積に対して、射影対が存在することを仮定する。

Axiom 21 (pair_of_projections) $\exists p : A \times B \rightarrow A, \exists q : A \times B \rightarrow B,$

$$p^\# \cdot q = \nabla_{AB} \wedge p \cdot p^\# \sqcap q \cdot q^\# = id_{A \times B}.$$

Definition *axiom21* :=

$\forall (A B : eqType), \exists (p : Rel (prod_eqType A B) A)(q : Rel (prod_eqType A B) B),$

$p^\# \cdot q = \nabla_{AB} \wedge (p \cdot p^\#) \sqcap (q \cdot q^\#) = Id (prod_eqType A B) \wedge univalent_r p$
 $\wedge univalent_r q.$

Axiom *pair_of_projections* : *axiom21*.

Chapter 3

Library `Basic_Notations_Set`

```
Require Export ssreflect eqtype bigop.
Require Export Logic.ClassicalFacts.
Require Import Logic.FunctionalExtensionality.
Require Import Logic.Classical_Prop.
Require Import Logic.IndefiniteDescription.
Require Import Logic.ProofIrrelevance.

Axiom prop_extensionality_ok : prop_extensionality.
```

3.1 定義

この章では、関係を集合論的に定義した場合の定義、およびその定義で諸公理が成立することを示す。公理名や記号などは `Basic_Notations` と同じものを使用するため、`Basic_Lemms` 以降ではその代わりにこのライブラリをインポートすることもできる。

```
Definition Rel (A B : eqType) := A → B → Prop.

Definition inverse {A B : eqType} (alpha : Rel A B) : Rel B A
:= (fun (b : B)(a : A) => alpha a b).

Notation "a #" := (inverse a) (at level 20).

Definition composite {A B C : eqType} (alpha : Rel A B) (beta : Rel B C) : Rel A C
:= (fun (a : A)(c : C) => ∃ b : B, alpha a b ∧ beta b c).

Notation "a ' · ' b" := (composite a b) (at level 50).

Definition residual {A B C : eqType} (alpha : Rel A B) (beta : Rel B C) : Rel A C
:= (fun (a : A)(c : C) => ∀ b : B, alpha a b → beta b c).

Notation "a ' ' b" := (residual a b) (at level 50).

Definition identity (A : eqType) : Rel A A := (fun a a0 : A => a = a0).

Notation "'Id'" := identity.

Definition empty (A B : eqType) : Rel A B := (fun (a : A)(b : B) => False).

Notation "' ' " := empty.

Definition universal (A B : eqType) : Rel A B := (fun (a : A)(b : B) => True).
```


Notation "''" := *universal*.

Definition *include* $\{A B : eqType\}$ ($\alpha \beta : Rel A B$) : **Prop**
:= $(\forall (a : A)(b : B), \alpha a b \rightarrow \beta a b)$.

Notation "a' ' b" := (*include a b*) (at level 50).

Definition *cupP* $\{A B C D : eqType\}$ ($P : Rel C D \rightarrow Prop$) ($f : Rel C D \rightarrow Rel A B$) : $Rel A B$
:= $(\text{fun } (a : A)(b : B) \Rightarrow \exists \alpha : Rel C D, P \alpha \wedge (f \alpha) a b)$.

Notation "'' -{' p '}' f" := (*cupP p f*) (at level 50).

Definition *capP* $\{A B C D : eqType\}$ ($P : Rel C D \rightarrow Prop$) ($f : Rel C D \rightarrow Rel A B$) : $Rel A B$
:= $(\text{fun } (a : A)(b : B) \Rightarrow \forall \alpha : Rel C D, P \alpha \rightarrow (f \alpha) a b)$.

Notation "'' -{' p '}' f" := (*capP p f*) (at level 50).

Definition *cup* $\{A B : eqType\}$ ($\alpha \beta : Rel A B$)
:= $\text{--}\{\text{fun } \gamma : Rel A B \Rightarrow \gamma = \alpha \vee \gamma = \beta\} id$.

Notation "a' ' b" := (*cup a b*) (at level 50).

Definition *cap* $\{A B : eqType\}$ ($\alpha \beta : Rel A B$)
:= $\text{--}\{\text{fun } \gamma : Rel A B \Rightarrow \gamma = \alpha \vee \gamma = \beta\} id$.

Notation "a' ' b" := (*cap a b*) (at level 50).

Definition *rpc* $\{A B : eqType\}$ ($\alpha \beta : Rel A B$) : $Rel A B$
:= $(\text{fun } (a : A)(b : B) \Rightarrow \alpha a b \rightarrow \beta a b)$.

Notation "a' » b" := (*rpc a b*) (at level 50).

Definition *complement* $\{A B : eqType\}$ ($\alpha : Rel A B$) := $\alpha \gg A B$.

Notation "a' ^" := (*complement a*) (at level 20).

Definition *difference* $\{A B : eqType\}$ ($\alpha \beta : Rel A B$) := $\alpha \beta ^$.

Notation "a - b" := (*difference a b*) (at level 50).

Notation "i'" := *unit_eqType*.

3.2 関数の定義

Definition *total_r* $\{A B : eqType\}$ ($\alpha : Rel A B$) := $(Id A) \quad (\alpha \cdot \alpha \#)$.

Definition *univalent_r* $\{A B : eqType\}$ ($\alpha : Rel A B$) := $(\alpha \# \cdot \alpha) \quad (Id B)$.

Definition *function_r* $\{A B : eqType\}$ ($\alpha : Rel A B$)
:= $(total_r \alpha) \wedge (univalent_r \alpha)$.

Definition *surjection_r* $\{A B : eqType\}$ ($\alpha : Rel A B$)
:= $(function_r \alpha) \wedge (total_r (\alpha \#))$.

Definition *injection_r* $\{A B : eqType\}$ ($\alpha : Rel A B$)
:= $(function_r \alpha) \wedge (univalent_r (\alpha \#))$.

Definition *bijection_r* $\{A B : eqType\}$ ($\alpha : Rel A B$)
:= $(function_r \alpha) \wedge (total_r (\alpha \#)) \wedge (univalent_r (\alpha \#))$.

3.3 関係の公理

今後の諸定理の証明は、原則以下の公理群、およびそれらから導かれる補題のみを用いて行っていくことにする。

3.3.1 Dedekind 圏の公理

Lemma 1 (`comp_id_l`, `comp_id_r`) *Let $\alpha : A \rightarrow B$. Then,*

$$id_A \cdot \alpha = \alpha \cdot id_B = \alpha.$$

Definition `axiom1a` := $\forall (A\ B : eqType)(\alpha : Rel\ A\ B),\ Id\ A \cdot \alpha = \alpha$.

Lemma `comp_id_l` : `axiom1a`.

Proof.

`move $\Rightarrow A\ B\ \alpha$.`

`apply functional_extensionality.`

`move $\Rightarrow a$.`

`apply functional_extensionality.`

`move $\Rightarrow b$.`

`apply prop_extensionality_ok.`

`split.`

`elim $\Rightarrow a0$.`

`elim $\Rightarrow H\ H0$.`

`rewrite H .`

`apply $H0$.`

`move $\Rightarrow H$.`

`$\exists\ a$.`

`split.`

`by [].`

`apply H .`

Qed.

Definition `axiom1b` := $\forall (A\ B : eqType)(\alpha : Rel\ A\ B),\ \alpha \cdot Id\ B = \alpha$.

Lemma `comp_id_r` : `axiom1b`.

Proof.

`move $\Rightarrow A\ B\ \alpha$.`

`apply functional_extensionality.`

`move $\Rightarrow a$.`

`apply functional_extensionality.`

`move $\Rightarrow b$.`

`apply prop_extensionality_ok.`

`split.`

`elim $\Rightarrow b0$.`

```

elim  $\Rightarrow$   $H$   $H0$ .
rewrite  $-H0$ .
apply  $H$ .
move  $\Rightarrow$   $H$ .
 $\exists$   $b$ .
split.
apply  $H$ .
by  $||$ .
Qed.

```

Lemma 2 (comp_assoc) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$, and $\gamma : C \rightarrow D$. Then,*

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

Definition *axiom2* :=

$\forall (A\ B\ C\ D : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ C\ D),$
 $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$

Lemma *comp_assoc* : *axiom2*.

Proof.

```

move  $\Rightarrow$   $A\ B\ C\ D\ \alpha\ \beta\ \gamma$ .
apply functional_extensionality.
move  $\Rightarrow$   $a$ .
apply functional_extensionality.
move  $\Rightarrow$   $d$ .
apply prop_extensionality_ok.
split.
elim  $\Rightarrow$   $c$ .
elim  $\Rightarrow$   $H\ H0$ .
elim  $H \Rightarrow$   $b\ H1$ .
 $\exists$   $b$ .
split.
apply  $H1$ .
 $\exists$   $c$ .
split.
apply  $H1$ .
apply  $H0$ .
elim  $\Rightarrow$   $b$ .
elim  $\Rightarrow$   $H$ .
elim  $\Rightarrow$   $c\ H0$ .
 $\exists$   $c$ .
split.
 $\exists$   $b$ .
split.

```

apply *H*.
 apply *H0*.
 apply *H0*.
 Qed.

Lemma 3 (inc_refl) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \alpha.$$

Definition *axiom3* := $\forall (A B : eqType)(\alpha : Rel A B), \alpha \sqsubseteq \alpha$.

Lemma *inc_refl* : *axiom3*.

Proof.

by [rewrite /*axiom3*/include].

Qed.

Lemma 4 (inc_trans) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma \Rightarrow \alpha \sqsubseteq \gamma.$$

Definition *axiom4* := $\forall (A B : eqType)(\alpha \beta \gamma : Rel A B),$

$\alpha \sqsubseteq \beta \rightarrow \beta \sqsubseteq \gamma \rightarrow \alpha \sqsubseteq \gamma$.

Lemma *inc_trans* : *axiom4*.

Proof.

move $\Rightarrow A B \alpha \beta \gamma H H0 a b H1$.

apply (*H0* _ _ (*H* _ _ *H1*)).

Qed.

Lemma 5 (inc_antisym) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \alpha \Rightarrow \alpha = \beta.$$

Definition *axiom5* := $\forall (A B : eqType)(\alpha \beta : Rel A B),$

$\alpha \sqsubseteq \beta \rightarrow \beta \sqsubseteq \alpha \rightarrow \alpha = \beta$.

Lemma *inc_antisym* : *axiom5*.

Proof.

move $\Rightarrow A B \alpha \beta H H0$.

apply *functional_extensionality*.

move $\Rightarrow a$.

apply *functional_extensionality*.

move $\Rightarrow b$.

apply *prop_extensionality_ok*.

split.

apply *H*.

apply *H0*.

Qed.

Lemma 6 (inc_empty_alpha) *Let $\alpha : A \rightarrow B$. Then,*

$$\phi_{AB} \sqsubseteq \alpha.$$

Definition *axiom6* := $\forall (A B : eqType)(\alpha : Rel A B), \quad A B \quad \alpha$.

Lemma *inc_empty_alpha* : *axiom6*.

Proof.

move $\Rightarrow A B \alpha a b$.

apply *False_ind*.

Qed.

Lemma 7 (inc_alpha_universal) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \nabla_{AB}.$$

Definition *axiom7* := $\forall (A B : eqType)(\alpha : Rel A B), \alpha \sqsubseteq \nabla_{AB}$.

Lemma *inc_alpha_universal* : *axiom7*.

Proof.

move $\Rightarrow A B \alpha a b H$.

apply *I*.

Qed.

Lemma 8 (inc_capP, inc_cap)

1. **inc_capP** : *Let $\alpha : A \rightarrow B$, $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and P : predicate. Then,*

$$\alpha \sqsubseteq (\sqcap_{P(\beta)} f(\beta)) \Leftrightarrow \forall \beta : C \rightarrow D, P(\beta) \Rightarrow \alpha \sqsubseteq f(\beta).$$

2. **inc_cap** : *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq (\beta \sqcap \gamma) \Leftrightarrow \alpha \sqsubseteq \beta \wedge \alpha \sqsubseteq \gamma.$$

Definition *axiom8a* :=

$\forall (A B C D : eqType)$

$(\alpha : Rel A B)(f : Rel C D \rightarrow Rel A B)(P : Rel C D \rightarrow \mathbf{Prop}),$

$\alpha \sqsubseteq (\sqcap_{\{P\}} f) \Leftrightarrow \forall \beta : Rel C D, P \beta \rightarrow \alpha \sqsubseteq f \beta.$

Lemma *inc_capP* : *axiom8a*.

Proof.

move $\Rightarrow A B C D \alpha f P$.

split; move $\Rightarrow H$.

```

move ⇒ beta H0 a b H1.
apply (H _ _ H1 _ H0).
move ⇒ a b H0 beta H1.
apply (H _ H1 _ _ H0).
Qed.
Definition axiom8b := ∀ (A B : eqType) (alpha beta gamma : Rel A B),
  alpha (beta gamma) ↔ (alpha beta) ∧ (alpha gamma).
Lemma inc_cap : axiom8b.
Proof.
move ⇒ A B alpha beta gamma.
rewrite inc_capP.
split; move ⇒ H.
split; apply H.
by [left].
by [right].
move ⇒ delta H0.
case H0 ⇒ H1; rewrite H1; apply H.
Qed.

```

Lemma 9 (inc_cupP, inc_cup)

1. **inc_cupP** : Let $\alpha : A \rightarrow B$, $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,

$$(\sqcup_{P(\beta)} f(\beta)) \sqsubseteq \alpha \Leftrightarrow \forall \beta : C \rightarrow D, P(\beta) \Rightarrow f(\beta) \sqsubseteq \alpha.$$

2. **inc_cup** : Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,

$$(\beta \sqcup \gamma) \sqsubseteq \alpha \Leftrightarrow \beta \sqsubseteq \alpha \wedge \gamma \sqsubseteq \alpha.$$

```

Definition axiom9a :=
  ∀ (A B C D : eqType)
  (alpha : Rel A B) (f : Rel C D → Rel A B) (P : Rel C D → Prop),
  ( _ {P} f ) alpha ↔ ∀ beta : Rel C D, P beta → f beta alpha.

```

Lemma inc_cupP : axiom9a.

Proof.

```

move ⇒ A B C D alpha f P.
split.
move ⇒ H beta H0 a b H1.
apply H.
∃ beta.
split.
apply H0.
apply H1.

```

```

move ⇒ H a b.
elim ⇒ beta.
elim ⇒ H0 H1.
apply (H beta H0 _ _ H1).
Qed.
Definition axiom9b := ∀ (A B : eqType) (alpha beta gamma : Rel A B),
  (beta gamma) alpha ↔ (beta alpha) ∧ (gamma alpha).
Lemma inc_cup : axiom9b.
Proof.
move ⇒ A B alpha beta gamma.
rewrite inc_cupP.
split; move ⇒ H.
split; apply H.
by [left].
by [right].
move ⇒ delta H0.
case H0 ⇒ H1; rewrite H1; apply H.
Qed.

```

Lemma 10 (inc_rpc) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq (\beta \Rightarrow \gamma) \Leftrightarrow (\alpha \sqcap \beta) \sqsubseteq \gamma.$$

```

Definition axiom10 := ∀ (A B : eqType) (alpha beta gamma : Rel A B),
  alpha (beta » gamma) ↔ (alpha beta) gamma.
Lemma inc_rpc : axiom10.
Proof.
move ⇒ A B alpha beta gamma.
split; move ⇒ H.
move ⇒ a b H0.
apply H.
apply H0.
by [left].
apply H0.
by [right].
move ⇒ a b H0 H1.
apply H.
move ⇒ delta.
case ⇒ H2; rewrite H2.
apply H0.
apply H1.
Qed.

```

Lemma 11 (inv_invol) *Let $\alpha : A \rightarrow B$. Then,*

$$(\alpha^\#)^\# = \alpha.$$

Definition *axiom11* := $\forall (A\ B : eqType)(\alpha : Rel\ A\ B), (\alpha^\#)^\# = \alpha$.

Lemma *inv_invol* : *axiom11*.

Proof.

by [move $\Rightarrow A\ B\ \alpha$].

Qed.

Lemma 12 (comp_inv) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then,*

$$(\alpha \cdot \beta)^\# = \beta^\# \cdot \alpha^\#.$$

Definition *axiom12* := $\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C), (\alpha \cdot \beta)^\# = (\beta^\# \cdot \alpha^\#)$.

Lemma *comp_inv* : *axiom12*.

Proof.

move $\Rightarrow A\ B\ C\ \alpha\ \beta$.

apply *functional_extensionality*.

move $\Rightarrow c$.

apply *functional_extensionality*.

move $\Rightarrow a$.

apply *prop_extensionality_ok*.

split; elim $\Rightarrow b$.

elim $\Rightarrow H\ H0$.

$\exists b$.

split.

apply *H0*.

apply *H*.

elim $\Rightarrow H\ H0$.

$\exists b$.

split.

apply *H0*.

apply *H*.

Qed.

Lemma 13 (inc_inv) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \beta \Rightarrow \alpha^\# \sqsubseteq \beta^\#.$$

Definition *axiom13* :=

$\forall (A\ B : eqType)(\alpha\ \beta : Rel\ A\ B), \alpha \sqsubseteq \beta \rightarrow \alpha^\# \sqsubseteq \beta^\#$.

Lemma *inc_inv* : *axiom13*.

Proof.

move $\Rightarrow A\ B\ \text{alpha}\ \text{beta}\ H\ b\ a\ H0$.

apply ($H\ _ _ H0$).

Qed.

Lemma 14 (dedekind) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$, and $\gamma : A \rightarrow C$. Then,*

$$(\alpha \cdot \beta) \sqcap \gamma \sqsubseteq (\alpha \sqcap (\gamma \cdot \beta^\#)) \cdot (\beta \sqcap (\alpha^\# \cdot \gamma)).$$

Definition *axiom14* :=

$\forall (A\ B\ C : \text{eqType})(\text{alpha} : \text{Rel}\ A\ B)(\text{beta} : \text{Rel}\ B\ C)(\text{gamma} : \text{Rel}\ A\ C),$
 $((\text{alpha} \cdot \text{beta}) \sqcap \text{gamma})$
 $((\text{alpha} \sqcap (\text{gamma} \cdot \text{beta}^\#)) \cdot (\text{beta} \sqcap (\text{alpha}^\# \cdot \text{gamma}))).$

Lemma *dedekind* : *axiom14*.

Proof.

move $\Rightarrow A\ B\ C\ \text{alpha}\ \text{beta}\ \text{gamma}\ a\ c\ H$.

assert ($\exists b : B, \text{alpha}\ a\ b \wedge \text{beta}\ b\ c$).

apply H .

by [left].

elim $H0 \Rightarrow b$.

elim $\Rightarrow H1\ H2$.

$\exists b$.

repeat split.

move $\Rightarrow \text{delta}\ H3$.

case $H3 \Rightarrow H4$; rewrite $H4$.

apply $H1$.

unfold *id*.

$\exists c$.

split.

apply H .

by [right].

apply $H2$.

move $\Rightarrow \text{delta}\ H3$.

case $H3 \Rightarrow H4$; rewrite $H4$.

apply $H2$.

unfold *id*.

$\exists a$.

split.

apply $H1$.

apply H .

by [right].

Qed.

Lemma 15 (inc_residual) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$, and $\gamma : A \rightarrow C$. Then,*

$$\gamma \sqsubseteq (\alpha \triangleright \beta) \Leftrightarrow \alpha^\# \cdot \gamma \sqsubseteq \beta.$$

Definition *axiom15* :=

$\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ A\ C),$
 $\gamma \sqsubseteq (\alpha \triangleright \beta) \Leftrightarrow \alpha^\# \cdot \gamma \sqsubseteq \beta.$

Lemma *inc_residual* : *axiom15*.

Proof.

move $\Rightarrow A\ B\ C\ \alpha\ \beta\ \gamma$.

split; move $\Rightarrow H$.

move $\Rightarrow b\ c$.

elim $\Rightarrow a\ H0$.

apply $(H\ a)$.

apply $H0$.

apply $H0$.

move $\Rightarrow a\ c\ H0\ b\ H1$.

apply H .

$\exists\ a$.

split.

apply $H1$.

apply $H0$.

Qed.

3.3.2 排中律

Dedekind 圏の公理のほかに、以下の“排中律”を仮定すれば、与えられる圏は Schröder 圏となり、Bool 代数の性質も満たされる。ちなみに剰余合成は補関係から定義可能なので、本来 Schröder 圏には剰余合成に関する公理は存在しない。

Lemma 16 (complement_classic) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqcup \alpha^- = \nabla_{AB}$$

Definition *axiom16* := $\forall (A\ B : eqType)(\alpha : Rel\ A\ B),$
 $\alpha \sqcup \alpha^- = \nabla_{AB}.$

Lemma *complement_classic* : *axiom16*.

Proof.

move $\Rightarrow A\ B\ \alpha$.

apply *functional_extensionality*.

move $\Rightarrow a$.

apply *functional_extensionality*.

```

move ⇒ b.
apply prop_extensionality_ok.
split; move ⇒ H.
apply I.
case (classic (alpha a b)) ⇒ H0.
∃ alpha.
split.
by [left].
apply H0.
∃ (fun (a0 : A) (b0 : B) ⇒ alpha a0 b0 → False).
split.
by [right].
apply H0.
Qed.

```

3.3.3 単域

1 点集合 I が単域となるための条件は

$$\phi_{II} \neq id_I \wedge id_I = \nabla_{II} \wedge \forall X, \nabla_{XI} \cdot \nabla_{IX} = \nabla_{XX}$$

だが, Rel の定義から左 2 つは証明できるため, 右の式だけ仮定する.

Lemma 17 (*unit_universal*)

$$\nabla_{AI} \cdot \nabla_{IA} = \nabla_{AA}$$

Definition *axiom17* := $\forall (A : eqType), \quad A \text{ i } \bullet \quad i A = \quad A A$.

Lemma *unit_universal* : *axiom17*.

Proof.

```

move ⇒ A.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ a0.
apply prop_extensionality_ok.
split; move ⇒ H.
apply I.
∃ tt.
by [].
Qed.

```

3.3.4 選択公理

この“選択公理”を仮定すれば、排中律と単域の存在 (厳密には全域性公理) を利用して点公理を導出できる。証明には集合論の選択公理を用いる。

Lemma 18 (axiom_of_choice) *Let $\alpha : A \rightarrow B$ be a total relation. Then,*

$$\exists \beta : A \rightarrow B, \beta \sqsubseteq \alpha.$$

Definition *axiom18* := $\forall (A B : \text{eqType})(\alpha : \text{Rel } A B),$
 $\text{total_r } \alpha \rightarrow \exists \text{beta} : \text{Rel } A B, \text{function_r beta} \wedge \text{beta} \sqsubseteq \alpha.$

Lemma *axiom_of_choice* : *axiom18*.

Proof.

move $\Rightarrow A B \alpha$.

rewrite $/\text{function_r}/\text{total_r}/\text{univalent_r}/\text{identity}/\text{include}/\text{composite}/\text{inverse}.$

move $\Rightarrow H.$

assert $(\forall a : A, \{b : B \mid \alpha a b\}).$

move $\Rightarrow a.$

apply *constructive_indefinite_description*.

move : $(H a a (\text{Logic.eq_refl } a)).$

elim $\Rightarrow b H0.$

$\exists b.$

apply *H0*.

$\exists (\text{fun } (a : A)(b : B) \Rightarrow b = \text{sval } (X a)).$

repeat split.

move $\Rightarrow a a0 H0.$

$\exists (\text{sval } (X a)).$

by [rewrite *H0*].

move $\Rightarrow b b0.$

elim $\Rightarrow a.$

elim $\Rightarrow H0 H1.$

by [rewrite *H0 H1*].

move $\Rightarrow a b H0.$

rewrite *H0*.

apply *proj2_sig*.

Qed.

3.3.5 関係の有理性

集合の選択公理 (`Logic.IndefiniteDescription`) や証明の一意性 (`Logic.ProofIrrelevance`) を仮定すれば, 集合論上ならごり押しで証明できる. 旧ライブラリの頃は無理だと諦めて `Axiom` を追加していたが, `Standard Library` のインポートだけで解けた. 正直びっくり.

Lemma 19 (rationality) *Let $\alpha : A \rightarrow B$. Then,*

$$\exists R, \exists f : R \rightarrow A, \exists g : R \rightarrow B, \alpha = f^\# \cdot g \wedge f \cdot f^\# \sqcap g \cdot g^\# = id_R.$$

この付近は, ごり押しのための補題. 命題の真偽を選択公理で `bool` 値に変換したり, 部分集合の元から上位集合の元を生成する `sval (proj1_sig)` の単射性を示したりしている.

Lemma `is_true_inv0` : $\forall P : \mathbf{Prop}, \exists b : \mathbf{bool}, P \leftrightarrow \mathit{is_true} \ b$.

Proof.

`move $\Rightarrow P$.`

`case (classic P); move $\Rightarrow H$.`

`$\exists \mathit{true}$.`

`split; move $\Rightarrow H0$.`

`by [].`

`apply H .`

`$\exists \mathit{false}$.`

`split; move $\Rightarrow H0$.`

`apply $\mathit{False_ind}$.`

`apply ($H \ H0$).`

`discriminate $H0$.`

Qed.

Definition `is_true_inv` : $\mathbf{Prop} \rightarrow \mathbf{bool}$.

`move $\Rightarrow P$.`

`move : (is_true_inv0 P) $\Rightarrow H$.`

`apply constructive_indefinite_description in H .`

`apply H .`

Defined.

Lemma `is_true_id` : $\forall P : \mathbf{Prop}, \mathit{is_true} (\mathit{is_true_inv} \ P) \leftrightarrow P$.

Proof.

`move $\Rightarrow P$.`

`unfold is_true_inv.`

`move : (constructive_indefinite_description (fun b : bool $\Rightarrow P \leftrightarrow \mathit{is_true} \ b$) (is_true_inv0 P)) $\Rightarrow x0$.`

`apply (@sig_ind bool (fun b $\Rightarrow (P \leftrightarrow \mathit{is_true} \ b)$) (fun y $\Rightarrow \mathit{is_true} (\mathit{let} (x, _) := y \mathit{ in } x) \leftrightarrow P$)).`

move $\Rightarrow x\ H$.

apply *iff_sym*.

apply *H*.

Qed.

Lemma *sval_inv* : $\forall (A : \text{Type})(P : A \rightarrow \text{Prop})(x : \text{sig } P)(a : A), a = \text{sval } x \rightarrow \exists (H : P\ a), x = \text{exist } P\ a\ H$.

Proof.

move $\Rightarrow A\ P\ x\ a\ H0$.

rewrite *H0*.

\exists (*proj2_sig* *x*).

apply (@*sig_ind* *A* *P* (**fun** *y* $\Rightarrow y = \text{exist } P\ (\text{sval } y)\ (\text{proj2_sig } y)$)).

move $\Rightarrow a0\ H$.

by [simpl].

Qed.

Lemma *sval_injective* : $\forall (A : \text{Type})(P : A \rightarrow \text{Prop})(x\ y : \text{sig } P), \text{sval } x = \text{sval } y \rightarrow x = y$.

Proof.

move $\Rightarrow A\ P\ x\ y\ H$.

move : (*sval_inv* *A* *P* *y* (*sval* *x*) *H*).

elim $\Rightarrow H0\ H1$.

rewrite *H1*.

assert (*H0* = *proj2_sig* *x*).

apply *proof_irrelevance*.

rewrite *H2*.

apply (@*sig_ind* *A* *P* (**fun** *y* $\Rightarrow y = \text{exist } P\ (\text{sval } y)\ (\text{proj2_sig } y)$)).

move $\Rightarrow a0\ H3$.

by [simpl].

Qed.

Definition *axiom19* := $\forall (A\ B : \text{eqType})(\alpha : \text{Rel } A\ B),$

$\exists (R : \text{eqType})(f : \text{Rel } R\ A)(g : \text{Rel } R\ B),$

$\text{function_r } f \wedge \text{function_r } g \wedge \alpha = f \# \cdot g \wedge ((f \cdot f \#) \quad (g \cdot g \#)) = \text{Id } R.$

Lemma *rationality* : *axiom19*.

Proof.

move $\Rightarrow A\ B\ \alpha$.

rewrite /*function_r*/total_r/*univalent_r*/*cap*/*capP*/*identity*/*composite*/*inverse*/*include*.

\exists (*sig_eqType* (**fun** *x* : *prod_eqType* *A* *B* $\Rightarrow \text{is_true_inv } (\alpha\ (\text{fst } x)\ (\text{snd } x))$)).

\exists (**fun** *x* *a* $\Rightarrow a = (\text{fst } (\text{sval } x))$).

\exists (**fun** *x* *b* $\Rightarrow b = (\text{snd } (\text{sval } x))$).

simpl.

repeat split.

move $\Rightarrow x\ x0\ H$.

\exists (*fst* (*sval* *x*)).

```
repeat split.
by [rewrite H].
move  $\Rightarrow$  a a0.
elim  $\Rightarrow$  x.
elim  $\Rightarrow$  H H0.
by [rewrite H H0].
move  $\Rightarrow$  x x0 H.
 $\exists$  (snd (sval x)).
repeat split.
by [rewrite H].
move  $\Rightarrow$  b b0.
elim  $\Rightarrow$  x.
elim  $\Rightarrow$  H H0.
by [rewrite H H0].
apply functional_extensionality.
move  $\Rightarrow$  a.
apply functional_extensionality.
move  $\Rightarrow$  b.
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
assert (is_true (is_true_inv (alpha (fst (a,b)) (snd (a,b))))).
simpl.
apply is_true_id.
apply H.
 $\exists$  (exist (fun x  $\Rightarrow$  (is_true (is_true_inv (alpha (fst x) (snd x)))))) (a,b) H0).
by [simpl].
elim H  $\Rightarrow$  x.
elim  $\Rightarrow$  H0 H1.
rewrite H0 H1.
apply is_true_id.
apply (@sig_ind (A  $\times$  B) (fun x  $\Rightarrow$  is_true (is_true_inv (alpha (fst x) (snd x)))) (fun x
 $\Rightarrow$  is_true (is_true_inv (alpha (fst (sval x)) (snd (sval x)))))).
simpl.
by [move  $\Rightarrow$  x0].
apply functional_extensionality.
move  $\Rightarrow$  y.
apply functional_extensionality.
move  $\Rightarrow$  y0.
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
apply sval_injective.
move : (H (fun a c : {x : A  $\times$  B | is_true (is_true_inv (alpha (fst x) (snd x))}))  $\Rightarrow$   $\exists$  b :
```

```

A, b = fst (sval a) ∧ b = fst (sval c)) (or_introl Logic.eq_refl)).
move : (H (fun a c : {x : A × B | is_true (is_true_inv (alpha (fst x) (snd x))))} ⇒ ∃ b :
B, b = snd (sval a) ∧ b = snd (sval c)) (or_intror Logic.eq_refl)).
unfold id.
clear H.
elim ⇒ b.
elim ⇒ H H0.
elim ⇒ a.
elim ⇒ H1 H2.
rewrite (surjective_pairing (sval y0)) -H0 -H2 H H1.
apply surjective_pairing.
rewrite H.
move ⇒ beta H0.
case H0 ⇒ H1; rewrite H1; unfold id.
∃ (fst (sval y0)).
repeat split.
∃ (snd (sval y0)).
repeat split.
Qed.

```

3.3.6 直和と直積

任意の直和に対して、入射対が存在することを仮定する。

Lemma 20 (pair_of_inclusions) $\exists j : A \rightarrow A + B, \exists k : B \rightarrow A + B,$

$$j \cdot j^\# = id_A \wedge k \cdot k^\# = id_B \wedge j \cdot k^\# = \phi_{AB} \wedge j^\# \cdot j \sqcup k^\# \cdot k = id_{A+B}.$$

Definition axiom20 :=

$\forall (A B : eqType), \exists (j : Rel A (sum_eqType A B))(k : Rel B (sum_eqType A B)),$
 $j \cdot j^\# = Id A \wedge k \cdot k^\# = Id B \wedge j \cdot k^\# = \phi_{AB} \wedge$
 $(j^\# \cdot j) \sqcup (k^\# \cdot k) = Id (sum_eqType A B).$

Lemma pair_of_inclusions : axiom20.

Proof.

```

move ⇒ A B.
∃ (fun (a : A)(x : sum_eqType A B) ⇒ x = inl a).
∃ (fun (b : B)(x : sum_eqType A B) ⇒ x = inr b).
repeat split.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ a0.

```



```
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
elim H  $\Rightarrow$  x.
elim  $\Rightarrow$  H0 H1.
rewrite H0 in H1.
by [injection H1].
 $\exists$  (inl a).
repeat split.
by [rewrite H].
apply functional_extensionality.
move  $\Rightarrow$  b.
apply functional_extensionality.
move  $\Rightarrow$  b0.
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
elim H  $\Rightarrow$  x.
elim  $\Rightarrow$  H0 H1.
rewrite H0 in H1.
by [injection H1].
 $\exists$  (inr b).
repeat split.
by [rewrite H].
apply functional_extensionality.
move  $\Rightarrow$  a.
apply functional_extensionality.
move  $\Rightarrow$  b.
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
elim H  $\Rightarrow$  x.
elim  $\Rightarrow$  H0 H1.
rewrite H0 in H1.
discriminate H1.
apply False_ind.
apply H.
apply functional_extensionality.
move  $\Rightarrow$  x.
apply functional_extensionality.
move  $\Rightarrow$  x0.
apply prop_extensionality_ok.
split.
elim  $\Rightarrow$  alpha.
elim  $\Rightarrow$  H0 H1.
```

```

case  $H0 \Rightarrow H2$ ; rewrite  $H2$  in  $H1$ .
elim  $H1 \Rightarrow a$ .
elim  $\Rightarrow H3 H4$ .
by [rewrite  $H3 H4$ ].
elim  $H1 \Rightarrow b$ .
elim  $\Rightarrow H3 H4$ .
by [rewrite  $H3 H4$ ].
assert  $((\exists a : A, x = \text{inl } a) \vee (\exists b : B, x = \text{inr } b))$ .
move :  $x$ .
apply sum_ind.
move  $\Rightarrow a$ .
left.
by  $[\exists a]$ .
move  $\Rightarrow b$ .
right.
by  $[\exists b]$ .
case  $H$ .
elim  $\Rightarrow a H0 H1$ .
 $\exists$  (fun  $x x0 \Rightarrow \exists a0 : A, (x = \text{inl } a0 \wedge x0 = \text{inl } a0)$ ).
split.
by [left].
 $\exists a$ .
by [rewrite - $H1 H0$ ].
elim  $\Rightarrow b H0 H1$ .
 $\exists$  (fun  $x x0 \Rightarrow \exists b0 : B, (x = \text{inr } b0 \wedge x0 = \text{inr } b0)$ ).
split.
by [right].
 $\exists b$ .
by [rewrite - $H1 H0$ ].
Qed.
    
```

任意の直積に対して、射影対が存在することを仮定する。

Lemma 21 (pair_of_projections) $\exists p : A \times B \rightarrow A, \exists q : A \times B \rightarrow B,$

$$p^\# \cdot q = \nabla_{AB} \wedge p \cdot p^\# \sqcap q \cdot q^\# = \text{id}_{A \times B}.$$

Definition *axiom21* :=

$\forall (A B : \text{eqType}), \exists (p : \text{Rel } (\text{prod_eqType } A B) A) (q : \text{Rel } (\text{prod_eqType } A B) B),$
 $p \# \cdot q = A B \wedge (p \cdot p \#) \quad (q \cdot q \#) = \text{Id } (\text{prod_eqType } A B) \wedge \text{univalent_r } p$
 $\wedge \text{univalent_r } q.$

Lemma *pair_of_projections* : *axiom21*.

Proof.

```
move  $\Rightarrow A\ B$ .
 $\exists$  (fun ( $x : \text{prod\_eqType } A\ B$ ))( $a : A$ )  $\Rightarrow a = (\text{fst } x)$ .
 $\exists$  (fun ( $x : \text{prod\_eqType } A\ B$ ))( $b : B$ )  $\Rightarrow b = (\text{snd } x)$ .
split.
apply functional_extensionality.
move  $\Rightarrow a$ .
apply functional_extensionality.
move  $\Rightarrow b$ .
apply prop_extensionality_ok.
split; move  $\Rightarrow H$ .
apply I.
 $\exists (a, b)$ .
by [simpl].
split.
apply functional_extensionality.
move  $\Rightarrow x$ .
apply functional_extensionality.
move  $\Rightarrow x0$ .
apply prop_extensionality_ok.
split; move  $\Rightarrow H$ .
move : ( $H$  (fun  $a\ c : \text{prod\_eqType } A\ B \Rightarrow \exists b : A, b = \text{fst } a \wedge b = \text{fst } c$ ) (or_introl
Logic.eq_refl)).
move : ( $H$  (fun  $a\ c : \text{prod\_eqType } A\ B \Rightarrow \exists b : B, b = \text{snd } a \wedge b = \text{snd } c$ ) (or_intror
Logic.eq_refl)).
unfold id.
clear H.
elim  $\Rightarrow b$ .
elim  $\Rightarrow H\ H0$ .
elim  $\Rightarrow a$ .
elim  $\Rightarrow H1\ H2$ .
rewrite (surjective_pairing  $x0$ ) -H0 -H2 H H1.
apply surjective_pairing.
rewrite H.
move  $\Rightarrow \alpha\ H0$ .
case H0  $\Rightarrow H1$ ; rewrite H1; unfold id.
 $\exists (\text{fst } x0)$ .
repeat split.
 $\exists (\text{snd } x0)$ .
repeat split.
split.
move  $\Rightarrow a\ a0$ .
elim  $\Rightarrow x$ .
```

```
elim  $\Rightarrow$   $H\ H0$ .  
by [rewrite  $H\ H0$ ].  
move  $\Rightarrow$   $b\ b0$ .  
elim  $\Rightarrow$   $x$ .  
elim  $\Rightarrow$   $H\ H0$ .  
by [rewrite  $H\ H0$ ].  
Qed.
```

Chapter 4

Library `Basic_Lemmas`

```
Require Import Basic_Notations.  
Require Import Logic.Classical_Prop.
```

4.1 束論に関する補題

4.1.1 和関係, 共通関係

Lemma 22 (cap_l) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqcap \beta \sqsubseteq \alpha.$$

```
Lemma cap_l {A B : eqType} {alpha beta : Rel A B}: (alpha beta) alpha.
```

Proof.

```
assert ((alpha beta) (alpha beta)).
```

```
apply inc_refl.
```

```
apply inc_cap in H.
```

```
apply H.
```

Qed.

Lemma 23 (cap_r) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqcap \beta \sqsubseteq \beta.$$

```
Lemma cap_r {A B : eqType} {alpha beta : Rel A B}: (alpha beta) beta.
```

Proof.

```
assert ((alpha beta) (alpha beta)).
```

```
apply inc_refl.
```

```
apply inc_cap in H.
```

```
apply H.
```

Qed.

Lemma 24 (cup_l) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \alpha \sqcup \beta.$$

Lemma cup_l {A B : eqType} {alpha beta : Rel A B}: alpha (alpha beta).

Proof.

assert ((alpha beta) (alpha beta)).

apply inc_refl.

apply inc_cup in H.

apply H.

Qed.

Lemma 25 (cup_r) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\beta \sqsubseteq \alpha \sqcup \beta.$$

Lemma cup_r {A B : eqType} {alpha beta : Rel A B}: beta (alpha beta).

Proof.

assert ((alpha beta) (alpha beta)).

apply inc_refl.

apply inc_cup in H.

apply H.

Qed.

Lemma 26 (inc_def1) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha = \alpha \sqcap \beta \Leftrightarrow \alpha \sqsubseteq \beta.$$

Lemma inc_def1 {A B : eqType} {alpha beta : Rel A B}:

alpha = alpha beta \leftrightarrow alpha beta.

Proof.

split; move \Rightarrow H.

assert (alpha (alpha beta)).

rewrite -H.

apply inc_refl.

apply inc_cap in H0.

apply H0.

apply inc_antisym.

apply inc_cap.

split.

apply inc_refl.

apply *H*.
 apply *cap_l*.
 Qed.

Lemma 27 (inc_def2) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\beta = \alpha \sqcup \beta \Leftrightarrow \alpha \sqsubseteq \beta.$$

Lemma inc_def2 $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:
 $beta = alpha \quad beta \leftrightarrow alpha \quad beta.$

Proof.

split; move \Rightarrow *H*.
 assert $((alpha \quad beta) \quad beta).$
 rewrite -*H*.
 apply *inc_refl*.
 apply *inc_cup* in *H0*.
 apply *H0*.
 apply *inc_antisym*.
 assert $((alpha \quad beta) \quad (alpha \quad beta)).$
 apply *inc_refl*.
 apply *cup_r*.
 apply *inc_cup*.
 split.
 apply *H*.
 apply *inc_refl*.
 Qed.

Lemma 28 (cap_assoc) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$(\alpha \sqcap \beta) \sqcap \gamma = \alpha \sqcap (\beta \sqcap \gamma).$$

Lemma cap_assoc $\{A\ B : eqType\} \{alpha\ beta\ gamma : Rel\ A\ B\}$:
 $(alpha \quad beta) \quad gamma = alpha \quad (beta \quad gamma).$

Proof.

apply *inc_antisym*.
 rewrite *inc_cap*.
 split.
 apply $(inc_trans _ _ (alpha \quad beta)).$
 apply *cap_l*.
 apply *cap_l*.
 rewrite *inc_cap*.
 split.
 apply $(inc_trans _ _ (alpha \quad beta)).$

```

apply cap_l.
apply cap_r.
apply cap_r.
rewrite inc_cap.
split.
rewrite inc_cap.
split.
apply cap_l.
apply (inc_trans _ _ _ (beta gamma)).
apply cap_r.
apply cap_l.
apply (inc_trans _ _ _ (beta gamma)).
apply cap_r.
apply cap_r.
Qed.

```

Lemma 29 (cup_assoc) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$(\alpha \sqcup \beta) \sqcup \gamma = \alpha \sqcup (\beta \sqcup \gamma).$$

Lemma cup_assoc $\{A B : eqType\} \{alpha\ beta\ gamma : Rel\ A\ B\}$:
 $(alpha\ \beta)\ \gamma = alpha\ (\beta\ \gamma).$

Proof.

```

apply inc_antisym.
rewrite inc_cup.
split.
rewrite inc_cup.
split.
apply cup_l.
apply (inc_trans _ _ _ (beta gamma)).
apply cup_l.
apply cup_r.
apply (inc_trans _ _ _ (beta gamma)).
apply cup_r.
apply cup_r.
rewrite inc_cup.
split.
apply (inc_trans _ _ _ (alpha beta)).
apply cup_l.
apply cup_l.
rewrite inc_cup.
split.
apply (inc_trans _ _ _ (alpha beta)).

```


apply *cup_r*.
 apply *cup_l*.
 apply *cup_r*.
 Qed.

Lemma 30 (cap_comm) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqcap \beta = \beta \sqcap \alpha.$$

Lemma *cap_comm* {*A B : eqType*} {*alpha beta : Rel A B*}: *alpha* *beta = beta* *alpha*.

Proof.

apply *inc_antisym*.
 rewrite *inc_cap*.
 split.
 apply *cap_r*.
 apply *cap_l*.
 rewrite *inc_cap*.
 split.
 apply *cap_r*.
 apply *cap_l*.
 Qed.

Lemma 31 (cup_comm) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqcup \beta = \beta \sqcup \alpha.$$

Lemma *cup_comm* {*A B : eqType*} {*alpha beta : Rel A B*}: *alpha* *beta = beta* *alpha*.

Proof.

apply *inc_antisym*.
 rewrite *inc_cup*.
 split.
 apply *cup_r*.
 apply *cup_l*.
 rewrite *inc_cup*.
 split.
 apply *cup_r*.
 apply *cup_l*.
 Qed.

Lemma 32 (cup_cap_abs) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqcup (\alpha \sqcap \beta) = \alpha.$$

CHAPTER 4. LIBRARY BASIC_LEMMAS

Lemma *cup_cap_abs* {*A B* : *eqType*} {*alpha beta* : *Rel A B*}:
alpha (alpha *beta*) = *alpha*.

Proof.

move : (@*cap_l* _ _ *alpha beta*) \Rightarrow *H*.

apply *inc_def2* in *H*.

by [rewrite *cup_comm -H*].

Qed.

Lemma 33 (*cap_cup_abs*) *Let* $\alpha, \beta : A \rightarrow B$. *Then,*

$$\alpha \sqcap (\alpha \sqcup \beta) = \alpha.$$

Lemma *cap_cup_abs* {*A B* : *eqType*} {*alpha beta* : *Rel A B*}:
alpha (alpha *beta*) = *alpha*.

Proof.

move : (@*cup_l* _ _ *alpha beta*) \Rightarrow *H*.

apply *inc_def1* in *H*.

by [rewrite *-H*].

Qed.

Lemma 34 (*cap_idem*) *Let* $\alpha : A \rightarrow B$. *Then,*

$$\alpha \sqcap \alpha = \alpha.$$

Lemma *cap_idem* {*A B* : *eqType*} {*alpha* : *Rel A B*}: *alpha* *alpha* = *alpha*.

Proof.

apply *inc_antisym*.

apply *cap_l*.

apply *inc_cap*.

split; apply *inc_refl*.

Qed.

Lemma 35 (*cup_idem*) *Let* $\alpha : A \rightarrow B$. *Then,*

$$\alpha \sqcup \alpha = \alpha.$$

Lemma *cup_idem* {*A B* : *eqType*} {*alpha* : *Rel A B*}: *alpha* *alpha* = *alpha*.

Proof.

apply *inc_antisym*.

apply *inc_cup*.

split; apply *inc_refl*.

apply *cup_l*.

Qed.

CHAPTER 4. LIBRARY BASIC_LEMMAS

Lemma 36 (cap_inc_compat) *Let $\alpha, \alpha', \beta, \beta' : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha' \sqcap \beta'.$$

Lemma `cap_inc_compat` $\{A\ B : eqType\} \{alpha\ alpha'\ beta\ beta' : Rel\ A\ B\}$:
`alpha alpha' → beta beta' → (alpha beta) (alpha' beta').`

Proof.

`move ⇒ H H0.`

`rewrite -inc_def1.`

`apply inc_def1 in H.`

`apply inc_def1 in H0.`

`rewrite cap_assoc -(@cap_assoc - - beta).`

`rewrite (@cap_comm - - beta).`

`rewrite cap_assoc -(@cap_assoc - - alpha).`

`by [rewrite -H -H0].`

Qed.

Lemma 37 (cap_inc_compat_l) *Let $\alpha, \beta, \beta' : A \rightarrow B$. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha \sqcap \beta'.$$

Lemma `cap_inc_compat_l` $\{A\ B : eqType\} \{alpha\ beta\ beta' : Rel\ A\ B\}$:
`beta beta' → (alpha beta) (alpha beta').`

Proof.

`move ⇒ H.`

`apply (@cap_inc_compat - - - - - (@inc_refl - - alpha) H).`

Qed.

Lemma 38 (cap_inc_compat_r) *Let $\alpha, \alpha', \beta : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha' \sqcap \beta.$$

Lemma `cap_inc_compat_r` $\{A\ B : eqType\} \{alpha\ alpha'\ beta : Rel\ A\ B\}$:
`alpha alpha' → (alpha beta) (alpha' beta).`

Proof.

`move ⇒ H.`

`apply (@cap_inc_compat - - - - - H (@inc_refl - - beta)).`

Qed.

Lemma 39 (cup_inc_compat) *Let $\alpha, \alpha', \beta, \beta' : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha' \sqcup \beta'.$$

CHAPTER 4. LIBRARY BASIC_LEMMAS

Lemma *cup_inc_compat* {*A B* : *eqType*} {*alpha alpha' beta beta' : Rel A B*}:
alpha alpha' → beta beta' → (alpha beta) (alpha' beta').

Proof.

move \Rightarrow *H H0*.

rewrite *-inc_def2*.

apply *inc_def2* in *H*.

apply *inc_def2* in *H0*.

rewrite *cup_assoc* -(@*cup_assoc* - - *beta*).

rewrite (@*cup_comm* - - *beta*).

rewrite *cup_assoc* -(@*cup_assoc* - - *alpha*).

by [rewrite *-H -H0*].

Qed.

Lemma 40 (*cup_inc_compat_l*) *Let $\alpha, \beta, \beta' : A \rightarrow B$. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha \sqcup \beta'.$$

Lemma *cup_inc_compat_l* {*A B* : *eqType*} {*alpha beta beta' : Rel A B*}:
beta beta' → (alpha beta) (alpha beta').

Proof.

move \Rightarrow *H*.

apply (@*cup_inc_compat* - - - - - (@*inc_refl* - - *alpha*) *H*).

Qed.

Lemma 41 (*cup_inc_compat_r*) *Let $\alpha, \alpha', \beta : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha' \sqcup \beta.$$

Lemma *cup_inc_compat_r* {*A B* : *eqType*} {*alpha alpha' beta : Rel A B*}:
alpha alpha' → (alpha beta) (alpha' beta).

Proof.

move \Rightarrow *H*.

apply (@*cup_inc_compat* - - - - - *H* (@*inc_refl* - - *beta*)).

Qed.

Lemma 42 (*cap_empty*) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqcap \phi_{AB} = \phi_{AB}.$$

Lemma *cap_empty* {*A B* : *eqType*} {*alpha : Rel A B*}: *alpha A B = A B*.

Proof.

apply *inc_antisym*.

apply *cap_r*.

apply *inc_empty_alpha*.

Qed.

Lemma 43 (cup_empty) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqcup \phi_{AB} = \alpha.$$

Lemma *cup_empty* {*A B : eqType*} {*alpha : Rel A B*}: *alpha* $A\ B = \alpha$.

Proof.

apply *inc_antisym*.

apply *inc_cup*.

split.

apply *inc_refl*.

apply *inc_empty_alpha*.

apply *cup_l*.

Qed.

Lemma 44 (cap_universal) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqcap \nabla_{AB} = \alpha.$$

Lemma *cap_universal* {*A B : eqType*} {*alpha : Rel A B*}: *alpha* $A\ B = \alpha$.

Proof.

apply *inc_antisym*.

apply *cap_l*.

apply *inc_cap*.

split.

apply *inc_refl*.

apply *inc_alpha_universal*.

Qed.

Lemma 45 (cup_universal) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqcup \nabla_{AB} = \nabla_{AB}.$$

Lemma *cup_universal* {*A B : eqType*} {*alpha : Rel A B*}: *alpha* $A\ B = \nabla_{AB}$.

Proof.

apply *inc_antisym*.

apply *inc_cup*.

split.

apply *inc_alpha_universal*.

apply *inc_refl*.

apply *cup_r*.

Qed.

Lemma 46 (inc_lower) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha = \beta \Leftrightarrow (\forall \gamma : A \rightarrow B, \gamma \sqsubseteq \alpha \Leftrightarrow \gamma \sqsubseteq \beta).$$

Lemma inc_lower $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:

$$alpha = beta \Leftrightarrow (\forall\ gamma : Rel\ A\ B, gamma\ alpha \Leftrightarrow gamma\ beta).$$

Proof.

split; move $\Rightarrow H$.

move $\Rightarrow gamma$.

by [rewrite H].

apply *inc_antisym*.

rewrite $-H$.

apply *inc_refl*.

rewrite H .

apply *inc_refl*.

Qed.

Lemma 47 (inc_upper) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha = \beta \Leftrightarrow (\forall \gamma : A \rightarrow B, \alpha \sqsubseteq \gamma \Leftrightarrow \beta \sqsubseteq \gamma).$$

Lemma inc_upper $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:

$$alpha = beta \Leftrightarrow (\forall\ gamma : Rel\ A\ B, alpha\ gamma \Leftrightarrow beta\ gamma).$$

Proof.

split; move $\Rightarrow H$.

move $\Rightarrow gamma$.

by [rewrite H].

apply *inc_antisym*.

rewrite H .

apply *inc_refl*.

rewrite $-H$.

apply *inc_refl*.

Qed.

4.1.2 分配法則

Lemma 48 (cap_cup_distr_l) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqcap (\beta \sqcup \gamma) = (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma).$$

CHAPTER 4. LIBRARY BASIC_LEMMAS

Lemma *cap_cup_distr_l* {A B : eqType} {alpha beta gamma : Rel A B}:
 $\alpha \text{ (beta gamma) } = (\alpha \text{ beta}) \text{ (alpha gamma)}$.

Proof.

apply *inc_upper*.
 move \Rightarrow *delta*.
 split; move \Rightarrow *H*.
 rewrite *cap_comm* (@*cap_comm* _ _ *gamma*).
 apply *inc_cup*.
 rewrite -*inc_rpc* -*inc_rpc*.
 apply *inc_cup*.
 rewrite *inc_rpc* *cap_comm*.
 apply *H*.
 rewrite *cap_comm* -*inc_rpc*.
 apply *inc_cup*.
 rewrite *inc_rpc* *inc_rpc*.
 apply *inc_cup*.
 rewrite *cap_comm* (@*cap_comm* _ _ *gamma*).
 apply *H*.

Qed.

Lemma 49 (cap_cup_distr_r) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$(\alpha \sqcup \beta) \sqcap \gamma = (\alpha \sqcap \gamma) \sqcup (\beta \sqcap \gamma).$$

Lemma *cap_cup_distr_r* {A B : eqType} {alpha beta gamma : Rel A B}:
 $(\alpha \text{ beta}) \text{ gamma} = (\alpha \text{ gamma}) \text{ (beta gamma)}$.

Proof.

rewrite (@*cap_comm* _ _ (*alpha* *beta*)) (@*cap_comm* _ _ *alpha*) (@*cap_comm* _ _ *beta*).
 apply *cap_cup_distr_l*.

Qed.

Lemma 50 (cup_cap_distr_l) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqcup (\beta \sqcap \gamma) = (\alpha \sqcup \beta) \sqcap (\alpha \sqcup \gamma).$$

Lemma *cup_cap_distr_l* {A B : eqType} {alpha beta gamma : Rel A B}:
 $\alpha \text{ (beta gamma) } = (\alpha \text{ beta}) \text{ (alpha gamma)}$.

Proof.

rewrite *cap_cup_distr_l*.
 rewrite (@*cap_comm* _ _ (*alpha* *beta*)) *cap_cup_abs* (@*cap_comm* _ _ (*alpha* *beta*)).
 rewrite *cap_cup_distr_l*.
 rewrite -*cup_assoc* (@*cap_comm* _ _ *gamma*) *cup_cap_abs*.
 by [rewrite *cap_comm*].

Qed.

Lemma 51 (cup_cap_distr_r) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$(\alpha \sqcap \beta) \sqcup \gamma = (\alpha \sqcup \gamma) \sqcap (\beta \sqcup \gamma).$$

Lemma cup_cap_distr_r {A B : eqType} {alpha beta gamma : Rel A B}:
(alpha beta) gamma = (alpha gamma) (beta gamma).

Proof.

rewrite (@cup_comm _ _ (alpha beta)) (@cup_comm _ _ alpha) (@cup_comm _ _ beta).
apply cup_cap_distr_l.

Qed.

Lemma 52 (cap_cup_unique) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqcap \beta = \alpha \sqcap \gamma \wedge \alpha \sqcup \beta = \alpha \sqcup \gamma \Rightarrow \beta = \gamma.$$

Lemma cap_cup_unique {A B : eqType} {alpha beta gamma : Rel A B}:
alpha beta = alpha gamma \rightarrow alpha beta = alpha gamma \rightarrow beta = gamma.

Proof.

move \Rightarrow H H0.
rewrite -(@cap_cup_abs _ _ beta alpha) cup_comm H0.
rewrite cap_cup_distr_l.
rewrite cap_comm H.
rewrite -cap_cup_distr_r.
rewrite H0 cap_comm cup_comm.
apply cap_cup_abs.

Qed.

4.1.3 原子性

空関係でない $\alpha : A \rightarrow B$ が, 任意の $\beta : A \rightarrow B$ について

$$\beta \sqsubseteq \alpha \Rightarrow \beta = \phi_{AB} \vee \beta = \alpha$$

を満たすとき, α は原子的 (atomic) であると言われる.

Definition atomic {A B : eqType} (alpha : Rel A B):=
alpha \neq $\phi_{AB} \wedge (\forall \beta : Rel A B, \beta \sqsubseteq \alpha \rightarrow \beta = \phi_{AB} \vee \beta = \alpha).$

CHAPTER 4. LIBRARY BASIC_LEMMAS

Lemma 53 (atomic_cap_empty) *Let $\alpha, \beta : A \rightarrow B$ are atomic and $\alpha \neq \beta$. Then,*

$$\alpha \sqcap \beta = \phi_{AB}.$$

Lemma *atomic_cap_empty* { $A B : eqType$ } { $\alpha \beta : Rel A B$ }:
 $atomic \ \alpha \rightarrow atomic \ \beta \rightarrow \alpha \neq \beta \rightarrow \alpha \sqcap \beta = \phi_{AB}.$

Proof.

```
move  $\Rightarrow$   $H \ H0$ .
apply or_to_imply.
case (classic ( $\alpha \sqcap \beta = \phi_{AB}$ )); move  $\Rightarrow$   $H1$ .
right.
apply  $H1$ .
left.
move  $\Rightarrow$   $H2$ .
apply  $H2$ .
apply inc_antisym.
apply inc_def1.
elim  $H \Rightarrow H3 \ H4$ .
case ( $H4 \ (\alpha \sqcap \beta) \ (cap\_l)$ ); move  $\Rightarrow$   $H5$ .
apply False_ind.
apply ( $H1 \ H5$ ).
by [rewrite  $H5$ ].
apply inc_def1.
elim  $H0 \Rightarrow H3 \ H4$ .
case ( $H4 \ (\alpha \sqcap \beta) \ (cap\_r)$ ); move  $\Rightarrow$   $H5$ .
apply False_ind.
apply ( $H1 \ H5$ ).
by [rewrite cap_comm  $H5$ ].
```

Qed.

Lemma 54 (atomic_cup) *Let $\alpha, \beta, \gamma : A \rightarrow B$ and α is atomic. Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Rightarrow \alpha \sqsubseteq \beta \vee \alpha \sqsubseteq \gamma.$$

Lemma *atomic_cup* { $A B : eqType$ } { $\alpha \beta \gamma : Rel A B$ }:
 $atomic \ \alpha \rightarrow \alpha \sqsubseteq \beta \sqcup \gamma \rightarrow \alpha \sqsubseteq \beta \vee \alpha \sqsubseteq \gamma.$

Proof.

```
move  $\Rightarrow$   $H \ H0$ .
apply inc_def1 in  $H0$ .
rewrite cap_cup_distr_l in  $H0$ .
elim  $H \Rightarrow H1 \ H2$ .
rewrite  $H0$  in  $H1$ .
assert ( $\alpha \sqsubseteq \beta \vee \alpha \sqsubseteq \gamma \neq \alpha \sqsubseteq \beta \sqcup \gamma$ ).
```

```

apply not_and_or.
elim  $\Rightarrow$   $H3$   $H4$ .
rewrite  $H3$   $H4$  in  $H1$ .
apply  $H1$ .
by [rewrite cup_empty].
case  $H3$ ; move  $\Rightarrow$   $H4$ .
left.
apply inc_def1.
case ( $H2$  ( $\alpha$   $\beta$ ) ( $cap\_l$ )); move  $\Rightarrow$   $H5$ .
apply False_ind.
apply ( $H4$   $H5$ ).
by [rewrite  $H5$ ].
right.
apply inc_def1.
case ( $H2$  ( $\alpha$   $\gamma$ ) ( $cap\_l$ )); move  $\Rightarrow$   $H5$ .
apply False_ind.
apply ( $H4$   $H5$ ).
by [rewrite  $H5$ ].
Qed.

```

4.2 Heyting 代数に関する補題

Lemma 55 (rpc_universal) *Let $\alpha : A \rightarrow B$. Then,*

$$(\alpha \Rightarrow \alpha) = \nabla_{AB}.$$

Lemma *rpc_universal* { A B : eqType} { α : Rel A B }: ($\alpha \gg \alpha$) = ∇_{AB} .

Proof.

```

apply inc_lower.
move  $\Rightarrow$   $\gamma$ .
split; move  $\Rightarrow$   $H$ .
apply inc_alpha_universal.
apply inc_rpc.
apply cap_r.
Qed.

```

Lemma 56 (rpc_r) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$(\alpha \Rightarrow \beta) \sqcap \beta = \beta.$$

Lemma *rpc_r* { A B : eqType} { α β : Rel A B }: ($\alpha \gg \beta$) \sqcap β = β .

Proof.

```
assert (beta (alpha » beta)).
apply inc_rpc.
apply cap_l.
apply inc_def1 in H.
by [rewrite cap_comm -H].
Qed.
```

Lemma 57 (inc_def3) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$(\alpha \Rightarrow \beta) = \nabla_{AB} \Leftrightarrow \alpha \sqsubseteq \beta.$$

Lemma inc_def3 $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:
 $(alpha \Rightarrow beta) = A\ B \leftrightarrow alpha\ beta.$

Proof.

```
split; move => H.
rewrite -(@rpc_universal _ _ alpha) in H.
assert ((alpha » alpha) (alpha » beta)).
rewrite H.
apply inc_refl.
apply inc_rpc in H0.
rewrite rpc_r in H0.
apply H0.
apply inc_antisym.
apply inc_alpha_universal.
rewrite -(@rpc_universal _ _ alpha).
apply inc_rpc.
rewrite rpc_r.
apply H.
Qed.
```

Lemma 58 (rpc_l) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqcap (\alpha \Rightarrow \beta) = \alpha \sqcap \beta.$$

Lemma rpc_l $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:
 $alpha\ (alpha \Rightarrow beta) = alpha\ beta.$

Proof.

```
apply inc_lower.
move => gamma.
split; move => H.
apply inc_cap.
apply inc_cap in H.
```

```

split.
apply H.
elim H ⇒ H0 H1.
apply inc_rpc in H1.
rewrite -(@cap_idem _ _ gamma).
apply (inc_trans _ _ _ (gamma alpha)).
apply cap_inc_compat.
apply inc_refl.
apply H0.
apply H1.
apply inc_cap.
apply inc_cap in H.
split.
apply H.
apply inc_rpc.
apply (inc_trans _ _ _ gamma).
apply cap_l.
apply H.
Qed.

```

Lemma 59 (rpc_inc_compat) *Let $\alpha, \alpha', \beta, \beta' : A \rightarrow B$. Then,*

$$\alpha' \sqsubseteq \alpha \wedge \beta \sqsubseteq \beta' \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha' \Rightarrow \beta').$$

Lemma *rpc_inc_compat* {A B : eqType} {alpha alpha' beta beta' : Rel A B}:
 alpha' alpha → beta beta' → (alpha » beta) (alpha' » beta').

Proof.

```

move ⇒ H H0.
apply inc_rpc.
apply (@inc_trans _ _ _ ((alpha » beta) alpha)).
apply (@cap_inc_compat_l _ _ _ _ H).
rewrite cap_comm rpc_l.
apply (@inc_trans _ _ _ beta).
apply cap_r.
apply H0.
Qed.

```

Lemma 60 (rpc_inc_compat_l) *Let $\alpha, \beta, \beta' : A \rightarrow B$. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha \Rightarrow \beta').$$

Lemma *rpc_inc_compat_l* {A B : eqType} {alpha beta beta' : Rel A B}:
 beta beta' → (alpha » beta) (alpha » beta').

Proof.

move $\Rightarrow H$.

apply (@rpc_inc_compat _ _ _ _ (@inc_refl _ alpha) H).

Qed.

Lemma 61 (rpc_inc_compat_r) *Let $\alpha, \alpha', \beta : A \rightarrow B$. Then,*

$$\alpha' \sqsubseteq \alpha \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha' \Rightarrow \beta).$$

Lemma *rpc_inc_compat_r* {A B : eqType} {alpha alpha' beta : Rel A B}:
 alpha' alpha \rightarrow (alpha » beta) (alpha' » beta).

Proof.

move $\Rightarrow H$.

apply (@rpc_inc_compat _ _ _ _ H (@inc_refl _ beta)).

Qed.

Lemma 62 (rpc_universal_alpha) *Let $\alpha : A \rightarrow B$. Then,*

$$\nabla_{AB} \Rightarrow \alpha = \alpha.$$

Lemma *rpc_universal_alpha* {A B : eqType} {alpha : Rel A B}: A B » alpha = alpha.

Proof.

apply inc_lower.

move \Rightarrow gamma.

split; move $\Rightarrow H$.

apply inc_rpc in H.

rewrite cap_universal in H.

apply H.

apply inc_rpc.

rewrite cap_universal.

apply H.

Qed.

Lemma 63 (rpc_lemma1) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$(\alpha \Rightarrow \beta) \sqsubseteq ((\alpha \sqcap \gamma) \Rightarrow (\beta \sqcap \gamma)).$$

Lemma *rpc_lemma1* {A B : eqType} {alpha beta gamma : Rel A B}:
 (alpha » beta) ((alpha gamma) » (beta gamma)).

Proof.

apply inc_rpc.

rewrite -cap_assoc (@cap_comm _ _ alpha).

rewrite rpc_l.

$$(\alpha \Rightarrow \beta) \sqcap (\alpha \Rightarrow \gamma) = (\alpha \Rightarrow (\beta \sqcap \gamma)).$$

Proof.

$$(\alpha \Rightarrow \beta) \sqcap (\beta \Rightarrow \gamma) \sqsubseteq ((\alpha \sqcup \beta) \Rightarrow (\beta \sqcap \gamma)).$$

Proof.

53

CHAPTER 4. LIBRARY BASIC_LEMMAS

Lemma 66 (rpc_lemma4) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$(\alpha \Rightarrow \beta) \sqcap (\beta \Rightarrow \gamma) \sqsubseteq (\alpha \Rightarrow \gamma).$$

Lemma *rpc_lemma4* {A B : eqType} {alpha beta gamma : Rel A B}:
 ((alpha » beta) (beta » gamma)) (alpha » gamma).

Proof.

apply (@inc_trans _ _ _ ((alpha beta) » (beta gamma))).

apply rpc_lemma3.

apply rpc_inc_compat.

apply cup_l.

apply cap_r.

Qed.

Lemma 67 (rpc_lemma5) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \Rightarrow (\beta \Rightarrow \gamma) = (\alpha \sqcap \beta) \Rightarrow \gamma.$$

Lemma *rpc_lemma5* {A B : eqType} {alpha beta gamma : Rel A B}:
 alpha » (beta » gamma) = (alpha beta) » gamma.

Proof.

apply inc_lower.

move => delta.

split; move => H.

apply inc_rpc.

rewrite -cap_assoc.

rewrite -inc_rpc -inc_rpc.

apply H.

rewrite inc_rpc inc_rpc.

rewrite cap_assoc.

apply inc_rpc.

apply H.

Qed.

Lemma 68 (rpc_lemma6) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \Rightarrow (\beta \Rightarrow \gamma) \sqsubseteq (\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma).$$

Lemma *rpc_lemma6* {A B : eqType} {alpha beta gamma : Rel A B}:
 (alpha » (beta » gamma)) ((alpha » beta) » (alpha » gamma)).

Proof.

rewrite inc_rpc inc_rpc.

rewrite cap_assoc (@cap_comm _ _ _ alpha).

```

rewrite rpc_l.
rewrite -cap_assoc (@cap_comm _ _ alpha).
rewrite rpc_l.
rewrite cap_assoc (@cap_comm _ _ beta).
rewrite rpc_l.
rewrite -cap_assoc.
apply cap_r.
Qed.

```

Lemma 69 (rpc_lemma7) *Let $\alpha, \beta, \gamma, \delta : A \rightarrow B$ and $\beta \sqsubseteq \alpha \sqsubseteq \gamma$. Then,*

$$(\alpha \sqcap \delta = \beta) \wedge (\alpha \sqcup \delta = \gamma) \Leftrightarrow (\gamma \sqsubseteq \alpha \sqcup (\alpha \Rightarrow \beta)) \wedge (\delta = \gamma \sqcap (\alpha \Rightarrow \beta)).$$

Lemma *rpc_lemma7* {A B : eqType} {alpha beta gamma delta : Rel A B}:
 beta alpha → alpha gamma → (alpha delta = beta ∧ alpha delta = gamma
 ↔ gamma (alpha (alpha » beta)) ∧ delta = gamma (alpha » beta)).

Proof.

```

move ⇒ H H0.
split; elim; move ⇒ H1 H2; split.
rewrite -H2.
apply cup_inc_compat_l.
apply inc_rpc.
rewrite cap_comm H1.
apply inc_refl.
rewrite -H2.
rewrite cap_cup_distr_r rpc_l.
assert (delta (alpha » beta)).
apply inc_rpc.
rewrite cap_comm H1.
apply inc_refl.
apply inc_def1 in H3.
rewrite -H3 -H1.
rewrite -cap_assoc cap_idem.
by [rewrite cap_comm cup_comm cup_cap_abs].
rewrite H2.
rewrite (@cap_comm _ _ gamma) -cap_assoc rpc_l.
apply inc_antisym.
apply (@inc_trans _ _ (beta gamma)).
apply cap_inc_compat_r.
apply cap_r.
apply cap_l.
move : (@inc_trans _ _ _ H H0) ⇒ H3.
apply inc_def1 in H.

```



```

apply inc_def1 in H3.
rewrite cap_comm in H.
rewrite -H -H3.
apply inc_refl.
rewrite H2.
rewrite cup_cap_distr_l.
apply inc_def2 in H0.
rewrite -H0.
apply inc_def1 in H1.
by [rewrite -H1].
Qed.

```

4.3 補関係に関する補題

Lemma 70 (complement_universal)

$$\nabla_{AB}^- = \phi_{AB}.$$

Lemma *complement_universal* $\{A\ B : eqType\}$: $A\ B\ ^\wedge = A\ B$.

Proof.

```

apply rpc_universal_alpha.

```

Qed.

Lemma 71 (complement_alpha_universal) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha^- = \nabla_{AB} \Leftrightarrow \alpha = \phi_{AB}.$$

Lemma *complement_alpha_universal* $\{A\ B : eqType\}\ \{\alpha : Rel\ A\ B\}$:
 $\alpha\ ^\wedge = A\ B \Leftrightarrow \alpha = A\ B$.

Proof.

```

split; move => H.
apply inc_antisym.
rewrite -(@cap_universal _ _ alpha) cap_comm.
apply inc_rpc.
rewrite -H.
apply inc_refl.
apply inc_empty_alpha.
apply inc_antisym.
apply inc_alpha_universal.
apply inc_rpc.
rewrite cap_comm cap_universal.
rewrite H.

```

apply *inc_refl*.

Qed.

Lemma 72 (complement_empty)

$$\phi_{AB}^- = \nabla_{AB}.$$

Lemma *complement_empty* {A B : eqType}: $\phi_{AB}^- = \nabla_{AB}$.

Proof.

by [apply *complement_alpha_universal*].

Qed.

Lemma 73 (complement_invol_inc) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq (\alpha^-)^-.$$

Lemma *complement_invol_inc* {A B : eqType} {alpha : Rel A B}: $\alpha \sqsubseteq (\alpha^-)^-$.

Proof.

apply *inc_rpc*.

rewrite *cap_comm*.

apply *inc_rpc*.

apply *inc_refl*.

Qed.

Lemma 74 (cap_complement_empty) *Let $\alpha : A \rightarrow B$. Then,*

$$\alpha \sqcap \alpha^- = \phi_{AB}.$$

Lemma *cap_complement_empty* {A B : eqType} {alpha : Rel A B}:

$$\alpha \sqcap \alpha^- = \phi_{AB}.$$

Proof.

apply *inc_antisym*.

rewrite *cap_comm*.

apply *inc_rpc*.

apply *inc_refl*.

apply *inc_empty_alpha*.

Qed.

Lemma 75 (complement_invol) *Let $\alpha : A \rightarrow B$. Then,*

$$(\alpha^-)^- = \alpha.$$

Lemma *complement_invol* {A B : eqType} {alpha : Rel A B}: $(\alpha^-)^- = \alpha$.

CHAPTER 4. LIBRARY BASIC_LEMMAS

Proof.

```
rewrite -(@cap_universal _ _ ((alpha ^) ^)).
rewrite -(@complement_classic _ _ alpha).
rewrite cap_cup_distr_l.
rewrite (@cap_comm _ _ (alpha ^)) cap_complement_empty.
rewrite cup_empty cap_comm.
apply Logic.eq_sym.
apply inc_def1.
apply complement_invol_inc.
```

Qed.

Lemma 76 (complement_move) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha = \beta^- \Leftrightarrow \alpha^- = \beta.$$

Lemma complement_move $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:
 $alpha = beta^{\wedge} \Leftrightarrow alpha^{\wedge} = beta.$

Proof.

```
split; move => H.
by [rewrite H complement_invol].
by [rewrite -H complement_invol].
```

Qed.

Lemma 77 (contraposition) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$(\alpha \Rightarrow \beta) = (\beta^- \Rightarrow \alpha^-).$$

Lemma contraposition $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:
 $alpha \gg beta = beta^{\wedge} \gg alpha^{\wedge}.$

Proof.

```
apply inc_antisym.
apply inc_rpc.
apply rpc_lemma4.
replace (alpha >> beta) with ((alpha ^) ^ >> (beta ^) ^).
apply inc_rpc.
apply rpc_lemma4.
by [rewrite complement_invol complement_invol].
```

Qed.

Lemma 78 (de_morgan1) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$(\alpha \sqcup \beta)^- = \alpha^- \sqcap \beta^-.$$

CHAPTER 4. LIBRARY BASIC_LEMMAS

Lemma *de_morgan1* {A B : eqType} {alpha beta : Rel A B}:
 $(\alpha \sqcap \beta)^\perp = \alpha^\perp \sqcup \beta^\perp$.

Proof.

apply *inc_lower*.
 move \Rightarrow *gamma*.
 split; move \Rightarrow *H*.
 apply *inc_cap*.
 rewrite *inc_rpc inc_rpc*.
 apply *inc_cup*.
 rewrite *-cap_cup_distr_l*.
 apply *inc_rpc*.
 apply *H*.
 apply *inc_rpc*.
 rewrite *cap_cup_distr_l*.
 apply *inc_cup*.
 rewrite *-inc_rpc -inc_rpc*.
 apply *inc_cap*.
 apply *H*.

Qed.

Lemma 79 (de_morgan2) Let $\alpha, \beta : A \rightarrow B$. Then,

$$(\alpha \sqcap \beta)^\perp = \alpha^\perp \sqcup \beta^\perp.$$

Lemma *de_morgan2* {A B : eqType} {alpha beta : Rel A B}:
 $(\alpha \sqcap \beta)^\perp = \alpha^\perp \sqcup \beta^\perp$.

Proof.

by [rewrite *-complement_move de_morgan1 complement_invol complement_invol*].

Qed.

Lemma 80 (cup_to_rpc) Let $\alpha, \beta : A \rightarrow B$. Then,

$$\alpha^\perp \sqcup \beta = (\alpha \Rightarrow \beta).$$

Lemma *cup_to_rpc* {A B : eqType} {alpha beta : Rel A B}:
 $\alpha^\perp \sqcup \beta = \alpha \gg \beta$.

Proof.

apply *inc_antisym*.
 apply *inc_rpc*.
 rewrite *cap_cup_distr_r cap_comm*.
 rewrite *cap_complement_empty cup_comm cup_empty*.
 apply *cap_l*.
 rewrite *-(@cap_universal _ _ (alpha >> beta)) cap_comm*.

CHAPTER 4. LIBRARY BASIC_LEMMAS

```
rewrite -(@complement_classic _ _ alpha).
rewrite cap_cup_distr_r cup_comm.
apply cup_inc_compat.
apply cap_l.
rewrite rpc_l.
apply cap_r.
Qed.
```

Lemma 81 (beta_contradiction) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$(\alpha \Rightarrow \beta) \sqcap (\alpha \Rightarrow \beta^-) = \alpha^-.$$

Lemma beta_contradiction $\{A B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:
 $(alpha \gg beta) \quad (alpha \gg beta^-) = alpha^-.$

Proof.

```
rewrite -cup_to_rpc -cup_to_rpc.
rewrite -cup_cap_distr_l.
by [rewrite cap_complement_empty cup_empty].
Qed.
```

4.4 Bool 代数に関する補題

Lemma 82 (bool_lemma1) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \beta \Leftrightarrow \nabla_{AB} = \alpha^- \sqcup \beta.$$

Lemma bool_lemma1 $\{A B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:
 $alpha \sqsubseteq beta \Leftrightarrow A\ B = alpha^- \sqcup beta.$

Proof.

```
split; move => H.
apply inc_antisym.
rewrite -(@complement_classic _ _ alpha) cup_comm.
apply cup_inc_compat_l.
apply H.
apply inc_alpha_universal.
apply inc_def3.
rewrite H.
apply (Logic.eq_sym cup_to_rpc).
Qed.
```

CHAPTER 4. LIBRARY BASIC_LEMMAS

Lemma 83 (bool_lemma2) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \beta \Leftrightarrow \alpha \sqcap \beta^- = \phi_{AB}.$$

Lemma *bool_lemma2* {*A B : eqType*} {*alpha beta : Rel A B*}:

alpha beta \leftrightarrow *alpha beta* \wedge = *A B*.

Proof.

split; move \Rightarrow *H*.

rewrite -(@cap_universal _ _ (*alpha beta* \wedge)).

apply *bool_lemma1* in *H*.

rewrite *H*.

rewrite *cap_cup_distr_l*.

rewrite (@cap_comm _ _ *alpha*) *cap_assoc cap_complement_empty cap_empty*.

rewrite *cap_comm -cap_assoc cap_complement_empty cap_comm cap_empty*.

by [rewrite *cup_empty*].

rewrite -(@cap_universal _ _ *alpha*).

rewrite -(@complement_classic _ _ *beta*).

rewrite *cap_cup_distr_l*.

rewrite *H cup_empty*.

apply *cap_r*.

Qed.

Lemma 84 (bool_lemma3) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Leftrightarrow \alpha \sqcap \beta^- \sqsubseteq \gamma.$$

Lemma *bool_lemma3* {*A B : eqType*} {*alpha beta gamma : Rel A B*}:

alpha (*beta gamma*) \leftrightarrow (*alpha beta* \wedge) *gamma*.

Proof.

split; move \Rightarrow *H*.

apply (@inc_trans _ _ _ ((*beta gamma*) *beta* \wedge)).

apply *cap_inc_compat_r*.

apply *H*.

rewrite *cap_cup_distr_r*.

rewrite *cap_complement_empty cup_comm cup_empty*.

apply *cap_l*.

apply (@inc_trans _ _ _ (*beta* (*alpha beta* \wedge))).

rewrite *cup_cap_distr_l*.

rewrite *complement_classic cap_universal*.

apply *cup_r*.

apply *cup_inc_compat_l*.

apply *H*.

Qed.

CHAPTER 4. LIBRARY BASIC_LEMMAS

Lemma 85 (bool_lemma4) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Leftrightarrow \beta^- \sqsubseteq \alpha^- \sqcup \gamma.$$

Lemma *bool_lemma4* {*A B : eqType*} {*alpha beta gamma : Rel A B*}:
alpha (beta gamma) ↔ beta ^ (alpha ^ gamma).

Proof.

rewrite *bool_lemma3*.

rewrite *cap_comm*.

apply *iff_sym*.

replace (*beta ^ alpha*) with (*beta ^ (alpha ^ ^)*).

apply *bool_lemma3*.

by [rewrite *complement_invol*].

Qed.

Lemma 86 (bool_lemma5) *Let $\alpha, \beta, \gamma : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Leftrightarrow \nabla_{AB} = \alpha^- \sqcup \beta \sqcup \gamma.$$

Lemma *bool_lemma5* {*A B : eqType*} {*alpha beta gamma : Rel A B*}:
alpha (beta gamma) ↔ A B = (alpha ^ beta) gamma.

Proof.

rewrite *bool_lemma1*.

by [rewrite *cup_assoc*].

Qed.

Chapter 5

Library **Relation_Properties**

```
Require Import Basic_Notations.  
Require Import Basic_Lemmas.  
Require Import Logic.FunctionalExtensionality.  
Require Import Logic.Classical_Prop.
```

5.1 関係計算の基本的な性質

Lemma 87 (RelAB_unique)

$$\phi_{AB} = \nabla_{AB} \Leftrightarrow \forall \alpha, \beta : A \rightarrow B, \alpha = \beta.$$

Lemma *RelAB_unique* {A B : eqType}:

$$A B = \quad A B \Leftrightarrow (\forall \text{ alpha beta} : \text{Rel } A B, \text{ alpha} = \text{ beta}).$$

Proof.

```
split; move => H.  
move => alpha beta.  
replace beta with ( A B).  
apply inc_antisym.  
rewrite H.  
apply inc_alpha_universal.  
apply inc_empty_alpha.  
apply inc_antisym.  
apply inc_empty_alpha.  
rewrite H.  
apply inc_alpha_universal.  
apply H.  
Qed.
```


Lemma 88 (either_empty)

$$\phi_{AB} = \nabla_{AB} \Leftrightarrow A = \emptyset \vee B = \emptyset.$$

Lemma *either_empty* {*A B : eqType*}: $A \ B = \ A \ B \Leftrightarrow (A \rightarrow \text{False}) \vee (B \rightarrow \text{False})$.

Proof.

```

rewrite RelAB_unique.
split; move  $\Rightarrow$  H.
case (classic ( $\exists \_ : A, \text{True}$ )).
elim  $\Rightarrow$  a H0.
right.
move  $\Rightarrow$  b.
remember (fun ( $\_ : A$ ) ( $\_ : B$ )  $\Rightarrow$  True) as T.
remember (fun ( $\_ : A$ ) ( $\_ : B$ )  $\Rightarrow$  False) as F.
move : (H T F)  $\Rightarrow$  H1.
assert (T a b = F a b).
by [rewrite H1].
rewrite HeqT HeqF in H2.
rewrite -H2.
apply I.
move  $\Rightarrow$  H0.
left.
move  $\Rightarrow$  a.
apply H0.
 $\exists$  a.
apply I.
move  $\Rightarrow$  alpha beta.
assert ( $A \rightarrow B \rightarrow \text{False}$ ).
move  $\Rightarrow$  a b.
case H; move  $\Rightarrow$  H0.
apply (H0 a).
apply (H0 b).
apply functional_extensionality.
move  $\Rightarrow$  a.
apply functional_extensionality.
move  $\Rightarrow$  b.
apply False_ind.
apply (H0 a b).
Qed.

```

Lemma 89 (unit_empty_not_universal)

$$\phi_{II} \neq \nabla_{II}.$$

Lemma *unit_empty_not_universal* : $\quad i \ i \neq \quad i \ i$.

Proof.

move $\Rightarrow H$.

apply *either_empty* in H .

case H ; move $\Rightarrow H0$.

apply ($H0 \ tt$).

apply ($H0 \ tt$).

Qed.

Lemma 90 (unit_empty_or_universal) *Let $\alpha : I \rightarrow I$. Then,*

$$\alpha = \phi_{II} \vee \alpha = \nabla_{II}.$$

Lemma *unit_empty_or_universal* {*alpha* : *Rel i i*} : *alpha* = $\quad i \ i \vee \alpha = \quad i \ i$.

Proof.

assert ($\forall \text{beta} : \text{Rel } i \ i, \text{beta} = (\text{fun } (_ _) : i) \Rightarrow \text{True}) \vee \text{beta} = (\text{fun } (_ _) : i) \Rightarrow \text{False}$).

move $\Rightarrow \text{beta}$.

case (*classic* (*beta tt tt*)); move $\Rightarrow H$.

left.

apply *functional_extensionality*.

induction x .

apply *functional_extensionality*.

induction x .

apply *prop_extensionality_ok*.

split; move $\Rightarrow H0$.

apply I .

apply H .

right.

apply *functional_extensionality*.

induction x .

apply *functional_extensionality*.

induction x .

apply *prop_extensionality_ok*.

split.

apply H .

apply *False_ind*.

assert ($(\text{fun } _ _ : i \Rightarrow \text{True}) \neq (\text{fun } _ _ : i \Rightarrow \text{False})$).

move $\Rightarrow H0$.

remember ($\text{fun } _ _ : i \Rightarrow \text{True}$) **as** T .

```

remember (fun _ _ : i ⇒ False) as F.
assert (T tt tt = F tt tt).
by [rewrite H0].
rewrite HeqT HeqF in H1.
rewrite -H1.
apply I.
case (H ( i i)); move ⇒ H1.
case (H ( i i)); move ⇒ H2.
apply False_ind.
apply unit_empty_not_universal.
by [rewrite H1 H2].
case (H alpha); move ⇒ H3.
left.
by [rewrite H3 H1].
right.
by [rewrite H3 H2].
case (H ( i i)); move ⇒ H2.
case (H alpha); move ⇒ H3.
right.
by [rewrite H3 H2].
left.
by [rewrite H3 H1].
apply False_ind.
apply unit_empty_not_universal.
by [rewrite H1 H2].
Qed.

```

Lemma 91 (unit_identity_is_universal)

$$id_I = \nabla_{II}.$$

Lemma *unit_identity_is_universal* : $Id\ i = \quad i\ i$.

Proof.

```

case (@unit_empty_or_universal (Id i)); move ⇒ H.
apply False_ind.
assert (Id i ( i i # i i)).
rewrite H.
apply inc_empty_alpha.
apply inc_residual in H0.
rewrite inv_invol_comp_id_r in H0.
apply unit_empty_not_universal.
apply inc_antisym.
apply inc_empty_alpha.

```

apply *H0*.
 apply *H*.
 Qed.

Lemma 92 (unit_identity_not_empty)

$$id_I \neq \phi_{II}.$$

Lemma *unit_identity_not_empty* : $Id\ i \neq\ i\ i$.

Proof.

move \Rightarrow *H*.
 apply *unit_empty_not_universal*.
 rewrite *-H*.
 apply *unit_identity_is_universal*.
 Qed.

Lemma 93 (cupP_False) *Let $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P(\alpha) := \text{"False"}$. Then,*

$$\sqcup_{P(\alpha)} f(\alpha) = \phi_{AB}.$$

Lemma *cupP_False* {*A B C D : eqType*} {*f : Rel C D \rightarrow Rel A B*}:

-{fun _ : Rel C D \Rightarrow False} f = A B.

Proof.

apply *inc_antisym*.
 apply *inc_cupP*.
 move \Rightarrow *beta*.
 apply *False_ind*.
 apply *inc_empty_alpha*.
 Qed.

Lemma 94 (capP_False) *Let $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P(\alpha) := \text{"False"}$. Then,*

$$\sqcap_{P(\alpha)} f(\alpha) = \nabla_{AB}.$$

Lemma *capP_False* {*A B C D : eqType*} {*f : Rel C D \rightarrow Rel A B*}:

-{fun _ : Rel C D \Rightarrow False} f = A B.

Proof.

apply *inc_antisym*.
 apply *inc_alpha_universal*.
 apply *inc_capP*.
 move \Rightarrow *beta*.
 apply *False_ind*.
 Qed.

Lemma 95 (cupP_eq) *Let $f, g : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$(\forall \alpha : C \rightarrow D, P(\alpha) \Rightarrow f(\alpha) = g(\alpha)) \Rightarrow \sqcup_{P(\alpha)} f(\alpha) = \sqcup_{P(\alpha)} g(\alpha).$$

Lemma cupP_eq $\{A\ B\ C\ D : \text{eqType}\}$
 $\{f\ g : \text{Rel}\ C\ D \rightarrow \text{Rel}\ A\ B\} \{P : \text{Rel}\ C\ D \rightarrow \text{Prop}\}:$
 $(\forall \text{alpha} : \text{Rel}\ C\ D, P\ \text{alpha} \rightarrow f\ \text{alpha} = g\ \text{alpha}) \rightarrow _ \{P\} f = _ \{P\} g.$

Proof.

move $\Rightarrow H$.
 apply *inc_antisym*.
 apply *inc_cupP*.
 move \Rightarrow **beta** *H0*.
 rewrite $(H _ H0)$.
 move : **beta** *H0*.
 apply *inc_cupP*.
 apply *inc_refl*.
 apply *inc_cupP*.
 move \Rightarrow **beta** *H0*.
 rewrite $-(H _ H0)$.
 move : **beta** *H0*.
 apply *inc_cupP*.
 apply *inc_refl*.
Qed.

Lemma 96 (capP_eq) *Let $f, g : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$(\forall \alpha : C \rightarrow D, P(\alpha) \Rightarrow f(\alpha) = g(\alpha)) \Rightarrow \sqcap_{P(\alpha)} f(\alpha) = \sqcap_{P(\alpha)} g(\alpha).$$

Lemma capP_eq $\{A\ B\ C\ D : \text{eqType}\}$
 $\{f\ g : \text{Rel}\ C\ D \rightarrow \text{Rel}\ A\ B\} \{P : \text{Rel}\ C\ D \rightarrow \text{Prop}\}:$
 $(\forall \text{alpha} : \text{Rel}\ C\ D, P\ \text{alpha} \rightarrow f\ \text{alpha} = g\ \text{alpha}) \rightarrow _ \{P\} f = _ \{P\} g.$

Proof.

move $\Rightarrow H$.
 apply *inc_antisym*.
 apply *inc_capP*.
 move \Rightarrow **beta** *H0*.
 rewrite $-(H _ H0)$.
 move : **beta** *H0*.
 apply *inc_capP*.
 apply *inc_refl*.
 apply *inc_capP*.
 move \Rightarrow **beta** *H0*.

```

rewrite (H - H0).
move : beta H0.
apply inc_capP.
apply inc_refl.
Qed.

```

Lemma 97 (cap_cupP_distr_l) *Let $\alpha : A \rightarrow B$, $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P :$ predicate. Then,*

$$\alpha \sqcap (\sqcup_{P(\beta)} f(\beta)) = \sqcup_{P(\beta)} (\alpha \sqcap f(\beta)).$$

Lemma cap_cupP_distr_l $\{A\ B\ C\ D : eqType\}$
 $\{\alpha : Rel\ A\ B\} \{f : Rel\ C\ D \rightarrow Rel\ A\ B\} \{P : Rel\ C\ D \rightarrow Prop\}$:
 $\alpha \sqcap (\sqcup_{P} f) = \sqcup_{P} (\alpha \sqcap f)$.

Proof.

```

apply inc_upper.
move => gamma.
split; move => H.
apply inc_cupP.
move => beta H0.
apply (@inc_trans _ _ _ (alpha _ {P} f)).
apply cap_inc_compat_l.
move : H0.
apply inc_cupP.
apply inc_refl.
apply H.
assert (forall beta : Rel C D, P beta -> (alpha (f beta) gamma)).
apply inc_cupP.
apply H.
assert (forall beta : Rel C D, P beta -> f beta (alpha » gamma)).
move => beta H1.
rewrite inc_rpc_cap_comm.
apply (H0 - H1).
rewrite cap_comm_inc_rpc.
apply inc_cupP.
apply H1.
Qed.

```

Lemma 98 (cap_cupP_distr_r) *Let $\beta : A \rightarrow B$, $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P :$ predicate. Then,*

$$(\sqcup_{P(\alpha)} f(\alpha)) \sqcap \beta = \sqcup_{P(\alpha)} (f(\alpha) \sqcap \beta).$$

Lemma cap_cupP_distr_r $\{A\ B\ C\ D : eqType\}$

$\{\text{beta} : \text{Rel } A \ B\} \{f : \text{Rel } C \ D \rightarrow \text{Rel } A \ B\} \{P : \text{Rel } C \ D \rightarrow \text{Prop}\}:$
 $(\ _ \{P\} f) \quad \text{beta} = \ _ \{P\} (\text{fun } \alpha : \text{Rel } C \ D \Rightarrow f \ \alpha \quad \text{beta}).$

Proof.

rewrite *cap_comm*.

replace (fun *alpha* : *Rel C D* \Rightarrow *f alpha* *beta*) with (fun *alpha* : *Rel C D* \Rightarrow *beta f alpha*).

apply *cap_cupP_distr_l*.

apply *functional_extensionality*.

move \Rightarrow *x*.

by [rewrite *cap_comm*].

Qed.

Lemma 99 (cup_capP_distr_l) *Let $\alpha : A \rightarrow B$, $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P :$ predicate. Then,*

$$\alpha \sqcup (\sqcap_{P(\beta)} f(\beta)) = \sqcap_{P(\beta)} (\alpha \sqcup f(\beta)).$$

Lemma *cup_capP_distr_l* {*A B C D* : *eqType*}

$\{\alpha : \text{Rel } A \ B\} \{f : \text{Rel } C \ D \rightarrow \text{Rel } A \ B\} \{P : \text{Rel } C \ D \rightarrow \text{Prop}\}:$
 $\alpha \quad (\ _ \{P\} f) = \ _ \{P\} (\text{fun } \beta : \text{Rel } C \ D \Rightarrow \alpha \quad f \ \beta).$

Proof.

apply *inc_lower*.

move \Rightarrow *gamma*.

split; move \Rightarrow *H*.

apply *inc_capP*.

move \Rightarrow *beta H0*.

apply (@*inc_trans* _ _ _ (*alpha* _ $_ \{P\} f$)).

apply *H*.

apply *cup_inc_compat_l*.

move : *H0*.

apply *inc_capP*.

apply *inc_refl*.

rewrite *bool_lemma3*.

assert ($\forall \beta : \text{Rel } C \ D, P \ \beta \rightarrow \text{gamma} \quad (\alpha \quad f \ \beta)$).

apply *inc_capP*.

apply *H*.

apply *inc_capP*.

move \Rightarrow *beta H1*.

rewrite *-bool_lemma3*.

apply (*H0* _ *H1*).

Qed.

CHAPTER 5. LIBRARY RELATION_PROPERTIES

Lemma 100 (cup_capP_distr_r) *Let $\beta : A \rightarrow B$, $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$(\sqcap_{P(\alpha)} f(\alpha)) \sqcup \beta = \sqcap_{P(\alpha)} (f(\alpha) \sqcup \beta).$$

Lemma cup_capP_distr_r $\{A\ B\ C\ D : \text{eqType}\}$
 $\{\text{beta} : \text{Rel}\ A\ B\} \{f : \text{Rel}\ C\ D \rightarrow \text{Rel}\ A\ B\} \{P : \text{Rel}\ C\ D \rightarrow \text{Prop}\}:$
 $(\ _ \{P\} f) \ \ \text{beta} = \ _ \{P\} (\text{fun alpha} : \text{Rel}\ C\ D \Rightarrow f\ \text{alpha} \ \ \text{beta}).$

Proof.

rewrite cup_comm.
 replace (fun alpha : Rel C D \Rightarrow f alpha beta) with (fun alpha : Rel C D \Rightarrow beta f alpha).
 apply cup_capP_distr_l.
 apply functional_extensionality.
 move \Rightarrow x.
 by [rewrite cup_comm].
Qed.

Lemma 101 (de_morgan3) *Let $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$(\sqcup_{P(\alpha)} f(\alpha))^- = (\sqcap_{P(\alpha)} f(\alpha))^-.$$

Lemma de_morgan3
 $\{A\ B\ C\ D : \text{eqType}\} \{f : \text{Rel}\ C\ D \rightarrow \text{Rel}\ A\ B\} \{P : \text{Rel}\ C\ D \rightarrow \text{Prop}\}:$
 $(\ _ \{P\} f) ^ = \ _ \{P\} (\text{fun alpha} : \text{Rel}\ C\ D \Rightarrow f\ \text{alpha} ^).$

Proof.

apply inc_lower.
 move \Rightarrow gamma.
 rewrite inc_capP.
 split; move \Rightarrow H.
 move \Rightarrow beta H0.
 rewrite bool_lemma1 -de_morgan2 complement_move complement_universal.
 apply bool_lemma2 in H.
 apply inc_antisym.
 apply inc_empty_alpha.
 rewrite -H complement_invol.
 apply cap_inc_compat_l.
 move : H0.
 apply inc_cupP.
 apply inc_refl.
 rewrite bool_lemma2 complement_invol.
 rewrite cap_cupP_distr_l.
 apply inc_antisym.

apply *inc_cupP*.
 move \Rightarrow **beta** *H0*.
 rewrite *-inc_rpc*.
 apply (*H* _ *H0*).
 apply *inc_empty_alpha*.
Qed.

Lemma 102 (de_morgan4) *Let $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$(\sqcap_{P(\alpha)} f(\alpha))^- = (\sqcup_{P(\alpha)} f(\alpha)^-).$$

Lemma de_morgan4

$\{A\ B\ C\ D : \text{eqType}\} \{f : \text{Rel } C\ D \rightarrow \text{Rel } A\ B\} \{P : \text{Rel } C\ D \rightarrow \text{Prop}\}:$
 $(\neg \{P\} f)^\wedge = \neg \{P\} (\text{fun } \alpha : \text{Rel } C\ D \Rightarrow f\ \alpha \wedge).$

Proof.

rewrite *-complement_move_de_morgan3*.
 replace (*fun* *alpha* : *Rel* *C* *D* \Rightarrow (*f alpha* \wedge) \wedge) **with** *f*.
 by [].
 apply *functional_extensionality*.
 move \Rightarrow *x*.
 by [rewrite *complement_invol*].
Qed.

Lemma 103 (cup_to_cupP) *Let $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$f(\alpha) \sqcup f(\beta) = \sqcup_{\gamma=\alpha \vee \gamma=\beta} f(\gamma).$$

Lemma cup_to_cupP

$\{A\ B\ C\ D : \text{eqType}\} \{\alpha\ \beta : \text{Rel } C\ D\} \{f : \text{Rel } C\ D \rightarrow \text{Rel } A\ B\}:$
 $(f\ \alpha \sqcup f\ \beta) = \neg \{\text{fun } \gamma : \text{Rel } C\ D \Rightarrow \gamma = \alpha \vee \gamma = \beta\} f.$

Proof.

apply *inc_upper*.
 move \Rightarrow **delta**.
 split; move \Rightarrow *H*.
 apply *inc_cupP*.
 apply *inc_cup* in *H*.
 move \Rightarrow *gamma* *H0*.
 case *H0* \Rightarrow *H1*.
 rewrite *H1*.
 apply *H*.
 rewrite *H1*.
 apply *H*.

```

apply inc_cup.
assert (∀ gamma : Rel C D, gamma = alpha ∨ gamma = beta → f gamma = delta).
apply inc_cupP.
apply H.
split.
apply (H0 alpha).
by [left].
apply (H0 beta).
by [right].
Qed.

```

Lemma 104 (cap_to_capP) *Let $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$f(\alpha) \sqcap f(\beta) = \sqcap_{\gamma=\alpha \vee \gamma=\beta} f(\gamma).$$

Lemma cap_to_capP

```

{A B C D : eqType} {alpha beta : Rel C D} {f : Rel C D → Rel A B}:
(f alpha    f beta) =    _{fun gamma : Rel C D ⇒ gamma = alpha ∨ gamma = beta}
f.

```

Proof.

```

apply inc_lower.
move ⇒ delta.
split; move ⇒ H.
apply inc_capP.
apply inc_cap in H.
move ⇒ gamma H0.
case H0 ⇒ H1.
rewrite H1.
apply H.
rewrite H1.
apply H.
apply inc_cap.
assert (∀ gamma : Rel C D, gamma = alpha ∨ gamma = beta → delta = f gamma).
apply inc_capP.
apply H.
split.
apply (H0 alpha).
by [left].
apply (H0 beta).
by [right].
Qed.

```

5.2 comp_inc_compat と派生補題

Lemma 105 (comp_inc_compat_ab_ab') *Let $\alpha : A \rightarrow B$ and $\beta, \beta' : B \rightarrow C$. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha \cdot \beta'.$$

Lemma comp_inc_compat_ab_ab'

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta\ beta' : Rel\ B\ C\} :$
 $beta\ \ beta' \rightarrow (alpha \cdot beta)\ \ (alpha \cdot beta').$

Proof.

move $\Rightarrow H$.

replace $(alpha \cdot beta)$ with $((alpha \ #) \ # \cdot beta)$.

apply *inc_residual*.

apply $(@inc_trans _ _ _ beta')$.

apply *H*.

apply *inc_residual*.

rewrite *inv_invol*.

apply *inc_refl*.

by [rewrite *inv_invol*].

Qed.

Lemma 106 (comp_inc_compat_ab_a'b) *Let $\alpha, \alpha' : A \rightarrow B$ and $\beta : B \rightarrow C$. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha' \cdot \beta.$$

Lemma comp_inc_compat_ab_a'b

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{beta : Rel\ B\ C\} :$
 $alpha\ \ alpha' \rightarrow (alpha \cdot beta)\ \ (alpha' \cdot beta).$

Proof.

move $\Rightarrow H$.

rewrite $-(@inv_invol _ _ (alpha \cdot beta))$.

rewrite $-(@inv_invol _ _ (alpha' \cdot beta))$.

apply *inc_inv*.

rewrite *comp_inv comp_inv*.

apply *comp_inc_compat_ab_ab'*.

apply *inc_inv*.

apply *H*.

Qed.

CHAPTER 5. LIBRARY RELATION_PROPERTIES

Lemma 107 (comp_inc_compat) *Let $\alpha, \alpha' : A \rightarrow B$ and $\beta, \beta' : B \rightarrow C$. Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha' \cdot \beta'.$$

Lemma *comp_inc_compat*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{beta\ beta' : Rel\ B\ C\} :$
 $alpha\ alpha' \rightarrow beta\ beta' \rightarrow (alpha \cdot beta)\ (alpha' \cdot beta').$

Proof.

move $\Rightarrow H\ H0$.

apply (@inc_trans _ _ _ (alpha' · beta)).

apply (@comp_inc_compat_ab_a'b _ _ _ _ _ H).

apply (@comp_inc_compat_ab_ab' _ _ _ _ _ H0).

Qed.

Lemma 108 (comp_inc_compat_ab_a) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow B$. Then,*

$$\beta \sqsubseteq id_B \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha.$$

Lemma *comp_inc_compat_ab_a* $\{A\ B : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ B\} :$
 $beta\ Id\ B \rightarrow (alpha \cdot beta)\ alpha.$

Proof.

move $\Rightarrow H$.

move : (@comp_inc_compat_ab_ab' _ _ _ alpha _ _ H) $\Rightarrow H0$.

rewrite comp_id_r in H0.

apply H0.

Qed.

Lemma 109 (comp_inc_compat_a_ab) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow B$. Then,*

$$id_B \sqsubseteq \beta \Rightarrow \beta \sqsubseteq \alpha \cdot \beta.$$

Lemma *comp_inc_compat_a_ab* $\{A\ B : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ B\} :$
 $Id\ B\ beta \rightarrow alpha\ (alpha \cdot beta).$

Proof.

move $\Rightarrow H$.

move : (@comp_inc_compat_ab_ab' _ _ _ alpha _ _ H) $\Rightarrow H0$.

rewrite comp_id_r in H0.

apply H0.

Qed.

Lemma 110 (comp_inc_compat_ab_b) *Let $\alpha : A \rightarrow A$ and $\beta : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq id_A \Rightarrow \alpha \cdot \beta \sqsubseteq \beta.$$

Lemma *comp_inc_compat_ab_b* { $A\ B : eqType$ } { $\alpha : Rel\ A\ A$ } { $\beta : Rel\ A\ B$ }:
 α $Id\ A \rightarrow (\alpha \cdot \beta)$ β .

Proof.

move $\Rightarrow H$.

move : (@comp_inc_compat_ab_a'b _ _ _ _ β H) $\Rightarrow H0$.

rewrite comp_id_l in $H0$.

apply $H0$.

Qed.

Lemma 111 (comp_inc_compat_b_ab) *Let $\alpha : A \rightarrow A$ and $\beta : A \rightarrow B$. Then,*

$$id_A \sqsubseteq \alpha \Rightarrow \beta \sqsubseteq \alpha \cdot \beta.$$

Lemma *comp_inc_compat_b_ab* { $A\ B : eqType$ } { $\alpha : Rel\ A\ A$ } { $\beta : Rel\ A\ B$ }:
 $Id\ A \rightarrow \beta (\alpha \cdot \beta)$.

Proof.

move $\Rightarrow H$.

move : (@comp_inc_compat_ab_a'b _ _ _ _ β H) $\Rightarrow H0$.

rewrite comp_id_l in $H0$.

apply $H0$.

Qed.

5.3 逆関係に関する補題

Lemma 112 (inv_move) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow A$. Then,*

$$\alpha = \beta^\# \Leftrightarrow \alpha^\# = \beta.$$

Lemma *inv_move* { $A\ B : eqType$ } { $\alpha : Rel\ A\ B$ } { $\beta : Rel\ B\ A$ }:
 $\alpha = \beta^\# \Leftrightarrow \alpha^\# = \beta$.

Proof.

split; move $\Rightarrow H$.

by [rewrite H inv_invol].

by [rewrite - H inv_invol].

Qed.

CHAPTER 5. LIBRARY RELATION_PROPERTIES

Lemma 113 (comp_inv_inv) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then,*

$$\alpha \cdot \beta = (\beta^\# \cdot \alpha^\#)^\#.$$

Lemma *comp_inv_inv* { $A\ B\ C : eqType$ } { $\alpha : Rel\ A\ B$ } { $\beta : Rel\ B\ C$ }:
 $\alpha \cdot \beta = (\beta^\# \cdot \alpha^\#)^\#$.

Proof.

apply *inv_move*.

apply *comp_inv*.

Qed.

Lemma 114 (inv_inc_move) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow A$. Then,*

$$\alpha \sqsubseteq \beta^\# \Leftrightarrow \alpha^\# \sqsubseteq \beta.$$

Lemma *inv_inc_move* { $A\ B : eqType$ } { $\alpha : Rel\ A\ B$ } { $\beta : Rel\ B\ A$ }:
 $\alpha \sqsubseteq \beta^\# \Leftrightarrow \alpha^\# \sqsubseteq \beta$.

Proof.

split; move $\Rightarrow H$.

rewrite $-(@inv_invol _ _ \beta)$.

apply *inc_inv*.

apply *H*.

rewrite $-(@inv_invol _ _ \alpha)$.

apply *inc_inv*.

apply *H*.

Qed.

Lemma 115 (inv_invol2) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha^\# = \beta^\# \Rightarrow \alpha = \beta.$$

Lemma *inv_invol2* { $A\ B : eqType$ } { $\alpha\ \beta : Rel\ A\ B$ }:
 $\alpha^\# = \beta^\# \Rightarrow \alpha = \beta$.

Proof.

move $\Rightarrow H$.

rewrite $-(@inv_invol _ _ \alpha) \text{ } -(@inv_invol _ _ \beta)$.

apply *f_equal*.

apply *H*.

Qed.

Lemma 116 (inv_inc_invol) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$\alpha^\# \sqsubseteq \beta^\# \Rightarrow \alpha \sqsubseteq \beta.$$

Lemma *inv_inc_invol* {A B : eqType} {alpha beta : Rel A B}:

alpha # beta # → alpha beta.

Proof.

move ⇒ H.

rewrite -(@inv_invol _ _ alpha) -(@inv_invol _ _ beta).

apply inc_inv.

apply H.

Qed.

Lemma 117 (inv_cupP_distr, inv_cup_distr) *Let $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$(\sqcup_{P(\alpha)} f(\alpha))^\# = (\sqcup_{P(\alpha)} f(\alpha)^\#).$$

Lemma *inv_cupP_distr* {A B C D : eqType} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:

(_ {P} f) # = (_ {P} (fun alpha : Rel C D ⇒ f alpha #)).

Proof.

apply inc_antisym.

rewrite -inv_inc_move.

apply inc_cupP.

assert (∀ beta : Rel C D, P beta → f beta # _ {P} (fun alpha : Rel C D ⇒ f alpha #)).

apply inc_cupP.

apply inc_refl.

move ⇒ beta H0.

rewrite inv_inc_move.

apply (H _ H0).

apply inc_cupP.

move ⇒ beta H0.

apply inc_inv.

move : H0.

apply inc_cupP.

apply inc_refl.

Qed.

Lemma *inv_cup_distr* {A B : eqType} {alpha beta : Rel A B}:

(alpha beta) # = alpha # beta #.

Proof.

by [rewrite cup_to_cupP -inv_cupP_distr -cup_to_cupP].

Qed.

Lemma 118 (inv_capP_distr, inv_cap_distr) *Let $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$(\sqcap_{P(\alpha)} f(\alpha))^{\#} = (\sqcap_{P(\alpha)} f(\alpha)^{\#}).$$

Lemma inv_capP_distr $\{A\ B\ C\ D : \text{eqType}\} \{f : \text{Rel } C\ D \rightarrow \text{Rel } A\ B\} \{P : \text{Rel } C\ D \rightarrow \text{Prop}\}$:

$$(_ \{P\} f)^{\#} = (_ \{P\} (\text{fun } \alpha : \text{Rel } C\ D \Rightarrow f\ \alpha\ \#)).$$

Proof.

apply *inc_antisym*.

apply *inc_capP*.

move \Rightarrow **beta** *H*.

apply *inc_inv*.

move : *H*.

apply *inc_capP*.

apply *inc_refl*.

rewrite *inv_inc_move*.

apply *inc_capP*.

assert $(\forall \text{beta} : \text{Rel } C\ D, P\ \text{beta} \rightarrow _ \{P\} (\text{fun } \alpha : \text{Rel } C\ D \Rightarrow f\ \alpha\ \#)) \quad f\ \text{beta}\ \#$.

apply *inc_capP*.

apply *inc_refl*.

move \Rightarrow **beta** *H0*.

rewrite *-inv_inc_move*.

apply $(H _ H0)$.

Qed.

Lemma inv_cap_distr $\{A\ B : \text{eqType}\} \{\alpha\ \text{beta} : \text{Rel } A\ B\}$:

$$(\alpha \ \text{beta})^{\#} = \alpha^{\#} \ \text{beta}^{\#}.$$

Proof.

by [rewrite *cap_to_capP -inv_capP_distr -cap_to_capP*].

Qed.

Lemma 119 (rpc_inv_distr) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$(\alpha \Rightarrow \beta)^{\#} = \alpha^{\#} \Rightarrow \beta^{\#}.$$

Lemma rpc_inv_distr $\{A\ B : \text{eqType}\} \{\alpha\ \text{beta} : \text{Rel } A\ B\}$:

$$(\alpha \gg \text{beta})^{\#} = \alpha^{\#} \gg \text{beta}^{\#}.$$

Proof.

apply *inc_lower*.

move \Rightarrow *gamma*.


```

split; move => H.
apply inc_rpc.
rewrite inv_inc_move inv_cap_distr inv_invol.
rewrite -inc_rpc -inv_inc_move.
apply H.
rewrite inv_inc_move inc_rpc.
rewrite -(@inv_invol _ _ alpha) -inv_cap_distr -inv_inc_move.
apply inc_rpc.
apply H.
Qed.

```

Lemma 120 (inv_empty)

$$\phi_{AB}^\# = \phi_{BA}.$$

Lemma *inv_empty* {A B : eqType}: A B # = B A.

Proof.

```

apply inc_antisym.
rewrite -inv_inc_move.
apply inc_empty_alpha.
apply inc_empty_alpha.
Qed.

```

Lemma 121 (inv_universal)

$$\nabla_{AB}^\# = \nabla_{BA}.$$

Lemma *inv_universal* {A B : eqType}: A B # = B A.

Proof.

```

apply inc_antisym.
apply inc_alpha_universal.
rewrite inv_inc_move.
apply inc_alpha_universal.
Qed.

```

Lemma 122 (inv_id)

$$id_A^\# = id_A.$$

Lemma *inv_id* {A : eqType}: (Id A) # = Id A.

Proof.

```

replace (Id A #) with ((Id A #) # • Id A #).
by [rewrite -comp_inv comp_id_l inv_invol].
by [rewrite inv_invol comp_id_l].
Qed.

```

Lemma 123 (inv_complement) *Let $\alpha : A \rightarrow B$. Then,*

$$(\alpha^-)^\# = (\alpha^\#)^-.$$

Lemma *inv_complement* { $A B : eqType$ } { $\alpha : Rel A B$ }: $(\alpha^\wedge)^\# = (\alpha^\#)^\wedge$.

Proof.

apply *inc_antisym*.

apply *inc_rpc*.

rewrite *-inv_cap_distr*.

rewrite *cap_comm -inv_inc_move inv_empty*.

rewrite *cap_complement_empty*.

apply *inc_refl*.

rewrite *inv_inc_move*.

apply *inc_rpc*.

replace $((\alpha^\#)^\wedge)^\#$ *alpha* with $((\alpha^\#)^\wedge)^\#$ $(\alpha^\#)^\#$.

rewrite *-inv_cap_distr*.

rewrite *cap_comm -inv_inc_move inv_empty*.

rewrite *cap_complement_empty*.

apply *inc_refl*.

by [rewrite *inv_invol*].

Qed.

Lemma 124 (inv_difference_distr) *Let $\alpha, \beta : A \rightarrow B$. Then,*

$$(\alpha - \beta)^\# = \alpha^\# - \beta^\#.$$

Lemma *inv_difference_distr* { $A B : eqType$ } { $\alpha \beta : Rel A B$ }:

$(\alpha - \beta)^\# = \alpha^\# - \beta^\#$.

Proof.

rewrite *inv_cap_distr*.

by [rewrite *inv_complement*].

Qed.

5.4 合成に関する補題

Lemma 125 (comp_cupP_distr_l, comp_cup_distr_l) *Let $\alpha : A \rightarrow B$, $f : (D \rightarrow E) \rightarrow (B \rightarrow C)$ and $P : predicate$. Then,*

$$\alpha \cdot (\sqcup_{P(\beta)} f(\beta)) = \sqcup_{P(\beta)} (\alpha \cdot f(\beta)).$$

Lemma *comp_cupP_distr_l* { $A B C D E : eqType$ }

$\{ \alpha : \text{Rel } A \ B \} \{ f : \text{Rel } D \ E \rightarrow \text{Rel } B \ C \} \{ P : \text{Rel } D \ E \rightarrow \text{Prop} \} :$
 $\alpha \cdot (_ \{ P \} f) = _ \{ P \} (\text{fun } \beta : \text{Rel } D \ E \Rightarrow (\alpha \cdot f \beta)) .$

Proof.

apply *inc_upper*.

move \Rightarrow *gamma*.

split; move \Rightarrow *H*.

rewrite $-(\text{@inv_inv} _ _ \alpha)$ in *H*.

apply *inc_residual* in *H*.

apply *inc_cupP*.

assert $(\forall \beta : \text{Rel } D \ E, P \beta \rightarrow f \beta \quad (\alpha \# \quad \text{gamma}))$.

apply *inc_cupP*.

apply *H*.

move \Rightarrow *beta H1*.

rewrite $-(\text{@inv_inv} _ _ \alpha)$.

apply *inc_residual*.

apply $(H0 _ H1)$.

rewrite $-(\text{@inv_inv} _ _ \alpha)$.

apply *inc_residual*.

apply *inc_cupP*.

assert $(\forall \beta : \text{Rel } D \ E, P \beta \rightarrow (\alpha \cdot f \beta) \quad \text{gamma})$.

apply *inc_cupP*.

apply *H*.

move \Rightarrow *beta H1*.

apply *inc_residual*.

rewrite *inv_inv*.

apply $(H0 _ H1)$.

Qed.

Lemma comp_cup_distr_l

$\{ A \ B \ C : \text{eqType} \} \{ \alpha : \text{Rel } A \ B \} \{ \beta \ \text{gamma} : \text{Rel } B \ C \} :$
 $\alpha \cdot (\beta \quad \text{gamma}) = (\alpha \cdot \beta) \quad (\alpha \cdot \text{gamma}) .$

Proof.

by [rewrite *cup_to_cupP -comp_cupP_distr_l -cup_to_cupP*].

Qed.

Lemma 126 (comp_cupP_distr_r, comp_cup_distr_r) *Let $f : (D \rightarrow E) \rightarrow (A \rightarrow B)$, $\beta : B \rightarrow C$ and $P : \text{predicate}$. Then,*

$$(\sqcup_{P(\alpha)} f(\alpha)) \cdot \beta = \sqcup_{P(\alpha)} (f(\alpha) \cdot \beta).$$

Lemma comp_cupP_distr_r $\{ A \ B \ C \ D \ E : \text{eqType} \}$

$\{ \beta : \text{Rel } B \ C \} \{ f : \text{Rel } D \ E \rightarrow \text{Rel } A \ B \} \{ P : \text{Rel } D \ E \rightarrow \text{Prop} \} :$
 $(_ \{ P \} f) \cdot \beta = _ \{ P \} (\text{fun } \alpha : \text{Rel } D \ E \Rightarrow (f \alpha \cdot \beta)) .$

Proof.

```

replace (fun alpha : Rel D E => f alpha · beta) with (fun alpha : Rel D E => (beta #
· f alpha #) #).
rewrite -inv_cupP_distr.
rewrite -comp_cupP_distr_l.
rewrite -inv_cupP_distr.
rewrite comp_inv.
by [rewrite inv_invol inv_invol].
apply functional_extensionality.
move => x.
rewrite comp_inv.
by [rewrite inv_invol inv_invol].
Qed.

```

Lemma comp_cup_distr_r

$\{A\ B\ C : eqType\} \{alpha\ beta : Rel\ A\ B\} \{gamma : Rel\ B\ C\} :$
 $(alpha\ \beta) \cdot gamma = (alpha \cdot gamma)\ (\beta \cdot gamma).$

Proof.

```

by [rewrite (@cup_to_cupP _ _ _ _ id) comp_cupP_distr_r -cup_to_cupP].
Qed.

```

Lemma 127 (comp_capP_distr) *Let $\alpha : A \rightarrow B$, $\gamma : C \rightarrow D$, $f : (E \rightarrow F) \rightarrow (B \rightarrow C)$ and $P : \text{predicate}$. Then,*

$$\alpha \cdot (\sqcap_{P(\beta)} f(\beta)) \cdot \gamma \sqsubseteq \sqcap_{P(\beta)} (\alpha \cdot f(\beta) \cdot \gamma).$$

Lemma comp_capP_distr $\{A\ B\ C\ D\ E\ F : eqType\}$

$\{alpha : Rel\ A\ B\} \{gamma : Rel\ C\ D\}$
 $\{f : Rel\ E\ F \rightarrow Rel\ B\ C\} \{P : Rel\ E\ F \rightarrow Prop\} :$
 $(alpha \cdot (_ \{P\} f)) \cdot gamma$
 $_ \{P\} (\text{fun } beta : Rel\ E\ F \Rightarrow ((alpha \cdot f\ beta) \cdot gamma)).$

Proof.

```

apply inc_capP.
move => beta H.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
move : H.
apply inc_capP.
apply inc_refl.
Qed.

```

Lemma 128 (comp_capP_distr_l, comp_cap_distr_l) *Let $\alpha : A \rightarrow B$, $f : (D \rightarrow E) \rightarrow (B \rightarrow C)$ and $P : \text{predicate}$. Then,*

$$\alpha \cdot (\sqcap_{P(\beta)} f(\beta)) \sqsubseteq \sqcap_{P(\beta)} (\alpha \cdot f(\beta)).$$

Lemma comp_capP_distr_l $\{A B C D E : \text{eqType}\}$
 $\{\alpha : \text{Rel } A B\} \{f : \text{Rel } D E \rightarrow \text{Rel } B C\} \{P : \text{Rel } D E \rightarrow \text{Prop}\}$:
 $(\alpha \cdot (_ \{P\} f)) _ \{P\} (\text{fun beta} : \text{Rel } D E \Rightarrow (\alpha \cdot f \text{ beta})).$

Proof.

move : (@comp_capP_distr _ _ _ _ _ alpha (Id C) f P) \Rightarrow H.

rewrite comp_id_r in H.

replace (fun beta : Rel D E \Rightarrow (alpha · f beta) · Id C) with (fun beta : Rel D E \Rightarrow (alpha · f beta)) in H.

apply H.

apply functional_extensionality.

move \Rightarrow x.

by [rewrite comp_id_r].

Qed.

Lemma comp_cap_distr_l
 $\{A B C : \text{eqType}\} \{\alpha : \text{Rel } A B\} \{\text{beta gamma} : \text{Rel } B C\}$:
 $(\alpha \cdot (\text{beta gamma})) ((\alpha \cdot \text{beta}) (\alpha \cdot \text{gamma})).$

Proof.

rewrite cap_to_capP (@cap_to_capP _ _ _ _ _ id).

apply comp_capP_distr_l.

Qed.

Lemma 129 (comp_capP_distr_r, comp_cap_distr_r) *Let $\beta : B \rightarrow C$, $f : (D \rightarrow E) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$(\sqcap_{P(\alpha)} f(\alpha)) \cdot \beta \sqsubseteq \sqcap_{P(\alpha)} (f(\alpha) \cdot \beta).$$

Lemma comp_capP_distr_r $\{A B C D E : \text{eqType}\} \{\text{beta} : \text{Rel } B C\} \{f : \text{Rel } D E \rightarrow \text{Rel } A B\} \{P : \text{Rel } D E \rightarrow \text{Prop}\}$:
 $((_ \{P\} f) \cdot \text{beta}) _ \{P\} (\text{fun alpha} : \text{Rel } D E \Rightarrow (f \text{ alpha} \cdot \text{beta})).$

Proof.

move : (@comp_capP_distr _ _ _ _ _ (Id A) beta f P) \Rightarrow H.

rewrite comp_id_l in H.

replace (fun alpha : Rel D E \Rightarrow (Id A · f alpha) · beta) with (fun alpha : Rel D E \Rightarrow f alpha · beta) in H.

apply H.

apply functional_extensionality.

move $\Rightarrow x$.
 by [rewrite *comp_id_l*].
 Qed.

Lemma *comp_cap_distr_r* $\{A\ B\ C : eqType\} \{alpha\ beta : Rel\ A\ B\} \{gamma : Rel\ B\ C\}$:
 $((alpha\ \beta) \cdot gamma) \quad ((alpha \cdot gamma) \quad (\beta \cdot gamma))$.

Proof.

rewrite (@cap_to_capP _ _ _ _ id) (@cap_to_capP _ _ _ _ (fun $x \Rightarrow x \cdot gamma$)).
 apply *comp_capP_distr_r*.
 Qed.

Lemma 130 (*comp_empty_l*, *comp_empty_r*) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$. Then,*

$$\alpha \cdot \phi_{BC} = \phi_{AB} \cdot \beta = \phi_{AC}.$$

Lemma *comp_empty_r* $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\}$: $alpha \cdot \quad B\ C = \quad A\ C$.

Proof.

apply *inc_antisym*.
 rewrite -(@inv_invol _ _ alpha).
 apply *inc_residual*.
 apply *inc_empty_alpha*.
 apply *inc_empty_alpha*.
 Qed.

Lemma *comp_empty_l* $\{A\ B\ C : eqType\} \{beta : Rel\ B\ C\}$: $A\ B \cdot beta = \quad A\ C$.

Proof.

rewrite -(@inv_invol _ _ ($A\ B \cdot beta$)).
 rewrite -inv_move comp_inv inv_empty inv_empty.
 apply *comp_empty_r*.
 Qed.

Lemma 131 (*comp_either_empty*) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$. Then,*

$$\alpha = \phi_{AB} \vee \beta = \phi_{BC} \Rightarrow \alpha \cdot \beta = \phi_{AC}.$$

Lemma *comp_either_empty* $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$:
 $alpha = \quad A\ B \vee beta = \quad B\ C \rightarrow alpha \cdot beta = \quad A\ C$.

Proof.

case; move $\Rightarrow H$.
 rewrite H .
 apply *comp_empty_l*.
 rewrite H .
 apply *comp_empty_r*.

Qed.

Lemma 132 (comp_neither_empty) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$. Then,*

$$\alpha \cdot \beta \neq \phi_{AC} \Rightarrow \alpha \neq \phi_{AB} \wedge \beta \neq \phi_{BC}.$$

Lemma *comp_neither_empty* { $A\ B\ C : eqType$ } { $\alpha : Rel\ A\ B$ } { $\beta : Rel\ B\ C$ }:
 $\alpha \cdot \beta \neq \phi_{AC} \rightarrow \alpha \neq \phi_{AB} \wedge \beta \neq \phi_{BC}.$

Proof.

move $\Rightarrow H$.

split; move $\Rightarrow H0$.

apply H .

rewrite $H0$.

apply *comp_empty_l*.

apply H .

rewrite $H0$.

apply *comp_empty_r*.

Qed.

5.5 単域と Tarski の定理

Lemma 133 (lemma_for_tarski1) *Let $\alpha : A \rightarrow B$ and $\alpha \neq \phi_{AB}$. Then,*

$$\nabla_{IA} \cdot \alpha \cdot \nabla_{BI} = id_I.$$

Lemma *lemma_for_tarski1* { $A\ B : eqType$ } { $\alpha : Rel\ A\ B$ }:
 $\alpha \neq \phi_{AB} \rightarrow ((\nabla_{IA} \cdot \alpha) \cdot \nabla_{BI}) = Id\ I.$

Proof.

move $\Rightarrow H$.

assert ((($\nabla_{IA} \cdot \alpha$) $\cdot \nabla_{BI}$) $\neq Id\ I$).

move $\Rightarrow H0$.

apply H .

apply *inc_antisym*.

apply (@*inc_trans* _ _ (($\nabla_{IA} \cdot \alpha$) $\cdot \nabla_{BI}$) $\cdot \nabla_{BI}$).

rewrite *comp_assoc comp_assoc unit_universal*.

rewrite *-comp_assoc -comp_assoc unit_universal*.

apply (@*inc_trans* _ _ (($Id\ A \cdot \alpha$) $\cdot Id\ B$)).

rewrite *comp_id_l comp_id_r*.

apply *inc_refl*.

apply *comp_inc_compat*.

apply *comp_inc_compat_ab_a'b*.

apply *inc_alpha_universal*.

```

apply inc_alpha_universal.
rewrite H0 comp_empty_r comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
case (@unit_empty_or_universal (( i A • alpha) • B i)); move => H1.
apply False_ind.
apply (H0 H1).
rewrite unit_identity_is_universal.
apply H1.
Qed.

```

Lemma 134 (lemma_for_tarski2)

$$\nabla_{AI} \cdot \nabla_{IB} = \nabla_{AB}.$$

Lemma lemma_for_tarski2 {A B : eqType}: A i • i B = A B.

Proof.

```

apply inc_antisym.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ ( A A • A B)).
apply (@inc_trans _ _ _ (Id A • A B)).
rewrite comp_id_l.
apply inc_refl.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
rewrite -(@unit_universal A) comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
Qed.

```

Lemma 135 (tarski) Let $\alpha : A \rightarrow B$ and $\alpha \neq \phi_{AB}$. Then,

$$\nabla_{AA} \cdot \alpha \cdot \nabla_{BB} = \nabla_{AB}.$$

Lemma tarski {A B : eqType} {alpha : Rel A B}:
 $\alpha \neq A B \rightarrow ((A A \cdot \alpha) \cdot B B) = A B.$

Proof.

```

move => H.
rewrite -(@unit_universal A) -(@unit_universal B).
move : (@lemma_for_tarski1 _ _ alpha H) => H0.
rewrite -comp_assoc (@comp_assoc _ _ _ _ ( A i)) (@comp_assoc _ _ _ _ ( A i)).
rewrite H0 comp_id_r.
apply lemma_for_tarski2.

```


Qed.

Lemma 136 (comp_universal1) *Let $B \neq \emptyset$. Then,*

$$\nabla_{AB} \cdot \nabla_{BC} = \nabla_{AC}.$$

Lemma comp_universal $\{A\ B\ C : eqType\} : B \rightarrow A\ B \cdot B\ C = A\ C$.

Proof.

move $\Rightarrow b$.

replace ($A\ B$) with ($A\ B \cdot B\ B$).

rewrite -(@lemma_for_tarski2 $A\ B$) -(@lemma_for_tarski2 $B\ C$).

rewrite (@comp_assoc _ _ _ ($A\ i$)) (@comp_assoc _ _ _ ($A\ i$)) -(@comp_assoc _ _ _ ($B\ i$)).

rewrite lemma_for_tarski1.

rewrite comp_id_l.

apply lemma_for_tarski2.

apply not_eq_sym.

move $\Rightarrow H$.

apply either_empty in H .

case H ; move $\Rightarrow H0$.

apply ($H0\ b$).

apply ($H0\ b$).

apply inc_antisym.

apply inc_alpha_universal.

apply (@inc_trans _ _ _ ($A\ B \cdot Id\ B$)).

rewrite comp_id_r.

apply inc_refl.

apply comp_inc_compat_ab_ab'.

apply inc_alpha_universal.

Qed.

Lemma 137 (comp_universal2)

$$\nabla_{IA}^\sharp \cdot \nabla_{IB} = \nabla_{AB}.$$

Lemma comp_universal2 $\{A\ B : eqType\} : i\ A\ \# \cdot i\ B = A\ B$.

Proof.

rewrite inv_universal.

apply lemma_for_tarski2.

Qed.

Lemma 138 (`empty_equivalence1`, `empty_equivalence2`, `empty_equivalence3`)

$$A = \emptyset \Leftrightarrow \nabla_{IA} = \phi_{IA} \Leftrightarrow \nabla_{AA} = \phi_{AA} \Leftrightarrow id_A = \phi_{AA}.$$

Lemma *empty_equivalence1* $\{A : eqType\} : (A \rightarrow False) \leftrightarrow \quad i \ A = \quad i \ A.$

Proof.

`move : (@either_empty i A) \Rightarrow H.`

`split; move \Rightarrow H0.`

`apply Logic.eq_sym.`

`apply H.`

`right.`

`apply H0.`

`apply Logic.eq_sym in H0.`

`apply H in H0.`

`case H0.`

`move \Rightarrow H1 H2.`

`apply H1.`

`apply tt.`

`by [].`

Qed.

Lemma *empty_equivalence2* $\{A : eqType\} : (A \rightarrow False) \leftrightarrow \quad A \ A = \quad A \ A.$

Proof.

`move : (@either_empty A A) \Rightarrow H.`

`split; move \Rightarrow H0.`

`apply Logic.eq_sym.`

`apply H.`

`left.`

`apply H0.`

`apply Logic.eq_sym in H0.`

`apply H in H0.`

`case H0.`

`by [].`

`by [].`

Qed.

Lemma *empty_equivalence3* $\{A : eqType\} : (A \rightarrow False) \leftrightarrow Id \ A = \quad A \ A.$

Proof.

`split; move \Rightarrow H.`

`assert ($A \ A = \quad A \ A$).`

`apply empty_equivalence2.`

`apply H.`

`apply RelAB.unique.`

`apply Logic.eq_sym.`

```
apply H0.
assert (  $\vdash A\ A = \vdash A\ A$  ).
by [rewrite -(@comp_id_r _ _ (  $\vdash A\ A$ )) H comp_empty_r].
apply either_empty in H0.
case H0.
by [].
by [].
Qed.
```

Chapter 6

Library **Functions_Mappings**

```
Require Import Basic_Notations.  
Require Import Basic_Lemmas.  
Require Import Relation_Properties.  
Require Import Logic.FunctionalExtensionality.
```

6.1 全域性, 一価性, 写像に関する補題

Lemma 139 (id_function) $id_A : A \rightarrow A$ is a function.

```
Lemma id_function {A : eqType}: function_r (Id A).  
Proof.  
rewrite /function_r/total_r/univalent_r.  
rewrite inv_id comp_id_l.  
split.  
apply inc_refl.  
apply inc_refl.  
Qed.
```

Lemma 140 (unit_function) $\nabla_{AI} : A \rightarrow I$ is a function.

```
Lemma unit_function {A : eqType}: function_r ( A i).  
Proof.  
rewrite /function_r/total_r/univalent_r.  
rewrite inv_universal lemma_for_tarski2 unit_identity_is_universal.  
split.  
apply inc_alpha_universal.  
apply inc_alpha_universal.  
Qed.
```

CHAPTER 6. LIBRARY FUNCTIONS_MAPPINGS

Lemma 141 (total_comp) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be total relations, then $\alpha \cdot \beta$ is also a total relation.*

Lemma `total_comp` $\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel } A\ B\} \{beta : \text{Rel } B\ C\}$:

`total_r alpha` \rightarrow `total_r beta` \rightarrow `total_r (alpha \cdot beta)`.

Proof.

`rewrite /total_r.`

`move \Rightarrow H H0.`

`rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ beta).`

`apply (@inc_trans _ _ _ _ H).`

`apply comp_inc_compat_ab_ab'.`

`apply comp_inc_compat_b_ab.`

`apply H0.`

Qed.

Lemma 142 (univalent_comp) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be univalent relations, then $\alpha \cdot \beta$ is also a univalent relation.*

Lemma `univalent_comp` $\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel } A\ B\} \{beta : \text{Rel } B\ C\}$:

`univalent_r alpha` \rightarrow `univalent_r beta` \rightarrow `univalent_r (alpha \cdot beta)`.

Proof.

`rewrite /univalent_r.`

`move \Rightarrow H H0.`

`rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ (alpha #)).`

`apply (fun H' \Rightarrow @inc_trans _ _ _ _ H' H0).`

`apply comp_inc_compat_ab_ab'.`

`apply comp_inc_compat_ab_b.`

`apply H.`

Qed.

Lemma 143 (function_comp) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be functions, then $\alpha \cdot \beta$ is also a function.*

Lemma `function_comp` $\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel } A\ B\} \{beta : \text{Rel } B\ C\}$:

`function_r alpha` \rightarrow `function_r beta` \rightarrow `function_r (alpha \cdot beta)`.

Proof.

`elim \Rightarrow H H0.`

`elim \Rightarrow H1 H2.`

`split.`

`apply (total_comp H H1).`

`apply (univalent_comp H0 H2).`

Qed.

CHAPTER 6. LIBRARY FUNCTIONS_MAPPINGS

Lemma 144 (total_comp2) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and $\alpha \cdot \beta$ be a total relation, then α is also a total relation.*

Lemma `total_comp2` $\{A\ B\ C : \text{eqType}\} \{ \text{alpha} : \text{Rel}\ A\ B\} \{ \text{beta} : \text{Rel}\ B\ C\}$:
`total_r (alpha · beta) → total_r alpha.`

Proof.

`move ⇒ H.`

`apply inc_def1 in H.`

`rewrite comp_inv cap_comm comp_assoc in H.`

`rewrite /total_r.`

`rewrite H.`

`apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).`

`apply comp_inc_compat.`

`apply cap_l.`

`rewrite comp_id_r.`

`apply cap_r.`

Qed.

Lemma 145 (univalent_comp2) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$, $\alpha \cdot \beta$ be a univalent relation and $\alpha^\#$ be a total relation, then β is a univalent relation.*

Lemma `univalent_comp2` $\{A\ B\ C : \text{eqType}\} \{ \text{alpha} : \text{Rel}\ A\ B\} \{ \text{beta} : \text{Rel}\ B\ C\}$:
`univalent_r (alpha · beta) → total_r (alpha #) → univalent_r beta.`

Proof.

`move ⇒ H H0.`

`apply (fun H' ⇒ @inc_trans _ _ _ _ H' H).`

`rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ alpha).`

`apply comp_inc_compat_ab_ab'.`

`rewrite /total_r in H0.`

`rewrite inv_invol in H0.`

`apply (comp_inc_compat_b_ab H0).`

Qed.

Lemma 146 (total_inc) *Let $\alpha : A \rightarrow B$ be a total relation and $\alpha \sqsubseteq \beta$, then β is also a total relation.*

Lemma `total_inc` $\{A\ B : \text{eqType}\} \{ \text{alpha}\ \text{beta} : \text{Rel}\ A\ B\}$:
`total_r alpha → alpha beta → total_r beta.`

Proof.

`move ⇒ H H0.`

`apply (@inc_trans _ _ _ _ H).`

`apply comp_inc_compat.`

`apply H0.`

apply (@inc_inv _ _ _ _ H0).

Qed.

Lemma 147 (univalent_inc) *Let $\alpha : A \rightarrow B$ be a univalent relation and $\beta \sqsubseteq \alpha$, then β is also a univalent relation.*

Lemma univalent_inc {A B : eqType} {alpha beta : Rel A B}:
 univalent_r alpha → beta alpha → univalent_r beta.

Proof.

move ⇒ H H0.

apply (fun H' ⇒ @inc_trans _ _ _ _ H' H).

apply comp_inc_compat.

apply (@inc_inv _ _ _ _ H0).

apply H0.

Qed.

Lemma 148 (function_inc) *Let $\alpha, \beta : A \rightarrow B$ be functions and $\alpha \sqsubseteq \beta$. Then,*

$$\alpha = \beta.$$

Lemma function_inc {A B : eqType} {alpha beta : Rel A B}:
 function_r alpha → function_r beta → alpha beta → alpha = beta.

Proof.

move ⇒ H H0 H1.

apply inc_antisym.

apply H1.

apply (@inc_trans _ _ _ ((alpha · alpha #) · beta)).

apply comp_inc_compat_b_ab.

apply H.

move : (@inc_inv _ _ _ _ H1) ⇒ H2.

apply (@inc_trans _ _ _ ((alpha · beta #) · beta)).

apply comp_inc_compat_ab_a'b.

apply comp_inc_compat_ab_ab'.

apply H2.

rewrite comp_assoc.

apply comp_inc_compat_ab_a.

apply H0.

Qed.

Lemma 149 (total_universal) *If ∇_{IB} be a total relation, then*

$$\nabla_{AB} \cdot \nabla_{BC} = \nabla_{AC}.$$

CHAPTER 6. LIBRARY FUNCTIONS_MAPPINGS

Lemma *total_universal* $\{A\ B\ C : eqType\}$:
 $total_r\ (_ \ i\ B) \rightarrow _ \ A\ B \cdot _ \ B\ C = _ \ A\ C.$

Proof.

move $\Rightarrow H$.

rewrite $-(@lemma_for_tarski2\ A\ B) \ -(@lemma_for_tarski2\ B\ C).$

rewrite *comp_assoc* $-(@comp_assoc\ _ \ _ \ _ \ (_ \ i\ B)).$

replace $(_ \ i\ B \cdot _ \ B\ i)$ with $(Id\ i).$

rewrite *comp_id_l*.

apply *lemma_for_tarski2*.

apply *inc_antisym*.

rewrite */total_r* in H .

rewrite *inv_universal* in H .

apply H .

rewrite *unit_identity_is_universal*.

apply *inc_alpha_universal*.

Qed.

Lemma 150 (function_rel_inv_rel) *Let $\alpha : A \rightarrow B$ be function. Then,*

$$\alpha \cdot \alpha^\# \cdot \alpha = \alpha.$$

Lemma *function_rel_inv_rel* $\{A\ B : eqType\} \{alpha : Rel\ A\ B\}$:
 $function_r\ alpha \rightarrow (alpha \cdot alpha^\#) \cdot alpha = alpha.$

Proof.

move $\Rightarrow H$.

apply *inc_antisym*.

rewrite *comp_assoc*.

apply *comp_inc_compat_ab_a*.

apply H .

apply *comp_inc_compat_b_ab*.

apply H .

Qed.

Lemma 151 (function_capP_distr) *Let $f : A \rightarrow B, g : D \rightarrow C$ be functions, $\theta : (E \rightarrow F) \rightarrow (B \rightarrow C)$ and $P : predicate$. Then,*

$$f \cdot (\sqcap_{P(\theta)} \theta(\alpha)) \cdot g^\# = \sqcap_{P(\alpha)} (f \cdot \theta(\alpha) \cdot g^\#).$$

Lemma *function_capP_distr* $\{A\ B\ C\ D\ E\ F : eqType\}$
 $\{f : Rel\ A\ B\} \{g : Rel\ D\ C\} \{theta : Rel\ E\ F \rightarrow Rel\ B\ C\} \{P : Rel\ E\ F \rightarrow Prop\}$:
 $function_r\ f \rightarrow function_r\ g \rightarrow$
 $(f \cdot (_ \{P\} \theta)) \cdot g^\# =$
 $_ \{P\} (\text{fun } alpha : Rel\ E\ F \Rightarrow (f \cdot \theta\ alpha) \cdot g^\#).$

Proof.

```

elim ⇒ H H0.
elim ⇒ H1 H2.
apply inc_antisym.
apply comp_capP_distr.
apply (@inc_trans _ _ _ (((f · f #) · _{P} (fun alpha : Rel E F ⇒ (f · theta alpha)
· g #))) · (g · g #))).
apply (@inc_trans _ _ _ ((f · f #) · ( _{P} (fun alpha : Rel E F ⇒ (f · theta alpha)
· g #)))).
apply (comp_inc_compat_b_ab H).
apply (comp_inc_compat_a_ab H1).
rewrite (@comp_assoc _ _ _ _ (f #)) comp_assoc - (@comp_assoc _ _ _ _ g) - comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ ( _{P} (fun alpha : Rel E F ⇒ (f # · ((f · theta alpha) · g
#)) · g))).
apply comp_capP_distr.
replace (fun alpha : Rel E F ⇒ (f # · ((f · theta alpha) · g #)) · g) with (fun alpha
: Rel E F ⇒ ((f # · f) · theta alpha) · (g # · g)).
apply inc_capP.
move ⇒ beta H3.
apply (@inc_trans _ _ _ ((f # · f) · theta beta)).
apply (@inc_trans _ _ _ (((f # · f) · theta beta) · (g # · g))).
move : beta H3.
apply inc_capP.
apply inc_refl.
apply (comp_inc_compat_ab_a H2).
apply (comp_inc_compat_ab_b H0).
apply functional_extensionality.
move ⇒ x.
by [rewrite comp_assoc comp_assoc comp_assoc comp_assoc comp_assoc].
Qed.
    
```

Lemma 152 (`function_cap_distr`, `function_cap_distr_l`, `function_cap_distr_r`)

Let $f : A \rightarrow B, g : D \rightarrow C$ be functions and $\alpha, \beta : B \rightarrow C$. Then,

$$f \cdot (\alpha \sqcap \beta) \cdot g^\# = (f \cdot \alpha \cdot g^\#) \sqcap (f \cdot \beta \cdot g^\#).$$

Lemma `function_cap_distr`

$\{A\ B\ C\ D : \text{eqType}\} \{f : \text{Rel } A\ B\} \{\alpha\ \beta : \text{Rel } B\ C\} \{g : \text{Rel } D\ C\} :$
 $\text{function_r } f \rightarrow \text{function_r } g \rightarrow$
 $(f \cdot (\alpha \sqcap \beta)) \cdot g^\# = ((f \cdot \alpha) \cdot g^\#) \sqcap ((f \cdot \beta) \cdot g^\#).$

Proof.

CHAPTER 6. LIBRARY FUNCTIONS_MAPPINGS

```
rewrite (@cap_to_capP _ _ _ _ id) (@cap_to_capP _ _ _ _ (fun x => (f • x) • g
#)).
apply function_capP_distr.
Qed.
```

Lemma *function_cap_distr_l*
 $\{A\ B\ C : \text{eqType}\} \{f : \text{Rel } A\ B\} \{\text{alpha } \text{beta} : \text{Rel } B\ C\} :$
 $\text{function_r } f \rightarrow$
 $f \cdot (\text{alpha } \text{beta}) = (f \cdot \text{alpha}) \quad (f \cdot \text{beta}).$

Proof.
 $\text{move} : (@\text{id_function } C) \Rightarrow H.$
 $\text{move} \Rightarrow H0.$
 $\text{apply } (@\text{function_cap_distr } _ _ _ f \text{ alpha } \text{beta}) \text{ in } H.$
 $\text{rewrite } \text{inv_id } \text{comp_id_r } \text{comp_id_r } \text{comp_id_r} \text{ in } H.$
 $\text{apply } H.$
 $\text{apply } H0.$
Qed.

Lemma *function_cap_distr_r*
 $\{B\ C\ D : \text{eqType}\} \{\text{alpha } \text{beta} : \text{Rel } B\ C\} \{g : \text{Rel } D\ C\} :$
 $\text{function_r } g \rightarrow$
 $(\text{alpha } \text{beta}) \cdot g \# = (\text{alpha } \cdot g \#) \quad (\text{beta } \cdot g \#).$

Proof.
 $\text{move} : (@\text{id_function } B) \Rightarrow H.$
 $\text{move} \Rightarrow H0.$
 $\text{apply } (@\text{function_cap_distr } _ _ _ _ \text{alpha } \text{beta } g) \text{ in } H.$
 $\text{rewrite } \text{comp_id_l } \text{comp_id_l } \text{comp_id_l} \text{ in } H.$
 $\text{apply } H.$
 $\text{apply } H0.$
Qed.

Lemma 153 (function_move1) *Let $\alpha : A \rightarrow B$ be a function, $\beta : B \rightarrow C$ and $\gamma : A \rightarrow C$. Then,*

$$\gamma \sqsubseteq \alpha \cdot \beta \Leftrightarrow \alpha^\# \cdot \gamma \sqsubseteq \beta.$$

Lemma *function_move1* $\{A\ B\ C : \text{eqType}\} \{\text{alpha} : \text{Rel } A\ B\} \{\text{beta} : \text{Rel } B\ C\} \{\text{gamma} : \text{Rel } A\ C\} :$
 $\text{function_r } \text{alpha} \rightarrow (\text{gamma } (\text{alpha } \cdot \text{beta}) \leftrightarrow (\text{alpha } \# \cdot \text{gamma}) \text{ beta}).$

Proof.
 $\text{move} \Rightarrow H.$
 $\text{split}; \text{move} \Rightarrow H0.$
 $\text{apply } (@\text{inc_trans } _ _ ((\text{alpha } \# \cdot \text{alpha}) \cdot \text{beta})).$
 $\text{rewrite } \text{comp_assoc}.$
 $\text{apply } (\text{comp_inc_compat_ab_ab'} H0).$

```

apply comp_inc_compat_ab_b.
apply H.
apply (@inc_trans _ _ _ ((alpha · alpha #) · gamma)).
apply comp_inc_compat_b_ab.
apply H.
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H0).
Qed.

```

Lemma 154 (function_move2) *Let $\beta : B \rightarrow C$ be a function, $\alpha : A \rightarrow B$ and $\gamma : A \rightarrow C$. Then,*

$$\alpha \cdot \beta \sqsubseteq \gamma \Leftrightarrow \alpha \sqsubseteq \gamma \cdot \beta^\#.$$

Lemma function_move2 $\{A\ B\ C : \text{eqType}\} \{\alpha : \text{Rel } A\ B\} \{\beta : \text{Rel } B\ C\} \{\gamma : \text{Rel } A\ C\}$:

$\text{function_r } \beta \rightarrow ((\alpha \cdot \beta) \quad \gamma \leftrightarrow \alpha \quad (\gamma \cdot \beta \#)).$

Proof.

```

move  $\Rightarrow$  H.
split; move  $\Rightarrow$  H0.
apply (@inc_trans _ _ _ ((alpha · beta) · beta #)).
rewrite comp_assoc.
apply comp_inc_compat_a_ab.
apply H.
apply (comp_inc_compat_ab_a'b H0).
apply (@inc_trans _ _ _ ((gamma · beta #) · beta)).
apply (comp_inc_compat_ab_a'b H0).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H.
Qed.

```

Lemma 155 (function_rpc_distr) *Let $f : A \rightarrow B, g : D \rightarrow C$ be functions and $\alpha, \beta : B \rightarrow C$. Then,*

$$f \cdot (\alpha \Rightarrow \beta) \cdot g^\# = (f \cdot \alpha \cdot g^\#) \Rightarrow (f \cdot \beta \cdot g^\#).$$

Lemma function_rpc_distr

$\{A\ B\ C\ D : \text{eqType}\} \{f : \text{Rel } A\ B\} \{\alpha\ \beta : \text{Rel } B\ C\} \{g : \text{Rel } D\ C\}$:

$\text{function_r } f \rightarrow \text{function_r } g \rightarrow$

$(f \cdot (\alpha \gg \beta)) \cdot g \# = ((f \cdot \alpha) \cdot g \#) \gg ((f \cdot \beta) \cdot g \#).$

Proof.

```

move  $\Rightarrow$  H H0.
apply inc_lower.

```

```

move  $\Rightarrow$  gamma.
split; move  $\Rightarrow$  H1.
apply inc_rpc.
apply (function_move2 H0).
apply (function_move1 H).
apply (@inc_trans _ _ _ (((f #  $\cdot$  gamma)  $\cdot$  g) ((f #  $\cdot$  ((f  $\cdot$  alpha)  $\cdot$  g #))  $\cdot$  g))).
rewrite -comp_assoc.
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (@comp_cap_distr_r _ _ _ _ _)).
apply comp_inc_compat_ab_a'b.
apply comp_cap_distr_l.
apply (function_move2 H0) in H1.
apply (function_move1 H) in H1.
rewrite -inc_rpc comp_assoc.
apply (@inc_trans _ _ _ _ H1).
apply rpc_inc_compat_r.
rewrite comp_assoc comp_assoc comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ (alpha  $\cdot$  (g #  $\cdot$  g))).
apply comp_inc_compat_ab_b.
apply H.
apply comp_inc_compat_ab_a.
apply H0.
apply (function_move2 H0).
apply (function_move1 H).
apply inc_rpc.
apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).
apply (@inc_trans _ _ _ (f #  $\cdot$  ((gamma  $\cdot$  g) ((f #) #  $\cdot$  alpha)))).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite inv_invol.
apply (@inc_trans _ _ _ ((f #  $\cdot$  (gamma ((f  $\cdot$  alpha)  $\cdot$  g #)))  $\cdot$  g)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).
apply comp_inc_compat_ab_ab'.
apply cap_l.
apply (function_move2 H0).
apply (function_move1 H).
rewrite -inc_rpc -comp_assoc.
apply H1.
Qed.

```

Lemma 156 (function_inv_rel1, function_inv_rel2) *Let $f : A \rightarrow B$ be a function. Then,*

$$f^\# \cdot f = id_B \sqcap f^\# \cdot \nabla_{AA} \cdot f = id_B \sqcap \nabla_{BA} \cdot f.$$

Lemma *function_inv_rel1* { $A\ B : eqType$ } { $f : Rel\ A\ B$ }:
function_r $f \rightarrow f \# \cdot f = Id\ B \quad ((f \# \cdot \quad A\ A) \cdot f).$

Proof.

move $\Rightarrow H$.
 apply *inc_antisym*.
 apply *inc_cap*.
 split.
 apply H .
 apply *comp_inc_compat_ab_a'b*.
 apply *comp_inc_compat_a_ab*.
 apply *inc_alpha_universal*.
 apply (@*inc_trans* _ _ _ (Id B ($B\ A \cdot f$))).
 apply *cap_inc_compat_l*.
 apply *comp_inc_compat_ab_a'b*.
 apply *inc_alpha_universal*.
 rewrite *cap_comm*.
 apply (@*inc_trans* _ _ _ _ (@*dedekind* _ _ _ _ _)).
 rewrite *comp_id_l comp_id_r cap_comm inv_universal*.
 rewrite *cap_universal cap_universal*.
 apply *inc_refl*.

Qed.

Lemma *function_inv_rel2* { $A\ B : eqType$ } { $f : Rel\ A\ B$ }:
function_r $f \rightarrow f \# \cdot f = Id\ B \quad (\quad B\ A \cdot f).$

Proof.

move $\Rightarrow H$.
 apply *inc_antisym*.
 rewrite (@*function_inv_rel1* _ _ _ H).
 apply *cap_inc_compat_l*.
 apply *comp_inc_compat_ab_a'b*.
 apply *inc_alpha_universal*.
 rewrite *cap_comm*.
 apply (@*inc_trans* _ _ _ _ (@*dedekind* _ _ _ _ _)).
 rewrite *comp_id_l comp_id_r cap_comm inv_universal*.
 rewrite *cap_universal cap_universal*.
 apply *inc_refl*.

Qed.

Lemma 157 (function_dedekind1, function_dedekind2) *Let $f : A \rightarrow B$ be a function, $\mu : C \rightarrow A$ and $\rho : C \rightarrow B$. Then,*

$$(\mu \sqcap \rho \cdot f^\#) \cdot f = \mu \cdot f \sqcap \rho \wedge \rho \cdot f^\# \cdot f = \nabla_{CA} \cdot f \sqcap \rho.$$

Lemma function_dedekind1

$\{A\ B\ C : \text{eqType}\} \{f : \text{Rel}\ A\ B\} \{\mu : \text{Rel}\ C\ A\} \{\rho : \text{Rel}\ C\ B\} :$
 $\text{function_r } f \rightarrow (\mu \quad (\rho \cdot f^\#)) \cdot f = (\mu \cdot f) \quad \rho.$

Proof.

move $\Rightarrow H$.

apply *inc_antisym*.

apply (*@inc_trans* _ _ _ _ (*comp_cap_distr_r*)).

apply *cap_inc_compat_l*.

rewrite *comp_assoc*.

apply *comp_inc_compat_ab_a*.

apply *H*.

apply (*@inc_trans* _ _ _ _ (*@dedekind* _ _ _ _ _)).

apply *comp_inc_compat_ab_ab'*.

apply *cap_l*.

Qed.

Lemma function_dedekind2 $\{A\ B\ C : \text{eqType}\} \{f : \text{Rel}\ A\ B\} \{\rho : \text{Rel}\ C\ B\} :$
 $\text{function_r } f \rightarrow (\rho \cdot f^\#) \cdot f = (\quad C\ A \cdot f) \quad \rho.$

Proof.

move $\Rightarrow H$.

move : (*@function_dedekind1* _ _ _ *f* (*C A*) *rho H*) $\Rightarrow H0$.

rewrite *cap_comm cap_universal* in *H0*.

apply *H0*.

Qed.

6.2 全射, 単射に関する補題

Lemma 158 (surjection_comp) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be surjections, then $\alpha \cdot \beta$ is also a surjection.*

Lemma surjection_comp $\{A\ B\ C : \text{eqType}\} \{\alpha : \text{Rel}\ A\ B\} \{\beta : \text{Rel}\ B\ C\} :$
 $\text{surjection_r } \alpha \rightarrow \text{surjection_r } \beta \rightarrow \text{surjection_r } (\alpha \cdot \beta).$

Proof.

rewrite */surjection_r*.

elim $\Rightarrow H\ H0$.

elim $\Rightarrow H1\ H2$.

split.

```

apply (function_comp H H1).
rewrite comp_inv.
apply (total_comp H2 H0).
Qed.

```

Lemma 159 (injection_comp) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be injections, then $\alpha \cdot \beta$ is also an injection.*

Lemma *injection_comp* $\{A B C : \text{eqType}\} \{\alpha : \text{Rel } A B\} \{\beta : \text{Rel } B C\}$:
 $\text{injection_r } \alpha \rightarrow \text{injection_r } \beta \rightarrow \text{injection_r } (\alpha \cdot \beta)$.

Proof.
rewrite /injection_r.
elim $\Rightarrow H H0$.
elim $\Rightarrow H1 H2$.
split.
apply (function_comp H H1).
rewrite comp_inv.
apply (univalent_comp H2 H0).
Qed.

Lemma 160 (bijection_comp) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$ be bijections, then $\alpha \cdot \beta$ is also a bijection.*

Lemma *bijection_comp* $\{A B C : \text{eqType}\} \{\alpha : \text{Rel } A B\} \{\beta : \text{Rel } B C\}$:
 $\text{bijection_r } \alpha \rightarrow \text{bijection_r } \beta \rightarrow \text{bijection_r } (\alpha \cdot \beta)$.

Proof.
rewrite /bijection_r.
elim $\Rightarrow H$.
elim $\Rightarrow H0 H1$.
elim $\Rightarrow H2$.
elim $\Rightarrow H3 H4$.
split.
apply (function_comp H H2).
rewrite comp_inv.
split.
apply (total_comp H3 H0).
apply (univalent_comp H4 H1).
Qed.

Lemma 161 (surjection_unique1) *Let $e : A \twoheadrightarrow B$ be a surjection, $f : A \rightarrow C$ be a function and $e \cdot e^\sharp \sqsubseteq f \cdot f^\sharp$, then there exists a unique function $g : B \rightarrow C$ s.t. $f = eg$.*

Lemma *surjection_unique1* $\{A B C : \text{eqType}\} \{e : \text{Rel } A B\} \{f : \text{Rel } A C\}$:

$surjection_r\ e \rightarrow function_r\ f \rightarrow (e \cdot e \#) \quad (f \cdot f \#) \rightarrow$
 $(\exists! g : Rel\ B\ C, function_r\ g \wedge f = e \cdot g).$

Proof.

```

rewrite /surjection_r/function_r/total_r/univalent_r.
elim.
elim  $\Rightarrow H\ H0\ H1$ .
elim  $\Rightarrow H2\ H3\ H4$ .
 $\exists (e \# \cdot f)$ .
repeat split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ f).
apply (@inc_trans _ _ _ _ H1).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H2.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ e).
apply (@inc_trans _ _ _ (f #  $\cdot ((f \cdot f \#) \cdot f)))$ .
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H4).
rewrite comp_assoc -comp_assoc.
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' H3).
apply (comp_inc_compat_ab_a H3).
apply function_inc.
split.
apply H2.
apply H3.
split.
rewrite /total_r.
rewrite comp_inv comp_inv inv_invol.
rewrite -(@comp_assoc _ _ _ e) (@comp_assoc _ _ _ e) (@comp_assoc _ _ _ f)
-(@comp_assoc _ _ _ f).
apply (@inc_trans _ _ _ _ H).
apply comp_inc_compat_a_ab.
apply (@inc_trans _ _ _ _ H2).
apply (comp_inc_compat_a_ab H).
rewrite /univalent_r.
rewrite comp_inv comp_inv inv_invol.
rewrite (@comp_assoc _ _ _ e) -(@comp_assoc _ _ _ e) comp_assoc -(@comp_assoc
_ _ _ _ f).
apply (@inc_trans _ _ _ (f #  $\cdot (((f \cdot f \#) \cdot (f \cdot f \#)) \cdot f)))$ .
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat.
    
```



```

apply H4.
apply H4.
rewrite comp_assoc (@comp_assoc _ _ _ _ f) - (@comp_assoc _ _ _ _ (f #)) - (@comp_assoc
_ _ _ _ (f #)) (@comp_assoc _ _ _ _ (f #)) - (@comp_assoc _ _ _ _ (f #)).
apply (fun H' => @inc_trans _ _ _ _ H' H3).
apply comp_inc_compat_ab_a.
apply (fun H' => @inc_trans _ _ _ _ H' H3).
apply (comp_inc_compat_ab_a H3).
rewrite -comp_assoc.
apply (comp_inc_compat_b_ab H).
move => g.
elim.
elim => H5 H6 H7.
replace g with (e # • (e • g)).
apply f_equal.
apply H7.
rewrite -comp_assoc.
apply inc_antisym.
apply (comp_inc_compat_ab_b H0).
rewrite inv_invol in H1.
apply (comp_inc_compat_b_ab H1).
Qed.

```

Lemma 162 (surjection_unique2) *Let $e : A \twoheadrightarrow B$ be a surjection, $f : A \rightarrow C$ be a function and $e \cdot e^\# = f \cdot f^\#$, then function $e^\# f$ is an injection.*

Lemma *surjection_unique2* {A B C : eqType} {e : Rel A B} {f : Rel A C}:
 surjection_r e → function_r f → (e • e #) = (f • f #) → injection_r (e # • f).

Proof.

```

rewrite /surjection_r/injection_r/function_r/total_r/univalent_r.
elim.
elim => H H0 H1.
elim => H2 H3 H4.
repeat split.
rewrite comp_inv comp_assoc - (@comp_assoc _ _ _ _ f).
apply (@inc_trans _ _ _ _ H1).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H2.
rewrite comp_inv inv_invol comp_assoc - (@comp_assoc _ _ _ _ e).
rewrite H4.
rewrite comp_assoc -comp_assoc.
apply (fun H' => @inc_trans _ _ _ _ H' H3).

```

```

apply (comp_inc_compat_ab_a H3).
rewrite inv_invol comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ f).
rewrite -H4.
rewrite comp_assoc -comp_assoc.
apply (fun H' => @inc_trans _ _ _ _ H' H0).
apply comp_inc_compat_ab_a.
apply H0.
Qed.

```

Lemma 163 (injection_unique1) *Let $m : B \rightarrowtail A$ be an injection, $f : C \rightarrow A$ be a function and $f^\# \cdot f \sqsubseteq m^\# \cdot m$, then there exists a unique function $g : C \rightarrow B$ s.t. $f = gm$.*

Lemma *injection_unique1* {A B C : eqType} {m : Rel B A} {f : Rel C A}:
 injection_r m → function_r f → (f # · f) (m # · m) →
 (∃! g : Rel C B, function_r g ∧ f = g · m).

Proof.

```

rewrite /injection_r/function_r/total_r/univalent_r.
elim.
elim => H H0 H1.
elim => H2 H3 H4.
∃ (f · m #).
repeat split.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ m).
apply (@inc_trans _ _ _ (f · ((f # · f) · f #))).
rewrite comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ _ H2).
apply (comp_inc_compat_a_ab H2).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H4).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ f).
apply (fun H' => @inc_trans _ _ _ _ H' H1).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).
rewrite comp_assoc.
apply Logic.eq-sym.
apply function_inc.
split.
rewrite /total_r.
rewrite comp_inv comp_inv inv_invol.
apply (@inc_trans _ _ _ _ H2).
apply comp_inc_compat.
apply (@inc_trans _ _ _ (f · (f # · f))).
rewrite -comp_assoc.

```

```

apply (comp_inc_compat_b_ab H2).
apply (comp_inc_compat_ab_ab' H4).
apply (@inc_trans _ _ _ ((f # · f) · f #)).
rewrite comp_assoc.
apply (comp_inc_compat_a_ab H2).
apply (comp_inc_compat_ab_a'b H4).
rewrite /univalent_r.
rewrite comp_inv comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ f).
apply (fun H' => @inc_trans _ _ _ _ H' H0).
apply comp_inc_compat_ab_a.
apply (fun H' => @inc_trans _ _ _ _ H' H3).
apply (comp_inc_compat_ab_a H0).
split.
apply H2.
apply H3.
apply (comp_inc_compat_ab_a H0).
move => g.
elim.
elim => H5 H6 H7.
rewrite H7 comp_assoc.
apply inc_antisym.
rewrite inv_invol in H1.
apply (comp_inc_compat_ab_a H1).
apply (comp_inc_compat_a_ab H).
Qed.

```

Lemma 164 (injection_unique2) *Let $m : B \rightarrowtail A$ be an injection, $f : C \rightarrow A$ be a function and $f^\# \cdot f = m^\# \cdot m$, then function $f \cdot m^\#$ is a surjection.*

Lemma *injection_unique2* {A B C : eqType} {m : Rel B A} {f : Rel C A}:
 injection_r m → function_r f → (f # · f) = (m # · m) → surjection_r (f · m #).

Proof.

```

rewrite /surjection_r/injection_r/function_r/total_r/univalent_r.
elim.
elim => H H0 H1.
elim => H2 H3 H4.
repeat split.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ m).
apply (@inc_trans _ _ _ (f · ((f # · f) · f #))).
rewrite comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ _ H2).
apply (comp_inc_compat_a_ab H2).
apply comp_inc_compat_ab_ab'.

```

```

rewrite  $H_4$ .
apply inc_refl.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _  $f$ ).
apply (fun  $H' \Rightarrow$  @inc_trans _ _ _ _  $H' H1$ ).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b  $H3$ ).
rewrite inv_invol comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _  $f$ ).
apply (@inc_trans _ _ _ _  $H$ ).
apply comp_inc_compat_ab_ab'.
rewrite  $H_4$  comp_assoc.
apply (comp_inc_compat_a_ab  $H$ ).
Qed.

```

Lemma 165 (*bijection_inv*) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow A$, $\alpha \cdot \beta = id_A$ and $\beta \cdot \alpha = id_B$, then α and β are bijections and $\beta = \alpha^\#$.*

Lemma *bijection_inv* { $A\ B : eqType$ } { $\alpha : Rel\ A\ B$ } { $\beta : Rel\ B\ A$ }:
 $\alpha \cdot \beta = Id\ A \rightarrow \beta \cdot \alpha = Id\ B \rightarrow$ *bijection_r* $\alpha \wedge$ *bijection_r* $\beta \wedge$
 $\beta = \alpha^\#$.

Proof.

```

move  $\Rightarrow$   $H\ H0$ .
move : (@id_function  $A$ )  $\Rightarrow$   $H1$ .
move : (@id_function  $B$ )  $\Rightarrow$   $H2$ .
assert (bijection_r  $\alpha \wedge$  bijection_r  $\beta$ ).
assert (total_r  $\alpha \wedge$  total_r ( $\alpha^\#$ )  $\wedge$  total_r  $\beta \wedge$  total_r ( $\beta^\#$ )).
repeat split.
apply (@total_comp2 _ _ _ _  $\beta$ ).
rewrite  $H$ .
apply  $H1$ .
apply (@total_comp2 _ _ _ _ ( $\beta^\#$ )).
rewrite -comp_inv  $H0$  inv_id.
apply  $H2$ .
apply (@total_comp2 _ _ _ _  $\alpha$ ).
rewrite  $H0$ .
apply  $H2$ .
apply (@total_comp2 _ _ _ _ ( $\alpha^\#$ )).
rewrite -comp_inv  $H$  inv_id.
apply  $H1$ .
repeat split.
apply  $H3$ .
apply (@univalent_comp2 _ _ _ _  $\beta$ ).
rewrite  $H0$ .
apply  $H2$ .

```

```

apply H3.
apply H3.
apply (@univalent_comp2 _ _ _ (beta #)).
rewrite -comp_inv H inv_id.
apply H1.
rewrite inv_invol.
apply H3.
apply H3.
apply (@univalent_comp2 _ _ _ alpha).
rewrite H.
apply H1.
apply H3.
apply H3.
apply (@univalent_comp2 _ _ _ (alpha #)).
rewrite -comp_inv H0 inv_id.
apply H2.
rewrite inv_invol.
apply H3.
split.
apply H3.
split.
apply H3.
rewrite -(@comp_id_r _ _ beta) -(@comp_id_l _ _ (alpha #)).
rewrite -H0 comp_assoc.
apply f_equal.
apply inc_antisym.
apply H3.
rewrite comp_inv_inv -inv_inc_move inv_id.
apply H3.
Qed.

```

Lemma 166 (bijection_inv_corollary) *Let $\alpha : A \rightarrow B$ be a bijection, then $\alpha^\#$ is also a bijection.*

Lemma *bijection_inv_corollary* $\{A\ B : \text{eqType}\} \{\alpha : \text{Rel } A\ B\}$:
bijection_r alpha \rightarrow bijection_r (alpha #).

Proof.

```

move : (@bijection_inv _ _ alpha (alpha #))  $\Rightarrow$  H.
move  $\Rightarrow$  H0.
rewrite /bijection_r/function_r/total_r/univalent_r in H0.
rewrite inv_invol in H0.
apply H.

```

apply *inc_antisym*.

apply *H0*.

apply *H0*.

apply *inc_antisym*.

apply *H0*.

apply *H0*.

Qed.

Chapter 7

Library Dedekind

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
```

7.1 Dedekind formula に関する補題

Lemma 167 (dedekind1) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and $\gamma : A \rightarrow C$. Then*

$$\alpha \cdot \beta \sqcap \gamma \sqsubseteq \alpha \cdot (\beta \sqcap \alpha^\# \cdot \gamma).$$

Lemma dedekind1

```
{A B C : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel A C}:
((alpha · beta) gamma) (alpha · (beta (alpha # · gamma)))
```

Proof.

```
apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).
apply comp_inc_compat_ab_a'b.
apply cap_l.
```

Qed.

Lemma 168 (dedekind2) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and $\gamma : A \rightarrow C$. Then*

$$\alpha \cdot \beta \sqcap \gamma \sqsubseteq (\alpha \sqcap \gamma \cdot \beta^\#) \cdot \beta.$$

Lemma dedekind2

```
{A B C : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel A C}:
((alpha · beta) gamma) ((alpha (gamma · beta #)) · beta)
```

Proof.

```
apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).
```

CHAPTER 7. LIBRARY DEDEKIND

apply *comp_inc_compat_ab_ab'*.
 apply *cap_l*.
 Qed.

Lemma 169 (relation_rel_inv_rel) *Let $\alpha : A \rightarrow B$. Then*

$$\alpha \sqsubseteq \alpha \cdot \alpha^\# \cdot \alpha.$$

Lemma *relation_rel_inv_rel* {*A B : eqType*} {*alpha : Rel A B*}:
alpha ((*alpha* · *alpha* #) · *alpha*).

Proof.

move : (@dedekind1 _ _ _ *alpha* (*Id B*) *alpha*) \Rightarrow *H*.
 rewrite *comp_id_r cap_idem* in *H*.
 apply (@*inc_trans* _ _ _ _ *H*).
 rewrite *comp_assoc*.
 apply *comp_inc_compat_ab_ab'*.
 apply *cap_r*.
 Qed.

7.2 Dedekind formula と全関係

Lemma 170 (dedekind_universal1) *Let $\alpha : B \rightarrow C$. Then*

$$\nabla_{AC} \cdot \alpha^\# \cdot \alpha = \nabla_{AB} \cdot \alpha.$$

Lemma *dedekind_universal1* {*A B C : eqType*} {*alpha : Rel B C*}:
 (*A C* · *alpha* #) · *alpha* = *A B* · *alpha*.

Proof.

apply *inc_antisym*.
 apply *comp_inc_compat_ab_a'b*.
 apply *inc_alpha_universal*.
 apply (@*inc_trans* _ _ _ (*A B* · ((*alpha* · *alpha* #) · *alpha*))).
 apply *comp_inc_compat_ab_ab'*.
 apply *relation_rel_inv_rel*.
 rewrite -*comp_assoc* -*comp_assoc*.
 apply *comp_inc_compat_ab_a'b*.
 apply *comp_inc_compat_ab_a'b*.
 apply *inc_alpha_universal*.
 Qed.

Lemma 171 (`dedekind_universal2a`, `dedekind_universal2b`, `dedekind_universal2c`) *Let $\alpha : A \rightarrow B$ and $\beta : C \rightarrow B$. Then*

$$\nabla_{IC} \cdot \beta \sqsubseteq \nabla_{IA} \cdot \alpha \Leftrightarrow \nabla_{CC} \cdot \beta \sqsubseteq \nabla_{CA} \cdot \alpha \Leftrightarrow \beta \sqsubseteq \beta \cdot \alpha^\# \cdot \alpha.$$

Lemma `dedekind_universal2a` $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ C\ B\} :$
 $(\ i\ C \cdot beta) \ (\ i\ A \cdot alpha) \rightarrow (\ C\ C \cdot beta) \ (\ C\ A \cdot alpha).$

Proof.

`move \Rightarrow H.`
`rewrite -unit_universal -(@lemma_for_tarski2 C A).`
`rewrite comp_assoc comp_assoc.`
`apply (comp_inc_compat_ab' H).`
Qed.

Lemma `dedekind_universal2b` $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ C\ B\} :$
 $(\ C\ C \cdot beta) \ (\ C\ A \cdot alpha) \rightarrow beta \ ((beta \cdot alpha \#) \cdot alpha).$

Proof.

`move \Rightarrow H.`
`apply (@inc_trans _ _ _ (beta (\ C\ C \cdot beta))).`
`apply inc_cap.`
`split.`
`apply inc_refl.`
`apply comp_inc_compat_b_ab.`
`apply inc_alpha_universal.`
`apply (@inc_trans _ _ _ (beta (\ C\ A \cdot alpha))).`
`apply (cap_inc_compat_l H).`
`rewrite cap_comm.`
`apply (@inc_trans _ _ _ _ (dedekind2)).`
`apply comp_inc_compat_ab_a'b.`
`apply cap_r.`
Qed.

Lemma `dedekind_universal2c` $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ C\ B\} :$
 $beta \ ((beta \cdot alpha \#) \cdot alpha) \rightarrow (\ i\ C \cdot beta) \ (\ i\ A \cdot alpha).$

Proof.

`move \Rightarrow H.`
`apply (@inc_trans _ _ _ (\ i\ C \cdot ((beta \cdot alpha \#) \cdot alpha))).`
`apply (comp_inc_compat_ab_ab' H).`
`rewrite -comp_assoc.`
`apply comp_inc_compat_ab_a'b.`
`apply inc_alpha_universal.`
Qed.

Lemma 172 (dedekind_universal3a, dedekind_universal3b) *Let $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$. Then*

$$\beta \cdot \nabla_{CI} \sqsubseteq \alpha \cdot \nabla_{BI} \Leftrightarrow \beta \cdot \nabla_{CC} \sqsubseteq \alpha \cdot \nabla_{BC} \Leftrightarrow \beta \sqsubseteq \alpha \cdot \alpha^\# \cdot \beta.$$

Lemma dedekind_universal3a $\{A\ B\ C : \text{eqType}\} \{\alpha : \text{Rel } A\ B\} \{\beta : \text{Rel } A\ C\} :$
 $(\beta \cdot C\ i) \quad (\alpha \cdot B\ i) \Leftrightarrow (\beta \cdot C\ C) \quad (\alpha \cdot B\ C).$

Proof.

split; move $\Rightarrow H$.
 apply inv_inc_invol.
 rewrite comp_inv comp_inv inv_universal inv_universal.
 apply dedekind_universal2a.
 apply inv_inc_invol.
 rewrite comp_inv comp_inv inv_invol inv_invol inv_universal inv_universal.
 apply H.
 apply inv_inc_invol.
 rewrite comp_inv comp_inv inv_universal inv_universal.
 apply dedekind_universal2c.
 apply dedekind_universal2b.
 apply inv_inc_invol.
 rewrite comp_inv comp_inv inv_invol inv_invol inv_universal inv_universal.
 apply H.

Qed.

Lemma dedekind_universal3b $\{A\ B\ C : \text{eqType}\} \{\alpha : \text{Rel } A\ B\} \{\beta : \text{Rel } A\ C\} :$
 $(\beta \cdot C\ i) \quad (\alpha \cdot B\ i) \Leftrightarrow \beta \quad ((\alpha \cdot \alpha^\#) \cdot \beta).$

Proof.

split; move $\Rightarrow H$.
 apply inv_inc_invol.
 rewrite comp_inv comp_inv -comp_assoc.
 apply dedekind_universal2b.
 apply dedekind_universal2a.
 apply inv_inc_invol.
 rewrite comp_inv comp_inv inv_invol inv_invol inv_universal inv_universal.
 apply H.
 apply inv_inc_invol.
 rewrite comp_inv comp_inv inv_universal inv_universal.
 apply dedekind_universal2c.
 rewrite -comp_inv -comp_inv -comp_assoc.
 apply inc_inv.
 apply H.

Qed.

Lemma 173 (universal_total) *Let $\alpha : A \rightarrow B$. Then*

$$\alpha \cdot \nabla_{BI} = \nabla_{AI} \Leftrightarrow \text{“}\alpha \text{ is total”}.$$

Lemma *universal_total* { $A B : eqType$ } { $\alpha : Rel A B$ }:
 $\alpha \cdot \nabla_{BI} = \nabla_{AI} \Leftrightarrow total_r \alpha$.

Proof.

```
move : (@dedekind_universal3b _ _ _ alpha (Id A)) => H.
rewrite comp_id_l comp_id_r in H.
rewrite /total_r.
rewrite -H.
split; move => H0.
rewrite H0.
apply inc_refl.
apply inc_antisym.
apply inc_alpha_universal.
apply H0.
Qed.
```

7.3 Dedekind formula と恒等関係

Lemma 174 (dedekind_id1) *Let $\alpha : A \rightarrow A$. Then*

$$\alpha \sqsubseteq id_A \Rightarrow \alpha^\# = \alpha.$$

Lemma *dedekind_id1* { $A : eqType$ } { $\alpha : Rel A A$ }: $\alpha \sqsubseteq Id A \rightarrow \alpha^\# = \alpha$.

Proof.

```
move => H.
assert (alpha # alpha).
move : (@dedekind1 _ _ _ (alpha #) (Id A) (Id A)) => H0.
rewrite comp_id_r comp_id_r inv_invol in H0.
replace (alpha # Id A) with (alpha #) in H0.
replace (Id A alpha) with alpha in H0.
apply (@inc_trans _ _ _ (alpha # • alpha)).
apply H0.
apply comp_inc_compat_ab_b.
rewrite -inv_inc_move.
rewrite inv_id.
apply H.
rewrite cap_comm.
apply inc_def1.
```

```

apply H.
apply inc_def1.
rewrite -inv_inc_move.
rewrite inv_id.
apply H.
apply inc_antisym.
apply H0.
apply inv_inc_move.
apply H0.
Qed.

```

Lemma 175 (dedekind_id2) *Let $\alpha : A \rightarrow A$. Then*

$$\alpha \sqsubseteq id_A \Rightarrow \alpha \cdot \alpha = \alpha.$$

Lemma dedekind_id2 $\{A : eqType\} \{alpha : Rel A A\}$:
 $alpha \quad Id A \rightarrow alpha \cdot alpha = alpha.$

Proof.

```

move  $\Rightarrow$  H.
apply inc_antisym.
apply (comp_inc_compat_ab_a H).
move : (dedekind_id1 H)  $\Rightarrow$  H0.
apply (@inc_trans _ _ _ ((alpha  $\cdot$  Id A) Id A)).
rewrite comp_id_r.
apply inc_cap.
split.
apply inc_refl.
apply H.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite H0 comp_id_r.
apply cap_r.
Qed.

```

Lemma 176 (dedekind_id3) *Let $\alpha, \beta : A \rightarrow A$. Then*

$$\alpha \sqsubseteq id_A \wedge \beta \sqsubseteq id_A \Rightarrow \alpha \cdot \beta = \alpha \sqcap \beta.$$

Lemma dedekind_id3 $\{A : eqType\} \{alpha \ beta : Rel A A\}$:
 $alpha \quad Id A \rightarrow \beta \quad Id A \rightarrow alpha \cdot \beta = alpha \sqcap \beta.$

Proof.

```

move  $\Rightarrow$  H H0.
apply inc_antisym.

```

```

apply inc_cap.
split.
apply (comp_inc_compat_ab_a H0).
apply (comp_inc_compat_ab_b H).
replace (alpha beta) with ((alpha beta) • (alpha beta)).
apply comp_inc_compat.
apply cap_l.
apply cap_r.
apply dedekind_id2.
apply (fun H' => @inc_trans _ _ _ _ H' H).
apply cap_l.
Qed.

```

Lemma 177 (dedekind_id4) *Let $\alpha, \beta : A \rightarrow A$. Then*

$$\alpha \sqsubseteq id_A \wedge \beta \sqsubseteq id_A \Rightarrow (\alpha \triangleright \beta) \sqcap id_A = (\alpha \Rightarrow \beta) \sqcap id_A.$$

Lemma dedekind_id4 $\{A : eqType\} \{alpha\ beta : Rel\ A\ A\}$:
 $alpha\ Id\ A \rightarrow beta\ Id\ A \rightarrow (alpha\ beta)\ Id\ A = (alpha \gg beta)\ Id\ A.$

Proof.

```

move => H H0.
apply inc_lower.
move => gamma.
rewrite inc_cap inc_cap.
split; elim => H1 H2.
split.
rewrite inc_rpc cap_comm.
rewrite -(@dedekind_id3 _ _ _ H H2).
rewrite -(@dedekind_id1 _ _ H).
apply inc_residual.
apply H1.
apply H2.
split.
rewrite inc_residual (@dedekind_id1 _ _ H) (@dedekind_id3 _ _ _ H H2).
rewrite cap_comm -inc_rpc.
apply H1.
apply H2.
Qed.

```

Chapter 8

Library Rationality

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
```

8.1 有理性から導かれる系

Lemma 178 (rationality_corollary1) *Let $u : A \rightarrow A$ and $u \sqsubseteq id_A$. Then,*

$$\exists R, \exists j : R \rightarrowtail A, u = j^\# \cdot j.$$

Lemma rationality_corollary1 $\{A : eqType\} \{u : Rel\ A\ A\}$:
 $u \sqsubseteq Id\ A \rightarrow \exists (R : eqType)(j : Rel\ R\ A), injection_r\ j \wedge u = j^\# \cdot j.$

Proof.

```
move : (rationality _ _ u).
elim => R.
elim => f.
elim => g.
elim => H.
elim => H0.
elim => H1 H2 H3.
exists R.
exists f.
assert (g = f).
apply (function_inc H0 H).
apply (@inc_trans _ _ _ ((f · f #) · g)).
apply comp_inc_compat_b_ab.
apply H.
rewrite comp_assoc -H1.
```

CHAPTER 8. LIBRARY RATIONALITY

```

apply (comp_inc_compat_ab_a H3).
rewrite H4 in H1.
rewrite H4 cap_idem in H2.
split.
split.
apply H.
rewrite /univalent_r.
rewrite inv_invol H2.
apply inc_refl.
apply H1.
Qed.

```

Lemma 179 (rationality_corollary2) *Let $f : A \rightarrow B$ be a function. Then,*

$$\exists e : A \rightarrow R, \exists m : R \rightarrow B, f = e \cdot m.$$

Lemma *rationality_corollary2* $\{A\ B : \text{eqType}\} \{f : \text{Rel}\ A\ B\}$:
 $\text{function_r } f \rightarrow \exists (R : \text{eqType})(e : \text{Rel}\ A\ R)(m : \text{Rel}\ R\ B), \text{surjection_r } e \wedge \text{injection_r } m.$

Proof.

```

elim  $\Rightarrow$  H H0.
move : (@rationality_corollary1 - (f #  $\cdot$  f) H0).
elim  $\Rightarrow$  R.
elim  $\Rightarrow$  m.
elim  $\Rightarrow$  H1 H2.
 $\exists$  R.
 $\exists$  (f  $\cdot$  m #).
 $\exists$  m.
split.
apply (injection_unique2 H1 (conj H H0) H2).
apply H1.
Qed.

```

Chapter 9

Library Conjugate

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
```

9.1 共役性の定義

条件 P を満たす関係 $\alpha : A \rightarrow B$ と条件 Q を満たす関係 $\beta : A' \rightarrow B'$ が変換 $\alpha = \phi(\beta), \beta = \psi(\alpha)$ によって, 1 対 1 (全射的) に対応することを, 図式

$$\frac{\alpha : A \rightarrow B \{P\} \quad \alpha = \phi(\beta)}{\beta : A' \rightarrow B' \{Q\} \quad \beta = \psi(\alpha)}$$

によって表す. また, Coq では以下のように表すことにする.

Definition *conjugate*

```
(A B C D : eqType) (P : Rel A B → Prop) (Q : Rel C D → Prop)
(phi : Rel C D → Rel A B) (psi : Rel A B → Rel C D) :=
(∀ alpha : Rel A B, P alpha → Q (psi alpha) ∧ phi (psi alpha) = alpha)
∧ (∀ beta : Rel C D, Q beta → P (phi beta) ∧ psi (phi beta) = beta).
```

さらに, 上の図式において条件 P または Q が不要な場合には, 以下の `True_r` を代入する.

Definition *True_r* {A B : eqType} := fun _ : Rel A B ⇒ True.

9.2 共役の例

Lemma 180 (inv_conjugate) *Inverse relation ($\#$) makes conjugate. That is,*

$$\frac{\alpha : A \rightarrow B \quad \alpha = \beta^\#}{\beta : B \rightarrow A \quad \beta = \alpha^\#}.$$

Lemma *inv_conjugate* {A B : eqType}:

conjugate A B B A True_r True_r (@inverse - -) (@inverse - -).

Proof.

split.

move \Rightarrow *alpha H*.

split.

by [].

apply *inv_invol*.

move \Rightarrow *beta H*.

split.

by [].

apply *inv_invol*.

Qed.

Lemma 181 (injection_conjugate) *Let $j : C \hookrightarrow B$ be an injection. Then,*

$$\frac{f : A \rightarrow B \quad \{f^\# \cdot f \sqsubseteq j^\# \cdot j\}}{h : A \rightarrow C} \quad \frac{f = h \cdot j}{h = f \cdot j^\#}$$

Lemma *injection_conjugate* {A B C : eqType} {j : Rel C B}:

injection_r j \rightarrow

conjugate A B A C (fun f : Rel A B \Rightarrow ((f # \cdot f) (j # \cdot j)) \wedge function_r f)

(fun h : Rel A C \Rightarrow function_r h) (fun h : Rel A C \Rightarrow h \cdot j) (fun f : Rel A B \Rightarrow f \cdot j #).

Proof.

elim.

elim \Rightarrow *H H0 H1*.

split.

move \Rightarrow *alpha*.

elim \Rightarrow *H2*.

elim \Rightarrow *H3 H4*.

assert (function_r (alpha \cdot j #)).

split.

apply (@inc_trans - - - - H3).

rewrite *comp_inv inv_invol comp_assoc* -(@comp_assoc - - - - j).

CHAPTER 9. LIBRARY CONJUGATE

```

apply (@inc_trans _ _ _ (alpha • ((alpha # • alpha) • alpha #))).
rewrite comp_assoc -comp_assoc.
apply (comp_inc_compat_a_ab H3).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H2).
apply (fun H' ⇒ @inc_trans _ _ _ _ H' H1).
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ alpha).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
apply (@inc_trans _ _ _ _ H2).
apply H0.
split.
apply H5.
apply function_inc.
apply function_comp.
apply H5.
split.
apply H.
apply H0.
split.
apply H3.
apply H4.
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H0.
move ⇒ beta.
elim ⇒ H2 H3.
assert (function_r (beta • j)).
split.
apply (@inc_trans _ _ _ _ H2).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ j).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H).
apply (fun H' ⇒ @inc_trans _ _ _ _ H' H0).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).
split.
split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).

```

```

apply  $H_4$ .
rewrite  $\text{comp\_assoc}$ .
replace  $(j \cdot j \#)$  with  $(\text{Id } C)$ .
apply  $\text{comp\_id\_r}$ .
apply  $\text{inc\_antisym}$ .
apply  $H$ .
rewrite  $/\text{univalent\_r}$  in  $H_1$ .
rewrite  $\text{inv\_inv}$  in  $H_1$ .
apply  $H_1$ .
Qed.

```

Lemma 182 (`injection_conjugate_corollary1`, `injection_conjugate_corollary2`)

Let $j : C \rightarrow B$ be an injection and $f : A \rightarrow B$ be a function. Then,

$$f^\# \cdot f \sqsubseteq j^\# \cdot j \Leftrightarrow (\exists! h : A \rightarrow C, f = h \cdot j) \Leftrightarrow (\exists h' : A \rightarrow C, f \sqsubseteq h' \cdot j).$$

Lemma `injection_conjugate_corollary1` $\{A \ B \ C : \text{eqType}\} \{f : \text{Rel } A \ B\} \{j : \text{Rel } C \ B\}$:
 $\text{injection_r } j \rightarrow \text{function_r } f \rightarrow$
 $((f \# \cdot f) \sqsubseteq (j \# \cdot j) \Leftrightarrow \exists! h : \text{Rel } A \ C, \text{function_r } h \wedge f = h \cdot j).$

Proof.

```

move  $\Rightarrow H \ H_0$ .
move :  $(@injection\_conjugate \ A \ \_ \ \_ \ H)$ .
elim  $\Rightarrow H_1 \ H_2$ .
split; move  $\Rightarrow H_3$ .
 $\exists (f \cdot j \#)$ .
split.
move :  $(H_1 \ f \ (\text{conj } H_3 \ H_0))$ .
elim  $\Rightarrow H_4 \ H_5$ .
split.
apply  $H_4$ .
by [rewrite  $H_5$ ].
move  $\Rightarrow h$ .
elim  $\Rightarrow H_4 \ H_5$ .
rewrite  $H_5 \ \text{comp\_assoc}$ .
replace  $(j \cdot j \#)$  with  $(\text{Id } C)$ .
apply  $\text{comp\_id\_r}$ .
rewrite  $/\text{injection\_r}/\text{function\_r}/\text{univalent\_r}$  in  $H$ .
rewrite  $\text{inv\_inv}$  in  $H$ .
apply  $\text{inc\_antisym}$ .
apply  $H$ .
apply  $H$ .
elim  $H_3 \Rightarrow h$ .
elim.

```

CHAPTER 9. LIBRARY CONJUGATE

```

elim ⇒ H4 H5 H6.
rewrite H5 comp_inv comp_assoc -(@comp_assoc _ _ _ _ h).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
apply H4.
Qed.

Lemma injection_conjugate_corollary2 {A B C : eqType} {f : Rel A B} {j : Rel C B}:
  injection_r j → function_r f →
  ((f # • f) (• j # • j) ↔ ∃ h' : Rel A C, f (• h' • j)).

Proof.
move ⇒ H H0.
split; move ⇒ H1.
apply (injection_conjugate_corollary1 H H0) in H1.
elim H1 ⇒ h.
elim.
elim ⇒ H2 H3 H4.
∃ h.
rewrite H3.
apply inc_refl.
elim H1 ⇒ h' H2.
replace (f # • f) with (f # • (f (• h' • j))).
apply (@inc_trans _ _ ((f # • f) • (j # • j))).
rewrite comp_assoc cap_comm -(@comp_assoc _ _ _ _ f).
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
apply cap_r.
apply comp_inc_compat_ab_b.
apply H0.
apply f_equal.
apply inc_def1 in H2.
by [rewrite -H2].
Qed.

```

Lemma 183 (surjection_conjugate) *Let $e : A \twoheadrightarrow C$ be a surjection. Then,*

$$\frac{f : A \rightarrow B \quad \{e \cdot e^\# \sqsubseteq f \cdot f^\#\}}{h : C \rightarrow B} \quad \frac{f = e \cdot h}{h = e^\# \cdot f}$$

Lemma *surjection_conjugate* {*A B C* : *eqType*} {*e* : *Rel A C*}:
surjection_r e →
conjugate A B C B (**fun** *f* : *Rel A B* ⇒ ((*e* • *e* #) (• *f* • *f* #)) ∧ *function_r f*)

CHAPTER 9. LIBRARY CONJUGATE

(**fun** $h : \text{Rel } C \ B \Rightarrow \text{function_r } h$) (**fun** $h : \text{Rel } C \ B \Rightarrow e \cdot h$) (**fun** $f : \text{Rel } A \ B \Rightarrow e \# \cdot f$).

Proof.

elim.

elim $\Rightarrow H \ H0 \ H1$.

split.

move $\Rightarrow \text{alpha}$.

elim $\Rightarrow H2$.

elim $\Rightarrow H3 \ H4$.

assert ($\text{function_r } (e \# \cdot \text{alpha})$).

split.

apply ($\text{@inc_trans } _ _ _ _ H1$).

rewrite $\text{comp_inv inv_invol comp_assoc } -(\text{@comp_assoc } _ _ _ _ \text{alpha})$.

apply $\text{comp_inc_compat_ab_ab'}$.

apply ($\text{comp_inc_compat_b_ab } H3$).

apply (**fun** $H' \Rightarrow \text{@inc_trans } _ _ _ _ H' \ H4$).

rewrite $\text{comp_inv inv_invol comp_assoc } -(\text{@comp_assoc } _ _ _ _ e)$.

apply ($\text{@inc_trans } _ _ _ (\text{alpha} \# \cdot ((\text{alpha} \cdot \text{alpha} \#) \cdot \text{alpha}))$).

apply $\text{comp_inc_compat_ab_ab'}$.

apply ($\text{comp_inc_compat_ab_a'b } H2$).

rewrite $\text{comp_assoc } -\text{comp_assoc}$.

apply ($\text{comp_inc_compat_ab_a } H4$).

split.

apply $H5$.

apply Logic.eq_sym .

apply function_inc .

split.

apply $H3$.

apply $H4$.

apply function_comp .

split.

apply H .

apply $H0$.

apply $H5$.

rewrite $-\text{comp_assoc}$.

apply $\text{comp_inc_compat_b_ab}$.

apply H .

move $\Rightarrow \text{beta}$.

elim $\Rightarrow H2 \ H3$.

assert ($\text{function_r } (e \cdot \text{beta})$).

split.

apply ($\text{@inc_trans } _ _ _ _ H$).

CHAPTER 9. LIBRARY CONJUGATE

```

rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H2).
apply (fun H' => @inc_trans _ _ _ _ H' H3).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ e).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H0).
split.
split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H2).
apply H4.
rewrite -comp_assoc.
replace (e # · e) with (Id C).
apply comp_id_l.
apply inc_antisym.
rewrite /total_r in H1.
rewrite inv_invol in H1.
apply H1.
apply H0.
Qed.

```

Lemma 184 (surjection_conjugate_corollary) *Let $e : A \twoheadrightarrow C$ be a surjection and $f : A \rightarrow B$ be a function. Then,*

$$e \cdot e^\# \sqsubseteq f \cdot f^\# \Leftrightarrow (\exists! h : C \rightarrow B, f = e \cdot h).$$

Lemma *surjection_conjugate_corollary* $\{A\ B\ C : \text{eqType}\} \{f : \text{Rel } A\ B\} \{e : \text{Rel } A\ C\}$:
 $\text{surjection_r } e \rightarrow \text{function_r } f \rightarrow$
 $((e \cdot e^\#) \sqsubseteq (f \cdot f^\#)) \leftrightarrow \exists! h : \text{Rel } C\ B, \text{function_r } h \wedge f = e \cdot h).$

Proof.

```

move => H H0.
move : (@surjection_conjugate _ B _ _ H).
elim => H1 H2.
split; move => H3.
exists (e # · f).
split.
move : (H1 f (conj H3 H0)).
elim => H4 H5.
split.
apply H4.
by [rewrite H5].

```

```

move ⇒ h.
elim ⇒ H4 H5.
rewrite H5 -comp_assoc.
replace (e # · e) with (Id C).
apply comp_id_l.
rewrite /surjection_r/function_r/total_r in H.
rewrite inv_invol in H.
apply inc_antisym.
apply H.
apply H.
elim H3 ⇒ h.
elim.
elim ⇒ H4 H5 H6.
rewrite H5 comp_inv comp_assoc -(@comp_assoc _ _ _ h).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H4.
Qed.

```

Lemma 185 (subid_conjugate) *Subidentity $u \sqsubseteq id_A$ corresponds $\rho : I \rightarrow A$. That is,*

$$\frac{\rho : I \rightarrow A}{u : A \rightarrow A \{u \sqsubseteq id_A\}} \quad \frac{\rho = \nabla_{IA} \cdot u}{u = id_A \sqcap \nabla_{AI} \cdot \rho}.$$

Lemma subid_conjugate $\{A : eqType\}$:
 $conjugate\ i\ A\ A\ A\ True_r\ (\text{fun } u : Rel\ A\ A \Rightarrow u \quad Id\ A)$
 $(\text{fun } u : Rel\ A\ A \Rightarrow i\ A \cdot u) (\text{fun } rho : Rel\ i\ A \Rightarrow Id\ A \quad (A\ i \cdot rho)).$

Proof.
 split.
 move ⇒ alpha H.
 split.
 apply cap_l.
 apply inc_antisym.
 apply (@inc_trans _ _ _ (i A · (A i · alpha))).
 apply comp_inc_compat_ab_ab'.
 apply cap_r.
 rewrite -comp_assoc.
 apply comp_inc_compat_ab_b.
 rewrite unit_identity_is_universal.
 apply inc_alpha_universal.
 rewrite -(@inv_universal i A).
 apply (fun H' ⇒ @inc_trans _ _ _ _ H' (dedekind1)).
 rewrite comp_id_r cap_comm cap_universal.

```

apply inc_refl.
move ⇒ beta H.
split.
by [].
apply inc_antisym.
rewrite cap_comm -comp_assoc lemma_for_tarski2.
apply (@inc_trans _ _ _ _ (dedekind2)).
rewrite comp_id_l cap_comm cap_universal.
apply comp_inc_compat_ab_b.
rewrite -inv_inc_move inv_id.
apply H.
apply inc_cap.
split.
apply H.
rewrite -comp_assoc.
apply comp_inc_compat_b_ab.
rewrite lemma_for_tarski2.
apply inc_alpha_universal.
Qed.

```

Lemma 186 (subid_conjugate_corollary1) *Let $u, v : A \rightarrow A$ and $u, v \sqsubseteq id_A$. Then,*

$$\nabla_{IA} \cdot u = \nabla_{IA} \cdot v \Rightarrow u = v.$$

Lemma subid_conjugate_corollary1 $\{A : eqType\} \{u \ v : Rel \ A \ A\}$:
 $u \quad Id \ A \rightarrow v \quad Id \ A \rightarrow \quad i \ A \cdot u = \quad i \ A \cdot v \rightarrow u = v.$

Proof.

```

move ⇒ H H0 H1.
move : (@subid_conjugate A).
elim ⇒ H2 H3.
move : (H3 u H).
elim ⇒ H4 H5.
rewrite -H5.
move : (H3 v H0).
elim ⇒ H6 H7.
rewrite -H7.
apply f_equal.
apply f_equal.
apply H1.
Qed.

```


Lemma 187 (subid_conjugate_corollary2) *Let $\rho, \rho' : I \rightarrow A$. Then,*

$$id_A \sqcap \nabla_{AI} \cdot \rho = id_A \sqcap \nabla_{AI} \cdot \rho' \Rightarrow \rho = \rho'.$$

Lemma *subid_conjugate_corollary2* $\{A : eqType\} \{rho\ rho' : Rel\ i\ A\}$:
 $Id\ A \quad (\quad A\ i \cdot rho) = Id\ A \quad (\quad A\ i \cdot rho') \rightarrow rho = rho'.$

Proof.

move $\Rightarrow H$.

move : (*@subid_conjugate A*).

elim $\Rightarrow H0\ H1$.

move : (*H0 rho I*).

elim $\Rightarrow H2\ H3$.

rewrite -*H3*.

move : (*H0 rho' I*).

elim $\Rightarrow H4\ H5$.

rewrite -*H5*.

apply f_equal.

apply *H*.

Qed.

Chapter 10

Library Domain

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Logic.FunctionalExtensionality.
```

10.1 定義域の定義

関係 $\alpha : A \rightarrow B$ に対して, その定義域 (関係) $[\alpha] : A \rightarrow A$ は,

$$[\alpha] = \alpha \cdot \alpha^\# \sqcap id_A$$

で表される. また, Coq では以下のように表すことにする.

Definition *domain* $\{A\ B : eqType\}$ ($\alpha : Rel\ A\ B$) := ($\alpha \cdot \alpha^\#$) \sqcap $Id\ A$.

10.2 定義域の性質

10.2.1 基本的な性質

Lemma 188 (*domain_another_def*) *Let* $\alpha : A \rightarrow B$. *Then,*

$$[\alpha] = \alpha \cdot \nabla_{BA} \sqcap id_A.$$

Lemma *domain_another_def* $\{A\ B : eqType\}$ $\{\alpha : Rel\ A\ B\}$:
 $domain\ \alpha = (\alpha \cdot \nabla_{BA}) \sqcap Id\ A$.

Proof.

CHAPTER 10. LIBRARY DOMAIN

```

apply inc_antisym.
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply inc_cap.
split.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm comp_id_r cap_universal.
apply inc_refl.
apply cap_r.
Qed.

```

Lemma 189 (domain_inv) *Let $\alpha : A \rightarrow B$. Then,*

$$\lfloor \alpha \rfloor^\# = \lfloor \alpha \rfloor.$$

Lemma domain_inv $\{A B : eqType\} \{alpha : Rel A B\}$:
 $(domain\ alpha) \# = domain\ alpha$.

Proof.

```

apply dedekind_id1.
apply cap_r.
Qed.

```

Lemma 190 (domain_comp_alpha1, domain_comp_alpha2) *Let $\alpha : A \rightarrow B$. Then,*

$$\lfloor \alpha \rfloor \cdot \alpha = \alpha \wedge \alpha^\# \cdot \lfloor \alpha \rfloor = \alpha^\#.$$

Lemma domain_comp_alpha1 $\{A B : eqType\} \{alpha : Rel A B\}$:
 $(domain\ alpha) \cdot alpha = alpha$.

Proof.

```

apply inc_antisym.
apply comp_inc_compat_ab_b.
apply cap_r.
rewrite /domain.
rewrite cap_comm.
apply (fun H' => @inc_trans _ _ _ _ H' (dedekind2)).
rewrite comp_id_l cap_idem.
apply inc_refl.
Qed.

```

Lemma domain_comp_alpha2 $\{A B : eqType\} \{alpha : Rel A B\}$:
 $alpha \# \cdot (domain\ alpha) = alpha \#$.

CHAPTER 10. LIBRARY DOMAIN

Proof.

```
rewrite -domain_inv -comp_inv.
apply f_equal.
apply domain_comp_alpha1.
Qed.
```

Lemma 191 (domain_inc_compat) *Let $\alpha, \alpha' : A \rightarrow B$. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \lfloor \alpha \rfloor \sqsubseteq \lfloor \alpha' \rfloor.$$

Lemma *domain_inc_compat* {A B : eqType} {alpha alpha' : Rel A B}:
 alpha alpha' → domain alpha domain alpha'.

Proof.

```
move ⇒ H.
apply cap_inc_compat_r.
apply comp_inc_compat.
apply H.
apply (@inc_inv _ _ _ H).
Qed.
```

Lemma 192 (domain_total) *Let $\alpha : A \rightarrow B$. Then,*

$$“\alpha \text{ is total}” \Leftrightarrow \lfloor \alpha \rfloor = id_A.$$

Lemma *domain_total* {A B : eqType} {alpha : Rel A B}:
 total_r alpha ↔ domain alpha = Id A.

Proof.

```
split; move ⇒ H.
rewrite /domain.
rewrite cap_comm.
apply Logic.eq_sym.
apply inc_def1.
apply H.
apply inc_def1.
rewrite /domain in H.
by [rewrite cap_comm H].
Qed.
```

Lemma 193 (domain_inc_id) *Let $u : A \rightarrow A$. Then,*

$$u \sqsubseteq id_A \Leftrightarrow \lfloor u \rfloor = u.$$

Lemma *domain_inc_id* {A : eqType} {u : Rel A A}: u Id A ↔ domain u = u.

Proof.

```
split; move => H.
rewrite /domain.
rewrite (dedekind_id1 H) (dedekind_id2 H).
apply inc_def1 in H.
by [rewrite -H].
rewrite -H.
apply cap_r.
Qed.
```

10.2.2 合成と定義域

Lemma 194 (comp_domain1, comp_domain2) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then,*

$$[\alpha \cdot \beta] = [\alpha \cdot [\beta]] \sqsubseteq [\alpha].$$

Lemma comp_domain1 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$:
 $domain\ (alpha \cdot beta) = domain\ alpha.$

Proof.

```
rewrite /domain.
rewrite comp_inv.
apply (@inc_trans _ _ _ ((alpha · ((beta · (beta # · alpha #)) alpha #)) Id A)).
replace (((alpha · beta) · (beta # · alpha #)) Id A) with (((alpha · beta) ·
(beta # · alpha #)) Id A) Id A.
apply cap_inc_compat_r.
rewrite comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind1)).
rewrite comp_id_r.
apply inc_refl.
by [rewrite cap_assoc cap_idem].
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply cap_r.
Qed.
```

Lemma comp_domain2 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$:
 $domain\ (alpha \cdot beta) = domain\ (alpha \cdot domain\ beta).$

Proof.

```
apply inc_antisym.
replace (domain (alpha · beta)) with (domain ((alpha · domain beta) · beta)).
apply comp_domain1.
by [rewrite comp_assoc domain_comp_alpha1].
apply (@inc_trans _ _ _ (domain (alpha · (beta · beta #)))).
```

CHAPTER 10. LIBRARY DOMAIN

```

apply domain_inc_compat.
apply comp_inc_compat_ab_ab'.
apply cap_l.
rewrite -comp_assoc.
apply comp_domain1.
Qed.

```

Lemma 195 (comp_domain3) *Let $\alpha : A \rightarrow B$ be a relation and $\beta : B \rightarrow C$ be a total relation. Then,*

$$\lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \rfloor.$$

Lemma comp_domain3 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$:
 $total_r\ beta \rightarrow domain\ (alpha \cdot beta) = domain\ alpha.$

Proof.

```

move => H.
apply inc_antisym.
apply comp_domain1.
rewrite /domain.
rewrite comp_inv_comp_assoc -(@comp_assoc _ _ _ beta).
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H).
Qed.

```

Lemma 196 (comp_domain4) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then,*

$$\lfloor \alpha^\# \rfloor \sqsubseteq \lfloor \beta \rfloor \Rightarrow \lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \rfloor.$$

Lemma comp_domain4 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$:
 $domain\ (alpha \#) \quad domain\ beta \rightarrow domain\ (alpha \cdot beta) = domain\ alpha.$

Proof.

```

move => H.
apply inc_antisym.
apply comp_domain1.
rewrite /domain.
rewrite -(@domain_comp_alpha1 _ _ (alpha #)) comp_inv_comp_assoc -(@comp_assoc _ _
_ _ beta).
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ _ H).
apply cap_l.
Qed.

```

Lemma 197 (comp_domain5) *Let $\alpha : A \rightarrow B$ be a univalent relation and $\beta : B \rightarrow C$. Then,*

$$\lfloor \alpha^\# \rfloor \sqsubseteq \lfloor \beta \rfloor \Leftrightarrow \lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \rfloor.$$

Lemma comp_domain5 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$:
 $univalent_r\ alpha \rightarrow$
 $(domain\ (alpha\ \#) \quad domain\ beta \leftrightarrow domain\ (alpha \cdot beta) = domain\ alpha).$

Proof.

move $\Rightarrow H$.
split; move $\Rightarrow H0$.
apply (comp_domain4 H0).
rewrite /domain.
rewrite inv_invol.
apply cap_inc_compat_r.
replace (alpha # · alpha) with (alpha # · (domain (alpha · beta) · alpha)).
rewrite /domain.
rewrite comp_inv.
apply (@inc_trans _ _ _ (alpha # · (((alpha · beta) · (beta # · alpha #)) · alpha))).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite comp_assoc comp_assoc comp_assoc -comp_assoc -(@comp_assoc _ _ _ beta).
apply (@inc_trans _ _ _ _ (comp_inc_compat_ab_b H)).
apply (comp_inc_compat_ab_a H).
by [rewrite H0 domain_comp_alpha1].
Qed.

Lemma 198 (comp_domain6) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then,*

$$\alpha \cdot \lfloor \beta \rfloor \sqsubseteq \lfloor \alpha \cdot \beta \rfloor \cdot \alpha.$$

Lemma comp_domain6 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$:
 $(alpha \cdot domain\ beta) \quad (domain\ (alpha \cdot beta) \cdot alpha).$

Proof.

apply (@inc_trans _ _ _ _ (@comp_cap_distr_l _ _ _ _)).
rewrite cap_comm.
replace (alpha · Id B) with (Id A · alpha).
apply (@inc_trans _ _ _ _ (dedekind2)).
rewrite cap_comm -comp_assoc comp_assoc -comp_inv.
apply inc_refl.
by [rewrite comp_id_l comp_id_r].
Qed.

CHAPTER 10. LIBRARY DOMAIN

Lemma 199 (comp_domain7) *Let $\alpha : A \rightarrow B$ be a univalent relation and $\beta : B \rightarrow C$. Then,*

$$\alpha \cdot \lfloor \beta \rfloor = \lfloor \alpha \cdot \beta \rfloor \cdot \alpha.$$

Lemma comp_domain7 $\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel } A\ B\} \{beta : \text{Rel } B\ C\}$:
 $\text{univalent_r } alpha \rightarrow alpha \cdot \text{domain } beta = \text{domain } (alpha \cdot beta) \cdot alpha.$

Proof.

```
move  $\Rightarrow$  H.
apply inc_antisym.
apply comp_domain6.
apply (@inc_trans _ _ _ _ (@comp_cap_distr_r _ _ _ _ _)).
rewrite comp_id_l comp_inv comp_assoc comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
apply (fun H'  $\Rightarrow$  cap_inc_compat H' H).
rewrite comp_assoc -comp_assoc.
apply (comp_inc_compat_ab_a H).
```

Qed.

Lemma 200 (comp_domain8) *Let $u : A \rightarrow A$, $\alpha : A \rightarrow B$ and $u \sqsubseteq id_A$. Then,*

$$\lfloor u \cdot \alpha \rfloor = u \cdot \lfloor \alpha \rfloor.$$

Lemma comp_domain8 $\{A\ B : \text{eqType}\} \{u : \text{Rel } A\ A\} \{alpha : \text{Rel } A\ B\}$:
 $u \text{ Id } A \rightarrow \text{domain } (u \cdot alpha) = u \cdot \text{domain } alpha.$

Proof.

```
move  $\Rightarrow$  H.
apply inc_antisym.
rewrite -(@cap_idem _ _ (domain (u · alpha))).
rewrite (dedekind_id3 H).
apply cap_inc_compat.
apply (@inc_trans _ _ _ _ (comp_domain1)).
apply domain_inc_id in H.
rewrite H.
apply inc_refl.
apply domain_inc_compat.
apply (comp_inc_compat_ab_b H).
apply cap_r.
apply (@inc_trans _ _ _ _ (comp_domain6)).
apply (comp_inc_compat_ab_a H).
```

Qed.

10.2.3 その他の性質

Lemma 201 (cap_domain) *Let $\alpha, \alpha' : A \rightarrow B$. Then,*

$$\lfloor \alpha \sqcap \alpha' \rfloor = \alpha \cdot \alpha'^{\#} \sqcap \text{id}_A.$$

Lemma *cap_domain* { $A B : \text{eqType}$ } { $\alpha \alpha' : \text{Rel } A B$ }:
 $\text{domain } (\alpha \sqcap \alpha') = (\alpha \cdot \alpha' \#) \sqcap \text{Id } A.$

Proof.

```

apply inc_antisym.
apply cap_inc_compat_r.
apply comp_inc_compat.
apply cap_l.
apply inc_inv.
apply cap_r.
rewrite (@cap_idem _ _ (Id A)) -cap_assoc.
apply cap_inc_compat_r.
apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).
rewrite inv_invol comp_id_l comp_id_r -inv_cap_distr (@cap_comm _ _ alpha').
apply inc_refl.

```

Qed.

Lemma 202 (cupP_domain_distr, cup_domain_distr) *Let $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$ and $P : \text{predicate}$. Then,*

$$\lfloor \sqcup_{P(\alpha)} f(\alpha) \rfloor = \sqcup_{P(\alpha)} \lfloor f(\alpha) \rfloor.$$

Lemma *cupP_domain_distr* { $A B C D : \text{eqType}$ } { $f : \text{Rel } C D \rightarrow \text{Rel } A B$ } { $P : \text{Rel } C D \rightarrow \text{Prop}$ }:
 $\text{domain } (\sqcup \{P\} f) = \sqcup \{P\} (\text{fun } \alpha : \text{Rel } C D \Rightarrow \text{domain } (f \alpha)).$

Proof.

```

rewrite /domain.
rewrite inv_cupP_distr comp_cupP_distr_l cap_cupP_distr_r.
apply cupP_eq.
move => alpha H.
rewrite -cap_domain -cap_domain.
apply f_equal.
rewrite cap_idem.
apply inc_antisym.
apply cap_r.
apply inc_cap.
split.

```

CHAPTER 10. LIBRARY DOMAIN

move : *alpha H*.
 apply *inc_cupP*.
 apply *inc_refl*.
 apply *inc_refl*.
 Qed.

Lemma *cup_domain_distr* {*A B : eqType*} {*alpha alpha' : Rel A B*}:
 $\text{domain } (\text{alpha} \quad \text{alpha}') = \text{domain } \text{alpha} \quad \text{domain } \text{alpha}'.$

Proof.

rewrite *cup_to_cupP* (@*cup_to_cupP* _ _ _ _ _ *id*).
 apply *cupP_domain_distr*.
 Qed.

Lemma 203 (domain_universal1) *Let $\alpha : A \rightarrow B$. Then,*

$$\lfloor \alpha \rfloor \cdot \nabla_{AC} = \alpha \cdot \nabla_{BC}.$$

Lemma *domain_universal1* {*A B C : eqType*} {*alpha : Rel A B*}:
 $\text{domain } \text{alpha} \cdot \quad A \ C = \text{alpha} \cdot \quad B \ C.$

Proof.

apply *inc_antisym*.
 apply (@*inc_trans* _ _ _ ((*alpha* · *alpha* #) · *A C*)).
 apply *comp_inc_compat_ab_a'b*.
 apply *cap_l*.
 rewrite *comp_assoc*.
 apply *comp_inc_compat_ab_ab'*.
 apply *inc_alpha_universal*.
 apply (@*inc_trans* _ _ _ ((*domain alpha* · *alpha*) · *B C*)).
 rewrite *domain_comp_alpha1*.
 apply *inc_refl*.
 rewrite *comp_assoc*.
 apply *comp_inc_compat_ab_ab'*.
 apply *inc_alpha_universal*.
 Qed.

Lemma 204 (domain_universal2) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then,*

$$\alpha \cdot \lfloor \beta \rfloor = \alpha \sqcap \nabla_{AC} \cdot \beta^\#.$$

Lemma *domain_universal2* {*A B C : eqType*} {*alpha : Rel A B*} {*beta : Rel B C*}:
 $\text{alpha} \cdot \text{domain } \text{beta} = \text{alpha} \quad (\quad A \ C \cdot \text{beta} \#).$

Proof.

apply *inc_antisym*.
 apply *inc_cap*.

```

split.
apply comp_inc_compat_ab_a.
apply cap_r.
apply (@inc_trans _ _ _ _ (comp_cap_distr_l)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
rewrite -inv_universal -comp_inv -domain_universal1.
rewrite comp_inv inv_universal domain_inv cap_comm.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
rewrite cap_comm cap_universal domain_inv.
apply comp_inc_compat_ab_a.
apply cap_r.
Qed.

```

Lemma 205 (domain_lemma1) *Let $\alpha, \beta : A \rightarrow B$ and β is univalent. Then,*

$$\alpha \sqsubseteq \beta \wedge \lfloor \alpha \rfloor = \lfloor \beta \rfloor \Rightarrow \alpha = \beta.$$

Lemma domain_lemma1 $\{A B : eqType\} \{alpha\ beta : Rel\ A\ B\}$:
 $univalent_r\ beta \rightarrow alpha\ \beta \rightarrow domain\ alpha = domain\ beta \rightarrow alpha = beta.$

Proof.

```

move => H H0 H1.
apply inc_antisym.
apply H0.
rewrite -(@domain_comp_alpha1 _ _ beta) -H1.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply (fun H' => @inc_trans _ _ _ _ H' H).
apply comp_inc_compat_ab_a'b.
apply (@inc_inv _ _ _ _ H0).
Qed.

```

Lemma 206 (domain_lemma2a, domain_lemma2b) *Let $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$. Then,*

$$\lfloor \alpha \rfloor \sqsubseteq \lfloor \beta \rfloor \Leftrightarrow \alpha \cdot \nabla_{BB} \sqsubseteq \beta \cdot \nabla_{CB} \Leftrightarrow \alpha \sqsubseteq \beta \cdot \beta^\# \cdot \alpha.$$

Lemma domain_lemma2a $\{A B C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ A\ C\}$:
 $domain\ alpha\ \ domain\ beta \Leftrightarrow (alpha \cdot \beta\ B) \ (beta \cdot C\ B).$

Proof.

```

split; move  $\Rightarrow H$ .
rewrite -(@domain_comp_alpha1 _ _ alpha) comp_assoc.
apply (@inc_trans _ _ _ _ (comp_inc_compat_ab_a'b H)).
apply (@inc_trans _ _ _ _ (comp_inc_compat_ab_a'b (cap_l))).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ (domain ((beta  $\cdot$  beta #)  $\cdot$  alpha))).
apply domain_inc_compat.
apply (@inc_trans _ _ _ (alpha (beta  $\cdot$  C B))).
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (cap_inc_compat_l H)).
replace (alpha (alpha  $\cdot$  B B)) with ((alpha  $\cdot$  Id B) (alpha  $\cdot$  B B)).
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (comp_cap_distr_l)).
rewrite cap_universal comp_id_r.
apply inc_refl.
by [rewrite comp_id_r].
rewrite cap_comm comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind1)).
rewrite cap_comm cap_universal.
apply inc_refl.
rewrite comp_assoc.
apply comp_domain1.

```

Qed.

Lemma domain_lemma2b {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:

$$\text{domain } \alpha \quad \text{domain } \beta \leftrightarrow \alpha \quad ((\beta \cdot \beta \#) \cdot \alpha).$$

Proof.

```

split; move  $\Rightarrow H$ .
apply domain_lemma2a in H.
apply (@inc_trans _ _ _ (alpha (beta  $\cdot$  C B))).
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (cap_inc_compat_l H)).
replace (alpha (alpha  $\cdot$  B B)) with ((alpha  $\cdot$  Id B) (alpha  $\cdot$  B B)).
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (comp_cap_distr_l)).
rewrite cap_universal comp_id_r.
apply inc_refl.
by [rewrite comp_id_r].
rewrite cap_comm comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind1)).
rewrite cap_comm cap_universal.
apply inc_refl.
apply domain_inc_compat in H.
apply (@inc_trans _ _ _ _ H).

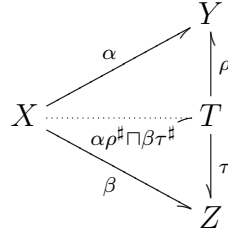
```

CHAPTER 10. LIBRARY DOMAIN

rewrite *comp_assoc*.
 apply *comp_domain1*.
 Qed.

Lemma 207 (domain_corollary1) *In below figure,*

“ α and β are total” $\wedge \alpha^\# \cdot \beta \sqsubseteq \rho^\# \cdot \tau \Rightarrow$ “ $\alpha \cdot \rho^\# \sqcap \beta \cdot \tau^\#$ is total”.



Lemma domain_corollary1 $\{X\ Y\ Z\ T : eqType\}$
 $\{alpha : Rel\ X\ Y\} \{beta : Rel\ X\ Z\} \{rho : Rel\ T\ Y\} \{tau : Rel\ T\ Z\} :$
 $total_r\ alpha \rightarrow total_r\ beta \rightarrow (alpha\ \# \cdot beta) \quad (rho\ \# \cdot tau) \rightarrow$
 $total_r\ ((alpha \cdot rho\ \#) \quad (beta \cdot tau\ \#)).$

Proof.

move $\Rightarrow H\ H0\ H1$.

move : $(comp_inc_compat\ H\ H0) \Rightarrow H2$.

rewrite *comp_id_l -comp_assoc* (@*comp_assoc* _ _ _ _ *alpha*) in *H2*.

rewrite /*total_r*.

replace (*Id X*) with $((alpha \cdot (rho\ \# \cdot tau)) \cdot beta\ \#) \quad Id\ X$.

rewrite *-comp_assoc comp_assoc*.

apply (@*inc_trans* _ _ _ _ (@*dedekind* _ _ _ _ _)).

rewrite *comp_id_l comp_id_r comp_inv comp_inv inv_invol inv_invol*.

rewrite *inv_cap_distr comp_inv comp_inv inv_invol inv_invol* (@*cap_comm* _ _ (*tau* · *beta* #)).

apply *inc_refl*.

apply *Logic.eq_sym*.

rewrite *cap_comm*.

apply *inc_def1*.

apply (@*inc_trans* _ _ _ _ *H2*).

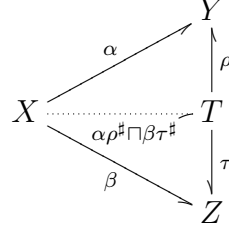
apply *comp_inc_compat_ab_a'b*.

apply (*comp_inc_compat_ab_ab' H1*).

Qed.

Lemma 208 (domain_corollary2) *In below figure,*

“ α and β are univalent” $\wedge \rho \cdot \rho^\sharp \sqcap \tau \cdot \tau^\sharp = \text{id}_T \Rightarrow$ “ $\alpha \cdot \rho^\sharp \sqcap \beta \cdot \tau^\sharp$ is univalent”.



Lemma domain_corollary2 $\{X\ Y\ Z\ T : \text{eqType}\}$
 $\{\text{alpha} : \text{Rel } X\ Y\} \{\text{beta} : \text{Rel } X\ Z\} \{\text{rho} : \text{Rel } T\ Y\} \{\text{tau} : \text{Rel } T\ Z\}$:
 $\text{univalent_r } \text{alpha} \rightarrow \text{univalent_r } \text{beta} \rightarrow (\text{rho} \cdot \text{rho}^\#) \quad (\text{tau} \cdot \text{tau}^\#) = \text{Id } T \rightarrow$
 $\text{univalent_r } ((\text{alpha} \cdot \text{rho}^\#) \quad (\text{beta} \cdot \text{tau}^\#)).$

Proof.

`move \Rightarrow H H0 H1.`

`rewrite /univalent_r.`

`rewrite -H1 inv_cap_distr.`

`apply (@inc_trans _ _ _ _ (comp_cap_distr_l)).`

`apply cap_inc_compat.`

`apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).`

`apply (@inc_trans _ _ _ _ (cap_l)).`

`rewrite comp_inv inv_invol -comp_assoc (@comp_assoc _ _ _ _ rho).`

`apply comp_inc_compat_ab_a'b.`

`apply (comp_inc_compat_ab_a H).`

`apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).`

`apply (@inc_trans _ _ _ _ (cap_r)).`

`rewrite comp_inv inv_invol -comp_assoc (@comp_assoc _ _ _ _ tau).`

`apply comp_inc_compat_ab_a'b.`

`apply (comp_inc_compat_ab_a H0).`

Qed.

10.2.4 矩形関係

$\alpha : A \rightarrow B$ が

$$\alpha \cdot \nabla_{BA} \cdot \alpha \sqsubseteq \alpha$$

を満たすとき, α は 矩形関係 (rectangular relation) であると言われる.

Definition *rectangular* $\{A\ B : \text{eqType}\} (\text{alpha} : \text{Rel } A\ B) :=$
 $((\text{alpha} \cdot \quad B\ A) \cdot \text{alpha}) \quad \text{alpha}.$

Lemma 209 (rectangular_inv) *Let $\alpha : A \rightarrow B$ is a rectangular relation, then $\alpha^\#$ is also a rectangular relation.*

Lemma *rectangular_inv* {A B : eqType} {alpha : Rel A B} :
rectangular alpha \rightarrow *rectangular (alpha #)*.

Proof.

move \Rightarrow *H*.

apply *inv_inc_move*.

rewrite *comp_inv comp_inv inv_invol inv_universal -comp_assoc*.

apply *H*.

Qed.

Lemma 210 (rectangular_capP, rectangular_cap) *Let $f(\alpha)$ is always a rectangular relation and P : predicate, then $\sqcap_{P(\beta)} f(\beta)$ is also a rectangular relation.*

Lemma *rectangular_capP* {A B C D : eqType} {f : Rel C D \rightarrow Rel A B} {P : Rel C D \rightarrow Prop} :

(\forall alpha : Rel C D, *P alpha* \rightarrow *rectangular (f alpha)*) \rightarrow *rectangular (_{P} f)*.

Proof.

move \Rightarrow *H*.

rewrite */rectangular*.

apply (@*inc_trans* _ _ _ (_{P} (fun alpha : Rel C D \Rightarrow (f alpha \cdot B A) \cdot f alpha)))).

apply (@*inc_trans* _ _ _ _ (comp_capP_distr_l)).

apply *inc_capP*.

move \Rightarrow **beta** *H0*.

apply (@*inc_trans* _ _ _ (((_{P} f) \cdot B A) \cdot f **beta**)).

move : **beta** *H0*.

apply *inc_capP*.

apply *inc_refl*.

apply *comp_inc_compat_ab_a'b*.

apply *comp_inc_compat_ab_a'b*.

move : *H0*.

apply *inc_capP*.

apply *inc_refl*.

apply *inc_capP*.

move \Rightarrow **beta** *H0*.

apply (fun *H'* \Rightarrow @*inc_trans* _ _ _ _ *H'* (*H beta H0*)).

move : **beta** *H0*.

apply *inc_capP*.

apply *inc_refl*.

Qed.

Lemma *rectangular_cap* {A B : eqType} {alpha beta : Rel A B} :

CHAPTER 10. LIBRARY DOMAIN

rectangular alpha \rightarrow *rectangular beta* \rightarrow *rectangular (alpha beta)*.

Proof.

move \Rightarrow *H H0*.

rewrite (@cap_to_capP _ _ _ _ _ id).

apply *rectangular_capP*.

move \Rightarrow *gamma*.

case \Rightarrow *H1*; rewrite *H1*.

apply *H*.

apply *H0*.

Qed.

Lemma 211 (rectangular_comp) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and α or β is a rectangular relation, then $\alpha \cdot \beta$ is also a rectangular relation.*

Lemma *rectangular_comp* {*A B C : eqType*} {*alpha : Rel A B*} {*beta : Rel B C*}:

rectangular alpha \vee *rectangular beta* \rightarrow *rectangular (alpha • beta)*.

Proof.

rewrite /*rectangular*.

case; move \Rightarrow *H*.

rewrite -*comp_assoc*.

apply *comp_inc_compat_ab_a'b*.

apply (fun *H'* \Rightarrow @*inc_trans* _ _ _ _ *H' H*).

apply *comp_inc_compat_ab_a'b*.

rewrite *comp_assoc*.

apply *comp_inc_compat_ab_ab'*.

apply *inc_alpha_universal*.

rewrite *comp_assoc comp_assoc*.

apply *comp_inc_compat_ab_ab'*.

apply (fun *H'* \Rightarrow @*inc_trans* _ _ _ _ *H' H*).

rewrite -*comp_assoc -comp_assoc*.

apply *comp_inc_compat_ab_a'b*.

rewrite *comp_assoc*.

apply *comp_inc_compat_ab_ab'*.

apply *inc_alpha_universal*.

Qed.

Lemma 212 (rectangular_unit) *Let $\alpha : A \rightarrow B$. Then,*

$$“\alpha \text{ is rectangular}” \Leftrightarrow \exists \mu : I \rightarrow A, \exists \rho : I \rightarrow B, \alpha = \rho^\# \cdot \mu.$$

Lemma *rectangular_unit* {*A B : eqType*} {*alpha : Rel A B*}:

rectangular alpha $\leftrightarrow \exists (\mu : Rel i A)(\rho : Rel i B), \alpha = \mu \# \cdot \rho$.

Proof.


```

split; move  $\Rightarrow H$ .
 $\exists (i B \cdot \alpha \#)$ .
 $\exists (i A \cdot \alpha)$ .
rewrite comp_inv inv_invol inv_universal.
rewrite -comp_assoc (@comp_assoc - - -  $\alpha$ ) lemma_for_tarski2.
apply inc_antisym.
apply (@inc_trans - - - (relation_rel_inv_rel)).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply  $H$ .
elim  $H \Rightarrow mu$ .
elim  $\Rightarrow \rho H0$ .
rewrite  $H0$ .
rewrite /rectangular.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_a.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
Qed.

```

Chapter 11

Library **Residual**

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Domain.
Require Import Logic.FunctionalExtensionality.
```

11.1 剰余合成関係の性質

11.1.1 基本的な性質

Lemma 213 (double_residual) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and $\gamma : C \rightarrow D$. Then*

$$\alpha \triangleright (\beta \triangleright \gamma) = (\alpha \cdot \beta) \triangleright \gamma.$$

Lemma *double_residual*

```
{A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel C D}:
alpha (beta gamma) = (alpha • beta) gamma.
```

Proof.

apply *inc_lower*.

move \Rightarrow **delta**.

split; move \Rightarrow *H*.

apply *inc_residual*.

rewrite *comp_inv comp_assoc*.

rewrite *-inc_residual -inc_residual*.

apply *H*.

rewrite *inc_residual inc_residual*.

rewrite *-comp_assoc -comp_inv*.

apply *inc_residual*.
 apply *H*.
 Qed.

Lemma 214 (residual_to_complement) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then*

$$\alpha \triangleright \beta = (\alpha \cdot \beta^-)^-.$$

Lemma *residual_to_complement* {*A B C : eqType*} {*alpha : Rel A B*} {*beta : Rel B C*}:
alpha beta = (*alpha* • *beta* ^) ^.

Proof.

apply *inc_lower*.
 move \Rightarrow *gamma*.
 split; move \Rightarrow *H*.
 rewrite *bool_lemma2 complement_invol cap_comm*.
 apply *inc_antisym*.
 apply (@*inc_trans* _ _ _ _ (*dedekind1*)).
 replace (*beta* ^ (*alpha* # • *gamma*)) with (*B C*).
 rewrite *comp_empty_r*.
 apply *inc_refl*.
 apply *Logic.eq_sym*.
 rewrite *cap_comm*.
 apply *bool_lemma2*.
 apply *inc_residual*.
 apply *H*.
 apply *inc_empty_alpha*.
 apply *inc_residual*.
 apply *bool_lemma2*.
 apply *inc_antisym*.
 apply (@*inc_trans* _ _ _ _ (*dedekind1*)).
 rewrite *inv_invol*.
 replace (*gamma* (*alpha* • *beta* ^)) with (*A C*).
 rewrite *comp_empty_r*.
 apply *inc_refl*.
 apply *Logic.eq_sym*.
 rewrite -(@*complement_invol* _ _ (*alpha* • *beta* ^)).
 apply *bool_lemma2*.
 apply *H*.
 apply *inc_empty_alpha*.
 Qed.

CHAPTER 11. LIBRARY RESIDUAL

Lemma 215 (inv_residual_inc) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then*

$$\alpha^\# \cdot (\alpha \triangleright \beta) \sqsubseteq \beta.$$

Lemma *inv_residual_inc* { $A\ B\ C : eqType$ } { $\alpha : Rel\ A\ B$ } { $\beta : Rel\ B\ C$ }:
 $\alpha \# \cdot (\alpha \quad \beta) \quad \beta.$

Proof.

apply *inc_residual*.

apply *inc_refl*.

Qed.

Lemma 216 (inc_residual_inv) *Let $\alpha : A \rightarrow B$ and $\gamma : A \rightarrow C$. Then*

$$\gamma \sqsubseteq \alpha \triangleright \alpha^\# \cdot \gamma.$$

Lemma *inc_residual_inv* { $A\ B\ C : eqType$ } { $\alpha : Rel\ A\ B$ } { $\gamma : Rel\ A\ C$ }:
 $\gamma \quad (\alpha \quad (\alpha \# \cdot \gamma)).$

Proof.

apply *inc_residual*.

apply *inc_refl*.

Qed.

Lemma 217 (id_inc_residual) *Let $\alpha : A \rightarrow B$. Then*

$$id_A \sqsubseteq \alpha \triangleright \alpha^\#.$$

Lemma *id_inc_residual* { $A\ B : eqType$ } { $\alpha : Rel\ A\ B$ }: $Id\ A \quad (\alpha \quad \alpha \#).$

Proof.

apply *inc_residual*.

rewrite *comp_id_r*.

apply *inc_refl*.

Qed.

Lemma 218 (residual_universal) *Let $\alpha : A \rightarrow B$. Then*

$$\alpha \triangleright \nabla_{BC} = \nabla_{AC}.$$

Lemma *residual_universal* { $A\ B\ C : eqType$ } { $\alpha : Rel\ A\ B$ }: $\alpha \quad B\ C = \quad A\ C.$

Proof.

apply *inc_antisym*.

apply *inc_alpha_universal*.

apply *inc_residual*.

apply *inc_alpha_universal*.

Qed.

11.1.2 単調性と分配法則

Lemma 219 (residual_inc_compat) *Let $\alpha, \alpha' : A \rightarrow B$ and $\beta, \beta' : B \rightarrow C$. Then*

$$\alpha' \sqsubseteq \alpha \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \triangleright \beta \sqsubseteq \alpha' \triangleright \beta'.$$

Lemma residual_inc_compat

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{beta\ beta' : Rel\ B\ C\}:$
 $alpha' \quad alpha \rightarrow beta \quad beta' \rightarrow (alpha \quad beta) \quad (alpha' \quad beta').$

Proof.

`move \Rightarrow H H0.`

`apply inc_residual.`

`apply (fun H' \Rightarrow @inc_trans _ _ _ _ H' H0).`

`move : (@inc_refl _ _ (alpha beta)) \Rightarrow H1.`

`apply inc_residual in H1.`

`apply (fun H' \Rightarrow @inc_trans _ _ _ _ H' H1).`

`apply comp_inc_compat_ab_a'b.`

`apply inc_inv.`

`apply H.`

Qed.

Lemma 220 (residual_inc_compat_l) *Let $\alpha : A \rightarrow B$ and $\beta, \beta' : B \rightarrow C$. Then*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \triangleright \beta \sqsubseteq \alpha \triangleright \beta'.$$

Lemma residual_inc_compat_l

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta\ beta' : Rel\ B\ C\}:$
 $beta \quad beta' \rightarrow (alpha \quad beta) \quad (alpha \quad beta').$

Proof.

`move \Rightarrow H.`

`apply (@residual_inc_compat _ _ _ _ _ (@inc_refl _ _ _)) H).`

Qed.

Lemma 221 (residual_inc_compat_r) *Let $\alpha, \alpha' : A \rightarrow B$ and $\beta : B \rightarrow C$. Then*

$$\alpha' \sqsubseteq \alpha \Rightarrow \alpha \triangleright \beta \sqsubseteq \alpha' \triangleright \beta.$$

Lemma residual_inc_compat_r

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{beta : Rel\ B\ C\}:$
 $alpha' \quad alpha \rightarrow (alpha \quad beta) \quad (alpha' \quad beta).$

Proof.

CHAPTER 11. LIBRARY RESIDUAL

move $\Rightarrow H$.
 apply (@residual_inc_compat _ _ _ _ _ H (@inc_refl _ _)).
 Qed.

Lemma 222 (residual_capP_distr_l, residual_cap_distr_l) *Let $\alpha : A \rightarrow B$, $f : (D \rightarrow E) \rightarrow (B \rightarrow C)$ and $P : \text{predicate}$. Then*

$$\alpha \triangleright (\sqcap_{P(\beta)} f(\beta)) = \sqcap_{P(\beta)} (\alpha \triangleright f(\beta)).$$

Lemma residual_capP_distr_l {A B C D E : eqType}
 {alpha : Rel A B} {f : Rel D E \rightarrow Rel B C} {P : Rel D E \rightarrow Prop}:
 alpha (_ {P} f) = _ {P} (fun beta : Rel D E \Rightarrow alpha f beta).

Proof.

apply inc_lower.
 move \Rightarrow gamma.
 split; move \Rightarrow H.
 apply inc_capP.
 move \Rightarrow beta H0.
 apply inc_residual.
 move : beta H0.
 apply inc_capP.
 apply inc_residual.
 apply H.
 apply inc_residual.
 apply inc_capP.
 move \Rightarrow beta H0.
 apply inc_residual.
 move : beta H0.
 apply inc_capP.
 apply H.
 Qed.

Lemma residual_cap_distr_l
 {A B C : eqType} {alpha : Rel A B} {beta gamma : Rel B C}:
 alpha (beta gamma) = (alpha beta) (alpha gamma).

Proof.

rewrite cap_to_capP (@cap_to_capP _ _ _ _ id).
 apply residual_capP_distr_l.
 Qed.

Lemma 223 (`residual_cupP_distr_r`, `residual_cup_distr_r`) *Let $f : (D \rightarrow E) \rightarrow (A \rightarrow B)$, $\beta : B \rightarrow C$ and $P : \text{predicate}$. Then*

$$(\sqcup_{P(\alpha)} f(\alpha)) \triangleright \beta = \sqcap_{P(\alpha)} (f(\alpha) \triangleright \beta).$$

Lemma `residual_cupP_distr_r` {*A B C D E : eqType*}
 {*beta : Rel B C*} {*f : Rel D E → Rel A B*} {*P : Rel D E → Prop*}:
 (*_* {*P*} *f*) *beta* = *_* {*P*} (*fun alpha : Rel D E ⇒ f alpha beta*).

Proof.

apply *inc_lower*.
 move ⇒ *gamma*.
 split; move ⇒ *H*.
 apply *inc_capP*.
 move ⇒ *alpha H0*.
 apply *inc_residual*.
 move : *alpha H0*.
 apply *inc_cupP*.
 rewrite *-comp_cupP_distr_r -inv_cupP_distr*.
 apply *inc_residual*.
 apply *H*.
 apply *inc_residual*.
 rewrite *inv_cupP_distr comp_cupP_distr_r*.
 apply *inc_cupP*.
 move ⇒ *alpha H0*.
 apply *inc_residual*.
 move : *alpha H0*.
 apply *inc_capP*.
 apply *H*.

Qed.

Lemma `residual_cup_distr_r`
 {*A B C : eqType*} {*alpha beta : Rel A B*} {*gamma : Rel B C*}:
 (*alpha beta*) *gamma* = (*alpha gamma*) (*beta gamma*).

Proof.

rewrite (*@cup_to_cupP _ _ _ _ id*) (*@cap_to_capP _ _ _ _ (fun x ⇒ x gamma)*).
 apply *residual_cupP_distr_r*.

Qed.

11.1.3 剰余合成と関数

Lemma 224 (total_residual) *Let $\alpha : A \rightarrow B$ be a total relation and $\beta : B \rightarrow C$. Then*

$$\alpha \triangleright \beta \sqsubseteq \alpha \cdot \beta.$$

Lemma *total_residual* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
total_r alpha → (alpha beta) (alpha · beta).

Proof.

move ⇒ H.

apply (@inc_trans _ _ _ ((alpha · alpha #) · (alpha beta))).

apply (comp_inc_compat_b_ab H).

rewrite comp_assoc.

apply comp_inc_compat_ab_ab'.

apply inv_residual_inc.

Qed.

Lemma 225 (univalent_residual) *Let $\alpha : A \rightarrow B$ be a univalent relation and $\beta : B \rightarrow C$. Then*

$$\alpha \cdot \beta \sqsubseteq \alpha \triangleright \beta.$$

Lemma *univalent_residual* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
univalent_r alpha → (alpha · beta) (alpha beta).

Proof.

move ⇒ H.

apply (@inc_trans _ _ _ _ (@inc_residual_inv _ _ alpha _)).

apply residual_inc_compat_l.

rewrite -comp_assoc.

apply (comp_inc_compat_ab_b H).

Qed.

Lemma 226 (function_residual1) *Let $\alpha : A \rightarrow B$ be a function and $\beta : B \rightarrow C$. Then*

$$\alpha \triangleright \beta = \alpha \cdot \beta.$$

Lemma *function_residual1* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
function_r alpha → alpha beta = alpha · beta.

Proof.

elim ⇒ H H0.

apply inc_antisym.

apply (total_residual H).

apply (univalent_residual H0).

Qed.

CHAPTER 11. LIBRARY RESIDUAL

Lemma 227 (residual_id) *Let $\alpha : A \rightarrow B$. Then*

$$id_A \triangleright \alpha = \alpha.$$

Lemma *residual_id* $\{A\ B : eqType\} \{alpha : Rel\ A\ B\}$:

Id A alpha = alpha.

Proof.

`move : (@function_residual1 _ _ _ (Id A) alpha (@id_function A)) \Rightarrow H.`

`rewrite comp_id_l in H.`

`apply H.`

Qed.

Lemma 228 (universal_residual) *Let $\alpha : A \rightarrow B$. Then*

$$\nabla_{AA} \triangleright \alpha \sqsubseteq \alpha.$$

Lemma *universal_residual* $\{A\ B : eqType\} \{alpha : Rel\ A\ B\}$:

A A alpha alpha.

Proof.

`apply (@inc_trans _ _ _ (Id A alpha)).`

`apply residual_inc_compat_r.`

`apply inc_alpha_universal.`

`rewrite residual_id.`

`apply inc_refl.`

Qed.

Lemma 229 (function_residual2) *Let $\alpha : A \rightarrow B$ be a function, $\beta : B \rightarrow C$ and $\gamma : C \rightarrow D$. Then*

$$\alpha \cdot (\beta \triangleright \gamma) = (\alpha \cdot \beta) \triangleright \gamma.$$

Lemma *function_residual2*

$\{A\ B\ C\ D : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\} \{gamma : Rel\ C\ D\}$:

function_r alpha \rightarrow alpha \cdot (beta gamma) = (alpha \cdot beta) gamma.

Proof.

`move \Rightarrow H.`

`rewrite -(@function_residual1 _ _ _ _ H).`

`apply double_residual.`

Qed.

CHAPTER 11. LIBRARY RESIDUAL

Lemma 230 (function_residual3) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ be relations and $\gamma : D \rightarrow C$ be a function. Then*

$$(\alpha \triangleright \beta) \cdot \gamma^\# = \alpha \triangleright (\beta \cdot \gamma^\#).$$

Lemma function_residual3

$\{A\ B\ C\ D : \text{eqType}\} \{alpha : \text{Rel}\ A\ B\} \{beta : \text{Rel}\ B\ C\} \{gamma : \text{Rel}\ D\ C\} :$
 $\text{function_r}\ gamma \rightarrow (alpha \ \ \ \ \ \ beta) \cdot gamma \ \# = alpha \ \ \ \ (\beta \cdot gamma \ \#).$

Proof.

move $\Rightarrow H$.
 apply inc_lower.
 move $\Rightarrow \text{delta}$.
 split; move $\Rightarrow H0$.
 apply inc_residual.
 rewrite -(@function_move2 _ _ _ _ _ H).
 rewrite comp_assoc.
 apply inc_residual.
 rewrite (@function_move2 _ _ _ _ _ H).
 apply H0.
 rewrite -(@function_move2 _ _ _ _ _ H).
 apply inc_residual.
 rewrite -comp_assoc.
 rewrite (@function_move2 _ _ _ _ _ H).
 apply inc_residual.
 apply H0.
Qed.

Lemma 231 (function_residual4) *Let $\alpha : A \rightarrow B$, $\gamma : C \rightarrow D$ be relations and $\beta : B \rightarrow C$ be a function. Then*

$$\alpha \cdot \beta \triangleright \gamma = \alpha \triangleright \beta \cdot \gamma.$$

Lemma function_residual4

$\{A\ B\ C\ D : \text{eqType}\} \{alpha : \text{Rel}\ A\ B\} \{beta : \text{Rel}\ B\ C\} \{gamma : \text{Rel}\ C\ D\} :$
 $\text{function_r}\ beta \rightarrow (alpha \cdot \ \ \ \ \ \ beta) \ \ \ \ gamma = alpha \ \ \ \ (\beta \cdot \ \ \ \ \ \ gamma).$

Proof.

move $\Rightarrow H$.
 rewrite -double_residual.
 by [rewrite (function_residual1 H)].
Qed.

11.2 Galois 同値とその系

Lemma 232 (galois) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and $\gamma : A \rightarrow C$. Then*

$$\gamma \sqsubseteq \alpha \triangleright \beta \Leftrightarrow \alpha \sqsubseteq \gamma \triangleright \beta^\#.$$

Lemma galois $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\} \{gamma : Rel\ A\ C\} :$
 $gamma \quad (alpha \quad beta) \leftrightarrow alpha \quad (gamma \quad beta \#).$

Proof.

split; move $\Rightarrow H$.
 apply inc_residual.
 apply inv_inc_move.
 rewrite comp_inv inv_invol.
 apply inc_residual.
 apply H.
 apply inc_residual.
 apply inv_inc_invol.
 rewrite comp_inv inv_invol.
 apply inc_residual.
 apply H.

Qed.

Lemma 233 (galois_corollary1) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then*

$$\alpha \sqsubseteq (\alpha \triangleright \beta) \triangleright \beta^\#.$$

Lemma galois_corollary1 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\} :$
 $alpha \quad ((alpha \quad beta) \quad beta \#).$

Proof.

rewrite -galois.
 apply inc_refl.

Qed.

Lemma 234 (galois_corollary2) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then*

$$((\alpha \triangleright \beta) \triangleright \beta^\#) \triangleright \beta = \alpha \triangleright \beta.$$

Lemma galois_corollary2 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\} :$
 $((alpha \quad beta) \quad beta \#) \quad beta = alpha \quad beta.$

Proof.

apply inc_antisym.
 apply residual_inc_compat_r.

CHAPTER 11. LIBRARY RESIDUAL

```

apply galois_corollary1.
move : (@galois_corollary1 _ _ _ (alpha beta) (beta #)) => H.
rewrite inv_invol in H.
apply H.
Qed.

```

Lemma 235 (galois_corollary3) *Let $\alpha : A \rightarrow B$ and $\beta : B \rightarrow C$. Then*

$$\alpha = (\alpha \triangleright \beta) \triangleright \beta^\# \Leftrightarrow \exists \gamma : A \rightarrow C, \alpha = \gamma \triangleright \beta^\#.$$

Lemma galois_corollary3 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$:
 $alpha = (alpha\ \beta)\ \beta^\# \Leftrightarrow (\exists\ gamma : Rel\ A\ C, alpha = gamma\ \beta^\#)$.
Proof.

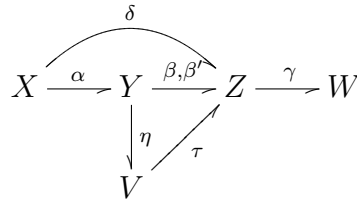
```

split; move => H.
exists (alpha beta).
apply H.
elim H => gamma H0.
rewrite H0.
move : (@galois_corollary2 _ _ _ gamma (beta #)) => H1.
rewrite inv_invol in H1.
by [rewrite H1].
Qed.

```

11.3 その他の性質

この節では、特記が無い限り、記号は以下の図式に従って割り振られるものとする。



Lemma 236 (residual_property1)

$$(\alpha \triangleright \beta) \cdot \gamma \sqsubseteq \alpha \triangleright \beta \cdot \gamma.$$

Lemma residual_property1
 $\{W\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{gamma : Rel\ Z\ W\}$:
 $((alpha\ \beta) \cdot gamma)\ (\alpha\ (\beta \cdot gamma))$.
Proof.

CHAPTER 11. LIBRARY RESIDUAL

```

apply (@inc_trans _ _ _ (alpha (alpha # • ((alpha beta) • gamma)))).
apply inc_residual_inv.
apply residual_inc_compat_l.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply inv_residual_inc.
Qed.

```

Lemma 237 (residual_property2)

$$(\alpha \triangleright \beta) \cdot (\beta^\# \triangleright \eta) \sqsubseteq \alpha \triangleright \eta.$$

Lemma residual_property2

```

{V X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {eta : Rel Y V}:
((alpha beta) • (beta # eta)) (alpha eta).

```

Proof.

```

apply (@inc_trans _ _ _ _ (residual_property1)).
apply residual_inc_compat_l.
move : (@inv_residual_inc _ _ _ (beta # eta)).
by [rewrite inv_invol].
Qed.

```

Lemma 238 (residual_property3)

$$\alpha \triangleright \beta \sqsubseteq \alpha \cdot \eta \triangleright \eta^\# \cdot \beta.$$

Lemma residual_property3

```

{V X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {eta : Rel Y V}:
(alpha beta) ((alpha • eta) (eta # • beta)).

```

Proof.

```

apply (@inc_trans _ _ _ _ (@inc_residual_inv _ _ _ (alpha • eta) (alpha beta)))).
apply residual_inc_compat_l.
rewrite comp_inv comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inv_residual_inc.
Qed.

```

Lemma 239 (residual_property4a, residual_property4b)

$$(\alpha \triangleright \beta) \cdot \gamma \sqsubseteq (\alpha \triangleright \beta \cdot \gamma) \sqcap \nabla_{XZ} \cdot \gamma \sqsubseteq (\alpha \triangleright \beta \cdot \gamma) \cdot \gamma^\# \cdot \gamma.$$

Lemma residual_property4a

```

{W X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {gamma : Rel Z W}:

```

CHAPTER 11. LIBRARY RESIDUAL

$((\alpha \quad \beta) \cdot \gamma) \quad ((\alpha \quad (\beta \cdot \gamma)) \quad (X \ Z \cdot \gamma)).$

Proof.

```
rewrite -(@cap_universal _ _ (alpha beta)).
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat_r.
apply residual_property1.
Qed.
```

Lemma residual_property4b

$\{W \ X \ Y \ Z : eqType\} \{alpha : Rel \ X \ Y\} \{beta : Rel \ Y \ Z\} \{gamma : Rel \ Z \ W\}:$
 $((\alpha \quad (\beta \cdot \gamma)) \quad (X \ Z \cdot \gamma)) \quad ((\alpha \quad (\beta \cdot \gamma)) \cdot$
 $(\gamma \# \cdot \gamma)).$

Proof.

```
rewrite cap_comm.
apply (@inc_trans _ _ _ _ (dedekind2)).
rewrite cap_comm cap_universal comp_assoc.
apply inc_refl.
Qed.
```

Lemma 240 (residual_property5) *Let τ be a univalent relation. Then,*

$$(\alpha \triangleright \beta) \cdot \tau^\# = (\alpha \triangleright \beta \cdot \tau^\#) \sqcap \nabla_{XZ} \cdot \tau^\#.$$

Lemma residual_property5

$\{V \ X \ Y \ Z : eqType\} \{alpha : Rel \ X \ Y\} \{beta : Rel \ Y \ Z\} \{tau : Rel \ V \ Z\}:$
 $univalent_r \ tau \rightarrow$
 $(\alpha \quad \beta) \cdot \tau \# = (\alpha \quad (\beta \cdot \tau \#)) \quad (X \ Z \cdot \tau \#).$

Proof.

```
move => H.
apply inc_antisym.
rewrite -(@cap_universal _ _ (alpha beta)).
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat_r.
apply residual_property1.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ (dedekind2)).
rewrite cap_comm cap_universal inv_invol.
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ _ (residual_property1)).
apply residual_inc_compat_l.
rewrite comp_assoc.
apply (comp_inc_compat_ab_a H).
Qed.
```

Lemma 241 (residual_property6)

$$\alpha \triangleright (\gamma^\# \triangleright \beta^\#)^\# = (\gamma^\# \triangleright (\alpha \triangleright \beta)^\#)^\#.$$

Lemma residual_property6

$\{W\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{gamma : Rel\ Z\ W\} :$
 $alpha\ (gamma\ \# \ \beta\ \#) \ \# = (gamma\ \# \ (alpha\ \beta) \ \#) \ \#.$

Proof.

apply *inc_lower*.
 move \Rightarrow *delta*.
 split; move \Rightarrow *H*.
 apply *inv_inc_move*.
 apply *inc_residual*.
 apply *inv_inc_move*.
 apply *inc_residual*.
 rewrite *comp_inv comp_assoc*.
 apply *inv_inc_move*.
 apply *inc_residual*.
 apply *inv_inc_invol*.
 rewrite *comp_inv inv_invol*.
 apply *inc_residual*.
 apply *H*.
 apply *inc_residual*.
 apply *inv_inc_move*.
 apply *inc_residual*.
 apply *inv_inc_move*.
 rewrite *comp_inv inv_invol inv_invol comp_assoc*.
 apply *inc_residual*.
 apply *inv_inc_invol*.
 rewrite *comp_inv*.
 apply *inc_residual*.
 apply *inv_inc_move*.
 apply *H*.

Qed.

Lemma 242 (residual_property7a, residual_property7b)

$$\alpha \triangleright (\beta \Rightarrow \beta') \sqsubseteq (\alpha \cdot \beta \Rightarrow \alpha \cdot \beta') \sqsubseteq \alpha \triangleright (\beta \Rightarrow \alpha^\# \cdot \alpha \cdot \beta').$$

Lemma residual_property7a $\{X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta\ beta' : Rel\ Y\ Z\} :$
 $(alpha\ (\beta \gg \beta')) \sqsubseteq ((alpha \cdot \beta) \gg (alpha \cdot \beta')).$

Proof.

CHAPTER 11. LIBRARY RESIDUAL

```

apply inc_rpc.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm.
apply inc_rpc.
apply inv_residual_inc.
Qed.

```

Lemma residual_property7b $\{X\ Y\ Z : \text{eqType}\} \{alpha : \text{Rel } X\ Y\} \{beta\ beta' : \text{Rel } Y\ Z\}$:
 $((alpha \cdot beta) \gg (alpha \cdot beta')) \quad (alpha \quad (beta \gg (alpha \# \cdot (alpha \cdot beta')))).$

Proof.

```

rewrite inc_residual inc_rpc.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite inv_invol -inc_rpc.
apply inc_refl.
Qed.

```

Lemma 243 (residual_property8) *Let α be a univalent relation. Then,*

$$\alpha \triangleright (\beta \Rightarrow \beta') = (\alpha \cdot \beta \Rightarrow \alpha \cdot \beta').$$

Lemma residual_property8 $\{X\ Y\ Z : \text{eqType}\} \{alpha : \text{Rel } X\ Y\} \{beta\ beta' : \text{Rel } Y\ Z\}$:
 $\text{univalent_r } alpha \rightarrow alpha \quad (beta \gg beta') = (alpha \cdot beta) \gg (alpha \cdot beta').$

Proof.

```

move => H.
apply inc_antisym.
apply residual_property7a.
apply (@inc_trans _ _ _ _ residual_property7b).
apply residual_inc_compat_l.
apply rpc_inc_compat_l.
rewrite -comp_assoc.
apply (comp_inc_compat_ab_b H).
Qed.

```

Lemma 244 (residual_property9) *Let α be a univalent relation. Then,*

$$\alpha \triangleright \beta = (\alpha \cdot \nabla_{YZ} \Rightarrow \alpha \cdot \beta).$$

Lemma residual_property9 $\{X\ Y\ Z : \text{eqType}\} \{alpha : \text{Rel } X\ Y\} \{beta : \text{Rel } Y\ Z\}$:
 $\text{univalent_r } alpha \rightarrow alpha \quad beta = (alpha \cdot \nabla_{YZ} \Rightarrow (alpha \cdot beta)).$

Proof.

```

move => H.

```


CHAPTER 11. LIBRARY RESIDUAL

by [rewrite -(residual_property8 H) rpc_universal_alpha].

Qed.

Lemma 245 (residual_property10) *Let α be a univalent relation. Then,*

$$\alpha \cdot \beta = \lfloor \alpha \rfloor \cdot (\alpha \triangleright \beta).$$

Lemma residual_property10 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:
 univalent_r alpha → alpha · beta = domain alpha · (alpha beta).

Proof.

move ⇒ H.

apply inc_antisym.

replace (alpha · beta) with (domain alpha · (alpha · beta)).

apply comp_inc_compat_ab_ab'.

rewrite inc_residual -comp_assoc.

apply (comp_inc_compat_ab_b H).

by [rewrite -comp_assoc domain_comp_alpha1].

apply (@inc_trans _ _ _ ((alpha · alpha #) · (alpha beta))).

apply comp_inc_compat_ab_a'b.

apply cap_l.

rewrite comp_assoc.

apply comp_inc_compat_ab_ab'.

apply inv_residual_inc.

Qed.

Lemma 246 (residual_property11)

$$(\alpha \cdot \beta \Rightarrow \delta) \sqsubseteq \alpha \triangleright (\beta \Rightarrow \alpha^\# \cdot \delta).$$

Lemma residual_property11

{X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {delta : Rel X Z}:
 ((alpha · beta) » delta) (alpha (beta » (alpha # · delta))).

Proof.

apply inc_residual.

apply inc_rpc.

apply (@inc_trans _ _ _ _ (dedekind1)).

rewrite inv_invol.

apply comp_inc_compat_ab_ab'.

apply inc_rpc.

apply inc_refl.

Qed.

Lemma 247 (residual_property12a, residual_property12b) *Let $u \sqsubseteq id_X$. Then,*

$$u \triangleright \alpha = u \cdot \nabla_{XY} \Rightarrow \alpha = u \triangleright u \cdot \alpha.$$

Lemma residual_property12a $\{X\ Y : eqType\} \{u : Rel\ X\ X\} \{alpha : Rel\ X\ Y\}$:
 $u \quad Id\ X \rightarrow u \quad alpha = (u \cdot \quad X\ Y) \gg alpha.$

Proof.

```
move  $\Rightarrow$   $H$ .
apply inc_antisym.
assert (univalent_r u).
apply (fun  $H' \Rightarrow @inc\_trans \_ \_ \_ \_ H' H$ ).
apply comp_inc_compat_ab_b.
rewrite -inv_id.
apply (@inc_inv \_ \_ \_  $H$ ).
rewrite (residual_property9  $H0$ ).
apply rpc_inc_compat_l.
apply (comp_inc_compat_ab_b  $H$ ).
apply (@inc_trans \_ \_ \_ \_ residual_property11).
apply residual_inc_compat_l.
rewrite rpc_universal_alpha.
apply comp_inc_compat_ab_b.
rewrite -inv_id.
apply (@inc_inv \_ \_ \_  $H$ ).
```

Qed.

Lemma residual_property12b $\{X\ Y : eqType\} \{u : Rel\ X\ X\} \{alpha : Rel\ X\ Y\}$:
 $u \quad Id\ X \rightarrow u \quad alpha = u \quad (u \cdot alpha).$

Proof.

```
move  $\Rightarrow$   $H$ .
apply inc_antisym.
rewrite (residual_property12a  $H$ ).
apply (@inc_trans \_ \_ \_ \_ residual_property11).
apply residual_inc_compat_l.
rewrite rpc_universal_alpha.
apply comp_inc_compat_ab_a'b.
rewrite (dedekind_id1  $H$ ).
apply inc_refl.
apply residual_inc_compat_l.
apply (comp_inc_compat_ab_b  $H$ ).
```

Qed.

Lemma 248 (residual_property13)

$$(\alpha \cdot \nabla_{YZ} \sqcap \delta) \triangleright \gamma = (\alpha \cdot \nabla_{YW} \Rightarrow (\delta \triangleright \gamma)).$$

Lemma residual_property13

$\{W\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{gamma : Rel\ Z\ W\} \{delta : Rel\ X\ Z\} :$
 $((alpha \cdot Y\ Z)\ delta)\ gamma = (alpha \cdot Y\ W) \gg (delta\ gamma).$

*Proof.*apply *inc_antisym*.rewrite *inc_rpc inc_residual*.remember (((*alpha* · *Y Z*) *delta*) *gamma*) as *sigma1*.apply (@*inc_trans* _ _ _ (((*alpha* · *Y Z*) *delta*) # · *sigma1*)).apply (@*inc_trans* _ _ _ (((*alpha* · *Y Z*) *delta*) # · (*sigma1* (*alpha* · *Y W*))))).assert ((*delta* # · (*sigma1* (*alpha* · *Y W*))) (*delta* # · *sigma1*)).apply *comp_inc_compat_ab_ab'*.apply *cap_l*.apply *inc_def1* in *H*.rewrite *H*.apply (@*inc_trans* _ _ _ _ (*dedekind2*)).apply *comp_inc_compat_ab_a'b*.rewrite (@*inv_cap_distr* _ _ _ *delta*) *cap_comm*.apply *cap_inc_compat_r*.rewrite *inv_cap_distr*.apply (@*inc_trans* _ _ _ _ (*comp_cap_distr_l*)).apply (@*inc_trans* _ _ _ _ (*cap_r*)).rewrite *comp_inv comp_inv-comp_assoc* (@*inv_universal* *Y Z*).apply *comp_inc_compat_ab_a'b*.apply *inc_alpha_universal*.apply *comp_inc_compat_ab_ab'*.apply *cap_l*.rewrite *Hsigma1*.apply *inc_residual*.apply *inc_refl*.rewrite *inc_residual*.remember ((*alpha* · *Y W*) » (*delta* *gamma*)) as *sigma2*.apply (@*inc_trans* _ _ _ (*delta* # · ((*alpha* · *Y W*) *sigma2*))).apply (@*inc_trans* _ _ _ (((*alpha* · *Y Z*) *delta*) # · ((*alpha* · *Y W*) *sigma2*))).assert ((((*alpha* · *Y Z*) *delta*) # · *sigma2*) (*delta* # · *sigma2*)).apply *comp_inc_compat_ab_a'b*.apply *inc_inv*.apply *cap_r*.

CHAPTER 11. LIBRARY RESIDUAL

```

apply inc_def1 in H.
rewrite H.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm_inv_invol.
apply cap_inc_compat_r.
apply (@inc_trans _ _ _ ((alpha · Y Z) · (delta # · sigma2))).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply comp_inc_compat_ab_a'b.
apply inc_inv.
apply cap_r.
rewrite Hegsigma2.
rewrite -inc_residual_cap_comm_inc_rpc.
apply inc_refl.
Qed.

```

Lemma 249 (residual_property14) *Let $\nabla_{XX} \cdot \alpha \sqsubseteq \alpha$. Then,*

$$\nabla_{XX} \cdot (\alpha \triangleright \beta) \sqsubseteq \alpha \triangleright \beta.$$

Lemma residual_property14 $\{X \ Y \ Z : eqType\} \{alpha : Rel \ X \ Y\} \{beta : Rel \ Y \ Z\} :$
 $(\nabla_{XX} \cdot alpha) \cdot alpha \rightarrow (\nabla_{XX} \cdot (alpha \cdot beta)) \cdot (alpha \cdot beta).$

Proof.

```

move  $\Rightarrow$  H.
apply (@inc_trans _ _ _ (  $\nabla_{XX} \cdot ( \nabla_{XX} \cdot (alpha \cdot beta))$ )).
apply comp_inc_compat_ab_ab'.
rewrite double_residual.
apply (residual_inc_compat_r H).
rewrite -inv_universal_inc_residual_inv_universal.
apply inc_refl.
Qed.

```

Lemma 250 (residual_property15) *Let $\beta \cdot \nabla_{ZZ} \sqsubseteq \beta$. Then,*

$$(\alpha \triangleright \beta) \cdot \nabla_{ZZ} \sqsubseteq \alpha \triangleright \beta.$$

Lemma residual_property15 $\{X \ Y \ Z : eqType\} \{alpha : Rel \ X \ Y\} \{beta : Rel \ Y \ Z\} :$
 $(beta \cdot \nabla_{ZZ} \cdot beta) \rightarrow ((alpha \cdot beta) \cdot \nabla_{ZZ} \cdot (alpha \cdot beta)).$

Proof.

move $\Rightarrow H$.
 apply (@inc_trans _ _ _ _ (residual_property1)).
 apply (residual_inc_compat_l H).
 Qed.

Lemma 251 (residual_property16)

$$id_X \sqsubseteq \alpha \triangleright \alpha^\# \wedge (\alpha \triangleright \alpha^\#) \cdot (\alpha \triangleright \alpha^\#) \sqsubseteq \alpha \triangleright \alpha^\#.$$

Lemma residual_property16 $\{X\ Y : eqType\} \{alpha : Rel\ X\ Y\}$:
 $Id\ X \quad (alpha \quad alpha^\#) \wedge$
 $((alpha \quad alpha^\#) \cdot (alpha \quad alpha^\#)) \quad (alpha \quad alpha^\#).$

Proof.
 split.
 rewrite inc_residual_comp_id_r.
 apply inc_refl.
 move : (@residual_property2 _ _ _ alpha (alpha #) (alpha #)) $\Rightarrow H$.
 rewrite inv_invol in H.
 apply H.
 Qed.

Lemma 252 (residual_property17) *Let $P(y) := “y : I \rightarrow Y$ is a function”. Then,*

$$\sqcup_{P(y)} y^\# \cdot y = id_Y \Rightarrow \alpha \triangleright \beta = \sqcap_{P(y)} (\alpha \cdot y^\# \cdot \nabla_{IZ} \Rightarrow \alpha \cdot y^\# \cdot y \cdot \beta).$$

Lemma residual_property17 $\{X\ Y\ Z : eqType\}$
 $\{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{P : Rel\ i\ Y \rightarrow Prop\}$:
 $P = (\text{fun } y : Rel\ i\ Y \Rightarrow function_r\ y) \rightarrow$
 $_ \{P\} (\text{fun } y : Rel\ i\ Y \Rightarrow y^\# \cdot y) = Id\ Y \rightarrow$
 $alpha \quad beta = _ \{P\} (\text{fun } y : Rel\ i\ Y \Rightarrow$
 $((alpha \cdot y^\#) \cdot _ i\ Z) \gg ((alpha \cdot y^\#) \cdot (y \cdot beta))).$

Proof.
 move $\Rightarrow H\ H0$.
 replace (alpha beta) with ((alpha · Id Y) beta).
 rewrite -H0_comp_cupP_distr_l residual_cupP_distr_r.
 apply capP_eq.
 move $\Rightarrow y\ H1$.
 rewrite H in H1.
 rewrite -comp_assoc (function_residual4 H1).
 apply residual_property9.
 rewrite /univalent_r.
 rewrite unit_identity_is_universal.
 apply inc_alpha_universal.

by [rewrite *comp_id_r*].

Qed.

11.4 順序の関係と左剰余合成

11.4.1 max, sup, min, inf

$\xi : X \rightarrow X$ を集合 X における順序と見なしたときの, 関係 $\rho : V \rightarrow X$ の 最大値 (max), 上限 (sup), 最小値 (min), 下限 (inf) はそれぞれ, 以下のように定義される.

- $\max(\rho, \xi) := \rho \sqcap (\rho \triangleright \xi)$
- $\sup(\rho, \xi) := (\rho \triangleright \xi) \sqcap ((\rho \triangleright \xi) \triangleright \xi^\#)$
- $\min(\rho, \xi) := \rho \sqcap (\rho \triangleright \xi^\#) (= \max(\rho, \xi^\#))$
- $\inf(\rho, \xi) := (\rho \triangleright \xi^\#) \sqcap ((\rho \triangleright \xi^\#) \triangleright \xi) (= \sup(\rho, \xi^\#))$

Definition $\max \{ V X : eqType \} (rho : Rel V X) (xi : Rel X X)$
 $:= rho \sqcap (rho \triangleright xi).$

Definition $\sup \{ V X : eqType \} (rho : Rel V X) (xi : Rel X X)$
 $:= (rho \triangleright xi) \sqcap ((rho \triangleright xi) \triangleright xi^\#).$

Definition $\min \{ V X : eqType \} (rho : Rel V X) (xi : Rel X X)$
 $:= rho \sqcap (rho \triangleright xi^\#).$

Definition $\inf \{ V X : eqType \} (rho : Rel V X) (xi : Rel X X)$
 $:= (rho \triangleright xi^\#) \sqcap ((rho \triangleright xi^\#) \triangleright xi).$

Lemma 253 (max-inc-sup) *Let $\rho : V \rightarrow X$ and $\xi : X \rightarrow X$. Then,*

$$\max(\rho, \xi) \sqsubseteq \sup(\rho, \xi).$$

Lemma $\max_inc_sup \{ V X : eqType \} \{ rho : Rel V X \} \{ xi : Rel X X \} :$
 $\max rho xi \sqsubseteq sup rho xi.$

Proof.

rewrite /max/sup.

rewrite cap-comm.

apply cap-inc-compatible_l.

apply galois_corollary1.

Qed.

CHAPTER 11. LIBRARY RESIDUAL

Lemma 254 (min_inc_inf) *Let $\rho : V \rightarrow X$ and $\xi : X \rightarrow X$. Then,*

$$\min(\rho, \xi) \sqsubseteq \inf(\rho, \xi).$$

Lemma *min_inc_inf* { $V\ X : eqType$ } { $\rho : Rel\ V\ X$ } { $\xi : Rel\ X\ X$ }:
 $\min\ \rho\ \xi \sqsubseteq \inf\ \rho\ \xi$.

Proof.

rewrite /min/inf.
 rewrite cap_comm.
 apply cap_inc_compat_l.
 move : (@galois_corollary1 _ _ _ rho (xi #)) => H.
 rewrite inv_invol in H.
 apply H.

Qed.

Lemma 255 (inf_to_sup) *Let $\rho : V \rightarrow X$ and $\xi : X \rightarrow X$. Then,*

$$\inf(\rho, \xi) = \sup(\rho \triangleright \xi^\sharp, \xi).$$

Lemma *inf_to_sup* { $V\ X : eqType$ } { $\rho : Rel\ V\ X$ } { $\xi : Rel\ X\ X$ }:
 $\inf\ \rho\ \xi = \sup\ (\rho \triangleright \xi^\sharp)\ \xi$.

Proof.

rewrite /sup/inf.
 rewrite cap_comm.
 move : (@galois_corollary2 _ _ _ rho (xi #)) => H.
 rewrite inv_invol in H.
 by [rewrite H].

Qed.

Lemma 256 (sup_to_inf) *Let $\rho : V \rightarrow X$ and $\xi : X \rightarrow X$. Then,*

$$\sup(\rho, \xi) = \inf(\rho \triangleright \xi, \xi).$$

Lemma *sup_to_inf* { $V\ X : eqType$ } { $\rho : Rel\ V\ X$ } { $\xi : Rel\ X\ X$ }:
 $\sup\ \rho\ \xi = \inf\ (\rho \triangleright \xi)\ \xi$.

Proof.

rewrite /sup/inf.
 rewrite cap_comm.
 by [rewrite galois_corollary2].

Qed.

CHAPTER 11. LIBRARY RESIDUAL

Lemma 257 (residual_inc_sup1, residual_inc_sup2) *Let $\rho : V \rightarrow X$ and $\xi : X \rightarrow X$. Then,*

$$\text{sup}(\rho, \xi) \sqsubseteq \rho \triangleright \xi \sqsubseteq \text{sup}(\rho, \xi) \triangleright \xi.$$

Lemma residual_inc_sup1 $\{V\ X : \text{eqType}\} \{rho : \text{Rel } V\ X\} \{xi : \text{Rel } X\ X\} :$
 $\text{sup } rho\ xi \quad (rho \quad xi).$

Proof.

apply *cap_l*.

Qed.

Lemma residual_inc_sup2 $\{V\ X : \text{eqType}\} \{rho : \text{Rel } V\ X\} \{xi : \text{Rel } X\ X\} :$
 $(rho \quad xi) \quad ((\text{sup } rho\ xi) \quad xi).$

Proof.

rewrite *galois*.

apply *cap_r*.

Qed.

Lemma 258 (max_inc_xi_cap) *Let $\rho : V \rightarrow X$ and $\xi : X \rightarrow X$. Then,*

$$(\text{max}(\rho, \xi))^{\#} \cdot \text{max}(\rho, \xi) \sqsubseteq \xi \sqcap \xi^{\#}.$$

Lemma max_inc_xi_cap $\{V\ X : \text{eqType}\} \{rho : \text{Rel } V\ X\} \{xi : \text{Rel } X\ X\} :$
 $(\text{max } rho\ xi \ \# \cdot \text{max } rho\ xi) \quad (xi \quad xi \ \#).$

Proof.

rewrite */max*.

rewrite *inv_cap_distr*.

apply (*@inc_trans - - - - (comp_cap_distr_r)*).

apply *cap_inc_compat*.

apply *inc_residual*.

apply *cap_r*.

apply *inv_inc_move*.

rewrite *comp_inv inv_invol*.

apply *inc_residual*.

apply *residual_inc_compat_r*.

apply *cap_l*.

Qed.

Lemma 259 (sup_inc_xi_cap) *Let $\rho : V \rightarrow X$ and $\xi : X \rightarrow X$. Then,*

$$(\text{sup}(\rho, \xi))^{\#} \cdot \text{sup}(\rho, \xi) \sqsubseteq \xi \sqcap \xi^{\#}.$$

Lemma sup_inc_xi_cap $\{V\ X : \text{eqType}\} \{rho : \text{Rel } V\ X\} \{xi : \text{Rel } X\ X\} :$
 $(\text{sup } rho\ xi \ \# \cdot \text{sup } rho\ xi) \quad (xi \quad xi \ \#).$

Proof.

move : (@max_inc_xi_cap _ _ (rho xi) (xi #)).

rewrite /max/sup.

by [rewrite inv_invol (@cap_comm _ _ xi)].

Qed.

Lemma 260 (transitive_sup1) *Let $\rho : V \rightarrow X$, $\xi : X \rightarrow X$ and $\xi \cdot \xi \sqsubseteq \xi$. Then,*

$$\text{sup}(\rho, \xi) \cdot (\xi \sqcap \xi^\#) = \text{sup}(\rho, \xi).$$

Lemma transitive_sup1 { V X : eqType } { rho : Rel V X } { xi : Rel X X }:

(xi • xi) xi → sup rho xi • (xi xi #) = sup rho xi.

Proof.

move ⇒ H.

apply inc_antisym.

rewrite /sup.

apply (@inc_trans _ _ _ _ (comp_cap_distr_l)).

apply cap_inc_compat.

apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).

apply (@inc_trans _ _ _ _ (cap_l)).

apply (@inc_trans _ _ _ _ (residual_property1)).

apply (residual_inc_compat_l H).

apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).

apply (@inc_trans _ _ _ _ (cap_r)).

apply (@inc_trans _ _ _ _ (residual_property1)).

apply residual_inc_compat_l.

rewrite -comp_inv inv_inc_move inv_invol.

apply H.

apply (@inc_trans _ _ _ _ (relation_rel_inv_rel)).

rewrite comp_assoc.

apply (comp_inc_compat_ab_ab' sup_inc_xi_cap).

Qed.

Lemma 261 (transitive_sup2) *Let $\rho : V \rightarrow X$, $\xi : X \rightarrow X$ and $\xi \cdot \xi \sqsubseteq \xi$. Then,*

$$\text{sup}(\rho, \xi) \cdot \xi = \lfloor \text{sup}(\rho, \xi) \rfloor \cdot (\rho \triangleright \xi).$$

Lemma transitive_sup2 { V X : eqType } { rho : Rel V X } { xi : Rel X X }:

(xi • xi) xi → sup rho xi • xi = domain (sup rho xi) • (rho xi).

Proof.

move ⇒ H.

apply inc_antisym.

replace (sup rho xi • xi) with (domain (sup rho xi) • (sup rho xi • xi)).

CHAPTER 11. LIBRARY RESIDUAL

```

apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ ((rho xi) • xi)).
apply (comp_inc_compat_ab_a'b cap_l).
apply (@inc_trans _ _ _ _ (residual_property1) (residual_inc_compat_l H)).
by [rewrite -comp_assoc domain_comp_alpha1].
apply (@inc_trans _ _ _ (domain (sup rho xi) • (sup rho xi xi))).
apply comp_inc_compat_ab_ab'.
apply galois.
apply cap_r.
rewrite /domain.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_residual.
apply inc_refl.
Qed.

```

Lemma 262 (domain_sup_inc) *Let $\rho : V \rightarrow X$ and $\xi : X \rightarrow X$. Then,*

$$\lfloor \text{sup}(\rho, \xi) \rfloor \cdot \rho \sqsubseteq \text{sup}(\rho, \xi) \cdot \xi^\sharp.$$

Lemma domain_sup_inc $\{V\ X : \text{eqType}\} \{\rho : \text{Rel } V\ X\} \{\xi : \text{Rel } X\ X\}$:
 $(\text{domain } (\text{sup } \rho \ \xi) \cdot \rho) \quad (\text{sup } \rho \ \xi \cdot \xi^\sharp).$

Proof.

```

apply (@inc_trans _ _ _ (domain (sup rho xi) • (sup rho xi xi))).
apply comp_inc_compat_ab_ab'.
rewrite -galois.
apply cap_l.
rewrite /domain.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_residual.
apply inc_refl.
Qed.

```

Lemma 263 (sup_function) *Let $\rho : V \rightarrow X$, $\xi : X \rightarrow X$ be relations and $f : W \rightarrow V$ be a function. Then,*

$$f \cdot \text{sup}(\rho, \xi) = \text{sup}(f \cdot \rho, \xi).$$

Lemma sup_function $\{V\ W\ X : \text{eqType}\} \{\rho : \text{Rel } V\ X\} \{\xi : \text{Rel } X\ X\} \{f : \text{Rel } W\ V\}$:

CHAPTER 11. LIBRARY RESIDUAL

$\text{function_r } f \rightarrow f \cdot \text{sup } \rho x i = \text{sup } (f \cdot \rho) x i.$

Proof.

`move` $\Rightarrow H$.

`rewrite` $/\text{sup}$.

`rewrite` $(\text{function_cap_distr_l } H)$.

`by` [`rewrite` $(\text{function_residual2 } H) (\text{function_residual2 } H) (\text{function_residual2 } H)$].

Qed.

Lemma 264 (max_univalent) *Let $\rho : V \rightarrow X$, $\xi : X \rightarrow X$ be relations and $\varphi : W \rightarrow V$ be a univalent relation. Then,*

$$\varphi \cdot \text{max}(\rho, \xi) = \text{max}(\varphi \cdot \rho, \xi).$$

Lemma $\text{max_univalent} \{V \ W \ X : \text{eqType}\}$
 $\{\rho : \text{Rel } V \ X\} \{xi : \text{Rel } X \ X\} \{phi : \text{Rel } W \ V\}:$
 $\text{univalent_r } phi \rightarrow phi \cdot \text{max } \rho x i = \text{max } (phi \cdot \rho) x i.$

Proof.

`move` $\Rightarrow H$.

`rewrite` $/\text{max}$.

`apply` inc_antisym .

`apply` $(@ \text{inc_trans} \text{ } _ _ _ _ (\text{comp_cap_distr_l}))$.

`apply` cap_inc_compat_l .

`apply` $(@ \text{inc_trans} \text{ } _ _ _ _ (\text{univalent_residual } H))$.

`rewrite` double_residual .

`apply` inc_refl .

`apply` $(@ \text{inc_trans} \text{ } _ _ _ _ (\text{dedekind1}))$.

`apply` $\text{comp_inc_compat_ab_ab'}$.

`apply` cap_inc_compat_l .

`rewrite` $-\text{inc_residual } \text{double_residual}$.

`apply` inc_refl .

Qed.

11.4.2 左剰余合成

関係 $\alpha : X \rightarrow Y$, $\beta : Y \rightarrow Z$ に対し, 左剰余合成を $\alpha \triangleleft \beta := (\beta^\# \triangleright \alpha^\#)^\#$ で定義する.

Definition $\text{leftres} \{X \ Y \ Z : \text{eqType}\} (\alpha : \text{Rel } X \ Y) (\beta : \text{Rel } Y \ Z)$
 $:= (\beta^\# \# \alpha^\#)^\#.$

CHAPTER 11. LIBRARY RESIDUAL

Lemma 265 (inc_leftres) *Let $\alpha : X \rightarrow Y$, $\beta : Y \rightarrow Z$ and $\delta : X \rightarrow Z$. Then,*

$$\delta \sqsubseteq \alpha \triangleleft \beta \Leftrightarrow \delta \cdot \beta^\# \sqsubseteq \alpha.$$

Lemma inc_leftres $\{X\ Y\ Z : eqType\}$
 $\{\alpha : Rel\ X\ Y\} \{\beta : Rel\ Y\ Z\} \{\delta : Rel\ X\ Z\}$:
 $\delta \sqsubseteq \alpha \quad leftres\ \alpha\ \beta \Leftrightarrow (\delta \cdot \beta^\#) \sqsubseteq \alpha$.

Proof.

rewrite /leftres.

by [rewrite inv_inc_move inc_residual -comp_inv inv_inc_move inv_invol].

Qed.

Lemma 266 (residual_leftres_assoc) *Let $\alpha : X \rightarrow Y$, $\beta : Y \rightarrow Z$ and $\gamma : Z \rightarrow W$. Then,*

$$(\alpha \triangleright \beta) \triangleleft \gamma = \alpha \triangleright (\beta \triangleleft \gamma).$$

Lemma residual_leftres_assoc $\{W\ X\ Y\ Z : eqType\}$
 $\{\alpha : Rel\ X\ Y\} \{\beta : Rel\ Y\ Z\} \{\gamma : Rel\ Z\ W\}$:
 $leftres\ (\alpha \triangleright \beta)\ \gamma = \alpha \triangleright leftres\ \beta\ \gamma$.

Proof.

apply inc_lower.

move \Rightarrow delta.

by [rewrite inc_leftres inc_residual -comp_assoc -inc_leftres -inc_residual].

Qed.

Chapter 12

Library Schroder

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Residual.
Require Import Logic.FunctionalExtensionality.
```

12.1 Schröder 圏の性質

この節では, 特記が無い限り, 記号は以下の図式に従って割り振られるものとする.

$$\begin{array}{ccccc}
 & & \delta & & \\
 & \nearrow & & \searrow & \\
 X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta, \beta', \beta_\lambda} & Z & \xrightarrow{\gamma} & W \\
 \uparrow \rho & & & & \uparrow \tau & & \\
 I & & & & V & &
 \end{array}$$

Lemma 267 (schroder_equivalence1, schroder_equivalence2)

$$\alpha \cdot \beta \sqsubseteq \delta \Leftrightarrow \alpha^\# \cdot \delta^- \sqsubseteq \beta^- \Leftrightarrow \delta^- \cdot \beta^\# \sqsubseteq \alpha^-.$$

Lemma *schroder_equivalence1*

```
{X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {delta : Rel X Z}:
(alpha · beta)    delta ↔ (alpha # · delta ^)    beta ^.
```

Proof.

```
split; move ⇒ H.
```

```
rewrite bool_lemma2 complement_invol.
```

```

apply inc_antisym.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply bool_lemma2 in H.
rewrite cap_comm inv_invol H comp_empty_r.
apply inc_refl.
apply inc_empty_alpha.
rewrite bool_lemma2.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply bool_lemma2 in H.
rewrite cap_comm -(@complement_invol _ _ beta) H comp_empty_r.
apply inc_refl.
apply inc_empty_alpha.
Qed.

```

Lemma *schroder_equivalence2*

$\{X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{delta : Rel\ X\ Z\} :$
 $(alpha \cdot beta) \quad delta \leftrightarrow (delta^\wedge \cdot beta^\#) \quad alpha^\wedge.$

Proof.

```

split; move => H.
rewrite bool_lemma2 complement_invol.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply bool_lemma2 in H.
rewrite cap_comm inv_invol H comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
rewrite bool_lemma2.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply bool_lemma2 in H.
rewrite cap_comm -(@complement_invol _ _ alpha) H comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
Qed.

```

Lemma 268 (function_inv_complement) *Let α and τ be functions. Then,*

$$(\alpha \cdot \beta \cdot \tau^\#)^- = \alpha \cdot \beta^- \cdot \tau^\#.$$

Lemma *function_inv_complement*

$\{V\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{tau : Rel\ V\ Z\} :$
 $function_r\ alpha \rightarrow function_r\ tau \rightarrow$

$$((\alpha \cdot \beta) \cdot \tau \#)^{\wedge} = (\alpha \cdot \beta^{\wedge}) \cdot \tau \#.$$

Proof.

move $\Rightarrow H \ H0$.

apply *inc_antisym*.

rewrite *bool_lemma1 complement_invol*.

apply *inc_antisym*.

rewrite *-comp_cup_distr_r -comp_cup_distr_l complement_classic*.

apply (@*inc_trans* _ _ _ ((($\alpha \cdot \alpha \#$) \cdot $X \ V$) \cdot ($\tau \cdot \tau \#$)))).

apply (@*inc_trans* _ _ _ (($\alpha \cdot \alpha \#$) \cdot $X \ V$)).

apply *comp_inc_compat_b_ab*.

apply *H*.

apply *comp_inc_compat_a_ab*.

apply *H0*.

rewrite *-comp_assoc (@comp_assoc _ _ _ α) (@comp_assoc _ _ _ α)*.

apply *comp_inc_compat_ab_a'b*.

apply *comp_inc_compat_ab_ab'*.

apply *inc_alpha_universal*.

apply *inc_alpha_universal*.

rewrite *bool_lemma2 complement_invol*.

apply *inc_antisym*.

rewrite *-(function_cap_distr H H0) cap_comm cap_complement_empty comp_empty_r comp_empty_l*.

apply *inc_refl*.

apply *inc_empty_alpha*.

Qed.

Lemma 269 (schroder_univalent1) *Let α be a univalent relation and $\beta \sqsubseteq \beta'$. Then,*

$$\alpha \cdot (\beta' \sqcap \beta^-) = \alpha \cdot \beta' \sqcap (\alpha \cdot \beta)^-.$$

Lemma *schroder_univalent1*

$\{X \ Y \ Z : \text{eqType}\} \{ \alpha : \text{Rel } X \ Y \} \{ \beta \ \beta' : \text{Rel } Y \ Z \} :$

univalent_r $\alpha \rightarrow \beta \ \beta' \rightarrow$

$\alpha \cdot (\beta' \ \beta^{\wedge}) = (\alpha \cdot \beta') \ (\alpha \cdot \beta)^{\wedge}.$

Proof.

move $\Rightarrow H \ H0$.

apply (@*cap_cup_unique* _ _ ($\alpha \cdot \beta$)).

replace (($\alpha \cdot \beta$) ($\alpha \cdot (\beta' \ \beta^{\wedge})$)) with ($X \ Z$).

rewrite (@*cap_comm* _ _ ($\alpha \cdot \beta'$)) *-cap_assoc*.

by [rewrite *cap_complement_empty cap_comm cap_empty*].

apply *inc_antisym*.

apply *inc_empty_alpha*.

apply (@*inc_trans* _ _ _ (($\alpha \cdot \beta$) (($\alpha \cdot \beta'$) ($\alpha \cdot \beta^{\wedge}$)))).

apply *cap_inc_compat_l*.

CHAPTER 12. LIBRARY SCHRODER

```

apply comp_cap_distr_l.
replace (X Z) with ((alpha • beta) (alpha • beta ^)).
apply cap_inc_compat_l.
apply cap_r.
apply inc_antisym.
move : (@univalent_residual _ _ _ beta H) ⇒ H1.
rewrite -inc_rpc.
rewrite residual_to_complement in H1.
apply H1.
apply inc_empty_alpha.
apply inc_def2 in H0.
rewrite -comp_cup_distr_l cup_cap_distr_l.
rewrite -H0 complement_classic cap_universal.
rewrite cup_cap_distr_l -comp_cup_distr_l.
by [rewrite -H0 complement_classic cap_universal].
Qed.

```

Lemma 270 (schroder_univalent2) *Let α be a univalent relation. Then,*

$$\alpha \cdot \beta^- = \alpha \cdot \nabla_{YZ} \sqcap (\alpha \cdot \beta)^-.$$

Lemma *schroder_univalent2* {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} :
univalent_r alpha → *alpha • beta ^* = (*alpha • Y Z*) (i>alpha • beta) ^.

Proof.

move ⇒ *H*.

move : (@schroder_univalent1 _ _ _ alpha beta (Y Z) H (@inc_alpha_universal _ _))
 ⇒ *H0*.

rewrite *cap_comm cap_universal* in *H0*.

apply *H0*.

Qed.

Lemma 271 (schroder_univalent3) *Let α be a univalent relation. Then,*

$$(\alpha \cdot \beta)^- = (\alpha \cdot \nabla_{YZ})^- \sqcup \alpha \cdot \beta^-.$$

Lemma *schroder_univalent3* {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} :
univalent_r alpha → (*alpha • beta*) ^ = (*alpha • Y Z*) ^ (i>alpha • beta) ^.

Proof.

move ⇒ *H*.

rewrite (*schroder_univalent2 H*).

rewrite *cup_cap_distr_l cup_comm complement_classic cap_comm cap_universal*.

apply *inc_def2*.

apply *rpc_inc_compat_r*.

CHAPTER 12. LIBRARY SCHRODER

apply *comp_inc_compat_ab_ab'*.
 apply *inc_alpha_universal*.
 Qed.

Lemma 272 (schroder_univalent4) *Let α be a univalent relation. Then,*

$$\alpha \triangleright \beta = (\alpha \cdot \nabla_{YZ})^- \sqcup \alpha \cdot \beta.$$

Lemma *schroder_univalent4* $\{X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\}$:
univalent_r alpha \rightarrow alpha beta = (alpha Y Z) ^ (alpha beta).

Proof.

move \Rightarrow *H*.
 rewrite (*residual_property9 H*).
 apply *Logic.eq_sym*.
 apply *cup_to_rpc*.
 Qed.

Lemma 273 (schroder_universal) *Let $\nabla_{XZ} \cdot \nabla_{ZW} = \nabla_{XW}$. Then,*

$$(\alpha \cdot \nabla_{YZ})^- \cdot \nabla_{ZW} = (\alpha \cdot \nabla_{YW})^-.$$

Lemma *schroder_universal* $\{W\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\}$:
 (X Z Z W) = X W \rightarrow
 (alpha Y Z) ^ Z W = (alpha Y W) ^.

Proof.

move \Rightarrow *H*.
 apply (@*cap_cup_unique* _ _ (alpha Y W)).
 rewrite *cap_complement_empty_cap_comm*.
 apply *inc_antisym*.
 apply (@*inc_trans* _ _ _ _ (dedekind2)).
 apply (@*inc_trans* _ _ _ (((alpha Y Z) ^ (alpha Y Z)) Z W)).
 apply *comp_inc_compat_ab_a'b*.
 apply *cap_inc_compat_l*.
 rewrite *comp_assoc*.
 apply *comp_inc_compat_ab_ab'*.
 apply *inc_alpha_universal*.
 rewrite *cap_comm_cap_complement_empty_comp_empty_l*.
 apply *inc_refl*.
 apply *inc_empty_alpha*.
 rewrite *complement_classic*.
 apply *inc_antisym*.
 apply *inc_alpha_universal*.
 rewrite -*H* -(@*complement_classic* _ _ (alpha Y Z)) *comp_cup_distr_r*.

apply *cup_inc_compat_r*.
 rewrite *comp_assoc*.
 apply *comp_inc_compat_ab_ab'*.
 apply *inc_alpha_universal*.
 Qed.

Lemma 274 (residual_inv)

$$(\alpha \triangleright \beta)^{\#} = \beta^{-\#} \triangleright \alpha^{-\#}.$$

Lemma *residual_inv* {*X Y Z* : *eqType*} {*alpha* : *Rel X Y*} {*beta* : *Rel Y Z*}:
 (*alpha* *beta*) $\#$ = (*beta* \wedge) $\#$ (*alpha* \wedge) $\#$.

Proof.

rewrite *residual_to_complement* *residual_to_complement*.
 by [rewrite -*inv_complement* *complement_invol* *inv_complement* *comp_inv*].
 Qed.

Lemma 275 (residual_cupP_distr_l, residual_cup_distr_l) *Let α be a univalent relation, $f : (V \rightarrow W) \rightarrow (Y \rightarrow Z)$ and $\exists \beta, P(\beta)$. Then,*

$$\alpha \triangleright (\sqcup_{P(\beta)} f(\beta)) = \sqcup_{P(\beta)} (\alpha \triangleright f(\beta)).$$

Lemma *residual_cupP_distr_l* {*V W X Y Z* : *eqType*}
 {*alpha* : *Rel X Y*} {*f* : *Rel V W* \rightarrow *Rel Y Z*} {*P* : *Rel V W* \rightarrow *Prop*}:
univalent_r *alpha* \rightarrow (\exists *beta'* : *Rel V W*, *P* *beta'*) \rightarrow
alpha ($_$ {*P*} *f*) = $_$ {*P*} (*fun* *beta* : *Rel V W* \Rightarrow *alpha* *f* *beta*).

Proof.

move \Rightarrow *H*.
 elim \Rightarrow *beta'* *H0*.
 rewrite (*schroder_univalent4* *H*) *comp_cupP_distr_l*.
 replace ($_$ {*P*} (*fun* *beta* : *Rel V W* \Rightarrow *alpha* *f* *beta*)) with ($_$ {*P*} (*fun* *beta* :
Rel V W \Rightarrow (*alpha* \cdot *Y Z*) \wedge (*alpha* \cdot *f* *beta*))).
 apply (@*cap_cup_unique* $_$ (*alpha* \cdot *Y Z*)).
 rewrite *cap_cup_distr_l* *cap_cupP_distr_l* *cap_complement_empty* *cup_comm* *cup_empty*.
 rewrite *cap_cupP_distr_l*.
 apply *cupP_eq*.
 move \Rightarrow *gamma* *H1*.
 by [rewrite *cap_cup_distr_l* *cap_complement_empty* *cup_comm* *cup_empty*].
 rewrite -*cup_assoc* *complement_classic* *cup_comm* *cup_universal*.
 rewrite -(@*complement_invol* $_$ (*alpha* \cdot *Y Z*)).
 apply *bool_lemma1*.
 rewrite *complement_invol*.
 apply (@*inc_trans* $_$ $_$ ((*alpha* \cdot *Y Z*) \wedge (*alpha* \cdot *f* *beta'*))).

```

apply cup_l.
move : beta' H0.
apply inc_cupP.
apply inc_refl.
apply cupP_eq.
move => gamma H1.
by [rewrite (schroder_univalent4 H)].
Qed.

```

Lemma residual_cup_distr_l
 $\{X \ Y \ Z : \text{eqType}\} \{alpha : \text{Rel } X \ Y\} \{beta \ beta' : \text{Rel } Y \ Z\} :$
 $\text{univalent_r } alpha \rightarrow$
 $alpha \quad (beta \quad beta') = (alpha \quad beta) \quad (alpha \quad beta').$

Proof.
 move => H.
 rewrite cup_to_cupP (@cup_to_cupP _ _ _ _ _ id).
 apply (residual_cupP_distr_l H).
 ∃ beta.
 by [left].
 Qed.

Lemma 276 (residual_capP_distr_r, residual_cap_distr_r) *Let $f : (Y \rightarrow Z) \rightarrow (I \rightarrow X)$ and $\exists \alpha, P(\alpha)$. Then,*

$$(\sqcap_{P(\alpha)} f(\alpha)^\#) \triangleright \rho = \sqcup_{P(\alpha)} (f(\alpha)^\# \triangleright \rho).$$

Lemma residual_capP_distr_r
 $\{X \ Y \ Z : \text{eqType}\} \{rho : \text{Rel } i \ X\} \{f : \text{Rel } Y \ Z \rightarrow \text{Rel } i \ X\} \{P : \text{Rel } Y \ Z \rightarrow \text{Prop}\} :$
 $(\exists \ alpha' : \text{Rel } Y \ Z, P \ alpha') \rightarrow$
 $(_ \{P\} (\text{fun } alpha : \text{Rel } Y \ Z \Rightarrow f \ alpha \ \#)) \quad rho = _ \{P\} (\text{fun } alpha : \text{Rel } Y \ Z \Rightarrow$
 $f \ alpha \ \# \quad rho).$

Proof.
 elim => alpha' H.
 rewrite residual_to_complement.
 rewrite -(@complement_invol _ _ (_ \{P\} (\text{fun } alpha : \text{Rel } Y \ Z \Rightarrow f \ alpha \ \# \quad rho))).
 apply f_equal.
 rewrite de_morgan3.
 replace (fun alpha : \text{Rel } Y \ Z \Rightarrow (f \ alpha \ \# \quad rho) ^) with (fun alpha : \text{Rel } Y \ Z \Rightarrow f \ alpha \ \# \cdot rho ^).
 apply inc_antisym.
 apply comp_capP_distr_r.
 apply (@inc_trans _ _ _ _ (relation_rel_inv_rel)).
 apply (@inc_trans _ _ _ (((_ \{P\} (\text{fun } alpha : \text{Rel } Y \ Z \Rightarrow f \ alpha \ \# \cdot rho ^)) \cdot (f

```

 $\alpha' \# \cdot \rho ^ \wedge \# ) \# ) \cdot (f \alpha' \# \cdot \rho ^ \wedge ))).$ 
apply comp_inc_compat.
apply comp_inc_compat_ab_ab'.
move :  $\alpha' H$ .
apply inc_capP.
rewrite inv_capP_distr.
apply inc_refl.
move :  $\alpha' H$ .
apply inc_capP.
apply inc_refl.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc.
apply (@inc_trans _ _ _ _ (comp_capP_distr_r)).
apply inc_capP.
move  $\Rightarrow$   $\beta H0$ .
apply (@inc_trans _ _ _ ((f  $\beta \# \cdot \rho ^ \wedge ) \cdot ((f \alpha' \# \cdot \rho ^ \wedge ) \# \cdot f \alpha' \# )))).$ 
move :  $\beta H0$ .
apply inc_capP.
apply inc_refl.
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
apply functional_extensionality.
move  $\Rightarrow x$ .
by [rewrite residual_to_complement complement_invol].
Qed.

```

Chapter 13

Library **Sum_Product**

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Conjugate.
Require Import Domain.
Require Import Logic.IndefiniteDescription.
```

13.1 関係の直和

13.1.1 入射対, 関係直和の定義

入射対の存在公理 (Axiom 23) で入射対が存在することまでは仮定済みなので, 実際に入射対 $j : A \rightarrow A + B, k : B \rightarrow A + B$ を定義する関数を定義する.

```
Definition sum_r (A B : eqType):
  {x : (Rel A (sum_eqType A B)) × (Rel B (sum_eqType A B)) |
    (fst x) • (fst x) # = Id A ∧ (snd x) • (snd x) # = Id B ∧
    (fst x) • (snd x) # = A B ∧
    ((fst x) # • (fst x)) ((snd x) # • (snd x)) = Id (sum_eqType A B)}.
apply constructive_indefinite_description.
elim (@pair_of_inclusions A B) ⇒ j.
elim ⇒ k H.
∃ (j,k).
simpl.
apply H.
Defined.
Definition inl_r (A B : eqType):= fst (sval (sum_r A B)).
```

CHAPTER 13. LIBRARY SUM_PRODUCT

Definition $\text{inr_r} (A B : \text{eqType}) := \text{snd} (\text{sval} (\text{sum_r} A B))$.

またこの定義による入射対が、入射対としての性質 (Axiom 23) $+\alpha$ を満たしていることも事前に証明しておく。

Lemma $\text{inl_id} \{A B : \text{eqType}\} : \text{inl_r} A B \cdot \text{inl_r} A B \# = \text{Id } A$.

Proof.

`apply (proj2_sig (sum_r A B)).`

Qed.

Lemma $\text{inr_id} \{A B : \text{eqType}\} : \text{inr_r} A B \cdot \text{inr_r} A B \# = \text{Id } B$.

Proof.

`apply (proj2_sig (sum_r A B)).`

Qed.

Lemma $\text{inl_inr_empty} \{A B : \text{eqType}\} : \text{inl_r} A B \cdot \text{inr_r} A B \# = A B$.

Proof.

`apply (proj2_sig (sum_r A B)).`

Qed.

Lemma $\text{inr_inl_empty} \{A B : \text{eqType}\} : \text{inr_r} A B \cdot \text{inl_r} A B \# = B A$.

Proof.

`apply inv_invol2.`

`rewrite comp_inv inv_invol inv_empty.`

`apply inl_inr_empty.`

Qed.

Lemma $\text{inl_inr_cup_id} \{A B : \text{eqType}\} :$

$(\text{inl_r} A B \# \cdot \text{inl_r} A B) (\text{inr_r} A B \# \cdot \text{inr_r} A B) = \text{Id} (\text{sum_eqType } A B)$.

Proof.

`apply (proj2_sig (sum_r A B)).`

Qed.

Lemma $\text{inl_function} \{A B : \text{eqType}\} : \text{function_r} (\text{inl_r} A B)$.

Proof.

`move : (proj2_sig (sum_r A B)).`

`elim $\Rightarrow H$.`

`elim $\Rightarrow H0$.`

`elim $\Rightarrow H1 H2$.`

`split.`

`rewrite /total_r.`

`rewrite H.`

`apply inc_refl.`

`rewrite /univalent_r.`

`rewrite -H2.`

`apply cup_l.`

Qed.

CHAPTER 13. LIBRARY SUM_PRODUCT

Lemma *inr_function* $\{A\ B : eqType\} : function_r\ (inr_r\ A\ B).$

Proof.

```
move : (proj2_sig (sum_r A B)).
elim  $\Rightarrow H$ .
elim  $\Rightarrow H0$ .
elim  $\Rightarrow H1\ H2$ .
split.
rewrite /total_r.
rewrite  $H0$ .
apply inc_refl.
rewrite /univalent_r.
rewrite - $H2$ .
apply cup_r.
Qed.
```

さらに $\alpha : A \rightarrow C$ と $\beta : B \rightarrow C$ の関係直和 $\alpha \perp \beta : A + B \rightarrow C$ を, $\alpha \perp \beta := j^\# \cdot \alpha \sqcup k^\# \cdot \beta$ で定義する.

Definition *Rel_sum* $\{A\ B\ C : eqType\} (alpha : Rel\ A\ C) (\mathbf{beta} : Rel\ B\ C) :=$
 $(inl_r\ A\ B \# \cdot alpha) \quad (inr_r\ A\ B \# \cdot \mathbf{beta}).$

13.1.2 関係直和の性質

Lemma 277 (sum_inc_compat) *Let $\alpha, \alpha' : A \rightarrow C$ and $\beta, \beta' : B \rightarrow C$. Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \perp \beta \sqsubseteq \alpha' \perp \beta'.$$

Lemma *sum_inc_compat*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ C\} \{\mathbf{beta}\ beta' : Rel\ B\ C\} :$
 $alpha \quad alpha' \rightarrow \mathbf{beta} \quad beta' \rightarrow Rel_sum\ alpha\ \mathbf{beta} \quad Rel_sum\ alpha'\ beta'.$

Proof.

```
move  $\Rightarrow H\ H0$ .
apply cup_inc_compat.
apply (comp_inc_compat_ab_ab' H).
apply (comp_inc_compat_ab_ab' H0).
Qed.
```

Lemma 278 (sum_inc_compat_l) *Let $\alpha : A \rightarrow C$ and $\beta, \beta' : B \rightarrow C$. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \perp \beta \sqsubseteq \alpha \perp \beta'.$$

Lemma *sum_inc_compat_l*

CHAPTER 13. LIBRARY SUM_PRODUCT

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta\ beta' : Rel\ B\ C\}:$
 $beta\ beta' \rightarrow Rel_sum\ alpha\ beta\ Rel_sum\ alpha\ beta'.$

Proof.

move $\Rightarrow H$.

apply (sum_inc_compat (@inc_refl _ _ alpha) H).

Qed.

Lemma 279 (sum_inc_compat_r) *Let $\alpha, \alpha' : A \rightarrow C$ and $\beta : B \rightarrow C$. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \perp \beta \sqsubseteq \alpha' \perp \beta.$$

Lemma sum_inc_compat_r

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ C\} \{beta : Rel\ B\ C\}:$
 $alpha\ alpha' \rightarrow Rel_sum\ alpha\ beta\ Rel_sum\ alpha'\ beta.$

Proof.

move $\Rightarrow H$.

apply (sum_inc_compat H (@inc_refl _ _ beta)).

Qed.

Lemma 280 (total_sum) *Let $\alpha : A \rightarrow C$ and $\beta : B \rightarrow C$ are total relations, then $\alpha \perp \beta$ is also a total relation.*

Lemma total_sum $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta : Rel\ B\ C\}:$
 $total_r\ alpha \rightarrow total_r\ beta \rightarrow total_r\ (Rel_sum\ alpha\ beta).$

Proof.

move $\Rightarrow H\ H0$.

rewrite /total_r/ Rel_sum.

rewrite -inl_inr_cup_id inv_cup_distr comp_cup_distr_l comp_cup_distr_r comp_cup_distr_r.

rewrite comp_inv comp_inv inv_invol inv_invol.

apply cup_inc_compat.

apply (fun H' \Rightarrow @inc_trans _ _ _ _ H' (cup_l)).

rewrite comp_assoc -(@comp_assoc _ _ _ _ alpha).

apply comp_inc_compat_ab_ab'.

apply (comp_inc_compat_b_ab H).

apply (fun H' \Rightarrow @inc_trans _ _ _ _ H' (cup_r)).

rewrite comp_assoc -(@comp_assoc _ _ _ _ beta).

apply comp_inc_compat_ab_ab'.

apply (comp_inc_compat_b_ab H0).

Qed.

Lemma 281 (univalent_sum) *Let $\alpha : A \rightarrow C$ and $\beta : B \rightarrow C$ are univalent relations, then $\alpha \perp \beta$ is also a univalent relation.*

CHAPTER 13. LIBRARY SUM_PRODUCT

Lemma *univalent_sum* $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta : Rel\ B\ C\}$:
 $univalent_r\ alpha \rightarrow univalent_r\ beta \rightarrow univalent_r\ (Rel_sum\ alpha\ beta)$.

Proof.

move $\Rightarrow H\ H0$.

rewrite $/univalent_r/Rel_sum$.

rewrite $inv_cup_distr\ comp_cup_distr_l\ comp_cup_distr_r\ comp_cup_distr_r$.

rewrite $comp_inv\ comp_inv\ inv_involver\ inv_involver$.

rewrite $comp_assoc\ -(@comp_assoc\ _ _ _ (inl_r\ A\ B))\ inl_id\ comp_id_l$.

rewrite $comp_assoc\ -(@comp_assoc\ _ _ _ (inr_r\ A\ B))\ inr_inl_empty\ comp_empty_l\ comp_empty_r\ cup_empty$.

rewrite $-cup_assoc\ comp_assoc\ -(@comp_assoc\ _ _ _ (inl_r\ A\ B))\ inl_inr_empty\ comp_empty_l\ comp_empty_r\ cup_empty$.

rewrite $comp_assoc\ -(@comp_assoc\ _ _ _ (inr_r\ A\ B))\ inr_id\ comp_id_l$.

apply inc_cup .

split.

apply H .

apply $H0$.

Qed.

Lemma 282 (function_sum) *Let $\alpha : A \rightarrow C$ and $\beta : B \rightarrow C$ be functions, then $\alpha \perp \beta$ is also a function.*

Lemma *function_sum* $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta : Rel\ B\ C\}$:
 $function_r\ alpha \rightarrow function_r\ beta \rightarrow function_r\ (Rel_sum\ alpha\ beta)$.

Proof.

elim $\Rightarrow H\ H0$.

elim $\Rightarrow H1\ H2$.

split.

apply $(total_sum\ H\ H1)$.

apply $(univalent_sum\ H0\ H2)$.

Qed.

Lemma 283 (sum_conjugate) *Let $\alpha : A \rightarrow C$, $\beta : B \rightarrow C$ and $\gamma : A + B \rightarrow C$ be relations, $j : A \rightarrow A + B$ and $k : B \rightarrow A + B$ be inclusions. Then,*

$$j \cdot \gamma = \alpha \wedge k \cdot \gamma = \beta \Leftrightarrow \gamma = \alpha \perp \beta.$$

Lemma *sum_conjugate*

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta : Rel\ B\ C\} \{gamma : Rel\ (sum_eqType\ A\ B)\ C\}$:

$inl_r\ A\ B \cdot gamma = alpha \wedge inr_r\ A\ B \cdot gamma = beta \Leftrightarrow gamma = Rel_sum\ alpha\ beta$.

Proof.

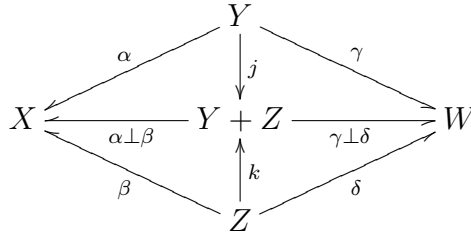
```

split; move => H.
elim H => H0 H1.
rewrite -(@comp_id_l _ _ gamma).
rewrite -inl_inr_cup_id comp_cup_distr_r comp_assoc comp_assoc.
by [rewrite H0 H1].
split.
rewrite H comp_cup_distr_l -comp_assoc -comp_assoc.
rewrite inl_id inl_inr_empty comp_id_l comp_empty_l.
by [rewrite cup_empty].
rewrite H comp_cup_distr_l -comp_assoc -comp_assoc.
rewrite inr_id inr_inl_empty comp_id_l comp_empty_l.
by [rewrite cup_comm cup_empty].
Qed.

```

Lemma 284 (sum_comp) *In below figure,*

$$(\alpha \perp \beta)^\# \cdot (\gamma \perp \delta) = \alpha^\# \cdot \gamma \sqcup \beta^\# \cdot \delta.$$



Lemma *sum_comp* { $W\ X\ Y\ Z : eqType$ }
 { $\alpha : Rel\ Y\ X$ } { $\beta : Rel\ Z\ X$ } { $\gamma : Rel\ Y\ W$ } { $\delta : Rel\ Z\ W$ }:
 (*Rel_sum* $\alpha\ \beta$) # • *Rel_sum* $\gamma\ \delta$ =
 ($\alpha\ \# \cdot \gamma$) ($\beta\ \# \cdot \delta$).

Proof.

```

rewrite /Rel_sum.
rewrite inv_cup_distr comp_cup_distr_l comp_cup_distr_r comp_cup_distr_r.
rewrite comp_inv comp_inv inv_invol inv_invol.
apply f_equal2.
rewrite comp_assoc -(@comp_assoc _ _ _ (inl_r Y Z)) inl_id comp_id_l.
by [rewrite comp_assoc -(@comp_assoc _ _ _ (inr_r Y Z)) inr_inl_empty comp_empty_l
    comp_empty_r cup_empty].
rewrite comp_assoc -(@comp_assoc _ _ _ (inl_r Y Z)) inl_inr_empty comp_empty_l
    comp_empty_r cup_comm cup_empty.
by [rewrite comp_assoc -(@comp_assoc _ _ _ (inr_r Y Z)) inr_id comp_id_l].
Qed.

```

13.1.3 分配法則

Lemma 285 (sum_cap_distr_l) *Let $\alpha : A \rightarrow C$ and $\beta, \beta' : B \rightarrow C$. Then,*

$$\alpha \perp (\beta \sqcap \beta') \sqsubseteq (\alpha \perp \beta) \sqcap (\alpha \perp \beta').$$

Lemma *sum_cap_distr_l*

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta\ beta' : Rel\ B\ C\}:$
 $Rel_sum\ alpha\ (beta\ \beta')\ (Rel_sum\ alpha\ beta\ Rel_sum\ alpha\ beta').$

Proof.

rewrite -cup_cap_distr_l.

apply cup_inc_compat_l.

apply comp_cap_distr_l.

Qed.

Lemma 286 (sum_cap_distr_r) *Let $\alpha, \alpha' : A \rightarrow C$ and $\beta : B \rightarrow C$. Then,*

$$(\alpha \sqcap \alpha') \perp \beta \sqsubseteq (\alpha \perp \beta) \sqcap (\alpha' \perp \beta).$$

Lemma *sum_cap_distr_r*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ C\} \{beta : Rel\ B\ C\}:$
 $Rel_sum\ (alpha\ alpha')\ beta\ (Rel_sum\ alpha\ beta\ Rel_sum\ alpha'\ beta').$

Proof.

rewrite -cup_cap_distr_r.

apply cup_inc_compat_r.

apply comp_cap_distr_l.

Qed.

Lemma 287 (sum_cup_distr_l) *Let $\alpha : A \rightarrow C$ and $\beta, \beta' : B \rightarrow C$. Then,*

$$\alpha \perp (\beta \sqcup \beta') = (\alpha \perp \beta) \sqcup (\alpha \perp \beta').$$

Lemma *sum_cup_distr_l*

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta\ beta' : Rel\ B\ C\}:$
 $Rel_sum\ alpha\ (beta\ \beta') = Rel_sum\ alpha\ beta\ Rel_sum\ alpha\ beta'.$

Proof.

rewrite -cup_assoc (@cup_comm _ _ (Rel_sum alpha beta)) -cup_assoc.

by [rewrite cup_idem cup_assoc -comp_cup_distr_l].

Qed.

CHAPTER 13. LIBRARY SUM_PRODUCT

Lemma 288 (sum_cup_distr_r) *Let $\alpha, \alpha' : A \rightarrow C$ and $\beta : B \rightarrow C$. Then,*

$$(\alpha \sqcup \alpha') \perp \beta = (\alpha \perp \beta) \sqcup (\alpha' \perp \beta).$$

Lemma *sum_cup_distr_r*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ C\} \{beta : Rel\ B\ C\} :$
 $Rel_sum\ (alpha\ \ alpha')\ beta = (Rel_sum\ alpha\ beta\ \ Rel_sum\ alpha'\ beta).$

Proof.

`rewrite cup_assoc (@cup_comm _ _ (inr_r A B # • beta)) cup_assoc.`

`by [rewrite cup_idem -cup_assoc -comp_cup_distr_l].`

Qed.

Lemma 289 (comp_sum_distr_r) *Let $\alpha : A \rightarrow C$, $\beta : B \rightarrow C$ and $\gamma : C \rightarrow D$. Then,*

$$(\alpha \perp \beta) \cdot \gamma = \alpha \cdot \gamma \perp \beta \cdot \gamma.$$

Lemma *comp_sum_distr_r*

$\{A\ B\ C\ D : eqType\} \{alpha : Rel\ A\ C\} \{beta : Rel\ B\ C\} \{gamma : Rel\ C\ D\} :$
 $(Rel_sum\ alpha\ beta) \cdot gamma = Rel_sum\ (alpha \cdot gamma)\ (beta \cdot gamma).$

Proof.

`by [rewrite comp_cup_distr_r comp_assoc comp_assoc].`

Qed.

13.2 関係の直積

13.2.1 射影対, 関係直積の定義

射影対の存在公理 (Axiom 24) で射影対が存在することまでは仮定済みなので, 実際に射影対 $p : A \times B \rightarrow A, k : A \times B \rightarrow B$ を定義する関数を定義する.

Definition *prod_r* ($A\ B : eqType$):

$\{x : (Rel\ (prod_eqType\ A\ B)\ A) \times (Rel\ (prod_eqType\ A\ B)\ B) \mid$
 $(fst\ x) \# \cdot (snd\ x) = A\ B \wedge$
 $((fst\ x) \cdot (fst\ x) \#) \cdot ((snd\ x) \cdot (snd\ x) \#) = Id\ (prod_eqType\ A\ B) \wedge$
 $univalent_r\ (fst\ x) \wedge univalent_r\ (snd\ x)\}.$

`apply constructive_indefinite_description.`

`elim (@pair_of_projections A B) => p.`

`elim => q H.`

`exists (p,q).`

`simpl.`

`apply H.`

Defined.

Definition $\text{fst}_r (A B : \text{eqType}) := \text{fst} (\text{sval} (\text{prod}_r A B))$.

Definition $\text{snd}_r (A B : \text{eqType}) := \text{snd} (\text{sval} (\text{prod}_r A B))$.

またこの定義による射影対が, 射影対としての性質 (Axiom 24) $+\alpha$ を満たしていることも事前に証明しておく.

Lemma $\text{fst_snd_universal} \{A B : \text{eqType}\} : \text{fst}_r A B \# \cdot \text{snd}_r A B = A B$.

Proof.

apply (proj2_sig (prod_r A B)).

Qed.

Lemma $\text{snd_fst_universal} \{A B : \text{eqType}\} : \text{snd}_r A B \# \cdot \text{fst}_r A B = B A$.

Proof.

apply inv_invol2.

rewrite comp_inv inv_invol inv_universal.

apply fst_snd_universal.

Qed.

Lemma $\text{fst_snd_cap_id} \{A B : \text{eqType}\} :$

$(\text{fst}_r A B \cdot \text{fst}_r A B \#) (\text{snd}_r A B \cdot \text{snd}_r A B \#) = \text{Id} (\text{prod_eqType} A B)$.

Proof.

apply (proj2_sig (prod_r A B)).

Qed.

Lemma $\text{fst_function} \{A B : \text{eqType}\} : \text{function}_r (\text{fst}_r A B)$.

Proof.

move : (proj2_sig (prod_r A B)).

elim $\Rightarrow H$.

elim $\Rightarrow H0 H1$.

split.

rewrite /total_r.

rewrite -H0.

apply cap_l.

apply H1.

Qed.

Lemma $\text{snd_function} \{A B : \text{eqType}\} : \text{function}_r (\text{snd}_r A B)$.

Proof.

move : (proj2_sig (prod_r A B)).

elim $\Rightarrow H$.

elim $\Rightarrow H0 H1$.

split.

rewrite /total_r.

rewrite -H0.

apply cap_r.

apply *H1*.

Qed.

さらに $\alpha : A \rightarrow B$ と $\beta : A \rightarrow C$ の関係直積 $\alpha \top \beta : A \rightarrow B \times C$ を, $\alpha \top \beta := \alpha \cdot p^\# \sqcap \beta \cdot q^\#$ で定義する.

Definition *Rel_prod* $\{A\ B\ C : eqType\}$ (*alpha* : *Rel* *A* *B*) (**beta** : *Rel* *A* *C*):=
 (*alpha* · *fst_r* *B* *C* #) (b**eta** · *snd_r* *B* *C* #).

13.2.2 関係直積の性質

Lemma 290 (prod_inc_compat) *Let* $\alpha, \alpha' : A \rightarrow B$ *and* $\beta, \beta' : A \rightarrow C$. *Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \top \beta \sqsubseteq \alpha' \top \beta'.$$

Lemma *prod_inc_compat*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{b\mathbf{eta}\ beta' : Rel\ A\ C\}:$
 $alpha\ alpha' \rightarrow b\mathbf{eta}\ beta' \rightarrow Rel_prod\ alpha\ b\mathbf{eta}\ Rel_prod\ alpha'\ beta'.$

Proof.

move \Rightarrow *H* *H0*.

apply *cap_inc_compat*.

apply (*comp_inc_compat_ab_a'b* *H*).

apply (*comp_inc_compat_ab_a'b* *H0*).

Qed.

Lemma 291 (prod_inc_compat_l) *Let* $\alpha : A \rightarrow B$ *and* $\beta, \beta' : A \rightarrow C$. *Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \top \beta \sqsubseteq \alpha \top \beta'.$$

Lemma *prod_inc_compat_l*

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{b\mathbf{eta}\ beta' : Rel\ A\ C\}:$
 $b\mathbf{eta}\ beta' \rightarrow Rel_prod\ alpha\ b\mathbf{eta}\ Rel_prod\ alpha\ beta'.$

Proof.

move \Rightarrow *H*.

apply (*prod_inc_compat* (@*inc_refl* _ _ *alpha*) *H*).

Qed.

Lemma 292 (prod_inc_compat_r) *Let* $\alpha, \alpha' : A \rightarrow B$ *and* $\beta : A \rightarrow C$. *Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \top \beta \sqsubseteq \alpha' \top \beta.$$

Lemma *prod_inc_compat_r*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{b\mathbf{eta} : Rel\ A\ C\}:$

CHAPTER 13. LIBRARY SUM_PRODUCT

α α' \rightarrow $\text{Rel_prod } \alpha$ β $\text{Rel_prod } \alpha' \beta$.

Proof.

move $\Rightarrow H$.

apply ($\text{prod_inc_compat } H$ ($\text{@inc_refl } _ _ \beta$)).

Qed.

Lemma 293 (total_prod) *Let $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$ are total relations, then $\alpha \top \beta$ is also a total relation.*

Lemma total_prod $\{A B C : \text{eqType}\} \{ \alpha : \text{Rel } A B \} \{ \beta : \text{Rel } A C \}$:
 $\text{total_r } \alpha \rightarrow \text{total_r } \beta \rightarrow \text{total_r } (\text{Rel_prod } \alpha \beta)$.

Proof.

move $\Rightarrow H H0$.

rewrite $\text{domain_total cap_domain cap_comm}$.

apply Logic.eq_sym .

apply inc_def1 .

apply ($\text{@inc_trans } _ _ _ _ H$).

rewrite $\text{comp_inv inv_invol comp_assoc}$.

apply $\text{comp_inc_compat_ab_ab'}$.

apply ($\text{@inc_trans } _ _ _ (\alpha \# \cdot (\beta \cdot \beta \#))$).

apply ($\text{comp_inc_compat_a_ab } H0$).

rewrite $\text{-comp_assoc -comp_assoc fst_snd_universal}$.

apply $\text{comp_inc_compat_ab_a'b}$.

apply $\text{inc_alpha_universal}$.

Qed.

Lemma 294 (univalent_prod) *Let $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$ are univalent relations, then $\alpha \top \beta$ is also a univalent relation.*

Lemma univalent_prod $\{A B C : \text{eqType}\} \{ \alpha : \text{Rel } A B \} \{ \beta : \text{Rel } A C \}$:
 $\text{univalent_r } \alpha \rightarrow \text{univalent_r } \beta \rightarrow \text{univalent_r } (\text{Rel_prod } \alpha \beta)$.

Proof.

move $\Rightarrow H H0$.

rewrite $/\text{univalent_r}/\text{Rel_prod}$.

rewrite $\text{inv_cap_distr comp_inv inv_invol comp_inv inv_invol}$.

apply ($\text{@inc_trans } _ _ _ _ (\text{comp_cap_distr_l})$).

rewrite -fst_snd_cap_id .

apply cap_inc_compat .

apply ($\text{@inc_trans } _ _ _ _ (\text{comp_cap_distr_r})$).

apply ($\text{@inc_trans } _ _ _ _ (\text{cap_l})$).

rewrite $\text{comp_assoc -(@comp_assoc } _ _ _ _ \alpha)$.

apply $\text{comp_inc_compat_ab_ab'}$.

apply ($\text{comp_inc_compat_ab_b } H$).

CHAPTER 13. LIBRARY SUM_PRODUCT

```

apply (@inc_trans - - - - (comp_cap_distr_r)).
apply (@inc_trans - - - - (cap_r)).
rewrite comp_assoc - (@comp_assoc - - - - beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H0).
Qed.

```

Lemma 295 (function_prod) *Let $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$ be functions, then $\alpha \top \beta$ is also a function.*

Lemma *function_prod* { $A B C : eqType$ } { $\alpha : Rel A B$ } { $\beta : Rel A C$ }:
function_r $\alpha \rightarrow$ *function_r* $\beta \rightarrow$ *function_r* (*Rel_prod* $\alpha \beta$).

Proof.

```

elim  $\Rightarrow$   $H H0$ .
elim  $\Rightarrow$   $H1 H2$ .
split.
apply (total_prod  $H H1$ ).
apply (univalent_prod  $H0 H2$ ).
Qed.

```

Lemma 296 (prod_fst_surjection) *Let $p : B \times C \rightarrow B$ be a projection. Then,*

$$“p \text{ is a surjection}” \Leftrightarrow \forall D, \nabla_{BD} = \nabla_{BC} \cdot \nabla_{CD}.$$

Lemma *prod_fst_surjection* { $B C : eqType$ }:
surjection_r (*fst_r* $B C$) $\Leftrightarrow \forall D : eqType, \quad B D = B C \cdot C D$.

Proof.

```

split; move  $\Rightarrow$   $H$ .
move  $\Rightarrow$   $D$ .
elim  $H \Rightarrow H0 H1$ .
apply inc_antisym.
apply (@inc_trans - - - ((fst_r  $B C \# \cdot$  (fst_r  $B C \#$ )  $\#$ )  $\cdot$   $B D$ )).
apply (comp_inc_compat_b_ab H1).
rewrite inv_invol.
apply (@inc_trans - - - (((fst_r  $B C \# \cdot$  snd_r  $B C$ )  $\cdot$  (snd_r  $B C \# \cdot$  fst_r  $B C$ ))  $\cdot$ 
 $B D$ )).
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc - (@comp_assoc - - - (snd_r  $B C$ )).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply snd_function.
rewrite (@comp_assoc - - - - - (  $B D$ )).
apply comp_inc_compat.

```



```

apply inc_alpha_universal.
apply inc_alpha_universal.
apply inc_alpha_universal.
split.
apply fst_function.
rewrite /total_r.
rewrite -(@cap_universal _ _ (Id B)) (H B) -(@fst_snd_universal B C) cap_comm comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite comp_id_r.
apply cap_r.
Qed.

```

Lemma 297 (prod_snd_surjection) *Let $q : B \times C \rightarrow C$ be a projection. Then,*

$$“q \text{ is a surjection}” \Leftrightarrow \forall D, \nabla_{CD} = \nabla_{CB} \cdot \nabla_{BD}.$$

Lemma *prod_snd_surjection* $\{B\ C : eqType\}$:
 $surjection_r\ (snd_r\ B\ C) \leftrightarrow \forall\ D : eqType, \quad C\ D = \quad C\ B \cdot \quad B\ D.$

Proof.

```

split; move => H.
move => D.
elim H => H0 H1.
apply inc_antisym.
apply (@inc_trans _ _ _ ((snd_r B C # · (snd_r B C #) #) · C D)).
apply (comp_inc_compat_b_ab H1).
rewrite inv_invol.
apply (@inc_trans _ _ _ (((snd_r B C # · fst_r B C) · (fst_r B C # · snd_r B C)) · C D)).
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -(@comp_assoc _ _ _ (fst_r B C)).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply fst_function.
rewrite (@comp_assoc _ _ _ _ _ (C D)).
apply comp_inc_compat.
apply inc_alpha_universal.
apply inc_alpha_universal.
apply inc_alpha_universal.
split.
apply snd_function.
rewrite /total_r.
rewrite -(@cap_universal _ _ (Id C)) (H C) -(@snd_fst_universal B C) cap_comm comp_assoc.

```

CHAPTER 13. LIBRARY SUM_PRODUCT

```

apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite comp_id_r.
apply cap_r.
Qed.

```

Lemma 298 (prod_fst_domain1) *Let $p : B \times C \rightarrow B$ be a projection, $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$. Then,*

$$(\alpha \top \beta) \cdot p = \lfloor \beta \rfloor \cdot \alpha.$$

Lemma prod_fst_domain1 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ A\ C\}$:
 $(Rel_prod\ alpha\ beta) \cdot fst_r\ B\ C = domain\ beta \cdot alpha$.

Proof.

```

rewrite (@comp_inv_inv A A) domain_inv.
rewrite domain_universal2 inv_cap_distr comp_inv inv_invol inv_invol inv_universal.
rewrite -snd_fst_universal.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
rewrite comp_assoc comp_assoc.
apply cap_inc_compat_r.
apply comp_inc_compat_ab_a.
apply fst_function.
rewrite cap_comm -comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind2)).
rewrite cap_comm.
apply inc_refl.
Qed.

```

Lemma 299 (prod_fst_domain2) *Let $p : B \times C \rightarrow B$ be a projection, $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$. Then,*

$$(\alpha \top \beta) \cdot p = \alpha \Leftrightarrow \lfloor \alpha \rfloor \sqsubseteq \lfloor \beta \rfloor.$$

Lemma prod_fst_domain2 $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ A\ C\}$:
 $(Rel_prod\ alpha\ beta) \cdot fst_r\ B\ C = alpha \Leftrightarrow domain\ alpha \sqsubseteq domain\ beta$.

Proof.

```

rewrite prod_fst_domain1.
split; move => H.
apply domain_lemma2b.
assert ((domain beta \cdot alpha) ((beta \cdot beta #) \cdot alpha)).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite H in H0.
apply H0.

```

```

apply inc_antisym.
apply comp_inc_compat_ab_b.
apply cap_r.
apply (@inc_trans _ _ _ (domain alpha · alpha)).
rewrite domain_comp_alpha1.
apply inc_refl.
apply (comp_inc_compat_ab_a'b H).
Qed.

```

Lemma 300 (prod_snd_domain1) *Let $q : B \times C \rightarrow C$ be a projection, $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$. Then,*

$$(\alpha \top \beta) \cdot q = \lfloor \alpha \rfloor \cdot \beta.$$

Lemma prod_snd_domain1 $\{A B C : eqType\} \{alpha : Rel A B\} \{beta : Rel A C\}$:
 $(Rel_prod\ alpha\ beta) \cdot snd_r\ B\ C = domain\ alpha \cdot beta.$

Proof.

```

rewrite (@comp_inv_inv A A) domain_inv.
rewrite domain_universal2 inv_cap_distr comp_inv inv_invol inv_invol inv_universal.
rewrite fst_snd_universal.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
rewrite comp_assoc comp_assoc cap_comm.
apply cap_inc_compat_r.
apply comp_inc_compat_ab_a.
apply snd_function.
rewrite cap_comm -comp_assoc.
apply dedekind2.
Qed.

```

Lemma 301 (prod_snd_domain2) *Let $q : B \times C \rightarrow C$ be a projection, $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$. Then,*

$$(\alpha \top \beta) \cdot q = \beta \Leftrightarrow \lfloor \beta \rfloor \sqsubseteq \lfloor \alpha \rfloor.$$

Lemma prod_snd_domain2 $\{A B C : eqType\} \{alpha : Rel A B\} \{beta : Rel A C\}$:
 $(Rel_prod\ alpha\ beta) \cdot snd_r\ B\ C = beta \leftrightarrow domain\ beta \sqsubseteq domain\ alpha.$

Proof.

```

rewrite prod_snd_domain1.
split; move => H.
apply domain_lemma2b.
assert ((domain alpha · beta) ((alpha · alpha #) · beta)).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite H in H0.

```

```

apply H0.
apply inc_antisym.
apply comp_inc_compat_ab_b.
apply cap_r.
apply (@inc_trans _ _ _ (domain beta • beta)).
rewrite domain_comp_alpha1.
apply inc_refl.
apply (comp_inc_compat_ab_a'b H).
Qed.

```

Lemma 302 (prod_to_cap) *Let $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$. Then,*

$$\lfloor \alpha \top \beta \rfloor = \lfloor \alpha \rfloor \sqcap \lfloor \beta \rfloor.$$

Lemma prod_to_cap $\{A B C : \text{eqType}\} \{\alpha : \text{Rel } A B\} \{\beta : \text{Rel } A C\}$:
 $\text{domain } (\text{Rel_prod } \alpha \beta) = \text{domain } \alpha \quad \text{domain } \beta.$

Proof.

```

replace (domain (Rel_prod alpha beta)) with (domain (Rel_prod alpha beta • snd_r B C)).
rewrite prod_snd_domain1 comp_domain8.
apply dedekind_id3.
apply cap_r.
apply cap_r.
apply cap_r.
apply comp_domain3.
apply snd_function.
Qed.

```

Lemma 303 (prod_conjugate1) *Let $\alpha : A \rightarrow B$ and $\beta : A \rightarrow C$ be functions, $p : B \times C \rightarrow B$ and $q : B \times C \rightarrow C$ be projections. Then,*

$$(\alpha \top \beta) \cdot p = \alpha \wedge (\alpha \top \beta) \cdot q = \beta.$$

Lemma prod_conjugate1 $\{A B C : \text{eqType}\} \{\alpha : \text{Rel } A B\} \{\beta : \text{Rel } A C\}$:
 $\text{function_r } \alpha \rightarrow \text{function_r } \beta \rightarrow$
 $\text{Rel_prod } \alpha \beta \cdot \text{fst_r } B C = \alpha \wedge \text{Rel_prod } \alpha \beta \cdot \text{snd_r } B C = \beta.$

Proof.

```

move => H H0.
split.
rewrite prod_fst_domain1.
elim H0 => H1 H2.
apply inc_def1 in H1.
rewrite /domain.

```

CHAPTER 13. LIBRARY SUM_PRODUCT

```
by [rewrite cap_comm -H1 comp_id_l].
rewrite prod_snd_domain1.
elim H ⇒ H1 H2.
apply inc_def1 in H1.
rewrite /domain.
by [rewrite cap_comm -H1 comp_id_l].
Qed.
```

Lemma 304 (prod_conjugate2) *Let $\gamma : A \rightarrow B \times C$ be a function, $p : B \times C \rightarrow B$ and $q : B \times C \rightarrow C$ be projections. Then,*

$$(\gamma \cdot p)^\top (\gamma \cdot q) = \gamma.$$

Lemma prod_conjugate2 $\{A\ B\ C : \text{eqType}\} \{ \text{gamma} : \text{Rel } A\ (\text{prod_eqType } B\ C) \}$:
 $\text{function_r gamma} \rightarrow \text{Rel_prod } (\text{gamma} \cdot \text{fst_r } B\ C) (\text{gamma} \cdot \text{snd_r } B\ C) = \text{gamma}.$

Proof.

```
move ⇒ H.
rewrite /Rel_prod.
rewrite comp_assoc comp_assoc -(function_cap_distr_l H).
by [rewrite fst_snd_cap_id comp_id_r].
Qed.
```

Lemma 305 (diagonal_conjugate) *Let $p : B \times C \rightarrow B$ and $q : B \times C \rightarrow C$ be projections. Then,*

$$\frac{\alpha : A \rightarrow B \quad \alpha = p^\# \cdot u \cdot q}{u \sqsubseteq \text{id}_{A \times B} \quad u = [p \cdot \alpha \sqcap q]}.$$

Lemma diagonal_conjugate $\{A\ B : \text{eqType}\} \{ \text{alpha} : \text{Rel } A\ B \}$:
 $\text{conjugate } A\ B\ (\text{prod_eqType } A\ B) (\text{prod_eqType } A\ B)$
 $\text{True_r } (\text{fun } u \Rightarrow u \quad \text{Id } (\text{prod_eqType } A\ B))$
 $(\text{fun } u \Rightarrow (\text{fst_r } A\ B \# \cdot u) \cdot \text{snd_r } A\ B)$
 $(\text{fun } \text{alpha} \Rightarrow \text{domain } ((\text{fst_r } A\ B \cdot \text{alpha}) \quad \text{snd_r } A\ B)).$

Proof.

```
split.
move ⇒ alpha0 H.
split.
apply cap_r.
rewrite cap_domain.
apply inc_antisym.
apply (@inc_trans _ _ ((fst_r A B # · ((fst_r A B · alpha0) · snd_r A B #)) · snd_r
A B)).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
```

```
apply cap_l.
rewrite comp_assoc comp_assoc -comp_assoc -(@comp_assoc _ _ _ (fst_r A B #)).
apply (@inc_trans _ _ _ ((fst_r A B # • fst_r A B) • alpha0)).
apply comp_inc_compat_ab_a.
apply snd_function.
apply comp_inc_compat_ab_b.
apply fst_function.
apply (@inc_trans _ _ _ (alpha0 ((fst_r A B # • Id (prod_eqType A B)) • snd_r A B))).
rewrite comp_id_r fst_snd_universal cap_universal.
apply inc_refl.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm inv_invol comp_assoc.
apply inc_refl.
move ⇒ u H.
split.
by [].
replace ((fst_r A B • ((fst_r A B # • u) • snd_r A B)) snd_r A B) with (u • snd_r A B).
apply domain_inc_id in H.
move : (@snd_function A B) ⇒ H0.
elim H0 ⇒ H1 H2.
by [rewrite (comp_domain3 H1) H].
rewrite comp_assoc -comp_assoc.
apply inc_antisym.
apply (@inc_trans _ _ _ ((u • snd_r A B) snd_r A B)).
apply inc_cap.
split.
apply inc_refl.
apply (comp_inc_compat_ab_b H).
apply cap_inc_compat_r.
apply comp_inc_compat_b_ab.
apply fst_function.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_b.
rewrite -fst_snd_cap_id.
apply cap_inc_compat_l.
apply comp_inc_compat_ab_ab'.
```

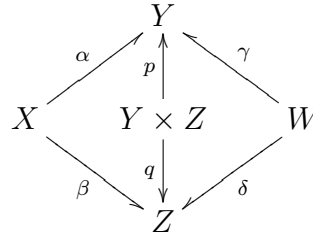
apply *inc_inv*.
 apply (*comp_inc_compat_ab_b* H).
 Qed.

13.2.3 鋭敏性

この節の補題は以下の 1 つのみだが、証明が異様に長いため単独の節を設ける。

Lemma 306 (sharpness) *In below figure,*

$$\alpha \cdot \gamma^\# \sqcap \beta \cdot \delta^\# = (\alpha \cdot p^\# \sqcap \beta \cdot q^\#) \cdot (p \cdot \gamma^\# \sqcap q \cdot \delta^\#).$$



Lemma sharpness {W X Y Z : eqType}
 {alpha : Rel X Y} {beta : Rel X Z} {gamma : Rel W Y} {delta : Rel W Z} :
 (alpha · gamma #) (beta · delta #) =
 ((alpha · fst_r Y Z #) (beta · snd_r Y Z #))
 · ((fst_r Y Z · gamma #) (snd_r Y Z · delta #)).

Proof.

apply *inc_antisym*.
 move : (rationality _ _ alpha) ⇒ H.
 move : (rationality _ _ beta) ⇒ H0.
 move : (rationality _ _ (gamma #)) ⇒ H1.
 move : (rationality _ _ (delta #)) ⇒ H2.
 elim H ⇒ R.
 elim ⇒ f0.
 elim ⇒ g0 H3.
 elim H0 ⇒ R0.
 elim ⇒ f1.
 elim ⇒ g1 H4.
 elim H1 ⇒ R1.
 elim ⇒ h0.
 elim ⇒ k0 H5.
 elim H2 ⇒ R2.
 elim ⇒ h1.
 elim ⇒ k1 H6.

CHAPTER 13. LIBRARY SUM_PRODUCT

```

move : (rationality _ _ (g0 · h0 #)) ⇒ H7.
move : (rationality _ _ (g1 · h1 #)) ⇒ H8.
move : (rationality _ _ ((alpha · gamma #) (beta · delta #))) ⇒ H9.
elim H7 ⇒ R3.
elim ⇒ s0.
elim ⇒ t0 H10.
elim H8 ⇒ R4.
elim ⇒ s1.
elim ⇒ t1 H11.
elim H9 ⇒ R5.
elim ⇒ x.
elim ⇒ z H12.
assert (alpha · gamma # = (f0 # · (s0 # · t0)) · k0).
replace alpha with (f0 # · g0).
replace (gamma #) with (h0 # · k0).
rewrite -comp_assoc (@comp_assoc _ _ _ (f0 #)).
apply f_equal2.
apply f_equal.
apply H10.
by [].
apply Logic.eq_sym.
apply H5.
apply Logic.eq_sym.
apply H3.
assert (beta · delta # = (f1 # · (s1 # · t1)) · k1).
replace beta with (f1 # · g1).
replace (delta #) with (h1 # · k1).
rewrite -comp_assoc (@comp_assoc _ _ _ (f1 #)).
apply f_equal2.
apply f_equal.
apply H11.
by [].
apply Logic.eq_sym.
apply H6.
apply Logic.eq_sym.
apply H4.
assert (t0 · h0 = s0 · g0).
apply function_inc.
apply function_comp.
apply H10.
apply H5.
apply function_comp.

```



```
apply H10.
apply H3.
apply (@inc_trans _ _ _ (s0 · ((s0 # · t0) · h0))).
rewrite comp_assoc -comp_assoc.
apply comp_inc_compat_b_ab.
apply H10.
apply comp_inc_compat_ab_ab'.
replace (s0 # · t0) with (g0 · h0 #).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H5.
apply H10.
assert (t1 · h1 = s1 · g1).
apply function_inc.
apply function_comp.
apply H11.
apply H6.
apply function_comp.
apply H11.
apply H4.
apply (@inc_trans _ _ _ (s1 · ((s1 # · t1) · h1))).
rewrite comp_assoc -comp_assoc.
apply comp_inc_compat_b_ab.
apply H11.
apply comp_inc_compat_ab_ab'.
replace (s1 # · t1) with (g1 · h1 #).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H6.
apply H11.
remember ((x · (s0 · f0) #) (z · (t0 · k0) #)) as m0.
remember ((x · (s1 · f1) #) (z · (t1 · k1) #)) as m1.
assert (total_r m0).
rewrite Heqm0.
apply domain_corollary1.
apply H12.
apply H12.
replace (x # · z) with ((alpha · gamma #) (beta · delta #)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite comp_inv H13 -comp_assoc comp_assoc.
apply inc_refl.
apply H12.
```

```
assert (total_r m1).
rewrite Heqm1.
apply domain_corollary1.
apply H12.
apply H12.
replace (x # • z) with ((alpha • gamma #) (beta • delta #)).
apply (@inc_trans - - - (cap_r)).
rewrite comp_inv H14 -comp_assoc comp_assoc.
apply inc_refl.
apply H12.
remember (m0 • (s0 • g0)) as n0.
remember (m1 • (s1 • g1)) as n1.
assert (total_r n0).
rewrite Heqn0.
apply (total_comp H17).
apply total_comp.
apply H10.
apply H3.
assert (total_r n1).
rewrite Heqn1.
apply (total_comp H18).
apply total_comp.
apply H11.
apply H4.
assert (total_r ((n0 • fst_r Y Z #) (n1 • snd_r Y Z #))).
apply (domain_corollary1 H19 H20).
rewrite fst_snd_universal.
apply inc_alpha_universal.
assert ((x # • n0) alpha).
replace alpha with (f0 # • g0).
rewrite Heqn0 Heqm0.
apply (@inc_trans - - - (((x # • x) • f0 #) • ((s0 # • s0) • g0))).
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -comp_inv.
apply cap_l.
apply comp_inc_compat.
apply comp_inc_compat_ab_b.
apply H12.
```

```
apply comp_inc_compat_ab_b.
apply H10.
apply Logic.eq_sym.
apply H3.
assert ((x # · n1) beta).
replace beta with (f1 # · g1).
rewrite Heqn1 Heqm1.
apply (@inc_trans _ _ _ (((x # · x) · f1 #) · ((s1 # · s1) · g1))).
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -comp_inv.
apply cap_l.
apply comp_inc_compat.
apply comp_inc_compat_ab_b.
apply H12.
apply comp_inc_compat_ab_b.
apply H11.
apply Logic.eq_sym.
apply H4.
assert ((n0 # · z) gamma #).
replace (gamma #) with (h0 # · k0).
rewrite Heqn0 Heqm0 -H15 comp_inv comp_inv inv_cap_distr.
apply (@inc_trans _ _ _ ((h0 # · (t0 # · t0)) · (k0 · (z # · z)))).
rewrite -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc comp_assoc comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc (@comp_inv _ _ _ z) inv_invol.
apply cap_r.
apply comp_inc_compat.
apply comp_inc_compat_ab_a.
apply H10.
apply comp_inc_compat_ab_a.
apply H12.
apply Logic.eq_sym.
apply H5.
assert ((n1 # · z) delta #).
replace (delta #) with (h1 # · k1).
```

CHAPTER 13. LIBRARY SUM_PRODUCT

```

rewrite Heqn1 Heqm1 -H16 comp_inv comp_inv inv_cap_distr.
apply (@inc_trans _ _ _ ((h1 # • (t1 # • t1)) • (k1 • (z # • z)))).
rewrite -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc comp_assoc comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc (@comp_inv _ _ _ z) inv_invol.
apply cap_r.
apply comp_inc_compat.
apply comp_inc_compat_ab_a.
apply H11.
apply comp_inc_compat_ab_a.
apply H12.
apply Logic.eq_sym.
apply H6.
replace ((alpha • gamma #) (beta • delta #)) with (x # • z).
apply (@inc_trans _ _ _ ((x # • (((n0 • fst_r Y Z #) (n1 • snd_r Y Z #)) • (((n0
• fst_r Y Z #) (n1 • snd_r Y Z #))) #)) • z)).
apply comp_inc_compat_ab_a'b.
apply (comp_inc_compat_a_ab H21).
rewrite -comp_assoc comp_assoc.
apply comp_inc_compat.
apply (@inc_trans _ _ _ _ (comp_cap_distr_l)).
apply cap_inc_compat.
rewrite -comp_assoc.
apply (comp_inc_compat_ab_a'b H22).
rewrite -comp_assoc.
apply (comp_inc_compat_ab_a'b H23).
rewrite inv_cap_distr comp_inv comp_inv inv_invol inv_invol.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat.
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H24).
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H25).
apply Logic.eq_sym.
apply H12.
apply (@inc_trans _ _ _ _ (comp_cap_distr_l)).
apply cap_inc_compat.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ (cap_l)).

```

CHAPTER 13. LIBRARY SUM_PRODUCT

```

rewrite -comp_assoc (@comp_assoc _ _ _ alpha).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a.
apply fst_function.
apply (@inc_trans _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ (cap_r)).
rewrite -comp_assoc (@comp_assoc _ _ _ beta).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a.
apply snd_function.
Qed.

```

13.2.4 分配法則

Lemma 307 (prod_cap_distr_l) *Let $\alpha : A \rightarrow B$ and $\beta, \beta' : A \rightarrow C$. Then,*

$$\alpha \top (\beta \sqcap \beta') = (\alpha \top \beta) \sqcap (\alpha \top \beta').$$

Lemma *prod_cap_distr_l* {A B C : eqType} {alpha : Rel A B} {beta beta' : Rel A C}:
Rel_prod alpha (beta beta') = Rel_prod alpha beta Rel_prod alpha beta'.

Proof.

```

rewrite /Rel_prod.
rewrite -cap_assoc (@cap_comm _ _ (alpha • fst_r B C #)) -cap_assoc cap_idem
cap_assoc.
apply f_equal.
apply function_cap_distr_r.
apply snd_function.
Qed.

```

Lemma 308 (prod_cap_distr_r) *Let $\alpha, \alpha' : A \rightarrow B$ and $\beta : A \rightarrow C$. Then,*

$$(\alpha \sqcap \alpha') \top \beta = (\alpha \top \beta) \sqcap (\alpha' \top \beta).$$

Lemma *prod_cap_distr_r* {A B C : eqType} {alpha alpha' : Rel A B} {beta : Rel A C}:
Rel_prod (alpha alpha') beta = Rel_prod alpha beta Rel_prod alpha' beta.

Proof.

```

rewrite /Rel_prod.
rewrite cap_assoc (@cap_comm _ _ (beta • snd_r B C #)) cap_assoc cap_idem -cap_assoc.
apply (@f_equal _ _ (fun x => @cap _ _ x (beta • snd_r B C #))).
apply function_cap_distr_r.
apply fst_function.
Qed.

```

Lemma 309 (prod_cup_distr_l) *Let $\alpha : A \rightarrow B$ and $\beta, \beta' : A \rightarrow C$. Then,*

$$\alpha \top (\beta \sqcup \beta') = (\alpha \top \beta) \sqcup (\alpha \top \beta').$$

Lemma *prod_cup_distr_l* {A B C : eqType} {alpha : Rel A B} {beta beta' : Rel A C}:
 $\text{Rel_prod } \alpha \text{ (beta beta')} = \text{Rel_prod } \alpha \text{ beta } \text{Rel_prod } \alpha \text{ beta'}.$

Proof.

by [rewrite -cap_cup_distr_l -comp_cup_distr_r].

Qed.

Lemma 310 (prod_cup_distr_r) *Let $\alpha, \alpha' : A \rightarrow B$ and $\beta : A \rightarrow C$. Then,*

$$(\alpha \sqcup \alpha') \top \beta = (\alpha \top \beta) \sqcup (\alpha' \top \beta).$$

Lemma *prod_cup_distr_r* {A B C : eqType} {alpha alpha' : Rel A B} {beta : Rel A C}:
 $\text{Rel_prod } (\alpha \sqcup \alpha') \text{ beta} = \text{Rel_prod } \alpha \text{ beta } \text{Rel_prod } \alpha' \text{ beta}.$

Proof.

by [rewrite -cap_cup_distr_r -comp_cup_distr_r].

Qed.

Lemma 311 (comp_prod_distr_l) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and $\gamma : B \rightarrow D$. Then,*

$$\alpha \cdot (\beta \top \gamma) \sqsubseteq \alpha \cdot \beta \top \alpha \cdot \gamma.$$

Lemma *comp_prod_distr_l*

{A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel B D}:
 $\alpha \cdot \text{Rel_prod } \beta \text{ gamma } \text{Rel_prod } (\alpha \cdot \beta) (\alpha \cdot \text{gamma}).$

Proof.

rewrite /Rel_prod.

rewrite comp_assoc comp_assoc.

apply comp_cap_distr_l.

Qed.

Lemma 312 (function_prod_distr_l) *Let $\alpha : A \rightarrow B$ be a function, $\beta : B \rightarrow C$ and $\gamma : B \rightarrow D$. Then,*

$$\alpha \cdot (\beta \top \gamma) = \alpha \cdot \beta \top \alpha \cdot \gamma.$$

Lemma *function_prod_distr_l*

{A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel B D}:
 $\text{function_r } \alpha \rightarrow \alpha \cdot \text{Rel_prod } \beta \text{ gamma} = \text{Rel_prod } (\alpha \cdot \beta) (\alpha \cdot \text{gamma}).$

Proof.

CHAPTER 13. LIBRARY SUM_PRODUCT

`move ⇒ H.`
`rewrite /Rel_prod.`
`rewrite comp_assoc comp_assoc.`
`apply (function_cap_distr_l H).`
`Qed.`

Lemma 313 (comp_prod_universal) *Let $\alpha : A \rightarrow B$, $\beta : B \rightarrow C$ and $\gamma : D \rightarrow E$. Then,*

$$\alpha \cdot (\beta \top \nabla_{BD} \cdot \gamma) = \alpha \cdot \beta \top \nabla_{AD} \cdot \gamma.$$

Lemma comp_prod_universal

$\{A\ B\ C\ D\ E : \text{eqType}\} \{alpha : \text{Rel}\ A\ B\} \{beta : \text{Rel}\ B\ C\} \{gamma : \text{Rel}\ D\ E\} :$
 $alpha \cdot \text{Rel_prod}\ beta\ (\quad B\ D \cdot gamma) = \text{Rel_prod}\ (alpha \cdot beta)\ (\quad A\ D \cdot gamma).$

Proof.

`apply inc_antisym.`
`apply (@inc_trans _ _ _ _ (comp_prod_distr_l)).`
`apply prod_inc_compat_l.`
`rewrite -comp_assoc.`
`apply comp_inc_compat_ab_a'b.`
`apply inc_alpha_universal.`
`rewrite /Rel_prod.`
`rewrite comp_assoc.`
`apply (@inc_trans _ _ _ _ (dedekind1)).`
`apply comp_inc_compat_ab_ab'.`
`apply cap_inc_compat_l.`
`rewrite comp_assoc comp_assoc -comp_assoc.`
`apply comp_inc_compat_ab_a'b.`
`apply inc_alpha_universal.`
`Qed.`

Lemma 314 (fst_cap_snd_distr) *Let $u, v : A \times B \rightarrow A \times B$ and $u, v \sqsubseteq id_{A \times B}$, $p : B \times C \rightarrow B$ and $q : B \times C \rightarrow C$ be projections. Then,*

$$p^\sharp \cdot (u \sqcap v) \cdot q = p^\sharp \cdot u \cdot q \sqcap p^\sharp \cdot v \cdot q.$$

Lemma fst_cap_snd_distr

$\{A\ B : \text{eqType}\} \{u\ v : \text{Rel}\ (prod_eqType\ A\ B)\ (prod_eqType\ A\ B)\} :$
 $u \quad Id\ (prod_eqType\ A\ B) \rightarrow v \quad Id\ (prod_eqType\ A\ B) \rightarrow$
 $fst_r\ A\ B\ \# \cdot (u \quad v) \cdot snd_r\ A\ B =$
 $((fst_r\ A\ B\ \# \cdot u) \cdot snd_r\ A\ B) \quad ((fst_r\ A\ B\ \# \cdot v) \cdot snd_r\ A\ B).$

Proof.

`move ⇒ H H0.`
`apply inc_antisym.`

CHAPTER 13. LIBRARY SUM_PRODUCT

```

apply (fun H' ⇒ @inc_trans _ _ _ _ H' (comp_cap_distr_r)).
apply comp_inc_compat_ab_a'b.
apply comp_cap_distr_l.
apply (@inc_trans _ _ _ _ (dedekind1)).
rewrite -(dedekind_id3 H H0) -(@comp_assoc _ _ _ _ u) (@comp_assoc _ _ _ _ (fst_r A
B # • u) v).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_b.
rewrite comp_inv comp_inv inv_invol -fst_snd_cap_id.
apply cap_inc_compat.
rewrite comp_assoc (dedekind_id1 H).
apply (comp_inc_compat_ab_b H).
rewrite -comp_assoc (dedekind_id1 H0).
apply (comp_inc_compat_ab_a H0).
Qed.

```


Chapter 14

Library **Point_Axiom**

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Logic.IndefiniteDescription.
```

14.1 I-点

14.1.1 I-点の定義

Dedekind 圏における域 X の I-点 x とは, 関数 $x : I \rightarrow X$ のことであり, 記号 $x \in X$ によって表される. また関係 $\rho : I \rightarrow X$ と I-点 $x : I \rightarrow X$ に対して, 記号 $x \in \rho$ で $x \sqsubseteq \rho$ を表すものとする.

ちなみに I-点の定義 $x \in X$ は $x \in \nabla_{IX}$ と言い換えることも可能である.

Definition *point_inc* $\{X : eqType\} (x \text{ rho} : Rel \ i \ X) := function_r \ x \wedge x \sqsubseteq \text{rho}.$

Definition *point* $\{X : eqType\} (x : Rel \ i \ X) := \text{point_inc} \ x \ (\ i \ X).$

14.1.2 I-点の性質

Lemma 315 (point_property1) *Let $x, y \in X$. Then,*

$$x = y \Leftrightarrow x \cdot y^\# = id_I.$$

Lemma *point_property1* $\{X : eqType\} \{x \ y : Rel \ i \ X\}:$
point $x \rightarrow \text{point} \ y \rightarrow (x = y \Leftrightarrow x \cdot y^\# = Id \ i).$

Proof.

```

move ⇒ H H0.
split; move ⇒ H1.
apply inc_antisym.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
rewrite H1.
apply H0.
apply Logic.eq_sym.
apply function_inc.
apply H0.
apply H.
rewrite -(@comp_id_l _ _ y) -H1 comp_assoc.
apply comp_inc_compat_ab_a.
apply H0.
Qed.

```

Lemma 316 (point_property2a, point_property2b) *Let $\rho : I \rightarrow X$ be a total relation. Then,*

$$\rho \cdot \rho^\# = \rho \cdot \nabla_{XI} = id_I.$$

Lemma point_property2a $\{X : eqType\} \{rho : Rel\ i\ X\}$:
total_r rho → rho · rho # = Id i.

Proof.

```

move ⇒ H.
apply inc_antisym.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
apply H.
Qed.

```

Lemma point_property2b $\{X : eqType\} \{rho : Rel\ i\ X\}$:
total_r rho → rho · rho # = rho · ∇_{XI} i.

Proof.

```

move ⇒ H.
apply inc_antisym.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
rewrite (point_property2a H) unit_identity_is_universal.
apply inc_alpha_universal.
Qed.

```

CHAPTER 14. LIBRARY POINT_AXIOM

Lemma 317 (point_property3) *Let $\rho : I \rightarrow X$. Then,*

$$\exists x \in \rho \Rightarrow “\rho \text{ is total}” \wedge \rho \neq \phi_{IX}.$$

Lemma *point_property3* $\{X : eqType\} \{rho : Rel\ i\ X\}$:
 $(\exists x : Rel\ i\ X, point_inc\ x\ rho) \rightarrow total_r\ rho \wedge rho \neq \phi_{IX}$.

Proof.

elim $\Rightarrow x\ H$.
 assert $(total_r\ rho)$.
 elim $H \Rightarrow H0\ H1$.
 elim $H0 \Rightarrow H2\ H3$.
 apply $(@inc_trans\ _ _ _ _ H2)$.
 apply *comp_inc_compat*.
 apply $H1$.
 apply $(@inc_inv\ _ _ _ _ H1)$.
 split.
 apply $H0$.
 move $\Rightarrow H1$.
 rewrite $/total_r$ in $H0$.
 rewrite $H1\ comp_empty_l$ in $H0$.
 apply *unit_identity_not_empty*.
 apply *inc_antisym*.
 apply $H0$.
 apply *inc_empty_alpha*.

Qed.

Lemma 318 (point_property4)

$$\exists x \in X \Rightarrow “\nabla_{IX} \text{ is total}” \wedge \nabla_{IX} \neq \phi_{IX}.$$

Lemma *point_property4* $\{X : eqType\}$:
 $(\exists x : Rel\ i\ X, point\ x) \rightarrow total_r\ (_ \ i\ X) \wedge (_ \ i\ X) \neq \phi_{IX}$.

Proof.

move $\Rightarrow H$.
 apply $(@point_property3\ _ (_ \ i\ X)\ H)$.

Qed.

14.2 I-点に関する諸公理

14.2.1 点公理

この“点公理”を使えば, I-点に関する様々な定理や補題が導出できる.

Lemma 319 (point_axiom) *Let $\rho : I \rightarrow X$. Then,*

$$\rho = \sqcup_{x \in \rho} x.$$

Lemma lemma_for_PA $\{X : eqType\} \{rho : Rel\ i\ X\}$:
 $((rho = _ i\ X) \rightarrow False) \rightarrow False \rightarrow rho = _ i\ X$.

Proof.

```
move => H.
case (@unit_empty_or_universal (rho · rho #)) => H0.
apply inc_antisym.
apply (@inc_trans _ _ _ (relation_rel_inv_rel)).
rewrite H0 comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
apply False_ind.
apply H.
move => H1.
rewrite H1 comp_empty_l in H0.
apply (unit_empty_not_universal H0).
```

Qed.

Lemma point_axiom $\{X : eqType\} \{rho : Rel\ i\ X\}$:
 $rho = _ \{fun\ x : Rel\ i\ X \Rightarrow point_inc\ x\ rho\} id$.

Proof.

```
apply inc_antisym.
apply bool_lemma2.
assert ((∃ x : Rel i X, point_inc x ((\_ {fun x : Rel i X => point_inc x rho} id)
( \_ {fun x : Rel i X => point_inc x rho} id) ^)) → False).
move => H.
move : (point_property3 H) => H0.
apply H0.
apply cap_complement_empty.
assert ((∃ x : Rel i X, point_inc x (rho ( \_ {fun x : Rel i X => point_inc x rho} id)
^)) → False).
move => H0.
apply H.
elim H0 => x H1.
```

```

∃ x.
split.
apply H1.
apply inc_cap.
split.
assert (point_inc x rho).
split.
apply H1.
elim H1 ⇒ H2 H3.
apply inc_cap in H3.
apply H3.
clear H1.
move : x H2.
apply inc_cupP.
apply inc_refl.
elim H1 ⇒ H2 H3.
apply inc_cap in H3.
apply H3.
apply lemma_for_PA.
move ⇒ H1.
apply H0.
apply axiom_of_choice.
rewrite /total_r.
remember (rho ( _ {fun x : Rel i X ⇒ point_inc x rho} id) ^) as rho'.
case (@unit_empty_or_universal (rho' • rho' #)) ⇒ H2.
apply False_ind.
apply H1.
apply inc_antisym.
apply (@inc_trans _ _ _ (relation_rel_inv_rel)).
rewrite H2 comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
rewrite H2.
apply inc_alpha_universal.
apply inc_cupP.
move ⇒ beta H.
apply H.
Qed.

```

Lemma 320 (PA_corollary1)

$$\nabla_{IX} = \sqcup_{x \in X} x.$$

CHAPTER 14. LIBRARY POINT_AXIOM

Lemma *PA_corollary1* {*X* : *eqType*}: $i\ X = _ \{point\}\ id$.

Proof.

apply *point_axiom*.

Qed.

Lemma 321 (PA_corollary2)

$$id_X = \sqcup_{x \in X} x^\# \cdot x.$$

Lemma *PA_corollary2* {*X* : *eqType*}:

$Id\ X = _ \{point\}\ (\text{fun } x : Rel\ i\ X \Rightarrow x \# \cdot x)$.

Proof.

rewrite -(@cap_universal _ _ (Id X)) -lemma_for_tarski2 *PA_corollary1*.

rewrite *comp_cupP_distr_l cap_cupP_distr_l*.

apply *cupP_eq*.

move \Rightarrow *alpha H*.

apply *inc_antisym*.

rewrite *cap_comm*.

apply (@inc_trans _ _ _ _ (dedekind2)).

rewrite *comp_id_l cap_comm cap_universal*.

apply *inc_refl*.

apply *inc_cap*.

split.

apply *H*.

apply *comp_inc_compat_ab_a'b*.

apply *inc_alpha_universal*.

Qed.

Lemma 322 (PA_corollary3) *Let* $\alpha, \beta : X \rightarrow Y$. *Then,*

$$(\forall x \in X, x \cdot \alpha = x \cdot \beta) \Rightarrow \alpha = \beta.$$

Lemma *PA_corollary3* {*X Y* : *eqType*} {*alpha beta* : *Rel X Y*}:

$(\forall x : Rel\ i\ X, point\ x \rightarrow x \cdot \alpha = x \cdot \beta) \rightarrow \alpha = \beta$.

Proof.

move \Rightarrow *H*.

rewrite -(@comp_id_l _ _ *alpha*) -(@comp_id_l _ _ *beta*) *PA_corollary2*.

rewrite *comp_cupP_distr_r comp_cupP_distr_r*.

apply *cupP_eq*.

move \Rightarrow *gamma H0*.

by [rewrite *comp_assoc comp_assoc (H gamma H0)*].

Qed.

Lemma 323 (PA_corollary4) *Let $\alpha : X \rightarrow Y$. Then,*

$$“\alpha \text{ is total}” \Leftrightarrow \forall x \in X, “x \cdot \alpha \text{ is total}”.$$

Lemma PA_corollary4 $\{X\ Y : eqType\} \{alpha : Rel\ X\ Y\}$:
 $total_r\ alpha \Leftrightarrow \forall x : Rel\ i\ X, point\ x \rightarrow total_r\ (x \cdot alpha).$

Proof.

split; move $\Rightarrow H$.
 move $\Rightarrow x\ H0$.
 apply $total_comp$.
 apply $H0$.
 apply H .
 rewrite $/total_r$.
 rewrite $PA_corollary2$.
 apply inc_cupP .
 move $\Rightarrow x\ H0$.
 move : $(H\ x\ H0) \Rightarrow H1$.
 apply $(@inc_trans\ _ _ _ ((x\ \# \cdot ((x \cdot alpha) \cdot (x \cdot alpha)\ \#))) \cdot x))$.
 apply $comp_inc_compat_ab_a'b$.
 apply $(comp_inc_compat_a_ab\ H1)$.
 rewrite $comp_inv\ -comp_assoc\ -comp_assoc\ -comp_assoc$.
 rewrite $comp_assoc\ (@comp_assoc\ _ _ _ _ (x\ \# \cdot x))$.
 apply $(@inc_trans\ _ _ _ ((x\ \# \cdot x) \cdot (alpha \cdot alpha\ \#)))$.
 apply $comp_inc_compat_ab_a$.
 apply $H0$.
 apply $comp_inc_compat_ab_b$.
 apply $H0$.

Qed.

Lemma 324 (PA_corollary5) *Let $\alpha : X \rightarrow Y$. Then,*

$$“\alpha \text{ is univalent}” \Leftrightarrow \forall x \in X, “x \cdot \alpha \text{ is univalent}”.$$

Lemma PA_corollary5 $\{X\ Y : eqType\} \{alpha : Rel\ X\ Y\}$:
 $univalent_r\ alpha \Leftrightarrow \forall x : Rel\ i\ X, point\ x \rightarrow univalent_r\ (x \cdot alpha).$

Proof.

split; move $\Rightarrow H$.
 move $\Rightarrow x\ H0$.
 apply $univalent_comp$.
 apply $H0$.
 apply H .
 rewrite $/univalent_r$.
 rewrite $-(@comp_id_r\ _ _ (alpha\ \#))\ PA_corollary2$.

```

rewrite comp_cupP_distr_l comp_cupP_distr_r.
apply inc_cupP.
move  $\Rightarrow x$  H0.
move : ( $H$   $x$  H0)  $\Rightarrow$  H1.
rewrite -comp_assoc -comp_inv comp_assoc.
apply H1.
Qed.

```

14.2.2 全域性公理

Lemma 325 (total_axiom) *Let $\rho : I \rightarrow X$. Then,*

$$\rho \neq \phi_{IX} \Rightarrow id_I = \rho \cdot \rho^\#.$$

Lemma *total_axiom* $\{X : eqType\} \{rho : Rel\ i\ X\}$:
 $rho \neq \quad i\ X \rightarrow Id\ i = rho \cdot rho \#$.

Proof.

```

move  $\Rightarrow H$ .
case (@unit_empty_or_universal (rho · rho #))  $\Rightarrow$  H0.
apply False_ind.
apply H.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (relation_rel_inv_rel)).
rewrite H0 comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
by [rewrite H0 unit_identity_is_universal].
Qed.

```

Lemma 326 (Tot_corollary1) *Let $\rho : I \rightarrow X$ and $x \in X$. Then,*

$$\rho \sqsubseteq x \Rightarrow \rho = \phi_{IX} \vee \rho = x.$$

Lemma *Tot_corollary1* $\{X : eqType\} \{rho\ x : Rel\ i\ X\}$:
 $point\ x \rightarrow rho \quad x \rightarrow rho = \quad i\ X \vee rho = x$.

Proof.

```

move  $\Rightarrow H$  H0.
case (@unit_empty_or_universal (rho · rho #))  $\Rightarrow$  H1.
left.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (relation_rel_inv_rel)).
rewrite H1 comp_empty_l.

```



```

apply inc_refl.
apply inc_empty_alpha.
right.
apply inc_antisym.
apply H0.
rewrite -(@comp_id_l _ _ x) unit_identity_is_universal -H1 comp_assoc.
apply (@inc_trans _ _ _ (rho • (x # • x))).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply (@inc_inv _ _ _ H0).
apply comp_inc_compat_ab_a.
apply H.
Qed.

```

Lemma 327 (Tot_corollary2) *Let $x, y \in X$. Then,*

$$x \neq y \Leftrightarrow x \cdot y^\# = \phi_I.$$

Lemma Tot_corollary2 $\{X : eqType\} \{x y : Rel\ i\ X\}$:
 $point\ x \rightarrow point\ y \rightarrow (x \neq y \Leftrightarrow x \cdot y^\# = \phi_I).$

Proof.

```

move => H H0.
assert (x = y <-> x • y # ≠ ϕ_I).
rewrite (point_property1 H H0).
split; move => H1.
rewrite H1.
apply unit_identity_not_empty.
case (@unit_empty_or_universal (x • y #)) => H2.
apply False_ind.
apply (H1 H2).
by [rewrite H2 unit_identity_is_universal].
rewrite H1.
split; move => H2.
apply (lemma_for_PA H2).
move => H3.
apply (H3 H2).
Qed.

```

Lemma 328 (Tot_corollary3) *Let $f : (I \rightarrow X) \rightarrow (I \rightarrow Y)$. Then,*

$$(\forall x \in X, "f(x) \text{ is a function}") \Rightarrow "\sqcup_{x \in X} x^\# \cdot f(x) \text{ is a function}."$$

Lemma Tot_corollary3 $\{X\ Y : eqType\} \{f : Rel\ i\ X \rightarrow Rel\ i\ Y\}$:

CHAPTER 14. LIBRARY POINT_AXIOM

$(\forall x : \text{Rel } i \ X, \text{point } x \rightarrow \text{function_r } (f \ x)) \rightarrow \text{function_r } (_ \{ \text{point} \} (\text{fun } x : \text{Rel } i \ X \Rightarrow x \# \cdot f \ x))$.

Proof.

move $\Rightarrow H$.

assert $(\forall x : \text{Rel } i \ X, \text{point } x \rightarrow x \cdot (_ \{ \text{point} \} (\text{fun } x0 : \text{Rel } i \ X \Rightarrow x0 \# \cdot f \ x0)) = f \ x)$.

move $\Rightarrow x \ H0$.

assert $(x \cdot x \# = \text{Id } i)$.

apply *inc_antisym*.

rewrite *unit_identity_is_universal*.

apply *inc_alpha_universal*.

apply *H0*.

rewrite $(\text{@comp_id_l } _ _ (f \ x)) \text{-} H1$.

apply *inc_antisym*.

rewrite *comp_cupP_distr_l*.

apply *inc_cupP*.

move $\Rightarrow y \ H2$.

rewrite *comp_assoc*.

case $(\text{@unit_empty_or_universal } (x \cdot y \#)) \Rightarrow H3$.

rewrite *H3 comp_empty_l*.

apply *inc_empty_alpha*.

rewrite *unit_identity_is_universal* in *H3*.

apply $(\text{point_property1 } H0 \ H2)$ in *H3*.

rewrite *H3*.

apply *inc_refl*.

rewrite *comp_assoc*.

apply *comp_inc_compat_ab_ab'*.

clear *H1*.

move : $x \ H0$.

apply *inc_cupP*.

apply *inc_refl*.

split.

rewrite *PA_corollary4*.

move $\Rightarrow x \ H1$.

rewrite $(H0 \ x \ H1)$.

apply $(H \ x \ H1)$.

rewrite *PA_corollary5*.

move $\Rightarrow x \ H1$.

rewrite $(H0 \ x \ H1)$.

apply $(H \ x \ H1)$.

Qed.

14.2.3 その他の公理

Lemma 329 (nonempty_axiom) *Let $\rho : I \rightarrow X$. Then,*

$$\rho \neq \phi_{IX} \Rightarrow \exists x \in \rho.$$

Lemma *nonempty_axiom* $\{X : eqType\} \{rho : Rel\ i\ X\}$:
 $rho \neq \phi_{IX} \rightarrow \exists x : Rel\ i\ X, point_inc\ x\ rho.$

Proof.

move : (@axiom_of_choice _ _ rho) \Rightarrow H .

move \Rightarrow $H0$.

apply H .

rewrite /total_r.

rewrite (total_axiom $H0$).

apply inc_refl.

Qed.

Lemma 330 (axiom_of_subobjects) *Let $\rho : I \rightarrow X$. Then,*

$$\exists S, \exists j : S \rightarrow X, \rho = \nabla_{IS} \cdot j \wedge j \cdot j^\# = id_S.$$

Lemma *axiom_of_subobjects* $\{X : eqType\} \{rho : Rel\ i\ X\}$:
 $\exists (S : eqType)(j : Rel\ S\ X), rho = \nabla_{IS} \cdot j \wedge j \cdot j^\# = Id\ S.$

Proof.

elim (@rationality _ _ rho) \Rightarrow R .

elim \Rightarrow f .

elim \Rightarrow g .

elim \Rightarrow H .

elim \Rightarrow $H0$.

elim \Rightarrow $H1\ H2$.

$\exists\ R$.

$\exists\ g$.

split.

rewrite $H1$.

apply inc_antisym.

apply comp_inc_compat_ab_a'b.

apply inc_alpha_universal.

apply comp_inc_compat_ab_a'b.

apply (@inc_trans _ _ _ (i R \cdot (f \cdot f^\#))).

apply comp_inc_compat_a_ab.

apply H .

rewrite -comp_assoc.

apply comp_inc_compat_ab_b.

```
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
rewrite -H2 cap_comm inc_def1.
assert ((f # • g) rho).
rewrite H1.
apply inc_refl.
apply (function_move1 H) in H3.
apply (@inc_trans _ _ _ ((f • rho) • (f • rho) #)).
apply comp_inc_compat.
apply H3.
apply (@inc_inv _ _ _ H3).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ rho).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
Qed.
```

Bibliography

- [1] R. Affeldt and M. Hagiwara. Formalization of Shannon 's Theorems in SSReflect-Coq. In 3rd Conference on Interactive Theorem Proving, LNCS 7406, 233–249, 2012.