

INSTITUTE OF MATHEMATICS FOR INDUSTRY,  
KYUSHU UNIVERSITY

LOGIC AND COMPUTATION PROJECT

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# Coq Modules for Relational Calculus

(Ver.0.1)

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**Repository:** <https://github.com/KyushuUniversityMathematics/RelationalCalculus>

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# Chapter 1

## Library `Basic_Notations`

### 1.1 このライブラリについて

- このライブラリは河原康雄先生の“関係の理論 - Dedekind 圏概説 -”をもとに制作されている。
- 現状サポートしているのは,
  - 1.4 節大半, 1.5 - 1.6 節全部
  - 2.1 - 2.3 節全部, 2.4 - 2.5 節大半, 2.6 節全部, 2.7 節大半, 2.8 節有理性
  - 4.2 - 4.3 節全部, 4.4 - 4.5 節大半, 4.6 節命題 4.6.1, 4.7 節大半, 4.8 節命題 4.8.1, 4.9 節全部
  - 5.1 節大半, 5.2 - 5.3 節一部

といったところである。

第 3 章以外でカバーしていない箇所は, 基礎的もしくは自明な補題, Coq での定式化が難しい定義や補題, 証明の正当性が示しきれなかった補題, 汎用性の低い一部の記号などである。

- 関係論で話を進めたい場合は, 下の行に `Require Export Basic_Notations_Rel.` を, 集合論で話を進めたい場合は, `Require Export Basic_Notations_Set.` を記述する。

`Require Export Basic_Notations_Rel.`

なお, 証明の書き方が悪いと, まれに“関係論では証明が通ったのに, 集合論では通らない”といったことも起こるようなので, ある程度注意しておく必要がある。

# Chapter 2

## Library `Basic_Notations_Rel`

`Require Export ssreflect eqtype bigop.`

`Require Export Logic.ClassicalFacts.`

`Axiom prop_extensionality_ok : prop_extensionality.`

### 2.1 定義

- $A, B$  を `eqType` として,  $A$  から  $B$  への関係の型を  $(\text{Rel } A B)$  と書き,  $A \rightarrow B \rightarrow \text{Prop}$  として定義する. 本文中では型  $(\text{Rel } A B)$  を  $A \rightarrow B$  と書く.
- 関係  $\alpha : A \rightarrow B$  の逆関係  $\alpha^\# : B \rightarrow A$  は  $(\text{inverse } \alpha)$  で, Coq では  $(\alpha \#)$  と記述する.
- 2つの関係  $\alpha : A \rightarrow B, \beta : B \rightarrow C$  の合成関係  $\alpha \cdot \beta$  (or  $\alpha\beta$ ) :  $A \rightarrow C$  は  $(\text{composite } \alpha \beta)$  で,  $(\alpha \cdot \beta)$  と記述する.
- 剰余合成関係  $\alpha \triangleright \beta : A \rightarrow C$  は  $(\text{residual } \alpha \beta)$  で,  $(\alpha \triangleright \beta)$  と記述する.
- 恒等関係  $\text{id}_A : A \rightarrow A$  は  $(\text{identity } A)$  で,  $(\text{Id } A)$  と記述する.
- 空関係  $\phi_{AB} : A \rightarrow B$  は  $(\text{empty } A B)$  で,  $(\phi_{AB})$  と記述する.
- 全関係  $\nabla_{AB} : A \rightarrow B$  は  $(\text{universal } A B)$  で,  $(\nabla_{AB})$  と記述する.
- 2つの関係  $\alpha : A \rightarrow B, \beta : A \rightarrow B$  の和関係  $\alpha \sqcup \beta : A \rightarrow B$  は  $(\text{cup } \alpha \beta)$  で,  $(\alpha \sqcup \beta)$  と記述する.
- 共通関係  $\alpha \sqcap \beta : A \rightarrow B$  は  $(\text{cap } \alpha \beta)$  で,  $(\alpha \sqcap \beta)$  と記述する.
- 相対擬補関係  $\alpha \Rightarrow \beta : A \rightarrow B$  は  $(\text{rpc } \alpha \beta)$  で,  $(\alpha \Rightarrow \beta)$  と記述する.
- 関係  $\alpha : A \rightarrow B$  の補関係  $\alpha^- : A \rightarrow B$  は  $(\text{complement } \alpha)$  で, Coq では  $(\alpha \sim)$  と記述する.

	数式	Coq	Notation
逆関係	$\alpha^\#$	(inverse $\alpha$ )	( $\alpha \#$ )
合成関係	$\alpha \cdot \beta$	(composite $\alpha \beta$ )	( $\alpha \cdot \beta$ )
剰余合成関係	$\alpha \triangleright \beta$	(residual $\alpha \beta$ )	( $\alpha \triangleright \beta$ )
恒等関係	$\text{id}_A$	(identity $A$ )	(Id $A$ )
空関係	$\phi_{AB}$	(empty $A B$ )	( $\emptyset A B$ )
全関係	$\nabla_{AB}$	(universal $A B$ )	( $\forall A B$ )
和関係	$\alpha \sqcup \beta$	(cup $\alpha \beta$ )	( $\alpha \sqcup \beta$ )
共通関係	$\alpha \sqcap \beta$	(cap $\alpha \beta$ )	( $\alpha \sqcap \beta$ )
相対擬補関係	$\alpha \Rightarrow \beta$	(rpc $\alpha \beta$ )	( $\alpha \Rightarrow \beta$ )
補関係	$\alpha^-$	(complement $\alpha$ )	( $\alpha^-$ )
差関係	$\alpha - \beta$	(difference $\alpha \beta$ )	( $\alpha - \beta$ )
添字付和関係	$\sqcup_{P(\alpha)} f(\alpha)$	(cupP $P f$ )	( $\sqcup_{\{P\}} f$ )
添字付共通関係	$\sqcap_{P(\alpha)} f(\alpha)$	(capP $P f$ )	( $\sqcap_{\{P\}} f$ )

Table 2.1: 関係の表記について

- 2 つの関係  $\alpha : A \rightarrow B$ ,  $\beta : A \rightarrow B$  の差関係  $\alpha - \beta : A \rightarrow B$  は (difference  $\alpha \beta$ ) で, ( $\alpha - \beta$ ) と記述する.
- (cupP) と (capP) は添字付の和関係と共通関係であり, 述語  $P$  に対し,  $\{f(\alpha) \mid P(\alpha)\}$  の和関係, 共通関係を表す.
- また, 1 点集合  $I = \{*\}$  は  $i$  と表記する.
- なお, 通常 of 共通関係, 和関係も添字付のもので表現することができるため, ここではそれを用いて表記する.
- 後で述べるように, 剰余合成  $\alpha \triangleright \beta$  も  $(\alpha \cdot \beta^-)^-$  のように表現することは可能だが, “剰余合成が存在すれば, それは  $(\alpha \cdot \beta^-)^-$  に等しい” というレベルのものであるため, 剰余合成に関する公理はやはり必要となる.

表 2.1 に関係の表記についてまとめる.

**Definition**  $Rel (A B : eqType) := A \rightarrow B \rightarrow Prop$ .

**Parameter**  $inverse : (\forall A B : eqType, Rel A B \rightarrow Rel B A)$ .

**Notation** " $a \#$ "  $:= (inverse \_ a)$  (at level 20).

**Parameter**  $composite : (\forall A B C : eqType, Rel A B \rightarrow Rel B C \rightarrow Rel A C)$ .

**Notation** " $a \cdot \cdot b$ "  $:= (composite \_ \_ a b)$  (at level 50).

**Parameter**  $residual : (\forall A B C : eqType, Rel A B \rightarrow Rel B C \rightarrow Rel A C)$ .

**Notation** " $a \triangleright b$ "  $:= (residual \_ \_ a b)$  (at level 50).

**Parameter**  $identity : (\forall A : eqType, Rel A A)$ .

**Notation** " $\text{Id}$ "  $:= identity$ .

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**Parameter** *empty* :  $(\forall A B : eqType, Rel A B)$ .

**Notation** "'''" := *empty*.

**Parameter** *universal* :  $(\forall A B : eqType, Rel A B)$ .

**Notation** "'''" := *universal*.

**Parameter** *include* :  $(\forall A B : eqType, Rel A B \rightarrow Rel A B \rightarrow Prop)$ .

**Notation** "a' ' b" := (*include* \_ \_ a b) (at level 50).

**Parameter** *cupP* :  $(\forall A B C D : eqType, (Rel C D \rightarrow Prop) \rightarrow (Rel C D \rightarrow Rel A B) \rightarrow Rel A B)$ .

**Notation** "'' \_{ ' p ' }' f" := (*cupP* \_ \_ \_ p f) (at level 50).

**Parameter** *capP* :  $(\forall A B C D : eqType, (Rel C D \rightarrow Prop) \rightarrow (Rel C D \rightarrow Rel A B) \rightarrow Rel A B)$ .

**Notation** "'' \_{ ' p ' }' f" := (*capP* \_ \_ \_ p f) (at level 50).

**Definition** *cup* {*A B* : *eqType*} (*alpha beta* : *Rel A B*)

:= \_{fun *gamma* : *Rel A B*  $\Rightarrow$  *gamma* = *alpha*  $\vee$  *gamma* = *beta*} *id*.

**Notation** "a' ' b" := (*cup* a b) (at level 50).

**Definition** *cap* {*A B* : *eqType*} (*alpha beta* : *Rel A B*)

:= \_{fun *gamma* : *Rel A B*  $\Rightarrow$  *gamma* = *alpha*  $\vee$  *gamma* = *beta*} *id*.

**Notation** "a' ' b" := (*cap* a b) (at level 50).

**Parameter** *rpc* :  $(\forall A B : eqType, Rel A B \rightarrow Rel A B \rightarrow Rel A B)$ .

**Notation** "a' »' b" := (*rpc* \_ \_ a b) (at level 50).

**Definition** *complement* {*A B* : *eqType*} (*alpha* : *Rel A B*) := *alpha* » *A B*.

**Notation** "a' ^'" := (*complement* a) (at level 20).

**Definition** *difference* {*A B* : *eqType*} (*alpha beta* : *Rel A B*) := *alpha* *beta* ^.

**Notation** "a - b" := (*difference* a b) (at level 50).

**Notation** "i'" := *unit\_eqType*.



## 2.2 関数の定義

$\alpha : A \rightarrow B$  に対し, 全域性 `total_r`, 一価性 `univalent_r`, 関数 `function_r`, 全射 `surjective_r`, 単射 `injective_r`, 全単射 `bijection_r` を以下のように定義する.

- `total_r` :  $id_A \sqsubseteq \alpha \cdot \alpha^\#$
- `univalent_r` :  $\alpha^\# \cdot \alpha \sqsubseteq id_B$
- `function_r` :  $id_A \sqsubseteq \alpha \cdot \alpha^\# \wedge \alpha^\# \cdot \alpha \sqsubseteq id_B$
- `surjection_r` :  $id_A \sqsubseteq \alpha \cdot \alpha^\# \wedge \alpha^\# \cdot \alpha = id_B$
- `injection_r` :  $id_A = \alpha \cdot \alpha^\# \wedge \alpha^\# \cdot \alpha \sqsubseteq id_B$
- `bijection_r` :  $id_A = \alpha \cdot \alpha^\# \wedge \alpha^\# \cdot \alpha = id_B$

**Definition** `total_r`  $\{A\ B : eqType\} (alpha : Rel\ A\ B) := (Id\ A) \quad (alpha \cdot alpha^\#)$ .

**Definition** `univalent_r`  $\{A\ B : eqType\} (alpha : Rel\ A\ B) := (alpha^\# \cdot alpha) \quad (Id\ B)$ .

**Definition** `function_r`  $\{A\ B : eqType\} (alpha : Rel\ A\ B)$   
 $:= (total\_r\ alpha) \wedge (univalent\_r\ alpha)$ .

**Definition** `surjection_r`  $\{A\ B : eqType\} (alpha : Rel\ A\ B)$   
 $:= (function\_r\ alpha) \wedge (total\_r\ (alpha^\#))$ .

**Definition** `injection_r`  $\{A\ B : eqType\} (alpha : Rel\ A\ B)$   
 $:= (function\_r\ alpha) \wedge (univalent\_r\ (alpha^\#))$ .

**Definition** `bijection_r`  $\{A\ B : eqType\} (alpha : Rel\ A\ B)$   
 $:= (function\_r\ alpha) \wedge (total\_r\ (alpha^\#)) \wedge (univalent\_r\ (alpha^\#))$ .

## 2.3 関係の公理

今後の諸定理の証明は, 原則以下の公理群, およびそれらから導かれる補題のみを用いて行っていくことにする.

### 2.3.1 Dedekind 圏の公理

**Axiom 1** (`comp_id_l`, `comp_id_r`) *Let*  $\alpha : A \rightarrow B$ . *Then,*

$$id_A \cdot \alpha = \alpha \cdot id_B = \alpha.$$

**Definition** `axiom1a`  $:= \forall (A\ B : eqType) (alpha : Rel\ A\ B), Id\ A \cdot alpha = alpha$ .

**Axiom** `comp_id_l` : `axiom1a`.

**Definition** `axiom1b`  $:= \forall (A\ B : eqType) (alpha : Rel\ A\ B), alpha \cdot Id\ B = alpha$ .

**Axiom** `comp_id_r` : `axiom1b`.

**Axiom 2 (comp\_assoc)** Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ , and  $\gamma : C \rightarrow D$ . Then,

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

**Definition** *axiom2* :=

$\forall (A\ B\ C\ D : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ C\ D),$   
 $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$

**Axiom** *comp\_assoc* : *axiom2*.

**Axiom 3 (inc\_refl)** Let  $\alpha : A \rightarrow B$ . Then,

$$\alpha \sqsubseteq \alpha.$$

**Definition** *axiom3* :=  $\forall (A\ B : eqType)(\alpha : Rel\ A\ B), \alpha \sqsubseteq \alpha.$

**Axiom** *inc\_refl* : *axiom3*.

**Axiom 4 (inc\_trans)** Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma \Rightarrow \alpha \sqsubseteq \gamma.$$

**Definition** *axiom4* :=  $\forall (A\ B : eqType)(\alpha\ \beta\ \gamma : Rel\ A\ B),$   
 $\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma \Rightarrow \alpha \sqsubseteq \gamma.$

**Axiom** *inc\_trans* : *axiom4*.

**Axiom 5 (inc\_antisym)** Let  $\alpha, \beta : A \rightarrow B$ . Then,

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \alpha \Rightarrow \alpha = \beta.$$

**Definition** *axiom5* :=  $\forall (A\ B : eqType)(\alpha\ \beta : Rel\ A\ B),$   
 $\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \alpha \Rightarrow \alpha = \beta.$

**Axiom** *inc\_antisym* : *axiom5*.

**Axiom 6 (inc\_empty\_alpha)** Let  $\alpha : A \rightarrow B$ . Then,

$$\phi_{AB} \sqsubseteq \alpha.$$

**Definition** *axiom6* :=  $\forall (A\ B : eqType)(\alpha : Rel\ A\ B), \phi_{AB} \sqsubseteq \alpha.$

**Axiom** *inc\_empty\_alpha* : *axiom6*.

**Axiom 7 (inc\_alpha\_universal)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \nabla_{AB}.$$

**Definition** *axiom7* :=  $\forall (A B : eqType)(\alpha : Rel A B), \alpha \sqsubseteq \nabla_{AB}$ .

**Axiom** *inc\_alpha\_universal* : *axiom7*.

**Axiom 8 (inc\_capP, inc\_cap)**

1. **inc\_capP** : *Let  $\alpha : A \rightarrow B$ ,  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P$  : predicate. Then,*

$$\alpha \sqsubseteq (\cap_{P(\beta)} f(\beta)) \Leftrightarrow \forall \beta : C \rightarrow D, P(\beta) \Rightarrow \alpha \sqsubseteq f(\beta).$$

2. **inc\_cap** : *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq (\beta \cap \gamma) \Leftrightarrow \alpha \sqsubseteq \beta \wedge \alpha \sqsubseteq \gamma.$$

**Definition** *axiom8a* :=

$\forall (A B C D : eqType)$   
 $(\alpha : Rel A B)(f : Rel C D \rightarrow Rel A B)(P : Rel C D \rightarrow Prop),$   
 $\alpha \sqsubseteq (\cap_{P} f) \Leftrightarrow \forall \beta : Rel C D, P \beta \rightarrow \alpha \sqsubseteq f \beta.$

**Axiom** *inc\_capP* : *axiom8a*.

**Definition** *axiom8b* :=  $\forall (A B : eqType)(\alpha \beta \gamma : Rel A B),$   
 $\alpha \sqsubseteq (\beta \cap \gamma) \Leftrightarrow (\alpha \sqsubseteq \beta) \wedge (\alpha \sqsubseteq \gamma).$

**Lemma** *inc\_cap* : *axiom8b*.

**Proof.**

move  $\Rightarrow A B \alpha \beta \gamma$ .

rewrite *inc\_capP*.

split; move  $\Rightarrow H$ .

split; apply *H*.

by [left].

by [right].

move  $\Rightarrow \delta H0$ .

case *H0*  $\Rightarrow H1$ ; rewrite *H1*; apply *H*.

**Qed.**

**Axiom 9 (inc\_cupP, inc\_cup)**

1. **inc\_cupP** : Let  $\alpha : A \rightarrow B$ ,  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,

$$(\sqcup_{P(\beta)} f(\beta)) \sqsubseteq \alpha \Leftrightarrow \forall \beta : C \rightarrow D, P(\beta) \Rightarrow f(\beta) \sqsubseteq \alpha.$$

2. **inc\_cup** : Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,

$$(\beta \sqcup \gamma) \sqsubseteq \alpha \Leftrightarrow \beta \sqsubseteq \alpha \wedge \gamma \sqsubseteq \alpha.$$

**Definition** *axiom9a* :=

$\forall (A\ B\ C\ D : \text{eqType})$   
 $(\alpha : \text{Rel } A\ B)(f : \text{Rel } C\ D \rightarrow \text{Rel } A\ B)(P : \text{Rel } C\ D \rightarrow \text{Prop}),$   
 $(\_ \{P\} f) \quad \alpha \leftrightarrow \forall \text{beta} : \text{Rel } C\ D, P\ \text{beta} \rightarrow f\ \text{beta} \quad \alpha.$

**Axiom** *inc\_cupP* : *axiom9a*.

**Definition** *axiom9b* :=  $\forall (A\ B : \text{eqType})(\alpha\ \text{beta}\ \gamma : \text{Rel } A\ B),$   
 $(\text{beta}\ \gamma) \quad \alpha \leftrightarrow (\text{beta}\ \alpha) \wedge (\gamma\ \alpha).$

**Lemma** *inc\_cup* : *axiom9b*.

**Proof.**

move  $\Rightarrow A\ B\ \alpha\ \text{beta}\ \gamma$ .

rewrite *inc\_cupP*.

split; move  $\Rightarrow H$ .

split; apply *H*.

by [left].

by [right].

move  $\Rightarrow \text{delta } H0$ .

case *H0*  $\Rightarrow H1$ ; rewrite *H1*; apply *H*.

**Qed.**

**Axiom 10 (inc\_rpc)** Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,

$$\alpha \sqsubseteq (\beta \Rightarrow \gamma) \Leftrightarrow (\alpha \sqcap \beta) \sqsubseteq \gamma.$$

**Definition** *axiom10* :=  $\forall (A\ B : \text{eqType})(\alpha\ \text{beta}\ \gamma : \text{Rel } A\ B),$   
 $\alpha \sqsubseteq (\text{beta} \gg \gamma) \leftrightarrow (\alpha\ \text{beta}) \sqsubseteq \gamma.$

**Axiom** *inc\_rpc* : *axiom10*.

**Axiom 11 (inv\_invol)** Let  $\alpha : A \rightarrow B$ . Then,

$$(\alpha^\#)^\# = \alpha.$$

**Definition** *axiom11* :=  $\forall (A\ B : \text{eqType})(\alpha : \text{Rel } A\ B), (\alpha^\#)^\# = \alpha.$

**Axiom** *inv\_invol* : *axiom11*.

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**Axiom 12 (comp\_inv)** Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then,

$$(\alpha \cdot \beta)^\# = \beta^\# \cdot \alpha^\#.$$

**Definition** *axiom12* :=  $\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C),$   
 $(\alpha \cdot \beta)^\# = (\beta^\# \cdot \alpha^\#).$

**Axiom** *comp\_inv* : *axiom12*.

**Axiom 13 (inc\_inv)** Let  $\alpha, \beta : A \rightarrow B$ . Then,

$$\alpha \sqsubseteq \beta \Rightarrow \alpha^\# \sqsubseteq \beta^\#.$$

**Definition** *axiom13* :=

$\forall (A\ B : eqType)(\alpha\ \beta : Rel\ A\ B), \alpha \rightarrow \beta \Rightarrow \alpha^\# \rightarrow \beta^\#.$

**Axiom** *inc\_inv* : *axiom13*.

**Axiom 14 (dedekind)** Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ , and  $\gamma : A \rightarrow C$ . Then,

$$(\alpha \cdot \beta) \sqcap \gamma \sqsubseteq (\alpha \sqcap (\gamma \cdot \beta^\#)) \cdot (\beta \sqcap (\alpha^\# \cdot \gamma)).$$

**Definition** *axiom14* :=

$\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ A\ C),$   
 $((\alpha \cdot \beta) \sqcap \gamma) \sqsubseteq ((\alpha \sqcap (\gamma \cdot \beta^\#)) \cdot (\beta \sqcap (\alpha^\# \cdot \gamma))).$

**Axiom** *dedekind* : *axiom14*.

**Axiom 15 (inc\_residual)** Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ , and  $\gamma : A \rightarrow C$ . Then,

$$\gamma \sqsubseteq (\alpha \triangleright \beta) \Leftrightarrow \alpha^\# \cdot \gamma \sqsubseteq \beta.$$

**Definition** *axiom15* :=

$\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ A\ C),$   
 $\gamma \sqsubseteq (\alpha \triangleright \beta) \Leftrightarrow (\alpha^\# \cdot \gamma) \sqsubseteq \beta.$

**Axiom** *inc\_residual* : *axiom15*.

## 2.3.2 排中律

Dedekind 圏の公理のほかに, 以下の“排中律”を仮定すれば, 与えられる圏は Schröder 圏となり, Bool 代数の性質も満たされる. ちなみに剰余合成は補関係から定義可能なので, 本来 Schröder 圏には剰余合成に関する公理は存在しない.

**Axiom 16 (complement\_classic)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqcup \alpha^- = \nabla_{AB}$$

**Definition** *axiom16* :=  $\forall (A B : eqType)(\alpha : Rel A B),$   
 $\alpha \sqcup \alpha^- = \nabla_{AB}.$

**Axiom** *complement\_classic* : *axiom16.*

## 2.3.3 単域

1 点集合  $I$  が単域となるための条件は

$$\phi_{II} \neq id_I \wedge id_I = \nabla_{II} \wedge \forall X, \nabla_{XI} \cdot \nabla_{IX} = \nabla_{XX}$$

だが, Rel の定義から左 2 つは証明できるため, 右の式だけ仮定する.

**Axiom 17 (unit\_universal)**

$$\nabla_{AI} \cdot \nabla_{IA} = \nabla_{AA}$$

**Definition** *axiom17* :=  $\forall (A : eqType), \nabla_{AI} \cdot \nabla_{IA} = \nabla_{AA}.$

**Axiom** *unit\_universal* : *axiom17.*

## 2.3.4 選択公理

この“選択公理”を仮定すれば, 排中律と単域の存在 (厳密には全域性公理) を利用して点公理を導出できる.

**Axiom 18 (axiom\_of\_choice)** *Let  $\alpha : A \rightarrow B$  be a total relation. Then,*

$$\exists \beta : A \rightarrow B, \beta \sqsubseteq \alpha.$$

**Definition** *axiom18* :=  $\forall (A B : eqType)(\alpha : Rel A B),$   
 $total\_r \alpha \rightarrow \exists \beta : Rel A B, function\_r \beta \wedge \beta \sqsubseteq \alpha.$

**Axiom** *axiom\_of\_choice* : *axiom18.*

## 2.3.5 関係の有理性

集合論では色々インポートしながら頑張って証明したので、できればそちらもご覧ください。

**Axiom 19 (rationality)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\exists R, \exists f : R \rightarrow A, \exists g : R \rightarrow B, \alpha = f^\# \cdot g \wedge f \cdot f^\# \sqcap g \cdot g^\# = id_R.$$

**Definition** *axiom19* :=  $\forall (A B : eqType)(alpha : Rel A B),$

$\exists (R : eqType)(f : Rel R A)(g : Rel R B),$

$function\_r f \wedge function\_r g \wedge alpha = f^\# \cdot g \wedge ((f \cdot f^\#) \sqcap (g \cdot g^\#)) = Id R.$

**Axiom** *rationality* : *axiom19*.

## 2.3.6 直和と直積

任意の直和に対して、入射対が存在することを仮定する。

**Axiom 20 (pair\_of\_inclusions)**  $\exists j : A \rightarrow A + B, \exists k : B \rightarrow A + B,$

$$j \cdot j^\# = id_A \wedge k \cdot k^\# = id_B \wedge j \cdot k^\# = \phi_{AB} \wedge j^\# \cdot j \sqcup k^\# \cdot k = id_{A+B}.$$

**Definition** *axiom20* :=

$\forall (A B : eqType), \exists (j : Rel A (sum\_eqType A B))(k : Rel B (sum\_eqType A B)),$

$j \cdot j^\# = Id A \wedge k \cdot k^\# = Id B \wedge j \cdot k^\# = \phi_{AB} \wedge$

$(j^\# \cdot j) \sqcup (k^\# \cdot k) = Id (sum\_eqType A B).$

**Axiom** *pair\_of\_inclusions* : *axiom20*.

任意の直積に対して、射影対が存在することを仮定する。

実は有理性公理 (Axiom 19) があれば直積の公理は必要ないのだが、Axiom 19 の準用では直積が “存在する” ことまでしか示してくれないので、“直積として  $prod\_eqType A B$  を用いてよい” ことを公理の中に含めたものを用意しておく。

**Axiom 21 (pair\_of\_projections)**  $\exists p : A \times B \rightarrow A, \exists q : A \times B \rightarrow B,$

$$p^\# \cdot q = \nabla_{AB} \wedge p \cdot p^\# \sqcap q \cdot q^\# = id_{A \times B}.$$

**Definition** *axiom21* :=

$\forall (A B : eqType), \exists (p : Rel (prod\_eqType A B) A)(q : Rel (prod\_eqType A B) B),$

$p^\# \cdot q = \nabla_{AB} \wedge (p \cdot p^\#) \sqcap (q \cdot q^\#) = Id (prod\_eqType A B) \wedge univalent\_r p$

$\wedge univalent\_r q.$

**Axiom** *pair\_of\_projections* : *axiom21*.

# Chapter 3

## Library `Basic_Notations_Set`

```
Require Export ssreflect eqtype bigop.
Require Export Logic.ClassicalFacts.
Require Import Logic.FunctionalExtensionality.
Require Import Logic.Classical_Prop.
Require Import Logic.IndefiniteDescription.
Require Import Logic.ProofIrrelevance.

Axiom prop_extensionality_ok : prop_extensionality.
```

### 3.1 定義

この章では、関係を集合論的に定義した場合の定義、およびその定義で諸公理が成立することを示す。公理名や記号などは `Basic_Notations` と同じものを使用するため、`Basic_Lemms` 以降ではその代わりにこのライブラリをインポートすることもできる。

```
Definition Rel (A B : eqType) := A → B → Prop.

Definition inverse {A B : eqType} (alpha : Rel A B) : Rel B A
:= (fun (b : B)(a : A) => alpha a b).

Notation "a #" := (inverse a) (at level 20).

Definition composite {A B C : eqType} (alpha : Rel A B) (beta : Rel B C) : Rel A C
:= (fun (a : A)(c : C) => ∃ b : B, alpha a b ∧ beta b c).

Notation "a ' · ' b" := (composite a b) (at level 50).

Definition residual {A B C : eqType} (alpha : Rel A B) (beta : Rel B C) : Rel A C
:= (fun (a : A)(c : C) => ∀ b : B, alpha a b → beta b c).

Notation "a ' ' b" := (residual a b) (at level 50).

Definition identity (A : eqType) : Rel A A := (fun a a0 : A => a = a0).

Notation "'Id'" := identity.

Definition empty (A B : eqType) : Rel A B := (fun (a : A)(b : B) => False).

Notation "' ' " := empty.

Definition universal (A B : eqType) : Rel A B := (fun (a : A)(b : B) => True).
```



**Notation** "''" := *universal*.

**Definition** *include*  $\{A B : eqType\}$  ( $\alpha \beta : Rel A B$ ) : **Prop**  
:=  $(\forall (a : A)(b : B), \alpha a b \rightarrow \beta a b)$ .

**Notation** "a' ' b" := (*include a b*) (at level 50).

**Definition** *cupP*  $\{A B C D : eqType\}$  ( $P : Rel C D \rightarrow Prop$ ) ( $f : Rel C D \rightarrow Rel A B$ ) :  $Rel A B$   
:=  $(\text{fun } (a : A)(b : B) \Rightarrow \exists \alpha : Rel C D, P \alpha \wedge (f \alpha) a b)$ .

**Notation** "'' -{' p '}' f" := (*cupP p f*) (at level 50).

**Definition** *capP*  $\{A B C D : eqType\}$  ( $P : Rel C D \rightarrow Prop$ ) ( $f : Rel C D \rightarrow Rel A B$ ) :  $Rel A B$   
:=  $(\text{fun } (a : A)(b : B) \Rightarrow \forall \alpha : Rel C D, P \alpha \rightarrow (f \alpha) a b)$ .

**Notation** "'' -{' p '}' f" := (*capP p f*) (at level 50).

**Definition** *cup*  $\{A B : eqType\}$  ( $\alpha \beta : Rel A B$ )  
:=  $\_ \{ \text{fun } \gamma : Rel A B \Rightarrow \gamma = \alpha \vee \gamma = \beta \} id$ .

**Notation** "a' ' b" := (*cup a b*) (at level 50).

**Definition** *cap*  $\{A B : eqType\}$  ( $\alpha \beta : Rel A B$ )  
:=  $\_ \{ \text{fun } \gamma : Rel A B \Rightarrow \gamma = \alpha \vee \gamma = \beta \} id$ .

**Notation** "a' ' b" := (*cap a b*) (at level 50).

**Definition** *rpc*  $\{A B : eqType\}$  ( $\alpha \beta : Rel A B$ ) :  $Rel A B$   
:=  $(\text{fun } (a : A)(b : B) \Rightarrow \alpha a b \rightarrow \beta a b)$ .

**Notation** "a' » b" := (*rpc a b*) (at level 50).

**Definition** *complement*  $\{A B : eqType\}$  ( $\alpha : Rel A B$ ) :=  $\alpha \gg A B$ .

**Notation** "a' ^" := (*complement a*) (at level 20).

**Definition** *difference*  $\{A B : eqType\}$  ( $\alpha \beta : Rel A B$ ) :=  $\alpha \beta ^$ .

**Notation** "a - b" := (*difference a b*) (at level 50).

**Notation** "i'" := *unit\_eqType*.

## 3.2 関数の定義

**Definition** *total\_r*  $\{A B : eqType\}$  ( $\alpha : Rel A B$ ) :=  $(Id A) \quad (\alpha \cdot \alpha \#)$ .

**Definition** *univalent\_r*  $\{A B : eqType\}$  ( $\alpha : Rel A B$ ) :=  $(\alpha \# \cdot \alpha) \quad (Id B)$ .

**Definition** *function\_r*  $\{A B : eqType\}$  ( $\alpha : Rel A B$ )  
:=  $(total\_r \alpha) \wedge (univalent\_r \alpha)$ .

**Definition** *surjection\_r*  $\{A B : eqType\}$  ( $\alpha : Rel A B$ )  
:=  $(function\_r \alpha) \wedge (total\_r (\alpha \#))$ .

**Definition** *injection\_r*  $\{A B : eqType\}$  ( $\alpha : Rel A B$ )  
:=  $(function\_r \alpha) \wedge (univalent\_r (\alpha \#))$ .

**Definition** *bijection\_r*  $\{A B : eqType\}$  ( $\alpha : Rel A B$ )  
:=  $(function\_r \alpha) \wedge (total\_r (\alpha \#)) \wedge (univalent\_r (\alpha \#))$ .

### 3.3 関係の公理

今後の諸定理の証明は、原則以下の公理群、およびそれらから導かれる補題のみを用いて行っていくことにする。

#### 3.3.1 Dedekind 圏の公理

**Lemma 1** (`comp_id_l`, `comp_id_r`) *Let  $\alpha : A \rightarrow B$ . Then,*

$$id_A \cdot \alpha = \alpha \cdot id_B = \alpha.$$

**Definition** `axiom1a` :=  $\forall (A\ B : eqType)(\alpha : Rel\ A\ B),\ Id\ A \cdot \alpha = \alpha$ .

**Lemma** `comp_id_l` : `axiom1a`.

**Proof.**

`move  $\Rightarrow A\ B\ \alpha$ .`

`apply functional_extensionality.`

`move  $\Rightarrow a$ .`

`apply functional_extensionality.`

`move  $\Rightarrow b$ .`

`apply prop_extensionality_ok.`

`split.`

`elim  $\Rightarrow a0$ .`

`elim  $\Rightarrow H\ H0$ .`

`rewrite  $H$ .`

`apply  $H0$ .`

`move  $\Rightarrow H$ .`

`$\exists\ a$ .`

`split.`

`by [].`

`apply  $H$ .`

**Qed.**

**Definition** `axiom1b` :=  $\forall (A\ B : eqType)(\alpha : Rel\ A\ B),\ \alpha \cdot Id\ B = \alpha$ .

**Lemma** `comp_id_r` : `axiom1b`.

**Proof.**

`move  $\Rightarrow A\ B\ \alpha$ .`

`apply functional_extensionality.`

`move  $\Rightarrow a$ .`

`apply functional_extensionality.`

`move  $\Rightarrow b$ .`

`apply prop_extensionality_ok.`

`split.`

`elim  $\Rightarrow b0$ .`

```

elim  $\Rightarrow$   $H$   $H0$ .
rewrite  $-H0$ .
apply  $H$ .
move  $\Rightarrow$   $H$ .
 $\exists$   $b$ .
split.
apply  $H$ .
by  $\square$ .
Qed.

```

**Lemma 2 (comp\_assoc)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ , and  $\gamma : C \rightarrow D$ . Then,*

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

**Definition** *axiom2* :=

$\forall (A\ B\ C\ D : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ C\ D),$   
 $(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$

**Lemma** *comp\_assoc* : *axiom2*.

**Proof.**

```

move  $\Rightarrow$   $A\ B\ C\ D\ \alpha\ \beta\ \gamma$ .
apply functional_extensionality.
move  $\Rightarrow$   $a$ .
apply functional_extensionality.
move  $\Rightarrow$   $d$ .
apply prop_extensionality_ok.
split.
elim  $\Rightarrow$   $c$ .
elim  $\Rightarrow$   $H\ H0$ .
elim  $H \Rightarrow$   $b\ H1$ .
 $\exists$   $b$ .
split.
apply  $H1$ .
 $\exists$   $c$ .
split.
apply  $H1$ .
apply  $H0$ .
elim  $\Rightarrow$   $b$ .
elim  $\Rightarrow$   $H$ .
elim  $\Rightarrow$   $c\ H0$ .
 $\exists$   $c$ .
split.
 $\exists$   $b$ .
split.

```

apply *H*.  
 apply *H0*.  
 apply *H0*.  
 Qed.

**Lemma 3 (inc\_refl)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \alpha.$$

**Definition** *axiom3* :=  $\forall (A B : eqType)(\alpha : Rel A B), \alpha \sqsubseteq \alpha$ .

**Lemma** *inc\_refl* : *axiom3*.

**Proof.**

by [rewrite /*axiom3*/include].

Qed.

**Lemma 4 (inc\_trans)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma \Rightarrow \alpha \sqsubseteq \gamma.$$

**Definition** *axiom4* :=  $\forall (A B : eqType)(\alpha \beta \gamma : Rel A B),$   
 $\alpha \sqsubseteq \beta \rightarrow \beta \sqsubseteq \gamma \rightarrow \alpha \sqsubseteq \gamma$ .

**Lemma** *inc\_trans* : *axiom4*.

**Proof.**

move  $\Rightarrow A B \alpha \beta \gamma H H0 a b H1$ .

apply (*H0* \_ \_ (*H* \_ \_ *H1*)).

Qed.

**Lemma 5 (inc\_antisym)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \alpha \Rightarrow \alpha = \beta.$$

**Definition** *axiom5* :=  $\forall (A B : eqType)(\alpha \beta : Rel A B),$   
 $\alpha \sqsubseteq \beta \rightarrow \beta \sqsubseteq \alpha \rightarrow \alpha = \beta$ .

**Lemma** *inc\_antisym* : *axiom5*.

**Proof.**

move  $\Rightarrow A B \alpha \beta H H0$ .

apply *functional\_extensionality*.

move  $\Rightarrow a$ .

apply *functional\_extensionality*.

move  $\Rightarrow b$ .

apply *prop\_extensionality\_ok*.

split.

apply *H*.

apply *H0*.

**Qed.**

**Lemma 6 (inc\_empty\_alpha)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\phi_{AB} \sqsubseteq \alpha.$$

**Definition** *axiom6* :=  $\forall (A\ B : eqType)(\alpha : Rel\ A\ B), \quad A\ B \quad \alpha$ .

**Lemma** *inc\_empty\_alpha* : *axiom6*.

**Proof.**

move  $\Rightarrow A\ B\ \alpha\ a\ b$ .

apply *False\_ind*.

**Qed.**

**Lemma 7 (inc\_alpha\_universal)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \nabla_{AB}.$$

**Definition** *axiom7* :=  $\forall (A\ B : eqType)(\alpha : Rel\ A\ B), \alpha \sqsubseteq \nabla_{AB}$ .

**Lemma** *inc\_alpha\_universal* : *axiom7*.

**Proof.**

move  $\Rightarrow A\ B\ \alpha\ a\ b\ H$ .

apply *I*.

**Qed.**

**Lemma 8 (inc\_capP, inc\_cap)**

1. **inc\_capP** : *Let  $\alpha : A \rightarrow B$ ,  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P$  : predicate. Then,*

$$\alpha \sqsubseteq (\sqcap_{P(\beta)} f(\beta)) \Leftrightarrow \forall \beta : C \rightarrow D, P(\beta) \Rightarrow \alpha \sqsubseteq f(\beta).$$

2. **inc\_cap** : *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq (\beta \sqcap \gamma) \Leftrightarrow \alpha \sqsubseteq \beta \wedge \alpha \sqsubseteq \gamma.$$

**Definition** *axiom8a* :=

$\forall (A\ B\ C\ D : eqType)$

$(\alpha : Rel\ A\ B)(f : Rel\ C\ D \rightarrow Rel\ A\ B)(P : Rel\ C\ D \rightarrow \mathbf{Prop}),$

$\alpha \sqsubseteq (\sqcap_{\{P\}} f) \Leftrightarrow \forall \beta : Rel\ C\ D, P\ \beta \rightarrow \alpha \sqsubseteq f\ \beta.$

**Lemma** *inc\_capP* : *axiom8a*.

**Proof.**

move  $\Rightarrow A\ B\ C\ D\ \alpha\ f\ P$ .

split; move  $\Rightarrow H$ .

```

move ⇒ beta H0 a b H1.
apply (H _ _ H1 _ H0).
move ⇒ a b H0 beta H1.
apply (H _ H1 _ _ H0).
Qed.
Definition axiom8b := ∀ (A B : eqType) (alpha beta gamma : Rel A B),
  alpha (beta gamma) ↔ (alpha beta) ∧ (alpha gamma).
Lemma inc_cap : axiom8b.
Proof.
move ⇒ A B alpha beta gamma.
rewrite inc_capP.
split; move ⇒ H.
split; apply H.
by [left].
by [right].
move ⇒ delta H0.
case H0 ⇒ H1; rewrite H1; apply H.
Qed.

```

#### Lemma 9 (inc\_cupP, inc\_cup)

1. **inc\_cupP** : Let  $\alpha : A \rightarrow B$ ,  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,

$$(\sqcup_{P(\beta)} f(\beta)) \sqsubseteq \alpha \Leftrightarrow \forall \beta : C \rightarrow D, P(\beta) \Rightarrow f(\beta) \sqsubseteq \alpha.$$

2. **inc\_cup** : Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,

$$(\beta \sqcup \gamma) \sqsubseteq \alpha \Leftrightarrow \beta \sqsubseteq \alpha \wedge \gamma \sqsubseteq \alpha.$$

```

Definition axiom9a :=
  ∀ (A B C D : eqType)
  (alpha : Rel A B) (f : Rel C D → Rel A B) (P : Rel C D → Prop),
  ( _ {P} f ) alpha ↔ ∀ beta : Rel C D, P beta → f beta alpha.

```

Lemma inc\_cupP : axiom9a.

Proof.

```

move ⇒ A B C D alpha f P.
split.
move ⇒ H beta H0 a b H1.
apply H.
∃ beta.
split.
apply H0.
apply H1.

```

```

move ⇒ H a b.
elim ⇒ beta.
elim ⇒ H0 H1.
apply (H beta H0 _ _ H1).
Qed.
Definition axiom9b := ∀ (A B : eqType) (alpha beta gamma : Rel A B),
  (beta gamma) alpha ↔ (beta alpha) ∧ (gamma alpha).
Lemma inc_cup : axiom9b.
Proof.
move ⇒ A B alpha beta gamma.
rewrite inc_cupP.
split; move ⇒ H.
split; apply H.
by [left].
by [right].
move ⇒ delta H0.
case H0 ⇒ H1; rewrite H1; apply H.
Qed.

```

**Lemma 10 (inc\_rpc)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq (\beta \Rightarrow \gamma) \Leftrightarrow (\alpha \sqcap \beta) \sqsubseteq \gamma.$$

```

Definition axiom10 := ∀ (A B : eqType) (alpha beta gamma : Rel A B),
  alpha (beta » gamma) ↔ (alpha beta) gamma.
Lemma inc_rpc : axiom10.
Proof.
move ⇒ A B alpha beta gamma.
split; move ⇒ H.
move ⇒ a b H0.
apply H.
apply H0.
by [left].
apply H0.
by [right].
move ⇒ a b H0 H1.
apply H.
move ⇒ delta.
case ⇒ H2; rewrite H2.
apply H0.
apply H1.
Qed.

```

## CHAPTER 3. LIBRARY BASIC\_NOTATIONS\_SET

**Lemma 11 (inv\_invol)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$(\alpha^\#)^\# = \alpha.$$

**Definition** *axiom11* :=  $\forall (A\ B : eqType)(\alpha : Rel\ A\ B), (\alpha^\#)^\# = \alpha$ .

**Lemma** *inv\_invol* : *axiom11*.

**Proof.**

by [move  $\Rightarrow A\ B\ \alpha$ ].

**Qed.**

**Lemma 12 (comp\_inv)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then,*

$$(\alpha \cdot \beta)^\# = \beta^\# \cdot \alpha^\#.$$

**Definition** *axiom12* :=  $\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C), (\alpha \cdot \beta)^\# = (\beta^\# \cdot \alpha^\#)$ .

**Lemma** *comp\_inv* : *axiom12*.

**Proof.**

move  $\Rightarrow A\ B\ C\ \alpha\ \beta$ .

apply *functional\_extensionality*.

move  $\Rightarrow c$ .

apply *functional\_extensionality*.

move  $\Rightarrow a$ .

apply *prop\_extensionality\_ok*.

split; elim  $\Rightarrow b$ .

elim  $\Rightarrow H\ H0$ .

$\exists b$ .

split.

apply *H0*.

apply *H*.

elim  $\Rightarrow H\ H0$ .

$\exists b$ .

split.

apply *H0*.

apply *H*.

**Qed.**

**Lemma 13 (inc\_inv)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \beta \Rightarrow \alpha^\# \sqsubseteq \beta^\#.$$

**Definition** *axiom13* :=

$\forall (A\ B : eqType)(\alpha\ \beta : Rel\ A\ B), \alpha \sqsubseteq \beta \rightarrow \alpha^\# \sqsubseteq \beta^\#$ .



**Lemma** *inc\_inv* : *axiom13*.

**Proof.**

move  $\Rightarrow A\ B\ \text{alpha}\ \text{beta}\ H\ b\ a\ H0$ .

apply ( $H\ \_ \_ H0$ ).

**Qed.**

**Lemma 14 (dedekind)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ , and  $\gamma : A \rightarrow C$ . Then,*

$$(\alpha \cdot \beta) \sqcap \gamma \sqsubseteq (\alpha \sqcap (\gamma \cdot \beta^\#)) \cdot (\beta \sqcap (\alpha^\# \cdot \gamma)).$$

**Definition** *axiom14* :=

$\forall (A\ B\ C : \text{eqType})(\text{alpha} : \text{Rel}\ A\ B)(\text{beta} : \text{Rel}\ B\ C)(\text{gamma} : \text{Rel}\ A\ C),$   
 $((\text{alpha} \cdot \text{beta}) \sqcap \text{gamma})$   
 $((\text{alpha} \sqcap (\text{gamma} \cdot \text{beta}^\#)) \cdot (\text{beta} \sqcap (\text{alpha}^\# \cdot \text{gamma}))).$

**Lemma** *dedekind* : *axiom14*.

**Proof.**

move  $\Rightarrow A\ B\ C\ \text{alpha}\ \text{beta}\ \text{gamma}\ a\ c\ H$ .

assert ( $\exists\ b : B, \text{alpha}\ a\ b \wedge \text{beta}\ b\ c$ ).

apply  $H$ .

by [left].

elim  $H0 \Rightarrow b$ .

elim  $\Rightarrow H1\ H2$ .

$\exists\ b$ .

repeat split.

move  $\Rightarrow \text{delta}\ H3$ .

case  $H3 \Rightarrow H4$ ; rewrite  $H4$ .

apply  $H1$ .

unfold *id*.

$\exists\ c$ .

split.

apply  $H$ .

by [right].

apply  $H2$ .

move  $\Rightarrow \text{delta}\ H3$ .

case  $H3 \Rightarrow H4$ ; rewrite  $H4$ .

apply  $H2$ .

unfold *id*.

$\exists\ a$ .

split.

apply  $H1$ .

apply  $H$ .

by [right].

**Qed.**

**Lemma 15 (inc\_residual)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ , and  $\gamma : A \rightarrow C$ . Then,*

$$\gamma \sqsubseteq (\alpha \triangleright \beta) \Leftrightarrow \alpha^\# \cdot \gamma \sqsubseteq \beta.$$

**Definition** *axiom15* :=

$\forall (A\ B\ C : eqType)(\alpha : Rel\ A\ B)(\beta : Rel\ B\ C)(\gamma : Rel\ A\ C),$   
 $\gamma \sqsubseteq (\alpha \triangleright \beta) \Leftrightarrow \alpha^\# \cdot \gamma \sqsubseteq \beta.$

**Lemma** *inc\_residual* : *axiom15*.

**Proof.**

move  $\Rightarrow A\ B\ C\ \alpha\ \beta\ \gamma$ .

split; move  $\Rightarrow H$ .

move  $\Rightarrow b\ c$ .

elim  $\Rightarrow a\ H0$ .

apply  $(H\ a)$ .

apply  $H0$ .

apply  $H0$ .

move  $\Rightarrow a\ c\ H0\ b\ H1$ .

apply  $H$ .

$\exists\ a$ .

split.

apply  $H1$ .

apply  $H0$ .

**Qed.**

### 3.3.2 排中律

Dedekind 圏の公理のほかに、以下の“排中律”を仮定すれば、与えられる圏は Schröder 圏となり、Bool 代数の性質も満たされる。ちなみに剰余合成は補関係から定義可能なので、本来 Schröder 圏には剰余合成に関する公理は存在しない。

**Lemma 16 (complement\_classic)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqcup \alpha^- = \nabla_{AB}$$

**Definition** *axiom16* :=  $\forall (A\ B : eqType)(\alpha : Rel\ A\ B),$   
 $\alpha \sqcup \alpha^- = \nabla_{AB}.$

**Lemma** *complement\_classic* : *axiom16*.

**Proof.**

move  $\Rightarrow A\ B\ \alpha$ .

apply *functional\_extensionality*.

move  $\Rightarrow a$ .

apply *functional\_extensionality*.

```

move ⇒ b.
apply prop_extensionality_ok.
split; move ⇒ H.
apply I.
case (classic (alpha a b)) ⇒ H0.
∃ alpha.
split.
by [left].
apply H0.
∃ (fun (a0 : A) (b0 : B) ⇒ alpha a0 b0 → False).
split.
by [right].
apply H0.
Qed.

```

### 3.3.3 単域

1 点集合  $I$  が単域となるための条件は

$$\phi_{II} \neq id_I \wedge id_I = \nabla_{II} \wedge \forall X, \nabla_{XI} \cdot \nabla_{IX} = \nabla_{XX}$$

だが, Rel の定義から左 2 つは証明できるため, 右の式だけ仮定する.

**Lemma 17** (*unit\_universal*)

$$\nabla_{AI} \cdot \nabla_{IA} = \nabla_{AA}$$

**Definition** *axiom17* :=  $\forall (A : eqType), \quad A \text{ i } \bullet \quad i A = \quad A A$ .

**Lemma** *unit\_universal* : *axiom17*.

**Proof.**

```

move ⇒ A.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ a0.
apply prop_extensionality_ok.
split; move ⇒ H.
apply I.
∃ tt.
by [].
Qed.

```

## 3.3.4 選択公理

この“選択公理”を仮定すれば、排中律と単域の存在（厳密には全域性公理）を利用して点公理を導出できる。証明には集合論の選択公理を用いる。

**Lemma 18 (axiom\_of\_choice)** *Let  $\alpha : A \rightarrow B$  be a total relation. Then,*

$$\exists \beta : A \rightarrow B, \beta \sqsubseteq \alpha.$$

**Definition** *axiom18* :=  $\forall (A B : \text{eqType})(\alpha : \text{Rel } A B),$   
 $\text{total\_r } \alpha \rightarrow \exists \text{beta} : \text{Rel } A B, \text{function\_r beta} \wedge \text{beta} \sqsubseteq \alpha.$

**Lemma** *axiom\_of\_choice* : *axiom18*.

**Proof.**

move  $\Rightarrow A B \alpha$ .

rewrite  $/\text{function\_r}/\text{total\_r}/\text{univalent\_r}/\text{identity}/\text{include}/\text{composite}/\text{inverse}.$

move  $\Rightarrow H.$

assert  $(\forall a : A, \{b : B \mid \alpha a b\}).$

move  $\Rightarrow a.$

apply *constructive\_indefinite\_description*.

move :  $(H a a (\text{Logic.eq\_refl } a)).$

elim  $\Rightarrow b H0.$

$\exists b.$

apply *H0*.

$\exists (\text{fun } (a : A)(b : B) \Rightarrow b = \text{sval } (X a)).$

repeat split.

move  $\Rightarrow a a0 H0.$

$\exists (\text{sval } (X a)).$

by [rewrite *H0*].

move  $\Rightarrow b b0.$

elim  $\Rightarrow a.$

elim  $\Rightarrow H0 H1.$

by [rewrite *H0 H1*].

move  $\Rightarrow a b H0.$

rewrite *H0*.

apply *proj2\_sig*.

**Qed.**

## 3.3.5 関係の有理性

集合の選択公理 (`Logic.IndefiniteDescription`) や証明の一意性 (`Logic.ProofIrrelevance`) を仮定すれば, 集合論上ならごり押しで証明できる. 旧ライブラリの頃は無理だと諦めて `Axiom` を追加していたが, `Standard Library` のインポートだけで解けた. 正直びっくり.

**Lemma 19 (rationality)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\exists R, \exists f : R \rightarrow A, \exists g : R \rightarrow B, \alpha = f^\# \cdot g \wedge f \cdot f^\# \sqcap g \cdot g^\# = id_R.$$

この付近は, ごり押しのための補題. 命題の真偽を選択公理で `bool` 値に変換したり, 部分集合の元から上位集合の元を生成する `sval (proj1_sig)` の単射性を示したりしている.

**Lemma `is_true_inv0`** :  $\forall P : \text{Prop}, \exists b : \text{bool}, P \leftrightarrow \text{is\_true } b$ .

**Proof.**

`move  $\Rightarrow P$ .`

`case (classic P); move  $\Rightarrow H$ .`

`$\exists \text{true}$ .`

`split; move  $\Rightarrow H0$ .`

`by [].`

`apply  $H$ .`

`$\exists \text{false}$ .`

`split; move  $\Rightarrow H0$ .`

`apply False_ind.`

`apply ( $H H0$ ).`

`discriminate  $H0$ .`

**Qed.**

**Definition `is_true_inv`** : `Prop`  $\rightarrow$  `bool`.

`move  $\Rightarrow P$ .`

`move : (is_true_inv0 P)  $\Rightarrow H$ .`

`apply constructive_indefinite_description in  $H$ .`

`apply  $H$ .`

**Defined.**

**Lemma `is_true_id`** :  $\forall P : \text{Prop}, \text{is\_true } (\text{is\_true\_inv } P) \leftrightarrow P$ .

**Proof.**

`move  $\Rightarrow P$ .`

`unfold is_true_inv.`

`move : (constructive_indefinite_description (fun b : bool  $\Rightarrow P \leftrightarrow \text{is\_true } b$ ) (is_true_inv0 P))  $\Rightarrow x0$ .`

`apply (@sig_ind bool (fun b  $\Rightarrow (P \leftrightarrow \text{is\_true } b)$ ) (fun y  $\Rightarrow \text{is\_true } (\text{let } (x, \_) := y \text{ in } x) \leftrightarrow P$ )).`

move  $\Rightarrow x\ H$ .

apply *iff\_sym*.

apply *H*.

**Qed.**

**Lemma** *sval\_inv* :  $\forall (A : \text{Type})(P : A \rightarrow \text{Prop})(x : \text{sig } P)(a : A), a = \text{sval } x \rightarrow \exists (H : P\ a), x = \text{exist } P\ a\ H$ .

**Proof.**

move  $\Rightarrow A\ P\ x\ a\ H0$ .

rewrite *H0*.

$\exists (proj2\_sig\ x)$ .

apply (@sig\_ind *A P* (fun *y*  $\Rightarrow y = \text{exist } P\ (\text{sval } y)\ (proj2\_sig\ y)$ )).

move  $\Rightarrow a0\ H$ .

by [simpl].

**Qed.**

**Lemma** *sval\_injective* :  $\forall (A : \text{Type})(P : A \rightarrow \text{Prop})(x\ y : \text{sig } P), \text{sval } x = \text{sval } y \rightarrow x = y$ .

**Proof.**

move  $\Rightarrow A\ P\ x\ y\ H$ .

move : (*sval\_inv A P y (sval x) H*).

elim  $\Rightarrow H0\ H1$ .

rewrite *H1*.

assert (*H0 = proj2\_sig x*).

apply *proof\_irrelevance*.

rewrite *H2*.

apply (@sig\_ind *A P* (fun *y*  $\Rightarrow y = \text{exist } P\ (\text{sval } y)\ (proj2\_sig\ y)$ )).

move  $\Rightarrow a0\ H3$ .

by [simpl].

**Qed.**

**Definition** *axiom19* :=  $\forall (A\ B : \text{eqType})(\alpha : \text{Rel } A\ B),$

$\exists (R : \text{eqType})(f : \text{Rel } R\ A)(g : \text{Rel } R\ B),$

$\text{function\_r } f \wedge \text{function\_r } g \wedge \alpha = f \# \cdot g \wedge ((f \cdot f \#) \quad (g \cdot g \#)) = \text{Id } R$ .

**Lemma** *rationality* : *axiom19*.

**Proof.**

move  $\Rightarrow A\ B\ \alpha$ .

rewrite /function\_r/total\_r/univalent\_r/cap/capP/identity/composite/inverse/include.

$\exists (\text{sig\_eqType } (\text{fun } x : \text{prod\_eqType } A\ B \Rightarrow \text{is\_true\_inv } (\alpha\ (\text{fst } x)\ (\text{snd } x))))$ .

$\exists (\text{fun } x\ a \Rightarrow a = (\text{fst } (\text{sval } x)))$ .

$\exists (\text{fun } x\ b \Rightarrow b = (\text{snd } (\text{sval } x)))$ .

simpl.

repeat split.

move  $\Rightarrow x\ x0\ H$ .

$\exists (\text{fst } (\text{sval } x))$ .

```

repeat split.
by [rewrite H].
move  $\Rightarrow$  a a0.
elim  $\Rightarrow$  x.
elim  $\Rightarrow$  H H0.
by [rewrite H H0].
move  $\Rightarrow$  x x0 H.
 $\exists$  (snd (sval x)).
repeat split.
by [rewrite H].
move  $\Rightarrow$  b b0.
elim  $\Rightarrow$  x.
elim  $\Rightarrow$  H H0.
by [rewrite H H0].
apply functional_extensionality.
move  $\Rightarrow$  a.
apply functional_extensionality.
move  $\Rightarrow$  b.
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
assert (is_true (is_true_inv (alpha (fst (a,b)) (snd (a,b))))).
simpl.
apply is_true_id.
apply H.
 $\exists$  (exist (fun x  $\Rightarrow$  (is_true (is_true_inv (alpha (fst x) (snd x)))))) (a,b) H0).
by [simpl].
elim H  $\Rightarrow$  x.
elim  $\Rightarrow$  H0 H1.
rewrite H0 H1.
apply is_true_id.
apply (@sig_ind (A  $\times$  B) (fun x  $\Rightarrow$  is_true (is_true_inv (alpha (fst x) (snd x)))) (fun x
 $\Rightarrow$  is_true (is_true_inv (alpha (fst (sval x)) (snd (sval x)))))).
simpl.
by [move  $\Rightarrow$  x0].
apply functional_extensionality.
move  $\Rightarrow$  y.
apply functional_extensionality.
move  $\Rightarrow$  y0.
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
apply sval_injective.
move : (H (fun a c : {x : A  $\times$  B | is_true (is_true_inv (alpha (fst x) (snd x))))}  $\Rightarrow$   $\exists$  b :
```

```

A, b = fst (sval a) ∧ b = fst (sval c)) (or_introl Logic.eq_refl)).
move : (H (fun a c : {x : A × B | is_true (is_true_inv (alpha (fst x) (snd x))))} ⇒ ∃ b :
B, b = snd (sval a) ∧ b = snd (sval c)) (or_intror Logic.eq_refl)).
unfold id.
clear H.
elim ⇒ b.
elim ⇒ H H0.
elim ⇒ a.
elim ⇒ H1 H2.
rewrite (surjective_pairing (sval y0)) -H0 -H2 H H1.
apply surjective_pairing.
rewrite H.
move ⇒ beta H0.
case H0 ⇒ H1; rewrite H1; unfold id.
∃ (fst (sval y0)).
repeat split.
∃ (snd (sval y0)).
repeat split.
Qed.

```

### 3.3.6 直和と直積

任意の直和に対して、入射対が存在することを仮定する.

**Lemma 20 (pair\_of\_inclusions)**  $\exists j : A \rightarrow A + B, \exists k : B \rightarrow A + B,$

$$j \cdot j^\# = id_A \wedge k \cdot k^\# = id_B \wedge j \cdot k^\# = \phi_{AB} \wedge j^\# \cdot j \sqcup k^\# \cdot k = id_{A+B}.$$

**Definition axiom20** :=

$\forall (A B : eqType), \exists (j : Rel A (sum_eqType A B))(k : Rel B (sum_eqType A B)),$   
 $j \cdot j^\# = Id A \wedge k \cdot k^\# = Id B \wedge j \cdot k^\# = \phi_{AB} \wedge$   
 $(j^\# \cdot j) \sqcup (k^\# \cdot k) = Id (sum_eqType A B).$

**Lemma pair\_of\_inclusions** : axiom20.

**Proof.**

```

move ⇒ A B.
∃ (fun (a : A)(x : sum_eqType A B) ⇒ x = inl a).
∃ (fun (b : B)(x : sum_eqType A B) ⇒ x = inr b).
repeat split.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ a0.

```



```
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
elim H  $\Rightarrow$  x.
elim  $\Rightarrow$  H0 H1.
rewrite H0 in H1.
by [injection H1].
 $\exists$  (inl a).
repeat split.
by [rewrite H].
apply functional_extensionality.
move  $\Rightarrow$  b.
apply functional_extensionality.
move  $\Rightarrow$  b0.
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
elim H  $\Rightarrow$  x.
elim  $\Rightarrow$  H0 H1.
rewrite H0 in H1.
by [injection H1].
 $\exists$  (inr b).
repeat split.
by [rewrite H].
apply functional_extensionality.
move  $\Rightarrow$  a.
apply functional_extensionality.
move  $\Rightarrow$  b.
apply prop_extensionality_ok.
split; move  $\Rightarrow$  H.
elim H  $\Rightarrow$  x.
elim  $\Rightarrow$  H0 H1.
rewrite H0 in H1.
discriminate H1.
apply False_ind.
apply H.
apply functional_extensionality.
move  $\Rightarrow$  x.
apply functional_extensionality.
move  $\Rightarrow$  x0.
apply prop_extensionality_ok.
split.
elim  $\Rightarrow$  alpha.
elim  $\Rightarrow$  H0 H1.
```

```

case  $H0 \Rightarrow H2$ ; rewrite  $H2$  in  $H1$ .
elim  $H1 \Rightarrow a$ .
elim  $\Rightarrow H3 H4$ .
by [rewrite  $H3 H4$ ].
elim  $H1 \Rightarrow b$ .
elim  $\Rightarrow H3 H4$ .
by [rewrite  $H3 H4$ ].
assert  $((\exists a : A, x = \text{inl } a) \vee (\exists b : B, x = \text{inr } b))$ .
move :  $x$ .
apply sum_ind.
move  $\Rightarrow a$ .
left.
by  $[\exists a]$ .
move  $\Rightarrow b$ .
right.
by  $[\exists b]$ .
case  $H$ .
elim  $\Rightarrow a H0 H1$ .
 $\exists$  (fun  $x x0 \Rightarrow \exists a0 : A, (x = \text{inl } a0 \wedge x0 = \text{inl } a0)$ ).
split.
by [left].
 $\exists a$ .
by [rewrite - $H1 H0$ ].
elim  $\Rightarrow b H0 H1$ .
 $\exists$  (fun  $x x0 \Rightarrow \exists b0 : B, (x = \text{inr } b0 \wedge x0 = \text{inr } b0)$ ).
split.
by [right].
 $\exists b$ .
by [rewrite - $H1 H0$ ].
Qed.
    
```

任意の直積に対して、射影対が存在することを仮定する。

実は有理性公理 (Axiom 19) があれば直積の公理は必要ないのだが、Axiom 19 の準用では直積が “存在する” ことまでしか示してくれないので、“直積として  $\text{prod\_eqType } A \ B$  を用いてよい” ことを公理の中に含めたものを用意しておく。

**Lemma 21 (pair\_of\_projections)**  $\exists p : A \times B \rightarrow A, \exists q : A \times B \rightarrow B,$

$$p^\# \cdot q = \nabla_{AB} \wedge p \cdot p^\# \sqcap q \cdot q^\# = \text{id}_{A \times B}.$$

**Definition** *axiom21* :=

$\forall (A \ B : \text{eqType}), \exists (p : \text{Rel } (\text{prod\_eqType } A \ B) \ A) (q : \text{Rel } (\text{prod\_eqType } A \ B) \ B),$   
 $p \ \# \cdot q = A \ B \wedge (p \cdot p \ \#) \quad (q \cdot q \ \#) = \text{Id } (\text{prod\_eqType } A \ B) \wedge \text{univalent\_r } p$

$\wedge$  *univalent\_r q*.

**Lemma** *pair\_of\_projections : axiom21*.

**Proof.**

move  $\Rightarrow A\ B$ .

$\exists$  (**fun** (*x* : *prod\_eqType A B*)(*a* : *A*)  $\Rightarrow a = (fst\ x)$ ).

$\exists$  (**fun** (*x* : *prod\_eqType A B*)(*b* : *B*)  $\Rightarrow b = (snd\ x)$ ).

split.

apply *functional\_extensionality*.

move  $\Rightarrow a$ .

apply *functional\_extensionality*.

move  $\Rightarrow b$ .

apply *prop\_extensionality\_ok*.

split; move  $\Rightarrow H$ .

apply *I*.

$\exists (a, b)$ .

by [simpl].

split.

apply *functional\_extensionality*.

move  $\Rightarrow x$ .

apply *functional\_extensionality*.

move  $\Rightarrow x0$ .

apply *prop\_extensionality\_ok*.

split; move  $\Rightarrow H$ .

move : (*H* (**fun** *a c* : *prod\_eqType A B*  $\Rightarrow \exists b : A, b = fst\ a \wedge b = fst\ c$ ) (*or\_introl Logic.eq\_refl*)).

move : (*H* (**fun** *a c* : *prod\_eqType A B*  $\Rightarrow \exists b : B, b = snd\ a \wedge b = snd\ c$ ) (*or\_intror Logic.eq\_refl*)).

unfold *id*.

clear *H*.

elim  $\Rightarrow b$ .

elim  $\Rightarrow H\ H0$ .

elim  $\Rightarrow a$ .

elim  $\Rightarrow H1\ H2$ .

rewrite (*surjective\_pairing x0*) -*H0* -*H2* *H H1*.

apply *surjective\_pairing*.

rewrite *H*.

move  $\Rightarrow \alpha\ H0$ .

case *H0*  $\Rightarrow H1$ ; rewrite *H1*; unfold *id*.

$\exists (fst\ x0)$ .

repeat split.

$\exists (snd\ x0)$ .

repeat split.

```
split.  
move  $\Rightarrow a\ a0$ .  
elim  $\Rightarrow x$ .  
elim  $\Rightarrow H\ H0$ .  
by [rewrite  $H\ H0$ ].  
move  $\Rightarrow b\ b0$ .  
elim  $\Rightarrow x$ .  
elim  $\Rightarrow H\ H0$ .  
by [rewrite  $H\ H0$ ].  
Qed.
```

# Chapter 4

## Library `Basic_Lemmas`

```
Require Import Basic_Notations.  
Require Import Logic.Classical_Prop.
```

### 4.1 束論に関する補題

#### 4.1.1 和関係, 共通関係

**Lemma 22 (cap\_l)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqcap \beta \sqsubseteq \alpha.$$

```
Lemma cap_l {A B : eqType} {alpha beta : Rel A B}: (alpha beta) alpha.
```

**Proof.**

```
assert ((alpha beta) (alpha beta)).
```

```
apply inc_refl.
```

```
apply inc_cap in H.
```

```
apply H.
```

**Qed.**

**Lemma 23 (cap\_r)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqcap \beta \sqsubseteq \beta.$$

```
Lemma cap_r {A B : eqType} {alpha beta : Rel A B}: (alpha beta) beta.
```

**Proof.**

```
assert ((alpha beta) (alpha beta)).
```

```
apply inc_refl.
```

```
apply inc_cap in H.
```

```
apply H.
```

**Qed.**

**Lemma 24 (cup\_l)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \alpha \sqcup \beta.$$

**Lemma cup\_l**  $\{A B : eqType\} \{alpha beta : Rel A B\} : alpha \sqsubseteq (alpha \sqcup beta).$

**Proof.**

assert  $((alpha \sqsubseteq beta) \rightarrow (alpha \sqsubseteq beta))$ .

apply *inc\_refl*.

apply *inc\_cup* in *H*.

apply *H*.

**Qed.**

**Lemma 25 (cup\_r)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\beta \sqsubseteq \alpha \sqcup \beta.$$

**Lemma cup\_r**  $\{A B : eqType\} \{alpha beta : Rel A B\} : beta \sqsubseteq (alpha \sqcup beta).$

**Proof.**

assert  $((alpha \sqsubseteq beta) \rightarrow (alpha \sqsubseteq beta))$ .

apply *inc\_refl*.

apply *inc\_cup* in *H*.

apply *H*.

**Qed.**

**Lemma 26 (inc\_def1)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha = \alpha \sqcap \beta \Leftrightarrow \alpha \sqsubseteq \beta.$$

**Lemma inc\_def1**  $\{A B : eqType\} \{alpha beta : Rel A B\} :$

$$alpha = alpha \sqcap beta \leftrightarrow alpha \sqsubseteq beta.$$

**Proof.**

split; move  $\Rightarrow$  *H*.

assert  $(alpha \sqsubseteq (alpha \sqcap beta))$ .

rewrite -*H*.

apply *inc\_refl*.

apply *inc\_cap* in *H0*.

apply *H0*.

apply *inc\_antisym*.

apply *inc\_cap*.

split.

apply *inc\_refl*.

apply *H*.  
 apply *cap\_l*.  
 Qed.

**Lemma 27 (inc\_def2)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\beta = \alpha \sqcup \beta \Leftrightarrow \alpha \sqsubseteq \beta.$$

**Lemma inc\_def2**  $\{A\ B : \text{eqType}\} \{alpha\ beta : \text{Rel } A\ B\}$ :  
 $beta = alpha \quad beta \leftrightarrow alpha \quad beta.$

**Proof.**

split; move  $\Rightarrow$  *H*.  
 assert  $((alpha \quad beta) \quad beta).$   
 rewrite -*H*.  
 apply *inc\_refl*.  
 apply *inc\_cup* in *H0*.  
 apply *H0*.  
 apply *inc\_antisym*.  
 assert  $((alpha \quad beta) \quad (alpha \quad beta)).$   
 apply *inc\_refl*.  
 apply *cup\_r*.  
 apply *inc\_cup*.  
 split.  
 apply *H*.  
 apply *inc\_refl*.  
 Qed.

**Lemma 28 (cap\_assoc)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$(\alpha \sqcap \beta) \sqcap \gamma = \alpha \sqcap (\beta \sqcap \gamma).$$

**Lemma cap\_assoc**  $\{A\ B : \text{eqType}\} \{alpha\ beta\ gamma : \text{Rel } A\ B\}$ :  
 $(alpha \quad beta) \quad gamma = alpha \quad (beta \quad gamma).$

**Proof.**

apply *inc\_antisym*.  
 rewrite *inc\_cap*.  
 split.  
 apply  $(inc\_trans \_ \_ (alpha \quad beta)).$   
 apply *cap\_l*.  
 apply *cap\_l*.  
 rewrite *inc\_cap*.  
 split.  
 apply  $(inc\_trans \_ \_ (alpha \quad beta)).$

```

apply cap_l.
apply cap_r.
apply cap_r.
rewrite inc_cap.
split.
rewrite inc_cap.
split.
apply cap_l.
apply (inc_trans _ _ _ (beta gamma)).
apply cap_r.
apply cap_l.
apply (inc_trans _ _ _ (beta gamma)).
apply cap_r.
apply cap_r.
Qed.

```

**Lemma 29 (cup\_assoc)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$(\alpha \sqcup \beta) \sqcup \gamma = \alpha \sqcup (\beta \sqcup \gamma).$$

**Lemma cup\_assoc**  $\{A B : eqType\} \{alpha \ beta \ gamma : Rel \ A \ B\}$ :  
 $(alpha \ \beta) \ \gamma = alpha \ (\beta \ \gamma).$

**Proof.**

```

apply inc_antisym.
rewrite inc_cup.
split.
rewrite inc_cup.
split.
apply cup_l.
apply (inc_trans _ _ _ (beta gamma)).
apply cup_l.
apply cup_r.
apply (inc_trans _ _ _ (beta gamma)).
apply cup_r.
apply cup_r.
rewrite inc_cup.
split.
apply (inc_trans _ _ _ (alpha beta)).
apply cup_l.
apply cup_l.
rewrite inc_cup.
split.
apply (inc_trans _ _ _ (alpha beta)).

```



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---

apply *cup\_r*.  
apply *cup\_l*.  
apply *cup\_r*.  
**Qed.**

**Lemma 30 (cap\_comm)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqcap \beta = \beta \sqcap \alpha.$$

**Lemma** *cap\_comm* {*A B : eqType*} {*alpha beta : Rel A B*}: *alpha*    **beta = beta**    *alpha*.

**Proof.**

apply *inc\_antisym*.  
rewrite *inc\_cap*.  
split.  
apply *cap\_r*.  
apply *cap\_l*.  
rewrite *inc\_cap*.  
split.  
apply *cap\_r*.  
apply *cap\_l*.  
**Qed.**

**Lemma 31 (cup\_comm)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqcup \beta = \beta \sqcup \alpha.$$

**Lemma** *cup\_comm* {*A B : eqType*} {*alpha beta : Rel A B*}: *alpha*    **beta = beta**    *alpha*.

**Proof.**

apply *inc\_antisym*.  
rewrite *inc\_cup*.  
split.  
apply *cup\_r*.  
apply *cup\_l*.  
rewrite *inc\_cup*.  
split.  
apply *cup\_r*.  
apply *cup\_l*.  
**Qed.**

**Lemma 32 (cup\_cap\_abs)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqcup (\alpha \sqcap \beta) = \alpha.$$

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**Lemma** *cup\_cap\_abs*  $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :  
 $alpha\ (alpha\ beta) = alpha$ .

**Proof.**

move : (@cap\_l \_ \_ alpha beta)  $\Rightarrow H$ .

apply *inc\_def2* in  $H$ .

by [rewrite *cup\_comm* - $H$ ].

**Qed.**

**Lemma 33** (*cap\_cup\_abs*) *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqcap (\alpha \sqcup \beta) = \alpha.$$

**Lemma** *cap\_cup\_abs*  $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :  
 $alpha\ (alpha\ beta) = alpha$ .

**Proof.**

move : (@cup\_l \_ \_ alpha beta)  $\Rightarrow H$ .

apply *inc\_def1* in  $H$ .

by [rewrite - $H$ ].

**Qed.**

**Lemma 34** (*cap\_idem*) *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqcap \alpha = \alpha.$$

**Lemma** *cap\_idem*  $\{A\ B : eqType\} \{alpha : Rel\ A\ B\}$ :  $alpha\ alpha = alpha$ .

**Proof.**

apply *inc\_antisym*.

apply *cap\_l*.

apply *inc\_cap*.

split; apply *inc\_refl*.

**Qed.**

**Lemma 35** (*cup\_idem*) *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqcup \alpha = \alpha.$$

**Lemma** *cup\_idem*  $\{A\ B : eqType\} \{alpha : Rel\ A\ B\}$ :  $alpha\ alpha = alpha$ .

**Proof.**

apply *inc\_antisym*.

apply *inc\_cup*.

split; apply *inc\_refl*.

apply *cup\_l*.

**Qed.**

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**Lemma 36 (cap\_inc\_compat)** *Let  $\alpha, \alpha', \beta, \beta' : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha' \sqcap \beta'.$$

**Lemma** `cap_inc_compat` {*A B : eqType*} {*alpha alpha' beta beta' : Rel A B*}:  
*alpha alpha' → beta beta' → (alpha beta) (alpha' beta').*

**Proof.**

`move ⇒ H H0.`

`rewrite -inc_def1.`

`apply inc_def1 in H.`

`apply inc_def1 in H0.`

`rewrite cap_assoc -(@cap_assoc - - beta).`

`rewrite (@cap_comm - - beta).`

`rewrite cap_assoc -(@cap_assoc - - alpha).`

`by [rewrite -H -H0].`

**Qed.**

**Lemma 37 (cap\_inc\_compat\_l)** *Let  $\alpha, \beta, \beta' : A \rightarrow B$ . Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha \sqcap \beta'.$$

**Lemma** `cap_inc_compat_l` {*A B : eqType*} {*alpha beta beta' : Rel A B*}:  
*beta beta' → (alpha beta) (alpha beta').*

**Proof.**

`move ⇒ H.`

`apply (@cap_inc_compat - - - - - (@inc_refl - - alpha) H).`

**Qed.**

**Lemma 38 (cap\_inc\_compat\_r)** *Let  $\alpha, \alpha', \beta : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha' \sqcap \beta.$$

**Lemma** `cap_inc_compat_r` {*A B : eqType*} {*alpha alpha' beta : Rel A B*}:  
*alpha alpha' → (alpha beta) (alpha' beta).*

**Proof.**

`move ⇒ H.`

`apply (@cap_inc_compat - - - - - H (@inc_refl - - beta)).`

**Qed.**

**Lemma 39 (cup\_inc\_compat)** *Let  $\alpha, \alpha', \beta, \beta' : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha' \sqcup \beta'.$$

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**Lemma** *cup\_inc\_compat*  $\{A\ B : \text{eqType}\} \{alpha\ alpha'\ beta\ beta' : \text{Rel } A\ B\}$ :  
 $alpha\ alpha' \rightarrow beta\ beta' \rightarrow (alpha\ beta)\ (alpha'\ beta')$ .

**Proof.**

move  $\Rightarrow H\ H0$ .

rewrite *inc\_def2*.

apply *inc\_def2* in *H*.

apply *inc\_def2* in *H0*.

rewrite *cup\_assoc* -(@*cup\_assoc* - - *beta*).

rewrite (@*cup\_comm* - - *beta*).

rewrite *cup\_assoc* -(@*cup\_assoc* - - *alpha*).

by [rewrite -*H* -*H0*].

**Qed.**

**Lemma 40** (*cup\_inc\_compat\_l*) *Let  $\alpha, \beta, \beta' : A \rightarrow B$ . Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha \sqcup \beta'.$$

**Lemma** *cup\_inc\_compat\_l*  $\{A\ B : \text{eqType}\} \{alpha\ beta\ beta' : \text{Rel } A\ B\}$ :  
 $beta\ beta' \rightarrow (alpha\ beta)\ (alpha\ beta')$ .

**Proof.**

move  $\Rightarrow H$ .

apply (@*cup\_inc\_compat* - - - - - (@*inc\_refl* - - *alpha*) *H*).

**Qed.**

**Lemma 41** (*cup\_inc\_compat\_r*) *Let  $\alpha, \alpha', \beta : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha' \sqcup \beta.$$

**Lemma** *cup\_inc\_compat\_r*  $\{A\ B : \text{eqType}\} \{alpha\ alpha'\ beta : \text{Rel } A\ B\}$ :  
 $alpha\ alpha' \rightarrow (alpha\ beta)\ (alpha'\ beta)$ .

**Proof.**

move  $\Rightarrow H$ .

apply (@*cup\_inc\_compat* - - - - - *H* (@*inc\_refl* - - *beta*)).

**Qed.**

**Lemma 42** (*cap\_empty*) *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqcap \phi_{AB} = \phi_{AB}.$$

**Lemma** *cap\_empty*  $\{A\ B : \text{eqType}\} \{alpha : \text{Rel } A\ B\}$ :  $alpha\ A\ B = A\ B$ .

**Proof.**

apply *inc\_antisym*.

apply *cap\_r*.

apply *inc\_empty\_alpha*.

**Qed.**

**Lemma 43 (cup\_empty)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqcup \phi_{AB} = \alpha.$$

**Lemma** *cup\_empty* {*A B : eqType*} {*alpha : Rel A B*}: *alpha*      *A B = alpha*.

**Proof.**

apply *inc\_antisym*.

apply *inc\_cup*.

split.

apply *inc\_refl*.

apply *inc\_empty\_alpha*.

apply *cup\_l*.

**Qed.**

**Lemma 44 (cap\_universal)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqcap \nabla_{AB} = \alpha.$$

**Lemma** *cap\_universal* {*A B : eqType*} {*alpha : Rel A B*}: *alpha*      *A B = alpha*.

**Proof.**

apply *inc\_antisym*.

apply *cap\_l*.

apply *inc\_cap*.

split.

apply *inc\_refl*.

apply *inc\_alpha\_universal*.

**Qed.**

**Lemma 45 (cup\_universal)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqcup \nabla_{AB} = \nabla_{AB}.$$

**Lemma** *cup\_universal* {*A B : eqType*} {*alpha : Rel A B*}: *alpha*      *A B =*      *A B*.

**Proof.**

apply *inc\_antisym*.

apply *inc\_cup*.

split.

apply *inc\_alpha\_universal*.

apply *inc\_refl*.

apply *cup\_r*.

**Qed.**

**Lemma 46 (inc\_lower)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha = \beta \Leftrightarrow (\forall \gamma : A \rightarrow B, \gamma \sqsubseteq \alpha \Leftrightarrow \gamma \sqsubseteq \beta).$$

**Lemma inc\_lower**  $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :

$$alpha = beta \Leftrightarrow (\forall\ gamma : Rel\ A\ B, gamma\ alpha \Leftrightarrow gamma\ beta).$$

**Proof.**

split; move  $\Rightarrow H$ .

move  $\Rightarrow gamma$ .

by [rewrite  $H$ ].

apply *inc\_antisym*.

rewrite  $-H$ .

apply *inc\_refl*.

rewrite  $H$ .

apply *inc\_refl*.

**Qed.**

**Lemma 47 (inc\_upper)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha = \beta \Leftrightarrow (\forall \gamma : A \rightarrow B, \alpha \sqsubseteq \gamma \Leftrightarrow \beta \sqsubseteq \gamma).$$

**Lemma inc\_upper**  $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :

$$alpha = beta \Leftrightarrow (\forall\ gamma : Rel\ A\ B, alpha\ gamma \Leftrightarrow beta\ gamma).$$

**Proof.**

split; move  $\Rightarrow H$ .

move  $\Rightarrow gamma$ .

by [rewrite  $H$ ].

apply *inc\_antisym*.

rewrite  $H$ .

apply *inc\_refl*.

rewrite  $-H$ .

apply *inc\_refl*.

**Qed.**

### 4.1.2 分配法則

**Lemma 48 (cap\_cup\_distr\_l)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqcap (\beta \sqcup \gamma) = (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma).$$

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**Lemma** *cap\_cup\_distr\_l* {A B : eqType} {alpha beta gamma : Rel A B}:  
 $\alpha \text{ (beta gamma) } = (\alpha \text{ beta}) \text{ (alpha gamma)}$ .

**Proof.**

apply *inc\_upper*.  
 move  $\Rightarrow$  *delta*.  
 split; move  $\Rightarrow$  *H*.  
 rewrite *cap\_comm* (@cap\_comm \_ \_ gamma).  
 apply *inc\_cup*.  
 rewrite -*inc\_rpc* -*inc\_rpc*.  
 apply *inc\_cup*.  
 rewrite *inc\_rpc* *cap\_comm*.  
 apply *H*.  
 rewrite *cap\_comm* -*inc\_rpc*.  
 apply *inc\_cup*.  
 rewrite *inc\_rpc* *inc\_rpc*.  
 apply *inc\_cup*.  
 rewrite *cap\_comm* (@cap\_comm \_ \_ gamma).  
 apply *H*.

**Qed.**

**Lemma 49 (cap\_cup\_distr\_r)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$(\alpha \sqcup \beta) \sqcap \gamma = (\alpha \sqcap \gamma) \sqcup (\beta \sqcap \gamma).$$

**Lemma** *cap\_cup\_distr\_r* {A B : eqType} {alpha beta gamma : Rel A B}:  
 $(\alpha \text{ beta}) \text{ gamma} = (\alpha \text{ gamma}) \text{ (beta gamma)}$ .

**Proof.**

rewrite (@cap\_comm \_ \_ (alpha beta)) (@cap\_comm \_ \_ alpha) (@cap\_comm \_ \_ beta).  
 apply *cap\_cup\_distr\_l*.

**Qed.**

**Lemma 50 (cup\_cap\_distr\_l)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqcup (\beta \sqcap \gamma) = (\alpha \sqcup \beta) \sqcap (\alpha \sqcup \gamma).$$

**Lemma** *cup\_cap\_distr\_l* {A B : eqType} {alpha beta gamma : Rel A B}:  
 $\alpha \text{ (beta gamma) } = (\alpha \text{ beta}) \text{ (alpha gamma)}$ .

**Proof.**

rewrite *cap\_cup\_distr\_l*.  
 rewrite (@cap\_comm \_ \_ (alpha beta)) *cap\_cup\_abs* (@cap\_comm \_ \_ (alpha beta)).  
 rewrite *cap\_cup\_distr\_l*.  
 rewrite -*cup\_assoc* (@cap\_comm \_ \_ gamma) *cup\_cap\_abs*.  
 by [rewrite *cap\_comm*].

*Qed.*

**Lemma 51 (cup\_cap\_distr\_r)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$(\alpha \sqcap \beta) \sqcup \gamma = (\alpha \sqcup \gamma) \sqcap (\beta \sqcup \gamma).$$

*Lemma cup\_cap\_distr\_r {A B : eqType} {alpha beta gamma : Rel A B}:*  
*(alpha beta) gamma = (alpha gamma) (beta gamma).*

*Proof.*

*rewrite (@cup\_comm \_ \_ (alpha beta)) (@cup\_comm \_ \_ alpha) (@cup\_comm \_ \_ beta).*  
*apply cup\_cap\_distr\_l.*

*Qed.*

**Lemma 52 (cap\_cup\_unique)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqcap \beta = \alpha \sqcap \gamma \wedge \alpha \sqcup \beta = \alpha \sqcup \gamma \Rightarrow \beta = \gamma.$$

*Lemma cap\_cup\_unique {A B : eqType} {alpha beta gamma : Rel A B}:*  
*alpha beta = alpha gamma  $\rightarrow$  alpha beta = alpha gamma  $\rightarrow$  beta = gamma.*

*Proof.*

*move  $\Rightarrow$  H H0.*  
*rewrite -(@cap\_cup\_abs \_ \_ beta alpha) cup\_comm H0.*  
*rewrite cap\_cup\_distr\_l.*  
*rewrite cap\_comm H.*  
*rewrite -cap\_cup\_distr\_r.*  
*rewrite H0 cap\_comm cup\_comm.*  
*apply cap\_cup\_abs.*

*Qed.*

### 4.1.3 原子性

空関係でない  $\alpha : A \rightarrow B$  が, 任意の  $\beta : A \rightarrow B$  について

$$\beta \sqsubseteq \alpha \Rightarrow \beta = \phi_{AB} \vee \beta = \alpha$$

を満たすとき,  $\alpha$  は原子的 (atomic) であると言われる.

*Definition atomic {A B : eqType} (alpha : Rel A B):=*  
*alpha  $\neq$   $\phi_{AB} \wedge (\forall \beta : Rel A B, \beta \sqsubseteq \alpha \rightarrow \beta = \phi_{AB} \vee \beta = \alpha).$*



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**Lemma 53 (atomic\_cap\_empty)** *Let  $\alpha, \beta : A \rightarrow B$  are atomic and  $\alpha \neq \beta$ . Then,*

$$\alpha \sqcap \beta = \phi_{AB}.$$

**Lemma** *atomic\_cap\_empty* { $A B : eqType$ } { $\alpha \beta : Rel A B$ }:  
 $atomic \ \alpha \rightarrow atomic \ \beta \rightarrow \alpha \neq \beta \rightarrow \alpha \sqcap \beta = \phi_{AB}.$

**Proof.**

```
move  $\Rightarrow$   $H \ H0$ .
apply or_to_imply.
case (classic ( $\alpha \sqcap \beta = \phi_{AB}$ )); move  $\Rightarrow$   $H1$ .
right.
apply  $H1$ .
left.
move  $\Rightarrow$   $H2$ .
apply  $H2$ .
apply inc_antisym.
apply inc_def1.
elim  $H \Rightarrow H3 \ H4$ .
case ( $H4 \ (\alpha \sqcap \beta) \ (cap\_l)$ ); move  $\Rightarrow$   $H5$ .
apply False_ind.
apply ( $H1 \ H5$ ).
by [rewrite  $H5$ ].
apply inc_def1.
elim  $H0 \Rightarrow H3 \ H4$ .
case ( $H4 \ (\alpha \sqcap \beta) \ (cap\_r)$ ); move  $\Rightarrow$   $H5$ .
apply False_ind.
apply ( $H1 \ H5$ ).
by [rewrite cap_comm  $H5$ ].
```

**Qed.**

**Lemma 54 (atomic\_cup)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$  and  $\alpha$  is atomic. Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Rightarrow \alpha \sqsubseteq \beta \vee \alpha \sqsubseteq \gamma.$$

**Lemma** *atomic\_cup* { $A B : eqType$ } { $\alpha \beta \gamma : Rel A B$ }:  
 $atomic \ \alpha \rightarrow \alpha \sqsubseteq \beta \sqcup \gamma \rightarrow \alpha \sqsubseteq \beta \vee \alpha \sqsubseteq \gamma.$

**Proof.**

```
move  $\Rightarrow$   $H \ H0$ .
apply inc_def1 in  $H0$ .
rewrite cap_cup_distr_l in  $H0$ .
elim  $H \Rightarrow H1 \ H2$ .
rewrite  $H0$  in  $H1$ .
assert ( $\alpha \sqcap \beta \neq \alpha \sqcap \gamma \vee \alpha \sqcap \beta \neq \alpha \sqcap \gamma$ ).
```

```

apply not_and_or.
elim  $\Rightarrow$  H3 H4.
rewrite H3 H4 in H1.
apply H1.
by [rewrite cup_empty].
case H3; move  $\Rightarrow$  H4.
left.
apply inc_def1.
case (H2 (alpha beta) (cap_l)); move  $\Rightarrow$  H5.
apply False_ind.
apply (H4 H5).
by [rewrite H5].
right.
apply inc_def1.
case (H2 (alpha gamma) (cap_l)); move  $\Rightarrow$  H5.
apply False_ind.
apply (H4 H5).
by [rewrite H5].
Qed.

```

## 4.2 Heyting 代数に関する補題

**Lemma 55 (rpc\_universal)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$(\alpha \Rightarrow \alpha) = \nabla_{AB}.$$

**Lemma** *rpc\_universal* {*A B* : *eqType*} {*alpha* : *Rel A B*}: (*alpha* » *alpha*) = *A B*.

**Proof.**

```

apply inc_lower.
move  $\Rightarrow$  gamma.
split; move  $\Rightarrow$  H.
apply inc_alpha_universal.
apply inc_rpc.
apply cap_r.
Qed.

```

**Lemma 56 (rpc\_r)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$(\alpha \Rightarrow \beta) \sqcap \beta = \beta.$$

**Lemma** *rpc\_r* {*A B* : *eqType*} {*alpha beta* : *Rel A B*}: (*alpha* » *beta*)  $\sqcap$  *beta* = *beta*.

**Proof.**

```
assert (beta (alpha » beta)).
apply inc_rpc.
apply cap_l.
apply inc_def1 in H.
by [rewrite cap_comm -H].
Qed.
```

**Lemma 57 (inc\_def3)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$(\alpha \Rightarrow \beta) = \nabla_{AB} \Leftrightarrow \alpha \sqsubseteq \beta.$$

**Lemma inc\_def3**  $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :  
 $(alpha \Rightarrow beta) = A\ B \leftrightarrow alpha\ beta.$

**Proof.**

```
split; move => H.
rewrite -(@rpc_universal _ _ alpha) in H.
assert ((alpha » alpha) (alpha » beta)).
rewrite H.
apply inc_refl.
apply inc_rpc in H0.
rewrite rpc_r in H0.
apply H0.
apply inc_antisym.
apply inc_alpha_universal.
rewrite -(@rpc_universal _ _ alpha).
apply inc_rpc.
rewrite rpc_r.
apply H.
Qed.
```

**Lemma 58 (rpc\_l)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqcap (\alpha \Rightarrow \beta) = \alpha \sqcap \beta.$$

**Lemma rpc\_l**  $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :  
 $alpha\ (alpha \Rightarrow beta) = alpha\ beta.$

**Proof.**

```
apply inc_lower.
move => gamma.
split; move => H.
apply inc_cap.
apply inc_cap in H.
```

```

split.
apply H.
elim H ⇒ H0 H1.
apply inc_rpc in H1.
rewrite -(@cap_idem _ _ gamma).
apply (inc_trans _ _ _ (gamma alpha)).
apply cap_inc_compat.
apply inc_refl.
apply H0.
apply H1.
apply inc_cap.
apply inc_cap in H.
split.
apply H.
apply inc_rpc.
apply (inc_trans _ _ _ gamma).
apply cap_l.
apply H.
Qed.

```

**Lemma 59 (rpc\_inc\_compat)** *Let  $\alpha, \alpha', \beta, \beta' : A \rightarrow B$ . Then,*

$$\alpha' \sqsubseteq \alpha \wedge \beta \sqsubseteq \beta' \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha' \Rightarrow \beta').$$

**Lemma** *rpc\_inc\_compat* {A B : eqType} {alpha alpha' beta beta' : Rel A B} :  
 $\alpha' \sqsubseteq \alpha \wedge \beta \sqsubseteq \beta' \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha' \Rightarrow \beta').$

**Proof.**

```

move ⇒ H H0.
apply inc_rpc.
apply (@inc_trans _ _ _ ((alpha » beta) alpha)).
apply (@cap_inc_compat_l _ _ _ _ H).
rewrite cap_comm rpc_l.
apply (@inc_trans _ _ _ beta).
apply cap_r.
apply H0.
Qed.

```

**Lemma 60 (rpc\_inc\_compat\_l)** *Let  $\alpha, \beta, \beta' : A \rightarrow B$ . Then,*

$$\beta \sqsubseteq \beta' \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha \Rightarrow \beta').$$

**Lemma** *rpc\_inc\_compat\_l* {A B : eqType} {alpha beta beta' : Rel A B} :  
 $\beta \sqsubseteq \beta' \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha \Rightarrow \beta').$

**Proof.**

move  $\Rightarrow H$ .

apply (@rpc\_inc\_compat \_ \_ \_ \_ (@inc\_refl \_ alpha) H).

**Qed.**

**Lemma 61 (rpc\_inc\_compat\_r)** *Let  $\alpha, \alpha', \beta : A \rightarrow B$ . Then,*

$$\alpha' \sqsubseteq \alpha \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha' \Rightarrow \beta).$$

**Lemma** *rpc\_inc\_compat\_r* {A B : eqType} {alpha alpha' beta : Rel A B}:  
 alpha' alpha  $\rightarrow$  (alpha » beta) (alpha' » beta).

**Proof.**

move  $\Rightarrow H$ .

apply (@rpc\_inc\_compat \_ \_ \_ \_ H (@inc\_refl \_ beta)).

**Qed.**

**Lemma 62 (rpc\_universal\_alpha)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\nabla_{AB} \Rightarrow \alpha = \alpha.$$

**Lemma** *rpc\_universal\_alpha* {A B : eqType} {alpha : Rel A B}: A B » alpha = alpha.

**Proof.**

apply inc\_lower.

move  $\Rightarrow$  gamma.

split; move  $\Rightarrow H$ .

apply inc\_rpc in H.

rewrite cap\_universal in H.

apply H.

apply inc\_rpc.

rewrite cap\_universal.

apply H.

**Qed.**

**Lemma 63 (rpc\_lemma1)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$(\alpha \Rightarrow \beta) \sqsubseteq ((\alpha \sqcap \gamma) \Rightarrow (\beta \sqcap \gamma)).$$

**Lemma** *rpc\_lemma1* {A B : eqType} {alpha beta gamma : Rel A B}:  
 (alpha » beta) ((alpha gamma) » (beta gamma)).

**Proof.**

apply inc\_rpc.

rewrite -cap\_assoc (@cap\_comm \_ \_ alpha).

rewrite rpc\_l.

$$(\alpha \Rightarrow \beta) \sqcap (\alpha \Rightarrow \gamma) = (\alpha \Rightarrow (\beta \sqcap \gamma)).$$

Proof.

$$(\alpha \Rightarrow \beta) \sqcap (\beta \Rightarrow \gamma) \sqsubseteq ((\alpha \sqcup \beta) \Rightarrow (\beta \sqcap \gamma)).$$

Proof.

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**Lemma 66 (rpc\_lemma4)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$(\alpha \Rightarrow \beta) \sqcap (\beta \Rightarrow \gamma) \sqsubseteq (\alpha \Rightarrow \gamma).$$

**Lemma** *rpc\_lemma4* {A B : eqType} {alpha beta gamma : Rel A B}:  
 ((alpha » beta) (beta » gamma)) (alpha » gamma).

**Proof.**

apply (@inc\_trans \_ \_ \_ ((alpha beta) » (beta gamma))).

apply rpc\_lemma3.

apply rpc\_inc\_compat.

apply cup\_l.

apply cap\_r.

**Qed.**

**Lemma 67 (rpc\_lemma5)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \Rightarrow (\beta \Rightarrow \gamma) = (\alpha \sqcap \beta) \Rightarrow \gamma.$$

**Lemma** *rpc\_lemma5* {A B : eqType} {alpha beta gamma : Rel A B}:  
 alpha » (beta » gamma) = (alpha beta) » gamma.

**Proof.**

apply inc\_lower.

move => delta.

split; move => H.

apply inc\_rpc.

rewrite -cap\_assoc.

rewrite -inc\_rpc -inc\_rpc.

apply H.

rewrite inc\_rpc inc\_rpc.

rewrite cap\_assoc.

apply inc\_rpc.

apply H.

**Qed.**

**Lemma 68 (rpc\_lemma6)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \Rightarrow (\beta \Rightarrow \gamma) \sqsubseteq (\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma).$$

**Lemma** *rpc\_lemma6* {A B : eqType} {alpha beta gamma : Rel A B}:  
 (alpha » (beta » gamma)) ((alpha » beta) » (alpha » gamma)).

**Proof.**

rewrite inc\_rpc inc\_rpc.

rewrite cap\_assoc (@cap\_comm \_ \_ \_ alpha).

```

rewrite rpc_l.
rewrite -cap_assoc (@cap_comm _ _ alpha).
rewrite rpc_l.
rewrite cap_assoc (@cap_comm _ _ beta).
rewrite rpc_l.
rewrite -cap_assoc.
apply cap_r.
Qed.

```

**Lemma 69 (rpc\_lemma7)** *Let  $\alpha, \beta, \gamma, \delta : A \rightarrow B$  and  $\beta \sqsubseteq \alpha \sqsubseteq \gamma$ . Then,*

$$(\alpha \sqcap \delta = \beta) \wedge (\alpha \sqcup \delta = \gamma) \Leftrightarrow (\gamma \sqsubseteq \alpha \sqcup (\alpha \Rightarrow \beta)) \wedge (\delta = \gamma \sqcap (\alpha \Rightarrow \beta)).$$

**Lemma** *rpc\_lemma7* {A B : eqType} {alpha beta gamma delta : Rel A B}:  
 beta alpha → alpha gamma → (alpha delta = beta ∧ alpha delta = gamma  
 ↔ gamma (alpha (alpha » beta)) ∧ delta = gamma (alpha » beta)).

**Proof.**

```

move ⇒ H H0.
split; elim; move ⇒ H1 H2; split.
rewrite -H2.
apply cup_inc_compat_l.
apply inc_rpc.
rewrite cap_comm H1.
apply inc_refl.
rewrite -H2.
rewrite cap_cup_distr_r rpc_l.
assert (delta (alpha » beta)).
apply inc_rpc.
rewrite cap_comm H1.
apply inc_refl.
apply inc_def1 in H3.
rewrite -H3 -H1.
rewrite -cap_assoc cap_idem.
by [rewrite cap_comm cup_comm cup_cap_abs].
rewrite H2.
rewrite (@cap_comm _ _ gamma) -cap_assoc rpc_l.
apply inc_antisym.
apply (@inc_trans _ _ _ (beta gamma)).
apply cap_inc_compat_r.
apply cap_r.
apply cap_l.
move : (@inc_trans _ _ _ _ H H0) ⇒ H3.
apply inc_def1 in H.

```



```

apply inc_def1 in H3.
rewrite cap_comm in H.
rewrite -H -H3.
apply inc_refl.
rewrite H2.
rewrite cup_cap_distr_l.
apply inc_def2 in H0.
rewrite -H0.
apply inc_def1 in H1.
by [rewrite -H1].
Qed.

```

### 4.3 補関係に関する補題

**Lemma 70 (complement\_universal)**

$$\nabla_{AB}^- = \phi_{AB}.$$

**Lemma** *complement\_universal*  $\{A\ B : eqType\}$ :  $A\ B\ ^\wedge = A\ B$ .

**Proof.**

```

apply rpc_universal_alpha.

```

**Qed.**

**Lemma 71 (complement\_alpha\_universal)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha^- = \nabla_{AB} \Leftrightarrow \alpha = \phi_{AB}.$$

**Lemma** *complement\_alpha\_universal*  $\{A\ B : eqType\}\ \{\alpha : Rel\ A\ B\}$ :  
 $\alpha\ ^\wedge = A\ B \Leftrightarrow \alpha = A\ B$ .

**Proof.**

```

split; move => H.
apply inc_antisym.
rewrite -(@cap_universal _ _ alpha) cap_comm.
apply inc_rpc.
rewrite -H.
apply inc_refl.
apply inc_empty_alpha.
apply inc_antisym.
apply inc_alpha_universal.
apply inc_rpc.
rewrite cap_comm cap_universal.
rewrite H.

```

apply *inc\_refl*.

**Qed.**

**Lemma 72 (complement\_empty)**

$$\phi_{AB}^- = \nabla_{AB}.$$

**Lemma** *complement\_empty* {A B : eqType}:  $\phi_{AB}^- = \nabla_{AB}$ .

**Proof.**

by [apply *complement\_alpha\_universal*].

**Qed.**

**Lemma 73 (complement\_invol\_inc)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq (\alpha^-)^-.$$

**Lemma** *complement\_invol\_inc* {A B : eqType} {alpha : Rel A B}:  $\alpha \sqsubseteq (\alpha^-)^-$ .

**Proof.**

apply *inc\_rpc*.

rewrite *cap\_comm*.

apply *inc\_rpc*.

apply *inc\_refl*.

**Qed.**

**Lemma 74 (cap\_complement\_empty)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\alpha \sqcap \alpha^- = \phi_{AB}.$$

**Lemma** *cap\_complement\_empty* {A B : eqType} {alpha : Rel A B}:

$$\alpha \sqcap \alpha^- = \phi_{AB}.$$

**Proof.**

apply *inc\_antisym*.

rewrite *cap\_comm*.

apply *inc\_rpc*.

apply *inc\_refl*.

apply *inc\_empty\_alpha*.

**Qed.**

**Lemma 75 (complement\_invol)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$(\alpha^-)^- = \alpha.$$

**Lemma** *complement\_invol* {A B : eqType} {alpha : Rel A B}:  $(\alpha^-)^- = \alpha$ .

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**Proof.**

```
rewrite -(@cap_universal _ _ ((alpha ^) ^)).
rewrite -(@complement_classic _ _ alpha).
rewrite cap_cup_distr_l.
rewrite (@cap_comm _ _ (alpha ^)) cap_complement_empty.
rewrite cup_empty cap_comm.
apply Logic.eq_sym.
apply inc_def1.
apply complement_invol_inc.
```

**Qed.**

**Lemma 76 (complement\_move)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha = \beta^- \Leftrightarrow \alpha^- = \beta.$$

**Lemma complement\_move**  $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :  
 $alpha = beta^{\wedge} \Leftrightarrow alpha^{\wedge} = beta.$

**Proof.**

```
split; move => H.
by [rewrite H complement_invol].
by [rewrite -H complement_invol].
```

**Qed.**

**Lemma 77 (contraposition)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$(\alpha \Rightarrow \beta) = (\beta^- \Rightarrow \alpha^-).$$

**Lemma contraposition**  $\{A\ B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :  
 $alpha \gg beta = beta^{\wedge} \gg alpha^{\wedge}.$

**Proof.**

```
apply inc_antisym.
apply inc_rpc.
apply rpc_lemma4.
replace (alpha >> beta) with ((alpha ^) ^ >> (beta ^) ^).
apply inc_rpc.
apply rpc_lemma4.
by [rewrite complement_invol complement_invol].
```

**Qed.**

**Lemma 78 (de\_morgan1)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$(\alpha \sqcup \beta)^- = \alpha^- \sqcap \beta^-.$$

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**Lemma** *de\_morgan1* {A B : eqType} {alpha beta : Rel A B}:  
 $(\alpha \sqcap \beta)^\wedge = \alpha^\wedge \sqcap \beta^\wedge$ .

**Proof.**

apply *inc\_lower*.  
 move  $\Rightarrow$  *gamma*.  
 split; move  $\Rightarrow$  *H*.  
 apply *inc\_cap*.  
 rewrite *inc\_rpc inc\_rpc*.  
 apply *inc\_cup*.  
 rewrite *-cap\_cup\_distr\_l*.  
 apply *inc\_rpc*.  
 apply *H*.  
 apply *inc\_rpc*.  
 rewrite *cap\_cup\_distr\_l*.  
 apply *inc\_cup*.  
 rewrite *-inc\_rpc -inc\_rpc*.  
 apply *inc\_cap*.  
 apply *H*.

**Qed.**

**Lemma 79 (de\_morgan2)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$(\alpha \sqcap \beta)^\neg = \alpha^\neg \sqcup \beta^\neg.$$

**Lemma** *de\_morgan2* {A B : eqType} {alpha beta : Rel A B}:  
 $(\alpha \sqcap \beta)^\wedge = \alpha^\wedge \sqcap \beta^\wedge$ .

**Proof.**

by [rewrite *-complement\_move de\_morgan1 complement\_invol complement\_invol*].

**Qed.**

**Lemma 80 (cup\_to\_rpc)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha^\neg \sqcup \beta = (\alpha \Rightarrow \beta).$$

**Lemma** *cup\_to\_rpc* {A B : eqType} {alpha beta : Rel A B}:  
 $\alpha^\wedge \sqcap \beta = \alpha \gg \beta$ .

**Proof.**

apply *inc\_antisym*.  
 apply *inc\_rpc*.  
 rewrite *cap\_cup\_distr\_r cap\_comm*.  
 rewrite *cap\_complement\_empty cup\_comm cup\_empty*.  
 apply *cap\_l*.  
 rewrite *-(@cap\_universal \_ \_ (alpha  $\gg$  beta)) cap\_comm*.

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```
rewrite -(@complement_classic _ _ alpha).
rewrite cap_cup_distr_r cup_comm.
apply cup_inc_compat.
apply cap_l.
rewrite rpc_l.
apply cap_r.
Qed.
```

**Lemma 81 (beta\_contradiction)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$(\alpha \Rightarrow \beta) \sqcap (\alpha \Rightarrow \beta^-) = \alpha^-.$$

**Lemma beta\_contradiction**  $\{A B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :  
 $(alpha \gg beta) \quad (alpha \gg beta^-) = alpha^-.$

**Proof.**

```
rewrite -cup_to_rpc -cup_to_rpc.
rewrite -cup_cap_distr_l.
by [rewrite cap_complement_empty cup_empty].
Qed.
```

## 4.4 Bool 代数に関する補題

**Lemma 82 (bool\_lemma1)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \beta \Leftrightarrow \nabla_{AB} = \alpha^- \sqcup \beta.$$

**Lemma bool\_lemma1**  $\{A B : eqType\} \{alpha\ beta : Rel\ A\ B\}$ :  
 $alpha \sqsubseteq beta \Leftrightarrow A\ B = alpha^- \sqcup beta.$

**Proof.**

```
split; move => H.
apply inc_antisym.
rewrite -(@complement_classic _ _ alpha) cup_comm.
apply cup_inc_compat_l.
apply H.
apply inc_alpha_universal.
apply inc_def3.
rewrite H.
apply (Logic.eq_sym cup_to_rpc).
Qed.
```

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**Lemma 83 (bool\_lemma2)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \beta \Leftrightarrow \alpha \sqcap \beta^- = \phi_{AB}.$$

**Lemma** *bool\_lemma2* {*A B : eqType*} {*alpha beta : Rel A B*}:

*alpha beta*  $\leftrightarrow$  *alpha beta* ^ = *A B*.

**Proof.**

split; move  $\Rightarrow$  *H*.

rewrite -(@cap\_universal \_ \_ (*alpha beta* ^)).

apply *bool\_lemma1* in *H*.

rewrite *H*.

rewrite *cap\_cup\_distr\_l*.

rewrite (@cap\_comm \_ \_ *alpha*) *cap\_assoc cap\_complement\_empty cap\_empty*.

rewrite *cap\_comm -cap\_assoc cap\_complement\_empty cap\_comm cap\_empty*.

by [rewrite *cup\_empty*].

rewrite -(@cap\_universal \_ \_ *alpha*).

rewrite -(@complement\_classic \_ \_ *beta*).

rewrite *cap\_cup\_distr\_l*.

rewrite *H cup\_empty*.

apply *cap\_r*.

**Qed.**

**Lemma 84 (bool\_lemma3)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Leftrightarrow \alpha \sqcap \beta^- \sqsubseteq \gamma.$$

**Lemma** *bool\_lemma3* {*A B : eqType*} {*alpha beta gamma : Rel A B*}:

*alpha* (*beta gamma*)  $\leftrightarrow$  (*alpha beta* ^) *gamma*.

**Proof.**

split; move  $\Rightarrow$  *H*.

apply (@inc\_trans \_ \_ \_ ((*beta gamma*) *beta* ^)).

apply *cap\_inc\_compat\_r*.

apply *H*.

rewrite *cap\_cup\_distr\_r*.

rewrite *cap\_complement\_empty cup\_comm cup\_empty*.

apply *cap\_l*.

apply (@inc\_trans \_ \_ \_ (*beta* (*alpha beta* ^))).

rewrite *cup\_cap\_distr\_l*.

rewrite *complement\_classic cap\_universal*.

apply *cup\_r*.

apply *cup\_inc\_compat\_l*.

apply *H*.

**Qed.**

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**Lemma 85 (bool\_lemma4)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Leftrightarrow \beta^- \sqsubseteq \alpha^- \sqcup \gamma.$$

**Lemma** *bool\_lemma4* {*A B : eqType*} {*alpha beta gamma : Rel A B*}:  
*alpha (beta gamma) ↔ beta ^ (alpha ^ gamma).*

**Proof.**

rewrite *bool\_lemma3*.

rewrite *cap\_comm*.

apply *iff\_sym*.

replace (*beta ^ alpha*) with (*beta ^ (alpha ^ ^)*).

apply *bool\_lemma3*.

by [rewrite *complement\_invol*].

**Qed.**

**Lemma 86 (bool\_lemma5)** *Let  $\alpha, \beta, \gamma : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Leftrightarrow \nabla_{AB} = \alpha^- \sqcup \beta \sqcup \gamma.$$

**Lemma** *bool\_lemma5* {*A B : eqType*} {*alpha beta gamma : Rel A B*}:  
*alpha (beta gamma) ↔ A B = (alpha ^ beta) gamma.*

**Proof.**

rewrite *bool\_lemma1*.

by [rewrite *cup\_assoc*].

**Qed.**

# Chapter 5

## Library **Relation\_Properties**

```
Require Import Basic_Notations.  
Require Import Basic_Lemmas.  
Require Import Logic.FunctionalExtensionality.  
Require Import Logic.Classical_Prop.
```

### 5.1 関係計算の基本的な性質

**Lemma 87 (RelAB\_unique)**

$$\phi_{AB} = \nabla_{AB} \Leftrightarrow \forall \alpha, \beta : A \rightarrow B, \alpha = \beta.$$

**Lemma** *RelAB\_unique* {A B : eqType}:

$$A B = A B \Leftrightarrow (\forall \text{ alpha beta} : \text{Rel } A B, \text{ alpha} = \text{ beta}).$$

**Proof.**

```
split; move => H.  
move => alpha beta.  
replace beta with ( A B).  
apply inc_antisym.  
rewrite H.  
apply inc_alpha_universal.  
apply inc_empty_alpha.  
apply inc_antisym.  
apply inc_empty_alpha.  
rewrite H.  
apply inc_alpha_universal.  
apply H.  
Qed.
```



**Lemma 88 (either\_empty)**

$$\phi_{AB} = \nabla_{AB} \Leftrightarrow A = \emptyset \vee B = \emptyset.$$

**Lemma** *either\_empty* {*A B : eqType*}:  $A \ B = \quad A \ B \Leftrightarrow (A \rightarrow \text{False}) \vee (B \rightarrow \text{False})$ .

**Proof.**

rewrite *RelAB\_unique*.

split; move  $\Rightarrow H$ .

case (*classic* ( $\exists \_ : A, \text{True}$ )).

elim  $\Rightarrow a \ H0$ .

right.

move  $\Rightarrow b$ .

*remember* (*fun* ( $\_ : A$ ) ( $\_ : B$ )  $\Rightarrow \text{True}$ ) **as** *T*.

*remember* (*fun* ( $\_ : A$ ) ( $\_ : B$ )  $\Rightarrow \text{False}$ ) **as** *F*.

move : (*H T F*)  $\Rightarrow H1$ .

assert (*T a b = F a b*).

by [rewrite *H1*].

rewrite *HeqT HeqF* in *H2*.

rewrite -*H2*.

apply *I*.

move  $\Rightarrow H0$ .

left.

move  $\Rightarrow a$ .

apply *H0*.

$\exists a$ .

apply *I*.

move  $\Rightarrow \text{alpha beta}$ .

assert ( $A \rightarrow B \rightarrow \text{False}$ ).

move  $\Rightarrow a \ b$ .

case *H*; move  $\Rightarrow H0$ .

apply (*H0 a*).

apply (*H0 b*).

apply *functional\_extensionality*.

move  $\Rightarrow a$ .

apply *functional\_extensionality*.

move  $\Rightarrow b$ .

apply *False\_ind*.

apply (*H0 a b*).

**Qed.**

**Lemma 89 (unit\_empty\_not\_universal)**

$$\phi_{II} \neq \nabla_{II}.$$

**Lemma** *unit\_empty\_not\_universal* :  $\phi_{II} \neq \nabla_{II}$ .

**Proof.**

move  $\Rightarrow H$ .

apply *either\_empty* in  $H$ .

case  $H$ ; move  $\Rightarrow H0$ .

apply ( $H0\ tt$ ).

apply ( $H0\ tt$ ).

**Qed.**

**Lemma 90 (unit\_empty\_or\_universal)** *Let  $\alpha : I \rightarrow I$ . Then,*

$$\alpha = \phi_{II} \vee \alpha = \nabla_{II}.$$

**Lemma** *unit\_empty\_or\_universal* { $\alpha : Rel\ i\ i$ }:  $\alpha = \phi_{II} \vee \alpha = \nabla_{II}$ .

**Proof.**

assert ( $\forall\ \text{beta} : Rel\ i\ i, \text{beta} = (\text{fun } (-) : i \Rightarrow True) \vee \text{beta} = (\text{fun } (-) : i \Rightarrow False)$ ).

move  $\Rightarrow \text{beta}$ .

case (*classic* ( $\text{beta}\ tt\ tt$ )); move  $\Rightarrow H$ .

left.

apply *functional\_extensionality*.

induction  $x$ .

apply *functional\_extensionality*.

induction  $x$ .

apply *prop\_extensionality\_ok*.

split; move  $\Rightarrow H0$ .

apply  $I$ .

apply  $H$ .

right.

apply *functional\_extensionality*.

induction  $x$ .

apply *functional\_extensionality*.

induction  $x$ .

apply *prop\_extensionality\_ok*.

split.

apply  $H$ .

apply *False\_ind*.

assert ( $(\text{fun } (-) : i \Rightarrow True) \neq (\text{fun } (-) : i \Rightarrow False)$ ).

move  $\Rightarrow H0$ .

*remember* ( $\text{fun } (-) : i \Rightarrow True$ ) **as**  $T$ .

```

remember (fun _ _ : i ⇒ False) as F.
assert (T tt tt = F tt tt).
by [rewrite H0].
rewrite HeqT HeqF in H1.
rewrite -H1.
apply I.
case (H ( i i)); move ⇒ H1.
case (H ( i i)); move ⇒ H2.
apply False_ind.
apply unit_empty_not_universal.
by [rewrite H1 H2].
case (H alpha); move ⇒ H3.
left.
by [rewrite H3 H1].
right.
by [rewrite H3 H2].
case (H ( i i)); move ⇒ H2.
case (H alpha); move ⇒ H3.
right.
by [rewrite H3 H2].
left.
by [rewrite H3 H1].
apply False_ind.
apply unit_empty_not_universal.
by [rewrite H1 H2].
Qed.

```

**Lemma 91 (unit\_identity\_is\_universal)**

$$id_I = \nabla_{II}.$$

**Lemma** *unit\_identity\_is\_universal* :  $Id\ i = \quad i\ i$ .

**Proof.**

```

case (@unit_empty_or_universal (Id i)); move ⇒ H.
apply False_ind.
assert (Id i ( i i # i i)).
rewrite H.
apply inc_empty_alpha.
apply inc_residual in H0.
rewrite inv_invol_comp_id_r in H0.
apply unit_empty_not_universal.
apply inc_antisym.
apply inc_empty_alpha.

```

apply *H0*.  
 apply *H*.  
 Qed.

**Lemma 92 (unit\_identity\_not\_empty)**

$$id_I \neq \phi_{II}.$$

**Lemma** *unit\_identity\_not\_empty* :  $Id\ i \neq\ i\ i$ .

**Proof.**

move  $\Rightarrow$  *H*.  
 apply *unit\_empty\_not\_universal*.  
 rewrite *-H*.  
 apply *unit\_identity\_is\_universal*.  
 Qed.

**Lemma 93 (cupP\_False)** *Let  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P(\alpha) := \text{"False"}$ . Then,*

$$\sqcup_{P(\alpha)} f(\alpha) = \phi_{AB}.$$

**Lemma** *cupP\_False* {*A B C D* : *eqType*} {*f* : *Rel C D*  $\rightarrow$  *Rel A B*}:

*-*{**fun** *-* : *Rel C D*  $\Rightarrow$  *False*} *f* = *A B*.

**Proof.**

apply *inc\_antisym*.  
 apply *inc\_cupP*.  
 move  $\Rightarrow$  **beta**.  
 apply *False\_ind*.  
 apply *inc\_empty\_alpha*.  
 Qed.

**Lemma 94 (capP\_False)** *Let  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P(\alpha) := \text{"False"}$ . Then,*

$$\sqcap_{P(\alpha)} f(\alpha) = \nabla_{AB}.$$

**Lemma** *capP\_False* {*A B C D* : *eqType*} {*f* : *Rel C D*  $\rightarrow$  *Rel A B*}:

*-*{**fun** *-* : *Rel C D*  $\Rightarrow$  *False*} *f* = *A B*.

**Proof.**

apply *inc\_antisym*.  
 apply *inc\_alpha\_universal*.  
 apply *inc\_capP*.  
 move  $\Rightarrow$  **beta**.  
 apply *False\_ind*.  
 Qed.

**Lemma 95 (cupP\_eq)** *Let  $f, g : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$(\forall \alpha : C \rightarrow D, P(\alpha) \Rightarrow f(\alpha) = g(\alpha)) \Rightarrow \sqcup_{P(\alpha)} f(\alpha) = \sqcup_{P(\alpha)} g(\alpha).$$

**Lemma cupP\_eq**  $\{A\ B\ C\ D : \text{eqType}\}$   
 $\{f\ g : \text{Rel}\ C\ D \rightarrow \text{Rel}\ A\ B\} \{P : \text{Rel}\ C\ D \rightarrow \text{Prop}\}:$   
 $(\forall \text{alpha} : \text{Rel}\ C\ D, P\ \text{alpha} \rightarrow f\ \text{alpha} = g\ \text{alpha}) \rightarrow \neg\{P\}\ f = \neg\{P\}\ g.$

**Proof.**

move  $\Rightarrow H$ .  
 apply *inc\_antisym*.  
 apply *inc\_cupP*.  
 move  $\Rightarrow$  **beta** *H0*.  
 rewrite  $(H \neg H0)$ .  
 move : **beta** *H0*.  
 apply *inc\_cupP*.  
 apply *inc\_refl*.  
 apply *inc\_cupP*.  
 move  $\Rightarrow$  **beta** *H0*.  
 rewrite  $\neg(H \neg H0)$ .  
 move : **beta** *H0*.  
 apply *inc\_cupP*.  
 apply *inc\_refl*.  
**Qed.**

**Lemma 96 (capP\_eq)** *Let  $f, g : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$(\forall \alpha : C \rightarrow D, P(\alpha) \Rightarrow f(\alpha) = g(\alpha)) \Rightarrow \sqcap_{P(\alpha)} f(\alpha) = \sqcap_{P(\alpha)} g(\alpha).$$

**Lemma capP\_eq**  $\{A\ B\ C\ D : \text{eqType}\}$   
 $\{f\ g : \text{Rel}\ C\ D \rightarrow \text{Rel}\ A\ B\} \{P : \text{Rel}\ C\ D \rightarrow \text{Prop}\}:$   
 $(\forall \text{alpha} : \text{Rel}\ C\ D, P\ \text{alpha} \rightarrow f\ \text{alpha} = g\ \text{alpha}) \rightarrow \neg\{P\}\ f = \neg\{P\}\ g.$

**Proof.**

move  $\Rightarrow H$ .  
 apply *inc\_antisym*.  
 apply *inc\_capP*.  
 move  $\Rightarrow$  **beta** *H0*.  
 rewrite  $\neg(H \neg H0)$ .  
 move : **beta** *H0*.  
 apply *inc\_capP*.  
 apply *inc\_refl*.  
 apply *inc\_capP*.  
 move  $\Rightarrow$  **beta** *H0*.

```

rewrite (H - H0).
move : beta H0.
apply inc_capP.
apply inc_refl.
Qed.

```

**Lemma 97 (cap\_cupP\_distr\_l)** *Let  $\alpha : A \rightarrow B$ ,  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P :$  predicate. Then,*

$$\alpha \sqcap (\sqcup_{P(\beta)} f(\beta)) = \sqcup_{P(\beta)} (\alpha \sqcap f(\beta)).$$

**Lemma cap\_cupP\_distr\_l**  $\{A B C D : eqType\}$   
 $\{\alpha : Rel A B\} \{f : Rel C D \rightarrow Rel A B\} \{P : Rel C D \rightarrow Prop\}$ :  
 $\alpha \sqcap (\sqcup_{P} f) = \sqcup_{P} (\alpha \sqcap f)$ .

**Proof.**

```

apply inc_upper.
move => gamma.
split; move => H.
apply inc_cupP.
move => beta H0.
apply (@inc_trans _ _ _ (alpha _ {P} f)).
apply cap_inc_compat_l.
move : H0.
apply inc_cupP.
apply inc_refl.
apply H.
assert (forall beta : Rel C D, P beta -> (alpha (f beta) gamma)).
apply inc_cupP.
apply H.
assert (forall beta : Rel C D, P beta -> f beta (alpha » gamma)).
move => beta H1.
rewrite inc_rpc cap_comm.
apply (H0 - H1).
rewrite cap_comm -inc_rpc.
apply inc_cupP.
apply H1.
Qed.

```

**Lemma 98 (cap\_cupP\_distr\_r)** *Let  $\beta : A \rightarrow B$ ,  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P :$  predicate. Then,*

$$(\sqcup_{P(\alpha)} f(\alpha)) \sqcap \beta = \sqcup_{P(\alpha)} (f(\alpha) \sqcap \beta).$$

**Lemma cap\_cupP\_distr\_r**  $\{A B C D : eqType\}$

```
{beta : Rel A B} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:
(  _{P} f)    beta =  _{P} (fun alpha : Rel C D ⇒ f alpha    beta).
```

**Proof.**

rewrite *cap\_comm*.

replace (fun alpha : Rel C D ⇒ f alpha beta) with (fun alpha : Rel C D ⇒ beta  
f alpha).

apply *cap\_cupP\_distr\_l*.

apply *functional\_extensionality*.

move ⇒ *x*.

by [rewrite *cap\_comm*].

**Qed.**

**Lemma 99 (cup\_capP\_distr\_l)** *Let  $\alpha : A \rightarrow B$ ,  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P :$   
predicate. Then,*

$$\alpha \sqcup (\sqcap_{P(\beta)} f(\beta)) = \sqcap_{P(\beta)} (\alpha \sqcup f(\beta)).$$

**Lemma** *cup\_capP\_distr\_l* {A B C D : eqType}

```
{alpha : Rel A B} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:
alpha    (  _{P} f) =  _{P} (fun beta : Rel C D ⇒ alpha    f beta).
```

**Proof.**

apply *inc\_lower*.

move ⇒ *gamma*.

split; move ⇒ *H*.

apply *inc\_capP*.

move ⇒ *beta H0*.

apply (@*inc\_trans* \_ \_ \_ (alpha \_{P} f)).

apply *H*.

apply *cup\_inc\_compat\_l*.

move : *H0*.

apply *inc\_capP*.

apply *inc\_refl*.

rewrite *bool\_lemma3*.

assert (∀ beta : Rel C D, P beta → *gamma* (alpha f beta)).

apply *inc\_capP*.

apply *H*.

apply *inc\_capP*.

move ⇒ *beta H1*.

rewrite *-bool\_lemma3*.

apply (*H0* - *H1*).

**Qed.**

## CHAPTER 5. LIBRARY RELATION\_PROPERTIES

**Lemma 100 (cup\_capP\_distr\_r)** *Let  $\beta : A \rightarrow B$ ,  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$(\sqcap_{P(\alpha)} f(\alpha)) \sqcup \beta = \sqcap_{P(\alpha)} (f(\alpha) \sqcup \beta).$$

**Lemma** *cup\_capP\_distr\_r* {A B C D : eqType}  
 {beta : Rel A B} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:  
 ( \_{P} f) beta = \_{P} (fun alpha : Rel C D ⇒ f alpha beta).

**Proof.**

rewrite *cup\_comm*.  
 replace (fun alpha : Rel C D ⇒ f alpha beta) with (fun alpha : Rel C D ⇒ beta f alpha).  
 apply *cup\_capP\_distr\_l*.  
 apply *functional\_extensionality*.  
 move ⇒ *x*.  
 by [rewrite *cup\_comm*].  
**Qed.**

**Lemma 101 (de\_morgan3)** *Let  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$(\sqcup_{P(\alpha)} f(\alpha))^- = (\sqcap_{P(\alpha)} f(\alpha)^-).$$

**Lemma** *de\_morgan3*  
 {A B C D : eqType} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:  
 ( \_{P} f) ^ = \_{P} (fun alpha : Rel C D ⇒ f alpha ^).

**Proof.**

apply *inc\_lower*.  
 move ⇒ *gamma*.  
 rewrite *inc\_capP*.  
 split; move ⇒ *H*.  
 move ⇒ **beta** *H0*.  
 rewrite *bool\_lemma1 -de\_morgan2 complement\_move complement\_universal*.  
 apply *bool\_lemma2* in *H*.  
 apply *inc\_antisym*.  
 apply *inc\_empty\_alpha*.  
 rewrite *-H complement\_invol*.  
 apply *cap\_inc\_compat\_l*.  
 move : *H0*.  
 apply *inc\_cupP*.  
 apply *inc\_refl*.  
 rewrite *bool\_lemma2 complement\_invol*.  
 rewrite *cap\_cupP\_distr\_l*.  
 apply *inc\_antisym*.



apply *inc\_cupP*.  
 move  $\Rightarrow$  **beta** *H0*.  
 rewrite *-inc\_rpc*.  
 apply (*H* \_ *H0*).  
 apply *inc\_empty\_alpha*.  
**Qed**.

**Lemma 102 (de\_morgan4)** *Let  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$(\sqcap_{P(\alpha)} f(\alpha))^- = (\sqcup_{P(\alpha)} f(\alpha)^-).$$

**Lemma de\_morgan4**

$\{A\ B\ C\ D : \text{eqType}\} \{f : \text{Rel } C\ D \rightarrow \text{Rel } A\ B\} \{P : \text{Rel } C\ D \rightarrow \text{Prop}\}:$   
 $(\neg \{P\} f)^\wedge = \neg \{P\} (\text{fun } \alpha : \text{Rel } C\ D \Rightarrow f\ \alpha \wedge).$

**Proof.**

rewrite *-complement\_move\_de\_morgan3*.  
 replace (*fun* *alpha* : *Rel* *C* *D*  $\Rightarrow$  (*f* *alpha*  $\wedge$ )  $\wedge$ ) **with** *f*.  
 by [].  
 apply *functional\_extensionality*.  
 move  $\Rightarrow$  *x*.  
 by [rewrite *complement\_invol*].  
**Qed**.

**Lemma 103 (cup\_to\_cupP)** *Let  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$f(\alpha) \sqcup f(\beta) = \sqcup_{\gamma=\alpha \vee \gamma=\beta} f(\gamma).$$

**Lemma cup\_to\_cupP**

$\{A\ B\ C\ D : \text{eqType}\} \{\alpha\ \beta : \text{Rel } C\ D\} \{f : \text{Rel } C\ D \rightarrow \text{Rel } A\ B\}:$   
 $(f\ \alpha \sqcup f\ \beta) = \neg \{\text{fun } \gamma : \text{Rel } C\ D \Rightarrow \gamma = \alpha \vee \gamma = \beta\} f.$

**Proof.**

apply *inc\_upper*.  
 move  $\Rightarrow$  **delta**.  
 split; move  $\Rightarrow$  *H*.  
 apply *inc\_cupP*.  
 apply *inc\_cup* in *H*.  
 move  $\Rightarrow$  *gamma* *H0*.  
 case *H0*  $\Rightarrow$  *H1*.  
 rewrite *H1*.  
 apply *H*.  
 rewrite *H1*.  
 apply *H*.

```

apply inc_cup.
assert (∀ gamma : Rel C D, gamma = alpha ∨ gamma = beta → f gamma = delta).
apply inc_cupP.
apply H.
split.
apply (H0 alpha).
by [left].
apply (H0 beta).
by [right].
Qed.

```

**Lemma 104 (cap\_to\_capP)** *Let  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$f(\alpha) \sqcap f(\beta) = \sqcap_{\gamma=\alpha \vee \gamma=\beta} f(\gamma).$$

**Lemma cap\_to\_capP**

```

{A B C D : eqType} {alpha beta : Rel C D} {f : Rel C D → Rel A B}:
(f alpha    f beta) =    _{fun gamma : Rel C D ⇒ gamma = alpha ∨ gamma = beta}
f.

```

**Proof.**

```

apply inc_lower.
move ⇒ delta.
split; move ⇒ H.
apply inc_capP.
apply inc_cap in H.
move ⇒ gamma H0.
case H0 ⇒ H1.
rewrite H1.
apply H.
rewrite H1.
apply H.
apply inc_cap.
assert (∀ gamma : Rel C D, gamma = alpha ∨ gamma = beta → delta = f gamma).
apply inc_capP.
apply H.
split.
apply (H0 alpha).
by [left].
apply (H0 beta).
by [right].
Qed.

```

## 5.2 comp\_inc\_compat と派生補題

**Lemma 105 (comp\_inc\_compat\_ab\_ab')** *Let  $\alpha : A \rightarrow B$  and  $\beta, \beta' : B \rightarrow C$ . Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha \cdot \beta'.$$

**Lemma comp\_inc\_compat\_ab\_ab'**

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta\ beta' : Rel\ B\ C\} :$   
 $beta\ \ beta' \rightarrow (alpha \cdot beta)\ \ (alpha \cdot beta').$

**Proof.**

move  $\Rightarrow H$ .

replace  $(alpha \cdot beta)$  with  $((alpha \ #) \ # \cdot beta)$ .

apply *inc\_residual*.

apply  $(@inc\_trans \_ \_ \_ beta')$ .

apply *H*.

apply *inc\_residual*.

rewrite *inv\_invol*.

apply *inc\_refl*.

by [rewrite *inv\_invol*].

**Qed.**

**Lemma 106 (comp\_inc\_compat\_ab\_a'b)** *Let  $\alpha, \alpha' : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha' \cdot \beta.$$

**Lemma comp\_inc\_compat\_ab\_a'b**

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{beta : Rel\ B\ C\} :$   
 $alpha\ \ alpha' \rightarrow (alpha \cdot beta)\ \ (alpha' \cdot beta).$

**Proof.**

move  $\Rightarrow H$ .

rewrite  $-(@inv\_invol \_ \_ (alpha \cdot beta))$ .

rewrite  $-(@inv\_invol \_ \_ (alpha' \cdot beta))$ .

apply *inc\_inv*.

rewrite *comp\_inv comp\_inv*.

apply *comp\_inc\_compat\_ab\_ab'*.

apply *inc\_inv*.

apply *H*.

**Qed.**

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**Lemma 107 (comp\_inc\_compat)** *Let  $\alpha, \alpha' : A \rightarrow B$  and  $\beta, \beta' : B \rightarrow C$ . Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha' \cdot \beta'.$$

**Lemma** *comp\_inc\_compat*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{beta\ beta' : Rel\ B\ C\} :$   
 $alpha\ alpha' \rightarrow beta\ beta' \rightarrow (alpha \cdot beta)\ (alpha' \cdot beta').$

**Proof.**

move  $\Rightarrow H\ H0$ .

apply (@inc\_trans \_ \_ \_ (alpha' · beta)).

apply (@comp\_inc\_compat\_ab\_a'b \_ \_ \_ \_ \_ H).

apply (@comp\_inc\_compat\_ab\_ab' \_ \_ \_ \_ \_ H0).

**Qed.**

**Lemma 108 (comp\_inc\_compat\_ab\_a)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow B$ . Then,*

$$\beta \sqsubseteq id_B \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha.$$

**Lemma** *comp\_inc\_compat\_ab\_a*  $\{A\ B : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ B\} :$   
 $beta\ Id\ B \rightarrow (alpha \cdot beta)\ alpha.$

**Proof.**

move  $\Rightarrow H$ .

move : (@comp\_inc\_compat\_ab\_ab' \_ \_ \_ alpha \_ \_ H)  $\Rightarrow H0$ .

rewrite comp\_id\_r in H0.

apply H0.

**Qed.**

**Lemma 109 (comp\_inc\_compat\_a\_ab)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow B$ . Then,*

$$id_B \sqsubseteq \beta \Rightarrow \beta \sqsubseteq \alpha \cdot \beta.$$

**Lemma** *comp\_inc\_compat\_a\_ab*  $\{A\ B : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ B\} :$   
 $Id\ B\ beta \rightarrow alpha\ (alpha \cdot beta).$

**Proof.**

move  $\Rightarrow H$ .

move : (@comp\_inc\_compat\_ab\_ab' \_ \_ \_ alpha \_ \_ H)  $\Rightarrow H0$ .

rewrite comp\_id\_r in H0.

apply H0.

**Qed.**

**Lemma 110 (comp\_inc\_compat\_ab\_b)** *Let  $\alpha : A \rightarrow A$  and  $\beta : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq id_A \Rightarrow \alpha \cdot \beta \sqsubseteq \beta.$$

**Lemma** *comp\_inc\_compat\_ab\_b* { $A\ B : eqType$ } { $\alpha : Rel\ A\ A$ } { $\beta : Rel\ A\ B$ }:  
 $\alpha$   $Id\ A \rightarrow (\alpha \cdot \beta)$   $\beta$ .

**Proof.**

move  $\Rightarrow H$ .

move : (@comp\_inc\_compat\_ab\_a'b \_ \_ \_ \_  $\beta$   $H$ )  $\Rightarrow H0$ .

rewrite comp\_id\_l in  $H0$ .

apply  $H0$ .

**Qed.**

**Lemma 111 (comp\_inc\_compat\_b\_ab)** *Let  $\alpha : A \rightarrow A$  and  $\beta : A \rightarrow B$ . Then,*

$$id_A \sqsubseteq \alpha \Rightarrow \beta \sqsubseteq \alpha \cdot \beta.$$

**Lemma** *comp\_inc\_compat\_b\_ab* { $A\ B : eqType$ } { $\alpha : Rel\ A\ A$ } { $\beta : Rel\ A\ B$ }:  
 $Id\ A \rightarrow \beta (\alpha \cdot \beta)$ .

**Proof.**

move  $\Rightarrow H$ .

move : (@comp\_inc\_compat\_ab\_a'b \_ \_ \_ \_  $\beta$   $H$ )  $\Rightarrow H0$ .

rewrite comp\_id\_l in  $H0$ .

apply  $H0$ .

**Qed.**

### 5.3 逆関係に関する補題

**Lemma 112 (inv\_move)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow A$ . Then,*

$$\alpha = \beta^\# \Leftrightarrow \alpha^\# = \beta.$$

**Lemma** *inv\_move* { $A\ B : eqType$ } { $\alpha : Rel\ A\ B$ } { $\beta : Rel\ B\ A$ }:  
 $\alpha = \beta^\# \Leftrightarrow \alpha^\# = \beta$ .

**Proof.**

split; move  $\Rightarrow H$ .

by [rewrite  $H$  inv\_invol].

by [rewrite - $H$  inv\_invol].

**Qed.**

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**Lemma 113 (comp\_inv\_inv)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then,*

$$\alpha \cdot \beta = (\beta^\# \cdot \alpha^\#)^\#.$$

**Lemma** *comp\_inv\_inv* {*A B C : eqType*} {*alpha : Rel A B*} {*beta : Rel B C*}:  
*alpha* • *beta* = (*beta* # • *alpha* #) #.

**Proof.**

apply *inv\_move*.

apply *comp\_inv*.

**Qed.**

**Lemma 114 (inv\_inc\_move)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow A$ . Then,*

$$\alpha \sqsubseteq \beta^\# \Leftrightarrow \alpha^\# \sqsubseteq \beta.$$

**Lemma** *inv\_inc\_move* {*A B : eqType*} {*alpha : Rel A B*} {*beta : Rel B A*}:  
*alpha* # *beta* #  $\leftrightarrow$  *alpha* # *beta*.

**Proof.**

split; move  $\Rightarrow$  *H*.

rewrite -(@*inv\_invol* \_ \_ *beta*).

apply *inc\_inv*.

apply *H*.

rewrite -(@*inv\_invol* \_ \_ *alpha*).

apply *inc\_inv*.

apply *H*.

**Qed.**

**Lemma 115 (inv\_invol2)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha^\# = \beta^\# \Rightarrow \alpha = \beta.$$

**Lemma** *inv\_invol2* {*A B : eqType*} {*alpha beta : Rel A B*}:  
*alpha* # = *beta* #  $\rightarrow$  *alpha* = *beta*.

**Proof.**

move  $\Rightarrow$  *H*.

rewrite -(@*inv\_invol* \_ \_ *alpha*) -(@*inv\_invol* \_ \_ *beta*).

apply f\_equal.

apply *H*.

**Qed.**

**Lemma 116 (inv\_inc\_invol)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$\alpha^\# \sqsubseteq \beta^\# \Rightarrow \alpha \sqsubseteq \beta.$$

**Lemma** *inv\_inc\_invol* {A B : eqType} {alpha beta : Rel A B}:

alpha # beta # → alpha beta.

**Proof.**

move ⇒ H.

rewrite -(@inv\_invol \_ \_ alpha) -(@inv\_invol \_ \_ beta).

apply inc\_inv.

apply H.

**Qed.**

**Lemma 117 (inv\_cupP\_distr, inv\_cup\_distr)** *Let  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$(\sqcup_{P(\alpha)} f(\alpha))^\# = (\sqcup_{P(\alpha)} f(\alpha)^\#).$$

**Lemma** *inv\_cupP\_distr* {A B C D : eqType} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:

( \_ {P} f) # = ( \_ {P} (fun alpha : Rel C D ⇒ f alpha #)).

**Proof.**

apply inc\_antisym.

rewrite -inv\_inc\_move.

apply inc\_cupP.

assert (∀ beta : Rel C D, P beta → f beta # \_ {P} (fun alpha : Rel C D ⇒ f alpha #)).

apply inc\_cupP.

apply inc\_refl.

move ⇒ beta H0.

rewrite inv\_inc\_move.

apply (H \_ H0).

apply inc\_cupP.

move ⇒ beta H0.

apply inc\_inv.

move : H0.

apply inc\_cupP.

apply inc\_refl.

**Qed.**

**Lemma** *inv\_cup\_distr* {A B : eqType} {alpha beta : Rel A B}:

(alpha beta) # = alpha # beta #.

**Proof.**

by [rewrite cup\_to\_cupP -inv\_cupP\_distr -cup\_to\_cupP].

**Qed.**

**Lemma 118 (inv\_capP\_distr, inv\_cap\_distr)** *Let  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$(\sqcap_{P(\alpha)} f(\alpha))^{\#} = (\sqcap_{P(\alpha)} f(\alpha)^{\#}).$$

**Lemma inv\_capP\_distr**  $\{A\ B\ C\ D : \text{eqType}\} \{f : \text{Rel } C\ D \rightarrow \text{Rel } A\ B\} \{P : \text{Rel } C\ D \rightarrow \text{Prop}\}$ :

$$(\_ \{P\} f)^{\#} = (\_ \{P\} (\text{fun } \alpha : \text{Rel } C\ D \Rightarrow f\ \alpha\ \#)).$$

**Proof.**

apply *inc\_antisym*.

apply *inc\_capP*.

move  $\Rightarrow$  **beta** *H*.

apply *inc\_inv*.

move : *H*.

apply *inc\_capP*.

apply *inc\_refl*.

rewrite *inv\_inc\_move*.

apply *inc\_capP*.

assert  $(\forall \text{beta} : \text{Rel } C\ D, P\ \text{beta} \rightarrow \_ \{P\} (\text{fun } \alpha : \text{Rel } C\ D \Rightarrow f\ \alpha\ \#) \quad f\ \text{beta}\ \#)$ .

apply *inc\_capP*.

apply *inc\_refl*.

move  $\Rightarrow$  **beta** *H0*.

rewrite *-inv\_inc\_move*.

apply  $(H \_ H0)$ .

**Qed.**

**Lemma inv\_cap\_distr**  $\{A\ B : \text{eqType}\} \{\alpha\ \text{beta} : \text{Rel } A\ B\}$ :

$$(\alpha \ \text{beta})^{\#} = \alpha^{\#} \ \text{beta}^{\#}.$$

**Proof.**

by [rewrite *cap\_to\_capP -inv\_capP\_distr -cap\_to\_capP*].

**Qed.**

**Lemma 119 (rpc\_inv\_distr)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$(\alpha \Rightarrow \beta)^{\#} = \alpha^{\#} \Rightarrow \beta^{\#}.$$

**Lemma rpc\_inv\_distr**  $\{A\ B : \text{eqType}\} \{\alpha\ \text{beta} : \text{Rel } A\ B\}$ :

$$(\alpha \gg \text{beta})^{\#} = \alpha^{\#} \gg \text{beta}^{\#}.$$

**Proof.**

apply *inc\_lower*.

move  $\Rightarrow$  *gamma*.



```

split; move => H.
apply inc_rpc.
rewrite inv_inc_move inv_cap_distr inv_invol.
rewrite -inc_rpc -inv_inc_move.
apply H.
rewrite inv_inc_move inc_rpc.
rewrite -(@inv_invol _ _ alpha) -inv_cap_distr -inv_inc_move.
apply inc_rpc.
apply H.
Qed.

```

**Lemma 120 (inv\_empty)**

$$\phi_{AB}^\# = \phi_{BA}.$$

**Lemma** *inv\_empty* {A B : eqType}:    A B # =    B A.

**Proof.**

```

apply inc_antisym.
rewrite -inv_inc_move.
apply inc_empty_alpha.
apply inc_empty_alpha.
Qed.

```

**Lemma 121 (inv\_universal)**

$$\nabla_{AB}^\# = \nabla_{BA}.$$

**Lemma** *inv\_universal* {A B : eqType}:    A B # =    B A.

**Proof.**

```

apply inc_antisym.
apply inc_alpha_universal.
rewrite inv_inc_move.
apply inc_alpha_universal.
Qed.

```

**Lemma 122 (inv\_id)**

$$id_A^\# = id_A.$$

**Lemma** *inv\_id* {A : eqType}: (Id A) # = Id A.

**Proof.**

```

replace (Id A #) with ((Id A #) # • Id A #).
by [rewrite -comp_inv comp_id_l inv_invol].
by [rewrite inv_invol comp_id_l].
Qed.

```

**Lemma 123 (inv\_complement)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$(\alpha^-)^\# = (\alpha^\#)^-.$$

**Lemma** *inv\_complement* { $A B : eqType$ } { $\alpha : Rel A B$ }:  $(\alpha^\wedge)^\# = (\alpha^\#)^\wedge$ .

**Proof.**

apply *inc\_antisym*.

apply *inc\_rpc*.

rewrite *-inv\_cap\_distr*.

rewrite *cap\_comm -inv\_inc\_move inv\_empty*.

rewrite *cap\_complement\_empty*.

apply *inc\_refl*.

rewrite *inv\_inc\_move*.

apply *inc\_rpc*.

replace  $((\alpha^\#)^\wedge)^\#$  *alpha* with  $((\alpha^\#)^\wedge)^\#$   $(\alpha^\#)^\#$ .

rewrite *-inv\_cap\_distr*.

rewrite *cap\_comm -inv\_inc\_move inv\_empty*.

rewrite *cap\_complement\_empty*.

apply *inc\_refl*.

by [rewrite *inv\_invol*].

**Qed.**

**Lemma 124 (inv\_difference\_distr)** *Let  $\alpha, \beta : A \rightarrow B$ . Then,*

$$(\alpha - \beta)^\# = \alpha^\# - \beta^\#.$$

**Lemma** *inv\_difference\_distr* { $A B : eqType$ } { $\alpha \beta : Rel A B$ }:  
 $(\alpha - \beta)^\# = \alpha^\# - \beta^\#$ .

**Proof.**

rewrite *inv\_cap\_distr*.

by [rewrite *inv\_complement*].

**Qed.**

## 5.4 合成に関する補題

**Lemma 125 (comp\_cupP\_distr\_l, comp\_cup\_distr\_l)** *Let  $\alpha : A \rightarrow B$ ,  $f : (D \rightarrow E) \rightarrow (B \rightarrow C)$  and  $P : predicate$ . Then,*

$$\alpha \cdot (\sqcup_{P(\beta)} f(\beta)) = \sqcup_{P(\beta)} (\alpha \cdot f(\beta)).$$

**Lemma** *comp\_cupP\_distr\_l* { $A B C D E : eqType$ }

$\{alpha : Rel\ A\ B\} \{f : Rel\ D\ E \rightarrow Rel\ B\ C\} \{P : Rel\ D\ E \rightarrow Prop\}$ :  
 $alpha \cdot (\_ \{P\} f) = \_ \{P\} (\text{fun } beta : Rel\ D\ E \Rightarrow (alpha \cdot f\ beta))$ .

**Proof.**

apply *inc\_upper*.

move  $\Rightarrow$  *gamma*.

split; move  $\Rightarrow$  *H*.

rewrite  $-(@inv\_involver\_alpha)$  in *H*.

apply *inc\_residual* in *H*.

apply *inc\_cupP*.

assert  $(\forall\ beta : Rel\ D\ E, P\ beta \rightarrow f\ beta \quad (alpha \# \quad gamma))$ .

apply *inc\_cupP*.

apply *H*.

move  $\Rightarrow$  *beta H1*.

rewrite  $-(@inv\_involver\_alpha)$ .

apply *inc\_residual*.

apply  $(H0 \_ H1)$ .

rewrite  $-(@inv\_involver\_alpha)$ .

apply *inc\_residual*.

apply *inc\_cupP*.

assert  $(\forall\ beta : Rel\ D\ E, P\ beta \rightarrow (alpha \cdot f\ beta) \quad gamma)$ .

apply *inc\_cupP*.

apply *H*.

move  $\Rightarrow$  *beta H1*.

apply *inc\_residual*.

rewrite *inv\_invol*.

apply  $(H0 \_ H1)$ .

**Qed.**

**Lemma comp\_cup\_distr\_l**

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta\ gamma : Rel\ B\ C\}$ :  
 $alpha \cdot (beta \quad gamma) = (alpha \cdot beta) \quad (alpha \cdot gamma)$ .

**Proof.**

by [rewrite *cup\_to\_cupP -comp\_cupP\_distr\_l -cup\_to\_cupP*].

**Qed.**

**Lemma 126 (comp\_cupP\_distr\_r, comp\_cup\_distr\_r)** *Let  $f : (D \rightarrow E) \rightarrow (A \rightarrow B)$ ,  $\beta : B \rightarrow C$  and  $P : predicate$ . Then,*

$$(\sqcup_{P(\alpha)} f(\alpha)) \cdot \beta = \sqcup_{P(\alpha)} (f(\alpha) \cdot \beta).$$

**Lemma comp\_cupP\_distr\_r**  $\{A\ B\ C\ D\ E : eqType\}$

$\{beta : Rel\ B\ C\} \{f : Rel\ D\ E \rightarrow Rel\ A\ B\} \{P : Rel\ D\ E \rightarrow Prop\}$ :  
 $(\_ \{P\} f) \cdot beta = \_ \{P\} (\text{fun } alpha : Rel\ D\ E \Rightarrow (f\ alpha \cdot beta))$ .

**Proof.**

```

replace (fun alpha : Rel D E => f alpha · beta) with (fun alpha : Rel D E => (beta #
· f alpha #) #).
rewrite -inv_cupP_distr.
rewrite -comp_cupP_distr_l.
rewrite -inv_cupP_distr.
rewrite comp_inv.
by [rewrite inv_invol inv_invol].
apply functional_extensionality.
move => x.
rewrite comp_inv.
by [rewrite inv_invol inv_invol].
Qed.

```

**Lemma comp\_cup\_distr\_r**

$\{A\ B\ C : eqType\} \{alpha\ beta : Rel\ A\ B\} \{gamma : Rel\ B\ C\} :$   
 $(alpha\ \beta) \cdot gamma = (alpha \cdot gamma)\ (\beta \cdot gamma).$

**Proof.**

```

by [rewrite (@cup_to_cupP _ _ _ _ id) comp_cupP_distr_r -cup_to_cupP].
Qed.

```

**Lemma 127 (comp\_capP\_distr)** *Let  $\alpha : A \rightarrow B$ ,  $\gamma : C \rightarrow D$ ,  $f : (E \rightarrow F) \rightarrow (B \rightarrow C)$  and  $P : \text{predicate}$ . Then,*

$$\alpha \cdot (\sqcap_{P(\beta)} f(\beta)) \cdot \gamma \sqsubseteq \sqcap_{P(\beta)} (\alpha \cdot f(\beta) \cdot \gamma).$$

**Lemma comp\_capP\_distr**  $\{A\ B\ C\ D\ E\ F : eqType\}$

$\{alpha : Rel\ A\ B\} \{gamma : Rel\ C\ D\}$

$\{f : Rel\ E\ F \rightarrow Rel\ B\ C\} \{P : Rel\ E\ F \rightarrow Prop\} :$

$(alpha \cdot (\_ \{P\} f)) \cdot gamma$

$\_ \{P\} (\text{fun } beta : Rel\ E\ F \Rightarrow ((alpha \cdot f\ beta) \cdot gamma)).$

**Proof.**

```

apply inc_capP.
move => beta H.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
move : H.
apply inc_capP.
apply inc_refl.
Qed.

```

**Lemma 128 (comp\_capP\_distr\_l, comp\_cap\_distr\_l)** *Let  $\alpha : A \rightarrow B$ ,  $f : (D \rightarrow E) \rightarrow (B \rightarrow C)$  and  $P : \text{predicate}$ . Then,*

$$\alpha \cdot (\sqcap_{P(\beta)} f(\beta)) \sqsubseteq \sqcap_{P(\beta)} (\alpha \cdot f(\beta)).$$

**Lemma comp\_capP\_distr\_l**  $\{A B C D E : \text{eqType}\}$   
 $\{\alpha : \text{Rel } A B\} \{f : \text{Rel } D E \rightarrow \text{Rel } B C\} \{P : \text{Rel } D E \rightarrow \text{Prop}\}$ :  
 $(\alpha \cdot (\_ \{P\} f)) \_ \{P\} (\text{fun beta} : \text{Rel } D E \Rightarrow (\alpha \cdot f \text{ beta})).$

**Proof.**

move : (@comp\_capP\_distr \_ \_ \_ \_ \_ alpha (Id C) f P)  $\Rightarrow$  H.

rewrite comp\_id\_r in H.

replace (fun beta : Rel D E  $\Rightarrow$  (alpha · f beta) · Id C) with (fun beta : Rel D E  $\Rightarrow$  (alpha · f beta)) in H.

apply H.

apply functional\_extensionality.

move  $\Rightarrow$  x.

by [rewrite comp\_id\_r].

**Qed.**

**Lemma comp\_cap\_distr\_l**  
 $\{A B C : \text{eqType}\} \{\alpha : \text{Rel } A B\} \{\text{beta gamma} : \text{Rel } B C\}$ :  
 $(\alpha \cdot (\text{beta gamma})) ((\alpha \cdot \text{beta}) (\alpha \cdot \text{gamma})).$

**Proof.**

rewrite cap\_to\_capP (@cap\_to\_capP \_ \_ \_ \_ \_ id).

apply comp\_capP\_distr\_l.

**Qed.**

**Lemma 129 (comp\_capP\_distr\_r, comp\_cap\_distr\_r)** *Let  $\beta : B \rightarrow C$ ,  $f : (D \rightarrow E) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$(\sqcap_{P(\alpha)} f(\alpha)) \cdot \beta \sqsubseteq \sqcap_{P(\alpha)} (f(\alpha) \cdot \beta).$$

**Lemma comp\_capP\_distr\_r**  $\{A B C D E : \text{eqType}\} \{\text{beta} : \text{Rel } B C\} \{f : \text{Rel } D E \rightarrow \text{Rel } A B\} \{P : \text{Rel } D E \rightarrow \text{Prop}\}$ :  
 $((\_ \{P\} f) \cdot \text{beta}) \_ \{P\} (\text{fun alpha} : \text{Rel } D E \Rightarrow (f \text{ alpha} \cdot \text{beta})).$

**Proof.**

move : (@comp\_capP\_distr \_ \_ \_ \_ \_ (Id A) beta f P)  $\Rightarrow$  H.

rewrite comp\_id\_l in H.

replace (fun alpha : Rel D E  $\Rightarrow$  (Id A · f alpha) · beta) with (fun alpha : Rel D E  $\Rightarrow$  f alpha · beta) in H.

apply H.

apply functional\_extensionality.

move  $\Rightarrow x$ .  
 by [rewrite *comp\_id\_l*].  
 Qed.

**Lemma** *comp\_cap\_distr\_r*  $\{A\ B\ C : eqType\} \{alpha\ beta : Rel\ A\ B\} \{gamma : Rel\ B\ C\}$ :  
 $((alpha\ \beta) \cdot gamma) \quad ((alpha \cdot gamma) \quad (\beta \cdot gamma))$ .

**Proof.**

rewrite (@cap\_to\_capP \_ \_ \_ \_ id) (@cap\_to\_capP \_ \_ \_ \_ (fun x  $\Rightarrow$  x  $\cdot$  gamma)).  
 apply *comp\_capP\_distr\_r*.  
 Qed.

**Lemma 130** (*comp\_empty\_l*, *comp\_empty\_r*) *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ . Then,*

$$\alpha \cdot \phi_{BC} = \phi_{AB} \cdot \beta = \phi_{AC}.$$

**Lemma** *comp\_empty\_r*  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\}$ :  $alpha \cdot \quad B\ C = \quad A\ C$ .

**Proof.**

apply *inc\_antisym*.  
 rewrite -(@inv\_invol \_ \_ alpha).  
 apply *inc\_residual*.  
 apply *inc\_empty\_alpha*.  
 apply *inc\_empty\_alpha*.  
 Qed.

**Lemma** *comp\_empty\_l*  $\{A\ B\ C : eqType\} \{beta : Rel\ B\ C\}$ :  $A\ B \cdot beta = \quad A\ C$ .

**Proof.**

rewrite -(@inv\_invol \_ \_ (  $A\ B \cdot beta$ )).  
 rewrite -inv\_move comp\_inv inv\_empty inv\_empty.  
 apply *comp\_empty\_r*.  
 Qed.

**Lemma 131** (*comp\_either\_empty*) *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ . Then,*

$$\alpha = \phi_{AB} \vee \beta = \phi_{BC} \Rightarrow \alpha \cdot \beta = \phi_{AC}.$$

**Lemma** *comp\_either\_empty*  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$ :  
 $alpha = \quad A\ B \vee beta = \quad B\ C \rightarrow alpha \cdot beta = \quad A\ C$ .

**Proof.**

case; move  $\Rightarrow H$ .  
 rewrite *H*.  
 apply *comp\_empty\_l*.  
 rewrite *H*.  
 apply *comp\_empty\_r*.

**Qed.**

**Lemma 132 (comp\_neither\_empty)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ . Then,*

$$\alpha \cdot \beta \neq \phi_{AC} \Rightarrow \alpha \neq \phi_{AB} \wedge \beta \neq \phi_{BC}.$$

**Lemma** *comp\_neither\_empty* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:  
 alpha • beta ≠ A C → alpha ≠ A B ∧ beta ≠ B C.

**Proof.**

move ⇒ H.

split; move ⇒ H0.

apply H.

rewrite H0.

apply comp\_empty\_l.

apply H.

rewrite H0.

apply comp\_empty\_r.

**Qed.**

## 5.5 単域と Tarski の定理

**Lemma 133 (lemma\_for\_tarski1)** *Let  $\alpha : A \rightarrow B$  and  $\alpha \neq \phi_{AB}$ . Then,*

$$\nabla_{IA} \cdot \alpha \cdot \nabla_{BI} = id_I.$$

**Lemma** *lemma\_for\_tarski1* {A B : eqType} {alpha : Rel A B}:  
 alpha ≠ A B → (( i A • alpha) • B i) = Id i.

**Proof.**

move ⇒ H.

assert ((( i A • alpha) • B i) ≠ i i).

move ⇒ H0.

apply H.

apply inc\_antisym.

apply (@inc\_trans \_ \_ (( A i • (( i A • alpha) • B i)) • i B)).

rewrite comp\_assoc comp\_assoc unit\_universal.

rewrite -comp\_assoc -comp\_assoc unit\_universal.

apply (@inc\_trans \_ \_ ((Id A • alpha) • Id B)).

rewrite comp\_id\_l comp\_id\_r.

apply inc\_refl.

apply comp\_inc\_compat.

apply comp\_inc\_compat\_ab\_a'b.

apply inc\_alpha\_universal.

```

apply inc_alpha_universal.
rewrite H0 comp_empty_r comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
case (@unit_empty_or_universal (( i A • alpha) • B i)); move => H1.
apply False_ind.
apply (H0 H1).
rewrite unit_identity_is_universal.
apply H1.
Qed.

```

**Lemma 134 (lemma\_for\_tarski2)**

$$\nabla_{AI} \cdot \nabla_{IB} = \nabla_{AB}.$$

**Lemma** lemma\_for\_tarski2 {A B : eqType}: A i • i B = A B.

**Proof.**

```

apply inc_antisym.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ ( A A • A B)).
apply (@inc_trans _ _ _ (Id A • A B)).
rewrite comp_id_l.
apply inc_refl.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
rewrite -(@unit_universal A) comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.

```

**Qed.**

**Lemma 135 (tarski)** Let  $\alpha : A \rightarrow B$  and  $\alpha \neq \phi_{AB}$ . Then,

$$\nabla_{AA} \cdot \alpha \cdot \nabla_{BB} = \nabla_{AB}.$$

**Lemma** tarski {A B : eqType} {alpha : Rel A B}:

alpha ≠ A B → (( A A • alpha) • B B) = A B.

**Proof.**

```

move => H.
rewrite -(@unit_universal A) -(@unit_universal B).
move : (@lemma_for_tarski1 _ _ alpha H) => H0.
rewrite -comp_assoc (@comp_assoc _ _ _ _ ( A i)) (@comp_assoc _ _ _ _ ( A i)).
rewrite H0 comp_id_r.
apply lemma_for_tarski2.

```



**Qed.**

**Lemma 136 (comp\_universal1)** *Let  $B \neq \emptyset$ . Then,*

$$\nabla_{AB} \cdot \nabla_{BC} = \nabla_{AC}.$$

**Lemma comp\_universal**  $\{A\ B\ C : eqType\} : B \rightarrow A\ B \cdot B\ C = A\ C$ .

**Proof.**

```

move  $\Rightarrow$   $b$ .
replace (  $A\ B$ ) with (  $A\ B \cdot B\ B$ ).
rewrite -(@lemma_for_tarski2  $A\ B$ ) -(@lemma_for_tarski2  $B\ C$ ).
rewrite (@comp_assoc _ _ _ (  $A\ i$ )) (@comp_assoc _ _ _ (  $A\ i$ )) -(@comp_assoc _
_ _ _ (  $B\ i$ )).
rewrite lemma_for_tarski1.
rewrite comp_id_l.
apply lemma_for_tarski2.
apply not_eq_sym.
move  $\Rightarrow$   $H$ .
apply either_empty in  $H$ .
case  $H$ ; move  $\Rightarrow$   $H0$ .
apply ( $H0\ b$ ).
apply ( $H0\ b$ ).
apply inc_antisym.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ (  $A\ B \cdot Id\ B$ )).
rewrite comp_id_r.
apply inc_refl.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
Qed.

```

**Lemma 137 (comp\_universal2)**

$$\nabla_{IA}^\# \cdot \nabla_{IB} = \nabla_{AB}.$$

**Lemma comp\_universal2**  $\{A\ B : eqType\} : i\ A\ \# \cdot i\ B = A\ B$ .

**Proof.**

```

rewrite inv_universal.
apply lemma_for_tarski2.
Qed.

```

**Lemma 138** (`empty_equivalence1`, `empty_equivalence2`, `empty_equivalence3`)

$$A = \emptyset \Leftrightarrow \nabla_{IA} = \phi_{IA} \Leftrightarrow \nabla_{AA} = \phi_{AA} \Leftrightarrow id_A = \phi_{AA}.$$

**Lemma** `empty_equivalence1`  $\{A : eqType\} : (A \rightarrow False) \leftrightarrow \quad i \ A = \quad i \ A$ .

**Proof.**

`move : (@either_empty i A)  $\Rightarrow$  H.`

`split; move  $\Rightarrow$  H0.`

`apply Logic.eq_sym.`

`apply H.`

`right.`

`apply H0.`

`apply Logic.eq_sym in H0.`

`apply H in H0.`

`case H0.`

`move  $\Rightarrow$  H1 H2.`

`apply H1.`

`apply tt.`

`by [].`

**Qed.**

**Lemma** `empty_equivalence2`  $\{A : eqType\} : (A \rightarrow False) \leftrightarrow \quad A \ A = \quad A \ A$ .

**Proof.**

`move : (@either_empty A A)  $\Rightarrow$  H.`

`split; move  $\Rightarrow$  H0.`

`apply Logic.eq_sym.`

`apply H.`

`left.`

`apply H0.`

`apply Logic.eq_sym in H0.`

`apply H in H0.`

`case H0.`

`by [].`

`by [].`

**Qed.**

**Lemma** `empty_equivalence3`  $\{A : eqType\} : (A \rightarrow False) \leftrightarrow Id \ A = \quad A \ A$ .

**Proof.**

`split; move  $\Rightarrow$  H.`

`assert (  $A \ A = \quad A \ A$  ).`

`apply empty_equivalence2.`

`apply H.`

`apply RelAB.unique.`

`apply Logic.eq_sym.`

```
apply H0.
assert (  $\vdash A\ A = \vdash A\ A$  ).
by [rewrite -(@comp_id_r _ _ (  $\vdash A\ A$ )) H comp_empty_r].
apply either_empty in H0.
case H0.
by [].
by [].
Qed.
```

# Chapter 6

## Library **Functions\_Mappings**

```
Require Import Basic_Notations.  
Require Import Basic_Lemmas.  
Require Import Relation_Properties.  
Require Import Logic.FunctionalExtensionality.
```

### 6.1 全域性, 一価性, 写像に関する補題

**Lemma 139 (id\_function)**  $id_A : A \rightarrow A$  is a function.

```
Lemma id_function {A : eqType}: function_r (Id A).  
Proof.  
rewrite /function_r/total_r/univalent_r.  
rewrite inv_id comp_id_l.  
split.  
apply inc_refl.  
apply inc_refl.  
Qed.
```

**Lemma 140 (unit\_function)**  $\nabla_{AI} : A \rightarrow I$  is a function.

```
Lemma unit_function {A : eqType}: function_r ( A i).  
Proof.  
rewrite /function_r/total_r/univalent_r.  
rewrite inv_universal lemma_for_tarski2 unit_identity_is_universal.  
split.  
apply inc_alpha_universal.  
apply inc_alpha_universal.  
Qed.
```

## CHAPTER 6. LIBRARY FUNCTIONS\_MAPPINGS

**Lemma 141 (total\_comp)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  be total relations, then  $\alpha \cdot \beta$  is also a total relation.*

**Lemma** *total\_comp* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:

*total\_r alpha*  $\rightarrow$  *total\_r beta*  $\rightarrow$  *total\_r (alpha • beta)*.

**Proof.**

*rewrite /total\_r.*

*move*  $\Rightarrow$  *H H0*.

*rewrite comp\_inv comp\_assoc* -(@comp\_assoc - - - - beta).

*apply* (@inc\_trans - - - - H).

*apply comp\_inc\_compat\_ab\_ab'.*

*apply comp\_inc\_compat\_b\_ab.*

*apply H0.*

**Qed.**

**Lemma 142 (univalent\_comp)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  be univalent relations, then  $\alpha \cdot \beta$  is also a univalent relation.*

**Lemma** *univalent\_comp* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:

*univalent\_r alpha*  $\rightarrow$  *univalent\_r beta*  $\rightarrow$  *univalent\_r (alpha • beta)*.

**Proof.**

*rewrite /univalent\_r.*

*move*  $\Rightarrow$  *H H0*.

*rewrite comp\_inv comp\_assoc* -(@comp\_assoc - - - - (alpha #)).

*apply* (fun H'  $\Rightarrow$  @inc\_trans - - - - H' H0).

*apply comp\_inc\_compat\_ab\_ab'.*

*apply comp\_inc\_compat\_ab\_b.*

*apply H.*

**Qed.**

**Lemma 143 (function\_comp)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  be functions, then  $\alpha \cdot \beta$  is also a function.*

**Lemma** *function\_comp* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:

*function\_r alpha*  $\rightarrow$  *function\_r beta*  $\rightarrow$  *function\_r (alpha • beta)*.

**Proof.**

*elim*  $\Rightarrow$  *H H0*.

*elim*  $\Rightarrow$  *H1 H2*.

*split.*

*apply* (total\_comp H H1).

*apply* (univalent\_comp H0 H2).

**Qed.**

## CHAPTER 6. LIBRARY FUNCTIONS\_MAPPINGS

**Lemma 144 (total\_comp2)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  and  $\alpha \cdot \beta$  be a total relation, then  $\alpha$  is also a total relation.*

**Lemma** `total_comp2`  $\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel } A\ B\} \{beta : \text{Rel } B\ C\}$ :  
`total_r (alpha · beta) → total_r alpha.`

**Proof.**

`move ⇒ H.`

`apply inc_def1 in H.`

`rewrite comp_inv cap_comm comp_assoc in H.`

`rewrite /total_r.`

`rewrite H.`

`apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).`

`apply comp_inc_compat.`

`apply cap_l.`

`rewrite comp_id_r.`

`apply cap_r.`

**Qed.**

**Lemma 145 (univalent\_comp2)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$ ,  $\alpha \cdot \beta$  be a univalent relation and  $\alpha^\#$  be a total relation, then  $\beta$  is a univalent relation.*

**Lemma** `univalent_comp2`  $\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel } A\ B\} \{beta : \text{Rel } B\ C\}$ :  
`univalent_r (alpha · beta) → total_r (alpha #) → univalent_r beta.`

**Proof.**

`move ⇒ H H0.`

`apply (fun H' ⇒ @inc_trans _ _ _ _ H' H).`

`rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ alpha).`

`apply comp_inc_compat_ab_ab'.`

`rewrite /total_r in H0.`

`rewrite inv_invol in H0.`

`apply (comp_inc_compat_b_ab H0).`

**Qed.**

**Lemma 146 (total\_inc)** *Let  $\alpha : A \rightarrow B$  be a total relation and  $\alpha \sqsubseteq \beta$ , then  $\beta$  is also a total relation.*

**Lemma** `total_inc`  $\{A\ B : \text{eqType}\} \{alpha\ beta : \text{Rel } A\ B\}$ :  
`total_r alpha → alpha beta → total_r beta.`

**Proof.**

`move ⇒ H H0.`

`apply (@inc_trans _ _ _ _ H).`

`apply comp_inc_compat.`

`apply H0.`

apply (@inc\_inv \_ \_ \_ \_ H0).

**Qed.**

**Lemma 147 (univalent\_inc)** *Let  $\alpha : A \rightarrow B$  be a univalent relation and  $\beta \sqsubseteq \alpha$ , then  $\beta$  is also a univalent relation.*

**Lemma univalent\_inc** {A B : eqType} {alpha beta : Rel A B}:  
 univalent\_r alpha  $\rightarrow$  beta    alpha  $\rightarrow$  univalent\_r beta.

**Proof.**

move  $\Rightarrow$  H H0.

apply (fun H'  $\Rightarrow$  @inc\_trans \_ \_ \_ \_ H' H).

apply comp\_inc\_compat.

apply (@inc\_inv \_ \_ \_ \_ H0).

apply H0.

**Qed.**

**Lemma 148 (function\_inc)** *Let  $\alpha, \beta : A \rightarrow B$  be functions and  $\alpha \sqsubseteq \beta$ . Then,*

$$\alpha = \beta.$$

**Lemma function\_inc** {A B : eqType} {alpha beta : Rel A B}:  
 function\_r alpha  $\rightarrow$  function\_r beta  $\rightarrow$  alpha    beta  $\rightarrow$  alpha = beta.

**Proof.**

move  $\Rightarrow$  H H0 H1.

apply inc\_antisym.

apply H1.

apply (@inc\_trans \_ \_ \_ ((alpha  $\cdot$  alpha #)  $\cdot$  beta)).

apply comp\_inc\_compat\_b\_ab.

apply H.

move : (@inc\_inv \_ \_ \_ \_ H1)  $\Rightarrow$  H2.

apply (@inc\_trans \_ \_ \_ ((alpha  $\cdot$  beta #)  $\cdot$  beta)).

apply comp\_inc\_compat\_ab\_a'b.

apply comp\_inc\_compat\_ab\_ab'.

apply H2.

rewrite comp\_assoc.

apply comp\_inc\_compat\_ab\_a.

apply H0.

**Qed.**

**Lemma 149 (total\_universal)** *If  $\nabla_{IB}$  be a total relation, then*

$$\nabla_{AB} \cdot \nabla_{BC} = \nabla_{AC}.$$

## CHAPTER 6. LIBRARY FUNCTIONS\_MAPPINGS

**Lemma** *total\_universal*  $\{A\ B\ C : eqType\}$ :  
 $total\_r\ (\ \ i\ B) \rightarrow\ \ A\ B \cdot\ \ B\ C =\ \ A\ C.$

**Proof.**

move  $\Rightarrow H$ .

rewrite  $-(@lemma\_for\_tarski2\ A\ B)\ -(@lemma\_for\_tarski2\ B\ C).$

rewrite *comp\_assoc*  $-(@comp\_assoc\ \_ \_ \_ (\ \ i\ B)).$

replace  $(\ \ i\ B \cdot\ \ B\ i)$  with  $(Id\ i).$

rewrite *comp\_id\_l*.

apply *lemma\_for\_tarski2*.

apply *inc\_antisym*.

rewrite */total\_r* in  $H$ .

rewrite *inv\_universal* in  $H$ .

apply  $H$ .

rewrite *unit\_identity\_is\_universal*.

apply *inc\_alpha\_universal*.

**Qed.**

**Lemma 150 (function\_rel\_inv\_rel)** *Let  $\alpha : A \rightarrow B$  be function. Then,*

$$\alpha \cdot \alpha^\# \cdot \alpha = \alpha.$$

**Lemma** *function\_rel\_inv\_rel*  $\{A\ B : eqType\}\ \{\alpha : Rel\ A\ B\}$ :  
 $function\_r\ \alpha \rightarrow (\alpha \cdot \alpha^\#) \cdot \alpha = \alpha.$

**Proof.**

move  $\Rightarrow H$ .

apply *inc\_antisym*.

rewrite *comp\_assoc*.

apply *comp\_inc\_compat\_ab\_a*.

apply  $H$ .

apply *comp\_inc\_compat\_b\_ab*.

apply  $H$ .

**Qed.**

**Lemma 151 (function\_capP\_distr)** *Let  $f : A \rightarrow B, g : D \rightarrow C$  be functions,  $\theta : (E \rightarrow F) \rightarrow (B \rightarrow C)$  and  $P : predicate$ . Then,*

$$f \cdot (\sqcap_{P(\theta)} \theta(\alpha)) \cdot g^\# = \sqcap_{P(\alpha)} (f \cdot \theta(\alpha) \cdot g^\#).$$

**Lemma** *function\_capP\_distr*  $\{A\ B\ C\ D\ E\ F : eqType\}$   
 $\{f : Rel\ A\ B\}\ \{g : Rel\ D\ C\}\ \{\theta : Rel\ E\ F \rightarrow Rel\ B\ C\}\ \{P : Rel\ E\ F \rightarrow Prop\}$ :  
 $function\_r\ f \rightarrow function\_r\ g \rightarrow$   
 $(f \cdot (\ \_{-}\{P\}\ \theta)) \cdot g^\# =$   
 $\_{-}\{P\}\ (\text{fun}\ \alpha : Rel\ E\ F \Rightarrow (f \cdot \theta\ \alpha) \cdot g^\#).$



**Proof.**

```

elim ⇒ H H0.
elim ⇒ H1 H2.
apply inc_antisym.
apply comp_capP_distr.
apply (@inc_trans _ _ _ (((f · f #) · _{P} (fun alpha : Rel E F ⇒ (f · theta alpha)
· g #))) · (g · g #))).
apply (@inc_trans _ _ _ ((f · f #) · ( _{P} (fun alpha : Rel E F ⇒ (f · theta alpha)
· g #)))).
apply (comp_inc_compat_b_ab H).
apply (comp_inc_compat_a_ab H1).
rewrite (@comp_assoc _ _ _ _ (f #)) comp_assoc - (@comp_assoc _ _ _ _ g) - comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ ( _{P} (fun alpha : Rel E F ⇒ (f # · ((f · theta alpha) · g
#)) · g))).
apply comp_capP_distr.
replace (fun alpha : Rel E F ⇒ (f # · ((f · theta alpha) · g #)) · g) with (fun alpha
: Rel E F ⇒ ((f # · f) · theta alpha) · (g # · g)).
apply inc_capP.
move ⇒ beta H3.
apply (@inc_trans _ _ _ ((f # · f) · theta beta)).
apply (@inc_trans _ _ _ (((f # · f) · theta beta) · (g # · g))).
move : beta H3.
apply inc_capP.
apply inc_refl.
apply (comp_inc_compat_ab_a H2).
apply (comp_inc_compat_ab_b H0).
apply functional_extensionality.
move ⇒ x.
by [rewrite comp_assoc comp_assoc comp_assoc comp_assoc comp_assoc].
Qed.

```

**Lemma 152** (function\_cap\_distr, function\_cap\_distr\_l, function\_cap\_distr\_r)

Let  $f : A \rightarrow B, g : D \rightarrow C$  be functions and  $\alpha, \beta : B \rightarrow C$ . Then,

$$f \cdot (\alpha \sqcap \beta) \cdot g^\# = (f \cdot \alpha \cdot g^\#) \sqcap (f \cdot \beta \cdot g^\#).$$

**Lemma** function\_cap\_distr

$\{A\ B\ C\ D : \text{eqType}\} \{f : \text{Rel } A\ B\} \{\alpha\ \beta : \text{Rel } B\ C\} \{g : \text{Rel } D\ C\} :$   
 $\text{function\_r } f \rightarrow \text{function\_r } g \rightarrow$   
 $(f \cdot (\alpha \sqcap \beta)) \cdot g^\# = ((f \cdot \alpha) \cdot g^\#) \sqcap ((f \cdot \beta) \cdot g^\#).$

**Proof.**

## CHAPTER 6. LIBRARY FUNCTIONS\_MAPPINGS

```
rewrite (@cap_to_capP _ _ _ _ _ id) (@cap_to_capP _ _ _ _ _ (fun x => (f • x) • g
#)).
apply function_capP_distr.
Qed.
```

**Lemma** *function\_cap\_distr\_l*  
 $\{A\ B\ C : \text{eqType}\} \{f : \text{Rel } A\ B\} \{\text{alpha } \text{beta} : \text{Rel } B\ C\} :$   
 $\text{function\_r } f \rightarrow$   
 $f \cdot (\text{alpha } \text{beta}) = (f \cdot \text{alpha}) \quad (f \cdot \text{beta}).$

**Proof.**  
 $\text{move} : (@\text{id\_function } C) \Rightarrow H.$   
 $\text{move} \Rightarrow H0.$   
 $\text{apply } (@\text{function\_cap\_distr } _ _ _ f \text{ alpha } \text{beta}) \text{ in } H.$   
 $\text{rewrite inv\_id comp\_id\_r comp\_id\_r comp\_id\_r in } H.$   
 $\text{apply } H.$   
 $\text{apply } H0.$   
**Qed.**

**Lemma** *function\_cap\_distr\_r*  
 $\{B\ C\ D : \text{eqType}\} \{\text{alpha } \text{beta} : \text{Rel } B\ C\} \{g : \text{Rel } D\ C\} :$   
 $\text{function\_r } g \rightarrow$   
 $(\text{alpha } \text{beta}) \cdot g \# = (\text{alpha } \cdot g \#) \quad (\text{beta } \cdot g \#).$

**Proof.**  
 $\text{move} : (@\text{id\_function } B) \Rightarrow H.$   
 $\text{move} \Rightarrow H0.$   
 $\text{apply } (@\text{function\_cap\_distr } _ _ _ _ \text{alpha } \text{beta } g) \text{ in } H.$   
 $\text{rewrite comp\_id\_l comp\_id\_l comp\_id\_l in } H.$   
 $\text{apply } H.$   
 $\text{apply } H0.$   
**Qed.**

**Lemma 153 (function\_move1)** *Let  $\alpha : A \rightarrow B$  be a function,  $\beta : B \rightarrow C$  and  $\gamma : A \rightarrow C$ . Then,*

$$\gamma \sqsubseteq \alpha \cdot \beta \Leftrightarrow \alpha^\# \cdot \gamma \sqsubseteq \beta.$$

**Lemma** *function\_move1*  $\{A\ B\ C : \text{eqType}\} \{\text{alpha} : \text{Rel } A\ B\} \{\text{beta} : \text{Rel } B\ C\} \{\text{gamma} : \text{Rel } A\ C\} :$   
 $\text{function\_r } \text{alpha} \rightarrow (\text{gamma } (\text{alpha } \cdot \text{beta}) \leftrightarrow (\text{alpha } \# \cdot \text{gamma}) \text{ beta}).$

**Proof.**  
 $\text{move} \Rightarrow H.$   
 $\text{split; move} \Rightarrow H0.$   
 $\text{apply } (@\text{inc\_trans } _ _ ((\text{alpha } \# \cdot \text{alpha}) \cdot \text{beta})).$   
 $\text{rewrite comp\_assoc.}$   
 $\text{apply } (\text{comp\_inc\_compat\_ab\_ab'} H0).$

```

apply comp_inc_compat_ab_b.
apply H.
apply (@inc_trans _ _ _ ((alpha · alpha #) · gamma)).
apply comp_inc_compat_b_ab.
apply H.
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H0).
Qed.

```

**Lemma 154 (function\_move2)** *Let  $\beta : B \rightarrow C$  be a function,  $\alpha : A \rightarrow B$  and  $\gamma : A \rightarrow C$ . Then,*

$$\alpha \cdot \beta \sqsubseteq \gamma \Leftrightarrow \alpha \sqsubseteq \gamma \cdot \beta^\#.$$

**Lemma function\_move2**  $\{A B C : \text{eqType}\} \{\alpha : \text{Rel } A B\} \{\beta : \text{Rel } B C\} \{\gamma : \text{Rel } A C\}$ :

$\text{function\_r } \beta \rightarrow ((\alpha \cdot \beta) \quad \gamma \leftrightarrow \alpha \quad (\gamma \cdot \beta \#)).$

**Proof.**

```

move  $\Rightarrow$  H.
split; move  $\Rightarrow$  H0.
apply (@inc_trans _ _ _ ((alpha · beta) · beta #)).
rewrite comp_assoc.
apply comp_inc_compat_a_ab.
apply H.
apply (comp_inc_compat_ab_a'b H0).
apply (@inc_trans _ _ _ ((gamma · beta #) · beta)).
apply (comp_inc_compat_ab_a'b H0).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H.
Qed.

```

**Lemma 155 (function\_rpc\_distr)** *Let  $f : A \rightarrow B, g : D \rightarrow C$  be functions and  $\alpha, \beta : B \rightarrow C$ . Then,*

$$f \cdot (\alpha \Rightarrow \beta) \cdot g^\# = (f \cdot \alpha \cdot g^\#) \Rightarrow (f \cdot \beta \cdot g^\#).$$

**Lemma function\_rpc\_distr**

$\{A B C D : \text{eqType}\} \{f : \text{Rel } A B\} \{\alpha \beta : \text{Rel } B C\} \{g : \text{Rel } D C\}$ :

$\text{function\_r } f \rightarrow \text{function\_r } g \rightarrow$

$(f \cdot (\alpha \gg \beta)) \cdot g \# = ((f \cdot \alpha) \cdot g \#) \gg ((f \cdot \beta) \cdot g \#).$

**Proof.**

```

move  $\Rightarrow$  H H0.
apply inc_lower.

```

```

move  $\Rightarrow$  gamma.
split; move  $\Rightarrow$  H1.
apply inc_rpc.
apply (function_move2 H0).
apply (function_move1 H).
apply (@inc_trans _ _ _ (((f #  $\cdot$  gamma)  $\cdot$  g) ((f #  $\cdot$  ((f  $\cdot$  alpha)  $\cdot$  g #))  $\cdot$  g))).
rewrite -comp_assoc.
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (@comp_cap_distr_r _ _ _ _ _)).
apply comp_inc_compat_ab_a'b.
apply comp_cap_distr_l.
apply (function_move2 H0) in H1.
apply (function_move1 H) in H1.
rewrite -inc_rpc comp_assoc.
apply (@inc_trans _ _ _ _ H1).
apply rpc_inc_compat_r.
rewrite comp_assoc comp_assoc comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ (alpha  $\cdot$  (g #  $\cdot$  g))).
apply comp_inc_compat_ab_b.
apply H.
apply comp_inc_compat_ab_a.
apply H0.
apply (function_move2 H0).
apply (function_move1 H).
apply inc_rpc.
apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).
apply (@inc_trans _ _ _ (f #  $\cdot$  ((gamma  $\cdot$  g) ((f #) #  $\cdot$  alpha)))).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite inv_invol.
apply (@inc_trans _ _ _ ((f #  $\cdot$  (gamma ((f  $\cdot$  alpha)  $\cdot$  g #)))  $\cdot$  g)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).
apply comp_inc_compat_ab_ab'.
apply cap_l.
apply (function_move2 H0).
apply (function_move1 H).
rewrite -inc_rpc -comp_assoc.
apply H1.
Qed.

```

**Lemma 156 (function\_inv\_rel1, function\_inv\_rel2)** *Let  $f : A \rightarrow B$  be a function. Then,*

$$f^\# \cdot f = id_B \sqcap f^\# \cdot \nabla_{AA} \cdot f = id_B \sqcap \nabla_{BA} \cdot f.$$

**Lemma** *function\_inv\_rel1* { $A\ B : eqType$ } { $f : Rel\ A\ B$ }:  
*function\_r*  $f \rightarrow f \# \cdot f = Id\ B \quad ((f \# \cdot \quad A\ A) \cdot f).$

**Proof.**

move  $\Rightarrow H$ .  
 apply *inc\_antisym*.  
 apply *inc\_cap*.  
 split.  
 apply  $H$ .  
 apply *comp\_inc\_compat\_ab\_a'b*.  
 apply *comp\_inc\_compat\_a\_ab*.  
 apply *inc\_alpha\_universal*.  
 apply (*@inc\_trans* \_ \_ \_ (*Id B* ( *B A*  $\cdot f$ ))).  
 apply *cap\_inc\_compat\_l*.  
 apply *comp\_inc\_compat\_ab\_a'b*.  
 apply *inc\_alpha\_universal*.  
 rewrite *cap\_comm*.  
 apply (*@inc\_trans* \_ \_ \_ \_ (*@dedekind* \_ \_ \_ \_ \_)).  
 rewrite *comp\_id\_l comp\_id\_r cap\_comm inv\_universal*.  
 rewrite *cap\_universal cap\_universal*.  
 apply *inc\_refl*.

**Qed.**

**Lemma** *function\_inv\_rel2* { $A\ B : eqType$ } { $f : Rel\ A\ B$ }:  
*function\_r*  $f \rightarrow f \# \cdot f = Id\ B \quad ( \quad B\ A \cdot f).$

**Proof.**

move  $\Rightarrow H$ .  
 apply *inc\_antisym*.  
 rewrite (*@function\_inv\_rel1* \_ \_ \_  $H$ ).  
 apply *cap\_inc\_compat\_l*.  
 apply *comp\_inc\_compat\_ab\_a'b*.  
 apply *inc\_alpha\_universal*.  
 rewrite *cap\_comm*.  
 apply (*@inc\_trans* \_ \_ \_ \_ \_ (*@dedekind* \_ \_ \_ \_ \_)).  
 rewrite *comp\_id\_l comp\_id\_r cap\_comm inv\_universal*.  
 rewrite *cap\_universal cap\_universal*.  
 apply *inc\_refl*.

**Qed.**

**Lemma 157 (function\_dedekind1, function\_dedekind2)** *Let  $f : A \rightarrow B$  be a function,  $\mu : C \rightarrow A$  and  $\rho : C \rightarrow B$ . Then,*

$$(\mu \sqcap \rho \cdot f^\#) \cdot f = \mu \cdot f \sqcap \rho \wedge \rho \cdot f^\# \cdot f = \nabla_{CA} \cdot f \sqcap \rho.$$

**Lemma function\_dedekind1**

$\{A\ B\ C : \text{eqType}\} \{f : \text{Rel}\ A\ B\} \{\mu : \text{Rel}\ C\ A\} \{\rho : \text{Rel}\ C\ B\} :$   
 $\text{function\_r } f \rightarrow (\mu \quad (\rho \cdot f^\#)) \cdot f = (\mu \cdot f) \quad \rho.$

**Proof.**

move  $\Rightarrow H$ .

apply *inc\_antisym*.

apply (*@inc\_trans* \_ \_ \_ \_ (*comp\_cap\_distr\_r*)).

apply *cap\_inc\_compat\_l*.

rewrite *comp\_assoc*.

apply *comp\_inc\_compat\_ab\_a*.

apply *H*.

apply (*@inc\_trans* \_ \_ \_ \_ (*@dedekind* \_ \_ \_ \_)).

apply *comp\_inc\_compat\_ab\_ab'*.

apply *cap\_l*.

**Qed.**

**Lemma function\_dedekind2**  $\{A\ B\ C : \text{eqType}\} \{f : \text{Rel}\ A\ B\} \{\rho : \text{Rel}\ C\ B\} :$   
 $\text{function\_r } f \rightarrow (\rho \cdot f^\#) \cdot f = (\quad C\ A \cdot f) \quad \rho.$

**Proof.**

move  $\Rightarrow H$ .

move : (*@function\_dedekind1* \_ \_ \_ *f* (*C A*) *rho H*)  $\Rightarrow H0$ .

rewrite *cap\_comm cap\_universal* in *H0*.

apply *H0*.

**Qed.**

## 6.2 全射, 単射に関する補題

**Lemma 158 (surjection\_comp)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  be surjections, then  $\alpha \cdot \beta$  is also a surjection.*

**Lemma surjection\_comp**  $\{A\ B\ C : \text{eqType}\} \{\alpha : \text{Rel}\ A\ B\} \{\beta : \text{Rel}\ B\ C\} :$   
 $\text{surjection\_r } \alpha \rightarrow \text{surjection\_r } \beta \rightarrow \text{surjection\_r } (\alpha \cdot \beta).$

**Proof.**

rewrite */surjection\_r*.

elim  $\Rightarrow H\ H0$ .

elim  $\Rightarrow H1\ H2$ .

split.

```

apply (function_comp H H1).
rewrite comp_inv.
apply (total_comp H2 H0).
Qed.

```

**Lemma 159 (injection\_comp)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  be injections, then  $\alpha \cdot \beta$  is also an injection.*

**Lemma** *injection\_comp*  $\{A\ B\ C : \text{eqType}\} \{\alpha : \text{Rel } A\ B\} \{\beta : \text{Rel } B\ C\}$ :  
 $\text{injection\_r } \alpha \rightarrow \text{injection\_r } \beta \rightarrow \text{injection\_r } (\alpha \cdot \beta)$ .

**Proof.**  
rewrite /injection\_r.  
elim  $\Rightarrow$   $H\ H0$ .  
elim  $\Rightarrow$   $H1\ H2$ .  
split.  
apply (function\_comp H H1).  
rewrite comp\_inv.  
apply (univalent\_comp H2 H0).  
Qed.

**Lemma 160 (bijection\_comp)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$  be bijections, then  $\alpha \cdot \beta$  is also a bijection.*

**Lemma** *bijection\_comp*  $\{A\ B\ C : \text{eqType}\} \{\alpha : \text{Rel } A\ B\} \{\beta : \text{Rel } B\ C\}$ :  
 $\text{bijection\_r } \alpha \rightarrow \text{bijection\_r } \beta \rightarrow \text{bijection\_r } (\alpha \cdot \beta)$ .

**Proof.**  
rewrite /bijection\_r.  
elim  $\Rightarrow$   $H$ .  
elim  $\Rightarrow$   $H0\ H1$ .  
elim  $\Rightarrow$   $H2$ .  
elim  $\Rightarrow$   $H3\ H4$ .  
split.  
apply (function\_comp H H2).  
rewrite comp\_inv.  
split.  
apply (total\_comp H3 H0).  
apply (univalent\_comp H4 H1).  
Qed.

**Lemma 161 (surjection\_unique1)** *Let  $e : A \twoheadrightarrow B$  be a surjection,  $f : A \rightarrow C$  be a function and  $e \cdot e^\sharp \sqsubseteq f \cdot f^\sharp$ , then there exists a unique function  $g : B \rightarrow C$  s.t.  $f = eg$ .*

**Lemma** *surjection\_unique1*  $\{A\ B\ C : \text{eqType}\} \{e : \text{Rel } A\ B\} \{f : \text{Rel } A\ C\}$ :

$surjection\_r\ e \rightarrow function\_r\ f \rightarrow (e \cdot e \#) \quad (f \cdot f \#) \rightarrow$   
 $(\exists! g : Rel\ B\ C, function\_r\ g \wedge f = e \cdot g).$

**Proof.**

```

rewrite /surjection_r/function_r/total_r/univalent_r.
elim.
elim  $\Rightarrow H\ H0\ H1$ .
elim  $\Rightarrow H2\ H3\ H4$ .
 $\exists (e \# \cdot f)$ .
repeat split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ f).
apply (@inc_trans _ _ _ _ H1).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H2.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ e).
apply (@inc_trans _ _ _ (f #  $\cdot ((f \cdot f \#) \cdot f)))$ .
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H4).
rewrite comp_assoc -comp_assoc.
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' H3).
apply (comp_inc_compat_ab_a H3).
apply function_inc.
split.
apply H2.
apply H3.
split.
rewrite /total_r.
rewrite comp_inv comp_inv inv_invol.
rewrite -(@comp_assoc _ _ _ e) (@comp_assoc _ _ _ e) (@comp_assoc _ _ _ f)
-(@comp_assoc _ _ _ f).
apply (@inc_trans _ _ _ _ H).
apply comp_inc_compat_a_ab.
apply (@inc_trans _ _ _ _ H2).
apply (comp_inc_compat_a_ab H).
rewrite /univalent_r.
rewrite comp_inv comp_inv inv_invol.
rewrite (@comp_assoc _ _ _ e) -(@comp_assoc _ _ _ e) comp_assoc -(@comp_assoc
_ _ _ _ f).
apply (@inc_trans _ _ _ (f #  $\cdot (((f \cdot f \#) \cdot (f \cdot f \#)) \cdot f)))$ .
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat.
    
```



```

apply H4.
apply H4.
rewrite comp_assoc (@comp_assoc _ _ _ _ f) - (@comp_assoc _ _ _ _ (f #)) - (@comp_assoc
_ _ _ _ (f #)) (@comp_assoc _ _ _ _ (f #)) - (@comp_assoc _ _ _ _ (f #)).
apply (fun H' => @inc_trans _ _ _ _ H' H3).
apply comp_inc_compat_ab_a.
apply (fun H' => @inc_trans _ _ _ _ H' H3).
apply (comp_inc_compat_ab_a H3).
rewrite -comp_assoc.
apply (comp_inc_compat_b_ab H).
move => g.
elim.
elim => H5 H6 H7.
replace g with (e # • (e • g)).
apply f_equal.
apply H7.
rewrite -comp_assoc.
apply inc_antisym.
apply (comp_inc_compat_ab_b H0).
rewrite inv_invol in H1.
apply (comp_inc_compat_b_ab H1).
Qed.

```

**Lemma 162 (surjection\_unique2)** *Let  $e : A \twoheadrightarrow B$  be a surjection,  $f : A \rightarrow C$  be a function and  $e \cdot e^\# = f \cdot f^\#$ , then function  $e^\# f$  is an injection.*

**Lemma** *surjection\_unique2* {A B C : eqType} {e : Rel A B} {f : Rel A C}:  
 surjection\_r e → function\_r f → (e • e #) = (f • f #) → injection\_r (e # • f).

**Proof.**

```

rewrite /surjection_r/injection_r/function_r/total_r/univalent_r.
elim.
elim => H H0 H1.
elim => H2 H3 H4.
repeat split.
rewrite comp_inv comp_assoc - (@comp_assoc _ _ _ _ f).
apply (@inc_trans _ _ _ _ H1).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H2.
rewrite comp_inv inv_invol comp_assoc - (@comp_assoc _ _ _ _ e).
rewrite H4.
rewrite comp_assoc -comp_assoc.
apply (fun H' => @inc_trans _ _ _ _ H' H3).

```

```

apply (comp_inc_compat_ab_a H3).
rewrite inv_invol comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ f).
rewrite -H4.
rewrite comp_assoc -comp_assoc.
apply (fun H' => @inc_trans _ _ _ _ H' H0).
apply comp_inc_compat_ab_a.
apply H0.
Qed.

```

**Lemma 163 (injection\_unique1)** *Let  $m : B \rightarrowtail A$  be an injection,  $f : C \rightarrow A$  be a function and  $f^\# \cdot f \sqsubseteq m^\# \cdot m$ , then there exists a unique function  $g : C \rightarrow B$  s.t.  $f = gm$ .*

**Lemma** *injection\_unique1* {A B C : eqType} {m : Rel B A} {f : Rel C A}:  
 injection\_r m → function\_r f → (f # · f) (m # · m) →  
 (∃! g : Rel C B, function\_r g ∧ f = g · m).

**Proof.**

```

rewrite /injection_r/function_r/total_r/univalent_r.
elim.
elim => H H0 H1.
elim => H2 H3 H4.
∃ (f · m #).
repeat split.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ m).
apply (@inc_trans _ _ _ (f · ((f # · f) · f #))).
rewrite comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ _ H2).
apply (comp_inc_compat_a_ab H2).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H4).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ f).
apply (fun H' => @inc_trans _ _ _ _ H' H1).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).
rewrite comp_assoc.
apply Logic.eq-sym.
apply function_inc.
split.
rewrite /total_r.
rewrite comp_inv comp_inv inv_invol.
apply (@inc_trans _ _ _ _ H2).
apply comp_inc_compat.
apply (@inc_trans _ _ _ (f · (f # · f))).
rewrite -comp_assoc.

```

```

apply (comp_inc_compat_b_ab H2).
apply (comp_inc_compat_ab_ab' H4).
apply (@inc_trans _ _ _ ((f # · f) · f #)).
rewrite comp_assoc.
apply (comp_inc_compat_a_ab H2).
apply (comp_inc_compat_ab_a'b H4).
rewrite /univalent_r.
rewrite comp_inv comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ f).
apply (fun H' ⇒ @inc_trans _ _ _ _ H' H0).
apply comp_inc_compat_ab_a.
apply (fun H' ⇒ @inc_trans _ _ _ _ H' H3).
apply (comp_inc_compat_ab_a H0).
split.
apply H2.
apply H3.
apply (comp_inc_compat_ab_a H0).
move ⇒ g.
elim.
elim ⇒ H5 H6 H7.
rewrite H7 comp_assoc.
apply inc_antisym.
rewrite inv_invol in H1.
apply (comp_inc_compat_ab_a H1).
apply (comp_inc_compat_a_ab H).
Qed.

```

**Lemma 164 (injection\_unique2)** *Let  $m : B \rightarrowtail A$  be an injection,  $f : C \rightarrow A$  be a function and  $f^\# \cdot f = m^\# \cdot m$ , then function  $f \cdot m^\#$  is a surjection.*

**Lemma** *injection\_unique2* {A B C : eqType} {m : Rel B A} {f : Rel C A}:  
 injection\_r m → function\_r f → (f # · f) = (m # · m) → surjection\_r (f · m #).

**Proof.**

```

rewrite /surjection_r/injection_r/function_r/total_r/univalent_r.
elim.
elim ⇒ H H0 H1.
elim ⇒ H2 H3 H4.
repeat split.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ m).
apply (@inc_trans _ _ _ (f · ((f # · f) · f #))).
rewrite comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ _ H2).
apply (comp_inc_compat_a_ab H2).
apply comp_inc_compat_ab_ab'.

```

```

rewrite  $H_4$ .
apply inc_refl.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _  $f$ ).
apply (fun  $H' \Rightarrow$  @inc_trans _ _ _ _  $H'$   $H1$ ).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b  $H3$ ).
rewrite inv_invol comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _  $f$ ).
apply (@inc_trans _ _ _ _  $H$ ).
apply comp_inc_compat_ab_ab'.
rewrite  $H_4$  comp_assoc.
apply (comp_inc_compat_a_ab  $H$ ).
Qed.

```

**Lemma 165 (*bijection\_inv*)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow A$ ,  $\alpha \cdot \beta = id_A$  and  $\beta \cdot \alpha = id_B$ , then  $\alpha$  and  $\beta$  are bijections and  $\beta = \alpha^\#$ .*

**Lemma *bijection\_inv***  $\{A\ B : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ A\}$ :  
 $alpha \cdot beta = Id\ A \rightarrow beta \cdot alpha = Id\ B \rightarrow bijection\_r\ alpha \wedge bijection\_r\ beta \wedge$   
 $beta = alpha\ \#$ .

**Proof.**

```

move  $\Rightarrow$   $H\ H0$ .
move : (@id_function  $A$ )  $\Rightarrow$   $H1$ .
move : (@id_function  $B$ )  $\Rightarrow$   $H2$ .
assert (bijection_r  $alpha$   $\wedge$  bijection_r  $beta$ ).
assert (total_r  $alpha$   $\wedge$  total_r ( $alpha\ \#$ )  $\wedge$  total_r  $beta$   $\wedge$  total_r ( $beta\ \#$ )).
repeat split.
apply (@total_comp2 _ _ _ _  $beta$ ).
rewrite  $H$ .
apply  $H1$ .
apply (@total_comp2 _ _ _ _ ( $beta\ \#$ )).
rewrite -comp_inv  $H0$  inv_id.
apply  $H2$ .
apply (@total_comp2 _ _ _ _  $alpha$ ).
rewrite  $H0$ .
apply  $H2$ .
apply (@total_comp2 _ _ _ _ ( $alpha\ \#$ )).
rewrite -comp_inv  $H$  inv_id.
apply  $H1$ .
repeat split.
apply  $H3$ .
apply (@univalent_comp2 _ _ _ _  $beta$ ).
rewrite  $H0$ .
apply  $H2$ .

```

```

apply H3.
apply H3.
apply (@univalent_comp2 _ _ _ (beta #)).
rewrite -comp_inv H inv_id.
apply H1.
rewrite inv_invol.
apply H3.
apply H3.
apply (@univalent_comp2 _ _ _ alpha).
rewrite H.
apply H1.
apply H3.
apply H3.
apply (@univalent_comp2 _ _ _ (alpha #)).
rewrite -comp_inv H0 inv_id.
apply H2.
rewrite inv_invol.
apply H3.
split.
apply H3.
split.
apply H3.
rewrite -(@comp_id_r _ _ beta) -(@comp_id_l _ _ (alpha #)).
rewrite -H0 comp_assoc.
apply f_equal.
apply inc_antisym.
apply H3.
rewrite comp_inv_inv -inv_inc_move inv_id.
apply H3.
Qed.

```

**Lemma 166 (bijection\_inv\_corollary)** *Let  $\alpha : A \rightarrow B$  be a bijection, then  $\alpha^\#$  is also a bijection.*

**Lemma** *bijection\_inv\_corollary*  $\{A\ B : \text{eqType}\} \{\alpha : \text{Rel } A\ B\}$ :  
*bijection\_r alpha  $\rightarrow$  bijection\_r (alpha #).*

**Proof.**

```

move : (@bijection_inv _ _ alpha (alpha #))  $\Rightarrow$  H.
move  $\Rightarrow$  H0.
rewrite /bijection_r/function_r/total_r/univalent_r in H0.
rewrite inv_invol in H0.
apply H.

```

apply *inc\_antisym*.

apply *H0*.

apply *H0*.

apply *inc\_antisym*.

apply *H0*.

apply *H0*.

**Qed.**

# Chapter 7

## Library Dedekind

Require Import Basic\_Notations.  
Require Import Basic\_Lemmas.  
Require Import Relation\_Properties.  
Require Import Functions\_Mappings.

### 7.1 Dedekind formula に関する補題

**Lemma 167 (dedekind1)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  and  $\gamma : A \rightarrow C$ . Then*

$$\alpha \cdot \beta \sqcap \gamma \sqsubseteq \alpha \cdot (\beta \sqcap \alpha^\# \cdot \gamma).$$

**Lemma dedekind1**

$\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel } A\ B\} \{beta : \text{Rel } B\ C\} \{gamma : \text{Rel } A\ C\} :$   
 $((alpha \cdot beta) \quad gamma) \quad (alpha \cdot (beta \quad (alpha \# \cdot gamma)))$ .

**Proof.**

apply (@inc\_trans \_ \_ \_ \_ (@dedekind \_ \_ \_ \_ \_)).  
apply comp\_inc\_compat\_ab\_a'b.  
apply cap\_l.

**Qed.**

**Lemma 168 (dedekind2)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  and  $\gamma : A \rightarrow C$ . Then*

$$\alpha \cdot \beta \sqcap \gamma \sqsubseteq (\alpha \sqcap \gamma \cdot \beta^\#) \cdot \beta.$$

**Lemma dedekind2**

$\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel } A\ B\} \{beta : \text{Rel } B\ C\} \{gamma : \text{Rel } A\ C\} :$   
 $((alpha \cdot beta) \quad gamma) \quad ((alpha \quad (gamma \cdot beta \#)) \cdot beta)$ .

**Proof.**

apply (@inc\_trans \_ \_ \_ \_ (@dedekind \_ \_ \_ \_ \_)).

apply *comp\_inc\_compat\_ab\_ab'*.  
 apply *cap\_l*.  
 Qed.

**Lemma 169 (relation\_rel\_inv\_rel)** *Let  $\alpha : A \rightarrow B$ . Then*

$$\alpha \sqsubseteq \alpha \cdot \alpha^\# \cdot \alpha.$$

**Lemma** *relation\_rel\_inv\_rel* {*A B : eqType*} {*alpha : Rel A B*}:  
*alpha* ((*alpha* · *alpha* #) · *alpha*).

**Proof.**

move : (@dedekind1 \_ \_ \_ *alpha* (*Id B*) *alpha*)  $\Rightarrow$  *H*.  
 rewrite *comp\_id\_r cap\_idem* in *H*.  
 apply (@*inc\_trans* \_ \_ \_ \_ *H*).  
 rewrite *comp\_assoc*.  
 apply *comp\_inc\_compat\_ab\_ab'*.  
 apply *cap\_r*.  
 Qed.

## 7.2 Dedekind formula と全関係

**Lemma 170 (dedekind\_universal1)** *Let  $\alpha : B \rightarrow C$ . Then*

$$\nabla_{AC} \cdot \alpha^\# \cdot \alpha = \nabla_{AB} \cdot \alpha.$$

**Lemma** *dedekind\_universal1* {*A B C : eqType*} {*alpha : Rel B C*}:  
 ( *A C* · *alpha* #) · *alpha* = *A B* · *alpha*.

**Proof.**

apply *inc\_antisym*.  
 apply *comp\_inc\_compat\_ab\_a'b*.  
 apply *inc\_alpha\_universal*.  
 apply (@*inc\_trans* \_ \_ \_ ( *A B* · ((*alpha* · *alpha* #) · *alpha*))).  
 apply *comp\_inc\_compat\_ab\_ab'*.  
 apply *relation\_rel\_inv\_rel*.  
 rewrite -*comp\_assoc* -*comp\_assoc*.  
 apply *comp\_inc\_compat\_ab\_a'b*.  
 apply *comp\_inc\_compat\_ab\_a'b*.  
 apply *inc\_alpha\_universal*.  
 Qed.



**Lemma 171** (`dedekind_universal2a`, `dedekind_universal2b`, `dedekind_universal2c`) *Let  $\alpha : A \rightarrow B$  and  $\beta : C \rightarrow B$ . Then*

$$\nabla_{IC} \cdot \beta \sqsubseteq \nabla_{IA} \cdot \alpha \Leftrightarrow \nabla_{CC} \cdot \beta \sqsubseteq \nabla_{CA} \cdot \alpha \Leftrightarrow \beta \sqsubseteq \beta \cdot \alpha^\# \cdot \alpha.$$

**Lemma** `dedekind_universal2a`  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ C\ B\} :$   
 $(\ i\ C \cdot beta) \ (\ i\ A \cdot alpha) \rightarrow (\ C\ C \cdot beta) \ (\ C\ A \cdot alpha).$

**Proof.**

`move  $\Rightarrow$  H.`  
`rewrite -unit_universal -(@lemma_for_tarski2 C A).`  
`rewrite comp_assoc comp_assoc.`  
`apply (comp_inc_compat_ab_ab' H).`  
**Qed.**

**Lemma** `dedekind_universal2b`  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ C\ B\} :$   
 $(\ C\ C \cdot beta) \ (\ C\ A \cdot alpha) \rightarrow beta \ ((beta \cdot alpha \#) \cdot alpha).$

**Proof.**

`move  $\Rightarrow$  H.`  
`apply (@inc_trans _ _ _ (beta (\ C\ C \cdot beta))).`  
`apply inc_cap.`  
`split.`  
`apply inc_refl.`  
`apply comp_inc_compat_b_ab.`  
`apply inc_alpha_universal.`  
`apply (@inc_trans _ _ _ (beta (\ C\ A \cdot alpha))).`  
`apply (cap_inc_compat_l H).`  
`rewrite cap_comm.`  
`apply (@inc_trans _ _ _ _ (dedekind2)).`  
`apply comp_inc_compat_ab_a'b.`  
`apply cap_r.`  
**Qed.**

**Lemma** `dedekind_universal2c`  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ C\ B\} :$   
 $beta \ ((beta \cdot alpha \#) \cdot alpha) \rightarrow (\ i\ C \cdot beta) \ (\ i\ A \cdot alpha).$

**Proof.**

`move  $\Rightarrow$  H.`  
`apply (@inc_trans _ _ _ (\ i\ C \cdot ((beta \cdot alpha \#) \cdot alpha))).`  
`apply (comp_inc_compat_ab_ab' H).`  
`rewrite -comp_assoc.`  
`apply comp_inc_compat_ab_a'b.`  
`apply inc_alpha_universal.`  
**Qed.**

**Lemma 172 (dedekind\_universal3a, dedekind\_universal3b)** *Let  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$ . Then*

$$\beta \cdot \nabla_{CI} \sqsubseteq \alpha \cdot \nabla_{BI} \Leftrightarrow \beta \cdot \nabla_{CC} \sqsubseteq \alpha \cdot \nabla_{BC} \Leftrightarrow \beta \sqsubseteq \alpha \cdot \alpha^\# \cdot \beta.$$

**Lemma dedekind\_universal3a**  $\{A\ B\ C : \text{eqType}\} \{\alpha : \text{Rel } A\ B\} \{\beta : \text{Rel } A\ C\} :$   
 $(\beta \cdot C\ i) \quad (\alpha \cdot B\ i) \Leftrightarrow (\beta \cdot C\ C) \quad (\alpha \cdot B\ C).$

**Proof.**

split; move  $\Rightarrow H$ .  
 apply inv\_inc\_invol.  
 rewrite comp\_inv comp\_inv inv\_universal inv\_universal.  
 apply dedekind\_universal2a.  
 apply inv\_inc\_invol.  
 rewrite comp\_inv comp\_inv inv\_invol inv\_invol inv\_universal inv\_universal.  
 apply H.  
 apply inv\_inc\_invol.  
 rewrite comp\_inv comp\_inv inv\_universal inv\_universal.  
 apply dedekind\_universal2c.  
 apply dedekind\_universal2b.  
 apply inv\_inc\_invol.  
 rewrite comp\_inv comp\_inv inv\_invol inv\_invol inv\_universal inv\_universal.  
 apply H.

**Qed.**

**Lemma dedekind\_universal3b**  $\{A\ B\ C : \text{eqType}\} \{\alpha : \text{Rel } A\ B\} \{\beta : \text{Rel } A\ C\} :$   
 $(\beta \cdot C\ i) \quad (\alpha \cdot B\ i) \Leftrightarrow \beta \quad ((\alpha \cdot \alpha^\#) \cdot \beta).$

**Proof.**

split; move  $\Rightarrow H$ .  
 apply inv\_inc\_invol.  
 rewrite comp\_inv comp\_inv -comp\_assoc.  
 apply dedekind\_universal2b.  
 apply dedekind\_universal2a.  
 apply inv\_inc\_invol.  
 rewrite comp\_inv comp\_inv inv\_invol inv\_invol inv\_universal inv\_universal.  
 apply H.  
 apply inv\_inc\_invol.  
 rewrite comp\_inv comp\_inv inv\_universal inv\_universal.  
 apply dedekind\_universal2c.  
 rewrite -comp\_inv -comp\_inv -comp\_assoc.  
 apply inc\_inv.  
 apply H.

**Qed.**

**Lemma 173 (universal\_total)** *Let  $\alpha : A \rightarrow B$ . Then*

$$\alpha \cdot \nabla_{BI} = \nabla_{AI} \Leftrightarrow \text{“}\alpha \text{ is total”}.$$

**Lemma** *universal\_total* { $A B : eqType$ } { $\alpha : Rel A B$ }:  
 $\alpha \cdot \nabla_{BI} = \nabla_{AI} \Leftrightarrow total\_r \alpha$ .

**Proof.**

```
move : (@dedekind_universal3b _ _ _ alpha (Id A)) => H.
rewrite comp_id_l comp_id_r in H.
rewrite /total_r.
rewrite -H.
split; move => H0.
rewrite H0.
apply inc_refl.
apply inc_antisym.
apply inc_alpha_universal.
apply H0.
Qed.
```

### 7.3 Dedekind formula と恒等関係

**Lemma 174 (dedekind\_id1)** *Let  $\alpha : A \rightarrow A$ . Then*

$$\alpha \sqsubseteq id_A \Rightarrow \alpha^\# = \alpha.$$

**Lemma** *dedekind\_id1* { $A : eqType$ } { $\alpha : Rel A A$ }:  $\alpha \sqsubseteq Id A \rightarrow \alpha^\# = \alpha$ .

**Proof.**

```
move => H.
assert (alpha # alpha).
move : (@dedekind1 _ _ _ (alpha #) (Id A) (Id A)) => H0.
rewrite comp_id_r comp_id_r inv_invol in H0.
replace (alpha # Id A) with (alpha #) in H0.
replace (Id A alpha) with alpha in H0.
apply (@inc_trans _ _ _ (alpha # • alpha)).
apply H0.
apply comp_inc_compat_ab_b.
rewrite -inv_inc_move.
rewrite inv_id.
apply H.
rewrite cap_comm.
apply inc_def1.
```

```

apply H.
apply inc_def1.
rewrite -inv_inc_move.
rewrite inv_id.
apply H.
apply inc_antisym.
apply H0.
apply inv_inc_move.
apply H0.
Qed.

```

**Lemma 175 (dedekind\_id2)** *Let  $\alpha : A \rightarrow A$ . Then*

$$\alpha \sqsubseteq id_A \Rightarrow \alpha \cdot \alpha = \alpha.$$

**Lemma dedekind\_id2**  $\{A : eqType\} \{alpha : Rel\ A\ A\}$ :  
 $alpha \quad Id\ A \rightarrow alpha \cdot alpha = alpha.$

**Proof.**

```

move  $\Rightarrow$  H.
apply inc_antisym.
apply (comp_inc_compat_ab_a H).
move : (dedekind_id1 H)  $\Rightarrow$  H0.
apply (@inc_trans _ _ _ ((alpha  $\cdot$  Id A) Id A)).
rewrite comp_id_r.
apply inc_cap.
split.
apply inc_refl.
apply H.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite H0 comp_id_r.
apply cap_r.
Qed.

```

**Lemma 176 (dedekind\_id3)** *Let  $\alpha, \beta : A \rightarrow A$ . Then*

$$\alpha \sqsubseteq id_A \wedge \beta \sqsubseteq id_A \Rightarrow \alpha \cdot \beta = \alpha \sqcap \beta.$$

**Lemma dedekind\_id3**  $\{A : eqType\} \{alpha\ beta : Rel\ A\ A\}$ :  
 $alpha \quad Id\ A \rightarrow beta \quad Id\ A \rightarrow alpha \cdot beta = alpha \sqcap beta.$

**Proof.**

```

move  $\Rightarrow$  H H0.
apply inc_antisym.

```

```

apply inc_cap.
split.
apply (comp_inc_compat_ab_a H0).
apply (comp_inc_compat_ab_b H).
replace (alpha beta) with ((alpha beta) • (alpha beta)).
apply comp_inc_compat.
apply cap_l.
apply cap_r.
apply dedekind_id2.
apply (fun H' => @inc_trans _ _ _ _ H' H).
apply cap_l.
Qed.

```

**Lemma 177 (dedekind\_id4)** *Let  $\alpha, \beta : A \rightarrow A$ . Then*

$$\alpha \sqsubseteq id_A \wedge \beta \sqsubseteq id_A \Rightarrow (\alpha \triangleright \beta) \sqcap id_A = (\alpha \Rightarrow \beta) \sqcap id_A.$$

**Lemma dedekind\_id4**  $\{A : eqType\} \{alpha\ beta : Rel\ A\ A\}$ :  
 $alpha\ Id\ A \rightarrow beta\ Id\ A \rightarrow (alpha\ beta)\ Id\ A = (alpha \gg beta)\ Id\ A.$

**Proof.**

```

move => H H0.
apply inc_lower.
move => gamma.
rewrite inc_cap inc_cap.
split; elim => H1 H2.
split.
rewrite inc_rpc cap_comm.
rewrite -(@dedekind_id3 _ _ _ H H2).
rewrite -(@dedekind_id1 _ _ H).
apply inc_residual.
apply H1.
apply H2.
split.
rewrite inc_residual (@dedekind_id1 _ _ H) (@dedekind_id3 _ _ _ H H2).
rewrite cap_comm -inc_rpc.
apply H1.
apply H2.
Qed.

```

# Chapter 8

## Library Rationality

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
```

### 8.1 有理性から導かれる系

**Lemma 178 (rationality\_corollary1)** *Let  $u : A \rightarrow A$  and  $u \sqsubseteq id_A$ . Then,*

$$\exists R, \exists j : R \rightarrowtail A, u = j^\# \cdot j.$$

**Lemma** *rationality\_corollary1* { $A : eqType$ } { $u : Rel\ A\ A$ }:  
 $u \sqsubseteq Id\ A \rightarrow \exists (R : eqType)(j : Rel\ R\ A), injection\_r\ j \wedge u = j^\# \cdot j.$

**Proof.**

```
move : (rationality _ _ u).
elim => R.
elim => f.
elim => g.
elim => H.
elim => H0.
elim => H1 H2 H3.
exists R.
exists f.
assert (g = f).
apply (function_inc H0 H).
apply (@inc_trans _ _ _ ((f · f #) · g)).
apply comp_inc_compat_b_ab.
apply H.
rewrite comp_assoc -H1.
```

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```

apply (comp_inc_compat_ab_a H3).
rewrite H4 in H1.
rewrite H4 cap_idem in H2.
split.
split.
apply H.
rewrite /univalent_r.
rewrite inv_invol H2.
apply inc_refl.
apply H1.
Qed.

```

**Lemma 179 (rationality\_corollary2)** *Let  $f : A \rightarrow B$  be a function. Then,*

$$\exists e : A \rightarrow R, \exists m : R \rightarrow B, f = e \cdot m.$$

**Lemma rationality\_corollary2**  $\{A\ B : eqType\} \{f : Rel\ A\ B\}$ :  
 $function\_r\ f \rightarrow \exists (R : eqType)(e : Rel\ A\ R)(m : Rel\ R\ B),\ surjection\_r\ e \wedge injection\_r\ m.$

**Proof.**

```

elim  $\Rightarrow$  H H0.
move : (@rationality_corollary1 _ (f # · f) H0).
elim  $\Rightarrow$  R.
elim  $\Rightarrow$  m.
elim  $\Rightarrow$  H1 H2.
 $\exists$  R.
 $\exists$  (f · m #).
 $\exists$  m.
split.
apply (injection_unique2 H1 (conj H H0) H2).
apply H1.
Qed.

```

**Lemma 180 (axiom\_of\_subobjects)** *Let  $u : A \rightarrow A$  and  $u \sqsubseteq id_A$ . Then,*

$$\exists R, \exists j : R \rightarrow A, j^\# \cdot j = u \wedge j \cdot j^\# = id_R.$$

**Lemma axiom\_of\_subobjects**  $\{A : eqType\} \{u : Rel\ A\ A\}$ :  
 $u\ Id\ A \rightarrow \exists (R : eqType)(j : Rel\ R\ A), j^\# \cdot j = u \wedge j \cdot j^\# = Id\ R.$

**Proof.**

```

move  $\Rightarrow$  H.
elim (rationality_corollary1 H)  $\Rightarrow$  R.
elim  $\Rightarrow$  j H0.

```

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```

 $\exists R.$ 
 $\exists j.$ 
split.
apply Logic.eq_sym.
apply H0.
apply inc_antisym.
replace ( $j \cdot j \#$ ) with  $((j \#) \# \cdot j \#)$ .
apply H0.
by [rewrite inv_invol].
apply H0.
Qed.

```

**Lemma 181 (square\_diagram)** *In below figure,*

$$f \cdot x = g \cdot y \Leftrightarrow f^\# \cdot g \sqsubseteq x \cdot y^\#.$$

$$\begin{array}{ccc}
 X & \xrightarrow{f} & A \\
 g \downarrow & & \downarrow x \\
 B & \xrightarrow{y} & D
 \end{array}$$

**Lemma** *square\_diagram*  $\{X \ A \ B \ D : eqType\}$   
 $\{f : Rel \ X \ A\} \{g : Rel \ X \ B\} \{x : Rel \ A \ D\} \{y : Rel \ B \ D\}$ :  
 $function\_r \ f \rightarrow function\_r \ g \rightarrow function\_r \ x \rightarrow function\_r \ y \rightarrow$   
 $(f \cdot x = g \cdot y \leftrightarrow (f \# \cdot g) \sqsubseteq (x \cdot y \#)).$

**Proof.**

```

move  $\Rightarrow$  H H0 H1 H2.
split; move  $\Rightarrow$  H3.
rewrite  $-(function\_move1 \ H) -comp\_assoc -(function\_move2 \ H2) \ H3$ .
apply inc_refl.
apply Logic.eq_sym.
apply function_inc.
apply  $(function\_comp \ H0 \ H2)$ .
apply  $(function\_comp \ H \ H1)$ .
rewrite  $(function\_move2 \ H2) \ comp\_assoc (function\_move1 \ H)$ .
apply H3.
Qed.

```



# Chapter 9

## Library Conjugate

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
```

### 9.1 共役性の定義

条件  $P$  を満たす関係  $\alpha : A \rightarrow B$  と条件  $Q$  を満たす関係  $\beta : A' \rightarrow B'$  が変換  $\alpha = \phi(\beta), \beta = \psi(\alpha)$  によって, 1 対 1 (全射的) に対応することを, 図式

$$\frac{\alpha : A \rightarrow B \ \{P\} \quad \alpha = \phi(\beta)}{\beta : A' \rightarrow B' \ \{Q\} \quad \beta = \psi(\alpha)}$$

によって表す. また, Coq では以下のように表すことにする.

**Definition** *conjugate*

```
(A B C D : eqType) (P : Rel A B → Prop) (Q : Rel C D → Prop)
(phi : Rel C D → Rel A B) (psi : Rel A B → Rel C D) :=
(∀ alpha : Rel A B, P alpha → Q (psi alpha) ∧ phi (psi alpha) = alpha)
∧ (∀ beta : Rel C D, Q beta → P (phi beta) ∧ psi (phi beta) = beta).
```

さらに, 上の図式において条件  $P$  または  $Q$  が不要な場合には, 以下の `True_r` を代入する.

**Definition** *True\_r* {A B : eqType} := fun \_ : Rel A B ⇒ True.

## 9.2 共役の例

**Lemma 182 (inv\_conjugate)** *Inverse relation ( $\#$ ) makes conjugate. That is,*

$$\frac{\alpha : A \rightarrow B}{\beta : B \rightarrow A} \frac{\alpha = \beta^\#}{\beta = \alpha^\#}.$$

**Lemma** *inv\_conjugate* {A B : eqType}:

*conjugate A B B A True\_r True\_r (@inverse -) (@inverse -).*

**Proof.**

split.

move  $\Rightarrow$  *alpha H*.

split.

by [].

apply *inv\_invol*.

move  $\Rightarrow$  *beta H*.

split.

by [].

apply *inv\_invol*.

**Qed.**

**Lemma 183 (injection\_conjugate)** *Let  $j : C \hookrightarrow B$  be an injection. Then,*

$$\frac{f : A \rightarrow B \quad \{f^\# \cdot f \sqsubseteq j^\# \cdot j\}}{h : A \rightarrow C} \quad \frac{f = h \cdot j}{h = f \cdot j^\#}$$

**Lemma** *injection\_conjugate* {A B C : eqType} {j : Rel C B}:

*injection\_r j  $\rightarrow$*

*conjugate A B A C (fun f : Rel A B  $\Rightarrow$  ((f #  $\cdot$  f) (j #  $\cdot$  j))  $\wedge$  function\_r f)*

*(fun h : Rel A C  $\Rightarrow$  function\_r h) (fun h : Rel A C  $\Rightarrow$  h  $\cdot$  j)*

*(fun f : Rel A B  $\Rightarrow$  f  $\cdot$  j #).*

**Proof.**

elim.

elim  $\Rightarrow$  *H H0 H1*.

split.

move  $\Rightarrow$  *alpha*.

elim  $\Rightarrow$  *H2*.

elim  $\Rightarrow$  *H3 H4*.

assert (function\_r (alpha  $\cdot$  j #)).

split.

apply (@inc\_trans - - - - H3).

rewrite *comp\_inv inv\_invol comp\_assoc* -(@comp\_assoc - - - - j).

```
apply (@inc_trans _ _ _ (alpha • ((alpha # • alpha) • alpha #))).
rewrite comp_assoc -comp_assoc.
apply (comp_inc_compat_a_ab H3).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H2).
apply (fun H' ⇒ @inc_trans _ _ _ _ H' H1).
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ alpha).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
apply (@inc_trans _ _ _ _ H2).
apply H0.
split.
apply H5.
apply function_inc.
apply function_comp.
apply H5.
split.
apply H.
apply H0.
split.
apply H3.
apply H4.
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H0.
move ⇒ beta.
elim ⇒ H2 H3.
assert (function_r (beta • j)).
split.
apply (@inc_trans _ _ _ _ H2).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ j).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H).
apply (fun H' ⇒ @inc_trans _ _ _ _ H' H0).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).
split.
split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).
```

```

apply H4.
rewrite comp_assoc.
replace (j • j #) with (Id C).
apply comp_id_r.
apply inc_antisym.
apply H.
rewrite /univalent_r in H1.
rewrite inv_invol in H1.
apply H1.
Qed.

```

**Lemma 184 (injection\_conjugate\_corollary1, injection\_conjugate\_corollary2)**  
*Let  $j : C \rightarrow B$  be an injection and  $f : A \rightarrow B$  be a function. Then,*

$$f^\# \cdot f \sqsubseteq j^\# \cdot j \Leftrightarrow (\exists! h : A \rightarrow C, f = h \cdot j) \Leftrightarrow (\exists h' : A \rightarrow C, f \sqsubseteq h' \cdot j).$$

**Lemma injection\_conjugate\_corollary1** {A B C : eqType} {f : Rel A B} {j : Rel C B}:  
*injection\_r j → function\_r f →*  
*((f # • f) (j # • j) ↔ ∃! h : Rel A C, function\_r h ∧ f = h • j).*

**Proof.**

```

move ⇒ H H0.
move : (@injection_conjugate A _ _ H).
elim ⇒ H1 H2.
split; move ⇒ H3.
∃ (f • j #).
split.
move : (H1 f (conj H3 H0)).
elim ⇒ H4 H5.
split.
apply H4.
by [rewrite H5].
move ⇒ h.
elim ⇒ H4 H5.
rewrite H5 comp_assoc.
replace (j • j #) with (Id C).
apply comp_id_r.
rewrite /injection_r/function_r/univalent_r in H.
rewrite inv_invol in H.
apply inc_antisym.
apply H.
apply H.
elim H3 ⇒ h.
elim.

```

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```

elim ⇒ H4 H5 H6.
rewrite H5 comp_inv comp_assoc -(@comp_assoc _ _ _ _ h).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
apply H4.
Qed.

Lemma injection_conjugate_corollary2 {A B C : eqType} {f : Rel A B} {j : Rel C B}:
  injection_r j → function_r f →
  ((f # • f) (• j # • j) ↔ ∃ h' : Rel A C, f (• h' • j)).

Proof.
move ⇒ H H0.
split; move ⇒ H1.
apply (injection_conjugate_corollary1 H H0) in H1.
elim H1 ⇒ h.
elim.
elim ⇒ H2 H3 H4.
∃ h.
rewrite H3.
apply inc_refl.
elim H1 ⇒ h' H2.
replace (f # • f) with (f # • (f (• h' • j))).
apply (@inc_trans _ _ ((f # • f) • (j # • j))).
rewrite comp_assoc cap_comm -(@comp_assoc _ _ _ _ f).
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
apply cap_r.
apply comp_inc_compat_ab_b.
apply H0.
apply f_equal.
apply inc_def1 in H2.
by [rewrite -H2].
Qed.

```

**Lemma 185 (surjection\_conjugate)** *Let  $e : A \twoheadrightarrow C$  be a surjection. Then,*

$$\frac{f : A \rightarrow B \quad \{e \cdot e^\# \sqsubseteq f \cdot f^\#\}}{h : C \rightarrow B} \quad \frac{f = e \cdot h}{h = e^\# \cdot f}$$

**Lemma** *surjection\_conjugate* {*A B C* : *eqType*} {*e* : *Rel A C*}:  
*surjection\_r e* →  
*conjugate A B C B* (**fun** *f* : *Rel A B* ⇒ ((*e* • *e* #) (• *f* • *f* #)) ∧ *function\_r f*)

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---

(**fun**  $h : \text{Rel } C \ B \Rightarrow \text{function\_r } h$ ) (**fun**  $h : \text{Rel } C \ B \Rightarrow e \cdot h$ ) (**fun**  $f : \text{Rel } A \ B \Rightarrow e \# \cdot f$ ).

**Proof.**

elim.

elim  $\Rightarrow H \ H0 \ H1$ .

split.

move  $\Rightarrow \text{alpha}$ .

elim  $\Rightarrow H2$ .

elim  $\Rightarrow H3 \ H4$ .

assert ( $\text{function\_r } (e \# \cdot \text{alpha})$ ).

split.

apply ( $\text{@inc\_trans } \_ \_ \_ \_ H1$ ).

rewrite  $\text{comp\_inv inv\_involver comp\_assoc } -(\text{@comp\_assoc } \_ \_ \_ \_ \text{alpha})$ .

apply  $\text{comp\_inc\_compat\_ab\_ab'}$ .

apply ( $\text{comp\_inc\_compat\_b\_ab } H3$ ).

apply (**fun**  $H' \Rightarrow \text{@inc\_trans } \_ \_ \_ \_ H' \ H4$ ).

rewrite  $\text{comp\_inv inv\_involver comp\_assoc } -(\text{@comp\_assoc } \_ \_ \_ \_ e)$ .

apply ( $\text{@inc\_trans } \_ \_ \_ (\text{alpha} \# \cdot ((\text{alpha} \cdot \text{alpha} \#) \cdot \text{alpha}))$ ).

apply  $\text{comp\_inc\_compat\_ab\_ab'}$ .

apply ( $\text{comp\_inc\_compat\_ab\_a'b } H2$ ).

rewrite  $\text{comp\_assoc } -\text{comp\_assoc}$ .

apply ( $\text{comp\_inc\_compat\_ab\_a } H4$ ).

split.

apply  $H5$ .

apply  $\text{Logic.eq\_sym}$ .

apply  $\text{function\_inc}$ .

split.

apply  $H3$ .

apply  $H4$ .

apply  $\text{function\_comp}$ .

split.

apply  $H$ .

apply  $H0$ .

apply  $H5$ .

rewrite  $-\text{comp\_assoc}$ .

apply  $\text{comp\_inc\_compat\_b\_ab}$ .

apply  $H$ .

move  $\Rightarrow \text{beta}$ .

elim  $\Rightarrow H2 \ H3$ .

assert ( $\text{function\_r } (e \cdot \text{beta})$ ).

split.

apply ( $\text{@inc\_trans } \_ \_ \_ \_ H$ ).

## CHAPTER 9. LIBRARY CONJUGATE

```

rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H2).
apply (fun H' => @inc_trans _ _ _ _ H' H3).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ e).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H0).
split.
split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H2).
apply H4.
rewrite -comp_assoc.
replace (e # • e) with (Id C).
apply comp_id_l.
apply inc_antisym.
rewrite /total_r in H1.
rewrite inv_invol in H1.
apply H1.
apply H0.
Qed.

```

**Lemma 186 (surjection\_conjugate\_corollary)** *Let  $e : A \twoheadrightarrow C$  be a surjection and  $f : A \rightarrow B$  be a function. Then,*

$$e \cdot e^\# \sqsubseteq f \cdot f^\# \Leftrightarrow (\exists! h : C \rightarrow B, f = e \cdot h).$$

**Lemma** *surjection\_conjugate\_corollary*  $\{A\ B\ C : \text{eqType}\} \{f : \text{Rel } A\ B\} \{e : \text{Rel } A\ C\}$ :  
 $\text{surjection\_r } e \rightarrow \text{function\_r } f \rightarrow$   
 $((e \cdot e^\#) \sqsubseteq (f \cdot f^\#)) \leftrightarrow \exists! h : \text{Rel } C\ B, \text{function\_r } h \wedge f = e \cdot h).$

**Proof.**

```

move => H H0.
move : (@surjection_conjugate _ B _ _ H).
elim => H1 H2.
split; move => H3.
exists (e # • f).
split.
move : (H1 f (conj H3 H0)).
elim => H4 H5.
split.
apply H4.
by [rewrite H5].

```

```

move ⇒ h.
elim ⇒ H4 H5.
rewrite H5 -comp_assoc.
replace (e # · e) with (Id C).
apply comp_id_l.
rewrite /surjection_r/function_r/total_r in H.
rewrite inv_invol in H.
apply inc_antisym.
apply H.
apply H.
elim H3 ⇒ h.
elim.
elim ⇒ H4 H5 H6.
rewrite H5 comp_inv comp_assoc -(@comp_assoc - - - h).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H4.
Qed.

```

**Lemma 187 (subid\_conjugate)** *Subidentity  $u \sqsubseteq id_A$  corresponds  $\rho : I \rightarrow A$ . That is,*

$$\frac{\rho : I \rightarrow A}{u : A \rightarrow A \{u \sqsubseteq id_A\}} \quad \frac{\rho = \nabla_{IA} \cdot u}{u = id_A \sqcap \nabla_{AI} \cdot \rho}.$$

**Lemma subid\_conjugate**  $\{A : eqType\}$ :  
*conjugate i A A A True\_r (fun u : Rel A A ⇒ u (Id A)*  
*(fun u : Rel A A ⇒ i A · u) (fun rho : Rel i A ⇒ Id A (A i · rho)).*

**Proof.**  
split.  
move ⇒ alpha H.  
split.  
apply cap\_l.  
apply inc\_antisym.  
apply (@inc\_trans - - - (i A · (A i · alpha))).  
apply comp\_inc\_compat\_ab\_ab'.  
apply cap\_r.  
rewrite -comp\_assoc.  
apply comp\_inc\_compat\_ab\_b.  
rewrite unit\_identity\_is\_universal.  
apply inc\_alpha\_universal.  
rewrite -(@inv\_universal i A).  
apply (fun H' ⇒ @inc\_trans - - - H' (dedekind1)).  
rewrite comp\_id\_r cap\_comm cap\_universal.



```

apply inc_refl.
move ⇒ beta H.
split.
by [].
apply inc_antisym.
rewrite cap_comm -comp_assoc lemma_for_tarski2.
apply (@inc_trans _ _ _ _ (dedekind2)).
rewrite comp_id_l cap_comm cap_universal.
apply comp_inc_compat_ab_b.
rewrite -inv_inc_move inv_id.
apply H.
apply inc_cap.
split.
apply H.
rewrite -comp_assoc.
apply comp_inc_compat_b_ab.
rewrite lemma_for_tarski2.
apply inc_alpha_universal.
Qed.

```

**Lemma 188 (subid\_conjugate\_corollary1)** *Let  $u, v : A \rightarrow A$  and  $u, v \sqsubseteq id_A$ . Then,*

$$\nabla_{IA} \cdot u = \nabla_{IA} \cdot v \Rightarrow u = v.$$

**Lemma subid\_conjugate\_corollary1**  $\{A : eqType\} \{u \ v : Rel \ A \ A\}$ :  
 $u \quad Id \ A \rightarrow v \quad Id \ A \rightarrow \quad i \ A \cdot u = \quad i \ A \cdot v \rightarrow u = v.$

**Proof.**

```

move ⇒ H H0 H1.
move : (@subid_conjugate A).
elim ⇒ H2 H3.
move : (H3 u H).
elim ⇒ H4 H5.
rewrite -H5.
move : (H3 v H0).
elim ⇒ H6 H7.
rewrite -H7.
apply f_equal.
apply f_equal.
apply H1.
Qed.

```

**Lemma 189 (subid\_conjugate\_corollary2)** *Let  $\rho, \rho' : I \rightarrow A$ . Then,*

$$id_A \sqcap \nabla_{AI} \cdot \rho = id_A \sqcap \nabla_{AI} \cdot \rho' \Rightarrow \rho = \rho'.$$

**Lemma** *subid\_conjugate\_corollary2*  $\{A : eqType\} \{rho\ rho' : Rel\ i\ A\}$ :  
 $Id\ A \quad ( \quad A\ i \cdot rho ) = Id\ A \quad ( \quad A\ i \cdot rho' ) \rightarrow rho = rho'.$

**Proof.**

move  $\Rightarrow H$ .

move : (*@subid\_conjugate A*).

elim  $\Rightarrow H0\ H1$ .

move : (*H0 rho I*).

elim  $\Rightarrow H2\ H3$ .

rewrite -*H3*.

move : (*H0 rho' I*).

elim  $\Rightarrow H4\ H5$ .

rewrite -*H5*.

apply f\_equal.

apply *H*.

**Qed.**

# Chapter 10

## Library Domain

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Logic.FunctionalExtensionality.
```

### 10.1 定義域の定義

関係  $\alpha : A \rightarrow B$  に対して, その定義域 (関係)  $[\alpha] : A \rightarrow A$  は,

$$[\alpha] = \alpha \cdot \alpha^\# \sqcap id_A$$

で表される. また, Coq では以下のように表すことにする.

**Definition** *domain*  $\{A\ B : eqType\}$  ( $alpha : Rel\ A\ B$ ) := ( $alpha \cdot alpha^\#$ )  $Id\ A$ .

### 10.2 定義域の性質

#### 10.2.1 基本的な性質

**Lemma 190** (*domain\_another\_def*) *Let*  $\alpha : A \rightarrow B$ . *Then,*

$$[\alpha] = \alpha \cdot \nabla_{BA} \sqcap id_A.$$

**Lemma** *domain\_another\_def*  $\{A\ B : eqType\}$   $\{alpha : Rel\ A\ B\}$ :  
 $domain\ alpha = (alpha \cdot \nabla_{BA}) \sqcap Id\ A$ .

**Proof.**

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```

apply inc_antisym.
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply inc_cap.
split.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm comp_id_r cap_universal.
apply inc_refl.
apply cap_r.
Qed.

```

**Lemma 191 (domain\_inv)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\lfloor \alpha \rfloor^\# = \lfloor \alpha \rfloor.$$

**Lemma domain\_inv**  $\{A B : eqType\} \{alpha : Rel A B\}$ :  
 $(domain\ alpha) \# = domain\ alpha$ .

**Proof.**

```

apply dedekind_id1.
apply cap_r.
Qed.

```

**Lemma 192 (domain\_comp\_alpha1, domain\_comp\_alpha2)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\lfloor \alpha \rfloor \cdot \alpha = \alpha \wedge \alpha^\# \cdot \lfloor \alpha \rfloor = \alpha^\#.$$

**Lemma domain\_comp\_alpha1**  $\{A B : eqType\} \{alpha : Rel A B\}$ :  
 $(domain\ alpha) \cdot alpha = alpha$ .

**Proof.**

```

apply inc_antisym.
apply comp_inc_compat_ab_b.
apply cap_r.
rewrite /domain.
rewrite cap_comm.
apply (fun H' => @inc_trans _ _ _ _ H' (dedekind2)).
rewrite comp_id_l cap_idem.
apply inc_refl.
Qed.

```

**Lemma domain\_comp\_alpha2**  $\{A B : eqType\} \{alpha : Rel A B\}$ :  
 $alpha \# \cdot (domain\ alpha) = alpha \#$ .

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**Proof.**

```
rewrite -domain_inv -comp_inv.
apply f_equal.
apply domain_comp_alpha1.
Qed.
```

**Lemma 193 (domain\_inc\_compat)** *Let  $\alpha, \alpha' : A \rightarrow B$ . Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \lfloor \alpha \rfloor \sqsubseteq \lfloor \alpha' \rfloor.$$

**Lemma** *domain\_inc\_compat* {A B : eqType} {alpha alpha' : Rel A B}:  
 $\text{alpha} \quad \text{alpha}' \rightarrow \text{domain alpha} \quad \text{domain alpha}'.$

**Proof.**

```
move => H.
apply cap_inc_compat_r.
apply comp_inc_compat.
apply H.
apply (@inc_inv _ _ _ H).
Qed.
```

**Lemma 194 (domain\_total)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$“\alpha \text{ is total}” \Leftrightarrow \lfloor \alpha \rfloor = \text{id}_A.$$

**Lemma** *domain\_total* {A B : eqType} {alpha : Rel A B}:  
 $\text{total\_r alpha} \leftrightarrow \text{domain alpha} = \text{Id A}.$

**Proof.**

```
split; move => H.
rewrite /domain.
rewrite cap_comm.
apply Logic.eq_sym.
apply inc_def1.
apply H.
apply inc_def1.
rewrite /domain in H.
by [rewrite cap_comm H].
Qed.
```

**Lemma 195 (domain\_inc\_id)** *Let  $u : A \rightarrow A$ . Then,*

$$u \sqsubseteq \text{id}_A \Leftrightarrow \lfloor u \rfloor = u.$$

**Lemma** *domain\_inc\_id* {A : eqType} {u : Rel A A}:  $u \quad \text{Id A} \leftrightarrow \text{domain } u = u.$

**Proof.**

```
split; move => H.
rewrite /domain.
rewrite (dedekind_id1 H) (dedekind_id2 H).
apply inc_def1 in H.
by [rewrite -H].
rewrite -H.
apply cap_r.
Qed.
```

### 10.2.2 合成と定義域

**Lemma 196 (comp\_domain1, comp\_domain2)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then,*

$$[\alpha \cdot \beta] = [\alpha \cdot [\beta]] \sqsubseteq [\alpha].$$

**Lemma comp\_domain1**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$ :  
 $domain\ (alpha \cdot beta) = domain\ alpha.$

**Proof.**

```
rewrite /domain.
rewrite comp_inv.
apply (@inc_trans _ _ _ ((alpha · ((beta · (beta # · alpha #)) alpha #)) Id A)).
replace (((alpha · beta) · (beta # · alpha #)) Id A) with (((alpha · beta) ·
(beta # · alpha #)) Id A) Id A.
apply cap_inc_compat_r.
rewrite comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind1)).
rewrite comp_id_r.
apply inc_refl.
by [rewrite cap_assoc cap_idem].
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply cap_r.
Qed.
```

**Lemma comp\_domain2**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$ :  
 $domain\ (alpha \cdot beta) = domain\ (alpha \cdot domain\ beta).$

**Proof.**

```
apply inc_antisym.
replace (domain (alpha · beta)) with (domain ((alpha · domain beta) · beta)).
apply comp_domain1.
by [rewrite comp_assoc domain_comp_alpha1].
apply (@inc_trans _ _ _ (domain (alpha · (beta · beta #)))).
```

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```

apply domain_inc_compat.
apply comp_inc_compat_ab_ab'.
apply cap_l.
rewrite -comp_assoc.
apply comp_domain1.
Qed.

```

**Lemma 197 (comp\_domain3)** *Let  $\alpha : A \rightarrow B$  be a relation and  $\beta : B \rightarrow C$  be a total relation. Then,*

$$\lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \rfloor.$$

**Lemma comp\_domain3**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$ :  
 $total\_r\ beta \rightarrow domain\ (alpha \cdot beta) = domain\ alpha.$

**Proof.**

```

move => H.
apply inc_antisym.
apply comp_domain1.
rewrite /domain.
rewrite comp_inv_comp_assoc -(@comp_assoc _ _ _ beta).
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H).
Qed.

```

**Lemma 198 (comp\_domain4)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then,*

$$\lfloor \alpha^\# \rfloor \sqsubseteq \lfloor \beta \rfloor \Rightarrow \lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \rfloor.$$

**Lemma comp\_domain4**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$ :  
 $domain\ (alpha \#) \quad domain\ beta \rightarrow domain\ (alpha \cdot beta) = domain\ alpha.$

**Proof.**

```

move => H.
apply inc_antisym.
apply comp_domain1.
rewrite /domain.
rewrite -(@domain_comp_alpha1 _ _ (alpha #)) comp_inv_comp_assoc -(@comp_assoc _ _
_ _ beta).
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ _ H).
apply cap_l.
Qed.

```

**Lemma 199 (comp\_domain5)** *Let  $\alpha : A \rightarrow B$  be a univalent relation and  $\beta : B \rightarrow C$ . Then,*

$$\lfloor \alpha^\# \rfloor \sqsubseteq \lfloor \beta \rfloor \Leftrightarrow \lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \rfloor.$$

**Lemma comp\_domain5**  $\{A\ B\ C : \text{eqType}\} \{ \text{alpha} : \text{Rel } A\ B \} \{ \text{beta} : \text{Rel } B\ C \}$ :  
 $\text{univalent\_r } \text{alpha} \rightarrow$   
 $(\text{domain } (\text{alpha } \#)) \quad \text{domain } \text{beta} \leftrightarrow \text{domain } (\text{alpha } \cdot \text{beta}) = \text{domain } \text{alpha}.$

**Proof.**

move  $\Rightarrow H$ .  
split; move  $\Rightarrow H0$ .  
apply (comp\_domain4 H0).  
rewrite /domain.  
rewrite inv\_invol.  
apply cap\_inc\_compat\_r.  
replace (alpha # · alpha) with (alpha # · (domain (alpha · beta) · alpha)).  
rewrite /domain.  
rewrite comp\_inv.  
apply (@inc\_trans \_ \_ \_ (alpha # · (((alpha · beta) · (beta # · alpha #)) · alpha))).  
apply comp\_inc\_compat\_ab\_ab'.  
apply comp\_inc\_compat\_ab\_a'b.  
apply cap\_l.  
rewrite comp\_assoc comp\_assoc comp\_assoc -comp\_assoc -(@comp\_assoc \_ \_ \_ beta).  
apply (@inc\_trans \_ \_ \_ \_ (comp\_inc\_compat\_ab\_b H)).  
apply (comp\_inc\_compat\_ab\_a H).  
by [rewrite H0 domain\_comp\_alpha1].  
**Qed.**

**Lemma 200 (comp\_domain6)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then,*

$$\alpha \cdot \lfloor \beta \rfloor \sqsubseteq \lfloor \alpha \cdot \beta \rfloor \cdot \alpha.$$

**Lemma comp\_domain6**  $\{A\ B\ C : \text{eqType}\} \{ \text{alpha} : \text{Rel } A\ B \} \{ \text{beta} : \text{Rel } B\ C \}$ :  
 $(\text{alpha } \cdot \text{domain } \text{beta}) \quad (\text{domain } (\text{alpha } \cdot \text{beta}) \cdot \text{alpha}).$

**Proof.**

apply (@inc\_trans \_ \_ \_ \_ (@comp\_cap\_distr\_l \_ \_ \_ \_)).  
rewrite cap\_comm.  
replace (alpha · Id B) with (Id A · alpha).  
apply (@inc\_trans \_ \_ \_ \_ (dedekind2)).  
rewrite cap\_comm -comp\_assoc comp\_assoc -comp\_inv.  
apply inc\_refl.  
by [rewrite comp\_id\_l comp\_id\_r].  
**Qed.**



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**Lemma 201 (comp\_domain7)** *Let  $\alpha : A \rightarrow B$  be a univalent relation and  $\beta : B \rightarrow C$ . Then,*

$$\alpha \cdot \lfloor \beta \rfloor = \lfloor \alpha \cdot \beta \rfloor \cdot \alpha.$$

**Lemma comp\_domain7**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$ :  
 $univalent\_r\ alpha \rightarrow alpha \cdot domain\ beta = domain\ (alpha \cdot beta) \cdot alpha.$

**Proof.**

move  $\Rightarrow H$ .  
 apply inc\_antisym.  
 apply comp\_domain6.  
 apply (@inc\_trans \_ \_ \_ \_ (@comp\_cap\_distr\_r \_ \_ \_ \_ \_)).  
 rewrite comp\_id\_l comp\_inv comp\_assoc comp\_assoc.  
 apply (@inc\_trans \_ \_ \_ \_ (dedekind1)).  
 apply comp\_inc\_compat\_ab\_ab'.  
 apply (fun H'  $\Rightarrow$  cap\_inc\_compat H' H).  
 rewrite comp\_assoc -comp\_assoc.  
 apply (comp\_inc\_compat\_ab\_a H).

**Qed.**

**Lemma 202 (comp\_domain8)** *Let  $u : A \rightarrow A$ ,  $\alpha : A \rightarrow B$  and  $u \sqsubseteq id_A$ . Then,*

$$\lfloor u \cdot \alpha \rfloor = u \cdot \lfloor \alpha \rfloor.$$

**Lemma comp\_domain8**  $\{A\ B : eqType\} \{u : Rel\ A\ A\} \{alpha : Rel\ A\ B\}$ :  
 $u\ Id\ A \rightarrow domain\ (u \cdot alpha) = u \cdot domain\ alpha.$

**Proof.**

move  $\Rightarrow H$ .  
 apply inc\_antisym.  
 rewrite -(@cap\_idem \_ \_ (domain (u · alpha))).  
 rewrite (dedekind\_id3 H).  
 apply cap\_inc\_compat.  
 apply (@inc\_trans \_ \_ \_ \_ (comp\_domain1)).  
 apply domain\_inc\_id in H.  
 rewrite H.  
 apply inc\_refl.  
 apply domain\_inc\_compat.  
 apply (comp\_inc\_compat\_ab\_b H).  
 apply cap\_r.  
 apply (@inc\_trans \_ \_ \_ \_ (comp\_domain6)).  
 apply (comp\_inc\_compat\_ab\_a H).

**Qed.**

## 10.2.3 その他の性質

**Lemma 203 (cap\_domain)** *Let  $\alpha, \alpha' : A \rightarrow B$ . Then,*

$$\lfloor \alpha \sqcap \alpha' \rfloor = \alpha \cdot \alpha'^{\#} \sqcap \text{id}_A.$$

**Lemma** *cap\_domain* { $A B : \text{eqType}$ } { $\alpha \alpha' : \text{Rel } A B$ }:  
 $\text{domain } (\alpha \sqcap \alpha') = (\alpha \cdot \alpha'^{\#}) \sqcap \text{Id } A.$

**Proof.**

```

apply inc_antisym.
apply cap_inc_compat_r.
apply comp_inc_compat.
apply cap_l.
apply inc_inv.
apply cap_r.
rewrite (@cap_idem _ _ (Id A)) -cap_assoc.
apply cap_inc_compat_r.
apply (@inc_trans _ _ _ _ (@dedekind _ _ _ _ _)).
rewrite inv_invol comp_id_l comp_id_r -inv_cap_distr (@cap_comm _ _ alpha').
apply inc_refl.

```

**Qed.**

**Lemma 204 (cupP\_domain\_distr, cup\_domain\_distr)** *Let  $f : (C \rightarrow D) \rightarrow (A \rightarrow B)$  and  $P : \text{predicate}$ . Then,*

$$\lfloor \sqcup_{P(\alpha)} f(\alpha) \rfloor = \sqcup_{P(\alpha)} \lfloor f(\alpha) \rfloor.$$

**Lemma** *cupP\_domain\_distr* { $A B C D : \text{eqType}$ } { $f : \text{Rel } C D \rightarrow \text{Rel } A B$ } { $P : \text{Rel } C D \rightarrow \text{Prop}$ }:

$\text{domain } (\sqcup_{P} f) = \sqcup_{P} (\text{fun } \alpha : \text{Rel } C D \Rightarrow \text{domain } (f \alpha)).$

**Proof.**

```

rewrite /domain.
rewrite inv_cupP_distr comp_cupP_distr_l cap_cupP_distr_r.
apply cupP_eq.
move => alpha H.
rewrite -cap_domain -cap_domain.
apply f_equal.
rewrite cap_idem.
apply inc_antisym.
apply cap_r.
apply inc_cap.
split.

```

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move : *alpha H*.  
 apply *inc\_cupP*.  
 apply *inc\_refl*.  
 apply *inc\_refl*.  
 Qed.

**Lemma** *cup\_domain\_distr* {*A B : eqType*} {*alpha alpha' : Rel A B*}:  
 $\text{domain } (\text{alpha} \quad \text{alpha}') = \text{domain } \text{alpha} \quad \text{domain } \text{alpha}'.$

**Proof.**

rewrite *cup\_to\_cupP* (@*cup\_to\_cupP* \_ \_ \_ \_ \_ id).  
 apply *cupP\_domain\_distr*.  
 Qed.

**Lemma 205 (domain\_universal1)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$\lfloor \alpha \rfloor \cdot \nabla_{AC} = \alpha \cdot \nabla_{BC}.$$

**Lemma** *domain\_universal1* {*A B C : eqType*} {*alpha : Rel A B*}:  
 $\text{domain } \text{alpha} \cdot \quad A \ C = \text{alpha} \cdot \quad B \ C.$

**Proof.**

apply *inc\_antisym*.  
 apply (@*inc\_trans* \_ \_ \_ ((*alpha* · *alpha* #) · *A C*)).  
 apply *comp\_inc\_compat\_ab\_a'b*.  
 apply *cap\_l*.  
 rewrite *comp\_assoc*.  
 apply *comp\_inc\_compat\_ab\_ab'*.  
 apply *inc\_alpha\_universal*.  
 apply (@*inc\_trans* \_ \_ \_ ((*domain alpha* · *alpha*) · *B C*)).  
 rewrite *domain\_comp\_alpha1*.  
 apply *inc\_refl*.  
 rewrite *comp\_assoc*.  
 apply *comp\_inc\_compat\_ab\_ab'*.  
 apply *inc\_alpha\_universal*.  
 Qed.

**Lemma 206 (domain\_universal2)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then,*

$$\alpha \cdot \lfloor \beta \rfloor = \alpha \sqcap \nabla_{AC} \cdot \beta^\#.$$

**Lemma** *domain\_universal2* {*A B C : eqType*} {*alpha : Rel A B*} {*beta : Rel B C*}:  
 $\text{alpha} \cdot \text{domain } \text{beta} = \text{alpha} \quad ( \quad A \ C \cdot \text{beta} \#).$

**Proof.**

apply *inc\_antisym*.  
 apply *inc\_cap*.

```

split.
apply comp_inc_compat_ab_a.
apply cap_r.
apply (@inc_trans _ _ _ _ (comp_cap_distr_l)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
rewrite -inv_universal -comp_inv -domain_universal1.
rewrite comp_inv inv_universal domain_inv cap_comm.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
rewrite cap_comm cap_universal domain_inv.
apply comp_inc_compat_ab_a.
apply cap_r.
Qed.

```

**Lemma 207 (domain\_lemma1)** *Let  $\alpha, \beta : A \rightarrow B$  and  $\beta$  is univalent. Then,*

$$\alpha \sqsubseteq \beta \wedge \lfloor \alpha \rfloor = \lfloor \beta \rfloor \Rightarrow \alpha = \beta.$$

**Lemma domain\_lemma1**  $\{A B : eqType\} \{alpha \ beta : Rel A B\}$ :  
 $univalent\_r \ beta \rightarrow alpha \quad \beta \rightarrow domain \ alpha = domain \ beta \rightarrow alpha = \beta$ .

**Proof.**

```

move => H H0 H1.
apply inc_antisym.
apply H0.
rewrite -(@domain_comp_alpha1 _ _ beta) -H1.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply (fun H' => @inc_trans _ _ _ _ H' H).
apply comp_inc_compat_ab_a'b.
apply (@inc_inv _ _ _ _ H0).
Qed.

```

**Lemma 208 (domain\_lemma2a, domain\_lemma2b)** *Let  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$ . Then,*

$$\lfloor \alpha \rfloor \sqsubseteq \lfloor \beta \rfloor \Leftrightarrow \alpha \cdot \nabla_{BB} \sqsubseteq \beta \cdot \nabla_{CB} \Leftrightarrow \alpha \sqsubseteq \beta \cdot \beta^\# \cdot \alpha.$$

**Lemma domain\_lemma2a**  $\{A B C : eqType\} \{alpha : Rel A B\} \{beta : Rel A C\}$ :  
 $domain \ alpha \quad domain \ beta \Leftrightarrow (alpha \cdot \quad B \ B) \quad (beta \cdot \quad C \ B)$ .

**Proof.**

```

split; move  $\Rightarrow H$ .
rewrite -(@domain_comp_alpha1 _ _ alpha) comp_assoc.
apply (@inc_trans _ _ _ _ (comp_inc_compat_ab_a'b H)).
apply (@inc_trans _ _ _ _ (comp_inc_compat_ab_a'b (cap_l))).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ (domain ((beta  $\cdot$  beta #)  $\cdot$  alpha))).
apply domain_inc_compat.
apply (@inc_trans _ _ _ (alpha (beta  $\cdot$  C B))).
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (cap_inc_compat_l H)).
replace (alpha (alpha  $\cdot$  B B)) with ((alpha  $\cdot$  Id B) (alpha  $\cdot$  B B)).
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (comp_cap_distr_l)).
rewrite cap_universal comp_id_r.
apply inc_refl.
by [rewrite comp_id_r].
rewrite cap_comm comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind1)).
rewrite cap_comm cap_universal.
apply inc_refl.
rewrite comp_assoc.
apply comp_domain1.

```

**Qed.**

**Lemma** domain\_lemma2b {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:

$$\text{domain } \alpha \quad \text{domain } \beta \leftrightarrow \alpha \quad ((\beta \cdot \beta \#) \cdot \alpha).$$

**Proof.**

```

split; move  $\Rightarrow H$ .
apply domain_lemma2a in H.
apply (@inc_trans _ _ _ (alpha (beta  $\cdot$  C B))).
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (cap_inc_compat_l H)).
replace (alpha (alpha  $\cdot$  B B)) with ((alpha  $\cdot$  Id B) (alpha  $\cdot$  B B)).
apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' (comp_cap_distr_l)).
rewrite cap_universal comp_id_r.
apply inc_refl.
by [rewrite comp_id_r].
rewrite cap_comm comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind1)).
rewrite cap_comm cap_universal.
apply inc_refl.
apply domain_inc_compat in H.
apply (@inc_trans _ _ _ _ H).

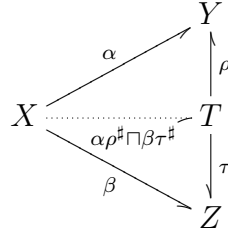
```

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rewrite *comp\_assoc*.  
 apply *comp\_domain1*.  
 Qed.

**Lemma 209 (domain\_corollary1)** *In below figure,*

*“ $\alpha$  and  $\beta$  are total”  $\wedge \alpha^\# \cdot \beta \sqsubseteq \rho^\# \cdot \tau \Rightarrow$  “ $\alpha \cdot \rho^\# \sqcap \beta \cdot \tau^\#$  is total”.*



**Lemma domain\_corollary1**  $\{X\ Y\ Z\ T : eqType\}$   
 $\{alpha : Rel\ X\ Y\} \{beta : Rel\ X\ Z\} \{rho : Rel\ T\ Y\} \{tau : Rel\ T\ Z\} :$   
 $total\_r\ alpha \rightarrow total\_r\ beta \rightarrow (alpha\ \# \cdot beta) \quad (rho\ \# \cdot tau) \rightarrow$   
 $total\_r\ ((alpha \cdot rho\ \#) \quad (beta \cdot tau\ \#)).$

**Proof.**

move  $\Rightarrow H\ H0\ H1$ .

move :  $(comp\_inc\_compat\ H\ H0) \Rightarrow H2$ .

rewrite *comp\_id\_l -comp\_assoc* (@*comp\_assoc* \_ \_ \_ \_ *alpha*) in *H2*.

rewrite /*total\_r*.

replace (*Id X*) with  $((alpha \cdot (rho\ \# \cdot tau)) \cdot beta\ \#) \quad Id\ X$ .

rewrite -*comp\_assoc comp\_assoc*.

apply (@*inc\_trans* \_ \_ \_ \_ (@*dedekind* \_ \_ \_ \_ \_)).

rewrite *comp\_id\_l comp\_id\_r comp\_inv comp\_inv inv\_invol inv\_invol*.

rewrite *inv\_cap\_distr comp\_inv comp\_inv inv\_invol inv\_invol* (@*cap\_comm* \_ \_ (*tau* · *beta* #)).

apply *inc\_refl*.

apply *Logic.eq\_sym*.

rewrite *cap\_comm*.

apply *inc\_def1*.

apply (@*inc\_trans* \_ \_ \_ \_ *H2*).

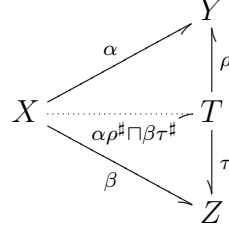
apply *comp\_inc\_compat\_ab\_a'b*.

apply (*comp\_inc\_compat\_ab\_ab' H1*).

Qed.

**Lemma 210 (domain\_corollary2)** *In below figure,*

*“ $\alpha$  and  $\beta$  are univalent”  $\wedge \rho \cdot \rho^\# \sqcap \tau \cdot \tau^\# = \text{id}_T \Rightarrow$  “ $\alpha \cdot \rho^\# \sqcap \beta \cdot \tau^\#$  is univalent”.*



**Lemma domain\_corollary2**  $\{X\ Y\ Z\ T : \text{eqType}\}$   
 $\{\text{alpha} : \text{Rel } X\ Y\} \{\text{beta} : \text{Rel } X\ Z\} \{\text{rho} : \text{Rel } T\ Y\} \{\text{tau} : \text{Rel } T\ Z\}$ :  
 $\text{univalent\_r } \text{alpha} \rightarrow \text{univalent\_r } \text{beta} \rightarrow (\text{rho} \cdot \text{rho}^\#) \quad (\text{tau} \cdot \text{tau}^\#) = \text{Id } T \rightarrow$   
 $\text{univalent\_r } ((\text{alpha} \cdot \text{rho}^\#) \quad (\text{beta} \cdot \text{tau}^\#)).$

**Proof.**

`move  $\Rightarrow$  H H0 H1.`

`rewrite /univalent_r.`

`rewrite -H1 inv_cap_distr.`

`apply (@inc_trans _ _ _ _ (comp_cap_distr_l)).`

`apply cap_inc_compat.`

`apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).`

`apply (@inc_trans _ _ _ _ (cap_l)).`

`rewrite comp_inv inv_invol -comp_assoc (@comp_assoc _ _ _ _ rho).`

`apply comp_inc_compat_ab_a'b.`

`apply (comp_inc_compat_ab_a H).`

`apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).`

`apply (@inc_trans _ _ _ _ (cap_r)).`

`rewrite comp_inv inv_invol -comp_assoc (@comp_assoc _ _ _ _ tau).`

`apply comp_inc_compat_ab_a'b.`

`apply (comp_inc_compat_ab_a H0).`

**Qed.**

## 10.2.4 矩形関係

$\alpha : A \rightarrow B$  が

$$\alpha \cdot \nabla_{BA} \cdot \alpha \sqsubseteq \alpha$$

を満たすとき,  $\alpha$  は 矩形関係 (rectangular relation) であると言われる.

**Definition** *rectangular*  $\{A\ B : \text{eqType}\} (\text{alpha} : \text{Rel } A\ B) :=$   
 $((\text{alpha} \cdot \quad B\ A) \cdot \text{alpha}) \quad \text{alpha}.$

**Lemma 211 (rectangular\_inv)** *Let  $\alpha : A \rightarrow B$  is a rectangular relation, then  $\alpha^\#$  is also a rectangular relation.*

**Lemma** *rectangular\_inv*  $\{A\ B : \text{eqType}\} \{\alpha : \text{Rel}\ A\ B\}$ :  
 $\text{rectangular\_inv}\ \alpha \rightarrow \text{rectangular}\ (\alpha^\#)$ .

**Proof.**

move  $\Rightarrow H$ .

apply *inv\_inc\_move*.

rewrite *comp\_inv comp\_inv inv\_invol inv\_universal -comp\_assoc*.

apply *H*.

**Qed.**

**Lemma 212 (rectangular\_capP, rectangular\_cap)** *Let  $f(\alpha)$  is always a rectangular relation and  $P$  : predicate, then  $\cap_{P(\beta)} f(\beta)$  is also a rectangular relation.*

**Lemma** *rectangular\_capP*  $\{A\ B\ C\ D : \text{eqType}\} \{f : \text{Rel}\ C\ D \rightarrow \text{Rel}\ A\ B\} \{P : \text{Rel}\ C\ D \rightarrow \text{Prop}\}$ :

$(\forall\ \alpha : \text{Rel}\ C\ D, P\ \alpha \rightarrow \text{rectangular}\ (f\ \alpha)) \rightarrow \text{rectangular}\ (\cap_{P}\ f)$ .

**Proof.**

move  $\Rightarrow H$ .

rewrite */rectangular*.

apply (*@inc\_trans*  $\_ \_ \_ (\cap_{P}\ f) (\text{fun}\ \alpha : \text{Rel}\ C\ D \Rightarrow (f\ \alpha \cdot \_ B\ A) \cdot f\ \alpha))$ ).

apply (*@inc\_trans*  $\_ \_ \_ \_ \_ (comp\_capP\_distr\_l)$ ).

apply *inc\_capP*.

move  $\Rightarrow$  **beta** *H0*.

apply (*@inc\_trans*  $\_ \_ \_ (((\cap_{P}\ f) \cdot \_ B\ A) \cdot f\ \text{beta}))$ ).

move : **beta** *H0*.

apply *inc\_capP*.

apply *inc\_refl*.

apply *comp\_inc\_compat\_ab\_a'b*.

apply *comp\_inc\_compat\_ab\_a'b*.

move : *H0*.

apply *inc\_capP*.

apply *inc\_refl*.

apply *inc\_capP*.

move  $\Rightarrow$  **beta** *H0*.

apply (*fun* *H'*  $\Rightarrow$  *@inc\_trans*  $\_ \_ \_ \_ \_ H' (H\ \text{beta}\ H0)$ ).

move : **beta** *H0*.

apply *inc\_capP*.

apply *inc\_refl*.

**Qed.**

**Lemma** *rectangular\_cap*  $\{A\ B : \text{eqType}\} \{\alpha\ \text{beta} : \text{Rel}\ A\ B\}$ :



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*rectangular alpha*  $\rightarrow$  *rectangular beta*  $\rightarrow$  *rectangular (alpha beta)*.

**Proof.**

move  $\Rightarrow$  *H H0*.

rewrite (@cap\_to\_capP \_ \_ \_ \_ \_ id).

apply *rectangular\_capP*.

move  $\Rightarrow$  *gamma*.

case  $\Rightarrow$  *H1*; rewrite *H1*.

apply *H*.

apply *H0*.

**Qed.**

**Lemma 213 (rectangular\_comp)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  and  $\alpha$  or  $\beta$  is a rectangular relation, then  $\alpha \cdot \beta$  is also a rectangular relation.*

**Lemma** *rectangular\_comp* {*A B C : eqType*} {*alpha : Rel A B*} {*beta : Rel B C*}:

*rectangular alpha*  $\vee$  *rectangular beta*  $\rightarrow$  *rectangular (alpha beta)*.

**Proof.**

rewrite /*rectangular*.

case; move  $\Rightarrow$  *H*.

rewrite *-comp\_assoc*.

apply *comp\_inc\_compat\_ab\_a'b*.

apply (fun *H'*  $\Rightarrow$  @*inc\_trans* \_ \_ \_ \_ *H' H*).

apply *comp\_inc\_compat\_ab\_a'b*.

rewrite *comp\_assoc*.

apply *comp\_inc\_compat\_ab\_ab'*.

apply *inc\_alpha\_universal*.

rewrite *comp\_assoc comp\_assoc*.

apply *comp\_inc\_compat\_ab\_ab'*.

apply (fun *H'*  $\Rightarrow$  @*inc\_trans* \_ \_ \_ \_ *H' H*).

rewrite *-comp\_assoc -comp\_assoc*.

apply *comp\_inc\_compat\_ab\_a'b*.

rewrite *comp\_assoc*.

apply *comp\_inc\_compat\_ab\_ab'*.

apply *inc\_alpha\_universal*.

**Qed.**

**Lemma 214 (rectangular\_unit)** *Let  $\alpha : A \rightarrow B$ . Then,*

$$“\alpha \text{ is rectangular}” \Leftrightarrow \exists \mu : I \rightarrow A, \exists \rho : I \rightarrow B, \alpha = \rho^\# \cdot \mu.$$

**Lemma** *rectangular\_unit* {*A B : eqType*} {*alpha : Rel A B*}:

*rectangular alpha*  $\leftrightarrow \exists (\mu : Rel i A)(\rho : Rel i B), \alpha = \mu \# \cdot \rho$ .

**Proof.**

```

split; move  $\Rightarrow H$ .
 $\exists (i B \cdot \alpha \#)$ .
 $\exists (i A \cdot \alpha)$ .
rewrite comp_inv inv_invol inv_universal.
rewrite -comp_assoc (@comp_assoc - - -  $\alpha$ ) lemma_for_tarski2.
apply inc_antisym.
apply (@inc_trans - - - (relation_rel_inv_rel)).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply  $H$ .
elim  $H \Rightarrow mu$ .
elim  $\Rightarrow \rho H0$ .
rewrite  $H0$ .
rewrite /rectangular.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_a.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
Qed.

```

# Chapter 11

## Library Residual

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Domain.
Require Import Logic.FunctionalExtensionality.
```

### 11.1 剰余合成関係の性質

#### 11.1.1 基本的な性質

**Lemma 215 (double\_residual)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  and  $\gamma : C \rightarrow D$ . Then*

$$\alpha \triangleright (\beta \triangleright \gamma) = (\alpha \cdot \beta) \triangleright \gamma.$$

**Lemma** *double\_residual*

```
{A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel C D}:
alpha (beta gamma) = (alpha • beta) gamma.
```

**Proof.**

apply *inc\_lower*.

move  $\Rightarrow$  *delta*.

split; move  $\Rightarrow$  *H*.

apply *inc\_residual*.

rewrite *comp\_inv comp\_assoc*.

rewrite *-inc\_residual -inc\_residual*.

apply *H*.

rewrite *inc\_residual inc\_residual*.

rewrite *-comp\_assoc -comp\_inv*.

apply *inc\_residual*.  
 apply *H*.  
 Qed.

**Lemma 216 (residual\_to\_complement)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then*

$$\alpha \triangleright \beta = (\alpha \cdot \beta^-)^-.$$

**Lemma** *residual\_to\_complement* {*A B C : eqType*} {*alpha : Rel A B*} {*beta : Rel B C*}:  
*alpha beta* = (*alpha* • *beta* ^) ^.

**Proof.**

apply *inc\_lower*.  
 move  $\Rightarrow$  *gamma*.  
 split; move  $\Rightarrow$  *H*.  
 rewrite *bool\_lemma2 complement\_invol cap\_comm*.  
 apply *inc\_antisym*.  
 apply (@*inc\_trans* \_ \_ \_ \_ (*dedekind1*)).  
 replace (*beta* ^ (*alpha* # • *gamma*)) with ( *B C*).  
 rewrite *comp\_empty\_r*.  
 apply *inc\_refl*.  
 apply *Logic.eq\_sym*.  
 rewrite *cap\_comm*.  
 apply *bool\_lemma2*.  
 apply *inc\_residual*.  
 apply *H*.  
 apply *inc\_empty\_alpha*.  
 apply *inc\_residual*.  
 apply *bool\_lemma2*.  
 apply *inc\_antisym*.  
 apply (@*inc\_trans* \_ \_ \_ \_ (*dedekind1*)).  
 rewrite *inv\_invol*.  
 replace (*gamma* ( *alpha* • *beta* ^)) with ( *A C*).  
 rewrite *comp\_empty\_r*.  
 apply *inc\_refl*.  
 apply *Logic.eq\_sym*.  
 rewrite -(@*complement\_invol* \_ \_ (*alpha* • *beta* ^)).  
 apply *bool\_lemma2*.  
 apply *H*.  
 apply *inc\_empty\_alpha*.  
 Qed.

## CHAPTER 11. LIBRARY RESIDUAL

**Lemma 217 (inv\_residual\_inc)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then*

$$\alpha^\# \cdot (\alpha \triangleright \beta) \sqsubseteq \beta.$$

**Lemma** *inv\_residual\_inc* { $A\ B\ C : eqType$ } { $\alpha : Rel\ A\ B$ } { $\beta : Rel\ B\ C$ }:  
 $\alpha \# \cdot (\alpha \quad \beta) \quad \beta.$

**Proof.**

apply *inc\_residual*.

apply *inc\_refl*.

**Qed.**

**Lemma 218 (inc\_residual\_inv)** *Let  $\alpha : A \rightarrow B$  and  $\gamma : A \rightarrow C$ . Then*

$$\gamma \sqsubseteq \alpha \triangleright \alpha^\# \cdot \gamma.$$

**Lemma** *inc\_residual\_inv* { $A\ B\ C : eqType$ } { $\alpha : Rel\ A\ B$ } { $\gamma : Rel\ A\ C$ }:  
 $\gamma \quad (\alpha \quad (\alpha \# \cdot \gamma)).$

**Proof.**

apply *inc\_residual*.

apply *inc\_refl*.

**Qed.**

**Lemma 219 (id\_inc\_residual)** *Let  $\alpha : A \rightarrow B$ . Then*

$$id_A \sqsubseteq \alpha \triangleright \alpha^\#.$$

**Lemma** *id\_inc\_residual* { $A\ B : eqType$ } { $\alpha : Rel\ A\ B$ }:  $Id\ A \quad (\alpha \quad \alpha \#).$

**Proof.**

apply *inc\_residual*.

rewrite *comp\_id\_r*.

apply *inc\_refl*.

**Qed.**

**Lemma 220 (residual\_universal)** *Let  $\alpha : A \rightarrow B$ . Then*

$$\alpha \triangleright \nabla_{BC} = \nabla_{AC}.$$

**Lemma** *residual\_universal* { $A\ B\ C : eqType$ } { $\alpha : Rel\ A\ B$ }:  $\alpha \quad B\ C = \quad A\ C.$

**Proof.**

apply *inc\_antisym*.

apply *inc\_alpha\_universal*.

apply *inc\_residual*.

apply *inc\_alpha\_universal*.

*Qed.*

### 11.1.2 単調性と分配法則

**Lemma 221 (residual\_inc\_compat)** *Let  $\alpha, \alpha' : A \rightarrow B$  and  $\beta, \beta' : B \rightarrow C$ . Then*

$$\alpha' \sqsubseteq \alpha \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \triangleright \beta \sqsubseteq \alpha' \triangleright \beta'.$$

*Lemma residual\_inc\_compat*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{beta\ beta' : Rel\ B\ C\}:$   
 $alpha' \quad alpha \rightarrow beta \quad beta' \rightarrow (alpha \quad beta) \quad (alpha' \quad beta').$

*Proof.*

`move  $\Rightarrow$  H H0.`

`apply inc_residual.`

`apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' H0).`

`move : (@inc_refl _ _ (alpha beta))  $\Rightarrow$  H1.`

`apply inc_residual in H1.`

`apply (fun H'  $\Rightarrow$  @inc_trans _ _ _ _ H' H1).`

`apply comp_inc_compat_ab_a'b.`

`apply inc_inv.`

`apply H.`

*Qed.*

**Lemma 222 (residual\_inc\_compat\_l)** *Let  $\alpha : A \rightarrow B$  and  $\beta, \beta' : B \rightarrow C$ . Then*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \triangleright \beta \sqsubseteq \alpha \triangleright \beta'.$$

*Lemma residual\_inc\_compat\_l*

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta\ beta' : Rel\ B\ C\}:$   
 $beta \quad beta' \rightarrow (alpha \quad beta) \quad (alpha \quad beta').$

*Proof.*

`move  $\Rightarrow$  H.`

`apply (@residual_inc_compat _ _ _ _ _ (@inc_refl _ _ _)) H).`

*Qed.*

**Lemma 223 (residual\_inc\_compat\_r)** *Let  $\alpha, \alpha' : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then*

$$\alpha' \sqsubseteq \alpha \Rightarrow \alpha \triangleright \beta \sqsubseteq \alpha' \triangleright \beta.$$

*Lemma residual\_inc\_compat\_r*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{beta : Rel\ B\ C\}:$   
 $alpha' \quad alpha \rightarrow (alpha \quad beta) \quad (alpha' \quad beta).$

*Proof.*

## CHAPTER 11. LIBRARY RESIDUAL

move  $\Rightarrow H$ .  
 apply (@residual\_inc\_compat \_ \_ \_ \_ \_ H (@inc\_refl \_ \_)).  
 Qed.

**Lemma 224** (residual\_capP\_distr\_l, residual\_cap\_distr\_l) *Let  $\alpha : A \rightarrow B$ ,  $f : (D \rightarrow E) \rightarrow (B \rightarrow C)$  and  $P : \text{predicate}$ . Then*

$$\alpha \triangleright (\sqcap_{P(\beta)} f(\beta)) = \sqcap_{P(\beta)} (\alpha \triangleright f(\beta)).$$

**Lemma** residual\_capP\_distr\_l {A B C D E : eqType}  
 {alpha : Rel A B} {f : Rel D E  $\rightarrow$  Rel B C} {P : Rel D E  $\rightarrow$  Prop}:  
 alpha ( \_ {P} f ) = \_ {P} (fun beta : Rel D E  $\Rightarrow$  alpha f beta).

**Proof.**

apply inc\_lower.  
 move  $\Rightarrow$  gamma.  
 split; move  $\Rightarrow$  H.  
 apply inc\_capP.  
 move  $\Rightarrow$  beta H0.  
 apply inc\_residual.  
 move : beta H0.  
 apply inc\_capP.  
 apply inc\_residual.  
 apply H.  
 apply inc\_residual.  
 apply inc\_capP.  
 move  $\Rightarrow$  beta H0.  
 apply inc\_residual.  
 move : beta H0.  
 apply inc\_capP.  
 apply H.  
 Qed.

**Lemma** residual\_cap\_distr\_l  
 {A B C : eqType} {alpha : Rel A B} {beta gamma : Rel B C}:  
 alpha (beta gamma) = (alpha beta) (alpha gamma).

**Proof.**

rewrite cap\_to\_capP (@cap\_to\_capP \_ \_ \_ \_ id).  
 apply residual\_capP\_distr\_l.  
 Qed.

**Lemma 225** (`residual_cupP_distr_r`, `residual_cup_distr_r`) *Let  $f : (D \rightarrow E) \rightarrow (A \rightarrow B)$ ,  $\beta : B \rightarrow C$  and  $P : \text{predicate}$ . Then*

$$(\sqcup_{P(\alpha)} f(\alpha)) \triangleright \beta = \sqcap_{P(\alpha)} (f(\alpha) \triangleright \beta).$$

**Lemma** `residual_cupP_distr_r` {*A B C D E : eqType*}  
 {*beta : Rel B C*} {*f : Rel D E → Rel A B*} {*P : Rel D E → Prop*}:  
 ( *\_* {*P*} *f*) *beta* = *\_* {*P*} (*fun alpha : Rel D E ⇒ f alpha beta*).

**Proof.**

apply *inc\_lower*.  
 move ⇒ *gamma*.  
 split; move ⇒ *H*.  
 apply *inc\_capP*.  
 move ⇒ *alpha H0*.  
 apply *inc\_residual*.  
 move : *alpha H0*.  
 apply *inc\_cupP*.  
 rewrite *-comp\_cupP\_distr\_r -inv\_cupP\_distr*.  
 apply *inc\_residual*.  
 apply *H*.  
 apply *inc\_residual*.  
 rewrite *inv\_cupP\_distr comp\_cupP\_distr\_r*.  
 apply *inc\_cupP*.  
 move ⇒ *alpha H0*.  
 apply *inc\_residual*.  
 move : *alpha H0*.  
 apply *inc\_capP*.  
 apply *H*.

**Qed.**

**Lemma** `residual_cup_distr_r`  
 {*A B C : eqType*} {*alpha beta : Rel A B*} {*gamma : Rel B C*}:  
 (*alpha beta*) *gamma* = (*alpha gamma*) (*beta gamma*).

**Proof.**

rewrite (*@cup\_to\_cupP \_ \_ \_ \_ id*) (*@cap\_to\_capP \_ \_ \_ \_ (fun x ⇒ x gamma)*).  
 apply *residual\_cupP\_distr\_r*.

**Qed.**



## 11.1.3 剰余合成と関数

**Lemma 226 (total\_residual)** *Let  $\alpha : A \rightarrow B$  be a total relation and  $\beta : B \rightarrow C$ . Then*

$$\alpha \triangleright \beta \sqsubseteq \alpha \cdot \beta.$$

**Lemma** *total\_residual* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:  
total\_r alpha → (alpha    beta)    (alpha · beta).

**Proof.**

move ⇒ H.

apply (@inc\_trans \_ \_ \_ ((alpha · alpha #) · (alpha    beta))).

apply (comp\_inc\_compat\_b\_ab H).

rewrite comp\_assoc.

apply comp\_inc\_compat\_ab\_ab'.

apply inv\_residual\_inc.

**Qed.**

**Lemma 227 (univalent\_residual)** *Let  $\alpha : A \rightarrow B$  be a univalent relation and  $\beta : B \rightarrow C$ . Then*

$$\alpha \cdot \beta \sqsubseteq \alpha \triangleright \beta.$$

**Lemma** *univalent\_residual* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:  
univalent\_r alpha → (alpha · beta)    (alpha    beta).

**Proof.**

move ⇒ H.

apply (@inc\_trans \_ \_ \_ \_ (@inc\_residual\_inv \_ \_ alpha \_)).

apply residual\_inc\_compat\_l.

rewrite -comp\_assoc.

apply (comp\_inc\_compat\_ab\_b H).

**Qed.**

**Lemma 228 (function\_residual1)** *Let  $\alpha : A \rightarrow B$  be a function and  $\beta : B \rightarrow C$ . Then*

$$\alpha \triangleright \beta = \alpha \cdot \beta.$$

**Lemma** *function\_residual1* {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:  
function\_r alpha → alpha    beta = alpha · beta.

**Proof.**

elim ⇒ H H0.

apply inc\_antisym.

apply (total\_residual H).

apply (univalent\_residual H0).

**Qed.**

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**Lemma 229 (residual\_id)** *Let  $\alpha : A \rightarrow B$ . Then*

$$id_A \triangleright \alpha = \alpha.$$

**Lemma** *residual\_id*  $\{A\ B : eqType\} \{alpha : Rel\ A\ B\}$ :

*Id A     alpha = alpha.*

**Proof.**

`move : (@function_residual1 _ _ _ (Id A) alpha (@id_function A))  $\Rightarrow$  H.`

`rewrite comp_id_l in H.`

`apply H.`

**Qed.**

**Lemma 230 (universal\_residual)** *Let  $\alpha : A \rightarrow B$ . Then*

$$\nabla_{AA} \triangleright \alpha \sqsubseteq \alpha.$$

**Lemma** *universal\_residual*  $\{A\ B : eqType\} \{alpha : Rel\ A\ B\}$ :

*A A     alpha     alpha.*

**Proof.**

`apply (@inc_trans _ _ _ (Id A     alpha)).`

`apply residual_inc_compat_r.`

`apply inc_alpha_universal.`

`rewrite residual_id.`

`apply inc_refl.`

**Qed.**

**Lemma 231 (function\_residual2)** *Let  $\alpha : A \rightarrow B$  be a function,  $\beta : B \rightarrow C$  and  $\gamma : C \rightarrow D$ . Then*

$$\alpha \cdot (\beta \triangleright \gamma) = (\alpha \cdot \beta) \triangleright \gamma.$$

**Lemma** *function\_residual2*

$\{A\ B\ C\ D : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\} \{gamma : Rel\ C\ D\}$ :

*function\_r alpha  $\rightarrow$  alpha  $\cdot$  (beta     gamma) = (alpha  $\cdot$  beta)     gamma.*

**Proof.**

`move  $\Rightarrow$  H.`

`rewrite -(@function_residual1 _ _ _ _ H).`

`apply double_residual.`

**Qed.**

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**Lemma 232 (function\_residual3)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  be relations and  $\gamma : D \rightarrow C$  be a function. Then*

$$(\alpha \triangleright \beta) \cdot \gamma^\# = \alpha \triangleright (\beta \cdot \gamma^\#).$$

**Lemma function\_residual3**

$\{A\ B\ C\ D : \text{eqType}\} \{alpha : \text{Rel}\ A\ B\} \{beta : \text{Rel}\ B\ C\} \{gamma : \text{Rel}\ D\ C\} :$   
 $\text{function\_r}\ gamma \rightarrow (alpha \quad beta) \cdot gamma \# = alpha \quad (beta \cdot gamma \#).$

**Proof.**

move  $\Rightarrow H$ .  
 apply inc\_lower.  
 move  $\Rightarrow \text{delta}$ .  
 split; move  $\Rightarrow H0$ .  
 apply inc\_residual.  
 rewrite -(@function\_move2 \_ \_ \_ \_ \_ H).  
 rewrite comp\_assoc.  
 apply inc\_residual.  
 rewrite (@function\_move2 \_ \_ \_ \_ \_ H).  
 apply H0.  
 rewrite -(@function\_move2 \_ \_ \_ \_ \_ H).  
 apply inc\_residual.  
 rewrite -comp\_assoc.  
 rewrite (@function\_move2 \_ \_ \_ \_ \_ H).  
 apply inc\_residual.  
 apply H0.  
**Qed.**

**Lemma 233 (function\_residual4)** *Let  $\alpha : A \rightarrow B$ ,  $\gamma : C \rightarrow D$  be relations and  $\beta : B \rightarrow C$  be a function. Then*

$$\alpha \cdot \beta \triangleright \gamma = \alpha \triangleright \beta \cdot \gamma.$$

**Lemma function\_residual4**

$\{A\ B\ C\ D : \text{eqType}\} \{alpha : \text{Rel}\ A\ B\} \{beta : \text{Rel}\ B\ C\} \{gamma : \text{Rel}\ C\ D\} :$   
 $\text{function\_r}\ beta \rightarrow (alpha \cdot beta) \quad gamma = alpha \quad (beta \cdot gamma).$

**Proof.**

move  $\Rightarrow H$ .  
 rewrite -double\_residual.  
 by [rewrite (function\_residual1 H)].  
**Qed.**

## 11.2 Galois 同値とその系

**Lemma 234 (galois)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  and  $\gamma : A \rightarrow C$ . Then*

$$\gamma \sqsubseteq \alpha \triangleright \beta \Leftrightarrow \alpha \sqsubseteq \gamma \triangleright \beta^\#.$$

**Lemma galois**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\} \{gamma : Rel\ A\ C\} :$   
 $gamma \quad (alpha \quad beta) \leftrightarrow alpha \quad (gamma \quad beta \#).$

**Proof.**

split; move  $\Rightarrow H$ .  
 apply inc\_residual.  
 apply inv\_inc\_move.  
 rewrite comp\_inv inv\_invol.  
 apply inc\_residual.  
 apply H.  
 apply inc\_residual.  
 apply inv\_inc\_invol.  
 rewrite comp\_inv inv\_invol.  
 apply inc\_residual.  
 apply H.

**Qed.**

**Lemma 235 (galois\_corollary1)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then*

$$\alpha \sqsubseteq (\alpha \triangleright \beta) \triangleright \beta^\#.$$

**Lemma galois\_corollary1**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\} :$   
 $alpha \quad ((alpha \quad beta) \quad beta \#).$

**Proof.**

rewrite -galois.  
 apply inc\_refl.

**Qed.**

**Lemma 236 (galois\_corollary2)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then*

$$((\alpha \triangleright \beta) \triangleright \beta^\#) \triangleright \beta = \alpha \triangleright \beta.$$

**Lemma galois\_corollary2**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\} :$   
 $((alpha \quad beta) \quad beta \#) \quad beta = alpha \quad beta.$

**Proof.**

apply inc\_antisym.  
 apply residual\_inc\_compat\_r.

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```

apply galois_corollary1.
move : (@galois_corollary1 _ _ _ (alpha beta) (beta #)) => H.
rewrite inv_invol in H.
apply H.
Qed.

```

**Lemma 237 (galois\_corollary3)** *Let  $\alpha : A \rightarrow B$  and  $\beta : B \rightarrow C$ . Then*

$$\alpha = (\alpha \triangleright \beta) \triangleright \beta^\# \Leftrightarrow \exists \gamma : A \rightarrow C, \alpha = \gamma \triangleright \beta^\#.$$

**Lemma galois\_corollary3**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ B\ C\}$ :  
 $alpha = (alpha\ \beta)\ \beta^\# \Leftrightarrow (\exists\ gamma : Rel\ A\ C, alpha = gamma\ \beta^\#)$ .  
**Proof.**

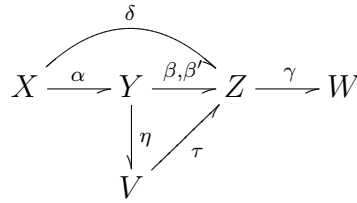
```

split; move => H.
exists (alpha beta).
apply H.
elim H => gamma H0.
rewrite H0.
move : (@galois_corollary2 _ _ _ gamma (beta #)) => H1.
rewrite inv_invol in H1.
by [rewrite H1].
Qed.

```

### 11.3 その他の性質

この節では、特記が無い限り、記号は以下の図式に従って割り振られるものとする。



**Lemma 238 (residual\_property1)**

$$(\alpha \triangleright \beta) \cdot \gamma \sqsubseteq \alpha \triangleright \beta \cdot \gamma.$$

**Lemma residual\_property1**  
 $\{W\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{gamma : Rel\ Z\ W\}$ :  
 $((alpha\ \beta) \cdot gamma)\ (alpha\ (\beta \cdot gamma))$ .  
**Proof.**

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apply (@inc\_trans \_ \_ \_ (alpha (alpha # • ((alpha beta) • gamma)))).  
 apply inc\_residual\_inv.  
 apply residual\_inc\_compat\_l.  
 rewrite -comp\_assoc.  
 apply comp\_inc\_compat\_ab\_a'b.  
 apply inv\_residual\_inc.  
 Qed.

### Lemma 239 (residual\_property2)

$$(\alpha \triangleright \beta) \cdot (\beta^\# \triangleright \eta) \sqsubseteq \alpha \triangleright \eta.$$

Lemma residual\_property2

$\{V\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{eta : Rel\ Y\ V\} :$   
 $((alpha\ \beta) \cdot (\beta^\# \eta)) \sqsubseteq (alpha\ eta).$

Proof.

apply (@inc\_trans \_ \_ \_ \_ (residual\_property1)).  
 apply residual\_inc\_compat\_l.  
 move : (@inv\_residual\_inc \_ \_ \_ (beta # eta)).  
 by [rewrite inv\_invol].  
 Qed.

### Lemma 240 (residual\_property3)

$$\alpha \triangleright \beta \sqsubseteq \alpha \cdot \eta \triangleright \eta^\# \cdot \beta.$$

Lemma residual\_property3

$\{V\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{eta : Rel\ Y\ V\} :$   
 $(alpha\ \beta) \sqsubseteq ((alpha \cdot eta) (eta^\# \cdot beta)).$

Proof.

apply (@inc\_trans \_ \_ \_ \_ (@inc\_residual\_inv \_ \_ \_ (alpha • eta) (alpha beta))).  
 apply residual\_inc\_compat\_l.  
 rewrite comp\_inv comp\_assoc.  
 apply comp\_inc\_compat\_ab\_ab'.  
 apply inv\_residual\_inc.  
 Qed.

### Lemma 241 (residual\_property4a, residual\_property4b)

$$(\alpha \triangleright \beta) \cdot \gamma \sqsubseteq (\alpha \triangleright \beta \cdot \gamma) \sqcap \nabla_{XZ} \cdot \gamma \sqsubseteq (\alpha \triangleright \beta \cdot \gamma) \cdot \gamma^\# \cdot \gamma.$$

Lemma residual\_property4a

$\{W\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{gamma : Rel\ Z\ W\} :$

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$((\alpha \quad \beta) \cdot \gamma) \quad ((\alpha \quad (\beta \cdot \gamma)) \quad (X \ Z \cdot \gamma)).$

**Proof.**

```
rewrite -(@cap_universal _ _ (alpha beta)).
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat_r.
apply residual_property1.
Qed.
```

**Lemma residual\_property4b**

$\{W \ X \ Y \ Z : \text{eqType}\} \{ \alpha : \text{Rel } X \ Y \} \{ \beta : \text{Rel } Y \ Z \} \{ \gamma : \text{Rel } Z \ W \} :$   
 $((\alpha \quad (\beta \cdot \gamma)) \quad (X \ Z \cdot \gamma)) \quad ((\alpha \quad (\beta \cdot \gamma)) \cdot (\gamma \# \cdot \gamma)).$

**Proof.**

```
rewrite cap_comm.
apply (@inc_trans _ _ _ _ (dedekind2)).
rewrite cap_comm cap_universal comp_assoc.
apply inc_refl.
Qed.
```

**Lemma 242 (residual\_property5)** *Let  $\tau$  be a univalent relation. Then,*

$$(\alpha \triangleright \beta) \cdot \tau^\# = (\alpha \triangleright \beta \cdot \tau^\#) \sqcap \nabla_{XZ} \cdot \tau^\#.$$

**Lemma residual\_property5**

$\{V \ X \ Y \ Z : \text{eqType}\} \{ \alpha : \text{Rel } X \ Y \} \{ \beta : \text{Rel } Y \ Z \} \{ \tau : \text{Rel } V \ Z \} :$   
 $\text{univalent}_r \tau \rightarrow$   
 $(\alpha \quad \beta) \cdot \tau \# = (\alpha \quad (\beta \cdot \tau \#)) \quad (X \ Z \cdot \tau \#).$

**Proof.**

```
move => H.
apply inc_antisym.
rewrite -(@cap_universal _ _ (alpha beta)).
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat_r.
apply residual_property1.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ (dedekind2)).
rewrite cap_comm cap_universal inv_invol.
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ _ (residual_property1)).
apply residual_inc_compat_l.
rewrite comp_assoc.
apply (comp_inc_compat_ab_a H).
Qed.
```

**Lemma 243 (residual\_property6)**

$$\alpha \triangleright (\gamma^\# \triangleright \beta^\#)^\# = (\gamma^\# \triangleright (\alpha \triangleright \beta)^\#)^\#.$$

*Lemma residual\_property6*

$\{W\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{gamma : Rel\ Z\ W\} :$   
 $alpha\ (gamma\ \# \ beta\ \#) \ \# = (gamma\ \# \ (alpha\ \beta) \ \#) \ \#.$

*Proof.*

apply *inc\_lower*.  
 move  $\Rightarrow$  *delta*.  
 split; move  $\Rightarrow$  *H*.  
 apply *inv\_inc\_move*.  
 apply *inc\_residual*.  
 apply *inv\_inc\_move*.  
 apply *inc\_residual*.  
 rewrite *comp\_inv comp\_assoc*.  
 apply *inv\_inc\_move*.  
 apply *inc\_residual*.  
 apply *inv\_inc\_invol*.  
 rewrite *comp\_inv inv\_invol*.  
 apply *inc\_residual*.  
 apply *H*.  
 apply *inc\_residual*.  
 apply *inv\_inc\_move*.  
 apply *inc\_residual*.  
 apply *inv\_inc\_move*.  
 rewrite *comp\_inv inv\_invol inv\_invol comp\_assoc*.  
 apply *inc\_residual*.  
 apply *inv\_inc\_invol*.  
 rewrite *comp\_inv*.  
 apply *inc\_residual*.  
 apply *inv\_inc\_move*.  
 apply *H*.

*Qed.*

**Lemma 244 (residual\_property7a, residual\_property7b)**

$$\alpha \triangleright (\beta \Rightarrow \beta') \sqsubseteq (\alpha \cdot \beta \Rightarrow \alpha \cdot \beta') \sqsubseteq \alpha \triangleright (\beta \Rightarrow \alpha^\# \cdot \alpha \cdot \beta').$$

*Lemma residual\_property7a*  $\{X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta\ beta' : Rel\ Y\ Z\} :$   
 $(alpha\ (\beta \gg \beta')) \sqsubseteq ((alpha \cdot \beta) \gg (alpha \cdot \beta')).$

*Proof.*



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```

apply inc_rpc.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm.
apply inc_rpc.
apply inv_residual_inc.
Qed.

```

**Lemma residual\_property7b**  $\{X\ Y\ Z : \text{eqType}\} \{alpha : \text{Rel } X\ Y\} \{beta\ beta' : \text{Rel } Y\ Z\}$ :  
 $((alpha \cdot beta) \gg (alpha \cdot beta')) \quad (alpha \quad (beta \gg (alpha \# \cdot (alpha \cdot beta')))).$

**Proof.**

```

rewrite inc_residual inc_rpc.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite inv_invol -inc_rpc.
apply inc_refl.
Qed.

```

**Lemma 245 (residual\_property8)** *Let  $\alpha$  be a univalent relation. Then,*

$$\alpha \triangleright (\beta \Rightarrow \beta') = (\alpha \cdot \beta \Rightarrow \alpha \cdot \beta').$$

**Lemma residual\_property8**  $\{X\ Y\ Z : \text{eqType}\} \{alpha : \text{Rel } X\ Y\} \{beta\ beta' : \text{Rel } Y\ Z\}$ :  
 $\text{univalent\_r } alpha \rightarrow alpha \quad (beta \gg beta') = (alpha \cdot beta) \gg (alpha \cdot beta').$

**Proof.**

```

move => H.
apply inc_antisym.
apply residual_property7a.
apply (@inc_trans _ _ _ _ residual_property7b).
apply residual_inc_compat_l.
apply rpc_inc_compat_l.
rewrite -comp_assoc.
apply (comp_inc_compat_ab_b H).
Qed.

```

**Lemma 246 (residual\_property9)** *Let  $\alpha$  be a univalent relation. Then,*

$$\alpha \triangleright \beta = (\alpha \cdot \nabla_{YZ} \Rightarrow \alpha \cdot \beta).$$

**Lemma residual\_property9**  $\{X\ Y\ Z : \text{eqType}\} \{alpha : \text{Rel } X\ Y\} \{beta : \text{Rel } Y\ Z\}$ :  
 $\text{univalent\_r } alpha \rightarrow alpha \quad beta = (alpha \cdot \nabla_{YZ} \Rightarrow (alpha \cdot beta)).$

**Proof.**

```

move => H.

```

by [rewrite -(residual\_property8 H) rpc\_universal\_alpha].  
**Qed.**

**Lemma 247 (residual\_property10)** *Let  $\alpha$  be a univalent relation. Then,*

$$\alpha \cdot \beta = \lfloor \alpha \rfloor \cdot (\alpha \triangleright \beta).$$

**Lemma residual\_property10** {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:  
 univalent\_r alpha → alpha · beta = domain alpha · (alpha beta).

**Proof.**

move ⇒ H.

apply inc\_antisym.

replace (alpha · beta) with (domain alpha · (alpha · beta)).

apply comp\_inc\_compat\_ab\_ab'.

rewrite inc\_residual\_comp\_assoc.

apply (comp\_inc\_compat\_ab\_b H).

by [rewrite -comp\_assoc domain\_comp\_alpha1].

apply (@inc\_trans \_ \_ \_ ((alpha · alpha #) · (alpha beta))).

apply comp\_inc\_compat\_ab\_a'b.

apply cap\_l.

rewrite comp\_assoc.

apply comp\_inc\_compat\_ab\_ab'.

apply inv\_residual\_inc.

**Qed.**

**Lemma 248 (residual\_property11)**

$$(\alpha \cdot \beta \Rightarrow \delta) \sqsubseteq \alpha \triangleright (\beta \Rightarrow \alpha^\# \cdot \delta).$$

**Lemma residual\_property11**

{X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {delta : Rel X Z}:  
 ((alpha · beta) » delta) (alpha (beta » (alpha # · delta))).

**Proof.**

apply inc\_residual.

apply inc\_rpc.

apply (@inc\_trans \_ \_ \_ \_ (dedekind1)).

rewrite inv\_invol.

apply comp\_inc\_compat\_ab\_ab'.

apply inc\_rpc.

apply inc\_refl.

**Qed.**

**Lemma 249 (residual\_property12a, residual\_property12b)** *Let  $u \sqsubseteq id_X$ . Then,*

$$u \triangleright \alpha = u \cdot \nabla_{XY} \Rightarrow \alpha = u \triangleright u \cdot \alpha.$$

**Lemma residual\_property12a**  $\{X\ Y : eqType\} \{u : Rel\ X\ X\} \{alpha : Rel\ X\ Y\}$ :  
 $u \quad Id\ X \rightarrow u \quad alpha = (u \cdot \quad X\ Y) \gg alpha.$

**Proof.**

move  $\Rightarrow H$ .  
 apply *inc\_antisym*.  
 assert (*univalent\_r*  $u$ ).  
 apply (**fun**  $H' \Rightarrow @inc\_trans \_ \_ \_ \_ H' H$ ).  
 apply *comp\_inc\_compat\_ab\_b*.  
 rewrite *inv\_id*.  
 apply (*@inc\_inv*  $\_ \_ \_ \_ H$ ).  
 rewrite (*residual\_property9*  $H0$ ).  
 apply *rpc\_inc\_compat\_l*.  
 apply (*comp\_inc\_compat\_ab\_b*  $H$ ).  
 apply (*@inc\_trans*  $\_ \_ \_ \_ residual\_property11$ ).  
 apply *residual\_inc\_compat\_l*.  
 rewrite *rpc\_universal\_alpha*.  
 apply *comp\_inc\_compat\_ab\_b*.  
 rewrite *inv\_id*.  
 apply (*@inc\_inv*  $\_ \_ \_ \_ H$ ).

**Qed.**

**Lemma residual\_property12b**  $\{X\ Y : eqType\} \{u : Rel\ X\ X\} \{alpha : Rel\ X\ Y\}$ :  
 $u \quad Id\ X \rightarrow u \quad alpha = u \quad (u \cdot alpha).$

**Proof.**

move  $\Rightarrow H$ .  
 apply *inc\_antisym*.  
 rewrite (*residual\_property12a*  $H$ ).  
 apply (*@inc\_trans*  $\_ \_ \_ \_ residual\_property11$ ).  
 apply *residual\_inc\_compat\_l*.  
 rewrite *rpc\_universal\_alpha*.  
 apply *comp\_inc\_compat\_ab\_a'b*.  
 rewrite (*dedekind\_id1*  $H$ ).  
 apply *inc\_refl*.  
 apply *residual\_inc\_compat\_l*.  
 apply (*comp\_inc\_compat\_ab\_b*  $H$ ).

**Qed.**

**Lemma 250 (residual\_property13)**

$$(\alpha \cdot \nabla_{YZ} \sqcap \delta) \triangleright \gamma = (\alpha \cdot \nabla_{YW} \Rightarrow (\delta \triangleright \gamma)).$$

*Lemma residual\_property13*

$\{W\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{gamma : Rel\ Z\ W\} \{delta : Rel\ X\ Z\} :$   
 $((alpha \cdot Y\ Z)\ delta)\ gamma = (alpha \cdot Y\ W) \gg (delta\ gamma).$

*Proof.*apply *inc\_antisym*.rewrite *inc\_rpc inc\_residual*.remember (((*alpha* · *Y Z*) *delta*) *gamma*) as *sigma1*.apply (@*inc\_trans* \_ \_ \_ (((*alpha* · *Y Z*) *delta*) # · *sigma1*)).apply (@*inc\_trans* \_ \_ \_ (((*alpha* · *Y Z*) *delta*) # · (*sigma1* (*alpha* · *Y W*)))).assert ((*delta* # · (*sigma1* (*alpha* · *Y W*))) (*delta* # · *sigma1*)).apply *comp\_inc\_compat\_ab\_ab'*.apply *cap\_l*.apply *inc\_def1* in *H*.rewrite *H*.apply (@*inc\_trans* \_ \_ \_ \_ (*dedekind2*)).apply *comp\_inc\_compat\_ab\_a'b*.rewrite (@*inv\_cap\_distr* \_ \_ \_ *delta*) *cap\_comm*.apply *cap\_inc\_compat\_r*.rewrite *inv\_cap\_distr*.apply (@*inc\_trans* \_ \_ \_ \_ (*comp\_cap\_distr\_l*)).apply (@*inc\_trans* \_ \_ \_ \_ (*cap\_r*)).rewrite *comp\_inv comp\_inv-comp\_assoc* (@*inv\_universal* *Y Z*).apply *comp\_inc\_compat\_ab\_a'b*.apply *inc\_alpha\_universal*.apply *comp\_inc\_compat\_ab\_ab'*.apply *cap\_l*.rewrite *Hesigma1*.apply *inc\_residual*.apply *inc\_refl*.rewrite *inc\_residual*.remember ((*alpha* · *Y W*) » (*delta* *gamma*)) as *sigma2*.apply (@*inc\_trans* \_ \_ \_ (*delta* # · ((*alpha* · *Y W*) *sigma2*))).apply (@*inc\_trans* \_ \_ \_ (((*alpha* · *Y Z*) *delta*) # · ((*alpha* · *Y W*) *sigma2*))).assert ((((*alpha* · *Y Z*) *delta*) # · *sigma2*) (*delta* # · *sigma2*)).apply *comp\_inc\_compat\_ab\_a'b*.apply *inc\_inv*.apply *cap\_r*.

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```

apply inc_def1 in H.
rewrite H.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm_inv_invol.
apply cap_inc_compat_r.
apply (@inc_trans _ _ _ ((alpha · Y Z) · (delta # · sigma2))).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply comp_inc_compat_ab_a'b.
apply inc_inv.
apply cap_r.
rewrite Hegsigma2.
rewrite -inc_residual_cap_comm_inc_rpc.
apply inc_refl.
Qed.

```

**Lemma 251 (residual\_property14)** *Let  $\nabla_{XX} \cdot \alpha \sqsubseteq \alpha$ . Then,*

$$\nabla_{XX} \cdot (\alpha \triangleright \beta) \sqsubseteq \alpha \triangleright \beta.$$

**Lemma residual\_property14**  $\{X \ Y \ Z : eqType\} \{alpha : Rel \ X \ Y\} \{beta : Rel \ Y \ Z\} :$   
 $(\nabla_{XX} \cdot alpha) \cdot alpha \rightarrow (\nabla_{XX} \cdot (alpha \cdot beta)) \cdot (alpha \cdot beta).$

**Proof.**

```

move  $\Rightarrow$  H.
apply (@inc_trans _ _ _ (  $\nabla_{XX} \cdot ( \nabla_{XX} \cdot (alpha \cdot beta) )$  )).
apply comp_inc_compat_ab_ab'.
rewrite double_residual.
apply (residual_inc_compat_r H).
rewrite -inv_universal_inc_residual_inv_universal.
apply inc_refl.
Qed.

```

**Lemma 252 (residual\_property15)** *Let  $\beta \cdot \nabla_{ZZ} \sqsubseteq \beta$ . Then,*

$$(\alpha \triangleright \beta) \cdot \nabla_{ZZ} \sqsubseteq \alpha \triangleright \beta.$$

**Lemma residual\_property15**  $\{X \ Y \ Z : eqType\} \{alpha : Rel \ X \ Y\} \{beta : Rel \ Y \ Z\} :$   
 $(beta \cdot \nabla_{ZZ} \cdot beta) \rightarrow ((alpha \cdot beta) \cdot \nabla_{ZZ} \cdot (alpha \cdot beta)).$

**Proof.**

move  $\Rightarrow H$ .  
 apply (@inc\_trans \_ \_ \_ \_ (residual\_property1)).  
 apply (residual\_inc\_compat\_l H).  
 Qed.

**Lemma 253 (residual\_property16)**

$$id_X \sqsubseteq \alpha \triangleright \alpha^\# \wedge (\alpha \triangleright \alpha^\#) \cdot (\alpha \triangleright \alpha^\#) \sqsubseteq \alpha \triangleright \alpha^\#.$$

**Lemma residual\_property16**  $\{X\ Y : eqType\} \{alpha : Rel\ X\ Y\}$ :  
 $Id\ X \quad (alpha \quad alpha^\#) \wedge$   
 $((alpha \quad alpha^\#) \cdot (alpha \quad alpha^\#)) \quad (alpha \quad alpha^\#).$

**Proof.**  
 split.  
 rewrite inc\_residual\_comp\_id\_r.  
 apply inc\_refl.  
 move : (@residual\_property2 \_ \_ \_ alpha (alpha #) (alpha #))  $\Rightarrow H$ .  
 rewrite inv\_invol in H.  
 apply H.  
 Qed.

**Lemma 254 (residual\_property17)** *Let  $P(y) := “y : I \rightarrow Y$  is a function”. Then,*

$$\sqcup_{P(y)} y^\# \cdot y = id_Y \Rightarrow \alpha \triangleright \beta = \sqcap_{P(y)} (\alpha \cdot y^\# \cdot \nabla_{IZ} \Rightarrow \alpha \cdot y^\# \cdot y \cdot \beta).$$

**Lemma residual\_property17**  $\{X\ Y\ Z : eqType\}$   
 $\{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{P : Rel\ i\ Y \rightarrow Prop\}$ :  
 $P = (\text{fun } y : Rel\ i\ Y \Rightarrow function\_r\ y) \rightarrow$   
 $\_ \{P\} (\text{fun } y : Rel\ i\ Y \Rightarrow y^\# \cdot y) = Id\ Y \rightarrow$   
 $alpha \quad beta = \_ \{P\} (\text{fun } y : Rel\ i\ Y \Rightarrow$   
 $((alpha \cdot y^\#) \cdot \_ i\ Z) \gg ((alpha \cdot y^\#) \cdot (y \cdot beta))).$

**Proof.**  
 move  $\Rightarrow H\ H0$ .  
 replace (alpha beta) with ((alpha · Id Y) beta).  
 rewrite -H0\_comp\_cupP\_distr\_l residual\_cupP\_distr\_r.  
 apply capP\_eq.  
 move  $\Rightarrow y\ H1$ .  
 rewrite H in H1.  
 rewrite -comp\_assoc (function\_residual4 H1).  
 apply residual\_property9.  
 rewrite /univalent\_r.  
 rewrite unit\_identity\_is\_universal.  
 apply inc\_alpha\_universal.

by [rewrite *comp\_id\_r*].

**Qed.**

## 11.4 順序の関係と左剰余合成

### 11.4.1 max, sup, min, inf

$\xi : X \rightarrow X$  を集合  $X$  における順序と見なしたときの, 関係  $\rho : V \rightarrow X$  の 最大値 (max), 上限 (sup), 最小値 (min), 下限 (inf) はそれぞれ, 以下のように定義される.

- $\max(\rho, \xi) := \rho \sqcap (\rho \triangleright \xi)$
- $\sup(\rho, \xi) := (\rho \triangleright \xi) \sqcap ((\rho \triangleright \xi) \triangleright \xi^\#)$
- $\min(\rho, \xi) := \rho \sqcap (\rho \triangleright \xi^\#) (= \max(\rho, \xi^\#))$
- $\inf(\rho, \xi) := (\rho \triangleright \xi^\#) \sqcap ((\rho \triangleright \xi^\#) \triangleright \xi) (= \sup(\rho, \xi^\#))$

**Definition**  $\max \{ V X : eqType \} (rho : Rel V X) (xi : Rel X X)$   
 $:= rho \sqcap (rho \triangleright xi).$

**Definition**  $\sup \{ V X : eqType \} (rho : Rel V X) (xi : Rel X X)$   
 $:= (rho \triangleright xi) \sqcap ((rho \triangleright xi) \triangleright xi^\#).$

**Definition**  $\min \{ V X : eqType \} (rho : Rel V X) (xi : Rel X X)$   
 $:= rho \sqcap (rho \triangleright xi^\#).$

**Definition**  $\inf \{ V X : eqType \} (rho : Rel V X) (xi : Rel X X)$   
 $:= (rho \triangleright xi^\#) \sqcap ((rho \triangleright xi^\#) \triangleright xi).$

**Lemma 255 (max-inc-sup)** *Let  $\rho : V \rightarrow X$  and  $\xi : X \rightarrow X$ . Then,*

$$\max(\rho, \xi) \sqsubseteq \sup(\rho, \xi).$$

**Lemma**  $\max\_inc\_sup \{ V X : eqType \} \{ rho : Rel V X \} \{ xi : Rel X X \} :$   
 $\max rho xi \sqsubseteq \sup rho xi.$

**Proof.**

rewrite /max/sup.

rewrite cap-comm.

apply cap-inc-compatible-l.

apply galois-corollary1.

**Qed.**

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**Lemma 256 (min\_inc\_inf)** *Let  $\rho : V \rightarrow X$  and  $\xi : X \rightarrow X$ . Then,*

$$\min(\rho, \xi) \sqsubseteq \inf(\rho, \xi).$$

**Lemma** *min\_inc\_inf* {  $V\ X : eqType$  } {  $\rho : Rel\ V\ X$  } {  $\xi : Rel\ X\ X$  }:  
 $\min\ \rho\ \xi \sqsubseteq \inf\ \rho\ \xi$ .

**Proof.**

rewrite /min/inf.  
 rewrite cap\_comm.  
 apply cap\_inc\_compat\_l.  
 move : (@galois\_corollary1 \_ \_ \_ rho (xi #)) => H.  
 rewrite inv\_invol in H.  
 apply H.

**Qed.**

**Lemma 257 (inf\_to\_sup)** *Let  $\rho : V \rightarrow X$  and  $\xi : X \rightarrow X$ . Then,*

$$\inf(\rho, \xi) = \sup(\rho \triangleright \xi^\sharp, \xi).$$

**Lemma** *inf\_to\_sup* {  $V\ X : eqType$  } {  $\rho : Rel\ V\ X$  } {  $\xi : Rel\ X\ X$  }:  
 $\inf\ \rho\ \xi = \sup\ (\rho \triangleright \xi^\sharp)\ \xi$ .

**Proof.**

rewrite /sup/inf.  
 rewrite cap\_comm.  
 move : (@galois\_corollary2 \_ \_ \_ rho (xi #)) => H.  
 rewrite inv\_invol in H.  
 by [rewrite H].

**Qed.**

**Lemma 258 (sup\_to\_inf)** *Let  $\rho : V \rightarrow X$  and  $\xi : X \rightarrow X$ . Then,*

$$\sup(\rho, \xi) = \inf(\rho \triangleright \xi, \xi).$$

**Lemma** *sup\_to\_inf* {  $V\ X : eqType$  } {  $\rho : Rel\ V\ X$  } {  $\xi : Rel\ X\ X$  }:  
 $\sup\ \rho\ \xi = \inf\ (\rho \triangleright \xi)\ \xi$ .

**Proof.**

rewrite /sup/inf.  
 rewrite cap\_comm.  
 by [rewrite galois\_corollary2].

**Qed.**



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**Lemma 259 (residual\_inc\_sup1, residual\_inc\_sup2)** *Let  $\rho : V \rightarrow X$  and  $\xi : X \rightarrow X$ . Then,*

$$\text{sup}(\rho, \xi) \sqsubseteq \rho \triangleright \xi \sqsubseteq \text{sup}(\rho, \xi) \triangleright \xi.$$

**Lemma residual\_inc\_sup1**  $\{V\ X : \text{eqType}\} \{rho : \text{Rel}\ V\ X\} \{xi : \text{Rel}\ X\ X\} :$   
 $\text{sup}\ rho\ xi \quad (rho \quad xi).$

**Proof.**

apply *cap\_l*.

**Qed.**

**Lemma residual\_inc\_sup2**  $\{V\ X : \text{eqType}\} \{rho : \text{Rel}\ V\ X\} \{xi : \text{Rel}\ X\ X\} :$   
 $(rho \quad xi) \quad ((\text{sup}\ rho\ xi) \quad xi).$

**Proof.**

rewrite *galois*.

apply *cap\_r*.

**Qed.**

**Lemma 260 (max\_inc\_xi\_cap)** *Let  $\rho : V \rightarrow X$  and  $\xi : X \rightarrow X$ . Then,*

$$(\text{max}(\rho, \xi))^{\#} \cdot \text{max}(\rho, \xi) \sqsubseteq \xi \sqcap \xi^{\#}.$$

**Lemma max\_inc\_xi\_cap**  $\{V\ X : \text{eqType}\} \{rho : \text{Rel}\ V\ X\} \{xi : \text{Rel}\ X\ X\} :$   
 $(\text{max}\ rho\ xi\ \# \cdot \text{max}\ rho\ xi) \quad (xi \quad xi\ \#).$

**Proof.**

rewrite */max*.

rewrite *inv\_cap\_distr*.

apply (*@inc\_trans - - - - (comp\_cap\_distr\_r)*).

apply *cap\_inc\_compat*.

apply *inc\_residual*.

apply *cap\_r*.

apply *inv\_inc\_move*.

rewrite *comp\_inv inv\_invol*.

apply *inc\_residual*.

apply *residual\_inc\_compat\_r*.

apply *cap\_l*.

**Qed.**

**Lemma 261 (sup\_inc\_xi\_cap)** *Let  $\rho : V \rightarrow X$  and  $\xi : X \rightarrow X$ . Then,*

$$(\text{sup}(\rho, \xi))^{\#} \cdot \text{sup}(\rho, \xi) \sqsubseteq \xi \sqcap \xi^{\#}.$$

**Lemma sup\_inc\_xi\_cap**  $\{V\ X : \text{eqType}\} \{rho : \text{Rel}\ V\ X\} \{xi : \text{Rel}\ X\ X\} :$   
 $(\text{sup}\ rho\ xi\ \# \cdot \text{sup}\ rho\ xi) \quad (xi \quad xi\ \#).$

**Proof.**

move : (@max\_inc\_xi\_cap \_ \_ (rho xi) (xi #)).

rewrite /max/sup.

by [rewrite inv\_invol (@cap\_comm \_ \_ xi)].

**Qed.**

**Lemma 262 (transitive\_sup1)** *Let  $\rho : V \rightarrow X$ ,  $\xi : X \rightarrow X$  and  $\xi \cdot \xi \sqsubseteq \xi$ . Then,*

$$\text{sup}(\rho, \xi) \cdot (\xi \sqcap \xi^\#) = \text{sup}(\rho, \xi).$$

**Lemma transitive\_sup1** { V X : eqType } { rho : Rel V X } { xi : Rel X X }:

(xi • xi) xi → sup rho xi • (xi xi #) = sup rho xi.

**Proof.**

move ⇒ H.

apply inc\_antisym.

rewrite /sup.

apply (@inc\_trans \_ \_ \_ \_ (comp\_cap\_distr\_l)).

apply cap\_inc\_compat.

apply (@inc\_trans \_ \_ \_ \_ (comp\_cap\_distr\_r)).

apply (@inc\_trans \_ \_ \_ \_ (cap\_l)).

apply (@inc\_trans \_ \_ \_ \_ (residual\_property1)).

apply (residual\_inc\_compat\_l H).

apply (@inc\_trans \_ \_ \_ \_ (comp\_cap\_distr\_r)).

apply (@inc\_trans \_ \_ \_ \_ (cap\_r)).

apply (@inc\_trans \_ \_ \_ \_ (residual\_property1)).

apply residual\_inc\_compat\_l.

rewrite -comp\_inv inv\_inc\_move inv\_invol.

apply H.

apply (@inc\_trans \_ \_ \_ \_ (relation\_rel\_inv\_rel)).

rewrite comp\_assoc.

apply (comp\_inc\_compat\_ab\_ab' sup\_inc\_xi\_cap).

**Qed.**

**Lemma 263 (transitive\_sup2)** *Let  $\rho : V \rightarrow X$ ,  $\xi : X \rightarrow X$  and  $\xi \cdot \xi \sqsubseteq \xi$ . Then,*

$$\text{sup}(\rho, \xi) \cdot \xi = \lfloor \text{sup}(\rho, \xi) \rfloor \cdot (\rho \triangleright \xi).$$

**Lemma transitive\_sup2** { V X : eqType } { rho : Rel V X } { xi : Rel X X }:

(xi • xi) xi → sup rho xi • xi = domain (sup rho xi) • (rho xi).

**Proof.**

move ⇒ H.

apply inc\_antisym.

replace (sup rho xi • xi) with (domain (sup rho xi) • (sup rho xi • xi)).

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```

apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ ((rho xi) • xi)).
apply (comp_inc_compat_ab_a'b cap_l).
apply (@inc_trans _ _ _ _ (residual_property1) (residual_inc_compat_l H)).
by [rewrite -comp_assoc domain_comp_alpha1].
apply (@inc_trans _ _ _ (domain (sup rho xi) • (sup rho xi xi))).
apply comp_inc_compat_ab_ab'.
apply galois.
apply cap_r.
rewrite /domain.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_residual.
apply inc_refl.
Qed.

```

**Lemma 264 (domain\_sup\_inc)** *Let  $\rho : V \rightarrow X$  and  $\xi : X \rightarrow X$ . Then,*

$$\lfloor \text{sup}(\rho, \xi) \rfloor \cdot \rho \sqsubseteq \text{sup}(\rho, \xi) \cdot \xi^\sharp.$$

**Lemma domain\_sup\_inc**  $\{V\ X : \text{eqType}\} \{\rho : \text{Rel } V\ X\} \{\xi : \text{Rel } X\ X\}$ :  
 $(\text{domain } (\text{sup } \rho \ \xi) \cdot \rho) \quad (\text{sup } \rho \ \xi \cdot \xi^\sharp).$

**Proof.**

```

apply (@inc_trans _ _ _ (domain (sup rho xi) • (sup rho xi xi))).
apply comp_inc_compat_ab_ab'.
rewrite -galois.
apply cap_l.
rewrite /domain.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_residual.
apply inc_refl.
Qed.

```

**Lemma 265 (sup\_function)** *Let  $\rho : V \rightarrow X$ ,  $\xi : X \rightarrow X$  be relations and  $f : W \rightarrow V$  be a function. Then,*

$$f \cdot \text{sup}(\rho, \xi) = \text{sup}(f \cdot \rho, \xi).$$

**Lemma sup\_function**  $\{V\ W\ X : \text{eqType}\} \{\rho : \text{Rel } V\ X\} \{\xi : \text{Rel } X\ X\} \{f : \text{Rel } W\ V\}$ :

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$\text{function\_r } f \rightarrow f \cdot \text{sup } \rho \text{ } xi = \text{sup } (f \cdot \rho) \text{ } xi.$

**Proof.**

$\text{move} \Rightarrow H.$

$\text{rewrite } / \text{sup}.$

$\text{rewrite } (\text{function\_cap\_distr\_l } H).$

$\text{by } [\text{rewrite } (\text{function\_residual2 } H) (\text{function\_residual2 } H) (\text{function\_residual2 } H)].$

**Qed.**

**Lemma 266 (max\_univalent)** *Let  $\rho : V \rightarrow X$ ,  $\xi : X \rightarrow X$  be relations and  $\varphi : W \rightarrow V$  be a univalent relation. Then,*

$$\varphi \cdot \text{max}(\rho, \xi) = \text{max}(\varphi \cdot \rho, \xi).$$

**Lemma**  $\text{max\_univalent } \{V \ W \ X : \text{eqType}\}$   
 $\{ \rho : \text{Rel } V \ X \} \{ xi : \text{Rel } X \ X \} \{ phi : \text{Rel } W \ V \} :$   
 $\text{univalent\_r } phi \rightarrow phi \cdot \text{max } \rho \text{ } xi = \text{max } (phi \cdot \rho) \text{ } xi.$

**Proof.**

$\text{move} \Rightarrow H.$

$\text{rewrite } / \text{max}.$

$\text{apply } \text{inc\_antisym}.$

$\text{apply } (@\text{inc\_trans } \_ \_ \_ \_ (\text{comp\_cap\_distr\_l})).$

$\text{apply } \text{cap\_inc\_compat\_l}.$

$\text{apply } (@\text{inc\_trans } \_ \_ \_ \_ (\text{univalent\_residual } H)).$

$\text{rewrite } \text{double\_residual}.$

$\text{apply } \text{inc\_refl}.$

$\text{apply } (@\text{inc\_trans } \_ \_ \_ \_ (\text{dedekind1})).$

$\text{apply } \text{comp\_inc\_compat\_ab\_ab'}.$

$\text{apply } \text{cap\_inc\_compat\_l}.$

$\text{rewrite } \text{-inc\_residual double\_residual}.$

$\text{apply } \text{inc\_refl}.$

**Qed.**

### 11.4.2 左剰余合成

関係  $\alpha : X \rightarrow Y$ ,  $\beta : Y \rightarrow Z$  に対し, 左剰余合成を  $\alpha \triangleleft \beta := (\beta^\# \triangleright \alpha^\#)^\#$  で定義する.

**Definition**  $\text{leftres } \{X \ Y \ Z : \text{eqType}\} (\alpha : \text{Rel } X \ Y) (\beta : \text{Rel } Y \ Z)$   
 $:= (\beta^\# \# \alpha^\#)^\#.$

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**Lemma 267 (inc\_leftres)** *Let  $\alpha : X \rightarrow Y$ ,  $\beta : Y \rightarrow Z$  and  $\delta : X \rightarrow Z$ . Then,*

$$\delta \sqsubseteq \alpha \triangleleft \beta \Leftrightarrow \delta \cdot \beta^\# \sqsubseteq \alpha.$$

**Lemma inc\_leftres**  $\{X\ Y\ Z : eqType\}$   
 $\{\alpha : Rel\ X\ Y\} \{\beta : Rel\ Y\ Z\} \{\delta : Rel\ X\ Z\}$ :  
 $\delta \sqsubseteq \alpha \quad leftres\ \alpha\ \beta \Leftrightarrow (\delta \cdot \beta^\#) \sqsubseteq \alpha$ .

**Proof.**

rewrite /leftres.

by [rewrite inv\_inc\_move inc\_residual -comp\_inv inv\_inc\_move inv\_invol].

**Qed.**

**Lemma 268 (residual\_leftres\_assoc)** *Let  $\alpha : X \rightarrow Y$ ,  $\beta : Y \rightarrow Z$  and  $\gamma : Z \rightarrow W$ . Then,*

$$(\alpha \triangleright \beta) \triangleleft \gamma = \alpha \triangleright (\beta \triangleleft \gamma).$$

**Lemma residual\_leftres\_assoc**  $\{W\ X\ Y\ Z : eqType\}$   
 $\{\alpha : Rel\ X\ Y\} \{\beta : Rel\ Y\ Z\} \{\gamma : Rel\ Z\ W\}$ :  
 $leftres\ (\alpha \triangleright \beta)\ \gamma = \alpha \triangleright leftres\ \beta\ \gamma$ .

**Proof.**

apply inc\_lower.

move  $\Rightarrow$  delta.

by [rewrite inc\_leftres inc\_residual -comp\_assoc -inc\_leftres -inc\_residual].

**Qed.**

# Chapter 12

## Library Schroder

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Residual.
Require Import Logic.FunctionalExtensionality.
```

### 12.1 Schröder 圏の性質

この節では, 特記が無い限り, 記号は以下の図式に従って割り振られるものとする.

$$\begin{array}{ccccc}
 & & \delta & & \\
 & \nearrow & & \searrow & \\
 X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta, \beta', \beta_\lambda} & Z & \xrightarrow{\gamma} & W \\
 \uparrow \rho & & & & \uparrow \tau & & \\
 I & & & & V & & 
 \end{array}$$

**Lemma 269** (schroder\_equivalence1, schroder\_equivalence2)

$$\alpha \cdot \beta \sqsubseteq \delta \Leftrightarrow \alpha^\# \cdot \delta^- \sqsubseteq \beta^- \Leftrightarrow \delta^- \cdot \beta^\# \sqsubseteq \alpha^-.$$

**Lemma** *schroder\_equivalence1*

```
{X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {delta : Rel X Z}:
(alpha · beta)    delta ↔ (alpha # · delta ^)    beta ^.
```

**Proof.**

```
split; move ⇒ H.
```

```
rewrite bool_lemma2 complement_invol.
```

```

apply inc_antisym.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply bool_lemma2 in H.
rewrite cap_comm inv_invol H comp_empty_r.
apply inc_refl.
apply inc_empty_alpha.
rewrite bool_lemma2.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply bool_lemma2 in H.
rewrite cap_comm -(@complement_invol _ _ beta) H comp_empty_r.
apply inc_refl.
apply inc_empty_alpha.
Qed.

```

**Lemma** *schroder\_equivalence2*

$\{X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{delta : Rel\ X\ Z\} :$   
 $(alpha \cdot beta) \quad delta \leftrightarrow (delta^\wedge \cdot beta^\#) \quad alpha^\wedge.$

**Proof.**

```

split; move => H.
rewrite bool_lemma2 complement_invol.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply bool_lemma2 in H.
rewrite cap_comm inv_invol H comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
rewrite bool_lemma2.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply bool_lemma2 in H.
rewrite cap_comm -(@complement_invol _ _ alpha) H comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
Qed.

```

**Lemma 270 (function\_inv\_complement)** *Let  $\alpha$  and  $\tau$  be functions. Then,*

$$(\alpha \cdot \beta \cdot \tau^\#)^- = \alpha \cdot \beta^- \cdot \tau^\#.$$

**Lemma** *function\_inv\_complement*

$\{V\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\} \{tau : Rel\ V\ Z\} :$   
 $function\_r\ alpha \rightarrow function\_r\ tau \rightarrow$

$$((\alpha \cdot \beta) \cdot \tau \#)^{\wedge} = (\alpha \cdot \beta^{\wedge}) \cdot \tau \#.$$

**Proof.**

move  $\Rightarrow H \ H0$ .

apply *inc\_antisym*.

rewrite *bool\_lemma1 complement\_invol*.

apply *inc\_antisym*.

rewrite *-comp\_cup\_distr\_r -comp\_cup\_distr\_l complement\_classic*.

apply (@*inc\_trans* \_ \_ \_ ((( $\alpha \cdot \alpha \#$ )  $\cdot$   $X \ V$ )  $\cdot$  ( $\tau \cdot \tau \#$ )))).

apply (@*inc\_trans* \_ \_ \_ ((( $\alpha \cdot \alpha \#$ )  $\cdot$   $X \ V$ ))).

apply *comp\_inc\_compat\_b\_ab*.

apply *H*.

apply *comp\_inc\_compat\_a\_ab*.

apply *H0*.

rewrite *-comp\_assoc (@comp\_assoc \_ \_ \_  $\alpha$ ) (@comp\_assoc \_ \_ \_  $\alpha$ )*.

apply *comp\_inc\_compat\_ab\_a'b*.

apply *comp\_inc\_compat\_ab\_ab'*.

apply *inc\_alpha\_universal*.

apply *inc\_alpha\_universal*.

rewrite *bool\_lemma2 complement\_invol*.

apply *inc\_antisym*.

rewrite *-(function\_cap\_distr H H0) cap\_comm cap\_complement\_empty comp\_empty\_r comp\_empty\_l*.

apply *inc\_refl*.

apply *inc\_empty\_alpha*.

**Qed.**

**Lemma 271 (schroder\_univalent1)** *Let  $\alpha$  be a univalent relation and  $\beta \sqsubseteq \beta'$ . Then,*

$$\alpha \cdot (\beta' \sqcap \beta^-) = \alpha \cdot \beta' \sqcap (\alpha \cdot \beta)^-.$$

**Lemma *schroder\_univalent1***

$\{X \ Y \ Z : \text{eqType}\} \ \{\alpha : \text{Rel } X \ Y\} \ \{\beta \ \beta' : \text{Rel } Y \ Z\} :$

*univalent\_r*  $\alpha \rightarrow \beta \ \beta' \rightarrow$

$\alpha \cdot (\beta' \ \beta^{\wedge}) = (\alpha \cdot \beta') \ (\alpha \cdot \beta)^{\wedge}.$

**Proof.**

move  $\Rightarrow H \ H0$ .

apply (@*cap\_cup\_unique* \_ \_ ( $\alpha \cdot \beta$ )).

replace (( $\alpha \cdot \beta$ ) ( $\alpha \cdot (\beta' \ \beta^{\wedge})$ )) with ( $X \ Z$ ).

rewrite (@*cap\_comm* \_ \_ ( $\alpha \cdot \beta'$ )) *-cap\_assoc*.

by [rewrite *cap\_complement\_empty cap\_comm cap\_empty*].

apply *inc\_antisym*.

apply *inc\_empty\_alpha*.

apply (@*inc\_trans* \_ \_ \_ ((( $\alpha \cdot \beta$ ) (( $\alpha \cdot \beta'$ ) ( $\alpha \cdot \beta^{\wedge}$ ))))).

apply *cap\_inc\_compat\_l*.



```

apply comp_cap_distr_l.
replace (X Z) with ((alpha • beta) (alpha • beta ^)).
apply cap_inc_compat_l.
apply cap_r.
apply inc_antisym.
move : (@univalent_residual _ _ _ beta H) ⇒ H1.
rewrite -inc_rpc.
rewrite residual_to_complement in H1.
apply H1.
apply inc_empty_alpha.
apply inc_def2 in H0.
rewrite -comp_cup_distr_l cup_cap_distr_l.
rewrite -H0 complement_classic cap_universal.
rewrite cup_cap_distr_l -comp_cup_distr_l.
by [rewrite -H0 complement_classic cap_universal].
Qed.

```

**Lemma 272 (schroder\_univalent2)** *Let  $\alpha$  be a univalent relation. Then,*

$$\alpha \cdot \beta^- = \alpha \cdot \nabla_{YZ} \sqcap (\alpha \cdot \beta)^-.$$

**Lemma** *schroder\_univalent2* {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} :  
*univalent\_r alpha* → *alpha • beta* ^ = (*alpha • Y Z*) (alpha • beta) ^.

**Proof.**

```

move ⇒ H.
move : (@schroder_univalent1 _ _ _ alpha beta (Y Z) H (@inc_alpha_universal _ _))
⇒ H0.
rewrite cap_comm cap_universal in H0.
apply H0.
Qed.

```

**Lemma 273 (schroder\_univalent3)** *Let  $\alpha$  be a univalent relation. Then,*

$$(\alpha \cdot \beta)^- = (\alpha \cdot \nabla_{YZ})^- \sqcup \alpha \cdot \beta^-.$$

**Lemma** *schroder\_univalent3* {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} :  
*univalent\_r alpha* → (*alpha • beta*) ^ = (*alpha • Y Z*) ^ (alpha • beta ^).

**Proof.**

```

move ⇒ H.
rewrite (schroder_univalent2 H).
rewrite cup_cap_distr_l cup_comm complement_classic cap_comm cap_universal.
apply inc_def2.
apply rpc_inc_compat_r.

```

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apply *comp\_inc\_compat\_ab\_ab'*.  
 apply *inc\_alpha\_universal*.  
 Qed.

**Lemma 274 (schroder\_univalent4)** *Let  $\alpha$  be a univalent relation. Then,*

$$\alpha \triangleright \beta = (\alpha \cdot \nabla_{YZ})^- \sqcup \alpha \cdot \beta.$$

**Lemma *schroder\_univalent4***  $\{X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\} \{beta : Rel\ Y\ Z\}$ :  
*univalent\_r alpha  $\rightarrow$  alpha    beta = (alpha    Y Z) ^    (alpha    beta).*

**Proof.**

move  $\Rightarrow$  *H*.  
 rewrite (*residual\_property9 H*).  
 apply *Logic.eq\_sym*.  
 apply *cup\_to\_rpc*.  
 Qed.

**Lemma 275 (schroder\_universal)** *Let  $\nabla_{XZ} \cdot \nabla_{ZW} = \nabla_{XW}$ . Then,*

$$(\alpha \cdot \nabla_{YZ})^- \cdot \nabla_{ZW} = (\alpha \cdot \nabla_{YW})^-.$$

**Lemma *schroder\_universal***  $\{W\ X\ Y\ Z : eqType\} \{alpha : Rel\ X\ Y\}$ :  
 (    X Z    Z W ) =    X W  $\rightarrow$   
 (alpha    Y Z) ^    Z W = (alpha    Y W) ^.

**Proof.**

move  $\Rightarrow$  *H*.  
 apply (@*cap\_cup\_unique* \_ \_ (alpha    Y W)).  
 rewrite *cap\_complement\_empty\_cap\_comm*.  
 apply *inc\_antisym*.  
 apply (@*inc\_trans* \_ \_ \_ \_ (dedekind2)).  
 apply (@*inc\_trans* \_ \_ \_ (((alpha    Y Z) ^    (alpha    Y Z))    Z W)).  
 apply *comp\_inc\_compat\_ab\_a'b*.  
 apply *cap\_inc\_compat\_l*.  
 rewrite *comp\_assoc*.  
 apply *comp\_inc\_compat\_ab\_ab'*.  
 apply *inc\_alpha\_universal*.  
 rewrite *cap\_comm\_cap\_complement\_empty\_comp\_empty\_l*.  
 apply *inc\_refl*.  
 apply *inc\_empty\_alpha*.  
 rewrite *complement\_classic*.  
 apply *inc\_antisym*.  
 apply *inc\_alpha\_universal*.  
 rewrite -*H* -(@*complement\_classic* \_ \_ (alpha    Y Z)) *comp\_cup\_distr\_r*.

apply *cup\_inc\_compat\_r*.  
 rewrite *comp\_assoc*.  
 apply *comp\_inc\_compat\_ab\_ab'*.  
 apply *inc\_alpha\_universal*.  
 Qed.

**Lemma 276 (residual\_inv)**

$$(\alpha \triangleright \beta)^{\#} = \beta^{-\#} \triangleright \alpha^{-\#}.$$

**Lemma** *residual\_inv* {*X Y Z* : *eqType*} {*alpha* : *Rel X Y*} {*beta* : *Rel Y Z*}:  
 (*alpha* *beta*)  $\#$  = (*beta*  $\wedge$ )  $\#$  (*alpha*  $\wedge$ )  $\#$ .

**Proof.**

rewrite *residual\_to\_complement* *residual\_to\_complement*.  
 by [rewrite -*inv\_complement* *complement\_invol* *inv\_complement* *comp\_inv*].  
 Qed.

**Lemma 277 (residual\_cupP\_distr\_l, residual\_cup\_distr\_l)** *Let  $\alpha$  be a univalent relation,  $f : (V \rightarrow W) \rightarrow (Y \rightarrow Z)$  and  $\exists \beta, P(\beta)$ . Then,*

$$\alpha \triangleright (\sqcup_{P(\beta)} f(\beta)) = \sqcup_{P(\beta)} (\alpha \triangleright f(\beta)).$$

**Lemma** *residual\_cupP\_distr\_l* {*V W X Y Z* : *eqType*}  
 {*alpha* : *Rel X Y*} {*f* : *Rel V W*  $\rightarrow$  *Rel Y Z*} {*P* : *Rel V W*  $\rightarrow$  *Prop*}:  
*univalent\_r* *alpha*  $\rightarrow$  ( $\exists$  *beta'* : *Rel V W*, *P* *beta'*)  $\rightarrow$   
*alpha* (  $\_$  {*P*} *f*) =  $\_$  {*P*} (*fun* *beta* : *Rel V W*  $\Rightarrow$  *alpha* *f* *beta*).

**Proof.**

move  $\Rightarrow$  *H*.  
 elim  $\Rightarrow$  *beta'* *H0*.  
 rewrite (*schroder\_univalent4* *H*) *comp\_cupP\_distr\_l*.  
 replace (  $\_$  {*P*} (*fun* *beta* : *Rel V W*  $\Rightarrow$  *alpha* *f* *beta*)) with (  $\_$  {*P*} (*fun* *beta* :  
*Rel V W*  $\Rightarrow$  (*alpha*  $\cdot$  *Y Z*)  $\wedge$  (*alpha*  $\cdot$  *f* *beta*))).  
 apply (@*cap\_cup\_unique*  $\_$  (*alpha*  $\cdot$  *Y Z*)).  
 rewrite *cap\_cup\_distr\_l* *cap\_cupP\_distr\_l* *cap\_complement\_empty* *cup\_comm* *cup\_empty*.  
 rewrite *cap\_cupP\_distr\_l*.  
 apply *cupP\_eq*.  
 move  $\Rightarrow$  *gamma* *H1*.  
 by [rewrite *cap\_cup\_distr\_l* *cap\_complement\_empty* *cup\_comm* *cup\_empty*].  
 rewrite -*cup\_assoc* *complement\_classic* *cup\_comm* *cup\_universal*.  
 rewrite -(@*complement\_invol*  $\_$  (*alpha*  $\cdot$  *Y Z*)).  
 apply *bool\_lemma1*.  
 rewrite *complement\_invol*.  
 apply (@*inc\_trans*  $\_$   $\_$  ((*alpha*  $\cdot$  *Y Z*)  $\wedge$  (*alpha*  $\cdot$  *f* *beta'*))).

```

apply cup_l.
move : beta' H0.
apply inc_cupP.
apply inc_refl.
apply cupP_eq.
move => gamma H1.
by [rewrite (schroder_univalent4 H)].
Qed.

```

**Lemma residual\_cup\_distr\_l**  
 $\{X \ Y \ Z : \text{eqType}\} \{alpha : \text{Rel } X \ Y\} \{beta \ beta' : \text{Rel } Y \ Z\} :$   
 $\text{univalent}_r \ alpha \rightarrow$   
 $alpha \quad (beta \quad beta') = (alpha \quad beta) \quad (alpha \quad beta').$

**Proof.**  
 move => H.  
 rewrite cup\_to\_cupP (@cup\_to\_cupP \_ \_ \_ \_ \_ id).  
 apply (residual\_cupP\_distr\_l H).  
 ∃ beta.  
 by [left].  
 Qed.

**Lemma 278 (residual\_capP\_distr\_r, residual\_cap\_distr\_r)** *Let  $f : (Y \rightarrow Z) \rightarrow (I \rightarrow X)$  and  $\exists \alpha, P(\alpha)$ . Then,*

$$(\sqcap_{P(\alpha)} f(\alpha)^\#) \triangleright \rho = \sqcup_{P(\alpha)} (f(\alpha)^\# \triangleright \rho).$$

**Lemma residual\_capP\_distr\_r**  
 $\{X \ Y \ Z : \text{eqType}\} \{rho : \text{Rel } i \ X\} \{f : \text{Rel } Y \ Z \rightarrow \text{Rel } i \ X\} \{P : \text{Rel } Y \ Z \rightarrow \text{Prop}\} :$   
 $(\exists \ alpha' : \text{Rel } Y \ Z, P \ alpha') \rightarrow$   
 $(\_ \{P\} (\text{fun } alpha : \text{Rel } Y \ Z \Rightarrow f \ alpha \ \#)) \quad rho = \_ \{P\} (\text{fun } alpha : \text{Rel } Y \ Z \Rightarrow$   
 $f \ alpha \ \# \quad rho).$

**Proof.**  
 elim => alpha' H.  
 rewrite residual\_to\_complement.  
 rewrite -(@complement\_invol \_ \_ ( \\_ \{P\} (\text{fun } alpha : \text{Rel } Y \ Z \Rightarrow f \ alpha \ \# \quad rho))).  
 apply f\_equal.  
 rewrite de\_morgan3.  
 replace (fun alpha : \text{Rel } Y \ Z \Rightarrow (f \ alpha \ \# \quad rho) ^) with (fun alpha : \text{Rel } Y \ Z \Rightarrow f \ alpha \ \# \cdot rho ^).  
 apply inc\_antisym.  
 apply comp\_capP\_distr\_r.  
 apply (@inc\_trans \_ \_ \_ \_ (relation\_rel\_inv\_rel)).  
 apply (@inc\_trans \_ \_ \_ ((( \\_ \{P\} (\text{fun } alpha : \text{Rel } Y \ Z \Rightarrow f \ alpha \ \# \cdot rho ^)) \cdot (f

```

 $\alpha'$  #  $\cdot$   $\rho$   $^$ ) #)  $\cdot$  ( $f$   $\alpha'$  #  $\cdot$   $\rho$   $^$ ))).
apply comp_inc_compat.
apply comp_inc_compat_ab_ab'.
move :  $\alpha'$   $H$ .
apply inc_capP.
rewrite inv_capP_distr.
apply inc_refl.
move :  $\alpha'$   $H$ .
apply inc_capP.
apply inc_refl.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc.
apply (@inc_trans _ _ _ _ (comp_capP_distr_r)).
apply inc_capP.
move  $\Rightarrow$   $\beta$   $H0$ .
apply (@inc_trans _ _ _ (( $f$   $\beta$  #  $\cdot$   $\rho$   $^$ )  $\cdot$  (( $f$   $\alpha'$  #  $\cdot$   $\rho$   $^$ ) #  $\cdot$   $f$   $\alpha'$  #))).
move :  $\beta$   $H0$ .
apply inc_capP.
apply inc_refl.
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
apply functional_extensionality.
move  $\Rightarrow$   $x$ .
by [rewrite residual_to_complement complement_invol].
Qed.

```

# Chapter 13

## Library **Sum\_Product**

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Conjugate.
Require Import Domain.
Require Import Logic.IndefiniteDescription.
```

### 13.1 関係の直和

#### 13.1.1 入射対, 関係直和の定義

入射対の存在公理 (Axiom 23) で入射対が存在することまでは仮定済みなので, 実際に入射対  $j : A \rightarrow A + B, k : B \rightarrow A + B$  を定義する関数を定義する.

```
Definition sum_r (A B : eqType):
  {x : (Rel A (sum_eqType A B)) × (Rel B (sum_eqType A B)) |
    (fst x) • (fst x) # = Id A ∧ (snd x) • (snd x) # = Id B ∧
    (fst x) • (snd x) # = A B ∧
    ((fst x) # • (fst x)) ((snd x) # • (snd x)) = Id (sum_eqType A B)}.
apply constructive_indefinite_description.
elim (@pair_of_inclusions A B) ⇒ j.
elim ⇒ k H.
∃ (j,k).
simpl.
apply H.
Defined.
Definition inl_r (A B : eqType):= fst (sval (sum_r A B)).
```

## CHAPTER 13. LIBRARY SUM\_PRODUCT

**Definition**  $\text{inr\_r} (A B : \text{eqType}) := \text{snd} (\text{sval} (\text{sum\_r} A B))$ .

またこの定義による入射対が、入射対としての性質 (Axiom 23)  $+\alpha$  を満たしていることも事前に証明しておく。

**Lemma**  $\text{inl\_id} \{A B : \text{eqType}\} : \text{inl\_r} A B \cdot \text{inl\_r} A B \# = \text{Id } A$ .

**Proof.**

`apply (proj2_sig (sum_r A B)).`

**Qed.**

**Lemma**  $\text{inr\_id} \{A B : \text{eqType}\} : \text{inr\_r} A B \cdot \text{inr\_r} A B \# = \text{Id } B$ .

**Proof.**

`apply (proj2_sig (sum_r A B)).`

**Qed.**

**Lemma**  $\text{inl\_inr\_empty} \{A B : \text{eqType}\} : \text{inl\_r} A B \cdot \text{inr\_r} A B \# = A B$ .

**Proof.**

`apply (proj2_sig (sum_r A B)).`

**Qed.**

**Lemma**  $\text{inr\_inl\_empty} \{A B : \text{eqType}\} : \text{inr\_r} A B \cdot \text{inl\_r} A B \# = B A$ .

**Proof.**

`apply inv_invol2.`

`rewrite comp_inv inv_invol inv_empty.`

`apply inl_inr_empty.`

**Qed.**

**Lemma**  $\text{inl\_inr\_cup\_id} \{A B : \text{eqType}\} :$

$(\text{inl\_r} A B \# \cdot \text{inl\_r} A B) (\text{inr\_r} A B \# \cdot \text{inr\_r} A B) = \text{Id} (\text{sum\_eqType } A B)$ .

**Proof.**

`apply (proj2_sig (sum_r A B)).`

**Qed.**

**Lemma**  $\text{inl\_function} \{A B : \text{eqType}\} : \text{function\_r} (\text{inl\_r} A B)$ .

**Proof.**

`move : (proj2_sig (sum_r A B)).`

`elim  $\Rightarrow H$ .`

`elim  $\Rightarrow H0$ .`

`elim  $\Rightarrow H1 H2$ .`

`split.`

`rewrite /total_r.`

`rewrite H.`

`apply inc_refl.`

`rewrite /univalent_r.`

`rewrite -H2.`

`apply cup_l.`

**Qed.**

## CHAPTER 13. LIBRARY SUM\_PRODUCT

**Lemma** *inr\_function*  $\{A\ B : eqType\} : function\_r\ (inr\_r\ A\ B).$

**Proof.**

```
move : (proj2_sig (sum_r A B)).
elim  $\Rightarrow$  H.
elim  $\Rightarrow$  H0.
elim  $\Rightarrow$  H1 H2.
split.
rewrite /total_r.
rewrite H0.
apply inc_refl.
rewrite /univalent_r.
rewrite -H2.
apply cup_r.
Qed.
```

さらに  $\alpha : A \rightarrow C$  と  $\beta : B \rightarrow C$  の関係直和  $\alpha \perp \beta : A + B \rightarrow C$  を,  $\alpha \perp \beta := j^\# \cdot \alpha \sqcup k^\# \cdot \beta$  で定義する.

**Definition** *Rel\_sum*  $\{A\ B\ C : eqType\} (alpha : Rel\ A\ C) (\mathbf{beta} : Rel\ B\ C) :=$   
 $(inl\_r\ A\ B \# \cdot alpha) \quad (inr\_r\ A\ B \# \cdot \mathbf{beta}).$

### 13.1.2 関係直和の性質

**Lemma 279 (sum\_inc\_compat)** *Let  $\alpha, \alpha' : A \rightarrow C$  and  $\beta, \beta' : B \rightarrow C$ . Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \perp \beta \sqsubseteq \alpha' \perp \beta'.$$

**Lemma** *sum\_inc\_compat*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ C\} \{\mathbf{beta}\ beta' : Rel\ B\ C\} :$   
 $alpha \quad alpha' \rightarrow \mathbf{beta} \quad beta' \rightarrow Rel\_sum\ alpha\ \mathbf{beta} \quad Rel\_sum\ alpha'\ beta'.$

**Proof.**

```
move  $\Rightarrow$  H H0.
apply cup_inc_compat.
apply (comp_inc_compat_ab_ab' H).
apply (comp_inc_compat_ab_ab' H0).
Qed.
```

**Lemma 280 (sum\_inc\_compat\_l)** *Let  $\alpha : A \rightarrow C$  and  $\beta, \beta' : B \rightarrow C$ . Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \perp \beta \sqsubseteq \alpha \perp \beta'.$$

**Lemma** *sum\_inc\_compat\_l*



## CHAPTER 13. LIBRARY SUM\_PRODUCT

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta\ beta' : Rel\ B\ C\}:$   
 $beta\ beta' \rightarrow Rel\_sum\ alpha\ beta\ Rel\_sum\ alpha\ beta'.$

**Proof.**

move  $\Rightarrow H$ .

apply (sum\_inc\_compat (@inc\_refl \_ \_ alpha) H).

**Qed.**

**Lemma 281 (sum\_inc\_compat\_r)** *Let  $\alpha, \alpha' : A \rightarrow C$  and  $\beta : B \rightarrow C$ . Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \perp \beta \sqsubseteq \alpha' \perp \beta.$$

**Lemma** sum\_inc\_compat\_r

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ C\} \{beta : Rel\ B\ C\}:$   
 $alpha\ alpha' \rightarrow Rel\_sum\ alpha\ beta\ Rel\_sum\ alpha'\ beta.$

**Proof.**

move  $\Rightarrow H$ .

apply (sum\_inc\_compat H (@inc\_refl \_ \_ beta)).

**Qed.**

**Lemma 282 (total\_sum)** *Let  $\alpha : A \rightarrow C$  and  $\beta : B \rightarrow C$  are total relations, then  $\alpha \perp \beta$  is also a total relation.*

**Lemma** total\_sum  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta : Rel\ B\ C\}:$   
 $total\_r\ alpha \rightarrow total\_r\ beta \rightarrow total\_r\ (Rel\_sum\ alpha\ beta).$

**Proof.**

move  $\Rightarrow H\ H0$ .

rewrite /total\_r/Rel\_sum.

rewrite -inl\_inr\_cup\_id inv\_cup\_distr comp\_cup\_distr\_l comp\_cup\_distr\_r comp\_cup\_distr\_r.

rewrite comp\_inv comp\_inv inv\_invol inv\_invol.

apply cup\_inc\_compat.

apply (fun H'  $\Rightarrow$  @inc\_trans \_ \_ \_ \_ H' (cup\_l)).

rewrite comp\_assoc -(@comp\_assoc \_ \_ \_ \_ alpha).

apply comp\_inc\_compat\_ab\_ab'.

apply (comp\_inc\_compat\_b\_ab H).

apply (fun H'  $\Rightarrow$  @inc\_trans \_ \_ \_ \_ H' (cup\_r)).

rewrite comp\_assoc -(@comp\_assoc \_ \_ \_ \_ beta).

apply comp\_inc\_compat\_ab\_ab'.

apply (comp\_inc\_compat\_b\_ab H0).

**Qed.**

**Lemma 283 (univalent\_sum)** *Let  $\alpha : A \rightarrow C$  and  $\beta : B \rightarrow C$  are univalent relations, then  $\alpha \perp \beta$  is also a univalent relation.*

## CHAPTER 13. LIBRARY SUM\_PRODUCT

**Lemma** *univalent\_sum*  $\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel}\ A\ C\} \{beta : \text{Rel}\ B\ C\}$ :  
 $\text{univalent\_r}\ alpha \rightarrow \text{univalent\_r}\ beta \rightarrow \text{univalent\_r}\ (\text{Rel\_sum}\ alpha\ beta).$

**Proof.**

`move  $\Rightarrow$  H H0.`

`rewrite /univalent_r/ Rel_sum.`

`rewrite inv_cup_distr comp_cup_distr_l comp_cup_distr_r comp_cup_distr_r.`

`rewrite comp_inv comp_inv inv_invol inv_invol.`

`rewrite comp_assoc -(@comp_assoc _ _ _ (inl_r A B)) inl_id comp_id_l.`

`rewrite comp_assoc -(@comp_assoc _ _ _ (inr_r A B)) inr_inl_empty comp_empty_l  
 comp_empty_r cup_empty.`

`rewrite -cup_assoc comp_assoc -(@comp_assoc _ _ _ (inl_r A B)) inl_inr_empty comp_empty_l  
 comp_empty_r cup_empty.`

`rewrite comp_assoc -(@comp_assoc _ _ _ (inr_r A B)) inr_id comp_id_l.`

`apply inc_cup.`

`split.`

`apply H.`

`apply H0.`

**Qed.**

**Lemma 284 (function\_sum)** *Let  $\alpha : A \rightarrow C$  and  $\beta : B \rightarrow C$  be functions, then  $\alpha \perp \beta$  is also a function.*

**Lemma** *function\_sum*  $\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel}\ A\ C\} \{beta : \text{Rel}\ B\ C\}$ :  
 $\text{function\_r}\ alpha \rightarrow \text{function\_r}\ beta \rightarrow \text{function\_r}\ (\text{Rel\_sum}\ alpha\ beta).$

**Proof.**

`elim  $\Rightarrow$  H H0.`

`elim  $\Rightarrow$  H1 H2.`

`split.`

`apply (total_sum H H1).`

`apply (univalent_sum H0 H2).`

**Qed.**

**Lemma 285 (sum\_conjugate)** *Let  $\alpha : A \rightarrow C$ ,  $\beta : B \rightarrow C$  and  $\gamma : A + B \rightarrow C$  be relations,  $j : A \rightarrow A + B$  and  $k : B \rightarrow A + B$  be inclusions. Then,*

$$j \cdot \gamma = \alpha \wedge k \cdot \gamma = \beta \Leftrightarrow \gamma = \alpha \perp \beta.$$

**Lemma** *sum\_conjugate*

$\{A\ B\ C : \text{eqType}\} \{alpha : \text{Rel}\ A\ C\} \{beta : \text{Rel}\ B\ C\} \{gamma : \text{Rel}\ (\text{sum\_eqType}\ A\ B)\ C\}$ :

$\text{inl\_r}\ A\ B \cdot \text{gamma} = \alpha \wedge \text{inr\_r}\ A\ B \cdot \text{gamma} = \text{beta} \Leftrightarrow$   
 $\text{gamma} = \text{Rel\_sum}\ alpha\ \text{beta}.$

**Proof.**

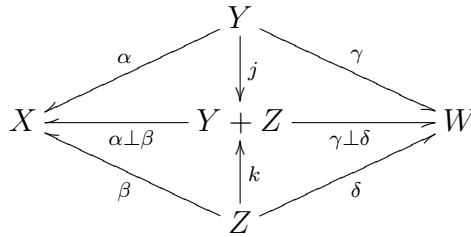
```

split; move => H.
elim H => H0 H1.
rewrite -(@comp_id_l _ _ gamma).
rewrite -inl_inr_cup_id comp_cup_distr_r comp_assoc comp_assoc.
by [rewrite H0 H1].
split.
rewrite H comp_cup_distr_l -comp_assoc -comp_assoc.
rewrite inl_id inl_inr_empty comp_id_l comp_empty_l.
by [rewrite cup_empty].
rewrite H comp_cup_distr_l -comp_assoc -comp_assoc.
rewrite inr_id inr_inl_empty comp_id_l comp_empty_l.
by [rewrite cup_comm cup_empty].
Qed.

```

**Lemma 286 (sum\_comp)** *In below figure,*

$$(\alpha \perp \beta)^\# \cdot (\gamma \perp \delta) = \alpha^\# \cdot \gamma \sqcup \beta^\# \cdot \delta.$$



**Lemma** *sum\_comp* {W X Y Z : eqType}  
 {alpha : Rel Y X} {beta : Rel Z X} {gamma : Rel Y W} {delta : Rel Z W}:  
 (Rel\_sum alpha beta) # • Rel\_sum gamma delta =  
 (alpha # • gamma) (beta # • delta).

**Proof.**

```

rewrite /Rel_sum.
rewrite inv_cup_distr comp_cup_distr_l comp_cup_distr_r comp_cup_distr_r.
rewrite comp_inv comp_inv inv_invol inv_invol.
apply f_equal2.
rewrite comp_assoc -(@comp_assoc _ _ _ (inl_r Y Z)) inl_id comp_id_l.
by [rewrite comp_assoc -(@comp_assoc _ _ _ (inr_r Y Z)) inr_inl_empty comp_empty_l
    comp_empty_r cup_empty].
rewrite comp_assoc -(@comp_assoc _ _ _ (inl_r Y Z)) inl_inr_empty comp_empty_l
    comp_empty_r cup_comm cup_empty.
by [rewrite comp_assoc -(@comp_assoc _ _ _ (inr_r Y Z)) inr_id comp_id_l].
Qed.

```

## 13.1.3 分配法則

**Lemma 287 (sum\_cap\_distr\_l)** *Let  $\alpha : A \rightarrow C$  and  $\beta, \beta' : B \rightarrow C$ . Then,*

$$\alpha \perp (\beta \sqcap \beta') \sqsubseteq (\alpha \perp \beta) \sqcap (\alpha \perp \beta').$$

**Lemma** *sum\_cap\_distr\_l*

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta\ beta' : Rel\ B\ C\}:$   
 $Rel\_sum\ alpha\ (beta\ \beta')\ (Rel\_sum\ alpha\ beta\ Rel\_sum\ alpha\ beta').$

**Proof.**

rewrite -cup\_cap\_distr\_l.

apply cup\_inc\_compat\_l.

apply comp\_cap\_distr\_l.

**Qed.**

**Lemma 288 (sum\_cap\_distr\_r)** *Let  $\alpha, \alpha' : A \rightarrow C$  and  $\beta : B \rightarrow C$ . Then,*

$$(\alpha \sqcap \alpha') \perp \beta \sqsubseteq (\alpha \perp \beta) \sqcap (\alpha' \perp \beta).$$

**Lemma** *sum\_cap\_distr\_r*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ C\} \{beta : Rel\ B\ C\}:$   
 $Rel\_sum\ (alpha\ alpha')\ beta\ (Rel\_sum\ alpha\ beta\ Rel\_sum\ alpha'\ beta').$

**Proof.**

rewrite -cup\_cap\_distr\_r.

apply cup\_inc\_compat\_r.

apply comp\_cap\_distr\_l.

**Qed.**

**Lemma 289 (sum\_cup\_distr\_l)** *Let  $\alpha : A \rightarrow C$  and  $\beta, \beta' : B \rightarrow C$ . Then,*

$$\alpha \perp (\beta \sqcup \beta') = (\alpha \perp \beta) \sqcup (\alpha \perp \beta').$$

**Lemma** *sum\_cup\_distr\_l*

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ C\} \{beta\ beta' : Rel\ B\ C\}:$   
 $Rel\_sum\ alpha\ (beta\ \beta') = Rel\_sum\ alpha\ beta\ Rel\_sum\ alpha\ beta'.$

**Proof.**

rewrite -cup\_assoc (@cup\_comm \_ \_ (Rel\_sum alpha beta)) -cup\_assoc.

by [rewrite cup\_idem cup\_assoc -comp\_cup\_distr\_l].

**Qed.**

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**Lemma 290 (sum\_cup\_distr\_r)** *Let  $\alpha, \alpha' : A \rightarrow C$  and  $\beta : B \rightarrow C$ . Then,*

$$(\alpha \sqcup \alpha') \perp \beta = (\alpha \perp \beta) \sqcup (\alpha' \perp \beta).$$

**Lemma** *sum\_cup\_distr\_r*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ C\} \{beta : Rel\ B\ C\} :$   
 $Rel\_sum\ (alpha\ \ alpha')\ beta = (Rel\_sum\ alpha\ beta\ \ Rel\_sum\ alpha'\ beta).$

**Proof.**

`rewrite cup_assoc (@cup_comm _ _ (inr_r A B # • beta)) cup_assoc.`

`by [rewrite cup_idem -cup_assoc -comp_cup_distr_l].`

**Qed.**

**Lemma 291 (comp\_sum\_distr\_r)** *Let  $\alpha : A \rightarrow C$ ,  $\beta : B \rightarrow C$  and  $\gamma : C \rightarrow D$ . Then,*

$$(\alpha \perp \beta) \cdot \gamma = \alpha \cdot \gamma \perp \beta \cdot \gamma.$$

**Lemma** *comp\_sum\_distr\_r*

$\{A\ B\ C\ D : eqType\} \{alpha : Rel\ A\ C\} \{beta : Rel\ B\ C\} \{gamma : Rel\ C\ D\} :$   
 $(Rel\_sum\ alpha\ beta) \cdot gamma = Rel\_sum\ (alpha \cdot gamma)\ (beta \cdot gamma).$

**Proof.**

`by [rewrite comp_cup_distr_r comp_assoc comp_assoc].`

**Qed.**

## 13.2 関係の直積

### 13.2.1 射影対, 関係直積の定義

射影対の存在公理 (Axiom 24) で射影対が存在することまでは仮定済みなので, 実際に射影対  $p : A \times B \rightarrow A, k : A \times B \rightarrow B$  を定義する関数を定義する.

**Definition** *prod\_r* ( $A\ B : eqType$ ):

$\{x : (Rel\ (prod\_eqType\ A\ B)\ A) \times (Rel\ (prod\_eqType\ A\ B)\ B) \mid$   
 $(fst\ x) \# \cdot (snd\ x) = A\ B \wedge$   
 $((fst\ x) \cdot (fst\ x) \#) \cdot ((snd\ x) \cdot (snd\ x) \#) = Id\ (prod\_eqType\ A\ B) \wedge$   
 $univalent\_r\ (fst\ x) \wedge univalent\_r\ (snd\ x)\}.$

`apply constructive_indefinite_description.`

`elim (@pair_of_projections A B) => p.`

`elim => q H.`

`∃ (p,q).`

`simpl.`

`apply H.`

**Defined.**

**Definition**  $\text{fst}_r (A B : \text{eqType}) := \text{fst} (\text{sva} (\text{prod}_r A B))$ .

**Definition**  $\text{snd}_r (A B : \text{eqType}) := \text{snd} (\text{sva} (\text{prod}_r A B))$ .

またこの定義による射影対が, 射影対としての性質 (Axiom 24)  $+\alpha$  を満たしていることも事前に証明しておく.

**Lemma**  $\text{fst\_snd\_universal} \{A B : \text{eqType}\} : \text{fst}_r A B \# \cdot \text{snd}_r A B = A B$ .

**Proof.**

$\text{apply} (\text{proj2\_sig} (\text{prod}_r A B))$ .

**Qed.**

**Lemma**  $\text{snd\_fst\_universal} \{A B : \text{eqType}\} : \text{snd}_r A B \# \cdot \text{fst}_r A B = B A$ .

**Proof.**

$\text{apply} \text{inv\_invol2}$ .

$\text{rewrite} \text{comp\_inv inv\_invol inv\_universal}$ .

$\text{apply} \text{fst\_snd\_universal}$ .

**Qed.**

**Lemma**  $\text{fst\_snd\_cap\_id} \{A B : \text{eqType}\} :$

$(\text{fst}_r A B \cdot \text{fst}_r A B \#) (\text{snd}_r A B \cdot \text{snd}_r A B \#) = \text{Id} (\text{prod\_eqType} A B)$ .

**Proof.**

$\text{apply} (\text{proj2\_sig} (\text{prod}_r A B))$ .

**Qed.**

**Lemma**  $\text{fst\_function} \{A B : \text{eqType}\} : \text{function}_r (\text{fst}_r A B)$ .

**Proof.**

$\text{move} : (\text{proj2\_sig} (\text{prod}_r A B))$ .

$\text{elim} \Rightarrow H$ .

$\text{elim} \Rightarrow H0 H1$ .

$\text{split}$ .

$\text{rewrite} / \text{total}_r$ .

$\text{rewrite} -H0$ .

$\text{apply} \text{cap}_l$ .

$\text{apply} H1$ .

**Qed.**

**Lemma**  $\text{snd\_function} \{A B : \text{eqType}\} : \text{function}_r (\text{snd}_r A B)$ .

**Proof.**

$\text{move} : (\text{proj2\_sig} (\text{prod}_r A B))$ .

$\text{elim} \Rightarrow H$ .

$\text{elim} \Rightarrow H0 H1$ .

$\text{split}$ .

$\text{rewrite} / \text{total}_r$ .

$\text{rewrite} -H0$ .

$\text{apply} \text{cap}_r$ .

apply *H1*.

**Qed.**

さらに  $\alpha : A \rightarrow B$  と  $\beta : A \rightarrow C$  の関係直積  $\alpha \top \beta : A \rightarrow B \times C$  を,  $\alpha \top \beta := \alpha \cdot p^\# \sqcap \beta \cdot q^\#$  で定義する.

**Definition** *Rel\_prod*  $\{A\ B\ C : eqType\}$  (*alpha* : *Rel A B*) (**beta** : *Rel A C*):=  
 (*alpha* · *fst\_r B C* #) (b**eta** · *snd\_r B C* #).

### 13.2.2 関係直積の性質

**Lemma 292 (prod\_inc\_compat)** *Let*  $\alpha, \alpha' : A \rightarrow B$  *and*  $\beta, \beta' : A \rightarrow C$ . *Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \top \beta \sqsubseteq \alpha' \top \beta'.$$

**Lemma** *prod\_inc\_compat*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{b\text{eta}\ beta' : Rel\ A\ C\}:$   
 $alpha\ alpha' \rightarrow b\text{eta}\ beta' \rightarrow Rel\_prod\ alpha\ b\text{eta}\ Rel\_prod\ alpha'\ beta'.$

**Proof.**

move  $\Rightarrow$  *H H0*.

apply *cap\_inc\_compat*.

apply (*comp\_inc\_compat\_ab\_a'b H*).

apply (*comp\_inc\_compat\_ab\_a'b H0*).

**Qed.**

**Lemma 293 (prod\_inc\_compat\_l)** *Let*  $\alpha : A \rightarrow B$  *and*  $\beta, \beta' : A \rightarrow C$ . *Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \top \beta \sqsubseteq \alpha \top \beta'.$$

**Lemma** *prod\_inc\_compat\_l*

$\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{b\text{eta}\ beta' : Rel\ A\ C\}:$   
 $b\text{eta}\ beta' \rightarrow Rel\_prod\ alpha\ b\text{eta}\ Rel\_prod\ alpha\ beta'.$

**Proof.**

move  $\Rightarrow$  *H*.

apply (*prod\_inc\_compat (@inc\_refl \_ alpha) H*).

**Qed.**

**Lemma 294 (prod\_inc\_compat\_r)** *Let*  $\alpha, \alpha' : A \rightarrow B$  *and*  $\beta : A \rightarrow C$ . *Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \top \beta \sqsubseteq \alpha' \top \beta.$$

**Lemma** *prod\_inc\_compat\_r*

$\{A\ B\ C : eqType\} \{alpha\ alpha' : Rel\ A\ B\} \{b\text{eta} : Rel\ A\ C\}:$

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*alpha*    *alpha'*  $\rightarrow$  *Rel\_prod* *alpha* **beta**    *Rel\_prod* *alpha'* **beta**.

**Proof.**

move  $\Rightarrow$  *H*.

apply (*prod\_inc\_compat* *H* (@*inc\_refl* \_ \_ **beta**)).

**Qed.**

**Lemma 295 (total\_prod)** *Let  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$  are total relations, then  $\alpha \top \beta$  is also a total relation.*

**Lemma** *total\_prod* {*A B C* : *eqType*} {*alpha* : *Rel A B*} {**beta** : *Rel A C*}:  
*total\_r* *alpha*  $\rightarrow$  *total\_r* **beta**  $\rightarrow$  *total\_r* (*Rel\_prod* *alpha* **beta**).

**Proof.**

move  $\Rightarrow$  *H H0*.

rewrite *domain\_total cap\_domain cap\_comm*.

apply *Logic.eq\_sym*.

apply *inc\_def1*.

apply (@*inc\_trans* \_ \_ \_ \_ *H*).

rewrite *comp\_inv inv\_invol comp\_assoc*.

apply *comp\_inc\_compat\_ab\_ab'*.

apply (@*inc\_trans* \_ \_ \_ (alpha # • (b beta #))).

apply (*comp\_inc\_compat\_a\_ab H0*).

rewrite -*comp\_assoc* -*comp\_assoc fst\_snd\_universal*.

apply *comp\_inc\_compat\_ab\_a'b*.

apply *inc\_alpha\_universal*.

**Qed.**

**Lemma 296 (univalent\_prod)** *Let  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$  are univalent relations, then  $\alpha \top \beta$  is also a univalent relation.*

**Lemma** *univalent\_prod* {*A B C* : *eqType*} {*alpha* : *Rel A B*} {**beta** : *Rel A C*}:  
*univalent\_r* *alpha*  $\rightarrow$  *univalent\_r* **beta**  $\rightarrow$  *univalent\_r* (*Rel\_prod* *alpha* **beta**).

**Proof.**

move  $\Rightarrow$  *H H0*.

rewrite /*univalent\_r*/ *Rel\_prod*.

rewrite *inv\_cap\_distr comp\_inv inv\_invol comp\_inv inv\_invol*.

apply (@*inc\_trans* \_ \_ \_ \_ (*comp\_cap\_distr\_l*)).

rewrite -*fst\_snd\_cap\_id*.

apply *cap\_inc\_compat*.

apply (@*inc\_trans* \_ \_ \_ \_ (*comp\_cap\_distr\_r*)).

apply (@*inc\_trans* \_ \_ \_ \_ (*cap\_l*)).

rewrite *comp\_assoc* -(@*comp\_assoc* \_ \_ \_ \_ *alpha*).

apply *comp\_inc\_compat\_ab\_ab'*.

apply (*comp\_inc\_compat\_ab\_b H*).



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```

apply (@inc_trans - - - - (comp_cap_distr_r)).
apply (@inc_trans - - - - (cap_r)).
rewrite comp_assoc - (@comp_assoc - - - - beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H0).
Qed.

```

**Lemma 297 (function\_prod)** *Let  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$  be functions, then  $\alpha \top \beta$  is also a function.*

**Lemma** *function\_prod* { $A B C : eqType$ } { $\alpha : Rel A B$ } { $\beta : Rel A C$ }:  
*function\_r*  $\alpha \rightarrow$  *function\_r*  $\beta \rightarrow$  *function\_r* (*Rel\_prod*  $\alpha \beta$ ).

**Proof.**

```

elim  $\Rightarrow$   $H H0$ .
elim  $\Rightarrow$   $H1 H2$ .
split.
apply (total_prod  $H H1$ ).
apply (univalent_prod  $H0 H2$ ).
Qed.

```

**Lemma 298 (prod\_fst\_surjection)** *Let  $p : B \times C \rightarrow B$  be a projection. Then,*

$$“p \text{ is a surjection}” \Leftrightarrow \forall D, \nabla_{BD} = \nabla_{BC} \cdot \nabla_{CD}.$$

**Lemma** *prod\_fst\_surjection* { $B C : eqType$ }:  
*surjection\_r* (*fst\_r*  $B C$ )  $\Leftrightarrow \forall D : eqType$ ,  $B D = B C \cdot C D$ .

**Proof.**

```

split; move  $\Rightarrow$   $H$ .
move  $\Rightarrow$   $D$ .
elim  $H \Rightarrow H0 H1$ .
apply inc_antisym.
apply (@inc_trans - - - ((fst_r  $B C \# \cdot$  (fst_r  $B C \#) \#) \cdot B D)).
apply (comp_inc_compat_b_ab H1).
rewrite inv_invol.
apply (@inc_trans - - - (((fst_r  $B C \# \cdot$  snd_r  $B C) \cdot$  (snd_r  $B C \# \cdot$  fst_r  $B C)) \cdot B D)).
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc - (@comp_assoc - - - (snd_r  $B C$ )).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply snd_function.
rewrite (@comp_assoc - - - - (  $B D$ )).
apply comp_inc_compat.$$ 
```

```

apply inc_alpha_universal.
apply inc_alpha_universal.
apply inc_alpha_universal.
split.
apply fst_function.
rewrite /total_r.
rewrite -(@cap_universal _ _ (Id B)) (H B) -(@fst_snd_universal B C) cap_comm comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite comp_id_r.
apply cap_r.
Qed.

```

**Lemma 299 (prod\_snd\_surjection)** *Let  $q : B \times C \rightarrow C$  be a projection. Then,*

$$“q \text{ is a surjection}” \Leftrightarrow \forall D, \nabla_{CD} = \nabla_{CB} \cdot \nabla_{BD}.$$

**Lemma** *prod\_snd\_surjection*  $\{B\ C : eqType\}$ :  
 $surjection\_r\ (snd\_r\ B\ C) \leftrightarrow \forall\ D : eqType, \quad C\ D = \quad C\ B \cdot \quad B\ D.$

**Proof.**

```

split; move => H.
move => D.
elim H => H0 H1.
apply inc_antisym.
apply (@inc_trans _ _ _ ((snd_r B C # · (snd_r B C #) #) · C D)).
apply (comp_inc_compat_b_ab H1).
rewrite inv_invol.
apply (@inc_trans _ _ _ (((snd_r B C # · fst_r B C) · (fst_r B C # · snd_r B C)) · C D)).
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -(@comp_assoc _ _ _ (fst_r B C)).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply fst_function.
rewrite (@comp_assoc _ _ _ _ _ (C D)).
apply comp_inc_compat.
apply inc_alpha_universal.
apply inc_alpha_universal.
apply inc_alpha_universal.
split.
apply snd_function.
rewrite /total_r.
rewrite -(@cap_universal _ _ (Id C)) (H C) -(@snd_fst_universal B C) cap_comm comp_assoc.

```

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```

apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite comp_id_r.
apply cap_r.
Qed.

```

**Lemma 300 (prod\_fst\_domain1)** *Let  $p : B \times C \rightarrow B$  be a projection,  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$ . Then,*

$$(\alpha \top \beta) \cdot p = \lfloor \beta \rfloor \cdot \alpha.$$

**Lemma prod\_fst\_domain1**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ A\ C\}$ :  
 $(Rel\_prod\ alpha\ beta) \cdot fst\_r\ B\ C = domain\ beta \cdot alpha$ .

**Proof.**

```

rewrite (@comp_inv_inv A A) domain_inv.
rewrite domain_universal2 inv_cap_distr comp_inv inv_invol inv_invol inv_universal.
rewrite -snd_fst_universal.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
rewrite comp_assoc comp_assoc.
apply cap_inc_compat_r.
apply comp_inc_compat_ab_a.
apply fst_function.
rewrite cap_comm -comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind2)).
rewrite cap_comm.
apply inc_refl.
Qed.

```

**Lemma 301 (prod\_fst\_domain2)** *Let  $p : B \times C \rightarrow B$  be a projection,  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$ . Then,*

$$(\alpha \top \beta) \cdot p = \alpha \Leftrightarrow \lfloor \alpha \rfloor \sqsubseteq \lfloor \beta \rfloor.$$

**Lemma prod\_fst\_domain2**  $\{A\ B\ C : eqType\} \{alpha : Rel\ A\ B\} \{beta : Rel\ A\ C\}$ :  
 $(Rel\_prod\ alpha\ beta) \cdot fst\_r\ B\ C = alpha \Leftrightarrow domain\ alpha \sqsubseteq domain\ beta$ .

**Proof.**

```

rewrite prod_fst_domain1.
split; move => H.
apply domain_lemma2b.
assert ((domain beta \cdot alpha) ((beta \cdot beta #) \cdot alpha)).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite H in H0.
apply H0.

```

```

apply inc_antisym.
apply comp_inc_compat_ab_b.
apply cap_r.
apply (@inc_trans _ _ _ (domain alpha · alpha)).
rewrite domain_comp_alpha1.
apply inc_refl.
apply (comp_inc_compat_ab_a'b H).
Qed.

```

**Lemma 302 (prod\_snd\_domain1)** *Let  $q : B \times C \rightarrow C$  be a projection,  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$ . Then,*

$$(\alpha \top \beta) \cdot q = \lfloor \alpha \rfloor \cdot \beta.$$

**Lemma prod\_snd\_domain1**  $\{A B C : eqType\} \{alpha : Rel A B\} \{beta : Rel A C\} :$   
 $(Rel\_prod\ alpha\ beta) \cdot snd\_r\ B\ C = domain\ alpha \cdot beta.$

**Proof.**

```

rewrite (@comp_inv_inv A A) domain_inv.
rewrite domain_universal2 inv_cap_distr comp_inv inv_invol inv_invol inv_universal.
rewrite fst_snd_universal.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
rewrite comp_assoc comp_assoc cap_comm.
apply cap_inc_compat_r.
apply comp_inc_compat_ab_a.
apply snd_function.
rewrite cap_comm -comp_assoc.
apply dedekind2.
Qed.

```

**Lemma 303 (prod\_snd\_domain2)** *Let  $q : B \times C \rightarrow C$  be a projection,  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$ . Then,*

$$(\alpha \top \beta) \cdot q = \beta \Leftrightarrow \lfloor \beta \rfloor \sqsubseteq \lfloor \alpha \rfloor.$$

**Lemma prod\_snd\_domain2**  $\{A B C : eqType\} \{alpha : Rel A B\} \{beta : Rel A C\} :$   
 $(Rel\_prod\ alpha\ beta) \cdot snd\_r\ B\ C = beta \leftrightarrow domain\ beta \sqsubseteq domain\ alpha.$

**Proof.**

```

rewrite prod_snd_domain1.
split; move => H.
apply domain_lemma2b.
assert ((domain alpha · beta) ((alpha · alpha #) · beta)).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite H in H0.

```

```

apply H0.
apply inc_antisym.
apply comp_inc_compat_ab_b.
apply cap_r.
apply (@inc_trans _ _ _ (domain beta • beta)).
rewrite domain_comp_alpha1.
apply inc_refl.
apply (comp_inc_compat_ab_a'b H).
Qed.

```

**Lemma 304 (prod\_to\_cap)** *Let  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$ . Then,*

$$\lfloor \alpha \top \beta \rfloor = \lfloor \alpha \rfloor \sqcap \lfloor \beta \rfloor.$$

**Lemma prod\_to\_cap**  $\{A B C : \text{eqType}\} \{\alpha : \text{Rel } A B\} \{\text{beta} : \text{Rel } A C\}$ :  
 $\text{domain } (\text{Rel\_prod } \alpha \text{ beta}) = \text{domain } \alpha \quad \text{domain } \text{beta}.$

**Proof.**

```

replace (domain (Rel_prod alpha beta)) with (domain (Rel_prod alpha beta • snd_r B C)).
rewrite prod_snd_domain1 comp_domain8.
apply dedekind_id3.
apply cap_r.
apply cap_r.
apply cap_r.
apply comp_domain3.
apply snd_function.
Qed.

```

**Lemma 305 (prod\_conjugate1)** *Let  $\alpha : A \rightarrow B$  and  $\beta : A \rightarrow C$  be functions,  $p : B \times C \rightarrow B$  and  $q : B \times C \rightarrow C$  be projections. Then,*

$$(\alpha \top \beta) \cdot p = \alpha \wedge (\alpha \top \beta) \cdot q = \beta.$$

**Lemma prod\_conjugate1**  $\{A B C : \text{eqType}\} \{\alpha : \text{Rel } A B\} \{\text{beta} : \text{Rel } A C\}$ :  
 $\text{function\_r } \alpha \rightarrow \text{function\_r } \text{beta} \rightarrow$   
 $\text{Rel\_prod } \alpha \text{ beta} \cdot \text{fst\_r } B C = \alpha \wedge \text{Rel\_prod } \alpha \text{ beta} \cdot \text{snd\_r } B C = \text{beta}.$

**Proof.**

```

move => H H0.
split.
rewrite prod_fst_domain1.
elim H0 => H1 H2.
apply inc_def1 in H1.
rewrite /domain.

```

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```
by [rewrite cap_comm -H1 comp_id_l].
rewrite prod_snd_domain1.
elim H ⇒ H1 H2.
apply inc_def1 in H1.
rewrite /domain.
by [rewrite cap_comm -H1 comp_id_l].
Qed.
```

**Lemma 306 (prod\_conjugate2)** *Let  $\gamma : A \rightarrow B \times C$  be a function,  $p : B \times C \rightarrow B$  and  $q : B \times C \rightarrow C$  be projections. Then,*

$$(\gamma \cdot p)^\top (\gamma \cdot q) = \gamma.$$

**Lemma prod\_conjugate2**  $\{A\ B\ C : \text{eqType}\} \{ \text{gamma} : \text{Rel } A\ (\text{prod\_eqType } B\ C) \}$ :  
 $\text{function\_r gamma} \rightarrow \text{Rel\_prod } (\text{gamma} \cdot \text{fst\_r } B\ C) (\text{gamma} \cdot \text{snd\_r } B\ C) = \text{gamma}.$

**Proof.**

```
move ⇒ H.
rewrite /Rel_prod.
rewrite comp_assoc comp_assoc -(function_cap_distr_l H).
by [rewrite fst_snd_cap_id comp_id_r].
Qed.
```

**Lemma 307 (diagonal\_conjugate)** *Let  $p : B \times C \rightarrow B$  and  $q : B \times C \rightarrow C$  be projections. Then,*

$$\frac{\alpha : A \rightarrow B \quad \alpha = p^\sharp \cdot u \cdot q}{u \sqsubseteq \text{id}_{A \times B} \quad u = [p \cdot \alpha \sqcap q]}.$$

**Lemma diagonal\_conjugate**  $\{A\ B : \text{eqType}\} \{ \text{alpha} : \text{Rel } A\ B \}$ :  
 $\text{conjugate } A\ B\ (\text{prod\_eqType } A\ B) (\text{prod\_eqType } A\ B)$   
 $\text{True\_r } (\text{fun } u \Rightarrow u \quad \text{Id } (\text{prod\_eqType } A\ B))$   
 $(\text{fun } u \Rightarrow (\text{fst\_r } A\ B \# \cdot u) \cdot \text{snd\_r } A\ B)$   
 $(\text{fun } \text{alpha} \Rightarrow \text{domain } ((\text{fst\_r } A\ B \cdot \text{alpha}) \quad \text{snd\_r } A\ B)).$

**Proof.**

```
split.
move ⇒ alpha0 H.
split.
apply cap_r.
rewrite cap_domain.
apply inc_antisym.
apply (@inc_trans _ _ ((fst_r A B # · ((fst_r A B · alpha0) · snd_r A B #)) · snd_r
A B)).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
```

```
apply cap_l.
rewrite comp_assoc comp_assoc -comp_assoc -(@comp_assoc _ _ _ (fst_r A B #)).
apply (@inc_trans _ _ _ ((fst_r A B # • fst_r A B) • alpha0)).
apply comp_inc_compat_ab_a.
apply snd_function.
apply comp_inc_compat_ab_b.
apply fst_function.
apply (@inc_trans _ _ _ (alpha0 ((fst_r A B # • Id (prod_eqType A B)) • snd_r A B))).
rewrite comp_id_r fst_snd_universal cap_universal.
apply inc_refl.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm inv_invol comp_assoc.
apply inc_refl.
move ⇒ u H.
split.
by [].
replace ((fst_r A B • ((fst_r A B # • u) • snd_r A B)) snd_r A B) with (u • snd_r A B).
apply domain_inc_id in H.
move : (@snd_function A B) ⇒ H0.
elim H0 ⇒ H1 H2.
by [rewrite (comp_domain3 H1) H].
rewrite comp_assoc -comp_assoc.
apply inc_antisym.
apply (@inc_trans _ _ _ ((u • snd_r A B) snd_r A B)).
apply inc_cap.
split.
apply inc_refl.
apply (comp_inc_compat_ab_b H).
apply cap_inc_compat_r.
apply comp_inc_compat_b_ab.
apply fst_function.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_b.
rewrite -fst_snd_cap_id.
apply cap_inc_compat_l.
apply comp_inc_compat_ab_ab'.
```

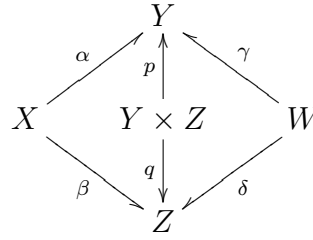
apply *inc\_inv*.  
 apply (*comp\_inc\_compat\_ab\_b* H).  
 Qed.

### 13.2.3 鋭敏性

この節の補題は以下の 1 つのみだが、証明が異様に長いため単独の節を設ける。

**Lemma 308 (sharpness)** *In below figure,*

$$\alpha \cdot \gamma^\# \sqcap \beta \cdot \delta^\# = (\alpha \cdot p^\# \sqcap \beta \cdot q^\#) \cdot (p \cdot \gamma^\# \sqcap q \cdot \delta^\#).$$



**Lemma sharpness** {W X Y Z : eqType}  
 {alpha : Rel X Y} {beta : Rel X Z} {gamma : Rel W Y} {delta : Rel W Z} :  
 (alpha · gamma #) (beta · delta #) =  
 ((alpha · fst\_r Y Z #) (beta · snd\_r Y Z #))  
 · ((fst\_r Y Z · gamma #) (snd\_r Y Z · delta #)).

**Proof.**

apply *inc\_antisym*.  
 move : (rationality \_ \_ alpha) ⇒ H.  
 move : (rationality \_ \_ beta) ⇒ H0.  
 move : (rationality \_ \_ (gamma #)) ⇒ H1.  
 move : (rationality \_ \_ (delta #)) ⇒ H2.  
 elim H ⇒ R.  
 elim ⇒ f0.  
 elim ⇒ g0 H3.  
 elim H0 ⇒ R0.  
 elim ⇒ f1.  
 elim ⇒ g1 H4.  
 elim H1 ⇒ R1.  
 elim ⇒ h0.  
 elim ⇒ k0 H5.  
 elim H2 ⇒ R2.  
 elim ⇒ h1.  
 elim ⇒ k1 H6.



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---

```

move : (rationality _ _ (g0 · h0 #)) ⇒ H7.
move : (rationality _ _ (g1 · h1 #)) ⇒ H8.
move : (rationality _ _ ((alpha · gamma #) (beta · delta #))) ⇒ H9.
elim H7 ⇒ R3.
elim ⇒ s0.
elim ⇒ t0 H10.
elim H8 ⇒ R4.
elim ⇒ s1.
elim ⇒ t1 H11.
elim H9 ⇒ R5.
elim ⇒ x.
elim ⇒ z H12.
assert (alpha · gamma # = (f0 # · (s0 # · t0)) · k0).
replace alpha with (f0 # · g0).
replace (gamma #) with (h0 # · k0).
rewrite -comp_assoc (@comp_assoc _ _ _ (f0 #)).
apply f_equal2.
apply f_equal.
apply H10.
by [].
apply Logic.eq_sym.
apply H5.
apply Logic.eq_sym.
apply H3.
assert (beta · delta # = (f1 # · (s1 # · t1)) · k1).
replace beta with (f1 # · g1).
replace (delta #) with (h1 # · k1).
rewrite -comp_assoc (@comp_assoc _ _ _ (f1 #)).
apply f_equal2.
apply f_equal.
apply H11.
by [].
apply Logic.eq_sym.
apply H6.
apply Logic.eq_sym.
apply H4.
assert (t0 · h0 = s0 · g0).
apply function_inc.
apply function_comp.
apply H10.
apply H5.
apply function_comp.

```

```
apply H10.
apply H3.
apply (@inc_trans _ _ _ (s0 · ((s0 # · t0) · h0))).
rewrite comp_assoc -comp_assoc.
apply comp_inc_compat_b_ab.
apply H10.
apply comp_inc_compat_ab_ab'.
replace (s0 # · t0) with (g0 · h0 #).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H5.
apply H10.
assert (t1 · h1 = s1 · g1).
apply function_inc.
apply function_comp.
apply H11.
apply H6.
apply function_comp.
apply H11.
apply H4.
apply (@inc_trans _ _ _ (s1 · ((s1 # · t1) · h1))).
rewrite comp_assoc -comp_assoc.
apply comp_inc_compat_b_ab.
apply H11.
apply comp_inc_compat_ab_ab'.
replace (s1 # · t1) with (g1 · h1 #).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H6.
apply H11.
remember ((x · (s0 · f0) #) (z · (t0 · k0) #)) as m0.
remember ((x · (s1 · f1) #) (z · (t1 · k1) #)) as m1.
assert (total_r m0).
rewrite Heqm0.
apply domain_corollary1.
apply H12.
apply H12.
replace (x # · z) with ((alpha · gamma #) (beta · delta #)).
apply (@inc_trans _ _ _ _ (cap_l)).
rewrite comp_inv H13 -comp_assoc comp_assoc.
apply inc_refl.
apply H12.
```

```
assert (total_r m1).
rewrite Heqm1.
apply domain_corollary1.
apply H12.
apply H12.
replace (x # • z) with ((alpha • gamma #) (beta • delta #)).
apply (@inc_trans - - - (cap_r)).
rewrite comp_inv H14 -comp_assoc comp_assoc.
apply inc_refl.
apply H12.
remember (m0 • (s0 • g0)) as n0.
remember (m1 • (s1 • g1)) as n1.
assert (total_r n0).
rewrite Heqn0.
apply (total_comp H17).
apply total_comp.
apply H10.
apply H3.
assert (total_r n1).
rewrite Heqn1.
apply (total_comp H18).
apply total_comp.
apply H11.
apply H4.
assert (total_r ((n0 • fst_r Y Z #) (n1 • snd_r Y Z #))).
apply (domain_corollary1 H19 H20).
rewrite fst_snd_universal.
apply inc_alpha_universal.
assert ((x # • n0) alpha).
replace alpha with (f0 # • g0).
rewrite Heqn0 Heqm0.
apply (@inc_trans - - - (((x # • x) • f0 #) • ((s0 # • s0) • g0))).
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -comp_inv.
apply cap_l.
apply comp_inc_compat.
apply comp_inc_compat_ab_b.
apply H12.
```

```
apply comp_inc_compat_ab_b.
apply H10.
apply Logic.eq_sym.
apply H3.
assert ((x # · n1) beta).
replace beta with (f1 # · g1).
rewrite Heqn1 Heqm1.
apply (@inc_trans _ _ _ (((x # · x) · f1 #) · ((s1 # · s1) · g1))).
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -comp_inv.
apply cap_l.
apply comp_inc_compat.
apply comp_inc_compat_ab_b.
apply H12.
apply comp_inc_compat_ab_b.
apply H11.
apply Logic.eq_sym.
apply H4.
assert ((n0 # · z) gamma #).
replace (gamma #) with (h0 # · k0).
rewrite Heqn0 Heqm0 -H15 comp_inv comp_inv inv_cap_distr.
apply (@inc_trans _ _ _ ((h0 # · (t0 # · t0)) · (k0 · (z # · z)))).
rewrite -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc comp_assoc comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc (@comp_inv _ _ _ z) inv_invol.
apply cap_r.
apply comp_inc_compat.
apply comp_inc_compat_ab_a.
apply H10.
apply comp_inc_compat_ab_a.
apply H12.
apply Logic.eq_sym.
apply H5.
assert ((n1 # · z) delta #).
replace (delta #) with (h1 # · k1).
```

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---

```

rewrite Heqn1 Heqm1 -H16 comp_inv comp_inv inv_cap_distr.
apply (@inc_trans _ _ _ ((h1 # · (t1 # · t1)) · (k1 · (z # · z)))).
rewrite -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc comp_assoc comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc (@comp_inv _ _ _ z) inv_invol.
apply cap_r.
apply comp_inc_compat.
apply comp_inc_compat_ab_a.
apply H11.
apply comp_inc_compat_ab_a.
apply H12.
apply Logic.eq_sym.
apply H6.
replace ((alpha · gamma #) (beta · delta #)) with (x # · z).
apply (@inc_trans _ _ _ ((x # · (((n0 · fst_r Y Z #) (n1 · snd_r Y Z #)) · (((n0
· fst_r Y Z #) (n1 · snd_r Y Z #))) #)) · z)).
apply comp_inc_compat_ab_a'b.
apply (comp_inc_compat_a_ab H21).
rewrite -comp_assoc comp_assoc.
apply comp_inc_compat.
apply (@inc_trans _ _ _ _ (comp_cap_distr_l)).
apply cap_inc_compat.
rewrite -comp_assoc.
apply (comp_inc_compat_ab_a'b H22).
rewrite -comp_assoc.
apply (comp_inc_compat_ab_a'b H23).
rewrite inv_cap_distr comp_inv comp_inv inv_invol inv_invol.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat.
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H24).
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H25).
apply Logic.eq_sym.
apply H12.
apply (@inc_trans _ _ _ _ (comp_cap_distr_l)).
apply cap_inc_compat.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ (cap_l)).

```

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```

rewrite -comp_assoc (@comp_assoc _ _ _ alpha).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a.
apply fst_function.
apply (@inc_trans _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ (cap_r)).
rewrite -comp_assoc (@comp_assoc _ _ _ beta).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a.
apply snd_function.
Qed.

```

### 13.2.4 分配法則

**Lemma 309 (prod\_cap\_distr\_l)** *Let  $\alpha : A \rightarrow B$  and  $\beta, \beta' : A \rightarrow C$ . Then,*

$$\alpha \top (\beta \sqcap \beta') = (\alpha \top \beta) \sqcap (\alpha \top \beta').$$

**Lemma prod\_cap\_distr\_l**  $\{A B C : eqType\} \{alpha : Rel A B\} \{beta beta' : Rel A C\}$ :  
 $Rel\_prod\ alpha\ (beta\ \beta') = Rel\_prod\ alpha\ beta\ Rel\_prod\ alpha\ beta'.$

**Proof.**

```

rewrite /Rel_prod.
rewrite -cap_assoc (@cap_comm _ _ _ (alpha • fst_r B C #)) -cap_assoc cap_idem
cap_assoc.
apply f_equal.
apply function_cap_distr_r.
apply snd_function.
Qed.

```

**Lemma 310 (prod\_cap\_distr\_r)** *Let  $\alpha, \alpha' : A \rightarrow B$  and  $\beta : A \rightarrow C$ . Then,*

$$(\alpha \sqcap \alpha') \top \beta = (\alpha \top \beta) \sqcap (\alpha' \top \beta).$$

**Lemma prod\_cap\_distr\_r**  $\{A B C : eqType\} \{alpha alpha' : Rel A B\} \{beta : Rel A C\}$ :  
 $Rel\_prod\ (alpha\ alpha')\ beta = Rel\_prod\ alpha\ beta\ Rel\_prod\ alpha'\ beta.$

**Proof.**

```

rewrite /Rel_prod.
rewrite cap_assoc (@cap_comm _ _ _ (beta • snd_r B C #)) cap_assoc cap_idem -cap_assoc.
apply (@f_equal _ _ (fun x => @cap _ _ x (beta • snd_r B C #))).
apply function_cap_distr_r.
apply fst_function.
Qed.

```

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**Lemma 311 (prod\_cup\_distr\_l)** *Let  $\alpha : A \rightarrow B$  and  $\beta, \beta' : A \rightarrow C$ . Then,*

$$\alpha \top (\beta \sqcup \beta') = (\alpha \top \beta) \sqcup (\alpha \top \beta').$$

**Lemma** *prod\_cup\_distr\_l* {A B C : eqType} {alpha : Rel A B} {beta beta' : Rel A C}:  
 $\text{Rel\_prod } \alpha \text{ (beta beta')} = \text{Rel\_prod } \alpha \text{ beta } \text{Rel\_prod } \alpha \text{ beta'}$ .

**Proof.**

by [rewrite -cap\_cup\_distr\_l -comp\_cup\_distr\_r].

**Qed.**

**Lemma 312 (prod\_cup\_distr\_r)** *Let  $\alpha, \alpha' : A \rightarrow B$  and  $\beta : A \rightarrow C$ . Then,*

$$(\alpha \sqcup \alpha') \top \beta = (\alpha \top \beta) \sqcup (\alpha' \top \beta).$$

**Lemma** *prod\_cup\_distr\_r* {A B C : eqType} {alpha alpha' : Rel A B} {beta : Rel A C}:  
 $\text{Rel\_prod } (\alpha \sqcup \alpha') \text{ beta} = \text{Rel\_prod } \alpha \text{ beta } \text{Rel\_prod } \alpha' \text{ beta}$ .

**Proof.**

by [rewrite -cap\_cup\_distr\_r -comp\_cup\_distr\_r].

**Qed.**

**Lemma 313 (comp\_prod\_distr\_l)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  and  $\gamma : B \rightarrow D$ . Then,*

$$\alpha \cdot (\beta \top \gamma) \sqsubseteq \alpha \cdot \beta \top \alpha \cdot \gamma.$$

**Lemma** *comp\_prod\_distr\_l*

{A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel B D}:  
 $\alpha \cdot \text{Rel\_prod } \beta \text{ gamma } \text{Rel\_prod } (\alpha \cdot \beta) (\alpha \cdot \text{gamma})$ .

**Proof.**

rewrite /Rel\_prod.

rewrite comp\_assoc comp\_assoc.

apply comp\_cap\_distr\_l.

**Qed.**

**Lemma 314 (function\_prod\_distr\_l)** *Let  $\alpha : A \rightarrow B$  be a function,  $\beta : B \rightarrow C$  and  $\gamma : B \rightarrow D$ . Then,*

$$\alpha \cdot (\beta \top \gamma) = \alpha \cdot \beta \top \alpha \cdot \gamma.$$

**Lemma** *function\_prod\_distr\_l*

{A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel B D}:  
 $\text{function\_r } \alpha \rightarrow \alpha \cdot \text{Rel\_prod } \beta \text{ gamma} = \text{Rel\_prod } (\alpha \cdot \beta) (\alpha \cdot \text{gamma})$ .

**Proof.**

## CHAPTER 13. LIBRARY SUM\_PRODUCT

`move ⇒ H.`  
`rewrite /Rel_prod.`  
`rewrite comp_assoc comp_assoc.`  
`apply (function_cap_distr_l H).`  
`Qed.`

**Lemma 315 (comp\_prod\_universal)** *Let  $\alpha : A \rightarrow B$ ,  $\beta : B \rightarrow C$  and  $\gamma : D \rightarrow E$ . Then,*

$$\alpha \cdot (\beta \top \nabla_{BD} \cdot \gamma) = \alpha \cdot \beta \top \nabla_{AD} \cdot \gamma.$$

**Lemma comp\_prod\_universal**

$\{A\ B\ C\ D\ E : \text{eqType}\} \{alpha : \text{Rel}\ A\ B\} \{beta : \text{Rel}\ B\ C\} \{gamma : \text{Rel}\ D\ E\} :$   
 $alpha \cdot \text{Rel\_prod}\ beta\ ( \quad B\ D \cdot gamma ) = \text{Rel\_prod}\ (alpha \cdot beta)\ ( \quad A\ D \cdot gamma ).$

**Proof.**

`apply inc_antisym.`  
`apply (@inc_trans _ _ _ _ (comp_prod_distr_l)).`  
`apply prod_inc_compat_l.`  
`rewrite -comp_assoc.`  
`apply comp_inc_compat_ab_a'b.`  
`apply inc_alpha_universal.`  
`rewrite /Rel_prod.`  
`rewrite comp_assoc.`  
`apply (@inc_trans _ _ _ _ (dedekind1)).`  
`apply comp_inc_compat_ab_ab'.`  
`apply cap_inc_compat_l.`  
`rewrite comp_assoc comp_assoc -comp_assoc.`  
`apply comp_inc_compat_ab_a'b.`  
`apply inc_alpha_universal.`  
`Qed.`

**Lemma 316 (fst\_cap\_snd\_distr)** *Let  $u, v : A \times B \rightarrow A \times B$  and  $u, v \sqsubseteq id_{A \times B}$ ,  $p : B \times C \rightarrow B$  and  $q : B \times C \rightarrow C$  be projections. Then,*

$$p^\sharp \cdot (u \sqcap v) \cdot q = p^\sharp \cdot u \cdot q \sqcap p^\sharp \cdot v \cdot q.$$

**Lemma fst\_cap\_snd\_distr**

$\{A\ B : \text{eqType}\} \{u\ v : \text{Rel}\ (prod\_eqType\ A\ B)\ (prod\_eqType\ A\ B)\} :$   
 $u \quad Id\ (prod\_eqType\ A\ B) \rightarrow v \quad Id\ (prod\_eqType\ A\ B) \rightarrow$   
 $fst\_r\ A\ B\ \# \cdot (u \quad v) \cdot snd\_r\ A\ B =$   
 $((fst\_r\ A\ B\ \# \cdot u) \cdot snd\_r\ A\ B) \quad ((fst\_r\ A\ B\ \# \cdot v) \cdot snd\_r\ A\ B).$

**Proof.**

`move ⇒ H H0.`  
`apply inc_antisym.`



## CHAPTER 13. LIBRARY SUM\_PRODUCT

---

```

apply (fun H' ⇒ @inc_trans _ _ _ _ H' (comp_cap_distr_r)).
apply comp_inc_compat_ab_a'b.
apply comp_cap_distr_l.
apply (@inc_trans _ _ _ _ (dedekind1)).
rewrite -(dedekind_id3 H H0) -(@comp_assoc _ _ _ _ u) (@comp_assoc _ _ _ _ (fst_r A
B # • u) v).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_b.
rewrite comp_inv comp_inv inv_invol -fst_snd_cap_id.
apply cap_inc_compat.
rewrite comp_assoc (dedekind_id1 H).
apply (comp_inc_compat_ab_b H).
rewrite -comp_assoc (dedekind_id1 H0).
apply (comp_inc_compat_ab_a H0).
Qed.

```

# Chapter 14

## Library **Point\_Axiom**

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Logic.IndefiniteDescription.
```

### 14.1 I-点

#### 14.1.1 I-点の定義

Dedekind 圏における域  $X$  の I-点  $x$  とは, 関数  $x : I \rightarrow X$  のことであり, 記号  $x \in X$  によって表される. また関係  $\rho : I \rightarrow X$  と I-点  $x : I \rightarrow X$  に対して, 記号  $x \in \rho$  で  $x \sqsubseteq \rho$  を表すものとする.

ちなみに I-点の定義  $x \in X$  は  $x \in \nabla_{IX}$  と言い換えることも可能である.

**Definition** *point\_inc*  $\{X : eqType\} (x \text{ rho} : Rel \ i \ X) := function\_r \ x \wedge x \text{ rho}$ .

**Definition** *point*  $\{X : eqType\} (x : Rel \ i \ X) := point\_inc \ x \ ( \ i \ X)$ .

#### 14.1.2 I-点の性質

**Lemma 317 (point\\_property1)** *Let  $x, y \in X$ . Then,*

$$x = y \Leftrightarrow x \cdot y^\# = id_I.$$

**Lemma** *point\\_property1*  $\{X : eqType\} \{x \ y : Rel \ i \ X\}$ :  
 $point \ x \rightarrow point \ y \rightarrow (x = y \Leftrightarrow x \cdot y^\# = Id \ i)$ .

**Proof.**

```

move  $\Rightarrow$   $H$   $H0$ .
split; move  $\Rightarrow$   $H1$ .
apply inc_antisym.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
rewrite  $H1$ .
apply  $H0$ .
apply Logic.eq_sym.
apply function_inc.
apply  $H0$ .
apply  $H$ .
rewrite  $(@comp\_id\_l \_ \_ y) - H1 \ comp\_assoc$ .
apply comp_inc_compat_ab_a.
apply  $H0$ .
Qed.

```

**Lemma 318 (point\_property2a, point\_property2b)** *Let  $\rho : I \rightarrow X$  be a total relation. Then,*

$$\rho \cdot \rho^\# = \rho \cdot \nabla_{XI} = id_I.$$

**Lemma point\_property2a**  $\{X : eqType\} \{rho : Rel\ i\ X\}$ :  
*total\_r rho  $\rightarrow$  rho  $\cdot$  rho  $\#$  = Id i.*

**Proof.**

```

move  $\Rightarrow$   $H$ .
apply inc_antisym.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
apply  $H$ .
Qed.

```

**Lemma point\_property2b**  $\{X : eqType\} \{rho : Rel\ i\ X\}$ :  
*total\_r rho  $\rightarrow$  rho  $\cdot$  rho  $\#$  = rho  $\cdot$   $\nabla_{XI}$  i.*

**Proof.**

```

move  $\Rightarrow$   $H$ .
apply inc_antisym.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
rewrite  $(point\_property2a\ H) \ unit\_identity\_is\_universal$ .
apply inc_alpha_universal.
Qed.

```

## CHAPTER 14. LIBRARY POINT\_AXIOM

**Lemma 319 (point\_property3)** *Let  $\rho : I \rightarrow X$ . Then,*

$$\exists x \in \rho \Rightarrow “\rho \text{ is total}” \wedge \rho \neq \phi_{IX}.$$

**Lemma** *point\_property3*  $\{X : eqType\} \{rho : Rel\ i\ X\}$ :  
 $(\exists x : Rel\ i\ X, point\_inc\ x\ rho) \rightarrow total\_r\ rho \wedge rho \neq \phi_{IX}$ .

**Proof.**

```
elim  $\Rightarrow x\ H$ .
assert (total_r rho).
elim  $H \Rightarrow H0\ H1$ .
elim  $H0 \Rightarrow H2\ H3$ .
apply (@inc_trans _ _ _ _ H2).
apply comp_inc_compat.
apply H1.
apply (@inc_inv _ _ _ _ H1).
split.
apply H0.
move  $\Rightarrow H1$ .
rewrite /total_r in H0.
rewrite H1 comp_empty_l in H0.
apply unit_identity_not_empty.
apply inc_antisym.
apply H0.
apply inc_empty_alpha.
```

**Qed.**

**Lemma 320 (point\_property4)**

$$\exists x \in X \Rightarrow “\nabla_{IX} \text{ is total}” \wedge \nabla_{IX} \neq \phi_{IX}.$$

**Lemma** *point\_property4*  $\{X : eqType\}$ :  
 $(\exists x : Rel\ i\ X, point\ x) \rightarrow total\_r\ (\phi_{IX}) \wedge (\phi_{IX} \neq \phi_{IX})$ .

**Proof.**

```
move  $\Rightarrow H$ .
apply (@point_property3 _ (\phi_{IX}) H).
```

**Qed.**

## 14.2 I-点に関する諸公理

### 14.2.1 点公理

この“点公理”を使えば, I-点に関する様々な定理や補題が導出できる.

**Lemma 321 (point\_axiom)** *Let  $\rho : I \rightarrow X$ . Then,*

$$\rho = \sqcup_{x \in \rho} x.$$

**Lemma** *lemma\_for\_PA*  $\{X : eqType\} \{rho : Rel\ i\ X\}$ :  
 $((\rho = \_ i\ X) \rightarrow False) \rightarrow False \rightarrow \rho = \_ i\ X$ .

**Proof.**

```
move => H.
case (@unit_empty_or_universal (rho · rho #)) => H0.
apply inc_antisym.
apply (@inc_trans _ _ _ (relation_rel_inv_rel)).
rewrite H0 comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
apply False_ind.
apply H.
move => H1.
rewrite H1 comp_empty_l in H0.
apply (unit_empty_not_universal H0).
Qed.
```

**Lemma** *point\_axiom*  $\{X : eqType\} \{rho : Rel\ i\ X\}$ :  
 $\rho = \_ \{ \text{fun } x : Rel\ i\ X \Rightarrow \text{point\_inc } x\ rho \} id$ .

**Proof.**

```
apply inc_antisym.
apply bool_lemma2.
assert ((∃ x : Rel i X, point_inc x (( _ {fun x : Rel i X => point_inc x rho} id)
( _ {fun x : Rel i X => point_inc x rho} id) ^)) → False).
move => H.
move : (point_property3 H) => H0.
apply H0.
apply cap_complement_empty.
assert ((∃ x : Rel i X, point_inc x (rho ( _ {fun x : Rel i X => point_inc x rho} id)
^)) → False).
move => H0.
apply H.
elim H0 => x H1.
```

```

∃ x.
split.
apply H1.
apply inc_cap.
split.
assert (point_inc x rho).
split.
apply H1.
elim H1 ⇒ H2 H3.
apply inc_cap in H3.
apply H3.
clear H1.
move : x H2.
apply inc_cupP.
apply inc_refl.
elim H1 ⇒ H2 H3.
apply inc_cap in H3.
apply H3.
apply lemma_for_PA.
move ⇒ H1.
apply H0.
apply axiom_of_choice.
rewrite /total_r.
remember (rho ( _ {fun x : Rel i X ⇒ point_inc x rho} id) ^) as rho'.
case (@unit_empty_or_universal (rho' • rho' #)) ⇒ H2.
apply False_ind.
apply H1.
apply inc_antisym.
apply (@inc_trans _ _ _ (relation_rel_inv_rel)).
rewrite H2 comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
rewrite H2.
apply inc_alpha_universal.
apply inc_cupP.
move ⇒ beta H.
apply H.
Qed.

```

**Lemma 322 (PA\_corollary1)**

$$\nabla_{IX} = \sqcup_{x \in X} x.$$

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**Lemma** *PA\_corollary1* {*X* : *eqType*}:  $i\ X = \_ \{point\}\ id$ .

**Proof.**

apply *point\_axiom*.

**Qed.**

**Lemma 323 (PA\_corollary2)**

$$id_X = \sqcup_{x \in X} x^\# \cdot x.$$

**Lemma** *PA\_corollary2* {*X* : *eqType*}:

$Id\ X = \_ \{point\}\ (\text{fun } x : Rel\ i\ X \Rightarrow x \# \cdot x)$ .

**Proof.**

rewrite -(@cap\_universal \_ \_ (Id X)) -lemma\_for\_tarski2 *PA\_corollary1*.

rewrite *comp\_cupP\_distr\_l cap\_cupP\_distr\_l*.

apply *cupP\_eq*.

move  $\Rightarrow$  *alpha H*.

apply *inc\_antisym*.

rewrite *cap\_comm*.

apply (@inc\_trans \_ \_ \_ \_ (dedekind2)).

rewrite *comp\_id\_l cap\_comm cap\_universal*.

apply *inc\_refl*.

apply *inc\_cap*.

split.

apply *H*.

apply *comp\_inc\_compat\_ab\_a'b*.

apply *inc\_alpha\_universal*.

**Qed.**

**Lemma 324 (PA\_corollary3)** *Let  $\alpha, \beta : X \rightarrow Y$ . Then,*

$$(\forall x \in X, x \cdot \alpha = x \cdot \beta) \Rightarrow \alpha = \beta.$$

**Lemma** *PA\_corollary3* {*X Y* : *eqType*} {*alpha beta* : *Rel X Y*}:

$(\forall x : Rel\ i\ X, point\ x \rightarrow x \cdot \alpha = x \cdot \beta) \rightarrow \alpha = \beta$ .

**Proof.**

move  $\Rightarrow$  *H*.

rewrite -(@comp\_id\_l \_ \_ *alpha*) -(@comp\_id\_l \_ \_ *beta*) *PA\_corollary2*.

rewrite *comp\_cupP\_distr\_r comp\_cupP\_distr\_r*.

apply *cupP\_eq*.

move  $\Rightarrow$  *gamma H0*.

by [rewrite *comp\_assoc comp\_assoc* (*H gamma H0*)].

**Qed.**

**Lemma 325 (PA\_corollary4)** *Let  $\alpha : X \rightarrow Y$ . Then,*

$$“\alpha \text{ is total}” \Leftrightarrow \forall x \in X, “x \cdot \alpha \text{ is total}”.$$

**Lemma PA\_corollary4**  $\{X\ Y : eqType\} \{alpha : Rel\ X\ Y\}$ :  
 $total\_r\ alpha \Leftrightarrow \forall x : Rel\ i\ X, point\ x \rightarrow total\_r\ (x \cdot alpha).$

**Proof.**

```
split; move  $\Rightarrow H$ .
move  $\Rightarrow x\ H0$ .
apply  $total\_comp$ .
apply  $H0$ .
apply  $H$ .
rewrite  $/total\_r$ .
rewrite  $PA\_corollary2$ .
apply  $inc\_cupP$ .
move  $\Rightarrow x\ H0$ .
move :  $(H\ x\ H0) \Rightarrow H1$ .
apply  $(@inc\_trans\ \_ \_ \_ ((x\ \# \cdot ((x \cdot alpha) \cdot (x \cdot alpha)\ \#))) \cdot x))$ .
apply  $comp\_inc\_compat\_ab\_a'b$ .
apply  $(comp\_inc\_compat\_a\_ab\ H1)$ .
rewrite  $comp\_inv\ -comp\_assoc\ -comp\_assoc\ -comp\_assoc$ .
rewrite  $comp\_assoc\ (@comp\_assoc\ \_ \_ \_ \_ (x\ \# \cdot x))$ .
apply  $(@inc\_trans\ \_ \_ \_ ((x\ \# \cdot x) \cdot (alpha \cdot alpha\ \#)))$ .
apply  $comp\_inc\_compat\_ab\_a$ .
apply  $H0$ .
apply  $comp\_inc\_compat\_ab\_b$ .
apply  $H0$ .
```

**Qed.**

**Lemma 326 (PA\_corollary5)** *Let  $\alpha : X \rightarrow Y$ . Then,*

$$“\alpha \text{ is univalent}” \Leftrightarrow \forall x \in X, “x \cdot \alpha \text{ is univalent}”.$$

**Lemma PA\_corollary5**  $\{X\ Y : eqType\} \{alpha : Rel\ X\ Y\}$ :  
 $univalent\_r\ alpha \Leftrightarrow \forall x : Rel\ i\ X, point\ x \rightarrow univalent\_r\ (x \cdot alpha).$

**Proof.**

```
split; move  $\Rightarrow H$ .
move  $\Rightarrow x\ H0$ .
apply  $univalent\_comp$ .
apply  $H0$ .
apply  $H$ .
rewrite  $/univalent\_r$ .
rewrite  $-(@comp\_id\_r\ \_ \_ (alpha\ \#))\ PA\_corollary2$ .
```



```

rewrite comp_cupP_distr_l comp_cupP_distr_r.
apply inc_cupP.
move  $\Rightarrow x$  H0.
move : ( $H$   $x$  H0)  $\Rightarrow$  H1.
rewrite -comp_assoc -comp_inv comp_assoc.
apply H1.
Qed.

```

### 14.2.2 全域性公理

**Lemma 327 (total\_axiom)** *Let  $\rho : I \rightarrow X$ . Then,*

$$\rho \neq \phi_{IX} \Rightarrow id_I = \rho \cdot \rho^\#.$$

**Lemma** *total\_axiom*  $\{X : eqType\} \{rho : Rel\ i\ X\}$ :  
 $rho \neq \quad i\ X \rightarrow Id\ i = rho \cdot rho \#.$

**Proof.**

```

move  $\Rightarrow H$ .
case (@unit_empty_or_universal (rho · rho #))  $\Rightarrow$  H0.
apply False_ind.
apply H.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (relation_rel_inv_rel)).
rewrite H0 comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
by [rewrite H0 unit_identity_is_universal].
Qed.

```

**Lemma 328 (Tot\_corollary1)** *Let  $\rho : I \rightarrow X$  and  $x \in X$ . Then,*

$$\rho \sqsubseteq x \Rightarrow \rho = \phi_{IX} \vee \rho = x.$$

**Lemma** *Tot\_corollary1*  $\{X : eqType\} \{rho\ x : Rel\ i\ X\}$ :  
 $point\ x \rightarrow rho \quad x \rightarrow rho = \quad i\ X \vee rho = x.$

**Proof.**

```

move  $\Rightarrow H$  H0.
case (@unit_empty_or_universal (rho · rho #))  $\Rightarrow$  H1.
left.
apply inc_antisym.
apply (@inc_trans _ _ _ _ (relation_rel_inv_rel)).
rewrite H1 comp_empty_l.

```

```

apply inc_refl.
apply inc_empty_alpha.
right.
apply inc_antisym.
apply H0.
rewrite -(@comp_id_l _ _ x) unit_identity_is_universal -H1 comp_assoc.
apply (@inc_trans _ _ _ (rho • (x # • x))).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply (@inc_inv _ _ _ H0).
apply comp_inc_compat_ab_a.
apply H.
Qed.

```

**Lemma 329 (Tot\_corollary2)** *Let  $x, y \in X$ . Then,*

$$x \neq y \Leftrightarrow x \cdot y^\# = \phi_I.$$

**Lemma Tot\_corollary2**  $\{X : eqType\} \{x y : Rel\ i\ X\}$ :  
 $point\ x \rightarrow point\ y \rightarrow (x \neq y \Leftrightarrow x \cdot y^\# = \phi_I).$

**Proof.**

```

move => H H0.
assert (x = y <=> x • y # ≠ ϕ_I).
rewrite (point_property1 H H0).
split; move => H1.
rewrite H1.
apply unit_identity_not_empty.
case (@unit_empty_or_universal (x • y #)) => H2.
apply False_ind.
apply (H1 H2).
by [rewrite H2 unit_identity_is_universal].
rewrite H1.
split; move => H2.
apply (lemma_for_PA H2).
move => H3.
apply (H3 H2).
Qed.

```

**Lemma 330 (Tot\_corollary3)** *Let  $f : (I \rightarrow X) \rightarrow (I \rightarrow Y)$ . Then,*

$$(\forall x \in X, "f(x) \text{ is a function}") \Rightarrow "\sqcup_{x \in X} x^\# \cdot f(x) \text{ is a function}."$$

**Lemma Tot\_corollary3**  $\{X\ Y : eqType\} \{f : Rel\ i\ X \rightarrow Rel\ i\ Y\}$ :

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$(\forall x : \text{Rel } i \ X, \text{point } x \rightarrow \text{function\_r } (f \ x)) \rightarrow \text{function\_r } ( \_ \{ \text{point} \} (\text{fun } x : \text{Rel } i \ X \Rightarrow x \# \cdot f \ x))$ .

**Proof.**

move  $\Rightarrow H$ .

assert  $(\forall x : \text{Rel } i \ X, \text{point } x \rightarrow x \cdot ( \_ \{ \text{point} \} (\text{fun } x0 : \text{Rel } i \ X \Rightarrow x0 \# \cdot f \ x0)) = f \ x)$ .

move  $\Rightarrow x \ H0$ .

assert  $(x \cdot x \# = \text{Id } i)$ .

apply *inc\_antisym*.

rewrite *unit\_identity\_is\_universal*.

apply *inc\_alpha\_universal*.

apply *H0*.

rewrite  $(\text{@comp\_id\_l } \_ \_ (f \ x)) \text{-} H1$ .

apply *inc\_antisym*.

rewrite *comp\_cupP\_distr\_l*.

apply *inc\_cupP*.

move  $\Rightarrow y \ H2$ .

rewrite *comp\_assoc*.

case  $(\text{@unit\_empty\_or\_universal } (x \cdot y \#)) \Rightarrow H3$ .

rewrite *H3 comp\_empty\_l*.

apply *inc\_empty\_alpha*.

rewrite *unit\_identity\_is\_universal* in *H3*.

apply  $(\text{point\_property1 } H0 \ H2)$  in *H3*.

rewrite *H3*.

apply *inc\_refl*.

rewrite *comp\_assoc*.

apply *comp\_inc\_compat\_ab\_ab'*.

clear *H1*.

move :  $x \ H0$ .

apply *inc\_cupP*.

apply *inc\_refl*.

split.

rewrite *PA\_corollary4*.

move  $\Rightarrow x \ H1$ .

rewrite  $(H0 \ x \ H1)$ .

apply  $(H \ x \ H1)$ .

rewrite *PA\_corollary5*.

move  $\Rightarrow x \ H1$ .

rewrite  $(H0 \ x \ H1)$ .

apply  $(H \ x \ H1)$ .

**Qed.**

## 14.2.3 その他の公理

**Lemma 331 (nonempty\_axiom)** *Let  $\rho : I \rightarrow X$ . Then,*

$$\rho \neq \phi_{IX} \Rightarrow \exists x \in \rho.$$

**Lemma** *nonempty\_axiom*  $\{X : eqType\} \{rho : Rel\ i\ X\}$ :  
 $rho \neq \phi_{IX} \rightarrow \exists x : Rel\ i\ X, point\_inc\ x\ rho.$

**Proof.**

move : (@axiom\_of\_choice \_ \_ rho)  $\Rightarrow$   $H$ .

move  $\Rightarrow$   $H0$ .

apply  $H$ .

rewrite /total\_r.

rewrite (total\_axiom  $H0$ ).

apply inc\_refl.

**Qed.**

**Lemma 332 (axiom\_of\_subobjects2)** *Let  $\rho : I \rightarrow X$ . Then,*

$$\exists S, \exists j : S \rightarrow X, \rho = \nabla_{IS} \cdot j \wedge j \cdot j^\# = id_S.$$

**Lemma** *axiom\_of\_subobjects2*  $\{X : eqType\} \{rho : Rel\ i\ X\}$ :  
 $\exists (S : eqType)(j : Rel\ S\ X), rho = \nabla_{IS} \cdot j \wedge j \cdot j^\# = Id\ S.$

**Proof.**

elim (@rationality \_ \_ rho)  $\Rightarrow$   $R$ .

elim  $\Rightarrow$   $f$ .

elim  $\Rightarrow$   $g$ .

elim  $\Rightarrow$   $H$ .

elim  $\Rightarrow$   $H0$ .

elim  $\Rightarrow$   $H1\ H2$ .

$\exists\ R$ .

$\exists\ g$ .

split.

rewrite  $H1$ .

apply inc\_antisym.

apply comp\_inc\_compat\_ab\_a'b.

apply inc\_alpha\_universal.

apply comp\_inc\_compat\_ab\_a'b.

apply (@inc\_trans \_ \_ \_ (i R \cdot (f \cdot f^\#))).

apply comp\_inc\_compat\_a\_ab.

apply  $H$ .

rewrite -comp\_assoc.

apply comp\_inc\_compat\_ab\_b.

```

rewrite unit_identity_is_universal.
apply inc_alpha_universal.
rewrite -H2 cap_comm inc_def1.
assert ((f # · g) rho).
rewrite H1.
apply inc_refl.
apply (function_move1 H) in H3.
apply (@inc_trans _ _ _ ((f · rho) · (f · rho) #)).
apply comp_inc_compat.
apply H3.
apply (@inc_inv _ _ _ H3).
rewrite comp_inv_comp_assoc -(@comp_assoc _ _ _ rho).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
Qed.

```

### 14.3 その他の補題

**Lemma 333 (point\_atomic)** *Let  $x \in X$ , then  $x$  is atomic.*

**Lemma** *point\_atomic* {X : eqType} {x : Rel i X}: point x → atomic x.

**Proof.**

```

move ⇒ H.
split.
move : (@point_property3 X x) ⇒ H0.
apply H0.
∃ x.
split.
apply H.
apply inc_refl.
move ⇒ beta.
apply (Tot_corollary1 H).
Qed.

```

**Lemma 334 (point\_atomic2)** *Let  $x \in X$  and  $y \in Y$ , then  $x^\# \cdot y$  is atomic.*

**Lemma** *point\_atomic2* {X Y : eqType} {x : Rel i X} {y : Rel i Y}:  
point x → point y → atomic (x # · y).

**Proof.**

```

move  $\Rightarrow$   $H\ H0$ .
split.
move  $\Rightarrow$   $H1$ .
assert ( $Id\ i = (x \cdot x \#) \cdot (y \cdot y \#)$ ).
apply inc_antisym.
rewrite  $-(@comp\_id\_l \_ \_ (Id\ i))$ .
apply comp\_inc\_compat.
apply  $H$ .
apply  $H0$ .
rewrite unit\_identity\_is\_universal.
apply inc\_alpha\_universal.
rewrite comp\_assoc  $-(@comp\_assoc \_ \_ \_ (x \#))$  in  $H2$ .
rewrite  $H1\ comp\_empty\_l\ comp\_empty\_r$  in  $H2$ .
apply (unit\_identity\_not\_empty  $H2$ ).
move  $\Rightarrow$  beta  $H1$ .
case (@unit\_empty\_or\_universal (( $\_ \_ i\ X \cdot \_ \_ \_ Y\ i$ ))  $\Rightarrow H2$ ).
left.
apply inc\_antisym.
replace ( $\_ \_ X\ Y$ ) with (( $\_ \_ X\ i \cdot \_ \_ i\ i$ )  $\cdot \_ \_ i\ Y$ ).
rewrite  $-H2\ -comp\_assoc\ -comp\_assoc\ unit\_universal$ .
rewrite comp\_assoc\ unit\_universal.
apply (@inc\_trans  $\_ \_ \_ (\_ \_ X\ X \cdot \_ \_ \_ \text{beta})$ ).
apply comp\_inc\_compat\_b\_ab.
apply inc\_alpha\_universal.
apply comp\_inc\_compat\_a\_ab.
apply inc\_alpha\_universal.
by [rewrite comp\_empty\_r\ comp\_empty\_l].
apply inc\_empty\_alpha.
right.
apply inc\_antisym.
apply  $H1$ .
assert (beta  $\neq \_ \_ X\ Y$ ).
move  $\Rightarrow$   $H3$ .
rewrite  $H3\ comp\_empty\_r\ comp\_empty\_l$  in  $H2$ .
apply (unit\_empty\_not\_universal  $H2$ ).
apply (@inc\_trans  $\_ \_ \_ (x \# \cdot (x \cdot \_ \_ \_ \text{beta}))$ ).
apply comp\_inc\_compat\_ab\_ab'.
assert (( $x \cdot \_ \_ \_ \text{beta}$ )  $\_ \_ y$ ).
apply (@inc\_trans  $\_ \_ \_ (x \cdot (x \# \cdot \_ \_ \_ y))$ ).
apply (comp\_inc\_compat\_ab\_ab'  $H1$ ).
rewrite  $-comp\_assoc$ .
apply comp\_inc\_compat\_ab\_b.

```

```
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
apply inc_def1 in H1.
rewrite H1 in H3.
assert ( $x \# \cdot ((x \cdot \text{beta}) \quad y) \neq \quad X \ Y$ ).
move  $\Rightarrow$  H5.
apply H3.
apply inc_antisym.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ (dedekind1)).
rewrite cap_comm inv_invol H5.
apply inc_refl.
apply inc_empty_alpha.
case (Tot_corollary1 H0 H4)  $\Rightarrow$  H6.
rewrite H6 cap_comm cap_empty comp_empty_r in H5.
apply False_ind.
by [apply H5].
rewrite H6.
apply inc_refl.
rewrite -comp_assoc.
apply comp_inc_compat_ab_b.
apply H.
Qed.
```

# Bibliography

- [1] R. Affeldt and M. Hagiwara. Formalization of Shannon 's Theorems in SSReflect-Coq. In 3rd Conference on Interactive Theorem Proving, LNCS 7406, 233–249, 2012.