INSTITUTE OF MATHEMATICS FOR INDUSTRY,
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LOGIC AND COMPUTATION PROJECT

Coq Modules for Relational Calculus

(Ver.0.1)

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Contents

Chapter 1

Library Basic_Notations

1.1 このライブラリについて

- このライブラリは河原康雄先生の “関係の理論 - Dedekind 圏概説 -” をもとに制作されている.
- 現状サポートしているのは,
  - 1.4 節大半, 1.5 - 1.6 節全部
  - 2.1 - 2.3 節全部, 2.4 - 2.5 節大半, 2.6 節全部, 2.7 節大半, 2.8 節有理性
  - 4.2 - 4.3 節全部, 4.4 - 4.5 節大半, 4.6 節命題 4.6.1, 4.7 節大半, 4.9 節全部
  - 4.8 節は部分域公理を用いるので, そちらが終わり次第

  といったところである.
- 関係論で話を進めたい場合は, 下の行に `Require Export Basic_Notations_Rel.` を, 集合論で話を進めたい場合は, `Require Export Basic_Notations_Set.` を記述する.

```
Require Export Basic_Notations_Set.
```

なお, 証明の書き方が悪いと, まれに “関係論では証明が通ったのに, 集合論では通らない” といったことも起こるようなので, ある程度注意しておく必要がある.

Chapter 2

Library Basic_Notations_Rel

```
Require Export ssreflect eqtype bigop.
Require Export Logic.ClassicalFacts.
Axiom prop_extensionality_ok : prop_extensionality.
```

2.1 定義

- A, B を eqType として, A から B への関係の型を (Rel A B) と書き, $A \to B \to$ Prop として定義する. 本文中では型 (Rel A B) を $A \rightharpoondown B$ と書く.
- 関係 $\alpha : A \rightharpoondown B$ の逆関係 $\alpha^\sharp : B \rightharpoondown A$ は (inverse α) で, Coq では (α #) と記述する.
- 2 つの関係 $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$ の合成関係 $\alpha{\cdot}\beta$(or $\alpha\beta$) : $A \rightharpoondown C$ は (composite α β) で, (α ・ β) と記述する.
- 剰余合成関係 $\alpha \rhd \beta : A \rightharpoondown C$ は (residual α β) で, (α △ β) と記述する.
- 恒等関係 $\mathrm{id}_A : A \rightharpoondown A$ は (identity A) で, (Id A) と記述する.
- 空関係 $\phi_{AB} : A \rightharpoondown B$ は (empty A B) で, (φ A B) と記述する.
- 全関係 $\nabla_{AB} : A \rightharpoondown B$ は (universal A B) で, (∇ A B) と記述する.
- 2 つの関係 $\alpha : A \rightharpoondown B$, $\beta : A \rightharpoondown B$ の和関係 $\alpha \sqcup \beta : A \rightharpoondown B$ は (cup α β) で, (α ∪ β) と記述する.
- 共通関係 $\alpha \sqcap \beta : A \rightharpoondown B$ は (cap α β) で, (α ∩ β) と記述する.
- 相対擬補関係 $\alpha \Rightarrow \beta : A \rightharpoondown B$ は (rpc α β) で, (α >> β) と記述する.
- 関係 $\alpha : A \rightharpoondown B$ の補関係 $\alpha^- : A \rightharpoondown B$ は (complement α) で, Coq では (α ^) と記述する.

	数式	Coq	Notation
逆関係	$\alpha^\sharp$	(`inverse` α)	(α `#`)
合成関係	$\alpha \cdot \beta$	(`composite` α β)	(α `•` β)
剰余合成関係	$\alpha \rhd \beta$	(`residual` α β)	(α △ β)
恒等関係	id_A	(`identity` A)	(`Id` A)
空関係	ϕ_{AB}	(`empty` A B)	(φ A B)
全関係	∇_{AB}	(`universal` A B)	(∇ A B)
和関係	$\alpha \sqcup \beta$	(`cup` α β)	(α ∪ β)
共通関係	$\alpha \sqcap \beta$	(`cap` α β)	(α ∩ β)
相対擬補関係	$\alpha \Rightarrow \beta$	(`rpc` α β)	(α `>>` β)
補関係	α^-	(`complement` α)	(α `^`)
差関係	$\alpha - \beta$	(`difference` α β)	(α `--` β)
添字付和関係	$\sqcup_{P(\alpha)} f(\alpha)$	(`cupP` P f)	(∪_{P} f)
添字付共通関係	$\sqcap_{P(\alpha)} f(\alpha)$	(`capP` P f)	(∩_{P} f)

Table 2.1: 関係の表記について

- 2 つの関係 $\alpha : A \rightharpoondown B$, $\beta : A \rightharpoondown B$ の差関係 $\alpha - \beta : A \rightharpoondown B$ は (`difference` α β) で, (α `--` β) と記述する.
- (`cupP`) と (`capP`) は添字付の和関係と共通関係であり, 述語 P に対し, $\{f(\alpha) \mid P(\alpha)\}$ の和関係, 共通関係を表す.
- また, 1 点集合 $I = \{*\}$ は `i` と表記する.
- なお, 通常の共通関係, 和関係も添字付のもので表現することができるため, ここではそれを用いて表記する.
- 後で述べるように, 剰余合成 $\alpha \rhd \beta$ も $(\alpha \cdot \beta^-)^-$ のように表現することは可能だが, "剰余合成が存在すれば, それは $(\alpha \cdot \beta^-)^-$ に等しい" というレベルのものであるため, 剰余合成に関する公理はやはり必要となる.

表 2.1 に関係の表記についてまとめる.

```
Definition Rel (A B : eqType) := A → B → Prop.

Parameter inverse : (∀ A B : eqType, Rel A B → Rel B A).
Notation "a #" := (inverse _ _ a) (at level 20).
Parameter composite : (∀ A B C : eqType, Rel A B → Rel B C → Rel A C).
Notation "a '•' b" := (composite _ _ _ a b) (at level 50).
Parameter residual : (∀ A B C : eqType, Rel A B → Rel B C → Rel A C).
Notation "a '△' b" := (residual _ _ _ a b) (at level 50).

Parameter identity : (∀ A : eqType, Rel A A).
Notation "'Id'" := identity.
```

```
Parameter empty : (∀ A B : eqType, Rel A B).
Notation "' φ '" := empty.
Parameter universal : (∀ A B : eqType, Rel A B).
Notation "' ∇ '" := universal.

Parameter include : (∀ A B : eqType, Rel A B → Rel A B → Prop).
Notation "a ' ⊆ ' b" := (include _ _ a b) (at level 50).

Parameter cupP : (∀ A B C D : eqType, (Rel C D → Prop) → (Rel C D → Rel A B) →
Rel A B).
Notation "' ∪_{' p '}' f" := (cupP _ _ _ _ p f) (at level 50).
Parameter capP : (∀ A B C D : eqType, (Rel C D → Prop) → (Rel C D → Rel A B) →
Rel A B).
Notation "' ∩_{' p '}' f" := (capP _ _ _ _ p f) (at level 50).

Definition cup {A B : eqType} (alpha beta : Rel A B)
 := ∪_{fun gamma : Rel A B ⇒ gamma = alpha ∨ gamma = beta} id.
Notation "a ' ∪ ' b" := (cup a b) (at level 50).
Definition cap {A B : eqType} (alpha beta : Rel A B)
 := ∩_{fun gamma : Rel A B ⇒ gamma = alpha ∨ gamma = beta} id.
Notation "a ' ∩ ' b" := (cap a b) (at level 50).
Parameter rpc : (∀ A B : eqType, Rel A B → Rel A B → Rel A B).
Notation "a '»' b" := (rpc _ _ a b) (at level 50).
Definition complement {A B : eqType} (alpha : Rel A B) := alpha » φ A B.
Notation "a '^'" := (complement a) (at level 20).
Definition difference {A B : eqType} (alpha beta : Rel A B) := alpha ∩ beta ^.
Notation "a - b" := (difference a b) (at level 50).

Notation "'i'" := unit_eqType.
```

2.2 関数の定義

$\alpha : A \rightharpoondown B$ に対し, 全域性 `total_r`, 一価性 `univalent_r`, 関数 `function_r`, 全射 `surjective_r`, 単射 `injective_r`, 全単射 `bijection_r` を以下のように定義する.

- `total_r` : $id_A \sqsubseteq \alpha \cdot \alpha^\sharp$
- `univalent_r` : $\alpha^\sharp \cdot \alpha \sqsubseteq id_B$
- `function_r` : $id_A \sqsubseteq \alpha \cdot \alpha^\sharp \wedge \alpha^\sharp \cdot \alpha \sqsubseteq id_B$
- `surjection_r` : $id_A \sqsubseteq \alpha \cdot \alpha^\sharp \wedge \alpha^\sharp \cdot \alpha = id_B$
- `injection_r` : $id_A = \alpha \cdot \alpha^\sharp \wedge \alpha^\sharp \cdot \alpha \sqsubseteq id_B$
- `bijection_r` : $id_A = \alpha \cdot \alpha^\sharp \wedge \alpha^\sharp \cdot \alpha = id_B$

```
Definition total_r {A B : eqType} (alpha : Rel A B) := (Id A) ⊆ (alpha • alpha #).
Definition univalent_r {A B : eqType} (alpha : Rel A B) := (alpha # • alpha) ⊆ (Id B).
Definition function_r {A B : eqType} (alpha : Rel A B)
 := (total_r alpha) ∧ (univalent_r alpha).
Definition surjection_r {A B : eqType} (alpha : Rel A B)
 := (function_r alpha) ∧ (total_r (alpha #)).
Definition injection_r {A B : eqType} (alpha : Rel A B)
 := (function_r alpha) ∧ (univalent_r (alpha #)).
Definition bijection_r {A B : eqType} (alpha : Rel A B)
 := (function_r alpha) ∧ (total_r (alpha #)) ∧ (univalent_r (alpha #)).
```

2.3 関係の公理

今後の諸定理の証明は, 原則以下の公理群, およびそれらから導かれる補題のみを用いて行っていくことにする.

2.3.1 Dedekind 圏の公理

Axiom 1 (comp_id_l, comp_id_r) *Let* $\alpha : A \rightharpoondown B$*. Then,*

$$id_A \cdot \alpha = \alpha \cdot id_B = \alpha.$$

```
Definition axiom1a := ∀ (A B : eqType)(alpha : Rel A B), Id A • alpha = alpha.
Axiom comp_id_l : axiom1a.
Definition axiom1b := ∀ (A B : eqType)(alpha : Rel A B), alpha • Id B = alpha.
Axiom comp_id_r : axiom1b.
```

Axiom 2 (comp_assoc) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$, and $\gamma : C \rightharpoondown D$. Then,*

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

```
Definition axiom2 :=
 ∀ (A B C D : eqType)(alpha : Rel A B)(beta : Rel B C)(gamma : Rel C D),
 (alpha ・ beta) ・ gamma = alpha ・ (beta ・ gamma).
Axiom comp_assoc : axiom2.
```

Axiom 3 (inc_refl) *Let $\alpha : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \alpha.$$

```
Definition axiom3 := ∀ (A B : eqType)(alpha : Rel A B), alpha ⊆ alpha.
Axiom inc_refl : axiom3.
```

Axiom 4 (inc_trans) *Let $\alpha, \beta, \gamma : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma \Rightarrow \alpha \sqsubseteq \gamma.$$

```
Definition axiom4 := ∀ (A B : eqType)(alpha beta gamma : Rel A B),
 alpha ⊆ beta → beta ⊆ gamma → alpha ⊆ gamma.
Axiom inc_trans : axiom4.
```

Axiom 5 (inc_antisym) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \alpha \Rightarrow \alpha = \beta.$$

```
Definition axiom5 := ∀ (A B : eqType)(alpha beta : Rel A B),
 alpha ⊆ beta → beta ⊆ alpha → alpha = beta.
Axiom inc_antisym : axiom5.
```

Axiom 6 (inc_empty_alpha) *Let $\alpha : A \rightharpoondown B$. Then,*

$$\phi_{AB} \sqsubseteq \alpha.$$

```
Definition axiom6 := ∀ (A B : eqType)(alpha : Rel A B), φ A B ⊆ alpha.
Axiom inc_empty_alpha : axiom6.
```

Axiom 7 (inc_alpha_universal) *Let $\alpha : A \rightharpoonup B$. Then,*

$$\alpha \sqsubseteq \nabla_{AB}.$$

```
Definition axiom7 := ∀ (A B : eqType)(alpha : Rel A B), alpha ⊆ ∇ A B.
Axiom inc_alpha_universal : axiom7.
```

Axiom 8 (inc_capP, inc_cap)

1. **inc_capP :** *Let $\alpha : A \rightharpoonup B$, $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$\alpha \sqsubseteq (\sqcap_{P(\beta)} f(\beta)) \Leftrightarrow \forall \beta : C \rightharpoonup D, P(\beta) \Rightarrow \alpha \sqsubseteq f(\beta).$$

2. **inc_cap :** *Let $\alpha, \beta, \gamma : A \rightharpoonup B$. Then,*

$$\alpha \sqsubseteq (\beta \sqcap \gamma) \Leftrightarrow \alpha \sqsubseteq \beta \wedge \alpha \sqsubseteq \gamma.$$

```
Definition axiom8a :=
 ∀ (A B C D : eqType)
 (alpha : Rel A B)(f : Rel C D → Rel A B)(P : Rel C D → Prop),
 alpha ⊆ (∩_{P} f) ↔ ∀ beta : Rel C D, P beta → alpha ⊆ f beta.
Axiom inc_capP : axiom8a.
Definition axiom8b := ∀ (A B : eqType)(alpha beta gamma : Rel A B),
 alpha ⊆ (beta ∩ gamma) ↔ (alpha ⊆ beta) ∧ (alpha ⊆ gamma).
Lemma inc_cap : axiom8b.
Proof.
move ⇒ A B alpha beta gamma.
rewrite inc_capP.
split; move ⇒ H.
split; apply H.
by [left].
by [right].
move ⇒ delta H0.
case H0 ⇒ H1; rewrite H1; apply H.
Qed.
```

Axiom 9 (inc_cupP, inc_cup)

1. **inc_cupP :** *Let* $\alpha : A \rightharpoondown B$, $f : (C \rightharpoondown D) \to (A \rightharpoondown B)$ *and* P : *predicate. Then,*

$$(\sqcup_{P(\beta)} f(\beta)) \sqsubseteq \alpha \Leftrightarrow \forall \beta : C \rightharpoondown D, P(\beta) \Rightarrow f(\beta) \sqsubseteq \alpha.$$

2. **inc_cup :** *Let* $\alpha, \beta, \gamma : A \rightharpoondown B$. *Then,*

$$(\beta \sqcup \gamma) \sqsubseteq \alpha \Leftrightarrow \beta \sqsubseteq \alpha \wedge \gamma \sqsubseteq \alpha.$$

```
Definition axiom9a :=
 ∀ (A B C D : eqType)
 (alpha : Rel A B)(f : Rel C D → Rel A B)(P : Rel C D → Prop),
 (∪_{P} f) ⊆ alpha ↔ ∀ beta : Rel C D, P beta → f beta ⊆ alpha.
Axiom inc_cupP : axiom9a.
Definition axiom9b := ∀ (A B : eqType)(alpha beta gamma : Rel A B),
 (beta ∪ gamma) ⊆ alpha ↔ (beta ⊆ alpha) ∧ (gamma ⊆ alpha).
Lemma inc_cup : axiom9b.
Proof.
move ⇒ A B alpha beta gamma.
rewrite inc_cupP.
split; move ⇒ H.
split; apply H.
by [left].
by [right].
move ⇒ delta H0.
case H0 ⇒ H1; rewrite H1; apply H.
Qed.
```

Axiom 10 (inc_rpc) *Let* $\alpha, \beta, \gamma : A \rightharpoondown B$. *Then,*

$$\alpha \sqsubseteq (\beta \Rightarrow \gamma) \Leftrightarrow (\alpha \sqcap \beta) \sqsubseteq \gamma.$$

```
Definition axiom10 := ∀ (A B : eqType)(alpha beta gamma : Rel A B),
 alpha ⊆ (beta » gamma) ↔ (alpha ∩ beta) ⊆ gamma.
Axiom inc_rpc : axiom10.
```

Axiom 11 (inv_invol) *Let* $\alpha : A \rightharpoondown B$. *Then,*

$$(\alpha^\sharp)^\sharp = \alpha.$$

```
Definition axiom11 := ∀ (A B : eqType)(alpha : Rel A B), (alpha #) # = alpha.
Axiom inv_invol : axiom11.
```

Axiom 12 (comp_inv) *Let $\alpha : A \rightharpoondown B$ and $\beta : B \rightharpoondown C$. Then,*

$$(\alpha \cdot \beta)^\sharp = \beta^\sharp \cdot \alpha^\sharp.$$

```
Definition axiom12 := ∀ (A B C : eqType)(alpha : Rel A B)(beta : Rel B C),
 (alpha ・ beta) # = (beta # ・ alpha #).
Axiom comp_inv : axiom12.
```

Axiom 13 (inc_inv) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \beta \Rightarrow \alpha^\sharp \sqsubseteq \beta^\sharp.$$

```
Definition axiom13 :=
 ∀ (A B : eqType)(alpha beta : Rel A B), alpha ⊆ beta → alpha # ⊆ beta #.
Axiom inc_inv : axiom13.
```

Axiom 14 (dedekind) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$, and $\gamma : A \rightharpoondown C$. Then,*

$$(\alpha \cdot \beta) \sqcap \gamma \sqsubseteq (\alpha \sqcap (\gamma \cdot \beta^\sharp)) \cdot (\beta \sqcap (\alpha^\sharp \cdot \gamma)).$$

```
Definition axiom14 :=
 ∀ (A B C : eqType)(alpha : Rel A B)(beta : Rel B C)(gamma : Rel A C),
 ((alpha ・ beta) ∩ gamma)
 ⊆ ((alpha ∩ (gamma ・ beta #)) ・ (beta ∩ (alpha # ・ gamma))).
Axiom dedekind : axiom14.
```

Axiom 15 (inc_residual) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$, and $\gamma : A \rightharpoondown C$. Then,*

$$\gamma \sqsubseteq (\alpha \rhd \beta) \Leftrightarrow \alpha^\sharp \cdot \gamma \sqsubseteq \beta.$$

```
Definition axiom15 :=
 ∀ (A B C : eqType)(alpha : Rel A B)(beta : Rel B C)(gamma : Rel A C),
 gamma ⊆ (alpha △ beta) ↔ (alpha # ・ gamma) ⊆ beta.
Axiom inc_residual : axiom15.
```

2.3.2 排中律

Dedekind 圏の公理のほかに, 以下の "排中律" を仮定すれば, 与えられる圏は Schröder 圏となり, Bool 代数の性質も満たされる. ちなみに剰余合成は補関係から定義可能なので, 本来 Schröder 圏には剰余合成に関する公理は存在しない.

Axiom 16 (complement_classic) *Let* $\alpha : A \rightharpoonup B$. *Then,*

$$\alpha \sqcup \alpha^{-} = \nabla_{AB}$$

```
Definition axiom16 := ∀ (A B : eqType)(alpha : Rel A B),
 alpha ∪ alpha ^ = ∇ A B.
Axiom complement_classic : axiom16.
```

2.3.3 単域

1 点集合 I が単域となるための条件は

$$\phi_{II} \neq id_I \wedge id_I = \nabla_{II} \wedge \forall X, \nabla_{XI} \cdot \nabla_{IX} = \nabla_{XX}$$

だが, Rel の定義から左 2 つは証明できるため, 右の式だけ仮定する.

Axiom 17 (unit_universal)

$$\nabla_{AI} \cdot \nabla_{IA} = \nabla_{AA}$$

```
Definition axiom17 := ∀ (A : eqType), ∇ A i • ∇ i A = ∇ A A.
Axiom unit_universal : axiom17.
```

2.3.4 弱選択公理

この "弱選択公理" を仮定すれば, 排中律と単域の存在 (厳密には全域性公理) を利用して点公理を導出できる.

Axiom 18 (weak_axiom_of_choice) *Let* $\alpha : I \rightharpoonup A$ *be a total relation. Then,*

$$\exists \beta : I \to A, \beta \sqsubseteq \alpha.$$

```
Definition axiom18 := ∀ (A : eqType)(alpha : Rel i A),
 total_r alpha → ∃ beta : Rel i A, function_r beta ∧ beta ⊆ alpha.
Axiom weak_axiom_of_choice : axiom18.
```

2.3.5 関係の有理性

集合論では色々インポートしながら頑張って証明したので, できればそちらもご覧ください.

Axiom 19 (rationality) *Let* $\alpha : A \rightharpoonup B$*. Then,*

$$\exists R, \exists f : R \to A, \exists g : R \to B, \alpha = f^\sharp \cdot g \wedge f \cdot f^\sharp \sqcap g \cdot g^\sharp = id_R.$$

```
Definition axiom19 := ∀ (A B : eqType)(alpha : Rel A B),
 ∃ (R : eqType)(f : Rel R A)(g : Rel R B),
 function_r f ∧ function_r g ∧ alpha = f # ・ g ∧ ((f ・ f #) ∩ (g ・ g #)) = Id R.
Axiom rationality : axiom19.
```

2.3.6 直和と直積

任意の直和に対して, 入射対が存在することを仮定する.

Axiom 20 (pair_of_inclusions) $\exists j : A \to A + B, \exists k : B \to A + B,$

$$j \cdot j^\sharp = id_A \wedge k \cdot k^\sharp = id_B \wedge j \cdot k^\sharp = \phi_{AB} \wedge j^\sharp \cdot j \sqcup k^\sharp \cdot k = id_{A+B}.$$

```
Definition axiom20 :=
 ∀ (A B : eqType), ∃ (j : Rel A (sum_eqType A B))(k : Rel B (sum_eqType A B)),
 j ・ j # = Id A ∧ k ・ k # = Id B ∧ j ・ k # = φ A B ∧
 (j # ・ j) ∪ (k # ・ k) = Id (sum_eqType A B).
Axiom pair_of_inclusions : axiom20.
```

任意の直積に対して, 射影対が存在することを仮定する.

Axiom 21 (pair_of_projections) $\exists p : A \times B \to A, \exists q : A \times B \to B,$

$$p^\sharp \cdot q = \nabla_{AB} \wedge p \cdot p^\sharp \sqcap q \cdot q^\sharp = id_{A\times B}.$$

```
Definition axiom21 :=
 ∀ (A B : eqType), ∃ (p : Rel (prod_eqType A B) A)(q : Rel (prod_eqType A B) B),
 p # ・ q = ∇ A B ∧ (p ・ p #) ∩ (q ・ q #) = Id (prod_eqType A B) ∧ univalent_r p
∧ univalent_r q.
Axiom pair_of_projections : axiom21.
```

Chapter 3

Library Basic_Notations_Set

```
Require Export ssreflect eqtype bigop.
Require Export Logic.ClassicalFacts.
Require Import Logic.FunctionalExtensionality.
Require Import Logic.Classical_Prop.
Require Import Logic.IndefiniteDescription.
Require Import Logic.ProofIrrelevance.
```

```
Axiom prop_extensionality_ok : prop_extensionality.
```

3.1 定義

この章では, 関係を集合論的に定義した場合の定義, およびその定義で諸公理が成立することを示す. 公理名や記号などは Basic_Notations と同じものを使用するため, Basic_Lemms 以降ではそれの代わりにこのライブラリをインポートすることもできる.

```
Definition Rel (A B : eqType) := A → B → Prop.
```

```
Definition inverse {A B : eqType} (alpha : Rel A B) : Rel B A
 := (fun (b : B)(a : A) ⇒ alpha a b).
Notation "a #" := (inverse a) (at level 20).
Definition composite {A B C : eqType} (alpha : Rel A B) (beta : Rel B C) : Rel A C
 := (fun (a : A)(c : C) ⇒ ∃ b : B, alpha a b ∧ beta b c).
Notation "a '・' b" := (composite a b) (at level 50).
Definition residual {A B C : eqType} (alpha : Rel A B) (beta : Rel B C) : Rel A C
 := (fun (a : A)(c : C) ⇒ ∀ b : B, alpha a b → beta b c).
Notation "a '△' b" := (residual a b) (at level 50).
```

```
Definition identity (A : eqType) : Rel A A := (fun a a0 : A ⇒ a = a0).
Notation "'Id'" := identity.
Definition empty (A B : eqType) : Rel A B := (fun (a : A)(b : B) ⇒ False).
Notation "'φ'" := empty.
Definition universal (A B : eqType) : Rel A B := (fun (a : A)(b : B) ⇒ True).
```

```
Notation "' ∇'" := universal.

Definition include {A B : eqType} (alpha beta : Rel A B) : Prop
 := (∀ (a : A)(b : B), alpha a b → beta a b).
Notation "a ' ⊆' b" := (include a b) (at level 50).

Definition cupP {A B C D : eqType} (P : Rel C D → Prop) (f : Rel C D → Rel A B)
: Rel A B
 := (fun (a : A)(b : B) ⇒ ∃ alpha : Rel C D, P alpha ∧ (f alpha) a b).
Notation "' ∪_{' p '}' f" := (cupP p f) (at level 50).
Definition capP {A B C D : eqType} (P : Rel C D → Prop) (f : Rel C D → Rel A B) :
Rel A B
 := (fun (a : A)(b : B) ⇒ ∀ alpha : Rel C D, P alpha → (f alpha) a b).
Notation "' ∩_{' p '}' f" := (capP p f) (at level 50).

Definition cup {A B : eqType} (alpha beta : Rel A B)
 := ∪_{fun gamma : Rel A B ⇒ gamma = alpha ∨ gamma = beta} id.
Notation "a ' ∪' b" := (cup a b) (at level 50).
Definition cap {A B : eqType} (alpha beta : Rel A B)
 := ∩_{fun gamma : Rel A B ⇒ gamma = alpha ∨ gamma = beta} id.
Notation "a ' ∩' b" := (cap a b) (at level 50).
Definition rpc {A B : eqType} (alpha beta : Rel A B) : Rel A B
 := (fun (a : A)(b : B) ⇒ alpha a b → beta a b).
Notation "a '»' b" := (rpc a b) (at level 50).
Definition complement {A B : eqType} (alpha : Rel A B) := alpha » ϕ A B.
Notation "a '^'" := (complement a) (at level 20).
Definition difference {A B : eqType} (alpha beta : Rel A B) := alpha ∩ beta ^.
Notation "a - b" := (difference a b) (at level 50).

Notation "'i'" := unit_eqType.
```

3.2 関数の定義

```
Definition total_r {A B : eqType} (alpha : Rel A B) := (Id A) ⊆ (alpha • alpha #).
Definition univalent_r {A B : eqType} (alpha : Rel A B) := (alpha # • alpha) ⊆ (Id B).
Definition function_r {A B : eqType} (alpha : Rel A B)
 := (total_r alpha) ∧ (univalent_r alpha).
Definition surjection_r {A B : eqType} (alpha : Rel A B)
 := (function_r alpha) ∧ (total_r (alpha #)).
Definition injection_r {A B : eqType} (alpha : Rel A B)
 := (function_r alpha) ∧ (univalent_r (alpha #)).
Definition bijection_r {A B : eqType} (alpha : Rel A B)
 := (function_r alpha) ∧ (total_r (alpha #)) ∧ (univalent_r (alpha #)).
```

3.3 関係の公理

今後の諸定理の証明は, 原則以下の公理群, およびそれらから導かれる補題のみを用いて行っていくことにする.

3.3.1 Dedekind 圏の公理

Lemma 1 (comp_id_l, comp_id_r) *Let* $\alpha : A \rightharpoondown B$. *Then,*

$$id_A \cdot \alpha = \alpha \cdot id_B = \alpha.$$

```
Definition axiom1a := ∀ (A B : eqType)(alpha : Rel A B), Id A • alpha = alpha.
Lemma comp_id_l : axiom1a.
Proof.
move ⇒ A B alpha.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ b.
apply prop_extensionality_ok.
split.
elim ⇒ a0.
elim ⇒ H H0.
rewrite H.
apply H0.
move ⇒ H.
∃ a.
split.
by [].
apply H.
Qed.
```

```
Definition axiom1b := ∀ (A B : eqType)(alpha : Rel A B), alpha • Id B = alpha.
Lemma comp_id_r : axiom1b.
Proof.
move ⇒ A B alpha.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ b.
apply prop_extensionality_ok.
split.
elim ⇒ b0.
```

```
elim ⇒ H H0.
rewrite -H0.
apply H.
move ⇒ H.
∃ b.
split.
apply H.
by [].
Qed.
```

Lemma 2 (comp_assoc) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$, and $\gamma : C \rightharpoondown D$. Then,*

$$(\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma).$$

```
Definition axiom2 :=
 ∀ (A B C D : eqType)(alpha : Rel A B)(beta : Rel B C)(gamma : Rel C D),
 (alpha • beta) • gamma = alpha • (beta • gamma).
Lemma comp_assoc : axiom2.
Proof.
move ⇒ A B C D alpha beta gamma.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ d.
apply prop_extensionality_ok.
split.
elim ⇒ c.
elim ⇒ H H0.
elim H ⇒ b H1.
∃ b.
split.
apply H1.
∃ c.
split.
apply H1.
apply H0.
elim ⇒ b.
elim ⇒ H.
elim ⇒ c H0.
∃ c.
split.
∃ b.
split.
```

```
apply H.
apply H0.
apply H0.
Qed.
```

Lemma 3 (inc_refl) *Let $\alpha : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \alpha.$$

```
Definition axiom3 := ∀ (A B : eqType)(alpha : Rel A B), alpha ⊆ alpha.
Lemma inc_refl : axiom3.
Proof.
by [rewrite /axiom3/include].
Qed.
```

Lemma 4 (inc_trans) *Let $\alpha, \beta, \gamma : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \gamma \Rightarrow \alpha \sqsubseteq \gamma.$$

```
Definition axiom4 := ∀ (A B : eqType)(alpha beta gamma : Rel A B),
 alpha ⊆ beta → beta ⊆ gamma → alpha ⊆ gamma.
Lemma inc_trans : axiom4.
Proof.
move ⇒ A B alpha beta gamma H H0 a b H1.
apply (H0 _ _ (H _ _ H1)).
Qed.
```

Lemma 5 (inc_antisym) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \beta \wedge \beta \sqsubseteq \alpha \Rightarrow \alpha = \beta.$$

```
Definition axiom5 := ∀ (A B : eqType)(alpha beta : Rel A B),
 alpha ⊆ beta → beta ⊆ alpha → alpha = beta.
Lemma inc_antisym : axiom5.
Proof.
move ⇒ A B alpha beta H H0.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ b.
apply prop_extensionality_ok.
split.
apply H.
```

```
apply H0.
Qed.
```

Lemma 6 (inc_empty_alpha) *Let $\alpha : A \rightharpoonup B$. Then,*

$$\phi_{AB} \sqsubseteq \alpha.$$

```
Definition axiom6 := ∀ (A B : eqType)(alpha : Rel A B), φ A B ⊆ alpha.
Lemma inc_empty_alpha : axiom6.
Proof.
move ⇒ A B alpha a b.
apply False_ind.
Qed.
```

Lemma 7 (inc_alpha_universal) *Let $\alpha : A \rightharpoonup B$. Then,*

$$\alpha \sqsubseteq \nabla_{AB}.$$

```
Definition axiom7 := ∀ (A B : eqType)(alpha : Rel A B), alpha ⊆ ∇ A B.
Lemma inc_alpha_universal : axiom7.
Proof.
move ⇒ A B alpha a b H.
apply I.
Qed.
```

Lemma 8 (inc_capP, inc_cap)

1. **inc_capP :** *Let $\alpha : A \rightharpoonup B$, $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$\alpha \sqsubseteq (\sqcap_{P(\beta)} f(\beta)) \Leftrightarrow \forall \beta : C \rightharpoonup D, P(\beta) \Rightarrow \alpha \sqsubseteq f(\beta).$$

2. **inc_cap :** *Let $\alpha, \beta, \gamma : A \rightharpoonup B$. Then,*

$$\alpha \sqsubseteq (\beta \sqcap \gamma) \Leftrightarrow \alpha \sqsubseteq \beta \wedge \alpha \sqsubseteq \gamma.$$

```
Definition axiom8a :=
 ∀ (A B C D : eqType)
 (alpha : Rel A B)(f : Rel C D → Rel A B)(P : Rel C D → Prop),
 alpha ⊆ (∩_{P} f) ↔ ∀ beta : Rel C D, P beta → alpha ⊆ f beta.
Lemma inc_capP : axiom8a.
Proof.
move ⇒ A B C D alpha f P.
split; move ⇒ H.
```

```
move ⇒ beta H0 a b H1.
apply (H _ _ H1 _ H0).
move ⇒ a b H0 beta H1.
apply (H _ H1 _ _ H0).
Qed.
Definition axiom8b := ∀ (A B : eqType)(alpha beta gamma : Rel A B),
 alpha ⊆ (beta ∩ gamma) ↔ (alpha ⊆ beta) ∧ (alpha ⊆ gamma).
Lemma inc_cap : axiom8b.
Proof.
move ⇒ A B alpha beta gamma.
rewrite inc_capP.
split; move ⇒ H.
split; apply H.
by [left].
by [right].
move ⇒ delta H0.
case H0 ⇒ H1; rewrite H1; apply H.
Qed.
```

Lemma 9 (inc_cupP, inc_cup)

1. **inc_cupP :** *Let* $\alpha : A \rightharpoonup B$, $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ *and* P *: predicate. Then,*
$$(\sqcup_{P(\beta)} f(\beta)) \sqsubseteq \alpha \Leftrightarrow \forall \beta : C \rightharpoonup D, P(\beta) \Rightarrow f(\beta) \sqsubseteq \alpha.$$

2. **inc_cup :** *Let* $\alpha, \beta, \gamma : A \rightharpoonup B$*. Then,*
$$(\beta \sqcup \gamma) \sqsubseteq \alpha \Leftrightarrow \beta \sqsubseteq \alpha \wedge \gamma \sqsubseteq \alpha.$$

```
Definition axiom9a :=
 ∀ (A B C D : eqType)
 (alpha : Rel A B)(f : Rel C D → Rel A B)(P : Rel C D → Prop),
 (∪_{P} f) ⊆ alpha ↔ ∀ beta : Rel C D, P beta → f beta ⊆ alpha.
Lemma inc_cupP : axiom9a.
Proof.
move ⇒ A B C D alpha f P.
split.
move ⇒ H beta H0 a b H1.
apply H.
∃ beta.
split.
apply H0.
apply H1.
```

```
move => H a b.
elim => beta.
elim => H0 H1.
apply (H beta H0 _ _ H1).
Qed.
Definition axiom9b := ∀ (A B : eqType)(alpha beta gamma : Rel A B),
 (beta ∪ gamma) ⊆ alpha ↔ (beta ⊆ alpha) ∧ (gamma ⊆ alpha).
Lemma inc_cup : axiom9b.
Proof.
move => A B alpha beta gamma.
rewrite inc_cupP.
split; move => H.
split; apply H.
by [left].
by [right].
move => delta H0.
case H0 => H1; rewrite H1; apply H.
Qed.
```

Lemma 10 (inc_rpc) *Let* $\alpha, \beta, \gamma : A \rightharpoonup B$. *Then,*

$$\alpha \sqsubseteq (\beta \Rightarrow \gamma) \Leftrightarrow (\alpha \sqcap \beta) \sqsubseteq \gamma.$$

```
Definition axiom10 := ∀ (A B : eqType)(alpha beta gamma : Rel A B),
 alpha ⊆ (beta » gamma) ↔ (alpha ∩ beta) ⊆ gamma.
Lemma inc_rpc : axiom10.
Proof.
move => A B alpha beta gamma.
split; move => H.
move => a b H0.
apply H.
apply H0.
by [left].
apply H0.
by [right].
move => a b H0 H1.
apply H.
move => delta.
case => H2; rewrite H2.
apply H0.
apply H1.
Qed.
```

Lemma 11 (inv_invol) *Let $\alpha : A \rightharpoonup B$. Then,*

$$(\alpha^\sharp)^\sharp = \alpha.$$

```
Definition axiom11 := ∀ (A B : eqType)(alpha : Rel A B), (alpha #) # = alpha.
Lemma inv_invol : axiom11.
Proof.
by [move ⇒ A B alpha].
Qed.
```

Lemma 12 (comp_inv) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$. Then,*

$$(\alpha \cdot \beta)^\sharp = \beta^\sharp \cdot \alpha^\sharp.$$

```
Definition axiom12 := ∀ (A B C : eqType)(alpha : Rel A B)(beta : Rel B C),
 (alpha • beta) # = (beta # • alpha #).
Lemma comp_inv : axiom12.
Proof.
move ⇒ A B C alpha beta.
apply functional_extensionality.
move ⇒ c.
apply functional_extensionality.
move ⇒ a.
apply prop_extensionality_ok.
split; elim ⇒ b.
elim ⇒ H H0.
∃ b.
split.
apply H0.
apply H.
elim ⇒ H H0.
∃ b.
split.
apply H0.
apply H.
Qed.
```

Lemma 13 (inc_inv) *Let $\alpha, \beta : A \rightharpoonup B$. Then,*

$$\alpha \sqsubseteq \beta \Rightarrow \alpha^\sharp \sqsubseteq \beta^\sharp.$$

```
Definition axiom13 :=
 ∀ (A B : eqType)(alpha beta : Rel A B), alpha ⊆ beta → alpha # ⊆ beta #.
```

```
Lemma inc_inv : axiom13.
Proof.
move ⇒ A B alpha beta H b a H0.
apply (H _ _ H0).
Qed.
```

Lemma 14 (dedekind) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$, and $\gamma : A \rightharpoondown C$. Then,*

$$(\alpha \cdot \beta) \sqcap \gamma \sqsubseteq (\alpha \sqcap (\gamma \cdot \beta^\sharp)) \cdot (\beta \sqcap (\alpha^\sharp \cdot \gamma)).$$

```
Definition axiom14 :=
 ∀ (A B C : eqType)(alpha : Rel A B)(beta : Rel B C)(gamma : Rel A C),
 ((alpha ・ beta) ∩ gamma)
 ⊆ ((alpha ∩ (gamma ・ beta #)) ・ (beta ∩ (alpha # ・ gamma))).
Lemma dedekind : axiom14.
Proof.
move ⇒ A B C alpha beta gamma a c H.
assert (∃ b : B, alpha a b ∧ beta b c).
apply H.
by [left].
elim H0 ⇒ b.
elim ⇒ H1 H2.
∃ b.
repeat split.
move ⇒ delta H3.
case H3 ⇒ H4; rewrite H4.
apply H1.
unfold id.
∃ c.
split.
apply H.
by [right].
apply H2.
move ⇒ delta H3.
case H3 ⇒ H4; rewrite H4.
apply H2.
unfold id.
∃ a.
split.
apply H1.
apply H.
by [right].
Qed.
```

Lemma 15 (inc_residual) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$, and $\gamma : A \rightharpoondown C$. Then,*

$$\gamma \sqsubseteq (\alpha \rhd \beta) \Leftrightarrow \alpha^\sharp \cdot \gamma \sqsubseteq \beta.$$

```
Definition axiom15 :=
 ∀ (A B C : eqType)(alpha : Rel A B)(beta : Rel B C)(gamma : Rel A C),
 gamma ⊆ (alpha △ beta) ↔ (alpha # • gamma) ⊆ beta.
Lemma inc_residual : axiom15.
Proof.
move ⇒ A B C alpha beta gamma.
split; move ⇒ H.
move ⇒ b c.
elim ⇒ a H0.
apply (H a).
apply H0.
apply H0.
move ⇒ a c H0 b H1.
apply H.
∃ a.
split.
apply H1.
apply H0.
Qed.
```

3.3.2 排中律

Dedekind 圏の公理のほかに, 以下の “排中律” を仮定すれば, 与えられる圏は Schröder 圏となり, Bool 代数の性質も満たされる. ちなみに剰余合成は補関係から定義可能なので, 本来 Schröder 圏には剰余合成に関する公理は存在しない.

Lemma 16 (complement_classic) *Let $\alpha : A \rightharpoondown B$. Then,*

$$\alpha \sqcup \alpha^- = \nabla_{AB}$$

```
Definition axiom16 := ∀ (A B : eqType)(alpha : Rel A B),
 alpha ∪ alpha ^ = ∇ A B.
Lemma complement_classic : axiom16.
Proof.
move ⇒ A B alpha.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
```

```
move ⇒ b.
apply prop_extensionality_ok.
split; move ⇒ H.
apply I.
case (classic (alpha a b)) ⇒ H0.
∃ alpha.
split.
by [left].
apply H0.
∃ (fun (a0 : A) (b0 : B) ⇒ alpha a0 b0 → False).
split.
by [right].
apply H0.
Qed.
```

3.3.3 単域

1 点集合 I が単域となるための条件は

$$\phi_{II} \neq id_I \wedge id_I = \nabla_{II} \wedge \forall X, \nabla_{XI} \cdot \nabla_{IX} = \nabla_{XX}$$

だが, Rel の定義から左 2 つは証明できるため, 右の式だけ仮定する.

Lemma 17 (unit_universal)

$$\nabla_{AI} \cdot \nabla_{IA} = \nabla_{AA}$$

```
Definition axiom17 := ∀ (A : eqType), ∇ A i • ∇ i A = ∇ A A.
Lemma unit_universal : axiom17.
Proof.
move ⇒ A.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ a0.
apply prop_extensionality_ok.
split; move ⇒ H.
apply I.
∃ tt.
by [].
Qed.
```

3.3.4 弱選択公理

この “弱選択公理” を仮定すれば, 排中律と単域の存在 (厳密には全域性公理) を利用して点公理を導出できる.

Lemma 18 (weak_axiom_of_choice) *Let $\alpha : I \rightharpoonup A$ be a total relation. Then,*

$$\exists \beta : I \to A, \beta \sqsubseteq \alpha.$$

```
Definition axiom18 := ∀ (A : eqType)(alpha : Rel i A),
 total_r alpha → ∃ beta : Rel i A, function_r beta ∧ beta ⊆ alpha.
Lemma weak_axiom_of_choice : axiom18.
Proof.
move ⇒ A alpha.
rewrite /function_r/total_r/univalent_r/identity/include/composite/inverse.
move ⇒ H.
move : (H tt tt (Logic.eq_refl tt)).
elim ⇒ a H0.
∃ (fun (_ : i)(a0 : A) ⇒ a = a0).
repeat split.
move ⇒ tt tt0 H1.
by [∃ a].
move ⇒ a0 a1.
elim ⇒ tt0.
elim ⇒ H1 H2.
by [rewrite -H1 -H2].
induction a0.
move ⇒ a0 H1.
rewrite -H1.
apply H0.
Qed.
```

3.3.5 関係の有理性

集合の選択公理 (`Logic.IndefiniteDescription`) や証明の一意性 (`Logic.ProofIrrelevance`) を仮定すれば, 集合論上ならごり押しで証明できる.
旧ライブラリの頃は無理だと諦めて Axiom を追加していたが, Standard Library のインポートだけで解けた. 正直びっくり.

Lemma 19 (rationality) *Let $\alpha : A \rightharpoonup B$. Then,*

$$\exists R, \exists f : R \to A, \exists g : R \to B, \alpha = f^\sharp \cdot g \wedge f \cdot f^\sharp \sqcap g \cdot g^\sharp = id_R.$$

この付近は, ごり押しのための補題. 命題の真偽を選択公理で bool 値に変換したり, 部分集合の元から上位集合の元を生成する `sval` (`proj1_sig`) の単射性を示したりしている.

```
Lemma is_true_inv0 : ∀ P : Prop, ∃ b : bool, P ↔ is_true b.
Proof.
move ⇒ P.
case (classic P); move ⇒ H.
∃ true.
split; move ⇒ H0.
by [].
apply H.
∃ false.
split; move ⇒ H0.
apply False_ind.
apply (H H0).
discriminate H0.
Qed.
Definition is_true_inv : Prop → bool.
move ⇒ P.
move : (is_true_inv0 P) ⇒ H.
apply constructive_indefinite_description in H.
apply H.
Defined.
Lemma is_true_id : ∀ P : Prop, is_true (is_true_inv P) ↔ P.
Proof.
move ⇒ P.
unfold is_true_inv.
move : (constructive_indefinite_description (fun b : bool ⇒ P ↔ is_true b) (is_true_inv0
P)) ⇒ x0.
apply (@sig_ind bool (fun b ⇒ (P ↔ is_true b)) (fun y ⇒ is_true (let (x, _) := y in x)
↔ P)).
move ⇒ x H.
apply iff_sym.
apply H.
Qed.
Lemma sval_inv : ∀ (A : Type)(P : A → Prop)(x : sig P)(a : A), a = sval x → ∃ (H : P a),
x = exist P a H.
Proof.
move ⇒ A P x a H0.
rewrite H0.
∃ (proj2_sig x).
apply (@sig_ind A P (fun y ⇒ y = exist P (sval y) (proj2_sig y))).
```

```
move ⇒ a0 H.
by [simpl].
Qed.
Lemma sval_injective : ∀ (A : Type)(P : A → Prop)(x y : sig P), sval x = sval y → x = y.
Proof.
move ⇒ A P x y H.
move : (sval_inv A P y (sval x) H).
elim ⇒ H0 H1.
rewrite H1.
assert (H0 = proj2_sig x).
apply proof_irrelevance.
rewrite H2.
apply (@sig_ind A P (fun y ⇒ y = exist P (sval y) (proj2_sig y))).
move ⇒ a0 H3.
by [simpl].
Qed.
```

```
Definition axiom19 := ∀ (A B : eqType)(alpha : Rel A B),
 ∃ (R : eqType)(f : Rel R A)(g : Rel R B),
 function_r f ∧ function_r g ∧ alpha = f # • g ∧ ((f • f #) ∩ (g • g #)) = Id R.
Lemma rationality : axiom19.
Proof.
move ⇒ A B alpha.
rewrite /function_r/total_r/univalent_r/cap/capP/identity/composite/inverse/include.
∃ (sig_eqType (fun x : prod_eqType A B ⇒ is_true_inv (alpha (fst x) (snd x)))).
∃ (fun x a ⇒ a = (fst (sval x))).
∃ (fun x b ⇒ b = (snd (sval x))).
simpl.
repeat split.
move ⇒ x x0 H.
∃ (fst (sval x)).
repeat split.
by [rewrite H].
move ⇒ a a0.
elim ⇒ x.
elim ⇒ H H0.
by [rewrite H H0].
move ⇒ x x0 H.
∃ (snd (sval x)).
repeat split.
by [rewrite H].
move ⇒ b b0.
```

```
elim ⇒ x.
elim ⇒ H H0.
by [rewrite H H0].
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ b.
apply prop_extensionality_ok.
split; move ⇒ H.
assert (is_true (is_true_inv (alpha (fst (a,b)) (snd (a,b))))).
simpl.
apply is_true_id.
apply H.
∃ (exist (fun x ⇒ (is_true (is_true_inv (alpha (fst x) (snd x))))) (a,b) H0).
by [simpl].
elim H ⇒ x.
elim ⇒ H0 H1.
rewrite H0 H1.
apply is_true_id.
apply (@sig_ind (A × B) (fun x ⇒ is_true (is_true_inv (alpha (fst x) (snd x)))) (fun x
⇒ is_true (is_true_inv (alpha (fst (sval x)) (snd (sval x)))))).
simpl.
by [move ⇒ x0].
apply functional_extensionality.
move ⇒ y.
apply functional_extensionality.
move ⇒ y0.
apply prop_extensionality_ok.
split; move ⇒ H.
apply sval_injective.
move : (H (fun a c : {x : A × B | is_true (is_true_inv (alpha (fst x) (snd x)))} ⇒ ∃ b :
A, b = fst (sval a) ∧ b = fst (sval c)) (or_introl Logic.eq_refl)).
move : (H (fun a c : {x : A × B | is_true (is_true_inv (alpha (fst x) (snd x)))} ⇒ ∃ b :
B, b = snd (sval a) ∧ b = snd (sval c)) (or_intror Logic.eq_refl)).
unfold id.
clear H.
elim ⇒ b.
elim ⇒ H H0.
elim ⇒ a.
elim ⇒ H1 H2.
rewrite (surjective_pairing (sval y0)) -H0 -H2 H H1.
apply surjective_pairing.
```

```
rewrite H.
move ⇒ beta H0.
case H0 ⇒ H1; rewrite H1; unfold id.
∃ (fst (sval y0)).
repeat split.
∃ (snd (sval y0)).
repeat split.
Qed.
```

3.3.6 直和と直積

任意の直和に対して, 入射対が存在することを仮定する.

Lemma 20 (pair_of_inclusions) $\exists j : A \to A+B, \exists k : B \to A+B,$

$$j \cdot j^\sharp = id_A \wedge k \cdot k^\sharp = id_B \wedge j \cdot k^\sharp = \phi_{AB} \wedge j^\sharp \cdot j \sqcup k^\sharp \cdot k = id_{A+B}.$$

```
Definition axiom20 :=
 ∀ (A B : eqType), ∃ (j : Rel A (sum_eqType A B))(k : Rel B (sum_eqType A B)),
 j • j # = Id A ∧ k • k # = Id B ∧ j • k # = ϕ A B ∧
 (j # • j) ∪ (k # • k) = Id (sum_eqType A B).
Lemma pair_of_inclusions : axiom20.
Proof.
move ⇒ A B.
∃ (fun (a : A)(x : sum_eqType A B) ⇒ x = inl a).
∃ (fun (b : B)(x : sum_eqType A B) ⇒ x = inr b).
repeat split.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ a0.
apply prop_extensionality_ok.
split; move ⇒ H.
elim H ⇒ x.
elim ⇒ H0 H1.
rewrite H0 in H1.
by [injection H1].
∃ (inl a).
repeat split.
by [rewrite H].
apply functional_extensionality.
move ⇒ b.
```

```
apply functional_extensionality.
move ⇒ b0.
apply prop_extensionality_ok.
split; move ⇒ H.
elim H ⇒ x.
elim ⇒ H0 H1.
rewrite H0 in H1.
by [injection H1].
∃ (inr b).
repeat split.
by [rewrite H].
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ b.
apply prop_extensionality_ok.
split; move ⇒ H.
elim H ⇒ x.
elim ⇒ H0 H1.
rewrite H0 in H1.
discriminate H1.
apply False_ind.
apply H.
apply functional_extensionality.
move ⇒ x.
apply functional_extensionality.
move ⇒ x0.
apply prop_extensionality_ok.
split.
elim ⇒ alpha.
elim ⇒ H0 H1.
case H0 ⇒ H2; rewrite H2 in H1.
elim H1 ⇒ a.
elim ⇒ H3 H4.
by [rewrite H3 H4].
elim H1 ⇒ b.
elim ⇒ H3 H4.
by [rewrite H3 H4].
assert ((∃ a : A, x = inl a) ∨ (∃ b : B, x = inr b)).
move : x.
apply sum_ind.
move ⇒ a.
```

```
left.
by [∃ a].
move ⇒ b.
right.
by [∃ b].
case H.
elim ⇒ a H0 H1.
∃ (fun x x0 ⇒ ∃ a0 : A, (x = inl a0 ∧ x0 = inl a0)).
split.
by [left].
∃ a.
by [rewrite -H1 H0].
elim ⇒ b H0 H1.
∃ (fun x x0 ⇒ ∃ b0 : B, (x = inr b0 ∧ x0 = inr b0)).
split.
by [right].
∃ b.
by [rewrite -H1 H0].
Qed.
```

任意の直積に対して, 射影対が存在することを仮定する.

Lemma 21 (pair_of_projections) $\exists p : A \times B \to A, \exists q : A \times B \to B,$

$$p^\sharp \cdot q = \nabla_{AB} \wedge p \cdot p^\sharp \sqcap q \cdot q^\sharp = id_{A\times B}.$$

```
Definition axiom21 :=
 ∀ (A B : eqType), ∃ (p : Rel (prod_eqType A B) A)(q : Rel (prod_eqType A B) B),
 p # • q = ∇ A B ∧ (p • p #) ∩ (q • q #) = Id (prod_eqType A B) ∧ univalent_r p
∧ univalent_r q.
Lemma pair_of_projections : axiom21.
Proof.
move ⇒ A B.
∃ (fun (x : prod_eqType A B)(a : A) ⇒ a = (fst x)).
∃ (fun (x : prod_eqType A B)(b : B) ⇒ b = (snd x)).
split.
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ b.
apply prop_extensionality_ok.
split; move ⇒ H.
apply I.
```

```
∃ (a,b).
by [simpl].
split.
apply functional_extensionality.
move ⇒ x.
apply functional_extensionality.
move ⇒ x0.
apply prop_extensionality_ok.
split; move ⇒ H.
move : (H (fun a c : prod_eqType A B ⇒ ∃ b : A, b = fst a ∧ b = fst c) (or_introl
Logic.eq_refl)).
move : (H (fun a c : prod_eqType A B ⇒ ∃ b : B, b = snd a ∧ b = snd c) (or_intror
Logic.eq_refl)).
unfold id.
clear H.
elim ⇒ b.
elim ⇒ H H0.
elim ⇒ a.
elim ⇒ H1 H2.
rewrite (surjective_pairing x0) -H0 -H2 H H1.
apply surjective_pairing.
rewrite H.
move ⇒ alpha H0.
case H0 ⇒ H1; rewrite H1; unfold id.
∃ (fst x0).
repeat split.
∃ (snd x0).
repeat split.
split.
move ⇒ a a0.
elim ⇒ x.
elim ⇒ H H0.
by [rewrite H H0].
move ⇒ b b0.
elim ⇒ x.
elim ⇒ H H0.
by [rewrite H H0].
Qed.
```

Chapter 4

Library Basic_Lemmas

```
Require Import Basic_Notations.
Require Import Logic.Classical_Prop.
```

4.1 束論に関する補題

4.1.1 和関係, 共通関係

Lemma 22 (cap_l) *Let* $\alpha, \beta : A \rightharpoondown B$*. Then,*

$$\alpha \sqcap \beta \sqsubseteq \alpha.$$

```
Lemma cap_l {A B : eqType} {alpha beta : Rel A B}: (alpha ∩ beta) ⊆ alpha.
Proof.
assert ((alpha ∩ beta) ⊆ (alpha ∩ beta)).
apply inc_refl.
apply inc_cap in H.
apply H.
Qed.
```

Lemma 23 (cap_r) *Let* $\alpha, \beta : A \rightharpoondown B$*. Then,*

$$\alpha \sqcap \beta \sqsubseteq \beta.$$

```
Lemma cap_r {A B : eqType} {alpha beta : Rel A B}: (alpha ∩ beta) ⊆ beta.
Proof.
assert ((alpha ∩ beta) ⊆ (alpha ∩ beta)).
apply inc_refl.
apply inc_cap in H.
apply H.
```

```
Qed.
```

Lemma 24 (cup_l) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \alpha \sqcup \beta.$$

```
Lemma cup_l {A B : eqType} {alpha beta : Rel A B}: alpha ⊆ (alpha ∪ beta).
Proof.
assert ((alpha ∪ beta) ⊆ (alpha ∪ beta)).
apply inc_refl.
apply inc_cup in H.
apply H.
Qed.
```

Lemma 25 (cup_r) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\beta \sqsubseteq \alpha \sqcup \beta.$$

```
Lemma cup_r {A B : eqType} {alpha beta : Rel A B}: beta ⊆ (alpha ∪ beta).
Proof.
assert ((alpha ∪ beta) ⊆ (alpha ∪ beta)).
apply inc_refl.
apply inc_cup in H.
apply H.
Qed.
```

Lemma 26 (inc_def1) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha = \alpha \sqcap \beta \Leftrightarrow \alpha \sqsubseteq \beta.$$

```
Lemma inc_def1 {A B : eqType} {alpha beta : Rel A B}:
 alpha = alpha ∩ beta ↔ alpha ⊆ beta.
Proof.
split; move ⇒ H.
assert (alpha ⊆ (alpha ∩ beta)).
rewrite -H.
apply inc_refl.
apply inc_cap in H0.
apply H0.
apply inc_antisym.
apply inc_cap.
split.
apply inc_refl.
```

```
apply H.
apply cap_l.
Qed.
```

Lemma 27 (inc_def2) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\beta = \alpha \sqcup \beta \Leftrightarrow \alpha \sqsubseteq \beta.$$

```
Lemma inc_def2 {A B : eqType} {alpha beta : Rel A B}:
 beta = alpha ∪ beta ↔ alpha ⊆ beta.
Proof.
split; move ⇒ H.
assert ((alpha ∪ beta) ⊆ beta).
rewrite -H.
apply inc_refl.
apply inc_cup in H0.
apply H0.
apply inc_antisym.
assert ((alpha ∪ beta) ⊆ (alpha ∪ beta)).
apply inc_refl.
apply cup_r.
apply inc_cup.
split.
apply H.
apply inc_refl.
Qed.
```

Lemma 28 (cap_assoc) *Let $\alpha, \beta, \gamma : A \rightharpoondown B$. Then,*

$$(\alpha \sqcap \beta) \sqcap \gamma = \alpha \sqcap (\beta \sqcap \gamma).$$

```
Lemma cap_assoc {A B : eqType} {alpha beta gamma : Rel A B}:
 (alpha ∩ beta) ∩ gamma = alpha ∩ (beta ∩ gamma).
Proof.
apply inc_antisym.
rewrite inc_cap.
split.
apply (inc_trans _ _ _ (alpha ∩ beta)).
apply cap_l.
apply cap_l.
rewrite inc_cap.
split.
apply (inc_trans _ _ _ (alpha ∩ beta)).
```

```
apply cap_l.
apply cap_r.
apply cap_r.
rewrite inc_cap.
split.
rewrite inc_cap.
split.
apply cap_l.
apply (inc_trans _ _ _ (beta ∩ gamma)).
apply cap_r.
apply cap_l.
apply (inc_trans _ _ _ (beta ∩ gamma)).
apply cap_r.
apply cap_r.
Qed.
```

Lemma 29 (cup_assoc) *Let $\alpha, \beta, \gamma : A \rightharpoondown B$. Then,*

$$(\alpha \sqcup \beta) \sqcup \gamma = \alpha \sqcup (\beta \sqcup \gamma).$$

```
Lemma cup_assoc {A B : eqType} {alpha beta gamma : Rel A B}:
 (alpha ∪ beta) ∪ gamma = alpha ∪ (beta ∪ gamma).
Proof.
apply inc_antisym.
rewrite inc_cup.
split.
rewrite inc_cup.
split.
apply cup_l.
apply (inc_trans _ _ _ (beta ∪ gamma)).
apply cup_l.
apply cup_r.
apply (inc_trans _ _ _ (beta ∪ gamma)).
apply cup_r.
apply cup_r.
rewrite inc_cup.
split.
apply (inc_trans _ _ _ (alpha ∪ beta)).
apply cup_l.
apply cup_l.
rewrite inc_cup.
split.
apply (inc_trans _ _ _ (alpha ∪ beta)).
```

```
apply cup_r.
apply cup_l.
apply cup_r.
Qed.
```

Lemma 30 (cap_comm) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha \sqcap \beta = \beta \sqcap \alpha.$$

```
Lemma cap_comm {A B : eqType} {alpha beta : Rel A B}: alpha ∩ beta = beta ∩ alpha.
Proof.
apply inc_antisym.
rewrite inc_cap.
split.
apply cap_r.
apply cap_l.
rewrite inc_cap.
split.
apply cap_r.
apply cap_l.
Qed.
```

Lemma 31 (cup_comm) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha \sqcup \beta = \beta \sqcup \alpha.$$

```
Lemma cup_comm {A B : eqType} {alpha beta : Rel A B}: alpha ∪ beta = beta ∪ alpha.
Proof.
apply inc_antisym.
rewrite inc_cup.
split.
apply cup_r.
apply cup_l.
rewrite inc_cup.
split.
apply cup_r.
apply cup_l.
Qed.
```

Lemma 32 (cup_cap_abs) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha \sqcup (\alpha \sqcap \beta) = \alpha.$$

```
Lemma cup_cap_abs {A B : eqType} {alpha beta : Rel A B}:
 alpha ∪ (alpha ∩ beta) = alpha.
Proof.
move : (@cap_l _ _ alpha beta) ⇒ H.
apply inc_def2 in H.
by [rewrite cup_comm -H].
Qed.
```

Lemma 33 (cap_cup_abs) *Let* $\alpha, \beta : A \rightharpoondown B$*. Then,*

$$\alpha \sqcap (\alpha \sqcup \beta) = \alpha.$$

```
Lemma cap_cup_abs {A B : eqType} {alpha beta : Rel A B}:
 alpha ∩ (alpha ∪ beta) = alpha.
Proof.
move : (@cup_l _ _ alpha beta) ⇒ H.
apply inc_def1 in H.
by [rewrite -H].
Qed.
```

Lemma 34 (cap_idem) *Let* $\alpha : A \rightharpoondown B$*. Then,*

$$\alpha \sqcap \alpha = \alpha.$$

```
Lemma cap_idem {A B : eqType} {alpha : Rel A B}: alpha ∩ alpha = alpha.
Proof.
apply inc_antisym.
apply cap_l.
apply inc_cap.
split; apply inc_refl.
Qed.
```

Lemma 35 (cup_idem) *Let* $\alpha : A \rightharpoondown B$*. Then,*

$$\alpha \sqcup \alpha = \alpha.$$

```
Lemma cup_idem {A B : eqType} {alpha : Rel A B}: alpha ∪ alpha = alpha.
Proof.
apply inc_antisym.
apply inc_cup.
split; apply inc_refl.
apply cup_l.
Qed.
```

Lemma 36 (cap_inc_compat) *Let* $\alpha, \alpha', \beta, \beta' : A \rightharpoondown B$. *Then,*

$$\alpha \sqsubseteq \alpha' \land \beta \sqsubseteq \beta' \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha' \sqcap \beta'.$$

```
Lemma cap_inc_compat {A B : eqType} {alpha alpha' beta beta' : Rel A B}:
 alpha ⊆ alpha' → beta ⊆ beta' → (alpha ∩ beta) ⊆ (alpha' ∩ beta').
Proof.
move ⇒ H H0.
rewrite -inc_def1.
apply inc_def1 in H.
apply inc_def1 in H0.
rewrite cap_assoc -(@cap_assoc _ _ beta).
rewrite (@cap_comm _ _ beta).
rewrite cap_assoc -(@cap_assoc _ _ alpha).
by [rewrite -H -H0].
Qed.
```

Lemma 37 (cap_inc_compat_l) *Let* $\alpha, \beta, \beta' : A \rightharpoondown B$. *Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha \sqcap \beta'.$$

```
Lemma cap_inc_compat_l {A B : eqType} {alpha beta beta' : Rel A B}:
 beta ⊆ beta' → (alpha ∩ beta) ⊆ (alpha ∩ beta').
Proof.
move ⇒ H.
apply (@cap_inc_compat _ _ _ _ _ _ (@inc_refl _ _ alpha) H).
Qed.
```

Lemma 38 (cap_inc_compat_r) *Let* $\alpha, \alpha', \beta : A \rightharpoondown B$. *Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \sqcap \beta \sqsubseteq \alpha' \sqcap \beta.$$

```
Lemma cap_inc_compat_r {A B : eqType} {alpha alpha' beta : Rel A B}:
 alpha ⊆ alpha' → (alpha ∩ beta) ⊆ (alpha' ∩ beta).
Proof.
move ⇒ H.
apply (@cap_inc_compat _ _ _ _ _ _ H (@inc_refl _ _ beta)).
Qed.
```

Lemma 39 (cup_inc_compat) *Let* $\alpha, \alpha', \beta, \beta' : A \rightharpoondown B$. *Then,*

$$\alpha \sqsubseteq \alpha' \land \beta \sqsubseteq \beta' \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha' \sqcup \beta'.$$

```
Lemma cup_inc_compat {A B : eqType} {alpha alpha' beta beta' : Rel A B}:
 alpha ⊆ alpha' → beta ⊆ beta' → (alpha ∪ beta) ⊆ (alpha' ∪ beta').
Proof.
move ⇒ H H0.
rewrite -inc_def2.
apply inc_def2 in H.
apply inc_def2 in H0.
rewrite cup_assoc -(@cup_assoc _ _ beta).
rewrite (@cup_comm _ _ beta).
rewrite cup_assoc -(@cup_assoc _ _ alpha).
by [rewrite -H -H0].
Qed.
```

Lemma 40 (cup_inc_compat_l) *Let $\alpha, \beta, \beta' : A \rightharpoondown B$. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha \sqcup \beta'.$$

```
Lemma cup_inc_compat_l {A B : eqType} {alpha beta beta' : Rel A B}:
 beta ⊆ beta' → (alpha ∪ beta) ⊆ (alpha ∪ beta').
Proof.
move ⇒ H.
apply (@cup_inc_compat _ _ _ _ _ _ (@inc_refl _ _ alpha) H).
Qed.
```

Lemma 41 (cup_inc_compat_r) *Let $\alpha, \alpha', \beta : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \sqcup \beta \sqsubseteq \alpha' \sqcup \beta.$$

```
Lemma cup_inc_compat_r {A B : eqType} {alpha alpha' beta : Rel A B}:
 alpha ⊆ alpha' → (alpha ∪ beta) ⊆ (alpha' ∪ beta).
Proof.
move ⇒ H.
apply (@cup_inc_compat _ _ _ _ _ _ H (@inc_refl _ _ beta)).
Qed.
```

Lemma 42 (cap_empty) *Let $\alpha : A \rightharpoondown B$. Then,*

$$\alpha \sqcap \phi_{AB} = \phi_{AB}.$$

```
Lemma cap_empty {A B : eqType} {alpha : Rel A B}: alpha ∩ φ A B = φ A B.
Proof.
apply inc_antisym.
apply cap_r.
```

apply *inc_empty_alpha*.
Qed.

Lemma 43 (cup_empty) *Let* $\alpha : A \rightharpoondown B$. *Then,*

$$\alpha \sqcup \phi_{AB} = \alpha.$$

Lemma *cup_empty* {*A B* : *eqType*} {*alpha* : *Rel A B*}: *alpha* ∪ φ *A B* = *alpha*.
Proof.
apply *inc_antisym*.
apply *inc_cup*.
split.
apply *inc_refl*.
apply *inc_empty_alpha*.
apply *cup_l*.
Qed.

Lemma 44 (cap_universal) *Let* $\alpha : A \rightharpoondown B$. *Then,*

$$\alpha \sqcap \nabla_{AB} = \alpha.$$

Lemma *cap_universal* {*A B* : *eqType*} {*alpha* : *Rel A B*}: *alpha* ∩ ∇ *A B* = *alpha*.
Proof.
apply *inc_antisym*.
apply *cap_l*.
apply *inc_cap*.
split.
apply *inc_refl*.
apply *inc_alpha_universal*.
Qed.

Lemma 45 (cup_universal) *Let* $\alpha : A \rightharpoondown B$. *Then,*

$$\alpha \sqcup \nabla_{AB} = \nabla_{AB}.$$

Lemma *cup_universal* {*A B* : *eqType*} {*alpha* : *Rel A B*}: *alpha* ∪ ∇ *A B* = ∇ *A B*.
Proof.
apply *inc_antisym*.
apply *inc_cup*.
split.
apply *inc_alpha_universal*.
apply *inc_refl*.
apply *cup_r*.

```
Qed.
```

Lemma 46 (inc_lower) *Let* $\alpha, \beta : A \rightharpoonup B$*. Then,*

$$\alpha = \beta \Leftrightarrow (\forall \gamma : A \rightharpoonup B, \gamma \sqsubseteq \alpha \Leftrightarrow \gamma \sqsubseteq \beta).$$

```
Lemma inc_lower {A B : eqType} {alpha beta : Rel A B}:
 alpha = beta ↔ (∀ gamma : Rel A B, gamma ⊆ alpha ↔ gamma ⊆ beta).
Proof.
split; move ⇒ H.
move ⇒ gamma.
by [rewrite H].
apply inc_antisym.
rewrite -H.
apply inc_refl.
rewrite H.
apply inc_refl.
Qed.
```

Lemma 47 (inc_upper) *Let* $\alpha, \beta : A \rightharpoonup B$*. Then,*

$$\alpha = \beta \Leftrightarrow (\forall \gamma : A \rightharpoonup B, \alpha \sqsubseteq \gamma \Leftrightarrow \beta \sqsubseteq \gamma).$$

```
Lemma inc_upper {A B : eqType} {alpha beta : Rel A B}:
 alpha = beta ↔ (∀ gamma : Rel A B, alpha ⊆ gamma ↔ beta ⊆ gamma).
Proof.
split; move ⇒ H.
move ⇒ gamma.
by [rewrite H].
apply inc_antisym.
rewrite H.
apply inc_refl.
rewrite -H.
apply inc_refl.
Qed.
```

4.1.2 分配法則

Lemma 48 (cap_cup_distr_l) *Let* $\alpha, \beta, \gamma : A \rightharpoonup B$*. Then,*

$$\alpha \sqcap (\beta \sqcup \gamma) = (\alpha \sqcap \beta) \sqcup (\alpha \sqcap \gamma).$$

```
Lemma cap_cup_distr_l {A B : eqType} {alpha beta gamma : Rel A B}:
 alpha ∩ (beta ∪ gamma) = (alpha ∩ beta) ∪ (alpha ∩ gamma).
Proof.
apply inc_upper.
move => delta.
split; move => H.
rewrite cap_comm (@cap_comm _ _ _ gamma).
apply inc_cup.
rewrite -inc_rpc -inc_rpc.
apply inc_cup.
rewrite inc_rpc cap_comm.
apply H.
rewrite cap_comm -inc_rpc.
apply inc_cup.
rewrite inc_rpc inc_rpc.
apply inc_cup.
rewrite cap_comm (@cap_comm _ _ gamma).
apply H.
Qed.
```

Lemma 49 (cap_cup_distr_r) *Let* $\alpha, \beta, \gamma : A \rightharpoondown B$. *Then,*

$$(\alpha \sqcup \beta) \sqcap \gamma = (\alpha \sqcap \gamma) \sqcup (\beta \sqcap \gamma).$$

```
Lemma cap_cup_distr_r {A B : eqType} {alpha beta gamma : Rel A B}:
 (alpha ∪ beta) ∩ gamma = (alpha ∩ gamma) ∪ (beta ∩ gamma).
Proof.
rewrite (@cap_comm _ _ (alpha ∪ beta)) (@cap_comm _ _ alpha) (@cap_comm _ _ beta).
apply cap_cup_distr_l.
Qed.
```

Lemma 50 (cup_cap_distr_l) *Let* $\alpha, \beta, \gamma : A \rightharpoondown B$. *Then,*

$$\alpha \sqcup (\beta \sqcap \gamma) = (\alpha \sqcup \beta) \sqcap (\alpha \sqcup \gamma).$$

```
Lemma cup_cap_distr_l {A B : eqType} {alpha beta gamma : Rel A B}:
 alpha ∪ (beta ∩ gamma) = (alpha ∪ beta) ∩ (alpha ∪ gamma).
Proof.
rewrite cap_cup_distr_l.
rewrite (@cap_comm _ _ (alpha ∪ beta)) cap_cup_abs (@cap_comm _ _ (alpha ∪ beta)).
rewrite cap_cup_distr_l.
rewrite -cup_assoc (@cap_comm _ _ gamma) cup_cap_abs.
by [rewrite cap_comm].
```

```
Qed.
```

Lemma 51 (cup_cap_distr_r) *Let $\alpha, \beta, \gamma : A \rightharpoondown B$. Then,*

$$(\alpha \sqcap \beta) \sqcup \gamma = (\alpha \sqcup \gamma) \sqcap (\beta \sqcup \gamma).$$

```
Lemma cup_cap_distr_r {A B : eqType} {alpha beta gamma : Rel A B}:
 (alpha ∩ beta) ∪ gamma = (alpha ∪ gamma) ∩ (beta ∪ gamma).
Proof.
rewrite (@cup_comm _ _ (alpha ∩ beta)) (@cup_comm _ _ alpha) (@cup_comm _ _ beta).
apply cup_cap_distr_l.
Qed.
```

Lemma 52 (cap_cup_unique) *Let $\alpha, \beta, \gamma : A \rightharpoondown B$. Then,*

$$\alpha \sqcap \beta = \alpha \sqcap \gamma \wedge \alpha \sqcup \beta = \alpha \sqcup \gamma \Rightarrow \beta = \gamma.$$

```
Lemma cap_cup_unique {A B : eqType} {alpha beta gamma : Rel A B}:
 alpha ∩ beta = alpha ∩ gamma → alpha ∪ beta = alpha ∪ gamma → beta = gamma.
Proof.
move ⇒ H H0.
rewrite -(@cap_cup_abs _ _ beta alpha) cup_comm H0.
rewrite cap_cup_distr_l.
rewrite cap_comm H.
rewrite -cap_cup_distr_r.
rewrite H0 cap_comm cup_comm.
apply cap_cup_abs.
Qed.
```

4.1.3 原子性

空関係でない $\alpha : A \rightharpoondown B$ が, 任意の $\beta : A \rightharpoondown B$ について

$$\beta \sqsubseteq \alpha \Rightarrow \beta = \phi_{AB} \vee \beta = \alpha$$

を満たすとき, α は原子的 (atomic) であると言われる.

```
Definition atomic {A B : eqType} (alpha : Rel A B):=
 alpha ≠ φ A B ∧ (∀ beta : Rel A B, beta ⊆ alpha → beta = φ A B ∨ beta = alpha).
```

Lemma 53 (atomic_cap_empty) *Let $\alpha, \beta : A \rightharpoonup B$ are atomic and $\alpha \neq \beta$. Then,*

$$\alpha \sqcap \beta = \phi_{AB}.$$

```
Lemma atomic_cap_empty {A B : eqType} {alpha beta : Rel A B}:
  atomic alpha → atomic beta → alpha ≠ beta → alpha ∩ beta = φ A B.
Proof.
move ⇒ H H0.
apply or_to_imply.
case (classic (alpha ∩ beta = φ A B)); move ⇒ H1.
right.
apply H1.
left.
move ⇒ H2.
apply H2.
apply inc_antisym.
apply inc_def1.
elim H ⇒ H3 H4.
case (H4 (alpha ∩ beta) (cap_l)); move ⇒ H5.
apply False_ind.
apply (H1 H5).
by [rewrite H5].
apply inc_def1.
elim H0 ⇒ H3 H4.
case (H4 (alpha ∩ beta) (cap_r)); move ⇒ H5.
apply False_ind.
apply (H1 H5).
by [rewrite cap_comm H5].
Qed.
```

Lemma 54 (atomic_cup) *Let $\alpha, \beta, \gamma : A \rightharpoonup B$ and α is atomic. Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Rightarrow \alpha \sqsubseteq \beta \vee \alpha \sqsubseteq \gamma.$$

```
Lemma atomic_cup {A B : eqType} {alpha beta gamma : Rel A B}:
  atomic alpha → alpha ⊆ (beta ∪ gamma) → alpha ⊆ beta ∨ alpha ⊆ gamma.
Proof.
move ⇒ H H0.
apply inc_def1 in H0.
rewrite cap_cup_distr_l in H0.
elim H ⇒ H1 H2.
rewrite H0 in H1.
assert (alpha ∩ beta ≠ φ A B ∨ alpha ∩ gamma ≠ φ A B).
```

```
apply not_and_or.
elim => H3 H4.
rewrite H3 H4 in H1.
apply H1.
by [rewrite cup_empty].
case H3; move => H4.
left.
apply inc_def1.
case (H2 (alpha ∩ beta) (cap_l)); move => H5.
apply False_ind.
apply (H4 H5).
by [rewrite H5].
right.
apply inc_def1.
case (H2 (alpha ∩ gamma) (cap_l)); move => H5.
apply False_ind.
apply (H4 H5).
by [rewrite H5].
Qed.
```

4.2 Heyting 代数に関する補題

Lemma 55 (rpc_universal) *Let $\alpha : A \rightharpoonup B$. Then,*

$$(\alpha \Rightarrow \alpha) = \nabla_{AB}.$$

```
Lemma rpc_universal {A B : eqType} {alpha : Rel A B}: (alpha » alpha) = ∇ A B.
Proof.
apply inc_lower.
move => gamma.
split; move => H.
apply inc_alpha_universal.
apply inc_rpc.
apply cap_r.
Qed.
```

Lemma 56 (rpc_r) *Let $\alpha, \beta : A \rightharpoonup B$. Then,*

$$(\alpha \Rightarrow \beta) \sqcap \beta = \beta.$$

```
Lemma rpc_r {A B : eqType} {alpha beta : Rel A B}: (alpha » beta) ∩ beta = beta.
```

```
Proof.
assert (beta ⊆ (alpha » beta)).
apply inc_rpc.
apply cap_l.
apply inc_def1 in H.
by [rewrite cap_comm -H].
Qed.
```

Lemma 57 (inc_def3) *Let $\alpha, \beta : A \rightharpoonup B$. Then,*

$$(\alpha \Rightarrow \beta) = \nabla_{AB} \Leftrightarrow \alpha \sqsubseteq \beta.$$

```
Lemma inc_def3 {A B : eqType} {alpha beta : Rel A B}:
 (alpha » beta) = ∇ A B ↔ alpha ⊆ beta.
Proof.
split; move ⇒ H.
rewrite -(@rpc_universal _ _ alpha) in H.
assert ((alpha » alpha) ⊆ (alpha » beta)).
rewrite H.
apply inc_refl.
apply inc_rpc in H0.
rewrite rpc_r in H0.
apply H0.
apply inc_antisym.
apply inc_alpha_universal.
rewrite -(@rpc_universal _ _ alpha).
apply inc_rpc.
rewrite rpc_r.
apply H.
Qed.
```

Lemma 58 (rpc_l) *Let $\alpha, \beta : A \rightharpoonup B$. Then,*

$$\alpha \sqcap (\alpha \Rightarrow \beta) = \alpha \sqcap \beta.$$

```
Lemma rpc_l {A B : eqType} {alpha beta : Rel A B}:
 alpha ∩ (alpha » beta) = alpha ∩ beta.
Proof.
apply inc_lower.
move ⇒ gamma.
split; move ⇒ H.
apply inc_cap.
apply inc_cap in H.
```

```
split.
apply H.
elim H => H0 H1.
apply inc_rpc in H1.
rewrite -(@cap_idem _ _ gamma).
apply (inc_trans _ _ _ (gamma ∩ alpha)).
apply cap_inc_compat.
apply inc_refl.
apply H0.
apply H1.
apply inc_cap.
apply inc_cap in H.
split.
apply H.
apply inc_rpc.
apply (inc_trans _ _ _ gamma).
apply cap_l.
apply H.
Qed.
```

Lemma 59 (rpc_inc_compat) *Let $\alpha, \alpha', \beta, \beta' : A \rightharpoonup B$. Then,*

$$\alpha' \sqsubseteq \alpha \wedge \beta \sqsubseteq \beta' \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha' \Rightarrow \beta').$$

```
Lemma rpc_inc_compat {A B : eqType} {alpha alpha' beta beta' : Rel A B}:
 alpha' ⊆ alpha → beta ⊆ beta' → (alpha » beta) ⊆ (alpha' » beta').
Proof.
move => H H0.
apply inc_rpc.
apply (@inc_trans _ _ _ ((alpha » beta) ∩ alpha)).
apply (@cap_inc_compat_l _ _ _ _ _ H).
rewrite cap_comm rpc_l.
apply (@inc_trans _ _ _ beta).
apply cap_r.
apply H0.
Qed.
```

Lemma 60 (rpc_inc_compat_l) *Let $\alpha, \beta, \beta' : A \rightharpoonup B$. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha \Rightarrow \beta').$$

```
Lemma rpc_inc_compat_l {A B : eqType} {alpha beta beta' : Rel A B}:
 beta ⊆ beta' → (alpha » beta) ⊆ (alpha » beta').
```

```
Proof.
move => H.
apply (@rpc_inc_compat _ _ _ _ _ _ (@inc_refl _ _ alpha) H).
Qed.
```

Lemma 61 (rpc_inc_compat_r) *Let $\alpha, \alpha', \beta : A \rightharpoondown B$. Then,*

$$\alpha' \sqsubseteq \alpha \Rightarrow (\alpha \Rightarrow \beta) \sqsubseteq (\alpha' \Rightarrow \beta).$$

```
Lemma rpc_inc_compat_r {A B : eqType} {alpha alpha' beta : Rel A B}:
 alpha' ⊆ alpha → (alpha » beta) ⊆ (alpha' » beta).
Proof.
move => H.
apply (@rpc_inc_compat _ _ _ _ _ _ H (@inc_refl _ _ beta)).
Qed.
```

Lemma 62 (rpc_universal_alpha) *Let $\alpha : A \rightharpoondown B$. Then,*

$$\nabla_{AB} \Rightarrow \alpha = \alpha.$$

```
Lemma rpc_universal_alpha {A B : eqType} {alpha : Rel A B}: ∇ A B » alpha = alpha.
Proof.
apply inc_lower.
move => gamma.
split; move => H.
apply inc_rpc in H.
rewrite cap_universal in H.
apply H.
apply inc_rpc.
rewrite cap_universal.
apply H.
Qed.
```

Lemma 63 (rpc_lemma1) *Let $\alpha, \beta, \gamma : A \rightharpoondown B$. Then,*

$$(\alpha \Rightarrow \beta) \sqsubseteq ((\alpha \sqcap \gamma) \Rightarrow (\beta \sqcap \gamma)).$$

```
Lemma rpc_lemma1 {A B : eqType} {alpha beta gamma : Rel A B}:
 (alpha » beta) ⊆ ((alpha ∩ gamma) » (beta ∩ gamma)).
Proof.
apply inc_rpc.
rewrite -cap_assoc (@cap_comm _ _ _ alpha).
rewrite rpc_l.
```

```
apply cap_inc_compat_r.
apply cap_r.
Qed.
```

Lemma 64 (rpc_lemma2) *Let* $\alpha, \beta, \gamma : A \rightharpoondown B$. *Then,*

$$(\alpha \Rightarrow \beta) \sqcap (\alpha \Rightarrow \gamma) = (\alpha \Rightarrow (\beta \sqcap \gamma)).$$

```
Lemma rpc_lemma2 {A B : eqType} {alpha beta gamma : Rel A B}:
 (alpha » beta) ∩ (alpha » gamma) = alpha » (beta ∩ gamma).
Proof.
apply inc_lower.
move ⇒ delta.
split; move ⇒ H.
rewrite inc_rpc.
apply inc_cap in H.
apply inc_cap.
rewrite -inc_rpc -inc_rpc.
apply H.
apply inc_cap.
rewrite inc_rpc inc_rpc.
apply inc_cap.
rewrite -inc_rpc.
apply H.
Qed.
```

Lemma 65 (rpc_lemma3) *Let* $\alpha, \beta, \gamma : A \rightharpoondown B$. *Then,*

$$(\alpha \Rightarrow \beta) \sqcap (\beta \Rightarrow \gamma) \sqsubseteq ((\alpha \sqcup \beta) \Rightarrow (\beta \sqcap \gamma)).$$

```
Lemma rpc_lemma3 {A B : eqType} {alpha beta gamma : Rel A B}:
 ((alpha » beta) ∩ (beta » gamma)) ⊆ ((alpha ∪ beta) » (beta ∩ gamma)).
Proof.
apply inc_rpc.
rewrite cap_cup_distr_l.
rewrite cap_comm -cap_assoc rpc_l.
rewrite (@cap_assoc _ _ _ _ beta) (@cap_comm _ _ (beta » gamma)) -cap_assoc rpc_r.
rewrite cap_assoc rpc_l.
apply inc_cup.
split.
apply cap_r.
apply inc_refl.
Qed.
```

Lemma 66 (rpc_lemma4) *Let* $\alpha, \beta, \gamma : A \rightharpoondown B$. *Then,*

$$(\alpha \Rightarrow \beta) \sqcap (\beta \Rightarrow \gamma) \sqsubseteq (\alpha \Rightarrow \gamma).$$

```
Lemma rpc_lemma4 {A B : eqType} {alpha beta gamma : Rel A B}:
 ((alpha » beta) ∩ (beta » gamma)) ⊆ (alpha » gamma).
Proof.
apply (@inc_trans _ _ _ ((alpha ∪ beta) » (beta ∩ gamma))).
apply rpc_lemma3.
apply rpc_inc_compat.
apply cup_l.
apply cap_r.
Qed.
```

Lemma 67 (rpc_lemma5) *Let* $\alpha, \beta, \gamma : A \rightharpoondown B$. *Then,*

$$\alpha \Rightarrow (\beta \Rightarrow \gamma) = (\alpha \sqcap \beta) \Rightarrow \gamma.$$

```
Lemma rpc_lemma5 {A B : eqType} {alpha beta gamma : Rel A B}:
 alpha » (beta » gamma) = (alpha ∩ beta) » gamma.
Proof.
apply inc_lower.
move => delta.
split; move => H.
apply inc_rpc.
rewrite -cap_assoc.
rewrite -inc_rpc -inc_rpc.
apply H.
rewrite inc_rpc inc_rpc.
rewrite cap_assoc.
apply inc_rpc.
apply H.
Qed.
```

Lemma 68 (rpc_lemma6) *Let* $\alpha, \beta, \gamma : A \rightharpoondown B$. *Then,*

$$\alpha \Rightarrow (\beta \Rightarrow \gamma) \sqsubseteq (\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma).$$

```
Lemma rpc_lemma6 {A B : eqType} {alpha beta gamma : Rel A B}:
 (alpha » (beta » gamma)) ⊆ ((alpha » beta) » (alpha » gamma)).
Proof.
rewrite inc_rpc inc_rpc.
rewrite cap_assoc (@cap_comm _ _ _ alpha).
```

```
rewrite rpc_l.
rewrite -cap_assoc (@cap_comm _ _ _ alpha).
rewrite rpc_l.
rewrite cap_assoc (@cap_comm _ _ _ beta).
rewrite rpc_l.
rewrite -cap_assoc.
apply cap_r.
Qed.
```

Lemma 69 (rpc_lemma7) *Let $\alpha, \beta, \gamma, \delta : A \rightharpoonup B$ and $\beta \sqsubseteq \alpha \sqsubseteq \gamma$. Then,*

$$(\alpha \sqcap \delta = \beta) \wedge (\alpha \sqcup \delta = \gamma) \Leftrightarrow (\gamma \sqsubseteq \alpha \sqcup (\alpha \Rightarrow \beta)) \wedge (\delta = \gamma \sqcap (\alpha \Rightarrow \beta)).$$

```
Lemma rpc_lemma7 {A B : eqType} {alpha beta gamma delta : Rel A B}:
beta ⊆ alpha → alpha ⊆ gamma → (alpha ∩ delta = beta ∧ alpha ∪ delta = gamma
 ↔ gamma ⊆ (alpha ∪ (alpha » beta)) ∧ delta = gamma ∩ (alpha » beta)).
Proof.
move ⇒ H H0.
split; elim; move ⇒ H1 H2; split.
rewrite -H2.
apply cup_inc_compat_l.
apply inc_rpc.
rewrite cap_comm H1.
apply inc_refl.
rewrite -H2.
rewrite cap_cup_distr_r rpc_l.
assert (delta ⊆ (alpha » beta)).
apply inc_rpc.
rewrite cap_comm H1.
apply inc_refl.
apply inc_def1 in H3.
rewrite -H3 -H1.
rewrite -cap_assoc cap_idem.
by [rewrite cap_comm cup_comm cup_cap_abs].
rewrite H2.
rewrite (@cap_comm _ _ gamma) -cap_assoc rpc_l.
apply inc_antisym.
apply (@inc_trans _ _ _ (beta ∩ gamma)).
apply cap_inc_compat_r.
apply cap_r.
apply cap_l.
move : (@inc_trans _ _ _ _ _ H H0) ⇒ H3.
apply inc_def1 in H.
```

```
apply inc_def1 in H3.
rewrite cap_comm in H.
rewrite -H -H3.
apply inc_refl.
rewrite H2.
rewrite cup_cap_distr_l.
apply inc_def2 in H0.
rewrite -H0.
apply inc_def1 in H1.
by [rewrite -H1].
Qed.
```

4.3 補関係に関する補題

Lemma 70 (complement_universal)

$$\nabla_{AB}{}^{-} = \phi_{AB}.$$

```
Lemma complement_universal {A B : eqType}: ∇ A B ^ = φ A B.
Proof.
apply rpc_universal_alpha.
Qed.
```

Lemma 71 (complement_alpha_universal) *Let* $\alpha : A \rightharpoondown B$. *Then,*

$$\alpha^{-} = \nabla_{AB} \Leftrightarrow \alpha = \phi_{AB}.$$

```
Lemma complement_alpha_universal {A B : eqType} {alpha : Rel A B}:
 alpha ^ = ∇ A B ↔ alpha = φ A B.
Proof.
split; move ⇒ H.
apply inc_antisym.
rewrite -(@cap_universal _ _ alpha) cap_comm.
apply inc_rpc.
rewrite -H.
apply inc_refl.
apply inc_empty_alpha.
apply inc_antisym.
apply inc_alpha_universal.
apply inc_rpc.
rewrite cap_comm cap_universal.
rewrite H.
```

```
apply inc_refl.
Qed.
```

Lemma 72 (complement_empty)

$$\phi_{AB}{}^{-} = \nabla_{AB}.$$

```
Lemma complement_empty {A B : eqType}: φ A B ^ = ∇ A B.
Proof.
by [apply complement_alpha_universal].
Qed.
```

Lemma 73 (complement_invol_inc) *Let $\alpha : A \rightharpoonup B$. Then,*

$$\alpha \sqsubseteq (\alpha^{-})^{-}.$$

```
Lemma complement_invol_inc {A B : eqType} {alpha : Rel A B}: alpha ⊆ (alpha ^) ^.
Proof.
apply inc_rpc.
rewrite cap_comm.
apply inc_rpc.
apply inc_refl.
Qed.
```

Lemma 74 (cap_complement_empty) *Let $\alpha : A \rightharpoonup B$. Then,*

$$\alpha \sqcap \alpha^{-} = \phi_{AB}.$$

```
Lemma cap_complement_empty {A B : eqType} {alpha : Rel A B}:
 alpha ∩ alpha ^ = φ A B.
Proof.
apply inc_antisym.
rewrite cap_comm.
apply inc_rpc.
apply inc_refl.
apply inc_empty_alpha.
Qed.
```

Lemma 75 (complement_invol) *Let $\alpha : A \rightharpoonup B$. Then,*

$$(\alpha^{-})^{-} = \alpha.$$

```
Lemma complement_invol {A B : eqType} {alpha : Rel A B}: (alpha ^) ^ = alpha.
```

```
Proof.
rewrite -(@cap_universal _ _ ((alpha ^) ^)).
rewrite -(@complement_classic _ _ alpha).
rewrite cap_cup_distr_l.
rewrite (@cap_comm _ _ _ (alpha ^)) cap_complement_empty.
rewrite cup_empty cap_comm.
apply Logic.eq_sym.
apply inc_def1.
apply complement_invol_inc.
Qed.
```

Lemma 76 (complement_move) *Let* $\alpha, \beta : A \rightharpoondown B$. *Then,*

$$\alpha = \beta^- \Leftrightarrow \alpha^- = \beta.$$

```
Lemma complement_move {A B : eqType} {alpha beta : Rel A B}:
  alpha = beta ^ <-> alpha ^ = beta.
Proof.
split; move => H.
by [rewrite H complement_invol].
by [rewrite -H complement_invol].
Qed.
```

Lemma 77 (contraposition) *Let* $\alpha, \beta : A \rightharpoondown B$. *Then,*

$$(\alpha \Rightarrow \beta) = (\beta^- \Rightarrow \alpha^-).$$

```
Lemma contraposition {A B : eqType} {alpha beta : Rel A B}:
  alpha » beta = beta ^ » alpha ^.
Proof.
apply inc_antisym.
apply inc_rpc.
apply rpc_lemma4.
replace (alpha » beta) with ((alpha ^) ^ » (beta ^) ^).
apply inc_rpc.
apply rpc_lemma4.
by [rewrite complement_invol complement_invol].
Qed.
```

Lemma 78 (de_morgan1) *Let* $\alpha, \beta : A \rightharpoondown B$. *Then,*

$$(\alpha \sqcup \beta)^- = \alpha^- \sqcap \beta^-.$$

```
Lemma de_morgan1 {A B : eqType} {alpha beta : Rel A B}:
 (alpha ∪ beta) ^ = alpha ^ ∩ beta ^.
Proof.
apply inc_lower.
move ⇒ gamma.
split; move ⇒ H.
apply inc_cap.
rewrite inc_rpc inc_rpc.
apply inc_cup.
rewrite -cap_cup_distr_l.
apply inc_rpc.
apply H.
apply inc_rpc.
rewrite cap_cup_distr_l.
apply inc_cup.
rewrite -inc_rpc -inc_rpc.
apply inc_cap.
apply H.
Qed.
```

Lemma 79 (de_morgan2) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$(\alpha \sqcap \beta)^- = \alpha^- \sqcup \beta^-.$$

```
Lemma de_morgan2 {A B : eqType} {alpha beta : Rel A B}:
 (alpha ∩ beta) ^ = alpha ^ ∪ beta ^.
Proof.
by [rewrite -complement_move de_morgan1 complement_invol complement_invol].
Qed.
```

Lemma 80 (cup_to_rpc) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha^- \sqcup \beta = (\alpha \Rightarrow \beta).$$

```
Lemma cup_to_rpc {A B : eqType} {alpha beta : Rel A B}:
 alpha ^ ∪ beta = alpha » beta.
Proof.
apply inc_antisym.
apply inc_rpc.
rewrite cap_cup_distr_r cap_comm.
rewrite cap_complement_empty cup_comm cup_empty.
apply cap_l.
rewrite -(@cap_universal _ _ (alpha » beta)) cap_comm.
```

```
rewrite -(@complement_classic _ _ alpha).
rewrite cap_cup_distr_r cup_comm.
apply cup_inc_compat.
apply cap_l.
rewrite rpc_l.
apply cap_r.
Qed.
```

Lemma 81 (beta_contradiction) *Let $\alpha, \beta : A \rightharpoonup B$. Then,*

$$(\alpha \Rightarrow \beta) \sqcap (\alpha \Rightarrow \beta^-) = \alpha^-.$$

```
Lemma beta_contradiction {A B : eqType} {alpha beta : Rel A B}:
 (alpha » beta) ∩ (alpha » beta ^) = alpha^.
Proof.
rewrite -cup_to_rpc -cup_to_rpc.
rewrite -cup_cap_distr_l.
by [rewrite cap_complement_empty cup_empty].
Qed.
```

4.4 Bool 代数に関する補題

Lemma 82 (bool_lemma1) *Let $\alpha, \beta : A \rightharpoonup B$. Then,*

$$\alpha \sqsubseteq \beta \Leftrightarrow \nabla_{AB} = \alpha^- \sqcup \beta.$$

```
Lemma bool_lemma1 {A B : eqType} {alpha beta : Rel A B}:
 alpha ⊆ beta ↔ ∇ A B = alpha ^ ∪ beta.
Proof.
split; move ⇒ H.
apply inc_antisym.
rewrite -(@complement_classic _ _ alpha) cup_comm.
apply cup_inc_compat_l.
apply H.
apply inc_alpha_universal.
apply inc_def3.
rewrite H.
apply (Logic.eq_sym cup_to_rpc).
Qed.
```

Lemma 83 (bool_lemma2) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \beta \Leftrightarrow \alpha \sqcap \beta^- = \phi_{AB}.$$

```
Lemma bool_lemma2 {A B : eqType} {alpha beta : Rel A B}:
 alpha ⊆ beta ↔ alpha ∩ beta ^ = φ A B.
Proof.
split; move ⇒ H.
rewrite -(@cap_universal _ _ (alpha ∩ beta ^)).
apply bool_lemma1 in H.
rewrite H.
rewrite cap_cup_distr_l.
rewrite (@cap_comm _ _ alpha) cap_assoc cap_complement_empty cap_empty.
rewrite cap_comm -cap_assoc cap_complement_empty cap_comm cap_empty.
by [rewrite cup_empty].
rewrite -(@cap_universal _ _ alpha).
rewrite -(@complement_classic _ _ beta).
rewrite cap_cup_distr_l.
rewrite H cup_empty.
apply cap_r.
Qed.
```

Lemma 84 (bool_lemma3) *Let $\alpha, \beta, \gamma : A \rightharpoondown B$. Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Leftrightarrow \alpha \sqcap \beta^- \sqsubseteq \gamma.$$

```
Lemma bool_lemma3 {A B : eqType} {alpha beta gamma : Rel A B}:
 alpha ⊆ (beta ∪ gamma) ↔ (alpha ∩ beta ^) ⊆ gamma.
Proof.
split; move ⇒ H.
apply (@inc_trans _ _ _ ((beta ∪ gamma) ∩ beta ^)).
apply cap_inc_compat_r.
apply H.
rewrite cap_cup_distr_r.
rewrite cap_complement_empty cup_comm cup_empty.
apply cap_l.
apply (@inc_trans _ _ _ (beta ∪ (alpha ∩ beta ^))).
rewrite cup_cap_distr_l.
rewrite complement_classic cap_universal.
apply cup_r.
apply cup_inc_compat_l.
apply H.
Qed.
```

Lemma 85 (bool_lemma4) *Let* $\alpha, \beta, \gamma : A \rightharpoonup B$. *Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Leftrightarrow \beta^{-} \sqsubseteq \alpha^{-} \sqcup \gamma.$$

```
Lemma bool_lemma4 {A B : eqType} {alpha beta gamma : Rel A B}:
 alpha ⊆ (beta ∪ gamma) ↔ beta ^ ⊆ (alpha ^ ∪ gamma).
Proof.
rewrite bool_lemma3.
rewrite cap_comm.
apply iff_sym.
replace (beta ^ ∩ alpha) with (beta ^ ∩ (alpha ^) ^).
apply bool_lemma3.
by [rewrite complement_invol].
Qed.
```

Lemma 86 (bool_lemma5) *Let* $\alpha, \beta, \gamma : A \rightharpoonup B$. *Then,*

$$\alpha \sqsubseteq \beta \sqcup \gamma \Leftrightarrow \nabla_{AB} = \alpha^{-} \sqcup \beta \sqcup \gamma.$$

```
Lemma bool_lemma5 {A B : eqType} {alpha beta gamma : Rel A B}:
 alpha ⊆ (beta ∪ gamma) ↔ ∇ A B = (alpha ^ ∪ beta) ∪ gamma.
Proof.
rewrite bool_lemma1.
by [rewrite cup_assoc].
Qed.
```

Chapter 5

Library Relation_Properties

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Logic.FunctionalExtensionality.
Require Import Logic.Classical_Prop.
```

5.1 関係計算の基本的な性質

Lemma 87 (RelAB_unique)

$$\phi_{AB} = \nabla_{AB} \Leftrightarrow \forall \alpha, \beta : A \rightharpoonup B, \alpha = \beta.$$

```
Lemma RelAB_unique {A B : eqType}:
φ A B = ∇ A B ↔ (∀ alpha beta : Rel A B, alpha = beta).
Proof.
split; move ⇒ H.
move ⇒ alpha beta.
replace beta with (φ A B).
apply inc_antisym.
rewrite H.
apply inc_alpha_universal.
apply inc_empty_alpha.
apply inc_antisym.
apply inc_empty_alpha.
rewrite H.
apply inc_alpha_universal.
apply H.
Qed.
```

Lemma 88 (either_empty)

$$\phi_{AB} = \nabla_{AB} \Leftrightarrow A = \emptyset \vee B = \emptyset.$$

```
Lemma either_empty {A B : eqType}: ϕ A B = ∇ A B ↔ (A → False) ∨ (B → False).
Proof.
rewrite RelAB_unique.
split; move ⇒ H.
case (classic (∃ _ : A, True)).
elim ⇒ a H0.
right.
move ⇒ b.
remember (fun (_ : A) (_ : B) ⇒ True) as T.
remember (fun (_ : A) (_ : B) ⇒ False) as F.
move : (H T F) ⇒ H1.
assert (T a b = F a b).
by [rewrite H1].
rewrite HeqT HeqF in H2.
rewrite -H2.
apply I.
move ⇒ H0.
left.
move ⇒ a.
apply H0.
∃ a.
apply I.
move ⇒ alpha beta.
assert (A → B → False).
move ⇒ a b.
case H; move ⇒ H0.
apply (H0 a).
apply (H0 b).
apply functional_extensionality.
move ⇒ a.
apply functional_extensionality.
move ⇒ b.
apply False_ind.
apply (H0 a b).
Qed.
```

Lemma 89 (unit_empty_not_universal)

$$\phi_{II} \neq \nabla_{II}.$$

```
Lemma unit_empty_not_universal : φ i i ≠ ∇ i i.
Proof.
move ⇒ H.
apply either_empty in H.
case H; move ⇒ H0.
apply (H0 tt).
apply (H0 tt).
Qed.
```

Lemma 90 (unit_empty_or_universal) *Let $\alpha : I \rightharpoondown I$. Then,*

$$\alpha = \phi_{II} \vee \alpha = \nabla_{II}.$$

```
Lemma unit_empty_or_universal {alpha : Rel i i}: alpha = φ i i ∨ alpha = ∇ i i.
Proof.
assert (∀ beta : Rel i i, beta = (fun (_ _ : i) ⇒ True) ∨ beta = (fun (_ _ : i) ⇒ False)).
move ⇒ beta.
case (classic (beta tt tt)); move ⇒ H.
left.
apply functional_extensionality.
induction x.
apply functional_extensionality.
induction x.
apply prop_extensionality_ok.
split; move ⇒ H0.
apply I.
apply H.
right.
apply functional_extensionality.
induction x.
apply functional_extensionality.
induction x.
apply prop_extensionality_ok.
split.
apply H.
apply False_ind.
assert ((fun _ _ : i ⇒ True) ≠ (fun _ _ : i ⇒ False)).
move ⇒ H0.
remember (fun _ _ : i ⇒ True) as T.
```

```
remember (fun _ _ : i ⇒ False) as F.
assert (T tt tt = F tt tt).
by [rewrite H0].
rewrite HeqT HeqF in H1.
rewrite -H1.
apply I.
case (H (φ i i)); move ⇒ H1.
case (H (∇ i i)); move ⇒ H2.
apply False_ind.
apply unit_empty_not_universal.
by [rewrite H1 H2].
case (H alpha); move ⇒ H3.
left.
by [rewrite H3 H1].
right.
by [rewrite H3 H2].
case (H (∇ i i)); move ⇒ H2.
case (H alpha); move ⇒ H3.
right.
by [rewrite H3 H2].
left.
by [rewrite H3 H1].
apply False_ind.
apply unit_empty_not_universal.
by [rewrite H1 H2].
Qed.
```

Lemma 91 (unit_identity_is_universal)

$$id_I = \nabla_{II}.$$

```
Lemma unit_identity_is_universal : Id i = ∇ i i.
Proof.
case (@unit_empty_or_universal (Id i)); move ⇒ H.
apply False_ind.
assert (Id i ⊆ (∇ i i # △ φ i i)).
rewrite H.
apply inc_empty_alpha.
apply inc_residual in H0.
rewrite inv_invol comp_id_r in H0.
apply unit_empty_not_universal.
apply inc_antisym.
apply inc_empty_alpha.
```

```
apply H0.
apply H.
Qed.
```

Lemma 92 (unit_identity_not_empty)

$$id_I \neq \phi_{II}.$$

```
Lemma unit_identity_not_empty : Id i ≠ φ i i.
Proof.
move ⇒ H.
apply unit_empty_not_universal.
rewrite -H.
apply unit_identity_is_universal.
Qed.
```

Lemma 93 (cupP_False) *Let $f : (C \rightharpoondown D) \to (A \rightharpoondown B)$ and $P(\alpha) :=$ "False". Then,*

$$\sqcup_{P(\alpha)} f(\alpha) = \phi_{AB}.$$

```
Lemma cupP_False {A B C D : eqType} {f : Rel C D → Rel A B}:
 ∪_{fun _ : Rel C D ⇒ False} f = φ A B.
Proof.
apply inc_antisym.
apply inc_cupP.
move ⇒ beta.
apply False_ind.
apply inc_empty_alpha.
Qed.
```

Lemma 94 (capP_False) *Let $f : (C \rightharpoondown D) \to (A \rightharpoondown B)$ and $P(\alpha) :=$ "False". Then,*

$$\sqcap_{P(\alpha)} f(\alpha) = \nabla_{AB}.$$

```
Lemma capP_False {A B C D : eqType} {f : Rel C D → Rel A B}:
 ∩_{fun _ : Rel C D ⇒ False} f = ∇ A B.
Proof.
apply inc_antisym.
apply inc_alpha_universal.
apply inc_capP.
move ⇒ beta.
apply False_ind.
Qed.
```

Lemma 95 (cupP_eq) *Let $f, g : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$(\forall \alpha : C \rightharpoonup D, P(\alpha) \Rightarrow f(\alpha) = g(\alpha)) \Rightarrow \sqcup_{P(\alpha)} f(\alpha) = \sqcup_{P(\alpha)} g(\alpha).$$

```
Lemma cupP_eq {A B C D : eqType}
 {f g : Rel C D → Rel A B} {P : Rel C D → Prop}:
 (∀ alpha : Rel C D, P alpha → f alpha = g alpha) → ∪_{P} f = ∪_{P} g.
Proof.
move ⇒ H.
apply inc_antisym.
apply inc_cupP.
move ⇒ beta H0.
rewrite (H _ H0).
move : beta H0.
apply inc_cupP.
apply inc_refl.
apply inc_cupP.
move ⇒ beta H0.
rewrite -(H _ H0).
move : beta H0.
apply inc_cupP.
apply inc_refl.
Qed.
```

Lemma 96 (capP_eq) *Let $f, g : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$(\forall \alpha : C \rightharpoonup D, P(\alpha) \Rightarrow f(\alpha) = g(\alpha)) \Rightarrow \sqcap_{P(\alpha)} f(\alpha) = \sqcap_{P(\alpha)} g(\alpha).$$

```
Lemma capP_eq {A B C D : eqType}
 {f g : Rel C D → Rel A B} {P : Rel C D → Prop}:
 (∀ alpha : Rel C D, P alpha → f alpha = g alpha) → ∩_{P} f = ∩_{P} g.
Proof.
move ⇒ H.
apply inc_antisym.
apply inc_capP.
move ⇒ beta H0.
rewrite -(H _ H0).
move : beta H0.
apply inc_capP.
apply inc_refl.
apply inc_capP.
move ⇒ beta H0.
```

```
rewrite (H _ H0).
move : beta H0.
apply inc_capP.
apply inc_refl.
Qed.
```

Lemma 97 (cap_cupP_distr_l) *Let $\alpha : A \rightharpoonup B$, $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$\alpha \sqcap (\sqcup_{P(\beta)} f(\beta)) = \sqcup_{P(\beta)} (\alpha \sqcap f(\beta)).$$

```
Lemma cap_cupP_distr_l {A B C D : eqType}
 {alpha : Rel A B} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:
 alpha ∩ (∪_{P} f) = ∪_{P} (fun beta : Rel C D ⇒ alpha ∩ f beta).
Proof.
apply inc_upper.
move ⇒ gamma.
split; move ⇒ H.
apply inc_cupP.
move ⇒ beta H0.
apply (@inc_trans _ _ _ (alpha ∩ ∪_{P} f)).
apply cap_inc_compat_l.
move : H0.
apply inc_cupP.
apply inc_refl.
apply H.
assert (∀ beta : Rel C D, P beta → (alpha ∩ f beta) ⊆ gamma).
apply inc_cupP.
apply H.
assert (∀ beta : Rel C D, P beta → f beta ⊆ (alpha » gamma)).
move ⇒ beta H1.
rewrite inc_rpc cap_comm.
apply (H0 _ H1).
rewrite cap_comm -inc_rpc.
apply inc_cupP.
apply H1.
Qed.
```

Lemma 98 (cap_cupP_distr_r) *Let $\beta : A \rightharpoonup B$, $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$(\sqcup_{P(\alpha)} f(\alpha)) \sqcap \beta = \sqcup_{P(\alpha)} (f(\alpha) \sqcap \beta).$$

```
Lemma cap_cupP_distr_r {A B C D : eqType}
```

```
 {beta : Rel A B} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:
 (∪_{P} f) ∩ beta = ∪_{P} (fun alpha : Rel C D ⇒ f alpha ∩ beta).
Proof.
rewrite cap_comm.
replace (fun alpha : Rel C D ⇒ f alpha ∩ beta) with (fun alpha : Rel C D ⇒ beta ∩
f alpha).
apply cap_cupP_distr_l.
apply functional_extensionality.
move ⇒ x.
by [rewrite cap_comm].
Qed.
```

Lemma 99 (cup_capP_distr_l) *Let $\alpha : A \rightharpoonup B$, $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$\alpha \sqcup (\sqcap_{P(\beta)} f(\beta)) = \sqcap_{P(\beta)} (\alpha \sqcup f(\beta)).$$

```
Lemma cup_capP_distr_l {A B C D : eqType}
 {alpha : Rel A B} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:
 alpha ∪ (∩_{P} f) = ∩_{P} (fun beta : Rel C D ⇒ alpha ∪ f beta).
Proof.
apply inc_lower.
move ⇒ gamma.
split; move ⇒ H.
apply inc_capP.
move ⇒ beta H0.
apply (@inc_trans _ _ _ (alpha ∪ ∩_{P} f)).
apply H.
apply cup_inc_compat_l.
move : H0.
apply inc_capP.
apply inc_refl.
rewrite bool_lemma3.
assert (∀ beta : Rel C D, P beta → gamma ⊆ (alpha ∪ f beta)).
apply inc_capP.
apply H.
apply inc_capP.
move ⇒ beta H1.
rewrite -bool_lemma3.
apply (H0 _ H1).
Qed.
```

Lemma 100 (cup_capP_distr_r) *Let $\beta : A \rightharpoondown B$, $f : (C \rightharpoondown D) \to (A \rightharpoondown B)$ and P : predicate. Then,*

$$(\sqcap_{P(\alpha)} f(\alpha)) \sqcup \beta = \sqcap_{P(\alpha)} (f(\alpha) \sqcup \beta).$$

```
Lemma cup_capP_distr_r {A B C D : eqType}
 {beta : Rel A B} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:
 (∩_{P} f) ∪ beta = ∩_{P} (fun alpha : Rel C D ⇒ f alpha ∪ beta).
Proof.
rewrite cup_comm.
replace (fun alpha : Rel C D ⇒ f alpha ∪ beta) with (fun alpha : Rel C D ⇒ beta ∪
f alpha).
apply cup_capP_distr_l.
apply functional_extensionality.
move ⇒ x.
by [rewrite cup_comm].
Qed.
```

Lemma 101 (de_morgan3) *Let $f : (C \rightharpoondown D) \to (A \rightharpoondown B)$ and P : predicate. Then,*

$$(\sqcup_{P(\alpha)} f(\alpha))^- = (\sqcap_{P(\alpha)} f(\alpha)^-).$$

```
Lemma de_morgan3
 {A B C D : eqType} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:
 (∪_{P} f) ^ = ∩_{P} (fun alpha : Rel C D ⇒ f alpha ^).
Proof.
apply inc_lower.
move ⇒ gamma.
rewrite inc_capP.
split; move ⇒ H.
move ⇒ beta H0.
rewrite bool_lemma1 -de_morgan2 complement_move complement_universal.
apply bool_lemma2 in H.
apply inc_antisym.
apply inc_empty_alpha.
rewrite -H complement_invol.
apply cap_inc_compat_l.
move : H0.
apply inc_cupP.
apply inc_refl.
rewrite bool_lemma2 complement_invol.
rewrite cap_cupP_distr_l.
apply inc_antisym.
```

```
apply inc_cupP.
move ⇒ beta H0.
rewrite -inc_rpc.
apply (H _ H0).
apply inc_empty_alpha.
Qed.
```

Lemma 102 (de_morgan4) *Let $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$(\sqcap_{P(\alpha)} f(\alpha))^- = (\sqcup_{P(\alpha)} f(\alpha)^-).$$

```
Lemma de_morgan4
 {A B C D : eqType} {f : Rel C D → Rel A B} {P : Rel C D → Prop}:
 (∩_{P} f) ^ = ∪_{P} (fun alpha : Rel C D ⇒ f alpha ^).
Proof.
rewrite -complement_move de_morgan3.
replace (fun alpha : Rel C D ⇒ (f alpha ^) ^) with f.
by [].
apply functional_extensionality.
move ⇒ x.
by [rewrite complement_invol].
Qed.
```

Lemma 103 (cup_to_cupP) *Let $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$f(\alpha) \sqcup f(\beta) = \sqcup_{\gamma=\alpha\vee\gamma=\beta} f(\gamma).$$

```
Lemma cup_to_cupP
 {A B C D : eqType} {alpha beta : Rel C D} {f : Rel C D → Rel A B}:
 (f alpha ∪ f beta) = ∪_{fun gamma : Rel C D ⇒ gamma = alpha ∨ gamma = beta}
f.
Proof.
apply inc_upper.
move ⇒ delta.
split; move ⇒ H.
apply inc_cupP.
apply inc_cup in H.
move ⇒ gamma H0.
case H0 ⇒ H1.
rewrite H1.
apply H.
rewrite H1.
apply H.
```

```
apply inc_cup.
assert (∀ gamma : Rel C D, gamma = alpha ∨ gamma = beta → f gamma ⊆ delta).
apply inc_cupP.
apply H.
split.
apply (H0 alpha).
by [left].
apply (H0 beta).
by [right].
Qed.
```

Lemma 104 (cap_to_capP) *Let $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$f(\alpha) \sqcap f(\beta) = \sqcap_{\gamma=\alpha\vee\gamma=\beta} f(\gamma).$$

```
Lemma cap_to_capP
 {A B C D : eqType} {alpha beta : Rel C D} {f : Rel C D → Rel A B}:
 (f alpha ∩ f beta) = ∩_{fun gamma : Rel C D ⇒ gamma = alpha ∨ gamma = beta}
f.
Proof.
apply inc_lower.
move ⇒ delta.
split; move ⇒ H.
apply inc_capP.
apply inc_cap in H.
move ⇒ gamma H0.
case H0 ⇒ H1.
rewrite H1.
apply H.
rewrite H1.
apply H.
apply inc_cap.
assert (∀ gamma : Rel C D, gamma = alpha ∨ gamma = beta → delta ⊆ f gamma).
apply inc_capP.
apply H.
split.
apply (H0 alpha).
by [left].
apply (H0 beta).
by [right].
Qed.
```

5.2 comp_inc_compat と派生補題

Lemma 105 (comp_inc_compat_ab_ab') *Let* $\alpha : A \rightharpoondown B$ *and* $\beta, \beta' : B \rightharpoondown C$*. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha \cdot \beta'.$$

```
Lemma comp_inc_compat_ab_ab'
 {A B C : eqType} {alpha : Rel A B} {beta beta' : Rel B C}:
 beta ⊆ beta' → (alpha ・ beta) ⊆ (alpha ・ beta').
Proof.
move ⇒ H.
replace (alpha ・ beta) with ((alpha #) # ・ beta).
apply inc_residual.
apply (@inc_trans _ _ _ beta').
apply H.
apply inc_residual.
rewrite inv_invol.
apply inc_refl.
by [rewrite inv_invol].
Qed.
```

Lemma 106 (comp_inc_compat_ab_a'b) *Let* $\alpha, \alpha' : A \rightharpoondown B$ *and* $\beta : B \rightharpoondown C$*. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha' \cdot \beta.$$

```
Lemma comp_inc_compat_ab_a'b
 {A B C : eqType} {alpha alpha' : Rel A B} {beta : Rel B C}:
 alpha ⊆ alpha' → (alpha ・ beta) ⊆ (alpha' ・ beta).
Proof.
move ⇒ H.
rewrite -(@inv_invol _ _ (alpha ・ beta)).
rewrite -(@inv_invol _ _ (alpha' ・ beta)).
apply inc_inv.
rewrite comp_inv comp_inv.
apply comp_inc_compat_ab_ab'.
apply inc_inv.
apply H.
Qed.
```

Lemma 107 (comp_inc_compat) *Let $\alpha, \alpha' : A \rightharpoonup B$ and $\beta, \beta' : B \rightharpoonup C$. Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \cdot \beta \sqsubseteq \alpha' \cdot \beta'.$$

```
Lemma comp_inc_compat
 {A B C : eqType} {alpha alpha' : Rel A B} {beta beta' : Rel B C}:
 alpha ⊆ alpha' → beta ⊆ beta' → (alpha • beta) ⊆ (alpha' • beta').
Proof.
move ⇒ H H0.
apply (@inc_trans _ _ _ (alpha' • beta)).
apply (@comp_inc_compat_ab_a'b _ _ _ _ _ _ _ H).
apply (@comp_inc_compat_ab_ab' _ _ _ _ _ _ _ H0).
Qed.
```

Lemma 108 (comp_inc_compat_ab_a) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup B$. Then,*

$$\beta \sqsubseteq id_B \Rightarrow \alpha \cdot \beta \sqsubseteq \beta.$$

```
Lemma comp_inc_compat_ab_a {A B : eqType} {alpha : Rel A B} {beta : Rel B B}:
 beta ⊆ Id B → (alpha • beta) ⊆ alpha.
Proof.
move ⇒ H.
move : (@comp_inc_compat_ab_ab' _ _ _ alpha _ _ H) ⇒ H0.
rewrite comp_id_r in H0.
apply H0.
Qed.
```

Lemma 109 (comp_inc_compat_a_ab) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup B$. Then,*

$$id_B \sqsubseteq \beta \Rightarrow \beta \sqsubseteq \alpha \cdot \beta.$$

```
Lemma comp_inc_compat_a_ab {A B : eqType} {alpha : Rel A B} {beta : Rel B B}:
 Id B ⊆ beta → alpha ⊆ (alpha • beta).
Proof.
move ⇒ H.
move : (@comp_inc_compat_ab_ab' _ _ _ alpha _ _ H) ⇒ H0.
rewrite comp_id_r in H0.
apply H0.
Qed.
```

Lemma 110 (comp_inc_compat_ab_b) *Let $\alpha : A \rightharpoonup A$ and $\beta : A \rightharpoonup B$. Then,*

$$\alpha \sqsubseteq id_A \Rightarrow \alpha \cdot \beta \sqsubseteq \beta.$$

```
Lemma comp_inc_compat_ab_b {A B : eqType} {alpha : Rel A A} {beta : Rel A B}:
  alpha ⊆ Id A → (alpha ・ beta) ⊆ beta.
Proof.
move ⇒ H.
move : (@comp_inc_compat_ab_a'b _ _ _ _ _ beta H) ⇒ H0.
rewrite comp_id_l in H0.
apply H0.
Qed.
```

Lemma 111 (comp_inc_compat_b_ab) *Let $\alpha : A \rightharpoonup A$ and $\beta : A \rightharpoonup B$. Then,*

$$id_A \sqsubseteq \alpha \Rightarrow \beta \sqsubseteq \alpha \cdot \beta.$$

```
Lemma comp_inc_compat_b_ab {A B : eqType} {alpha : Rel A A} {beta : Rel A B}:
  Id A ⊆ alpha → beta ⊆ (alpha ・ beta).
Proof.
move ⇒ H.
move : (@comp_inc_compat_ab_a'b _ _ _ _ _ beta H) ⇒ H0.
rewrite comp_id_l in H0.
apply H0.
Qed.
```

5.3 逆関係に関する補題

Lemma 112 (inv_move) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup A$. Then,*

$$\alpha = \beta^\sharp \Leftrightarrow \alpha^\sharp = \beta.$$

```
Lemma inv_move {A B : eqType} {alpha : Rel A B} {beta : Rel B A}:
  alpha = beta # ↔ alpha # = beta.
Proof.
split; move ⇒ H.
by [rewrite H inv_invol].
by [rewrite -H inv_invol].
Qed.
```

Lemma 113 (comp_inv_inv) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$. Then,*

$$\alpha \cdot \beta = (\beta^\sharp \cdot \alpha^\sharp)^\sharp.$$

```
Lemma comp_inv_inv {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 alpha • beta = (beta # • alpha #) #.
Proof.
apply inv_move.
apply comp_inv.
Qed.
```

Lemma 114 (inv_inc_move) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup A$. Then,*

$$\alpha \sqsubseteq \beta^\sharp \Leftrightarrow \alpha^\sharp \sqsubseteq \beta.$$

```
Lemma inv_inc_move {A B : eqType} {alpha : Rel A B} {beta : Rel B A}:
 alpha ⊆ beta # ↔ alpha # ⊆ beta.
Proof.
split; move ⇒ H.
rewrite -(@inv_invol _ _ beta).
apply inc_inv.
apply H.
rewrite -(@inv_invol _ _ alpha).
apply inc_inv.
apply H.
Qed.
```

Lemma 115 (inv_invol2) *Let $\alpha, \beta : A \rightharpoonup B$. Then,*

$$\alpha^\sharp = \beta^\sharp \Rightarrow \alpha = \beta.$$

```
Lemma inv_invol2 {A B : eqType} {alpha beta : Rel A B}:
 alpha # = beta # → alpha = beta.
Proof.
move ⇒ H.
rewrite -(@inv_invol _ _ alpha) -(@inv_invol _ _ beta).
apply f_equal.
apply H.
Qed.
```

Lemma 116 (inv_inc_invol) *Let $\alpha, \beta : A \rightharpoonup B$. Then,*

$$\alpha^\sharp \sqsubseteq \beta^\sharp \Rightarrow \alpha \sqsubseteq \beta.$$

```
Lemma inv_inc_invol {A B : eqType} {alpha beta : Rel A B}:
 alpha # ⊆ beta # → alpha ⊆ beta.
Proof.
move ⇒ H.
rewrite -(@inv_invol _ _ alpha) -(@inv_invol _ _ beta).
apply inc_inv.
apply H.
Qed.
```

Lemma 117 (inv_cupP_distr, inv_cup_distr) *Let $f : (C \rightharpoonup D) \to (A \rightharpoonup B)$ and P : predicate. Then,*

$$(\sqcup_{P(\alpha)} f(\alpha))^\sharp = (\sqcup_{P(\alpha)} f(\alpha)^\sharp).$$

```
Lemma inv_cupP_distr {A B C D : eqType} {f : Rel C D → Rel A B} {P : Rel C D →
Prop}:
 (∪_{P} f) # = (∪_{P} (fun alpha : Rel C D ⇒ f alpha #)).
Proof.
apply inc_antisym.
rewrite -inv_inc_move.
apply inc_cupP.
assert (∀ beta : Rel C D, P beta → f beta # ⊆ ∪_{P} (fun alpha : Rel C D ⇒ f
alpha #)).
apply inc_cupP.
apply inc_refl.
move ⇒ beta H0.
rewrite inv_inc_move.
apply (H _ H0).
apply inc_cupP.
move ⇒ beta H0.
apply inc_inv.
move : H0.
apply inc_cupP.
apply inc_refl.
Qed.
```

```
Lemma inv_cup_distr {A B : eqType} {alpha beta : Rel A B}:
 (alpha ∪ beta) # = alpha # ∪ beta #.
Proof.
by [rewrite cup_to_cupP -inv_cupP_distr -cup_to_cupP].
```

Qed.

Lemma 118 (inv_capP_distr, inv_cap_distr) *Let $f : (C \rightharpoondown D) \to (A \rightharpoondown B)$ and P : predicate. Then,*

$$(\sqcap_{P(\alpha)} f(\alpha))^\sharp = (\sqcap_{P(\alpha)} f(\alpha)^\sharp).$$

```
Lemma inv_capP_distr {A B C D : eqType} {f : Rel C D → Rel A B} {P : Rel C D →
Prop}:
 (∩_{P} f) # = (∩_{P} (fun alpha : Rel C D ⇒ f alpha #)).
Proof.
apply inc_antisym.
apply inc_capP.
move ⇒ beta H.
apply inc_inv.
move : H.
apply inc_capP.
apply inc_refl.
rewrite inv_inc_move.
apply inc_capP.
assert (∀ beta : Rel C D, P beta → ∩_{P} (fun alpha : Rel C D ⇒ f alpha #) ⊆ f
beta #).
apply inc_capP.
apply inc_refl.
move ⇒ beta H0.
rewrite -inv_inc_move.
apply (H _ H0).
Qed.
```

```
Lemma inv_cap_distr {A B : eqType} {alpha beta : Rel A B}:
 (alpha ∩ beta) # = alpha # ∩ beta #.
Proof.
by [rewrite cap_to_capP -inv_capP_distr -cap_to_capP].
Qed.
```

Lemma 119 (rpc_inv_distr) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$(\alpha \Rightarrow \beta)^\sharp = \alpha^\sharp \Rightarrow \beta^\sharp.$$

```
Lemma rpc_inv_distr {A B : eqType} {alpha beta : Rel A B}:
 (alpha » beta) # = alpha # » beta #.
Proof.
apply inc_lower.
move ⇒ gamma.
```

split; move ⇒ *H*.
apply *inc_rpc*.
rewrite *inv_inc_move inv_cap_distr inv_invol*.
rewrite -*inc_rpc* -*inv_inc_move*.
apply *H*.
rewrite *inv_inc_move inc_rpc*.
rewrite -(@*inv_invol _ _ alpha*) -*inv_cap_distr* -*inv_inc_move*.
apply *inc_rpc*.
apply *H*.
Qed.

Lemma 120 (inv_empty)

$$\phi_{AB}{}^{\sharp} = \phi_{BA}.$$

Lemma *inv_empty* {*A B* : *eqType*}: ϕ *A B* # = ϕ *B A*.
Proof.
apply *inc_antisym*.
rewrite -*inv_inc_move*.
apply *inc_empty_alpha*.
apply *inc_empty_alpha*.
Qed.

Lemma 121 (inv_universal)

$$\nabla_{AB}{}^{\sharp} = \nabla_{BA}.$$

Lemma *inv_universal* {*A B* : *eqType*}: ∇ *A B* # = ∇ *B A*.
Proof.
apply *inc_antisym*.
apply *inc_alpha_universal*.
rewrite *inv_inc_move*.
apply *inc_alpha_universal*.
Qed.

Lemma 122 (inv_id)

$$id_A{}^{\sharp} = id_A.$$

Lemma *inv_id* {*A* : *eqType*}: (*Id A*) # = *Id A*.
Proof.
replace (*Id A* #) with ((*Id A* #) # • *Id A* #).
by [rewrite -*comp_inv comp_id_l inv_invol*].
by [rewrite *inv_invol comp_id_l*].
Qed.

Lemma 123 (inv_complement) *Let $\alpha : A \rightharpoondown B$. Then,*

$$(\alpha^-)^\sharp = (\alpha^\sharp)^-.$$

```
Lemma inv_complement {A B : eqType} {alpha : Rel A B}: (alpha ^) # = (alpha #) ^.
Proof.
apply inc_antisym.
apply inc_rpc.
rewrite -inv_cap_distr.
rewrite cap_comm -inv_inc_move inv_empty.
rewrite cap_complement_empty.
apply inc_refl.
rewrite inv_inc_move.
apply inc_rpc.
replace (((alpha #) ^) # ∩ alpha) with (((alpha #) ^) # ∩ (alpha #) #).
rewrite -inv_cap_distr.
rewrite cap_comm -inv_inc_move inv_empty.
rewrite cap_complement_empty.
apply inc_refl.
by [rewrite inv_invol].
Qed.
```

Lemma 124 (inv_difference_distr) *Let $\alpha, \beta : A \rightharpoondown B$. Then,*

$$(\alpha - \beta)^\sharp = \alpha^\sharp - \beta^\sharp.$$

```
Lemma inv_difference_distr {A B : eqType} {alpha beta : Rel A B}:
 (alpha - beta) # = alpha # - beta #.
Proof.
rewrite inv_cap_distr.
by [rewrite inv_complement].
Qed.
```

5.4 合成に関する補題

Lemma 125 (comp_cupP_distr_l, comp_cup_distr_l) *Let $\alpha : A \rightharpoondown B$, $f : (D \rightharpoondown E) \to (B \rightharpoondown C)$ and P : predicate. Then,*

$$\alpha \cdot (\sqcup_{P(\beta)} f(\beta)) = \sqcup_{P(\beta)} (\alpha \cdot f(\beta)).$$

```
Lemma comp_cupP_distr_l {A B C D E : eqType}
```

```
 {alpha : Rel A B} {f : Rel D E → Rel B C} {P : Rel D E → Prop}:
 alpha · (∪_{P} f) = ∪_{P} (fun beta : Rel D E ⇒ (alpha · f beta)).
Proof.
apply inc_upper.
move ⇒ gamma.
split; move ⇒ H.
rewrite -(@inv_invol _ _ alpha) in H.
apply inc_residual in H.
apply inc_cupP.
assert (∀ beta : Rel D E, P beta → f beta ⊆ (alpha # △ gamma)).
apply inc_cupP.
apply H.
move ⇒ beta H1.
rewrite -(@inv_invol _ _ alpha).
apply inc_residual.
apply (H0 _ H1).
rewrite -(@inv_invol _ _ alpha).
apply inc_residual.
apply inc_cupP.
assert (∀ beta : Rel D E, P beta → (alpha · f beta) ⊆ gamma).
apply inc_cupP.
apply H.
move ⇒ beta H1.
apply inc_residual.
rewrite inv_invol.
apply (H0 _ H1).
Qed.
```

```
Lemma comp_cup_distr_l
 {A B C : eqType} {alpha : Rel A B} {beta gamma : Rel B C}:
 alpha · (beta ∪ gamma) = (alpha · beta) ∪ (alpha · gamma).
Proof.
by [rewrite cup_to_cupP -comp_cupP_distr_l -cup_to_cupP].
Qed.
```

Lemma 126 (comp_cupP_distr_r, comp_cup_distr_r) *Let* $f : (D \rightharpoonup E) \to (A \rightharpoonup B)$, $\beta : B \rightharpoonup C$ *and* P *: predicate. Then,*

$$(\sqcup_{P(\alpha)} f(\alpha)) \cdot \beta = \sqcup_{P(\alpha)} (f(\alpha) \cdot \beta).$$

```
Lemma comp_cupP_distr_r {A B C D E : eqType}
 {beta : Rel B C} {f : Rel D E → Rel A B} {P : Rel D E → Prop}:
 (∪_{P} f) · beta = ∪_{P} (fun alpha : Rel D E ⇒ (f alpha · beta)).
```

```
Proof.
replace (fun alpha : Rel D E ⇒ f alpha • beta) with (fun alpha : Rel D E ⇒ (beta #
• f alpha #) #).
rewrite -inv_cupP_distr.
rewrite -comp_cupP_distr_l.
rewrite -inv_cupP_distr.
rewrite comp_inv.
by [rewrite inv_invol inv_invol].
apply functional_extensionality.
move ⇒ x.
rewrite comp_inv.
by [rewrite inv_invol inv_invol].
Qed.
```

```
Lemma comp_cup_distr_r
 {A B C : eqType} {alpha beta : Rel A B} {gamma : Rel B C}:
 (alpha ∪ beta) • gamma = (alpha • gamma) ∪ (beta • gamma).
Proof.
by [rewrite (@cup_to_cupP _ _ _ _ _ _ id) comp_cupP_distr_r -cup_to_cupP].
Qed.
```

Lemma 127 (comp_capP_distr) *Let* $\alpha : A \rightharpoondown B$, $\gamma : C \rightharpoondown D$, $f : (E \rightharpoondown F) \to (B \rightharpoondown C)$ *and* P *: predicate. Then,*

$$\alpha \cdot (\sqcap_{P(\beta)} f(\beta)) \cdot \gamma \sqsubseteq \sqcap_{P(\beta)} (\alpha \cdot f(\beta) \cdot \gamma).$$

```
Lemma comp_capP_distr {A B C D E F : eqType}
 {alpha : Rel A B} {gamma : Rel C D}
 {f : Rel E F → Rel B C} {P : Rel E F → Prop}:
 (alpha • (∩_{P} f)) • gamma
 ⊆ ∩_{P} (fun beta : Rel E F ⇒ ((alpha • f beta) • gamma)).
Proof.
apply inc_capP.
move ⇒ beta H.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
move : H.
apply inc_capP.
apply inc_refl.
Qed.
```

Lemma 128 (comp_capP_distr_l, comp_cap_distr_l) *Let $\alpha : A \rightharpoondown B$, $f : (D \rightharpoondown E) \to (B \rightharpoondown C)$ and P : predicate. Then,*

$$\alpha \cdot (\sqcap_{P(\beta)} f(\beta)) \sqsubseteq \sqcap_{P(\beta)} (\alpha \cdot f(\beta)).$$

```
Lemma comp_capP_distr_l {A B C D E : eqType}
 {alpha : Rel A B} {f : Rel D E → Rel B C} {P : Rel D E → Prop}:
 (alpha ・ (∩_{P} f)) ⊆ ∩_{P} (fun beta : Rel D E ⇒ (alpha ・ f beta)).
Proof.
move : (@comp_capP_distr _ _ _ _ _ _ alpha (Id C) f P) ⇒ H.
rewrite comp_id_r in H.
replace (fun beta : Rel D E ⇒ (alpha ・ f beta) ・ Id C) with (fun beta : Rel D E ⇒
(alpha ・ f beta)) in H.
apply H.
apply functional_extensionality.
move ⇒ x.
by [rewrite comp_id_r].
Qed.

Lemma comp_cap_distr_l
 {A B C : eqType} {alpha : Rel A B} {beta gamma : Rel B C}:
 (alpha ・ (beta ∩ gamma)) ⊆ ((alpha ・ beta) ∩ (alpha ・ gamma)).
Proof.
rewrite cap_to_capP (@cap_to_capP _ _ _ _ _ _ id).
apply comp_capP_distr_l.
Qed.
```

Lemma 129 (comp_capP_distr_r, comp_cap_distr_r) *Let $\beta : B \rightharpoondown C$, $f : (D \rightharpoondown E) \to (A \rightharpoondown B)$ and P : predicate. Then,*

$$(\sqcap_{P(\alpha)} f(\alpha)) \cdot \beta \sqsubseteq \sqcap_{P(\alpha)} (f(\alpha) \cdot \beta).$$

```
Lemma comp_capP_distr_r
 {A B C D E : eqType} {beta : Rel B C} {f : Rel D E → Rel A B} {P : Rel D E →
Prop}:
 ((∩_{P} f) ・ beta) ⊆ ∩_{P} (fun alpha : Rel D E ⇒ (f alpha ・ beta)).
Proof.
move : (@comp_capP_distr _ _ _ _ _ _ (Id A) beta f P) ⇒ H.
rewrite comp_id_l in H.
replace (fun alpha : Rel D E ⇒ (Id A ・ f alpha) ・ beta) with (fun alpha : Rel D E
⇒ f alpha ・ beta) in H.
apply H.
apply functional_extensionality.
```

```
move => x.
by [rewrite comp_id_l].
Qed.

Lemma comp_cap_distr_r
 {A B C : eqType} {alpha beta : Rel A B} {gamma : Rel B C}:
 ((alpha ∩ beta) • gamma) ⊆ ((alpha • gamma) ∩ (beta • gamma)).
Proof.
rewrite (@cap_to_capP _ _ _ _ _ _ id) (@cap_to_capP _ _ _ _ _ _ (fun x => x • gamma)).
apply comp_capP_distr_r.
Qed.
```

Lemma 130 (comp_empty_l, comp_empty_r) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$. Then,*

$$\alpha \cdot \phi_{BC} = \phi_{AB} \cdot \beta = \phi_{AC}.$$

```
Lemma comp_empty_r {A B C : eqType} {alpha : Rel A B}: alpha • φ B C = φ A C.
Proof.
apply inc_antisym.
rewrite -(@inv_invol _ _ alpha).
apply inc_residual.
apply inc_empty_alpha.
apply inc_empty_alpha.
Qed.

Lemma comp_empty_l {A B C : eqType} {beta : Rel B C}: φ A B • beta = φ A C.
Proof.
rewrite -(@inv_invol _ _ (φ A B • beta)).
rewrite -inv_move comp_inv inv_empty inv_empty.
apply comp_empty_r.
Qed.
```

Lemma 131 (comp_either_empty) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$. Then,*

$$\alpha = \phi_{AB} \vee \beta = \phi_{BC} \Rightarrow \alpha \cdot \beta = \phi_{AC}.$$

```
Lemma comp_either_empty {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 alpha = φ A B ∨ beta = φ B C → alpha • beta = φ A C.
Proof.
case; move => H.
rewrite H.
apply comp_empty_l.
rewrite H.
apply comp_empty_r.
```

```
Qed.
```

Lemma 132 (comp_neither_empty) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$. Then,*

$$\alpha \cdot \beta \neq \phi_{AC} \Rightarrow \alpha \neq \phi_{AB} \wedge \beta \neq \phi_{BC}.$$

```
Lemma comp_neither_empty {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 alpha • beta ≠ φ A C → alpha ≠ φ A B ∧ beta ≠ φ B C.
Proof.
move ⇒ H.
split; move ⇒ H0.
apply H.
rewrite H0.
apply comp_empty_l.
apply H.
rewrite H0.
apply comp_empty_r.
Qed.
```

5.5 単域と Tarski の定理

Lemma 133 (lemma_for_tarski1) *Let $\alpha : A \rightharpoondown B$ and $\alpha \neq \phi_{AB}$. Then,*

$$\nabla_{IA} \cdot \alpha \cdot \nabla_{BI} = id_I.$$

```
Lemma lemma_for_tarski1 {A B : eqType} {alpha : Rel A B}:
 alpha ≠ φ A B → ((∇ i A • alpha) • ∇ B i) = Id i.
Proof.
move ⇒ H.
assert (((∇ i A • alpha) • ∇ B i) ≠ φ i i).
move ⇒ H0.
apply H.
apply inc_antisym.
apply (@inc_trans _ _ _ ((∇ A i • ((∇ i A • alpha) • ∇ B i)) • ∇ i B)).
rewrite comp_assoc comp_assoc unit_universal.
rewrite -comp_assoc -comp_assoc unit_universal.
apply (@inc_trans _ _ _ ((Id A • alpha) • Id B)).
rewrite comp_id_l comp_id_r.
apply inc_refl.
apply comp_inc_compat.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
```

```
apply inc_alpha_universal.
rewrite H0 comp_empty_r comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
case (@unit_empty_or_universal ((∇ i A • alpha) • ∇ B i)); move ⇒ H1.
apply False_ind.
apply (H0 H1).
rewrite unit_identity_is_universal.
apply H1.
Qed.
```

Lemma 134 (lemma_for_tarski2)

$$\nabla_{AI} \cdot \nabla_{IB} = \nabla_{AB}.$$

```
Lemma lemma_for_tarski2 {A B : eqType}: ∇ A i • ∇ i B = ∇ A B.
Proof.
apply inc_antisym.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ (∇ A A • ∇ A B)).
apply (@inc_trans _ _ _ (Id A • ∇ A B)).
rewrite comp_id_l.
apply inc_refl.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
rewrite -(@unit_universal A) comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
Qed.
```

Lemma 135 (tarski) *Let $\alpha : A \rightharpoonup B$ and $\alpha \neq \phi_{AB}$. Then,*

$$\nabla_{AA} \cdot \alpha \cdot \nabla_{BB} = \nabla_{AB}.$$

```
Lemma tarski {A B : eqType} {alpha : Rel A B}:
 alpha ≠ φ A B → ((∇ A A • alpha) • ∇ B B) = ∇ A B.
Proof.
move ⇒ H.
rewrite -(@unit_universal A) -(@unit_universal B).
move : (@lemma_for_tarski1 _ _ alpha H) ⇒ H0.
rewrite -comp_assoc (@comp_assoc _ _ _ _ (∇ A i)) (@comp_assoc _ _ _ _ (∇ A i)).
rewrite H0 comp_id_r.
apply lemma_for_tarski2.
```

Qed.

Lemma 136 (comp_universal1) *Let $B \neq \emptyset$. Then,*

$$\nabla_{AB} \cdot \nabla_{BC} = \nabla_{AC}.$$

```
Lemma comp_universal {A B C : eqType} : B → ∇ A B • ∇ B C = ∇ A C.
Proof.
move ⇒ b.
replace (∇ A B) with (∇ A B • ∇ B B).
rewrite -(@lemma_for_tarski2 A B) -(@lemma_for_tarski2 B C).
rewrite (@comp_assoc _ _ _ _ (∇ A i)) (@comp_assoc _ _ _ _ (∇ A i)) -(@comp_assoc _
_ _ _ _ (∇ B i)).
rewrite lemma_for_tarski1.
rewrite comp_id_l.
apply lemma_for_tarski2.
apply not_eq_sym.
move ⇒ H.
apply either_empty in H.
case H; move ⇒ H0.
apply (H0 b).
apply (H0 b).
apply inc_antisym.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ (∇ A B • Id B)).
rewrite comp_id_r.
apply inc_refl.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
Qed.
```

Lemma 137 (comp_universal2)

$$\nabla_{IA}{}^{\sharp} \cdot \nabla_{IB} = \nabla_{AB}.$$

```
Lemma comp_universal2 {A B : eqType}: ∇ i A # • ∇ i B = ∇ A B.
Proof.
rewrite inv_universal.
apply lemma_for_tarski2.
Qed.
```

Lemma 138 (empty_equivalence1, empty_equivalence2, empty_equivalence3)

$$A = \emptyset \Leftrightarrow \nabla_{IA} = \phi_{IA} \Leftrightarrow \nabla_{AA} = \phi_{AA} \Leftrightarrow id_A = \phi_{AA}.$$

```
Lemma empty_equivalence1 {A : eqType}: (A → False) ↔ ∇ i A = φ i A.
Proof.
move : (@either_empty i A) ⇒ H.
split; move ⇒ H0.
apply Logic.eq_sym.
apply H.
right.
apply H0.
apply Logic.eq_sym in H0.
apply H in H0.
case H0.
move ⇒ H1 H2.
apply H1.
apply tt.
by [].
Qed.
```

```
Lemma empty_equivalence2 {A : eqType}: (A → False) ↔ ∇ A A = φ A A.
Proof.
move : (@either_empty A A) ⇒ H.
split; move ⇒ H0.
apply Logic.eq_sym.
apply H.
left.
apply H0.
apply Logic.eq_sym in H0.
apply H in H0.
case H0.
by [].
by [].
Qed.
```

```
Lemma empty_equivalence3 {A : eqType}: (A → False) ↔ Id A = φ A A.
Proof.
split; move ⇒ H.
assert (∇ A A = φ A A).
apply empty_equivalence2.
apply H.
apply RelAB_unique.
apply Logic.eq_sym.
```

```
apply H0.
assert (φ A A = ∇ A A).
by [rewrite -(@comp_id_r _ _ (∇ A A)) H comp_empty_r].
apply either_empty in H0.
case H0.
by [].
by [].
Qed.
```

Chapter 6

Library Functions_Mappings

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Logic.FunctionalExtensionality.
```

6.1 全域性, 一価性, 写像に関する補題

Lemma 139 (id_function) *$id_A : A \rightharpoondown A$ is a function.*

```
Lemma id_function {A : eqType}: function_r (Id A).
Proof.
rewrite /function_r/total_r/univalent_r.
rewrite inv_id comp_id_l.
split.
apply inc_refl.
apply inc_refl.
Qed.
```

Lemma 140 (unit_function) *$\nabla_{AI} : A \rightharpoondown I$ is a function.*

```
Lemma unit_function {A : eqType}: function_r (∇ A i).
Proof.
rewrite /function_r/total_r/univalent_r.
rewrite inv_universal lemma_for_tarski2 unit_identity_is_universal.
split.
apply inc_alpha_universal.
apply inc_alpha_universal.
Qed.
```

Lemma 141 (total_comp) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$ be total relations, then $\alpha \cdot \beta$ is also a total relation.*

```
Lemma total_comp {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 total_r alpha → total_r beta → total_r (alpha ・ beta).
Proof.
rewrite /total_r.
move ⇒ H H0.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ beta).
apply (@inc_trans _ _ _ _ _ H).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H0.
Qed.
```

Lemma 142 (univalent_comp) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$ be univalent relations, then $\alpha \cdot \beta$ is also a univalent relation.*

```
Lemma univalent_comp {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 univalent_r alpha → univalent_r beta → univalent_r (alpha ・ beta).
Proof.
rewrite /univalent_r.
move ⇒ H H0.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ (alpha #)).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H0).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
apply H.
Qed.
```

Lemma 143 (function_comp) *Let $\alpha : A \to B$ and $\beta : B \to C$ be functions, then $\alpha \cdot \beta$ is also a function.*

```
Lemma function_comp {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 function_r alpha → function_r beta → function_r (alpha ・ beta).
Proof.
elim ⇒ H H0.
elim ⇒ H1 H2.
split.
apply (total_comp H H1).
apply (univalent_comp H0 H2).
Qed.
```

Lemma 144 (total_comp2) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$ and $\alpha \cdot \beta$ be a total relation, then α is also a total relation.*

```
Lemma total_comp2 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 total_r (alpha ・ beta) → total_r alpha.
Proof.
move ⇒ H.
apply inc_def1 in H.
rewrite comp_inv cap_comm comp_assoc in H.
rewrite /total_r.
rewrite H.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
apply comp_inc_compat.
apply cap_l.
rewrite comp_id_r.
apply cap_r.
Qed.
```

Lemma 145 (univalent_comp2) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$, $\alpha \cdot \beta$ be a univalent relation and $\alpha^\sharp$ be a total relation, then β is a univalent relation.*

```
Lemma univalent_comp2 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 univalent_r (alpha ・ beta) → total_r (alpha #) → univalent_r beta.
Proof.
move ⇒ H H0.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ _ alpha).
apply comp_inc_compat_ab_ab'.
rewrite /total_r in H0.
rewrite inv_invol in H0.
apply (comp_inc_compat_b_ab H0).
Qed.
```

Lemma 146 (total_inc) *Let $\alpha : A \rightharpoondown B$ be a total relation and $\alpha \sqsubseteq \beta$, then β is also a total relation.*

```
Lemma total_inc {A B : eqType} {alpha beta : Rel A B}:
 total_r alpha → alpha ⊆ beta → total_r beta.
Proof.
move ⇒ H H0.
apply (@inc_trans _ _ _ _ _ H).
apply comp_inc_compat.
apply H0.
```

```
apply (@inc_inv _ _ _ _ H0).
Qed.
```

Lemma 147 (univalent_inc) *Let $\alpha : A \rightharpoonup B$ be a univalent relation and $\beta \sqsubseteq \alpha$, then β is also a univalent relation.*

```
Lemma univalent_inc {A B : eqType} {alpha beta : Rel A B}:
 univalent_r alpha → beta ⊆ alpha → univalent_r beta.
Proof.
move ⇒ H H0.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H).
apply comp_inc_compat.
apply (@inc_inv _ _ _ _ H0).
apply H0.
Qed.
```

Lemma 148 (function_inc) *Let $\alpha, \beta : A \to B$ be functions and $\alpha \sqsubseteq \beta$. Then,*

$$\alpha = \beta.$$

```
Lemma function_inc {A B : eqType} {alpha beta : Rel A B}:
 function_r alpha → function_r beta → alpha ⊆ beta → alpha = beta.
Proof.
move ⇒ H H0 H1.
apply inc_antisym.
apply H1.
apply (@inc_trans _ _ _ ((alpha • alpha #) • beta)).
apply comp_inc_compat_b_ab.
apply H.
move : (@inc_inv _ _ _ _ H1) ⇒ H2.
apply (@inc_trans _ _ _ ((alpha • beta #) • beta)).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
apply H2.
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H0.
Qed.
```

Lemma 149 (total_universal) *If ∇_{IB} be a total relation, then*

$$\nabla_{AB} \cdot \nabla_{BC} = \nabla_{AC}.$$

```
Lemma total_universal {A B C : eqType}:
 total_r (∇ i B) → ∇ A B • ∇ B C = ∇ A C.
Proof.
move ⇒ H.
rewrite -(@lemma_for_tarski2 A B) -(@lemma_for_tarski2 B C).
rewrite comp_assoc -(@comp_assoc _ _ _ _ (∇ i B)).
replace (∇ i B • ∇ B i) with (Id i).
rewrite comp_id_l.
apply lemma_for_tarski2.
apply inc_antisym.
rewrite /total_r in H.
rewrite inv_universal in H.
apply H.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
Qed.
```

Lemma 150 (function_rel_inv_rel) *Let $\alpha : A \to B$ be function. Then,*

$$\alpha \cdot \alpha^\sharp \cdot \alpha = \alpha.$$

```
Lemma function_rel_inv_rel {A B : eqType} {alpha : Rel A B}:
 function_r alpha → (alpha • alpha #) • alpha = alpha.
Proof.
move ⇒ H.
apply inc_antisym.
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H.
apply comp_inc_compat_b_ab.
apply H.
Qed.
```

Lemma 151 (function_capP_distr) *Let $f : A \to B, g : D \to C$ be functions, $\theta : (E \rightharpoondown F) \to (B \rightharpoondown C)$ and P : predicate. Then,*

$$f \cdot (\sqcap_{P(\theta)}\theta(\alpha)) \cdot g^\sharp = \sqcap_{P(\alpha)}(f \cdot \theta(\alpha) \cdot g^\sharp).$$

```
Lemma function_capP_distr {A B C D E F : eqType}
 {f : Rel A B} {g : Rel D C} {theta : Rel E F → Rel B C} {P : Rel E F → Prop}:
 function_r f → function_r g →
 (f • (∩_{P} theta)) • g # =
 ∩_{P} (fun alpha : Rel E F ⇒ (f • theta alpha) • g #).
```

```
Proof.
elim => H H0.
elim => H1 H2.
apply inc_antisym.
apply comp_capP_distr.
apply (@inc_trans _ _ _ (((f ・ f #) ・ ∩_{P} (fun alpha : Rel E F => (f ・ theta alpha)
・ g #)) ・ (g ・ g #))).
apply (@inc_trans _ _ _ ((f ・ f #) ・ (∩_{P} (fun alpha : Rel E F => (f ・ theta alpha)
・ g #)))).
apply (comp_inc_compat_b_ab H).
apply (comp_inc_compat_a_ab H1).
rewrite (@comp_assoc _ _ _ _ _ (f #)) comp_assoc -(@comp_assoc _ _ _ _ _ g) -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ (∩_{P} (fun alpha : Rel E F => (f # ・ ((f ・ theta alpha) ・ g
#)) ・ g))).
apply comp_capP_distr.
replace (fun alpha : Rel E F => (f # ・ ((f ・ theta alpha) ・ g #)) ・ g) with (fun alpha
: Rel E F => ((f # ・ f) ・ theta alpha) ・ (g # ・ g)).
apply inc_capP.
move => beta H3.
apply (@inc_trans _ _ _ ((f # ・ f) ・ theta beta)).
apply (@inc_trans _ _ _ (((f # ・ f) ・ theta beta) ・ (g # ・ g))).
move : beta H3.
apply inc_capP.
apply inc_refl.
apply (comp_inc_compat_ab_a H2).
apply (comp_inc_compat_ab_b H0).
apply functional_extensionality.
move => x.
by [rewrite comp_assoc comp_assoc comp_assoc comp_assoc comp_assoc].
Qed.
```

Lemma 152 (function_cap_distr, function_cap_distr_l, function_cap_distr_r) *Let $f : A \to B, g : D \to C$ be functions and $\alpha, \beta : B \rightharpoondown C$. Then,*

$$f \cdot (\alpha \sqcap \beta) \cdot g^\sharp = (f \cdot \alpha \cdot g^\sharp) \sqcap (f \cdot \beta \cdot g^\sharp).$$

```
Lemma function_cap_distr
 {A B C D : eqType} {f : Rel A B} {alpha beta : Rel B C} {g : Rel D C}:
 function_r f -> function_r g ->
 (f ・ (alpha ∩ beta)) ・ g # = ((f ・ alpha) ・ g #) ∩ ((f ・ beta) ・ g #).
Proof.
```

```
rewrite (@cap_to_capP _ _ _ _ _ _ id) (@cap_to_capP _ _ _ _ _ _ (fun x ⇒ (f • x) • g
#)).
apply function_capP_distr.
Qed.
```

```
Lemma function_cap_distr_l
 {A B C : eqType} {f : Rel A B} {alpha beta : Rel B C}:
 function_r f →
 f • (alpha ∩ beta) = (f • alpha) ∩ (f • beta).
Proof.
move : (@id_function C) ⇒ H.
move ⇒ H0.
apply (@function_cap_distr _ _ _ _ f alpha beta) in H.
rewrite inv_id comp_id_r comp_id_r comp_id_r in H.
apply H.
apply H0.
Qed.
```

```
Lemma function_cap_distr_r
 {B C D : eqType} {alpha beta : Rel B C} {g : Rel D C}:
 function_r g →
 (alpha ∩ beta) • g # = (alpha • g #) ∩ (beta • g #).
Proof.
move : (@id_function B) ⇒ H.
move ⇒ H0.
apply (@function_cap_distr _ _ _ _ _ alpha beta g) in H.
rewrite comp_id_l comp_id_l comp_id_l in H.
apply H.
apply H0.
Qed.
```

Lemma 153 (function_move1) *Let $\alpha : A \to B$ be a function, $\beta : B \rightharpoondown C$ and $\gamma : A \rightharpoondown C$. Then,*

$$\gamma \sqsubseteq \alpha \cdot \beta \Leftrightarrow \alpha^\sharp \cdot \gamma \sqsubseteq \beta.$$

```
Lemma function_move1 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma :
Rel A C}:
 function_r alpha → (gamma ⊆ (alpha • beta) ↔ (alpha # • gamma) ⊆ beta).
Proof.
move ⇒ H.
split; move ⇒ H0.
apply (@inc_trans _ _ _ ((alpha # • alpha) • beta)).
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H0).
```

```
apply comp_inc_compat_ab_b.
apply H.
apply (@inc_trans _ _ _ ((alpha · alpha #) · gamma)).
apply comp_inc_compat_b_ab.
apply H.
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H0).
Qed.
```

Lemma 154 (function_move2) *Let $\beta : B \to C$ be a function, $\alpha : A \rightharpoonup B$ and $\gamma : A \rightharpoonup C$. Then,*

$$\alpha \cdot \beta \sqsubseteq \gamma \Leftrightarrow \alpha \sqsubseteq \gamma \cdot \beta^\sharp.$$

```
Lemma function_move2 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma :
Rel A C}:
 function_r beta → ((alpha · beta) ⊆ gamma ↔ alpha ⊆ (gamma · beta #)).
Proof.
move ⇒ H.
split; move ⇒ H0.
apply (@inc_trans _ _ _ ((alpha · beta) · beta #)).
rewrite comp_assoc.
apply comp_inc_compat_a_ab.
apply H.
apply (comp_inc_compat_ab_a'b H0).
apply (@inc_trans _ _ _ ((gamma · beta #) · beta)).
apply (comp_inc_compat_ab_a'b H0).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H.
Qed.
```

Lemma 155 (function_rpc_distr) *Let $f : A \to B, g : D \to C$ be functions and $\alpha, \beta : B \rightharpoonup C$. Then,*

$$f \cdot (\alpha \Rightarrow \beta) \cdot g^\sharp = (f \cdot \alpha \cdot g^\sharp) \Rightarrow (f \cdot \beta \cdot g^\sharp).$$

```
Lemma function_rpc_distr
 {A B C D : eqType} {f : Rel A B} {alpha beta : Rel B C} {g : Rel D C}:
 function_r f → function_r g →
 (f · (alpha » beta)) · g # = ((f · alpha) · g #) » ((f · beta) · g #).
Proof.
move ⇒ H H0.
apply inc_lower.
```

```
move => gamma.
split; move => H1.
apply inc_rpc.
apply (function_move2 H0).
apply (function_move1 H).
apply (@inc_trans _ _ _ (((f # · gamma) · g) ∩ ((f # · ((f · alpha) · g #)) · g))).
rewrite -comp_assoc.
apply (fun H' => @inc_trans _ _ _ _ _ H' (@comp_cap_distr_r _ _ _ _ _ _)).
apply comp_inc_compat_ab_a'b.
apply comp_cap_distr_l.
apply (function_move2 H0) in H1.
apply (function_move1 H) in H1.
rewrite -inc_rpc comp_assoc.
apply (@inc_trans _ _ _ _ _ H1).
apply rpc_inc_compat_r.
rewrite comp_assoc comp_assoc comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ (alpha · (g # · g))).
apply comp_inc_compat_ab_b.
apply H.
apply comp_inc_compat_ab_a.
apply H0.
apply (function_move2 H0).
apply (function_move1 H).
apply inc_rpc.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
apply (@inc_trans _ _ _ (f # · ((gamma · g) ∩ ((f #) # · alpha)))).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite inv_invol.
apply (@inc_trans _ _ _ ((f # · (gamma ∩ ((f · alpha) · g #))) · g)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
apply comp_inc_compat_ab_ab'.
apply cap_l.
apply (function_move2 H0).
apply (function_move1 H).
rewrite -inc_rpc -comp_assoc.
apply H1.
Qed.
```

Lemma 156 (function_inv_rel1, function_inv_rel2) *Let $f : A \to B$ be a function. Then,*

$$f^\sharp \cdot f = id_B \sqcap f^\sharp \cdot \nabla_{AA} \cdot f = id_B \sqcap \nabla_{BA} \cdot f.$$

```
Lemma function_inv_rel1 {A B : eqType} {f : Rel A B}:
 function_r f → f # • f = Id B ∩ ((f # • ∇ A A) • f).
Proof.
move ⇒ H.
apply inc_antisym.
apply inc_cap.
split.
apply H.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_a_ab.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ (Id B ∩ (∇ B A • f))).
apply cap_inc_compat_l.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
rewrite comp_id_l comp_id_r cap_comm inv_universal.
rewrite cap_universal cap_universal.
apply inc_refl.
Qed.
```

```
Lemma function_inv_rel2 {A B : eqType} {f : Rel A B}:
 function_r f → f # • f = Id B ∩ (∇ B A • f).
Proof.
move ⇒ H.
apply inc_antisym.
rewrite (@function_inv_rel1 _ _ _ H).
apply cap_inc_compat_l.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
rewrite comp_id_l comp_id_r cap_comm inv_universal.
rewrite cap_universal cap_universal.
apply inc_refl.
Qed.
```

Lemma 157 (function_dedekind1, function_dedekind2) *Let $f : A \to B$ be a function, $\mu : C \rightharpoondown A$ and $\rho : C \rightharpoondown B$. Then,*

$$(\mu \sqcap \rho \cdot f^\sharp) \cdot f = \mu \cdot f \sqcap \rho \wedge \rho \cdot f^\sharp \cdot f = \nabla_{CA} \cdot f \sqcap \rho.$$

```
Lemma function_dedekind1
 {A B C : eqType} {f : Rel A B} {mu : Rel C A} {rho : Rel C B}:
 function_r f → (mu ∩ (rho • f #)) • f = (mu • f) ∩ rho.
Proof.
move ⇒ H.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat_l.
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
apply comp_inc_compat_ab_ab'.
apply cap_l.
Qed.
```

```
Lemma function_dedekind2 {A B C : eqType} {f : Rel A B} {rho : Rel C B}:
 function_r f → (rho • f #) • f = (∇ C A • f) ∩ rho.
Proof.
move ⇒ H.
move : (@function_dedekind1 _ _ _ f (∇ C A) rho H) ⇒ H0.
rewrite cap_comm cap_universal in H0.
apply H0.
Qed.
```

6.2 全射, 単射に関する補題

Lemma 158 (surjection_comp) *Let $\alpha : A \rightharpoondown B$ and $\beta : B \rightharpoondown C$ be surjections, then $\alpha \cdot \beta$ is also a surjection.*

```
Lemma surjection_comp {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 surjection_r alpha → surjection_r beta → surjection_r (alpha • beta).
Proof.
rewrite /surjection_r.
elim ⇒ H H0.
elim ⇒ H1 H2.
split.
```

```
apply (function_comp H H1).
rewrite comp_inv.
apply (total_comp H2 H0).
Qed.
```

Lemma 159 (injection_comp) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$ be injections, then $\alpha \cdot \beta$ is also an injection.*

```
Lemma injection_comp {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 injection_r alpha → injection_r beta → injection_r (alpha • beta).
Proof.
rewrite /injection_r.
elim ⇒ H H0.
elim ⇒ H1 H2.
split.
apply (function_comp H H1).
rewrite comp_inv.
apply (univalent_comp H2 H0).
Qed.
```

Lemma 160 (bijection_comp) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$ be bijections, then $\alpha \cdot \beta$ is also a bijection.*

```
Lemma bijection_comp {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 bijection_r alpha → bijection_r beta → bijection_r (alpha • beta).
Proof.
rewrite /bijection_r.
elim ⇒ H.
elim ⇒ H0 H1.
elim ⇒ H2.
elim ⇒ H3 H4.
split.
apply (function_comp H H2).
rewrite comp_inv.
split.
apply (total_comp H3 H0).
apply (univalent_comp H4 H1).
Qed.
```

Lemma 161 (surjection_unique1) *Let $e : A \twoheadrightarrow B$ be a surjection, $f : A \to C$ be a function and $e \cdot e^\sharp \sqsubseteq f \cdot f^\sharp$, then there exists a unique function $g : B \to C$ s.t. $f = eg$.*

```
Lemma surjection_unique1 {A B C : eqType} {e : Rel A B} {f : Rel A C}:
```

```
 surjection_r e → function_r f → (e • e #) ⊆ (f • f #) →
 (∃! g : Rel B C, function_r g ∧ f = e • g).
Proof.
rewrite /surjection_r/function_r/total_r/univalent_r.
elim.
elim ⇒ H H0 H1.
elim ⇒ H2 H3 H4.
∃ (e # • f).
repeat split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ f).
apply (@inc_trans _ _ _ _ _ H1).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H2.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ e).
apply (@inc_trans _ _ _ (f # • ((f • f #) • f))).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H4).
rewrite comp_assoc -comp_assoc.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H3).
apply (comp_inc_compat_ab_a H3).
apply function_inc.
split.
apply H2.
apply H3.
split.
rewrite /total_r.
rewrite comp_inv comp_inv inv_invol.
rewrite -(@comp_assoc _ _ _ _ e) (@comp_assoc _ _ _ _ _ e) (@comp_assoc _ _ _ _ _ f)
-(@comp_assoc _ _ _ _ f).
apply (@inc_trans _ _ _ _ _ H).
apply comp_inc_compat_a_ab.
apply (@inc_trans _ _ _ _ _ H2).
apply (comp_inc_compat_a_ab H).
rewrite /univalent_r.
rewrite comp_inv comp_inv inv_invol.
rewrite (@comp_assoc _ _ _ _ _ e) -(@comp_assoc _ _ _ _ e) comp_assoc -(@comp_assoc
_ _ _ _ _ _ f).
apply (@inc_trans _ _ _ (f # • (((f • f #) • (f • f #)) • f))).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat.
```

```
apply H4.
apply H4.
rewrite comp_assoc (@comp_assoc _ _ _ _ _ _ f) -(@comp_assoc _ _ _ _ (f #)) -(@comp_assoc
_ _ _ _ (f #)) (@comp_assoc _ _ _ _ _ _ (f #)) -(@comp_assoc _ _ _ _ (f #)).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H3).
apply comp_inc_compat_ab_a.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H3).
apply (comp_inc_compat_ab_a H3).
rewrite -comp_assoc.
apply (comp_inc_compat_b_ab H).
move ⇒ g.
elim.
elim ⇒ H5 H6 H7.
replace g with (e # · (e · g)).
apply f_equal.
apply H7.
rewrite -comp_assoc.
apply inc_antisym.
apply (comp_inc_compat_ab_b H0).
rewrite inv_invol in H1.
apply (comp_inc_compat_b_ab H1).
Qed.
```

Lemma 162 (surjection_unique2) *Let $e : A \twoheadrightarrow B$ be a surjection, $f : A \to C$ be a function and $e \cdot e^\sharp = f \cdot f^\sharp$, then function $e^\sharp f$ is an injection.*

```
Lemma surjection_unique2 {A B C : eqType} {e : Rel A B} {f : Rel A C}:
 surjection_r e → function_r f → (e · e #) = (f · f #) → injection_r (e # · f).
Proof.
rewrite /surjection_r/injection_r/function_r/total_r/univalent_r.
elim.
elim ⇒ H H0 H1.
elim ⇒ H2 H3 H4.
repeat split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ f).
apply (@inc_trans _ _ _ _ _ H1).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H2.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ e).
rewrite H4.
rewrite comp_assoc -comp_assoc.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H3).
```

```
apply (comp_inc_compat_ab_a H3).
rewrite inv_invol comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ f).
rewrite -H4.
rewrite comp_assoc -comp_assoc.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H0).
apply comp_inc_compat_ab_a.
apply H0.
Qed.
```

Lemma 163 (injection_unique1) *Let $m : B \rightarrowtail A$ be an injection, $f : C \to A$ be a function and $f^\sharp \cdot f \sqsubseteq m^\sharp \cdot m$, then there exists a unique function $g : C \to B$ s.t. $f = gm$.*

```
Lemma injection_unique1 {A B C : eqType} {m : Rel B A} {f : Rel C A}:
 injection_r m → function_r f → (f # • f) ⊆ (m # • m) →
 (∃! g : Rel C B, function_r g ∧ f = g • m).
Proof.
rewrite /injection_r/function_r/total_r/univalent_r.
elim.
elim ⇒ H H0 H1.
elim ⇒ H2 H3 H4.
∃ (f • m #).
repeat split.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ _ m).
apply (@inc_trans _ _ _ (f • ((f # • f) • f #))).
rewrite comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ _ _ H2).
apply (comp_inc_compat_a_ab H2).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H4).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ _ f).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H1).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).
rewrite comp_assoc.
apply Logic.eq_sym.
apply function_inc.
split.
rewrite /total_r.
rewrite comp_inv comp_inv inv_invol.
apply (@inc_trans _ _ _ _ _ H2).
apply comp_inc_compat.
apply (@inc_trans _ _ _ (f • (f # • f))).
rewrite -comp_assoc.
```

```
apply (comp_inc_compat_b_ab H2).
apply (comp_inc_compat_ab_ab' H4).
apply (@inc_trans _ _ _ ((f # • f) • f #)).
rewrite comp_assoc.
apply (comp_inc_compat_a_ab H2).
apply (comp_inc_compat_ab_a'b H4).
rewrite /univalent_r.
rewrite comp_inv comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ _ f).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H0).
apply comp_inc_compat_ab_a.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H3).
apply (comp_inc_compat_ab_a H0).
split.
apply H2.
apply H3.
apply (comp_inc_compat_ab_a H0).
move ⇒ g.
elim.
elim ⇒ H5 H6 H7.
rewrite H7 comp_assoc.
apply inc_antisym.
rewrite inv_invol in H1.
apply (comp_inc_compat_ab_a H1).
apply (comp_inc_compat_a_ab H).
Qed.
```

Lemma 164 (injection_unique2) *Let $m : B \rightarrowtail A$ be an injection, $f : C \to A$ be a function and $f^\sharp \cdot f = m^\sharp \cdot m$, then function $f \cdot m^\sharp$ is a surjection.*

```
Lemma injection_unique2 {A B C : eqType} {m : Rel B A} {f : Rel C A}:
 injection_r m → function_r f → (f # • f) = (m # • m) → surjection_r (f • m #).
Proof.
rewrite /surjection_r/injection_r/function_r/total_r/univalent_r.
elim.
elim ⇒ H H0 H1.
elim ⇒ H2 H3 H4.
repeat split.
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ _ m).
apply (@inc_trans _ _ _ (f • ((f # • f) • f #))).
rewrite comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ _ _ H2).
apply (comp_inc_compat_a_ab H2).
apply comp_inc_compat_ab_ab'.
```

```
rewrite H4.
apply inc_refl.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ _ f).
apply (fun H' => @inc_trans _ _ _ _ _ H' H1).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).
rewrite inv_invol comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ _ f).
apply (@inc_trans _ _ _ _ _ H).
apply comp_inc_compat_ab_ab'.
rewrite H4 comp_assoc.
apply (comp_inc_compat_a_ab H).
Qed.
```

Lemma 165 (bijection_inv) *Let $\alpha : A \rightharpoonup B$, $\beta : B \rightharpoonup A$, $\alpha \cdot \beta = id_A$ and $\beta \cdot \alpha = id_B$, then α and β are bijections and $\beta = \alpha^\sharp$.*

```
Lemma bijection_inv {A B : eqType} {alpha : Rel A B} {beta : Rel B A}:
 alpha • beta = Id A → beta • alpha = Id B → bijection_r alpha ∧ bijection_r beta ∧
beta = alpha #.
Proof.
move => H H0.
move : (@id_function A) => H1.
move : (@id_function B) => H2.
assert (bijection_r alpha ∧ bijection_r beta).
assert (total_r alpha ∧ total_r (alpha #) ∧ total_r beta ∧ total_r (beta #)).
repeat split.
apply (@total_comp2 _ _ _ _ beta).
rewrite H.
apply H1.
apply (@total_comp2 _ _ _ _ (beta #)).
rewrite -comp_inv H0 inv_id.
apply H2.
apply (@total_comp2 _ _ _ _ alpha).
rewrite H0.
apply H2.
apply (@total_comp2 _ _ _ _ (alpha #)).
rewrite -comp_inv H inv_id.
apply H1.
repeat split.
apply H3.
apply (@univalent_comp2 _ _ _ beta).
rewrite H0.
apply H2.
```

```
apply H3.
apply H3.
apply (@univalent_comp2 _ _ _ (beta #)).
rewrite -comp_inv H inv_id.
apply H1.
rewrite inv_invol.
apply H3.
apply H3.
apply (@univalent_comp2 _ _ _ alpha).
rewrite H.
apply H1.
apply H3.
apply H3.
apply (@univalent_comp2 _ _ _ (alpha #)).
rewrite -comp_inv H0 inv_id.
apply H2.
rewrite inv_invol.
apply H3.
split.
apply H3.
split.
apply H3.
rewrite -(@comp_id_r _ _ beta) -(@comp_id_l _ _ (alpha #)).
rewrite -H0 comp_assoc.
apply f_equal.
apply inc_antisym.
apply H3.
rewrite comp_inv_inv -inv_inc_move inv_id.
apply H3.
Qed.
```

Lemma 166 (bijection_inv_corollary) *Let $\alpha : A \rightharpoondown B$ be a bijection, then $\alpha^\sharp$ is also a bijection.*

```
Lemma bijection_inv_corollary {A B : eqType} {alpha : Rel A B}:
 bijection_r alpha → bijection_r (alpha #).
Proof.
move : (@bijection_inv _ _ alpha (alpha #)) ⇒ H.
move ⇒ H0.
rewrite /bijection_r/function_r/total_r/univalent_r in H0.
rewrite inv_invol in H0.
apply H.
```

```
apply inc_antisym.
apply H0.
apply H0.
apply inc_antisym.
apply H0.
apply H0.
Qed.
```

Chapter 7

Library Dedekind

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
```

7.1 Dedekind formula に関する補題

Lemma 167 (dedekind1) *Let $\alpha : A \rightharpoonup B$, $\beta : B \rightharpoonup C$ and $\gamma : A \rightharpoonup C$. Then*

$$\alpha \cdot \beta \sqcap \gamma \sqsubseteq \alpha \cdot (\beta \sqcap \alpha^\sharp \cdot \gamma).$$

```
Lemma dedekind1
 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel A C}:
 ((alpha ・ beta) ∩ gamma) ⊆ (alpha ・ (beta ∩ (alpha # ・ gamma))).
Proof.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
apply comp_inc_compat_ab_a'b.
apply cap_l.
Qed.
```

Lemma 168 (dedekind2) *Let $\alpha : A \rightharpoonup B$, $\beta : B \rightharpoonup C$ and $\gamma : A \rightharpoonup C$. Then*

$$\alpha \cdot \beta \sqcap \gamma \sqsubseteq (\alpha \sqcap \gamma \cdot \beta^\sharp) \cdot \beta.$$

```
Lemma dedekind2
 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel A C}:
 ((alpha ・ beta) ∩ gamma) ⊆ ((alpha ∩ (gamma ・ beta #)) ・ beta).
Proof.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
```

```
apply comp_inc_compat_ab_ab'.
apply cap_l.
Qed.
```

Lemma 169 (relation_rel_inv_rel) *Let $\alpha : A \rightharpoondown B$. Then*

$$\alpha \sqsubseteq \alpha \cdot \alpha^\sharp \cdot \alpha.$$

```
Lemma relation_rel_inv_rel {A B : eqType} {alpha : Rel A B}:
 alpha ⊆ ((alpha • alpha #) • alpha).
Proof.
move : (@dedekind1 _ _ _ alpha (Id B) alpha) ⇒ H.
rewrite comp_id_r cap_idem in H.
apply (@inc_trans _ _ _ _ _ H).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply cap_r.
Qed.
```

7.2 Dedekind formula と全関係

Lemma 170 (dedekind_universal1) *Let $\alpha : B \rightharpoondown C$. Then*

$$\nabla_{AC} \cdot \alpha^\sharp \cdot \alpha = \nabla_{AB} \cdot \alpha.$$

```
Lemma dedekind_universal1 {A B C : eqType} {alpha : Rel B C}:
 (∇ A C • alpha #) • alpha = ∇ A B • alpha.
Proof.
apply inc_antisym.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ (∇ A B • ((alpha • alpha #) • alpha))).
apply comp_inc_compat_ab_ab'.
apply relation_rel_inv_rel.
rewrite -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
Qed.
```

Lemma 171 (dedekind_universal2a, dedekind_universal2b, dedekind_universal2c) *Let $\alpha : A \rightharpoondown B$ and $\beta : C \rightharpoondown B$. Then*

$$\nabla_{IC} \cdot \beta \sqsubseteq \nabla_{IA} \cdot \alpha \Leftrightarrow \nabla_{CC} \cdot \beta \sqsubseteq \nabla_{CA} \cdot \alpha \Leftrightarrow \beta \sqsubseteq \beta \cdot \alpha^\sharp \cdot \alpha.$$

```
Lemma dedekind_universal2a {A B C : eqType} {alpha : Rel A B} {beta : Rel C B}:
 (∇ i C ・ beta) ⊆ (∇ i A ・ alpha) → (∇ C C ・ beta) ⊆ (∇ C A ・ alpha).
Proof.
move ⇒ H.
rewrite -unit_universal -(@lemma_for_tarski2 C A).
rewrite comp_assoc comp_assoc.
apply (comp_inc_compat_ab_ab' H).
Qed.
```

```
Lemma dedekind_universal2b {A B C : eqType} {alpha : Rel A B} {beta : Rel C B}:
 (∇ C C ・ beta) ⊆ (∇ C A ・ alpha) → beta ⊆ ((beta ・ alpha #) ・ alpha).
Proof.
move ⇒ H.
apply (@inc_trans _ _ _ (beta ∩ (∇ C C ・ beta))).
apply inc_cap.
split.
apply inc_refl.
apply comp_inc_compat_b_ab.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ (beta ∩ (∇ C A ・ alpha))).
apply (cap_inc_compat_l H).
rewrite cap_comm.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
apply cap_r.
Qed.
```

```
Lemma dedekind_universal2c {A B C : eqType} {alpha : Rel A B} {beta : Rel C B}:
 beta ⊆ ((beta ・ alpha #) ・ alpha) → (∇ i C ・ beta) ⊆ (∇ i A ・ alpha).
Proof.
move ⇒ H.
apply (@inc_trans _ _ _ (∇ i C ・ ((beta ・ alpha #) ・ alpha))).
apply (comp_inc_compat_ab_ab' H).
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
Qed.
```

Lemma 172 (dedekind_universal3a, dedekind_universal3b) *Let* $\alpha : A \rightharpoondown B$ *and* $\beta : A \rightharpoondown C$*. Then*

$$\beta \cdot \nabla_{CI} \sqsubseteq \alpha \cdot \nabla_{BI} \Leftrightarrow \beta \cdot \nabla_{CC} \sqsubseteq \alpha \cdot \nabla_{BC} \Leftrightarrow \beta \sqsubseteq \alpha \cdot \alpha^{\sharp} \cdot \beta.$$

```
Lemma dedekind_universal3a {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 (beta ・ ∇ C i) ⊆ (alpha ・ ∇ B i) ↔ (beta ・ ∇ C C) ⊆ (alpha ・ ∇ B C).
Proof.
split; move ⇒ H.
apply inv_inc_invol.
rewrite comp_inv comp_inv inv_universal inv_universal.
apply dedekind_universal2a.
apply inv_inc_invol.
rewrite comp_inv comp_inv inv_invol inv_invol inv_universal inv_universal.
apply H.
apply inv_inc_invol.
rewrite comp_inv comp_inv inv_universal inv_universal.
apply dedekind_universal2c.
apply dedekind_universal2b.
apply inv_inc_invol.
rewrite comp_inv comp_inv inv_invol inv_invol inv_universal inv_universal.
apply H.
Qed.
```

```
Lemma dedekind_universal3b {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 (beta ・ ∇ C i) ⊆ (alpha ・ ∇ B i) ↔ beta ⊆ ((alpha ・ alpha #) ・ beta).
Proof.
split; move ⇒ H.
apply inv_inc_invol.
rewrite comp_inv comp_inv -comp_assoc.
apply dedekind_universal2b.
apply dedekind_universal2a.
apply inv_inc_invol.
rewrite comp_inv comp_inv inv_invol inv_invol inv_universal inv_universal.
apply H.
apply inv_inc_invol.
rewrite comp_inv comp_inv inv_universal inv_universal.
apply dedekind_universal2c.
rewrite -comp_inv -comp_inv -comp_assoc.
apply inc_inv.
apply H.
Qed.
```

Lemma 173 (universal_total) *Let $\alpha : A \rightharpoondown B$. Then*

$$\alpha \cdot \nabla_{BI} = \nabla_{AI} \Leftrightarrow \text{“}\alpha \textit{ is total”}.$$

```
Lemma universal_total {A B : eqType} {alpha : Rel A B}:
 alpha • ∇ B i = ∇ A i ↔ total_r alpha.
Proof.
move : (@dedekind_universal3b _ _ _ alpha (Id A)) ⇒ H.
rewrite comp_id_l comp_id_r in H.
rewrite /total_r.
rewrite -H.
split; move ⇒ H0.
rewrite H0.
apply inc_refl.
apply inc_antisym.
apply inc_alpha_universal.
apply H0.
Qed.
```

7.3 Dedekind formula と恒等関係

Lemma 174 (dedekind_id1) *Let $\alpha : A \rightharpoondown A$. Then*

$$\alpha \sqsubseteq id_A \Rightarrow \alpha^\sharp = \alpha.$$

```
Lemma dedekind_id1 {A : eqType} {alpha : Rel A A}: alpha ⊆ Id A → alpha # = alpha.
Proof.
move ⇒ H.
assert (alpha # ⊆ alpha).
move : (@dedekind1 _ _ _ (alpha #) (Id A) (Id A)) ⇒ H0.
rewrite comp_id_r comp_id_r inv_invol in H0.
replace (alpha # ∩ Id A) with (alpha #) in H0.
replace (Id A ∩ alpha) with alpha in H0.
apply (@inc_trans _ _ _ (alpha # • alpha)).
apply H0.
apply comp_inc_compat_ab_b.
rewrite -inv_inc_move.
rewrite inv_id.
apply H.
rewrite cap_comm.
apply inc_def1.
```

```
apply H.
apply inc_def1.
rewrite -inv_inc_move.
rewrite inv_id.
apply H.
apply inc_antisym.
apply H0.
apply inv_inc_move.
apply H0.
Qed.
```

Lemma 175 (dedekind_id2) *Let $\alpha : A \rightharpoondown A$. Then*

$$\alpha \sqsubseteq id_A \Rightarrow \alpha \cdot \alpha = \alpha.$$

```
Lemma dedekind_id2 {A : eqType} {alpha : Rel A A}:
 alpha ⊆ Id A → alpha • alpha = alpha.
Proof.
move ⇒ H.
apply inc_antisym.
apply (comp_inc_compat_ab_a H).
move : (dedekind_id1 H) ⇒ H0.
apply (@inc_trans _ _ _ ((alpha • Id A) ∩ Id A)).
rewrite comp_id_r.
apply inc_cap.
split.
apply inc_refl.
apply H.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite H0 comp_id_r.
apply cap_r.
Qed.
```

Lemma 176 (dedekind_id3) *Let $\alpha, \beta : A \rightharpoondown A$. Then*

$$\alpha \sqsubseteq id_A \wedge \beta \sqsubseteq id_A \Rightarrow \alpha \cdot \beta = \alpha \sqcap \beta.$$

```
Lemma dedekind_id3 {A : eqType} {alpha beta : Rel A A}:
 alpha ⊆ Id A → beta ⊆ Id A → alpha • beta = alpha ∩ beta.
Proof.
move ⇒ H H0.
apply inc_antisym.
```

```
apply inc_cap.
split.
apply (comp_inc_compat_ab_a H0).
apply (comp_inc_compat_ab_b H).
replace (alpha ∩ beta) with ((alpha ∩ beta) • (alpha ∩ beta)).
apply comp_inc_compat.
apply cap_l.
apply cap_r.
apply dedekind_id2.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H).
apply cap_l.
Qed.
```

Lemma 177 (dedekind_id4) *Let $\alpha, \beta : A \rightharpoondown A$. Then*

$$\alpha \sqsubseteq id_A \wedge \beta \sqsubseteq id_A \Rightarrow (\alpha \rhd \beta) \sqcap id_A = (\alpha \Rightarrow \beta) \sqcap id_A.$$

```
Lemma dedekind_id4 {A : eqType} {alpha beta : Rel A A}:
 alpha ⊆ Id A → beta ⊆ Id A → (alpha △ beta) ∩ Id A = (alpha » beta) ∩ Id A.
Proof.
move ⇒ H H0.
apply inc_lower.
move ⇒ gamma.
rewrite inc_cap inc_cap.
split; elim ⇒ H1 H2.
split.
rewrite inc_rpc cap_comm.
rewrite -(@dedekind_id3 _ _ _ H H2).
rewrite -(@dedekind_id1 _ _ H).
apply inc_residual.
apply H1.
apply H2.
split.
rewrite inc_residual (@dedekind_id1 _ _ H) (@dedekind_id3 _ _ _ H H2).
rewrite cap_comm -inc_rpc.
apply H1.
apply H2.
Qed.
```

Chapter 8

Library Rationality

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
```

8.1 有理性から導かれる系

Lemma 178 (rationality_corollary1) *Let $u : A \rightharpoonup A$ and $u \sqsubseteq id_A$. Then,*

$$\exists R, \exists j : R \rightarrowtail A, u = j^\sharp \cdot j.$$

```
Lemma rationality_corollary1 {A : eqType} {u : Rel A A}:
 u ⊆ Id A → ∃ (R : eqType)(j : Rel R A), injection_r j ∧ u = j # • j.
Proof.
move : (rationality _ _ u).
elim ⇒ R.
elim ⇒ f.
elim ⇒ g.
elim ⇒ H.
elim ⇒ H0.
elim ⇒ H1 H2 H3.
∃ R.
∃ f.
assert (g = f).
apply (function_inc H0 H).
apply (@inc_trans _ _ _ ((f • f #) • g)).
apply comp_inc_compat_b_ab.
apply H.
rewrite comp_assoc -H1.
```

```
apply (comp_inc_compat_ab_a H3).
rewrite H4 in H1.
rewrite H4 cap_idem in H2.
split.
split.
apply H.
rewrite /univalent_r.
rewrite inv_invol H2.
apply inc_refl.
apply H1.
Qed.
```

Lemma 179 (rationality_corollary2) *Let $f : A \to B$ be a function. Then,*

$$\exists e : A \twoheadrightarrow R, \exists m : R \rightarrowtail B, f = e \cdot m.$$

```
Lemma rationality_corollary2 {A B : eqType} {f : Rel A B}:
 function_r f → ∃ (R : eqType)(e : Rel A R)(m : Rel R B), surjection_r e ∧ injection_r
m.
Proof.
elim ⇒ H H0.
move : (@rationality_corollary1 _ (f # • f) H0).
elim ⇒ R.
elim ⇒ m.
elim ⇒ H1 H2.
∃ R.
∃ (f • m #).
∃ m.
split.
apply (injection_unique2 H1 (conj H H0) H2).
apply H1.
Qed.
```

Chapter 9

Library Conjugate

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
```

9.1 共役性の定義

条件 P を満たす関係 $\alpha : A \rightharpoondown B$ と条件 Q を満たす関係 $\beta : A' \rightharpoondown B'$ が変換 $\alpha = \phi(\beta), \beta = \psi(\alpha)$ によって, 1 対 1 (全射的) に対応することを, 図式

$$\frac{\alpha : A \rightharpoondown B\ \{P\}}{\beta : A' \rightharpoondown B'\ \{Q\}}\ \frac{\alpha = \phi(\beta)}{\beta = \psi(\alpha)}$$

によって表す. また, Coq では以下のように表すことにする.

```
Definition conjugate
 (A B C D : eqType) (P : Rel A B → Prop) (Q : Rel C D → Prop)
 (phi : Rel C D → Rel A B) (psi : Rel A B → Rel C D):=
 (∀ alpha : Rel A B, P alpha → Q (psi alpha) ∧ phi (psi alpha) = alpha)
 ∧ (∀ beta : Rel C D, Q beta → P (phi beta) ∧ psi (phi beta) = beta).
```

さらに, 上の図式において条件 P または Q が不要な場合には, 以下の `True_r` を代入する.

```
Definition True_r {A B : eqType} := fun _ : Rel A B ⇒ True.
```

9.2 共役の例

Lemma 180 (inv_conjugate) *Inverse relation* ($^\sharp$) *makes conjugate. That is,*

$$\frac{\alpha : A \rightharpoondown B}{\beta : B \rightharpoondown A} \; \frac{\alpha = \beta^\sharp}{\beta = \alpha^\sharp}.$$

```
Lemma inv_conjugate {A B : eqType}:
 conjugate A B B A True_r True_r (@inverse _ _) (@inverse _ _).
Proof.
split.
move => alpha H.
split.
by [].
apply inv_invol.
move => beta H.
split.
by [].
apply inv_invol.
Qed.
```

Lemma 181 (injection_conjugate) *Let* $j : C \rightarrowtail B$ *be an injection. Then,*

$$\frac{f : A \to B \; \{f^\sharp \cdot f \sqsubseteq j^\sharp \cdot j\}}{h : A \to C} \; \frac{f = h \cdot j}{h = f \cdot j^\sharp}$$

```
Lemma injection_conjugate {A B C : eqType} {j : Rel C B}:
 injection_r j →
 conjugate A B A C (fun f : Rel A B ⇒ ((f # • f) ⊆ (j # • j)) ∧ function_r f)
 (fun h : Rel A C ⇒ function_r h) (fun h : Rel A C ⇒ h • j) (fun f : Rel A B ⇒ f •
j #).
Proof.
elim.
elim => H H0 H1.
split.
move => alpha.
elim => H2.
elim => H3 H4.
assert (function_r (alpha • j #)).
split.
apply (@inc_trans _ _ _ _ _ H3).
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ _ j).
```

```
apply (@inc_trans _ _ _ (alpha · ((alpha # · alpha) · alpha #))).
rewrite comp_assoc -comp_assoc.
apply (comp_inc_compat_a_ab H3).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H2).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H1).
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ _ alpha).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
apply (@inc_trans _ _ _ _ _ H2).
apply H0.
split.
apply H5.
apply function_inc.
apply function_comp.
apply H5.
split.
apply H.
apply H0.
split.
apply H3.
apply H4.
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H0.
move ⇒ beta.
elim ⇒ H2 H3.
assert (function_r (beta · j)).
split.
apply (@inc_trans _ _ _ _ _ H2).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ j).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H0).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).
split.
split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H3).
```

```
apply H4.
rewrite comp_assoc.
replace (j • j #) with (Id C).
apply comp_id_r.
apply inc_antisym.
apply H.
rewrite /univalent_r in H1.
rewrite inv_invol in H1.
apply H1.
Qed.
```

Lemma 182 (injection_conjugate_corollary1, injection_conjugate_corollary2) *Let $j : C \rightarrowtail B$ be an injection and $f : A \to B$ be a function. Then,*

$$f^\sharp \cdot f \sqsubseteq j^\sharp \cdot j \Leftrightarrow (\exists! h : A \to C, f = h \cdot j) \Leftrightarrow (\exists h' : A \rightharpoonup C, f \sqsubseteq h' \cdot j).$$

```
Lemma injection_conjugate_corollary1 {A B C : eqType} {f : Rel A B} {j : Rel C B}:
 injection_r j → function_r f →
 ((f # • f) ⊆ (j # • j) ↔ ∃! h : Rel A C, function_r h ∧ f = h • j).
Proof.
move ⇒ H H0.
move : (@injection_conjugate A _ _ _ H).
elim ⇒ H1 H2.
split; move ⇒ H3.
∃ (f • j #).
split.
move : (H1 f (conj H3 H0)).
elim ⇒ H4 H5.
split.
apply H4.
by [rewrite H5].
move ⇒ h.
elim ⇒ H4 H5.
rewrite H5 comp_assoc.
replace (j • j #) with (Id C).
apply comp_id_r.
rewrite /injection_r/function_r/univalent_r in H.
rewrite inv_invol in H.
apply inc_antisym.
apply H.
apply H.
elim H3 ⇒ h.
elim.
```

```
elim => H4 H5 H6.
rewrite H5 comp_inv comp_assoc -(@comp_assoc _ _ _ _ _ h).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
apply H4.
Qed.
```

```
Lemma injection_conjugate_corollary2 {A B C : eqType} {f : Rel A B} {j : Rel C B}:
 injection_r j -> function_r f ->
 ((f # ・ f) ⊆ (j # ・ j) <-> ∃ h' : Rel A C, f ⊆ (h' ・ j)).
Proof.
move => H H0.
split; move => H1.
apply (injection_conjugate_corollary1 H H0) in H1.
elim H1 => h.
elim.
elim => H2 H3 H4.
∃ h.
rewrite H3.
apply inc_refl.
elim H1 => h' H2.
replace (f # ・ f) with (f # ・ (f ∩ (h' ・ j))).
apply (@inc_trans _ _ _ ((f # ・ f) ・ (j # ・ j))).
rewrite comp_assoc cap_comm -(@comp_assoc _ _ _ _ f).
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
apply cap_r.
apply comp_inc_compat_ab_b.
apply H0.
apply f_equal.
apply inc_def1 in H2.
by [rewrite -H2].
Qed.
```

Lemma 183 (surjection_conjugate) *Let $e : A \twoheadrightarrow C$ be a surjection. Then,*

$$\frac{f : A \to B \ \{e \cdot e^\sharp \sqsubseteq f \cdot f^\sharp\}}{h : C \to B} \quad \frac{f = e \cdot h}{h = e^\sharp \cdot f}$$

```
Lemma surjection_conjugate {A B C : eqType} {e : Rel A C}:
 surjection_r e ->
 conjugate A B C B (fun f : Rel A B => ((e ・ e #) ⊆ (f ・ f #)) ∧ function_r f)
```

```
 (fun h : Rel C B ⇒ function_r h) (fun h : Rel C B ⇒ e • h) (fun f : Rel A B ⇒ e #
• f).
Proof.
elim.
elim ⇒ H H0 H1.
split.
move ⇒ alpha.
elim ⇒ H2.
elim ⇒ H3 H4.
assert (function_r (e # • alpha)).
split.
apply (@inc_trans _ _ _ _ _ H1).
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ alpha).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H3).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H4).
rewrite comp_inv inv_invol comp_assoc -(@comp_assoc _ _ _ _ e).
apply (@inc_trans _ _ _ (alpha # • ((alpha • alpha #) • alpha))).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_a'b H2).
rewrite comp_assoc -comp_assoc.
apply (comp_inc_compat_ab_a H4).
split.
apply H5.
apply Logic.eq_sym.
apply function_inc.
split.
apply H3.
apply H4.
apply function_comp.
split.
apply H.
apply H0.
apply H5.
rewrite -comp_assoc.
apply comp_inc_compat_b_ab.
apply H.
move ⇒ beta.
elim ⇒ H2 H3.
assert (function_r (e • beta)).
split.
apply (@inc_trans _ _ _ _ _ H).
```

```
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H2).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H3).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ _ e).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H0).
split.
split.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H2).
apply H4.
rewrite -comp_assoc.
replace (e # • e) with (Id C).
apply comp_id_l.
apply inc_antisym.
rewrite /total_r in H1.
rewrite inv_invol in H1.
apply H1.
apply H0.
Qed.
```

Lemma 184 (surjection_conjugate_corollary) *Let $e : A \twoheadrightarrow C$ be a surjection and $f : A \to B$ be a function. Then,*

$$e \cdot e^\sharp \sqsubseteq f \cdot f^\sharp \Leftrightarrow (\exists! h : C \to B, f = e \cdot h).$$

```
Lemma surjection_conjugate_corollary {A B C : eqType} {f : Rel A B} {e : Rel A C}:
 surjection_r e → function_r f →
 ((e • e #) ⊆ (f • f #) ↔ ∃! h : Rel C B, function_r h ∧ f = e • h).
Proof.
move ⇒ H H0.
move : (@surjection_conjugate _ B _ _ H).
elim ⇒ H1 H2.
split; move ⇒ H3.
∃ (e # • f).
split.
move : (H1 f (conj H3 H0)).
elim ⇒ H4 H5.
split.
apply H4.
by [rewrite H5].
```

```
move => h.
elim => H4 H5.
rewrite H5 -comp_assoc.
replace (e # • e) with (Id C).
apply comp_id_l.
rewrite /surjection_r/function_r/total_r in H.
rewrite inv_invol in H.
apply inc_antisym.
apply H.
apply H.
elim H3 => h.
elim.
elim => H4 H5 H6.
rewrite H5 comp_inv comp_assoc -(@comp_assoc _ _ _ _ h).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply H4.
Qed.
```

Lemma 185 (subid_conjugate) *Subidentity $u \sqsubseteq id_A$ corresponds $\rho : I \rightharpoondown A$. That is,*

$$\frac{\rho : I \rightharpoondown A}{u : A \rightharpoondown A \ \{u \sqsubseteq id_A\}} \ \frac{\rho = \nabla_{IA} \cdot u}{u = id_A \sqcap \nabla_{AI} \cdot \rho}.$$

```
Lemma subid_conjugate {A : eqType}:
 conjugate i A A A True_r (fun u : Rel A A => u ⊆ Id A)
 (fun u : Rel A A => ∇ i A • u) (fun rho : Rel i A => Id A ∩ (∇ A i • rho)).
Proof.
split.
move => alpha H.
split.
apply cap_l.
apply inc_antisym.
apply (@inc_trans _ _ _ (∇ i A • (∇ A i • alpha))).
apply comp_inc_compat_ab_ab'.
apply cap_r.
rewrite -comp_assoc.
apply comp_inc_compat_ab_b.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
rewrite -(@inv_universal i A).
apply (fun H' => @inc_trans _ _ _ _ _ H' (dedekind1)).
rewrite comp_id_r cap_comm cap_universal.
```

```
apply inc_refl.
move => beta H.
split.
by [].
apply inc_antisym.
rewrite cap_comm -comp_assoc lemma_for_tarski2.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
rewrite comp_id_l cap_comm cap_universal.
apply comp_inc_compat_ab_b.
rewrite -inv_inc_move inv_id.
apply H.
apply inc_cap.
split.
apply H.
rewrite -comp_assoc.
apply comp_inc_compat_b_ab.
rewrite lemma_for_tarski2.
apply inc_alpha_universal.
Qed.
```

Lemma 186 (subid_conjugate_corollary1) *Let $u, v : A \rightharpoondown A$ and $u, v \sqsubseteq id_A$. Then,*

$$\nabla_{IA} \cdot u = \nabla_{IA} \cdot v \Rightarrow u = v.$$

```
Lemma subid_conjugate_corollary1 {A : eqType} {u v : Rel A A}:
 u ⊆ Id A → v ⊆ Id A → ∇ i A • u = ∇ i A • v → u = v.
Proof.
move => H H0 H1.
move : (@subid_conjugate A).
elim => H2 H3.
move : (H3 u H).
elim => H4 H5.
rewrite -H5.
move : (H3 v H0).
elim => H6 H7.
rewrite -H7.
apply f_equal.
apply f_equal.
apply H1.
Qed.
```

Lemma 187 (subid_conjugate_corollary2) *Let $\rho, \rho' : I \rightharpoonup A$. Then,*

$$id_A \sqcap \nabla_{AI} \cdot \rho = id_A \sqcap \nabla_{AI} \cdot \rho' \Rightarrow \rho = \rho'.$$

```
Lemma subid_conjugate_corollary2 {A : eqType} {rho rho' : Rel i A}:
 Id A ∩ (∇ A i • rho) = Id A ∩ (∇ A i • rho') → rho = rho'.
Proof.
move ⇒ H.
move : (@subid_conjugate A).
elim ⇒ H0 H1.
move : (H0 rho I).
elim ⇒ H2 H3.
rewrite -H3.
move : (H0 rho' I).
elim ⇒ H4 H5.
rewrite -H5.
apply f_equal.
apply H.
Qed.
```

Chapter 10

Library Domain

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Logic.FunctionalExtensionality.
```

10.1 定義域の定義

関係 $\alpha : A \rightharpoonup B$ に対して, その定義域 (関係) $\lfloor \alpha \rfloor : A \rightharpoonup A$ は,

$$\lfloor \alpha \rfloor = \alpha \cdot \alpha^\sharp \sqcap id_A$$

で表される. また, Coq では以下のように表すことにする.

```
Definition domain {A B : eqType} (alpha : Rel A B):= (alpha • alpha #) ∩ Id A.
```

10.2 定義域の性質

10.2.1 基本的な性質

Lemma 188 (domain_another_def) *Let* $\alpha : A \rightharpoonup B$. *Then,*

$$\lfloor \alpha \rfloor = \alpha \cdot \nabla_{BA} \sqcap id_A.$$

```
Lemma domain_another_def {A B : eqType} {alpha : Rel A B}:
 domain alpha = (alpha • ∇ B A) ∩ Id A.
Proof.
```

```
apply inc_antisym.
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply inc_cap.
split.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm comp_id_r cap_universal.
apply inc_refl.
apply cap_r.
Qed.
```

Lemma 189 (domain_inv) *Let* $\alpha : A \rightharpoondown B$*. Then,*

$$\lfloor \alpha \rfloor^\sharp = \lfloor \alpha \rfloor.$$

```
Lemma domain_inv {A B : eqType} {alpha : Rel A B}:
 (domain alpha) # = domain alpha.
Proof.
apply dedekind_id1.
apply cap_r.
Qed.
```

Lemma 190 (domain_comp_alpha1, domain_comp_alpha2) *Let* $\alpha : A \rightharpoondown B$*. Then,*

$$\lfloor \alpha \rfloor \cdot \alpha = \alpha \wedge \alpha^\sharp \cdot \lfloor \alpha \rfloor = \alpha^\sharp.$$

```
Lemma domain_comp_alpha1 {A B : eqType} {alpha : Rel A B}:
 (domain alpha) • alpha = alpha.
Proof.
apply inc_antisym.
apply comp_inc_compat_ab_b.
apply cap_r.
rewrite /domain.
rewrite cap_comm.
apply (fun H' => @inc_trans _ _ _ _ _ H' (dedekind2)).
rewrite comp_id_l cap_idem.
apply inc_refl.
Qed.
```

```
Lemma domain_comp_alpha2 {A B : eqType} {alpha : Rel A B}:
 alpha # • (domain alpha) = alpha #.
```

```
Proof.
rewrite -domain_inv -comp_inv.
apply f_equal.
apply domain_comp_alpha1.
Qed.
```

Lemma 191 (domain_inc_compat) *Let* $\alpha, \alpha' : A \rightharpoonup B$*. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \lfloor \alpha \rfloor \sqsubseteq \lfloor \alpha' \rfloor.$$

```
Lemma domain_inc_compat {A B : eqType} {alpha alpha' : Rel A B}:
  alpha ⊆ alpha' → domain alpha ⊆ domain alpha'.
Proof.
move ⇒ H.
apply cap_inc_compat_r.
apply comp_inc_compat.
apply H.
apply (@inc_inv _ _ _ _ H).
Qed.
```

Lemma 192 (domain_total) *Let* $\alpha : A \rightharpoonup B$*. Then,*

$$\text{“}\alpha \textit{ is total”} \Leftrightarrow \lfloor \alpha \rfloor = id_A.$$

```
Lemma domain_total {A B : eqType} {alpha : Rel A B}:
  total_r alpha ↔ domain alpha = Id A.
Proof.
split; move ⇒ H.
rewrite /domain.
rewrite cap_comm.
apply Logic.eq_sym.
apply inc_def1.
apply H.
apply inc_def1.
rewrite /domain in H.
by [rewrite cap_comm H].
Qed.
```

Lemma 193 (domain_inc_id) *Let* $u : A \rightharpoonup A$*. Then,*

$$u \sqsubseteq id_A \Leftrightarrow \lfloor u \rfloor = u.$$

```
Lemma domain_inc_id {A : eqType} {u : Rel A A}: u ⊆ Id A ↔ domain u = u.
```

```
Proof.
split; move => H.
rewrite /domain.
rewrite (dedekind_id1 H) (dedekind_id2 H).
apply inc_def1 in H.
by [rewrite -H].
rewrite -H.
apply cap_r.
Qed.
```

10.2.2 合成と定義域

Lemma 194 (comp_domain1, comp_domain2) *Let $\alpha : A \rightharpoondown B$ and $\beta : B \rightharpoondown C$. Then,*

$$\lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \cdot \lfloor \beta \rfloor \rfloor \sqsubseteq \lfloor \alpha \rfloor.$$

```
Lemma comp_domain1 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 domain (alpha • beta) ⊆ domain alpha.
Proof.
rewrite /domain.
rewrite comp_inv.
apply (@inc_trans _ _ _ ((alpha • ((beta • (beta # • alpha #)) ∩ alpha #)) ∩ Id A)).
replace (((alpha • beta) • (beta # • alpha #)) ∩ Id A) with ((((alpha • beta) •
(beta # • alpha #)) ∩ Id A) ∩ Id A).
apply cap_inc_compat_r.
rewrite comp_assoc.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
rewrite comp_id_r.
apply inc_refl.
by [rewrite cap_assoc cap_idem].
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply cap_r.
Qed.
```

```
Lemma comp_domain2 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 domain (alpha • beta) = domain (alpha • domain beta).
Proof.
apply inc_antisym.
replace (domain (alpha • beta)) with (domain ((alpha • domain beta) • beta)).
apply comp_domain1.
by [rewrite comp_assoc domain_comp_alpha1].
apply (@inc_trans _ _ _ (domain (alpha • (beta • beta #)))).
```

```
apply domain_inc_compat.
apply comp_inc_compat_ab_ab'.
apply cap_l.
rewrite -comp_assoc.
apply comp_domain1.
Qed.
```

Lemma 195 (comp_domain3) *Let $\alpha : A \rightharpoonup B$ be a relation and $\beta : B \rightharpoonup C$ be a total relation. Then,*

$$\lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \rfloor.$$

```
Lemma comp_domain3 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 total_r beta → domain (alpha · beta) = domain alpha.
Proof.
move ⇒ H.
apply inc_antisym.
apply comp_domain1.
rewrite /domain.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ beta).
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H).
Qed.
```

Lemma 196 (comp_domain4) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$. Then,*

$$\lfloor \alpha^\sharp \rfloor \sqsubseteq \lfloor \beta \rfloor \Rightarrow \lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \rfloor.$$

```
Lemma comp_domain4 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 domain (alpha #) ⊆ domain beta → domain (alpha · beta) = domain alpha.
Proof.
move ⇒ H.
apply inc_antisym.
apply comp_domain1.
rewrite /domain.
rewrite -(@domain_comp_alpha1 _ _ (alpha #)) comp_inv comp_assoc -(@comp_assoc _ _
_ _ beta).
apply cap_inc_compat_r.
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ _ _ H).
apply cap_l.
Qed.
```

Lemma 197 (comp_domain5) *Let $\alpha : A \rightharpoonup B$ be a univalent relation and $\beta : B \rightharpoonup C$. Then,*

$$\lfloor \alpha^\sharp \rfloor \sqsubseteq \lfloor \beta \rfloor \Leftrightarrow \lfloor \alpha \cdot \beta \rfloor = \lfloor \alpha \rfloor.$$

```
Lemma comp_domain5 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 univalent_r alpha →
 (domain (alpha #) ⊆ domain beta ↔ domain (alpha • beta) = domain alpha).
Proof.
move ⇒ H.
split; move ⇒ H0.
apply (comp_domain4 H0).
rewrite /domain.
rewrite inv_invol.
apply cap_inc_compat_r.
replace (alpha # • alpha) with (alpha # • (domain (alpha • beta) • alpha)).
rewrite /domain.
rewrite comp_inv.
apply (@inc_trans _ _ _ (alpha # • (((alpha • beta) • (beta # • alpha #)) • alpha))).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite comp_assoc comp_assoc comp_assoc -comp_assoc -(@comp_assoc _ _ _ _ beta).
apply (@inc_trans _ _ _ _ _ (comp_inc_compat_ab_b H)).
apply (comp_inc_compat_ab_a H).
by [rewrite H0 domain_comp_alpha1].
Qed.
```

Lemma 198 (comp_domain6) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$. Then,*

$$\alpha \cdot \lfloor \beta \rfloor \sqsubseteq \lfloor \alpha \cdot \beta \rfloor \cdot \alpha.$$

```
Lemma comp_domain6 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 (alpha • domain beta) ⊆ (domain (alpha • beta) • alpha).
Proof.
apply (@inc_trans _ _ _ _ _ (@comp_cap_distr_l _ _ _ _ _ _ )).
rewrite cap_comm.
replace (alpha • Id B) with (Id A • alpha).
apply (@inc_trans _ _ _ _ _ (dedekind2)).
rewrite cap_comm -comp_assoc comp_assoc -comp_inv.
apply inc_refl.
by [rewrite comp_id_l comp_id_r].
Qed.
```

Lemma 199 (comp_domain7) *Let $\alpha : A \rightharpoondown B$ be a univalent relation and $\beta : B \rightharpoondown C$. Then,*

$$\alpha \cdot \lfloor \beta \rfloor = \lfloor \alpha \cdot \beta \rfloor \cdot \alpha.$$

```
Lemma comp_domain7 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 univalent_r alpha → alpha • domain beta = domain (alpha • beta) • alpha.
Proof.
move ⇒ H.
apply inc_antisym.
apply comp_domain6.
apply (@inc_trans _ _ _ _ _ (@comp_cap_distr_r _ _ _ _ _ _ _)).
rewrite comp_id_l comp_inv comp_assoc comp_assoc.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
apply (fun H' ⇒ cap_inc_compat H' H).
rewrite comp_assoc -comp_assoc.
apply (comp_inc_compat_ab_a H).
Qed.
```

Lemma 200 (comp_domain8) *Let $u : A \rightharpoondown A$, $\alpha : A \rightharpoondown B$ and $u \sqsubseteq id_A$. Then,*

$$\lfloor u \cdot \alpha \rfloor = u \cdot \lfloor \alpha \rfloor.$$

```
Lemma comp_domain8 {A B : eqType} {u : Rel A A} {alpha : Rel A B}:
 u ⊆ Id A → domain (u • alpha) = u • domain alpha.
Proof.
move ⇒ H.
apply inc_antisym.
rewrite -(@cap_idem _ _ (domain (u • alpha))).
rewrite (dedekind_id3 H).
apply cap_inc_compat.
apply (@inc_trans _ _ _ _ _ (comp_domain1)).
apply domain_inc_id in H.
rewrite H.
apply inc_refl.
apply domain_inc_compat.
apply (comp_inc_compat_ab_b H).
apply cap_r.
apply (@inc_trans _ _ _ _ _ (comp_domain6)).
apply (comp_inc_compat_ab_a H).
Qed.
```

10.2.3 その他の性質

Lemma 201 (cap_domain) *Let $\alpha, \alpha' : A \rightharpoondown B$. Then,*

$$\lfloor \alpha \sqcap \alpha' \rfloor = \alpha \cdot \alpha'^{\sharp} \sqcap id_A.$$

```
Lemma cap_domain {A B : eqType} {alpha alpha' : Rel A B}:
 domain (alpha ∩ alpha') = (alpha • alpha' #) ∩ Id A.
Proof.
apply inc_antisym.
apply cap_inc_compat_r.
apply comp_inc_compat.
apply cap_l.
apply inc_inv.
apply cap_r.
rewrite -(@cap_idem _ _ (Id A)) -cap_assoc.
apply cap_inc_compat_r.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
rewrite inv_invol comp_id_l comp_id_r -inv_cap_distr (@cap_comm _ _ alpha').
apply inc_refl.
Qed.
```

Lemma 202 (cupP_domain_distr, cup_domain_distr) *Let $f : (C \rightharpoondown D) \to (A \rightharpoondown B)$ and P : predicate. Then,*

$$\lfloor \sqcup_{P(\alpha)} f(\alpha) \rfloor = \sqcup_{P(\alpha)} \lfloor f(\alpha) \rfloor.$$

```
Lemma cupP_domain_distr {A B C D : eqType} {f : Rel C D → Rel A B} {P : Rel C D
→ Prop}:
 domain (∪_{P} f) = ∪_{P} (fun alpha : Rel C D ⇒ domain (f alpha)).
Proof.
rewrite /domain.
rewrite inv_cupP_distr comp_cupP_distr_l cap_cupP_distr_r.
apply cupP_eq.
move ⇒ alpha H.
rewrite -cap_domain -cap_domain.
apply f_equal.
rewrite cap_idem.
apply inc_antisym.
apply cap_r.
apply inc_cap.
split.
```

```
move : alpha H.
apply inc_cupP.
apply inc_refl.
apply inc_refl.
Qed.
```

```
Lemma cup_domain_distr {A B : eqType} {alpha alpha' : Rel A B}:
 domain (alpha ∪ alpha') = domain alpha ∪ domain alpha'.
Proof.
rewrite cup_to_cupP (@cup_to_cupP _ _ _ _ _ _ id).
apply cupP_domain_distr.
Qed.
```

Lemma 203 (domain_universal1) *Let $\alpha : A \rightharpoondown B$. Then,*

$$\lfloor \alpha \rfloor \cdot \nabla_{AC} = \alpha \cdot \nabla_{BC}.$$

```
Lemma domain_universal1 {A B C : eqType} {alpha : Rel A B}:
 domain alpha • ∇ A C = alpha • ∇ B C.
Proof.
apply inc_antisym.
apply (@inc_trans _ _ _ ((alpha • alpha #) • ∇ A C)).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ ((domain alpha • alpha) • ∇ B C)).
rewrite domain_comp_alpha1.
apply inc_refl.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
Qed.
```

Lemma 204 (domain_universal2) *Let $\alpha : A \rightharpoondown B$ and $\beta : B \rightharpoondown C$. Then,*

$$\alpha \cdot \lfloor \beta \rfloor = \alpha \sqcap \nabla_{AC} \cdot \beta^\sharp.$$

```
Lemma domain_universal2 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 alpha • domain beta = alpha ∩ (∇ A C • beta #).
Proof.
apply inc_antisym.
apply inc_cap.
```

```
split.
apply comp_inc_compat_ab_a.
apply cap_r.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_l)).
apply (@inc_trans _ _ _ _ _ (cap_l)).
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
rewrite -inv_universal -comp_inv -domain_universal1.
rewrite comp_inv inv_universal domain_inv cap_comm.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
rewrite cap_comm cap_universal domain_inv.
apply comp_inc_compat_ab_a.
apply cap_r.
Qed.
```

Lemma 205 (domain_lemma1) *Let $\alpha, \beta : A \rightharpoonup B$ and β is univalent. Then,*

$$\alpha \sqsubseteq \beta \wedge \lfloor \alpha \rfloor = \lfloor \beta \rfloor \Rightarrow \alpha = \beta.$$

```
Lemma domain_lemma1 {A B : eqType} {alpha beta : Rel A B}:
 univalent_r beta → alpha ⊆ beta → domain alpha = domain beta → alpha = beta.
Proof.
move ⇒ H H0 H1.
apply inc_antisym.
apply H0.
rewrite -(@domain_comp_alpha1 _ _ beta) -H1.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_l)).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H).
apply comp_inc_compat_ab_a'b.
apply (@inc_inv _ _ _ _ H0).
Qed.
```

Lemma 206 (domain_lemma2a, domain_lemma2b) *Let $\alpha : A \rightharpoonup B$ and $\beta : A \rightharpoonup C$. Then,*

$$\lfloor \alpha \rfloor \sqsubseteq \lfloor \beta \rfloor \Leftrightarrow \alpha \cdot \nabla_{BB} \sqsubseteq \beta \cdot \nabla_{CB} \Leftrightarrow \alpha \sqsubseteq \beta \cdot \beta^\sharp \cdot \alpha.$$

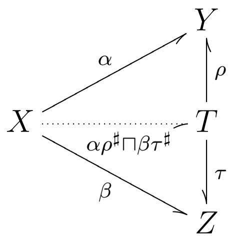
```
Lemma domain_lemma2a {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 domain alpha ⊆ domain beta ↔ (alpha • ∇ B B) ⊆ (beta • ∇ C B).
```

```
Proof.
split; move => H.
rewrite -(@domain_comp_alpha1 _ _ alpha) comp_assoc.
apply (@inc_trans _ _ _ _ _ (comp_inc_compat_ab_a'b H)).
apply (@inc_trans _ _ _ _ _ (comp_inc_compat_ab_a'b (cap_l))).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply (@inc_trans _ _ _ (domain ((beta • beta #) • alpha))).
apply domain_inc_compat.
apply (@inc_trans _ _ _ (alpha ∩ (beta • ∇ C B))).
apply (fun H' => @inc_trans _ _ _ _ _ H' (cap_inc_compat_l H)).
replace (alpha ∩ (alpha • ∇ B B)) with ((alpha • Id B) ∩ (alpha • ∇ B B)).
apply (fun H' => @inc_trans _ _ _ _ _ H' (comp_cap_distr_l)).
rewrite cap_universal comp_id_r.
apply inc_refl.
by [rewrite comp_id_r].
rewrite cap_comm comp_assoc.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
rewrite cap_comm cap_universal.
apply inc_refl.
rewrite comp_assoc.
apply comp_domain1.
Qed.
Lemma domain_lemma2b {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 domain alpha ⊆ domain beta ↔ alpha ⊆ ((beta • beta #) • alpha).
Proof.
split; move => H.
apply domain_lemma2a in H.
apply (@inc_trans _ _ _ (alpha ∩ (beta • ∇ C B))).
apply (fun H' => @inc_trans _ _ _ _ _ H' (cap_inc_compat_l H)).
replace (alpha ∩ (alpha • ∇ B B)) with ((alpha • Id B) ∩ (alpha • ∇ B B)).
apply (fun H' => @inc_trans _ _ _ _ _ H' (comp_cap_distr_l)).
rewrite cap_universal comp_id_r.
apply inc_refl.
by [rewrite comp_id_r].
rewrite cap_comm comp_assoc.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
rewrite cap_comm cap_universal.
apply inc_refl.
apply domain_inc_compat in H.
apply (@inc_trans _ _ _ _ _ H).
```

```
rewrite comp_assoc.
apply comp_domain1.
Qed.
```

Lemma 207 (domain_corollary1) *In below figure,*

$$\text{“}\alpha \text{ and } \beta \text{ are total”} \wedge \alpha^\sharp \cdot \beta \sqsubseteq \rho^\sharp \cdot \tau \Rightarrow \text{“}\alpha \cdot \rho^\sharp \sqcap \beta \cdot \tau^\sharp \text{ is total”}.$$




```
Lemma domain_corollary1 {X Y Z T : eqType}
 {alpha : Rel X Y} {beta : Rel X Z} {rho : Rel T Y} {tau : Rel T Z}:
 total_r alpha → total_r beta → (alpha # • beta) ⊆ (rho # • tau) →
 total_r ((alpha • rho #) ∩ (beta • tau #)).
Proof.
move ⇒ H H0 H1.
move : (comp_inc_compat H H0) ⇒ H2.
rewrite comp_id_l -comp_assoc (@comp_assoc _ _ _ _ alpha) in H2.
rewrite /total_r.
replace (Id X) with (((alpha • (rho # • tau)) • beta #) ∩ Id X).
rewrite -comp_assoc comp_assoc.
apply (@inc_trans _ _ _ _ _ (@dedekind _ _ _ _ _ _)).
rewrite comp_id_l comp_id_r comp_inv comp_inv inv_invol inv_invol.
rewrite inv_cap_distr comp_inv comp_inv inv_invol inv_invol (@cap_comm _ _ (tau •
beta #)).
apply inc_refl.
apply Logic.eq_sym.
rewrite cap_comm.
apply inc_def1.
apply (@inc_trans _ _ _ _ _ H2).
apply comp_inc_compat_ab_a'b.
apply (comp_inc_compat_ab_ab' H1).
Qed.
```

Lemma 208 (domain_corollary2) *In below figure,*

$$\text{“}\alpha \text{ and } \beta \text{ are univalent”} \wedge \rho \cdot \rho^\sharp \sqcap \tau \cdot \tau^\sharp = id_T \Rightarrow \text{“}\alpha \cdot \rho^\sharp \sqcap \beta \cdot \tau^\sharp \text{ is univalent”}.$$

$$\begin{tikzcd}
 & Y \\
X \arrow[ur, "\alpha"] \arrow[r, dotted, "\alpha\rho^\sharp\sqcap\beta\tau^\sharp"'] \arrow[dr, "\beta"'] & T \arrow[u, "\rho"'] \arrow[d, "\tau"] \\
 & Z
\end{tikzcd}$$

```
Lemma domain_corollary2 {X Y Z T : eqType}
 {alpha : Rel X Y} {beta : Rel X Z} {rho : Rel T Y} {tau : Rel T Z}:
 univalent_r alpha → univalent_r beta → (rho • rho #) ∩ (tau • tau #) = Id T →
 univalent_r ((alpha • rho #) ∩ (beta • tau #)).
Proof.
move ⇒ H H0 H1.
rewrite /univalent_r.
rewrite -H1 inv_cap_distr.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_l)).
apply cap_inc_compat.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_l)).
rewrite comp_inv inv_invol -comp_assoc (@comp_assoc _ _ _ _ rho).
apply comp_inc_compat_ab_a'b.
apply (comp_inc_compat_ab_a H).
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_r)).
rewrite comp_inv inv_invol -comp_assoc (@comp_assoc _ _ _ _ tau).
apply comp_inc_compat_ab_a'b.
apply (comp_inc_compat_ab_a H0).
Qed.
```

10.2.4 矩形関係

$\alpha : A \rightharpoonup B$ が

$$\alpha \cdot \nabla_{BA} \cdot \alpha \sqsubseteq \alpha$$

を満たすとき, α は 矩形関係 (rectangular relation) であると言われる.

```
Definition rectangular {A B : eqType} (alpha : Rel A B):=
 ((alpha • ∇ B A) • alpha) ⊆ alpha.
```

Lemma 209 (rectangular_inv) *Let $\alpha : A \rightharpoondown B$ is a rectangular relation, then $\alpha^\sharp$ is also a rectangular relation.*

```
Lemma rectangular_inv {A B : eqType} {alpha : Rel A B}:
 rectangular alpha -> rectangular (alpha #).
Proof.
move => H.
apply inv_inc_move.
rewrite comp_inv comp_inv inv_invol inv_universal -comp_assoc.
apply H.
Qed.
```

Lemma 210 (rectangular_capP, rectangular_cap) *Let $f(\alpha)$ is always a rectangular relation and P : predicate, then $\sqcap_{P(\beta)} f(\beta)$ is also a rectangular relation.*

```
Lemma rectangular_capP {A B C D : eqType} {f : Rel C D -> Rel A B} {P : Rel C D ->
Prop}:
 (∀ alpha : Rel C D, P alpha -> rectangular (f alpha)) -> rectangular (∩_{P} f).
Proof.
move => H.
rewrite /rectangular.
apply (@inc_trans _ _ _ (∩_{P} (fun alpha : Rel C D => (f alpha • ∇ B A) • f alpha))).
apply (@inc_trans _ _ _ _ _ (comp_capP_distr_l)).
apply inc_capP.
move => beta H0.
apply (@inc_trans _ _ _ (((∩_{P} f) • ∇ B A) • f beta)).
move : beta H0.
apply inc_capP.
apply inc_refl.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a'b.
move : H0.
apply inc_capP.
apply inc_refl.
apply inc_capP.
move => beta H0.
apply (fun H' => @inc_trans _ _ _ _ _ H' (H beta H0)).
move : beta H0.
apply inc_capP.
apply inc_refl.
Qed.
```

```
Lemma rectangular_cap {A B : eqType} {alpha beta : Rel A B}:
```

```
 rectangular alpha → rectangular beta → rectangular (alpha ∩ beta).
Proof.
move ⇒ H H0.
rewrite (@cap_to_capP _ _ _ _ _ _ id).
apply rectangular_capP.
move ⇒ gamma.
case ⇒ H1; rewrite H1.
apply H.
apply H0.
Qed.
```

Lemma 211 (rectangular_comp) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$ and α or β is a rectangular relation, then $\alpha \cdot \beta$ is also a rectangular relation.*

```
Lemma rectangular_comp {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 rectangular alpha ∨ rectangular beta → rectangular (alpha • beta).
Proof.
rewrite /rectangular.
case; move ⇒ H.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H).
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H).
rewrite -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
Qed.
```

Lemma 212 (rectangular_unit) *Let $\alpha : A \rightharpoondown B$. Then,*

$$\text{“}\alpha \textit{ is rectangular”} \Leftrightarrow \exists \mu : I \rightharpoondown A, \exists \rho : I \rightharpoondown B, \alpha = \rho^\sharp \cdot \mu.$$

```
Lemma rectangular_unit {A B : eqType} {alpha : Rel A B}:
 rectangular alpha ↔ ∃ (mu : Rel i A)(rho : Rel i B), alpha = mu # • rho.
Proof.
```

```
split; move => H.
∃ (∇ i B • alpha #).
∃ (∇ i A • alpha).
rewrite comp_inv inv_invol inv_universal.
rewrite -comp_assoc (@comp_assoc _ _ _ _ alpha) lemma_for_tarski2.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (relation_rel_inv_rel)).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply H.
elim H => mu.
elim => rho H0.
rewrite H0.
rewrite /rectangular.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_a.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
Qed.
```

Chapter 11

Library Residual

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Domain.
Require Import Logic.FunctionalExtensionality.
```

11.1 剰余合成関係の性質

11.1.1 基本的な性質

Lemma 213 (double_residual) *Let $\alpha : A \rightharpoondown B$, $\beta : B \rightharpoondown C$ and $\gamma : C \rightharpoondown D$. Then*

$$\alpha \rhd (\beta \rhd \gamma) = (\alpha \cdot \beta) \rhd \gamma.$$

```
Lemma double_residual
 {A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel C D}:
 alpha △ (beta △ gamma) = (alpha ・ beta) △ gamma.
Proof.
apply inc_lower.
move => delta.
split; move => H.
apply inc_residual.
rewrite comp_inv comp_assoc.
rewrite -inc_residual -inc_residual.
apply H.
rewrite inc_residual inc_residual.
rewrite -comp_assoc -comp_inv.
```

```
apply inc_residual.
apply H.
Qed.
```

Lemma 214 (residual_to_complement) *Let $\alpha : A \rightharpoondown B$ and $\beta : B \rightharpoondown C$. Then*

$$\alpha \rhd \beta = (\alpha \cdot \beta^{-})^{-}.$$

```
Lemma residual_to_complement {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 alpha △ beta = (alpha • beta ^) ^.
Proof.
apply inc_lower.
move ⇒ gamma.
split; move ⇒ H.
rewrite bool_lemma2 complement_invol cap_comm.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
replace (beta ^ ∩ (alpha # • gamma)) with (φ B C).
rewrite comp_empty_r.
apply inc_refl.
apply Logic.eq_sym.
rewrite cap_comm.
apply bool_lemma2.
apply inc_residual.
apply H.
apply inc_empty_alpha.
apply inc_residual.
apply bool_lemma2.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
rewrite inv_invol.
replace (gamma ∩ (alpha • beta ^)) with (φ A C).
rewrite comp_empty_r.
apply inc_refl.
apply Logic.eq_sym.
rewrite -(@complement_invol _ _ (alpha • beta ^)).
apply bool_lemma2.
apply H.
apply inc_empty_alpha.
Qed.
```

Lemma 215 (inv_residual_inc) *Let* $\alpha : A \rightharpoondown B$ *and* $\beta : B \rightharpoondown C$. *Then*

$$\alpha^\sharp \cdot (\alpha \rhd \beta) \sqsubseteq \beta.$$

```
Lemma inv_residual_inc {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
  alpha # • (alpha △ beta) ⊆ beta.
Proof.
apply inc_residual.
apply inc_refl.
Qed.
```

Lemma 216 (inc_residual_inv) *Let* $\alpha : A \rightharpoondown B$ *and* $\gamma : A \rightharpoondown C$. *Then*

$$\gamma \sqsubseteq \alpha \rhd \alpha^\sharp \cdot \gamma.$$

```
Lemma inc_residual_inv {A B C : eqType} {alpha : Rel A B} {gamma : Rel A C}:
  gamma ⊆ (alpha △ (alpha # • gamma)).
Proof.
apply inc_residual.
apply inc_refl.
Qed.
```

Lemma 217 (id_inc_residual) *Let* $\alpha : A \rightharpoondown B$. *Then*

$$id_A \sqsubseteq \alpha \rhd \alpha^\sharp.$$

```
Lemma id_inc_residual {A B : eqType} {alpha : Rel A B}: Id A ⊆ (alpha △ alpha #).
Proof.
apply inc_residual.
rewrite comp_id_r.
apply inc_refl.
Qed.
```

Lemma 218 (residual_universal) *Let* $\alpha : A \rightharpoondown B$. *Then*

$$\alpha \rhd \nabla_{BC} = \nabla_{AC}.$$

```
Lemma residual_universal {A B C : eqType} {alpha : Rel A B}: alpha △ ∇ B C = ∇ A C.
Proof.
apply inc_antisym.
apply inc_alpha_universal.
apply inc_residual.
apply inc_alpha_universal.
```

```
Qed.
```

11.1.2 単調性と分配法則

Lemma 219 (residual_inc_compat) *Let $\alpha, \alpha' : A \rightharpoondown B$ and $\beta, \beta' : B \rightharpoondown C$. Then*

$$\alpha' \sqsubseteq \alpha \land \beta \sqsubseteq \beta' \Rightarrow \alpha \rhd \beta \sqsubseteq \alpha' \rhd \beta'.$$

```
Lemma residual_inc_compat
 {A B C : eqType} {alpha alpha' : Rel A B} {beta beta' : Rel B C}:
 alpha' ⊆ alpha → beta ⊆ beta' → (alpha △ beta) ⊆ (alpha' △ beta').
Proof.
move ⇒ H H0.
apply inc_residual.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H0).
move : (@inc_refl _ _ (alpha △ beta)) ⇒ H1.
apply inc_residual in H1.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H1).
apply comp_inc_compat_ab_a'b.
apply inc_inv.
apply H.
Qed.
```

Lemma 220 (residual_inc_compat_l) *Let $\alpha : A \rightharpoondown B$ and $\beta, \beta' : B \rightharpoondown C$. Then*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \rhd \beta \sqsubseteq \alpha \rhd \beta'.$$

```
Lemma residual_inc_compat_l
 {A B C : eqType} {alpha : Rel A B} {beta beta' : Rel B C}:
 beta ⊆ beta' → (alpha △ beta) ⊆ (alpha △ beta').
Proof.
move ⇒ H.
apply (@residual_inc_compat _ _ _ _ _ _ _ (@inc_refl _ _ _) H).
Qed.
```

Lemma 221 (residual_inc_compat_r) *Let $\alpha, \alpha' : A \rightharpoondown B$ and $\beta : B \rightharpoondown C$. Then*

$$\alpha' \sqsubseteq \alpha \Rightarrow \alpha \rhd \beta \sqsubseteq \alpha' \rhd \beta.$$

```
Lemma residual_inc_compat_r
 {A B C : eqType} {alpha alpha' : Rel A B} {beta : Rel B C}:
 alpha' ⊆ alpha → (alpha △ beta) ⊆ (alpha' △ beta).
Proof.
```

```
move => H.
apply (@residual_inc_compat _ _ _ _ _ _ _ H (@inc_refl _ _ _)).
Qed.
```

Lemma 222 (residual_capP_distr_l, residual_cap_distr_l) *Let $\alpha : A \rightharpoonup B$, $f : (D \rightharpoonup E) \to (B \rightharpoonup C)$ and P : predicate. Then*

$$\alpha \rhd (\sqcap_{P(\beta)} f(\beta)) = \sqcap_{P(\beta)} (\alpha \rhd f(\beta)).$$

```
Lemma residual_capP_distr_l {A B C D E : eqType}
 {alpha : Rel A B} {f : Rel D E -> Rel B C} {P : Rel D E -> Prop}:
 alpha △ (∩_{P} f) = ∩_{P} (fun beta : Rel D E => alpha △ f beta).
Proof.
apply inc_lower.
move => gamma.
split; move => H.
apply inc_capP.
move => beta H0.
apply inc_residual.
move : beta H0.
apply inc_capP.
apply inc_residual.
apply H.
apply inc_residual.
apply inc_capP.
move => beta H0.
apply inc_residual.
move : beta H0.
apply inc_capP.
apply H.
Qed.
```

```
Lemma residual_cap_distr_l
 {A B C : eqType} {alpha : Rel A B} {beta gamma : Rel B C}:
 alpha △ (beta ∩ gamma) = (alpha △ beta) ∩ (alpha △ gamma).
Proof.
rewrite cap_to_capP (@cap_to_capP _ _ _ _ _ _ id).
apply residual_capP_distr_l.
Qed.
```

Lemma 223 (residual_cupP_distr_r, residual_cup_distr_r) *Let* $f : (D \rightharpoondown E) \to (A \rightharpoondown B)$, $\beta : B \rightharpoondown C$ *and* P *: predicate. Then*

$$(\sqcup_{P(\alpha)} f(\alpha)) \rhd \beta = \sqcap_{P(\alpha)} (f(\alpha) \rhd \beta).$$

```
Lemma residual_cupP_distr_r {A B C D E : eqType}
 {beta : Rel B C} {f : Rel D E → Rel A B} {P : Rel D E → Prop}:
 (∪_{P} f) △ beta = ∩_{P} (fun alpha : Rel D E ⇒ f alpha △ beta).
Proof.
apply inc_lower.
move ⇒ gamma.
split; move ⇒ H.
apply inc_capP.
move ⇒ alpha H0.
apply inc_residual.
move : alpha H0.
apply inc_cupP.
rewrite -comp_cupP_distr_r -inv_cupP_distr.
apply inc_residual.
apply H.
apply inc_residual.
rewrite inv_cupP_distr comp_cupP_distr_r.
apply inc_cupP.
move ⇒ alpha H0.
apply inc_residual.
move : alpha H0.
apply inc_capP.
apply H.
Qed.
```

```
Lemma residual_cup_distr_r
 {A B C : eqType} {alpha beta : Rel A B} {gamma : Rel B C}:
 (alpha ∪ beta) △ gamma = (alpha △ gamma) ∩ (beta △ gamma).
Proof.
rewrite (@cup_to_cupP _ _ _ _ _ _ id) (@cap_to_capP _ _ _ _ _ _ (fun x ⇒ x △ gamma)).
apply residual_cupP_distr_r.
Qed.
```

11.1.3 剰余合成と関数

Lemma 224 (total_residual) *Let $\alpha : A \to B$ be a total relation and $\beta : B \rightharpoondown C$. Then*

$$\alpha \rhd \beta \sqsubseteq \alpha \cdot \beta.$$

```
Lemma total_residual {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 total_r alpha -> (alpha △ beta) ⊆ (alpha • beta).
Proof.
move => H.
apply (@inc_trans _ _ _ ((alpha • alpha #) • (alpha △ beta))).
apply (comp_inc_compat_b_ab H).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inv_residual_inc.
Qed.
```

Lemma 225 (univalent_residual) *Let $\alpha : A \to B$ be a univalent relation and $\beta : B \rightharpoondown C$. Then*

$$\alpha \cdot \beta \sqsubseteq \alpha \rhd \beta.$$

```
Lemma univalent_residual {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 univalent_r alpha -> (alpha • beta) ⊆ (alpha △ beta).
Proof.
move => H.
apply (@inc_trans _ _ _ _ _ (@inc_residual_inv _ _ _ alpha _)).
apply residual_inc_compat_l.
rewrite -comp_assoc.
apply (comp_inc_compat_ab_b H).
Qed.
```

Lemma 226 (function_residual1) *Let $\alpha : A \to B$ be a function and $\beta : B \rightharpoondown C$. Then*

$$\alpha \rhd \beta = \alpha \cdot \beta.$$

```
Lemma function_residual1 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
 function_r alpha -> alpha △ beta = alpha • beta.
Proof.
elim => H H0.
apply inc_antisym.
apply (total_residual H).
apply (univalent_residual H0).
Qed.
```

Lemma 227 (residual_id) *Let* $\alpha : A \to B$*. Then*

$$id_A \rhd \alpha = \alpha.$$

```
Lemma residual_id {A B : eqType} {alpha : Rel A B}:
 Id A △ alpha = alpha.
Proof.
move : (@function_residual1 _ _ _ (Id A) alpha (@id_function A)) ⇒ H.
rewrite comp_id_l in H.
apply H.
Qed.
```

Lemma 228 (universal_residual) *Let* $\alpha : A \to B$*. Then*

$$\nabla_{AA} \rhd \alpha \sqsubseteq \alpha.$$

```
Lemma universal_residual {A B : eqType} {alpha : Rel A B}:
 ∇ A A △ alpha ⊆ alpha.
Proof.
apply (@inc_trans _ _ _ (Id A △ alpha)).
apply residual_inc_compat_r.
apply inc_alpha_universal.
rewrite residual_id.
apply inc_refl.
Qed.
```

Lemma 229 (function_residual2) *Let* $\alpha : A \to B$ *be a function,* $\beta : B \rightharpoondown C$ *and* $\gamma : C \rightharpoondown D$*. Then*

$$\alpha \cdot (\beta \rhd \gamma) = (\alpha \cdot \beta) \rhd \gamma.$$

```
Lemma function_residual2
 {A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel C D}:
 function_r alpha → alpha • (beta △ gamma) = (alpha • beta) △ gamma.
Proof.
move ⇒ H.
rewrite -(@function_residual1 _ _ _ _ _ H).
apply double_residual.
Qed.
```

Lemma 230 (function_residual3) *Let $\alpha : A \rightharpoonup B$, $\beta : B \rightharpoonup C$ be relations and $\gamma : D \to C$ be a function. Then*

$$(\alpha \rhd \beta) \cdot \gamma^\sharp = \alpha \rhd (\beta \cdot \gamma^\sharp).$$

```
Lemma function_residual3
 {A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel D C}:
 function_r gamma → (alpha △ beta) • gamma # = alpha △ (beta • gamma #).
Proof.
move ⇒ H.
apply inc_lower.
move ⇒ delta.
split; move ⇒ H0.
apply inc_residual.
rewrite -(@function_move2 _ _ _ _ _ _ H).
rewrite comp_assoc.
apply inc_residual.
rewrite (@function_move2 _ _ _ _ _ _ H).
apply H0.
rewrite -(@function_move2 _ _ _ _ _ _ H).
apply inc_residual.
rewrite -comp_assoc.
rewrite (@function_move2 _ _ _ _ _ _ H).
apply inc_residual.
apply H0.
Qed.
```

Lemma 231 (function_residual4) *Let $\alpha : A \rightharpoonup B$, $\gamma : C \rightharpoonup D$ be relations and $\beta : B \to C$ be a function. Then*

$$\alpha \cdot \beta \rhd \gamma = \alpha \rhd \beta \cdot \gamma.$$

```
Lemma function_residual4
 {A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel C D}:
 function_r beta → (alpha • beta) △ gamma = alpha △ (beta • gamma).
Proof.
move ⇒ H.
rewrite -double_residual.
by [rewrite (function_residual1 H)].
Qed.
```

11.2 Galois 同値とその系

Lemma 232 (galois) *Let $\alpha : A \rightharpoonup B$, $\beta : B \rightharpoonup C$ and $\gamma : A \rightharpoonup C$. Then*

$$\gamma \sqsubseteq \alpha \rhd \beta \Leftrightarrow \alpha \sqsubseteq \gamma \rhd \beta^\sharp.$$

```
Lemma galois {A B C : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel A C}:
  gamma ⊆ (alpha △ beta) ↔ alpha ⊆ (gamma △ beta #).
Proof.
split; move ⇒ H.
apply inc_residual.
apply inv_inc_move.
rewrite comp_inv inv_invol.
apply inc_residual.
apply H.
apply inc_residual.
apply inv_inc_invol.
rewrite comp_inv inv_invol.
apply inc_residual.
apply H.
Qed.
```

Lemma 233 (galois_corollary1) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$. Then*

$$\alpha \sqsubseteq (\alpha \rhd \beta) \rhd \beta^\sharp.$$

```
Lemma galois_corollary1 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
  alpha ⊆ ((alpha △ beta) △ beta #).
Proof.
rewrite -galois.
apply inc_refl.
Qed.
```

Lemma 234 (galois_corollary2) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$. Then*

$$((\alpha \rhd \beta) \rhd \beta^\sharp) \rhd \beta = \alpha \rhd \beta.$$

```
Lemma galois_corollary2 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
  ((alpha △ beta) △ beta #) △ beta = alpha △ beta.
Proof.
apply inc_antisym.
apply residual_inc_compat_r.
```

```
apply galois_corollary1.
move : (@galois_corollary1 _ _ _ (alpha △ beta) (beta #)) ⇒ H.
rewrite inv_invol in H.
apply H.
Qed.
```

Lemma 235 (galois_corollary3) *Let $\alpha : A \rightharpoonup B$ and $\beta : B \rightharpoonup C$. Then*

$$\alpha = (\alpha \rhd \beta) \rhd \beta^\sharp \Leftrightarrow \exists \gamma : A \rightharpoonup C, \alpha = \gamma \rhd \beta^\sharp.$$

```
Lemma galois_corollary3 {A B C : eqType} {alpha : Rel A B} {beta : Rel B C}:
  alpha = (alpha △ beta) △ beta # ↔ (∃ gamma : Rel A C, alpha = gamma △ beta #).
Proof.
split; move ⇒ H.
∃ (alpha △ beta).
apply H.
elim H ⇒ gamma H0.
rewrite H0.
move : (@galois_corollary2 _ _ _ gamma (beta #)) ⇒ H1.
rewrite inv_invol in H1.
by [rewrite H1].
Qed.
```

11.3 その他の性質

この節では, 特記が無い限り, 記号は以下の図式に従って割り振られるものとする.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[rr, bend left, "\delta"] & Y \arrow[r, "\beta,\beta'"] \arrow[d, "\eta"] & Z \arrow[r, "\gamma"] & W \\
& V \arrow[ur, "\tau"'] & &
\end{tikzcd}$$

Lemma 236 (residual_property1)

$$(\alpha \rhd \beta) \cdot \gamma \sqsubseteq \alpha \rhd \beta \cdot \gamma.$$

```
Lemma residual_property1
  {W X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {gamma : Rel Z W}:
  ((alpha △ beta) · gamma) ⊆ (alpha △ (beta · gamma)).
Proof.
```

```
apply (@inc_trans _ _ _ (alpha △ (alpha # • ((alpha △ beta) • gamma)))).
apply inc_residual_inv.
apply residual_inc_compat_l.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply inv_residual_inc.
Qed.
```

Lemma 237 (residual_property2)

$$(\alpha \rhd \beta) \cdot (\beta^\sharp \rhd \eta) \sqsubseteq \alpha \rhd \eta.$$

```
Lemma residual_property2
 {V X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {eta : Rel Y V}:
 ((alpha △ beta) • (beta # △ eta)) ⊆ (alpha △ eta).
Proof.
apply (@inc_trans _ _ _ _ _ (residual_property1)).
apply residual_inc_compat_l.
move : (@inv_residual_inc _ _ _ (beta #) eta).
by [rewrite inv_invol].
Qed.
```

Lemma 238 (residual_property3)

$$\alpha \rhd \beta \sqsubseteq \alpha \cdot \eta \rhd \eta^\sharp \cdot \beta.$$

```
Lemma residual_property3
 {V X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {eta : Rel Y V}:
 (alpha △ beta) ⊆ ((alpha • eta) △ (eta # • beta)).
Proof.
apply (@inc_trans _ _ _ _ _ (@inc_residual_inv _ _ _ (alpha • eta) (alpha △ beta))).
apply residual_inc_compat_l.
rewrite comp_inv comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inv_residual_inc.
Qed.
```

Lemma 239 (residual_property4a, residual_property4b)

$$(\alpha \rhd \beta) \cdot \gamma \sqsubseteq (\alpha \rhd \beta \cdot \gamma) \sqcap \nabla_{XZ} \cdot \gamma \sqsubseteq (\alpha \rhd \beta \cdot \gamma) \cdot \gamma^\sharp \cdot \gamma.$$

```
Lemma residual_property4a
 {W X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {gamma : Rel Z W}:
```

```
 ((alpha △ beta) ・ gamma) ⊆ ((alpha △ (beta ・ gamma)) ∩ (∇ X Z ・ gamma)).
Proof.
rewrite -(@cap_universal _ _ (alpha △ beta)).
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat_r.
apply residual_property1.
Qed.
```

```
Lemma residual_property4b
 {W X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {gamma : Rel Z W}:
 ((alpha △ (beta ・ gamma)) ∩ (∇ X Z ・ gamma)) ⊆ ((alpha △ (beta ・ gamma)) ・
(gamma # ・ gamma)).
Proof.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
rewrite cap_comm cap_universal comp_assoc.
apply inc_refl.
Qed.
```

Lemma 240 (residual_property5) *Let τ be a univalent relation. Then,*

$$(\alpha \rhd \beta) \cdot \tau^\sharp = (\alpha \rhd \beta \cdot \tau^\sharp) \sqcap \nabla_{XZ} \cdot \tau^\sharp.$$

```
Lemma residual_property5
 {V X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {tau : Rel V Z}:
 univalent_r tau →
 (alpha △ beta) ・ tau # = (alpha △ (beta ・ tau #)) ∩ (∇ X Z ・ tau #).
Proof.
move ⇒ H.
apply inc_antisym.
rewrite -(@cap_universal _ _ (alpha △ beta)).
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat_r.
apply residual_property1.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
rewrite cap_comm cap_universal inv_invol.
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ _ _ (residual_property1)).
apply residual_inc_compat_l.
rewrite comp_assoc.
apply (comp_inc_compat_ab_a H).
Qed.
```

Lemma 241 (residual_property6)

$$\alpha \rhd (\gamma^\sharp \rhd \beta^\sharp)^\sharp = (\gamma^\sharp \rhd (\alpha \rhd \beta)^\sharp)^\sharp.$$

```
Lemma residual_property6
 {W X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {gamma : Rel Z W}:
 alpha △ (gamma # △ beta #) # = (gamma # △ (alpha △ beta) #) #.
Proof.
apply inc_lower.
move ⇒ delta.
split; move ⇒ H.
apply inv_inc_move.
apply inc_residual.
apply inv_inc_move.
apply inc_residual.
rewrite -comp_inv comp_assoc.
apply inv_inc_move.
apply inc_residual.
apply inv_inc_invol.
rewrite comp_inv inv_invol.
apply inc_residual.
apply H.
apply inc_residual.
apply inv_inc_move.
apply inc_residual.
apply inv_inc_move.
rewrite comp_inv inv_invol inv_invol comp_assoc.
apply inc_residual.
apply inv_inc_invol.
rewrite comp_inv.
apply inc_residual.
apply inv_inc_move.
apply H.
Qed.
```

Lemma 242 (residual_property7a, residual_property7b)

$$\alpha \rhd (\beta \Rightarrow \beta') \sqsubseteq (\alpha \cdot \beta \Rightarrow \alpha \cdot \beta') \sqsubseteq \alpha \rhd (\beta \Rightarrow \alpha^\sharp \cdot \alpha \cdot \beta').$$

```
Lemma residual_property7a {X Y Z : eqType} {alpha : Rel X Y} {beta beta' : Rel Y Z}:
 (alpha △ (beta » beta')) ⊆ ((alpha • beta) » (alpha • beta')).
Proof.
```

```
apply inc_rpc.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm.
apply inc_rpc.
apply inv_residual_inc.
Qed.
```

```
Lemma residual_property7b {X Y Z : eqType} {alpha : Rel X Y} {beta beta' : Rel Y Z}:
 ((alpha • beta) » (alpha • beta')) ⊆ (alpha △ (beta » (alpha # • (alpha • beta')))).
Proof.
rewrite inc_residual inc_rpc.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite inv_invol -inc_rpc.
apply inc_refl.
Qed.
```

Lemma 243 (residual_property8) *Let α be a univalent relation. Then,*

$$\alpha \rhd (\beta \Rightarrow \beta') = (\alpha \cdot \beta \Rightarrow \alpha \cdot \beta').$$

```
Lemma residual_property8 {X Y Z : eqType} {alpha : Rel X Y} {beta beta' : Rel Y Z}:
 univalent_r alpha → alpha △ (beta » beta') = (alpha • beta) » (alpha • beta').
Proof.
move ⇒ H.
apply inc_antisym.
apply residual_property7a.
apply (@inc_trans _ _ _ _ _ residual_property7b).
apply residual_inc_compat_l.
apply rpc_inc_compat_l.
rewrite -comp_assoc.
apply (comp_inc_compat_ab_b H).
Qed.
```

Lemma 244 (residual_property9) *Let α be a univalent relation. Then,*

$$\alpha \rhd \beta = (\alpha \cdot \nabla_{YZ} \Rightarrow \alpha \cdot \beta).$$

```
Lemma residual_property9 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:
 univalent_r alpha → alpha △ beta = (alpha • ∇ Y Z) » (alpha • beta).
Proof.
move ⇒ H.
```

```
by [rewrite -(residual_property8 H) rpc_universal_alpha].
Qed.
```

Lemma 245 (residual_property10) *Let α be a univalent relation. Then,*

$$\alpha \cdot \beta = \lfloor \alpha \rfloor \cdot (\alpha \rhd \beta).$$

```
Lemma residual_property10 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:
 univalent_r alpha → alpha • beta = domain alpha • (alpha △ beta).
Proof.
move ⇒ H.
apply inc_antisym.
replace (alpha • beta) with (domain alpha • (alpha • beta)).
apply comp_inc_compat_ab_ab'.
rewrite inc_residual -comp_assoc.
apply (comp_inc_compat_ab_b H).
by [rewrite -comp_assoc domain_comp_alpha1].
apply (@inc_trans _ _ _ ((alpha • alpha #) • (alpha △ beta))).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inv_residual_inc.
Qed.
```

Lemma 246 (residual_property11)

$$(\alpha \cdot \beta \Rightarrow \delta) \sqsubseteq \alpha \rhd (\beta \Rightarrow \alpha^\sharp \cdot \delta).$$

```
Lemma residual_property11
 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {delta : Rel X Z}:
 ((alpha • beta) » delta) ⊆ (alpha △ (beta » (alpha # • delta))).
Proof.
apply inc_residual.
apply inc_rpc.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
rewrite inv_invol.
apply comp_inc_compat_ab_ab'.
apply inc_rpc.
apply inc_refl.
Qed.
```

Lemma 247 (residual_property12a, residual_property12b) *Let $u \sqsubseteq id_X$. Then,*

$$u \rhd \alpha = u \cdot \nabla_{XY} \Rightarrow \alpha = u \rhd u \cdot \alpha.$$

```
Lemma residual_property12a {X Y : eqType} {u : Rel X X} {alpha : Rel X Y}:
  u ⊆ Id X → u △ alpha = (u • ∇ X Y) » alpha.
Proof.
move ⇒ H.
apply inc_antisym.
assert (univalent_r u).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' H).
apply comp_inc_compat_ab_b.
rewrite -inv_id.
apply (@inc_inv _ _ _ _ H).
rewrite (residual_property9 H0).
apply rpc_inc_compat_l.
apply (comp_inc_compat_ab_b H).
apply (@inc_trans _ _ _ _ _ residual_property11).
apply residual_inc_compat_l.
rewrite rpc_universal_alpha.
apply comp_inc_compat_ab_b.
rewrite -inv_id.
apply (@inc_inv _ _ _ _ H).
Qed.
```

```
Lemma residual_property12b {X Y : eqType} {u : Rel X X} {alpha : Rel X Y}:
  u ⊆ Id X → u △ alpha = u △ (u • alpha).
Proof.
move ⇒ H.
apply inc_antisym.
rewrite (residual_property12a H).
apply (@inc_trans _ _ _ _ _ residual_property11).
apply residual_inc_compat_l.
rewrite rpc_universal_alpha.
apply comp_inc_compat_ab_a'b.
rewrite (dedekind_id1 H).
apply inc_refl.
apply residual_inc_compat_l.
apply (comp_inc_compat_ab_b H).
Qed.
```

Lemma 248 (residual_property13)

$$(\alpha \cdot \nabla_{YZ} \sqcap \delta) \rhd \gamma = (\alpha \cdot \nabla_{YW} \Rightarrow (\delta \rhd \gamma)).$$

```
Lemma residual_property13
 {W X Y Z : eqType} {alpha : Rel X Y} {gamma : Rel Z W} {delta : Rel X Z}:
 ((alpha ・ ∇ Y Z) ∩ delta) △ gamma = (alpha ・ ∇ Y W) » (delta △ gamma).
Proof.
apply inc_antisym.
rewrite inc_rpc inc_residual.
remember (((alpha ・ ∇ Y Z) ∩ delta) △ gamma) as sigma1.
apply (@inc_trans _ _ _ (((alpha ・ ∇ Y Z) ∩ delta) # ・ sigma1)).
apply (@inc_trans _ _ _ (((alpha ・ ∇ Y Z) ∩ delta) # ・ (sigma1 ∩ (alpha ・ ∇ Y
W)))).
assert ((delta # ・ (sigma1 ∩ (alpha ・ ∇ Y W))) ⊆ (delta # ・ sigma1)).
apply comp_inc_compat_ab_ab'.
apply cap_l.
apply inc_def1 in H.
rewrite H.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
rewrite (@inv_cap_distr _ _ _ delta) cap_comm.
apply cap_inc_compat_r.
rewrite inv_cap_distr.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_l)).
apply (@inc_trans _ _ _ _ _ (cap_r)).
rewrite comp_inv comp_inv -comp_assoc (@inv_universal Y Z).
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
apply comp_inc_compat_ab_ab'.
apply cap_l.
rewrite Heqsigma1.
apply inc_residual.
apply inc_refl.
rewrite inc_residual.
remember ((alpha ・ ∇ Y W) » (delta △ gamma)) as sigma2.
apply (@inc_trans _ _ _ (delta # ・ ((alpha ・ ∇ Y W) ∩ sigma2))).
apply (@inc_trans _ _ _ (((alpha ・ ∇ Y Z) ∩ delta) # ・ ((alpha ・ ∇ Y W) ∩ sigma2))).
assert ((((alpha ・ ∇ Y Z) ∩ delta) # ・ sigma2) ⊆ (delta # ・ sigma2)).
apply comp_inc_compat_ab_a'b.
apply inc_inv.
apply cap_r.
```

```
apply inc_def1 in H.
rewrite H.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm inv_invol.
apply cap_inc_compat_r.
apply (@inc_trans _ _ _ ((alpha • ∇ Y Z) • (delta # • sigma2))).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply comp_inc_compat_ab_a'b.
apply inc_inv.
apply cap_r.
rewrite Heqsigma2.
rewrite -inc_residual cap_comm -inc_rpc.
apply inc_refl.
Qed.
```

Lemma 249 (residual_property14) *Let* $\nabla_{XX} \cdot \alpha \sqsubseteq \alpha$. *Then,*

$$\nabla_{XX} \cdot (\alpha \rhd \beta) \sqsubseteq \alpha \rhd \beta.$$

```
Lemma residual_property14 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:
 (∇ X X • alpha) ⊆ alpha → (∇ X X • (alpha △ beta)) ⊆ (alpha △ beta).
Proof.
move ⇒ H.
apply (@inc_trans _ _ _ (∇ X X • (∇ X X △ (alpha △ beta)))).
apply comp_inc_compat_ab_ab'.
rewrite double_residual.
apply (residual_inc_compat_r H).
rewrite -inv_universal -inc_residual inv_universal.
apply inc_refl.
Qed.
```

Lemma 250 (residual_property15) *Let* $\beta \cdot \nabla_{ZZ} \sqsubseteq \beta$. *Then,*

$$(\alpha \rhd \beta) \cdot \nabla_{ZZ} \sqsubseteq \alpha \rhd \beta.$$

```
Lemma residual_property15 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:
 (beta • ∇ Z Z) ⊆ beta → ((alpha △ beta) • ∇ Z Z) ⊆ (alpha △ beta).
Proof.
```

```
move => H.
apply (@inc_trans _ _ _ _ _ (residual_property1)).
apply (residual_inc_compat_l H).
Qed.
```

Lemma 251 (residual_property16)

$$id_X \sqsubseteq \alpha \rhd \alpha^\sharp \land (\alpha \rhd \alpha^\sharp) \cdot (\alpha \rhd \alpha^\sharp) \sqsubseteq \alpha \rhd \alpha^\sharp.$$

```
Lemma residual_property16 {X Y : eqType} {alpha : Rel X Y}:
 Id X ⊆ (alpha △ alpha #) ∧
 ((alpha △ alpha #) • (alpha △ alpha #)) ⊆ (alpha △ alpha #).
Proof.
split.
rewrite inc_residual comp_id_r.
apply inc_refl.
move : (@residual_property2 _ _ _ _ alpha (alpha #) (alpha #)) => H.
rewrite inv_invol in H.
apply H.
Qed.
```

Lemma 252 (residual_property17) *Let $P(y) :=$ "$y : I \rightharpoondown Y$ is a function". Then,*

$$\sqcup_{P(y)} y^\sharp \cdot y = id_Y \Rightarrow \alpha \rhd \beta = \sqcap_{P(y)}(\alpha \cdot y^\sharp \cdot \nabla_{IZ} \Rightarrow \alpha \cdot y^\sharp \cdot y \cdot \beta).$$

```
Lemma residual_property17 {X Y Z : eqType}
 {alpha : Rel X Y} {beta : Rel Y Z} {P : Rel i Y → Prop}:
 P = (fun y : Rel i Y => function_r y) →
 ∪_{P} (fun y : Rel i Y => y # • y) = Id Y →
 alpha △ beta = ∩_{P} (fun y : Rel i Y =>
  ((alpha • y #) • ∇ i Z) » ((alpha • y #) • (y • beta))).
Proof.
move => H H0.
replace (alpha △ beta) with ((alpha • Id Y) △ beta).
rewrite -H0 comp_cupP_distr_l residual_cupP_distr_r.
apply capP_eq.
move => y H1.
rewrite H in H1.
rewrite -comp_assoc (function_residual4 H1).
apply residual_property9.
rewrite /univalent_r.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
```

```
by [rewrite comp_id_r].
Qed.
```

11.4 順序の関係と左剰余合成

11.4.1 max, sup, min, inf

$\xi : X \rightharpoonup X$ を集合 X における順序と見なしたときの, 関係 $\rho : V \rightharpoonup X$ の 最大値 (max), 上限 (sup), 最小値 (min), 下限 (inf) はそれぞれ, 以下のように定義される.

- $max(\rho, \xi) := \rho \sqcap (\rho \rhd \xi)$
- $sup(\rho, \xi) := (\rho \rhd \xi) \sqcap ((\rho \rhd \xi) \rhd \xi^\sharp)$
- $min(\rho, \xi) := \rho \sqcap (\rho \rhd \xi^\sharp)(= max(\rho, \xi^\sharp))$
- $inf(\rho, \xi) := (\rho \rhd \xi^\sharp) \sqcap ((\rho \rhd \xi^\sharp) \rhd \xi)(= sup(\rho, \xi^\sharp))$

```
Definition max {V X : eqType} (rho : Rel V X) (xi : Rel X X)
 := rho ∩ (rho △ xi).
Definition sup {V X : eqType} (rho : Rel V X) (xi : Rel X X)
 := (rho △ xi) ∩ ((rho △ xi) △ xi #).
Definition min {V X : eqType} (rho : Rel V X) (xi : Rel X X)
 := rho ∩ (rho △ xi #).
Definition inf {V X : eqType} (rho : Rel V X) (xi : Rel X X)
 := (rho △ xi #) ∩ ((rho △ xi #) △ xi).
```

Lemma 253 (max_inc_sup) *Let $\rho : V \rightharpoonup X$ and $\xi : X \rightharpoonup X$. Then,*

$$max(\rho, \xi) \sqsubseteq sup(\rho, \xi).$$

```
Lemma max_inc_sup {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
 max rho xi ⊆ sup rho xi.
Proof.
rewrite /max/sup.
rewrite cap_comm.
apply cap_inc_compat_l.
apply galois_corollary1.
Qed.
```

Lemma 254 (min_inc_inf) *Let* $\rho : V \rightharpoonup X$ *and* $\xi : X \rightharpoonup X$. *Then,*

$$min(\rho, \xi) \sqsubseteq inf(\rho, \xi).$$

```
Lemma min_inc_inf {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
  min rho xi ⊆ inf rho xi.
Proof.
rewrite /min/inf.
rewrite cap_comm.
apply cap_inc_compat_l.
move : (@galois_corollary1 _ _ _ rho (xi #)) ⇒ H.
rewrite inv_invol in H.
apply H.
Qed.
```

Lemma 255 (inf_to_sup) *Let* $\rho : V \rightharpoonup X$ *and* $\xi : X \rightharpoonup X$. *Then,*

$$inf(\rho, \xi) = sup(\rho \rhd \xi^\sharp, \xi).$$

```
Lemma inf_to_sup {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
  inf rho xi = sup (rho △ xi #) xi.
Proof.
rewrite /sup/inf.
rewrite cap_comm.
move : (@galois_corollary2 _ _ _ rho (xi #)) ⇒ H.
rewrite inv_invol in H.
by [rewrite H].
Qed.
```

Lemma 256 (sup_to_inf) *Let* $\rho : V \rightharpoonup X$ *and* $\xi : X \rightharpoonup X$. *Then,*

$$sup(\rho, \xi) = inf(\rho \rhd \xi, \xi).$$

```
Lemma sup_to_inf {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
  sup rho xi = inf (rho △ xi) xi.
Proof.
rewrite /sup/inf.
rewrite cap_comm.
by [rewrite galois_corollary2].
Qed.
```

Lemma 257 (residual_inc_sup1, residual_inc_sup2) *Let $\rho : V \rightharpoonup X$ and $\xi : X \rightharpoonup X$. Then,*

$$sup(\rho, \xi) \sqsubseteq \rho \rhd \xi \sqsubseteq sup(\rho, \xi) \rhd \xi.$$

```
Lemma residual_inc_sup1 {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
 sup rho xi ⊆ (rho △ xi).
Proof.
apply cap_l.
Qed.
```

```
Lemma residual_inc_sup2 {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
 (rho △ xi) ⊆ ((sup rho xi) △ xi).
Proof.
rewrite galois.
apply cap_r.
Qed.
```

Lemma 258 (max_inc_xi_cap) *Let $\rho : V \rightharpoonup X$ and $\xi : X \rightharpoonup X$. Then,*

$$(max(\rho, \xi))^\sharp \cdot max(\rho, \xi) \sqsubseteq \xi \sqcap \xi^\sharp.$$

```
Lemma max_inc_xi_cap {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
 (max rho xi # • max rho xi) ⊆ (xi ∩ xi #).
Proof.
rewrite /max.
rewrite inv_cap_distr.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat.
apply inc_residual.
apply cap_r.
apply inv_inc_move.
rewrite comp_inv inv_invol.
apply inc_residual.
apply residual_inc_compat_r.
apply cap_l.
Qed.
```

Lemma 259 (sup_inc_xi_cap) *Let $\rho : V \rightharpoonup X$ and $\xi : X \rightharpoonup X$. Then,*

$$(sup(\rho, \xi))^\sharp \cdot sup(\rho, \xi) \sqsubseteq \xi \sqcap \xi^\sharp.$$

```
Lemma sup_inc_xi_cap {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
 (sup rho xi # • sup rho xi) ⊆ (xi ∩ xi #).
```

```
Proof.
move : (@max_inc_xi_cap _ _ (rho △ xi) (xi #)).
rewrite /max/sup.
by [rewrite inv_invol (@cap_comm _ _ xi)].
Qed.
```

Lemma 260 (transitive_sup1) *Let $\rho : V \rightharpoondown X$, $\xi : X \rightharpoondown X$ and $\xi \cdot \xi \sqsubseteq \xi$. Then,*

$$sup(\rho, \xi) \cdot (\xi \sqcap \xi^\sharp) = sup(\rho, \xi).$$

```
Lemma transitive_sup1 {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
 (xi • xi) ⊆ xi → sup rho xi • (xi ∩ xi #) = sup rho xi.
Proof.
move ⇒ H.
apply inc_antisym.
rewrite /sup.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_l)).
apply cap_inc_compat.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_l)).
apply (@inc_trans _ _ _ _ _ (residual_property1)).
apply (residual_inc_compat_l H).
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_r)).
apply (@inc_trans _ _ _ _ _ (residual_property1)).
apply residual_inc_compat_l.
rewrite -comp_inv inv_inc_move inv_invol.
apply H.
apply (@inc_trans _ _ _ _ _ (relation_rel_inv_rel)).
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' sup_inc_xi_cap).
Qed.
```

Lemma 261 (transitive_sup2) *Let $\rho : V \rightharpoondown X$, $\xi : X \rightharpoondown X$ and $\xi \cdot \xi \sqsubseteq \xi$. Then,*

$$sup(\rho, \xi) \cdot \xi = \lfloor sup(\rho, \xi) \rfloor \cdot (\rho \rhd \xi).$$

```
Lemma transitive_sup2 {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
 (xi • xi) ⊆ xi → sup rho xi • xi = domain (sup rho xi) • (rho △ xi).
Proof.
move ⇒ H.
apply inc_antisym.
replace (sup rho xi • xi) with (domain (sup rho xi) • (sup rho xi • xi)).
```

```
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ ((rho △ xi) • xi)).
apply (comp_inc_compat_ab_a'b cap_l).
apply (@inc_trans _ _ _ _ _ (residual_property1) (residual_inc_compat_l H)).
by [rewrite -comp_assoc domain_comp_alpha1].
apply (@inc_trans _ _ _ (domain (sup rho xi) • (sup rho xi △ xi))).
apply comp_inc_compat_ab_ab'.
apply galois.
apply cap_r.
rewrite /domain.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_l)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_residual.
apply inc_refl.
Qed.
```

Lemma 262 (domain_sup_inc) *Let* $\rho : V \rightharpoondown X$ *and* $\xi : X \rightharpoondown X$*. Then,*

$$\lfloor sup(\rho, \xi) \rfloor \cdot \rho \sqsubseteq sup(\rho, \xi) \cdot \xi^\sharp.$$

```
Lemma domain_sup_inc {V X : eqType} {rho : Rel V X} {xi : Rel X X}:
 (domain (sup rho xi) • rho) ⊆ (sup rho xi • xi #).
Proof.
apply (@inc_trans _ _ _ (domain (sup rho xi) • (sup rho xi △ xi #))).
apply comp_inc_compat_ab_ab'.
rewrite -galois.
apply cap_l.
rewrite /domain.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_l)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_residual.
apply inc_refl.
Qed.
```

Lemma 263 (sup_function) *Let* $\rho : V \rightharpoondown X$, $\xi : X \rightharpoondown X$ *be relations and* $f : W \to V$ *be a function. Then,*

$$f \cdot sup(\rho, \xi) = sup(f \cdot \rho, \xi).$$

```
Lemma sup_function {V W X : eqType} {rho : Rel V X} {xi : Rel X X} {f : Rel W V}:
```

```
 function_r f → f • sup rho xi = sup (f • rho) xi.
Proof.
move ⇒ H.
rewrite /sup.
rewrite (function_cap_distr_l H).
by [rewrite (function_residual2 H) (function_residual2 H) (function_residual2 H)].
Qed.
```

Lemma 264 (max_univalent) *Let $\rho : V \rightharpoonup X$, $\xi : X \rightharpoonup X$ be relations and $\varphi : W \rightharpoonup V$ be a univalent relation. Then,*

$$\varphi \cdot max(\rho, \xi) = max(\varphi \cdot \rho, \xi).$$

```
Lemma max_univalent {V W X : eqType}
 {rho : Rel V X} {xi : Rel X X} {phi : Rel W V}:
 univalent_r phi → phi • max rho xi = max (phi • rho) xi.
Proof.
move ⇒ H.
rewrite /max.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_l)).
apply cap_inc_compat_l.
apply (@inc_trans _ _ _ _ _ (univalent_residual H)).
rewrite double_residual.
apply inc_refl.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
apply cap_inc_compat_l.
rewrite -inc_residual double_residual.
apply inc_refl.
Qed.
```

11.4.2 左剰余合成

関係 $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$ に対し, 左剰余合成を $\alpha \lhd \beta := (\beta^\sharp \rhd \alpha^\sharp)^\sharp$ で定義する.

```
Definition leftres {X Y Z : eqType} (alpha : Rel X Y) (beta : Rel Y Z)
 := (beta # △ alpha #) #.
```

Lemma 265 (inc_leftres) *Let* $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$ *and* $\delta : X \rightharpoonup Z$. *Then,*

$$\delta \sqsubseteq \alpha \lhd \beta \Leftrightarrow \delta \cdot \beta^\sharp \sqsubseteq \alpha.$$

```
Lemma inc_leftres {X Y Z : eqType}
 {alpha : Rel X Y} {beta : Rel Y Z} {delta : Rel X Z}:
 delta ⊆ leftres alpha beta ↔ (delta • beta #) ⊆ alpha.
Proof.
rewrite /leftres.
by [rewrite inv_inc_move inc_residual -comp_inv inv_inc_move inv_invol].
Qed.
```

Lemma 266 (residual_leftres_assoc) *Let* $\alpha : X \rightharpoonup Y$, $\beta : Y \rightharpoonup Z$ *and* $\gamma : Z \rightharpoonup W$. *Then,*

$$(\alpha \rhd \beta) \lhd \gamma = \alpha \rhd (\beta \lhd \gamma).$$

```
Lemma residual_leftres_assoc {W X Y Z : eqType}
 {alpha : Rel X Y} {beta : Rel Y Z} {gamma : Rel Z W}:
 leftres (alpha △ beta) gamma = alpha △ leftres beta gamma.
Proof.
apply inc_lower.
move ⇒ delta.
by [rewrite inc_leftres inc_residual -comp_assoc -inc_leftres -inc_residual].
Qed.
```

Chapter 12

Library Schroder

Require Import *Basic_Notations*.
Require Import *Basic_Lemmas*.
Require Import *Relation_Properties*.
Require Import *Functions_Mappings*.
Require Import *Dedekind*.
Require Import *Residual*.
Require Import *Logic.FunctionalExtensionality*.

12.1 Schröder 圏の性質

この節では, 特記が無い限り, 記号は以下の図式に従って割り振られるものとする.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[rr, bend left, "\delta"] & Y \arrow[r, "\beta{,}\beta'{,}\beta_\lambda"'] & Z \arrow[r, "\gamma"] & W \\
I \arrow[u, "\rho"'] & & V \arrow[u, "\tau"'] &
\end{tikzcd}$$

Lemma 267 (schroder_equivalence1, schroder_equivalence2)

$$\alpha \cdot \beta \sqsubseteq \delta \Leftrightarrow \alpha^\sharp \cdot \delta^- \sqsubseteq \beta^- \Leftrightarrow \delta^- \cdot \beta^\sharp \sqsubseteq \alpha^-.$$

Lemma *schroder_equivalence1*
{*X Y Z* : *eqType*} {*alpha* : *Rel X Y*} {beta : *Rel Y Z*} {delta : *Rel X Z*}:
(*alpha* ・ beta) ⊆ delta ↔ (*alpha* # ・ delta ^) ⊆ beta ^.
Proof.
split; move ⇒ *H*.
rewrite *bool_lemma2 complement_invol*.

```
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply bool_lemma2 in H.
rewrite cap_comm inv_invol H comp_empty_r.
apply inc_refl.
apply inc_empty_alpha.
rewrite bool_lemma2.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply bool_lemma2 in H.
rewrite cap_comm -(@complement_invol _ _ beta) H comp_empty_r.
apply inc_refl.
apply inc_empty_alpha.
Qed.
```

```
Lemma schroder_equivalence2
 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {delta : Rel X Z}:
 (alpha ・ beta) ⊆ delta ↔ (delta ^ ・ beta #) ⊆ alpha ^.
Proof.
split; move ⇒ H.
rewrite bool_lemma2 complement_invol.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply bool_lemma2 in H.
rewrite cap_comm inv_invol H comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
rewrite bool_lemma2.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply bool_lemma2 in H.
rewrite cap_comm -(@complement_invol _ _ alpha) H comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
Qed.
```

Lemma 268 (function_inv_complement) *Let α and τ be functions. Then,*

$$(\alpha \cdot \beta \cdot \tau^\sharp)^- = \alpha \cdot \beta^- \cdot \tau^\sharp.$$

```
Lemma function_inv_complement
 {V X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z} {tau : Rel V Z}:
 function_r alpha → function_r tau →
```

```
((alpha • beta) • tau #) ^ = (alpha • beta ^) • tau #.
Proof.
move => H H0.
apply inc_antisym.
rewrite bool_lemma1 complement_invol.
apply inc_antisym.
rewrite -comp_cup_distr_r -comp_cup_distr_l complement_classic.
apply (@inc_trans _ _ _ (((alpha • alpha #) • ∇ X V) • (tau • tau #))).
apply (@inc_trans _ _ _ ((alpha • alpha #) • ∇ X V)).
apply comp_inc_compat_b_ab.
apply H.
apply comp_inc_compat_a_ab.
apply H0.
rewrite -comp_assoc (@comp_assoc _ _ _ _ alpha) (@comp_assoc _ _ _ _ alpha).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
apply inc_alpha_universal.
rewrite bool_lemma2 complement_invol.
apply inc_antisym.
rewrite -(function_cap_distr H H0) cap_comm cap_complement_empty comp_empty_r comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
Qed.
```

Lemma 269 (schroder_univalent1) *Let α be a univalent relation and $\beta \sqsubseteq \beta'$. Then,*

$$\alpha \cdot (\beta' \sqcap \beta^{-}) = \alpha \cdot \beta' \sqcap (\alpha \cdot \beta)^{-}.$$

```
Lemma schroder_univalent1
 {X Y Z : eqType} {alpha : Rel X Y} {beta beta' : Rel Y Z}:
 univalent_r alpha → beta ⊆ beta' →
 alpha • (beta' ∩ beta ^) = (alpha • beta') ∩ (alpha • beta) ^.
Proof.
move => H H0.
apply (@cap_cup_unique _ _ (alpha • beta)).
replace ((alpha • beta) ∩ (alpha • (beta' ∩ beta ^))) with (φ X Z).
rewrite (@cap_comm _ _ (alpha • beta')) -cap_assoc.
by [rewrite cap_complement_empty cap_comm cap_empty].
apply inc_antisym.
apply inc_empty_alpha.
apply (@inc_trans _ _ _ ((alpha • beta) ∩ ((alpha • beta') ∩ (alpha • beta ^)))).
apply cap_inc_compat_l.
```

```
apply comp_cap_distr_l.
replace (φ X Z) with ((alpha • beta) ∩ (alpha • beta ^)).
apply cap_inc_compat_l.
apply cap_r.
apply inc_antisym.
move : (@univalent_residual _ _ _ _ beta H) ⇒ H1.
rewrite -inc_rpc.
rewrite residual_to_complement in H1.
apply H1.
apply inc_empty_alpha.
apply inc_def2 in H0.
rewrite -comp_cup_distr_l cup_cap_distr_l.
rewrite -H0 complement_classic cap_universal.
rewrite cup_cap_distr_l -comp_cup_distr_l.
by [rewrite -H0 complement_classic cap_universal].
Qed.
```

Lemma 270 (schroder_univalent2) *Let α be a univalent relation. Then,*

$$\alpha \cdot \beta^{-} = \alpha \cdot \nabla_{YZ} \sqcap (\alpha \cdot \beta)^{-}.$$

```
Lemma schroder_univalent2 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:
  univalent_r alpha → alpha • beta ^ = (alpha • ∇ Y Z) ∩ (alpha • beta) ^.
Proof.
move ⇒ H.
move : (@schroder_univalent1 _ _ _ alpha beta (∇ Y Z) H (@inc_alpha_universal _ _ _))
⇒ H0.
rewrite cap_comm cap_universal in H0.
apply H0.
Qed.
```

Lemma 271 (schroder_univalent3) *Let α be a univalent relation. Then,*

$$(\alpha \cdot \beta)^{-} = (\alpha \cdot \nabla_{YZ})^{-} \sqcup \alpha \cdot \beta^{-}.$$

```
Lemma schroder_univalent3 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:
  univalent_r alpha → (alpha • beta) ^ = (alpha • ∇ Y Z) ^ ∪ (alpha • beta ^).
Proof.
move ⇒ H.
rewrite (schroder_univalent2 H).
rewrite cup_cap_distr_l cup_comm complement_classic cap_comm cap_universal.
apply inc_def2.
apply rpc_inc_compat_r.
```

```
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
Qed.
```

Lemma 272 (schroder_univalent4) *Let α be a univalent relation. Then,*

$$\alpha \rhd \beta = (\alpha \cdot \nabla_{YZ})^{-} \sqcup \alpha \cdot \beta.$$

```
Lemma schroder_univalent4 {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:
 univalent_r alpha → alpha △ beta = (alpha • ∇ Y Z) ^ ∪ (alpha • beta).
Proof.
move ⇒ H.
rewrite (residual_property9 H).
apply Logic.eq_sym.
apply cup_to_rpc.
Qed.
```

Lemma 273 (schroder_universal) *Let $\nabla_{XZ} \cdot \nabla_{ZW} = \nabla_{XW}$. Then,*

$$(\alpha \cdot \nabla_{YZ})^{-} \cdot \nabla_{ZW} = (\alpha \cdot \nabla_{YW})^{-}.$$

```
Lemma schroder_universal {W X Y Z : eqType} {alpha : Rel X Y}:
 (∇ X Z • ∇ Z W) = ∇ X W →
 (alpha • ∇ Y Z) ^ • ∇ Z W = (alpha • ∇ Y W) ^.
Proof.
move ⇒ H.
apply (@cap_cup_unique _ _ (alpha • ∇ Y W)).
rewrite cap_complement_empty cap_comm.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply (@inc_trans _ _ _ (((alpha • ∇ Y Z) ^ ∩ (alpha • ∇ Y Z)) • ∇ Z W)).
apply comp_inc_compat_ab_a'b.
apply cap_inc_compat_l.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
rewrite cap_comm cap_complement_empty comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
rewrite complement_classic.
apply inc_antisym.
apply inc_alpha_universal.
rewrite -H -(@complement_classic _ _ (alpha • ∇ Y Z)) comp_cup_distr_r.
```

```
apply cup_inc_compat_r.
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
Qed.
```

Lemma 274 (residual_inv)

$$(\alpha \rhd \beta)^\sharp = \beta^{-\sharp} \rhd \alpha^{-\sharp}.$$

```
Lemma residual_inv {X Y Z : eqType} {alpha : Rel X Y} {beta : Rel Y Z}:
 (alpha △ beta) # = (beta ^) # △ (alpha ^) #.
Proof.
rewrite residual_to_complement residual_to_complement.
by [rewrite -inv_complement complement_invol inv_complement comp_inv].
Qed.
```

Lemma 275 (residual_cupP_distr_l, residual_cup_distr_l) *Let α be a univalent relation, $f : (V \rightharpoondown W) \to (Y \rightharpoondown Z)$ and $\exists \beta, P(\beta)$. Then,*

$$\alpha \rhd (\sqcup_{P(\beta)} f(\beta)) = \sqcup_{P(\beta)} (\alpha \rhd f(\beta)).$$

```
Lemma residual_cupP_distr_l {V W X Y Z : eqType}
 {alpha : Rel X Y} {f : Rel V W → Rel Y Z} {P : Rel V W → Prop}:
 univalent_r alpha → (∃ beta' : Rel V W, P beta') →
 alpha △ (∪_{P} f) = ∪_{P} (fun beta : Rel V W ⇒ alpha △ f beta).
Proof.
move ⇒ H.
elim ⇒ beta' H0.
rewrite (schroder_univalent4 H) comp_cupP_distr_l.
replace (∪_{P} (fun beta : Rel V W ⇒ alpha △ f beta)) with (∪_{P} (fun beta :
Rel V W ⇒ (alpha • ∇ Y Z) ^ ∪ (alpha • f beta))).
apply (@cap_cup_unique _ _ (alpha • ∇ Y Z)).
rewrite cap_cup_distr_l cap_cupP_distr_l cap_complement_empty cup_comm cup_empty.
rewrite cap_cupP_distr_l.
apply cupP_eq.
move ⇒ gamma H1.
by [rewrite cap_cup_distr_l cap_complement_empty cup_comm cup_empty].
rewrite -cup_assoc complement_classic cup_comm cup_universal.
rewrite -(@complement_invol _ _ (alpha • ∇ Y Z)).
apply bool_lemma1.
rewrite complement_invol.
apply (@inc_trans _ _ _ ((alpha • ∇ Y Z) ^ ∪ (alpha • f beta'))).
```

```
apply cup_l.
move : beta' H0.
apply inc_cupP.
apply inc_refl.
apply cupP_eq.
move ⇒ gamma H1.
by [rewrite (schroder_univalent4 H)].
Qed.
```

```
Lemma residual_cup_distr_l
 {X Y Z : eqType} {alpha : Rel X Y} {beta beta' : Rel Y Z}:
 univalent_r alpha →
 alpha △ (beta ∪ beta') = (alpha △ beta) ∪ (alpha △ beta').
Proof.
move ⇒ H.
rewrite cup_to_cupP (@cup_to_cupP _ _ _ _ _ _ id).
apply (residual_cupP_distr_l H).
∃ beta.
by [left].
Qed.
```

Lemma 276 (residual_capP_distr_r, residual_cap_distr_r) *Let* $f : (Y \rightharpoondown Z) \to (I \rightharpoondown X)$ *and* $\exists \alpha, P(\alpha)$. *Then,*

$$(\sqcap_{P(\alpha)} f(\alpha)^\sharp) \rhd \rho = \sqcup_{P(\alpha)} (f(\alpha)^\sharp \rhd \rho).$$

```
Lemma residual_capP_distr_r
 {X Y Z : eqType} {rho : Rel i X} {f : Rel Y Z → Rel i X} {P : Rel Y Z → Prop}:
 (∃ alpha' : Rel Y Z, P alpha') →
 (∩_{P} (fun alpha : Rel Y Z ⇒ f alpha #)) △ rho = ∪_{P} (fun alpha : Rel Y Z ⇒
f alpha # △ rho).
Proof.
elim ⇒ alpha' H.
rewrite residual_to_complement.
rewrite -(@complement_invol _ _ (∪_{P} (fun alpha : Rel Y Z ⇒ f alpha # △ rho))).
apply f_equal.
rewrite de_morgan3.
replace (fun alpha : Rel Y Z ⇒ (f alpha # △ rho) ^) with (fun alpha : Rel Y Z ⇒ f
alpha # • rho ^).
apply inc_antisym.
apply comp_capP_distr_r.
apply (@inc_trans _ _ _ _ _ (relation_rel_inv_rel)).
apply (@inc_trans _ _ _ (((∩_{P} (fun alpha : Rel Y Z ⇒ f alpha # • rho ^)) • (f
```

```
alpha' # • rho ^) #) • (f alpha' # • rho ^))).
apply comp_inc_compat.
apply comp_inc_compat_ab_ab'.
move : alpha' H.
apply inc_capP.
rewrite inv_capP_distr.
apply inc_refl.
move : alpha' H.
apply inc_capP.
apply inc_refl.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc.
apply (@inc_trans _ _ _ _ _ (comp_capP_distr_r)).
apply inc_capP.
move => beta H0.
apply (@inc_trans _ _ _ ((f beta # • rho ^) • ((f alpha' # • rho ^) # • f alpha' #))).
move : beta H0.
apply inc_capP.
apply inc_refl.
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
apply functional_extensionality.
move => x.
by [rewrite residual_to_complement complement_invol].
Qed.
```

Chapter 13

Library Sum_Product

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Conjugate.
Require Import Domain.
Require Import Logic.IndefiniteDescription.
```

13.1 関係の直和

13.1.1 入射対, 関係直和の定義

入射対の存在公理 (Axiom 23) で入射対が存在することまでは仮定済みなので, 実際に入射対 $j : A \rightharpoonup A + B, k : B \rightharpoonup A + B$ を定義する関数を定義する.

```
Definition sum_r (A B : eqType):
 {x : (Rel A (sum_eqType A B)) × (Rel B (sum_eqType A B)) |
 (fst x) • (fst x) # = Id A ∧ (snd x) • (snd x) # = Id B ∧
 (fst x) • (snd x) # = φ A B ∧
 ((fst x) # • (fst x)) ∪ ((snd x) # • (snd x)) = Id (sum_eqType A B)}.
apply constructive_indefinite_description.
elim (@pair_of_inclusions A B) ⇒ j.
elim ⇒ k H.
∃ (j,k).
simpl.
apply H.
Defined.
Definition inl_r (A B : eqType):= fst (sval (sum_r A B)).
```

```
Definition inr_r (A B : eqType):= snd (sval (sum_r A B)).
```

またこの定義による入射対が, 入射対としての性質 (Axiom 23) +α を満たしていることも事前に証明しておく.

```
Lemma inl_id {A B : eqType}: inl_r A B · inl_r A B # = Id A.
Proof.
apply (proj2_sig (sum_r A B)).
Qed.
Lemma inr_id {A B : eqType}: inr_r A B · inr_r A B # = Id B.
Proof.
apply (proj2_sig (sum_r A B)).
Qed.
Lemma inl_inr_empty {A B : eqType}: inl_r A B · inr_r A B # = φ A B.
Proof.
apply (proj2_sig (sum_r A B)).
Qed.
Lemma inr_inl_empty {A B : eqType}: inr_r A B · inl_r A B # = φ B A.
Proof.
apply inv_invol2.
rewrite comp_inv inv_invol inv_empty.
apply inl_inr_empty.
Qed.
Lemma inl_inr_cup_id {A B : eqType}:
 (inl_r A B # · inl_r A B) ∪ (inr_r A B # · inr_r A B) = Id (sum_eqType A B).
Proof.
apply (proj2_sig (sum_r A B)).
Qed.
Lemma inl_function {A B : eqType}: function_r (inl_r A B).
Proof.
move : (proj2_sig (sum_r A B)).
elim ⇒ H.
elim ⇒ H0.
elim ⇒ H1 H2.
split.
rewrite /total_r.
rewrite H.
apply inc_refl.
rewrite /univalent_r.
rewrite -H2.
apply cup_l.
Qed.
```

```
Lemma inr_function {A B : eqType}: function_r (inr_r A B).
Proof.
move : (proj2_sig (sum_r A B)).
elim => H.
elim => H0.
elim => H1 H2.
split.
rewrite /total_r.
rewrite H0.
apply inc_refl.
rewrite /univalent_r.
rewrite -H2.
apply cup_r.
Qed.
```

さらに $\alpha : A \rightharpoondown C$ と $\beta : B \rightharpoondown C$ の関係直和 $\alpha\perp\beta : A+B \rightharpoondown C$ を, $\alpha\perp\beta := j^\sharp \cdot \alpha \sqcup k^\sharp \cdot \beta$ で定義する.

```
Definition Rel_sum {A B C : eqType} (alpha : Rel A C) (beta : Rel B C):=
 (inl_r A B # • alpha) ∪ (inr_r A B # • beta).
```

13.1.2 関係直和の性質

Lemma 277 (sum_inc_compat) *Let $\alpha, \alpha' : A \rightharpoondown C$ and $\beta, \beta' : B \rightharpoondown C$. Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha\perp\beta \sqsubseteq \alpha'\perp\beta'.$$

```
Lemma sum_inc_compat
 {A B C : eqType} {alpha alpha' : Rel A C} {beta beta' : Rel B C}:
 alpha ⊆ alpha' → beta ⊆ beta' → Rel_sum alpha beta ⊆ Rel_sum alpha' beta'.
Proof.
move => H H0.
apply cup_inc_compat.
apply (comp_inc_compat_ab_ab' H).
apply (comp_inc_compat_ab_ab' H0).
Qed.
```

Lemma 278 (sum_inc_compat_l) *Let $\alpha : A \rightharpoondown C$ and $\beta, \beta' : B \rightharpoondown C$. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha\perp\beta \sqsubseteq \alpha\perp\beta'.$$

```
Lemma sum_inc_compat_l
```

```
 {A B C : eqType} {alpha : Rel A C} {beta beta' : Rel B C}:
 beta ⊆ beta' → Rel_sum alpha beta ⊆ Rel_sum alpha beta'.
Proof.
move ⇒ H.
apply (sum_inc_compat (@inc_refl _ _ alpha) H).
Qed.
```

Lemma 279 (sum_inc_compat_r) *Let $\alpha, \alpha' : A \rightharpoonup C$ and $\beta : B \rightharpoonup C$. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \perp \beta \sqsubseteq \alpha' \perp \beta.$$

```
Lemma sum_inc_compat_r
 {A B C : eqType} {alpha alpha' : Rel A C} {beta : Rel B C}:
 alpha ⊆ alpha' → Rel_sum alpha beta ⊆ Rel_sum alpha' beta.
Proof.
move ⇒ H.
apply (sum_inc_compat H (@inc_refl _ _ beta)).
Qed.
```

Lemma 280 (total_sum) *Let $\alpha : A \rightharpoonup C$ and $\beta : B \rightharpoonup C$ are total relations, then $\alpha \perp \beta$ is also a total relation.*

```
Lemma total_sum {A B C : eqType} {alpha : Rel A C} {beta : Rel B C}:
 total_r alpha → total_r beta → total_r (Rel_sum alpha beta).
Proof.
move ⇒ H H0.
rewrite /total_r/Rel_sum.
rewrite -inl_inr_cup_id inv_cup_distr comp_cup_distr_l comp_cup_distr_r comp_cup_distr_r.
rewrite comp_inv comp_inv inv_invol inv_invol.
apply cup_inc_compat.
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' (cup_l)).
rewrite comp_assoc -(@comp_assoc _ _ _ _ alpha).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H).
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' (cup_r)).
rewrite comp_assoc -(@comp_assoc _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H0).
Qed.
```

Lemma 281 (univalent_sum) *Let $\alpha : A \rightharpoonup C$ and $\beta : B \rightharpoonup C$ are univalent relations, then $\alpha \perp \beta$ is also a univalent relation.*

```
Lemma univalent_sum {A B C : eqType} {alpha : Rel A C} {beta : Rel B C}:
 univalent_r alpha → univalent_r beta → univalent_r (Rel_sum alpha beta).
Proof.
move ⇒ H H0.
rewrite /univalent_r/Rel_sum.
rewrite inv_cup_distr comp_cup_distr_l comp_cup_distr_r comp_cup_distr_r.
rewrite comp_inv comp_inv inv_invol inv_invol.
rewrite comp_assoc -(@comp_assoc _ _ _ _ (inl_r A B)) inl_id comp_id_l.
rewrite comp_assoc -(@comp_assoc _ _ _ _ (inr_r A B)) inr_inl_empty comp_empty_l
comp_empty_r cup_empty.
rewrite -cup_assoc comp_assoc -(@comp_assoc _ _ _ _ (inl_r A B)) inl_inr_empty comp_empty_l
comp_empty_r cup_empty.
rewrite comp_assoc -(@comp_assoc _ _ _ _ (inr_r A B)) inr_id comp_id_l.
apply inc_cup.
split.
apply H.
apply H0.
Qed.
```

Lemma 282 (function_sum) *Let $\alpha : A \to C$ and $\beta : B \to C$ are functions, then $\alpha \perp \beta$ is also a function.*

```
Lemma function_sum {A B C : eqType} {alpha : Rel A C} {beta : Rel B C}:
 function_r alpha → function_r beta → function_r (Rel_sum alpha beta).
Proof.
elim ⇒ H H0.
elim ⇒ H1 H2.
split.
apply (total_sum H H1).
apply (univalent_sum H0 H2).
Qed.
```

Lemma 283 (sum_conjugate) *Let $\alpha : A \rightharpoondown C$, $\beta : B \rightharpoondown C$ and $\gamma : A + B \rightharpoondown C$ be relations, $j : A \to A + B$ and $k : B \to A + B$ be inclusions. Then,*

$$j \cdot \gamma = \alpha \wedge k \cdot \gamma = \beta \Leftrightarrow \gamma = \alpha \perp \beta.$$

```
Lemma sum_conjugate
 {A B C : eqType} {alpha : Rel A C} {beta : Rel B C} {gamma : Rel (sum_eqType A B)
C}:
 inl_r A B • gamma = alpha ∧ inr_r A B • gamma = beta ↔
 gamma = Rel_sum alpha beta.
Proof.
```

```
split; move ⇒ H.
elim H ⇒ H0 H1.
rewrite -(@comp_id_l _ _ gamma).
rewrite -inl_inr_cup_id comp_cup_distr_r comp_assoc comp_assoc.
by [rewrite H0 H1].
split.
rewrite H comp_cup_distr_l -comp_assoc -comp_assoc.
rewrite inl_id inl_inr_empty comp_id_l comp_empty_l.
by [rewrite cup_empty].
rewrite H comp_cup_distr_l -comp_assoc -comp_assoc.
rewrite inr_id inr_inl_empty comp_id_l comp_empty_l.
by [rewrite cup_comm cup_empty].
Qed.
```

Lemma 284 (sum_comp) *In below figure,*

$$(\alpha \perp \beta)^\sharp \cdot (\gamma \perp \delta) = \alpha^\sharp \cdot \gamma \sqcup \beta^\sharp \cdot \delta.$$

$$\begin{tikzcd}
 & Y \arrow[dl, "\alpha"'] \arrow[dr, "\gamma"] \arrow[d, "j"] & \\
X & Y+Z \arrow[l, "\alpha\perp\beta"] \arrow[r, "\gamma\perp\delta"'] & W \\
 & Z \arrow[ul, "\beta"] \arrow[ur, "\delta"'] \arrow[u, "k"'] &
\end{tikzcd}$$

```
Lemma sum_comp {W X Y Z : eqType}
 {alpha : Rel Y X} {beta : Rel Z X} {gamma : Rel Y W} {delta : Rel Z W}:
 (Rel_sum alpha beta) # • Rel_sum gamma delta =
 (alpha # • gamma) ∪ (beta # • delta).
Proof.
rewrite /Rel_sum.
rewrite inv_cup_distr comp_cup_distr_l comp_cup_distr_r comp_cup_distr_r.
rewrite comp_inv comp_inv inv_invol inv_invol.
apply f_equal2.
rewrite comp_assoc -(@comp_assoc _ _ _ _ (inl_r Y Z)) inl_id comp_id_l.
by [rewrite comp_assoc -(@comp_assoc _ _ _ _ (inr_r Y Z)) inr_inl_empty comp_empty_l
comp_empty_r cup_empty].
rewrite comp_assoc -(@comp_assoc _ _ _ _ (inl_r Y Z)) inl_inr_empty comp_empty_l
comp_empty_r cup_comm cup_empty.
by [rewrite comp_assoc -(@comp_assoc _ _ _ _ (inr_r Y Z)) inr_id comp_id_l].
Qed.
```

13.1.3 分配法則

Lemma 285 (sum_cap_distr_l) *Let $\alpha : A \rightharpoondown C$ and $\beta, \beta' : B \rightharpoondown C$. Then,*

$$\alpha \perp (\beta \sqcap \beta') \sqsubseteq (\alpha \perp \beta) \sqcap (\alpha \perp \beta').$$

```
Lemma sum_cap_distr_l
 {A B C : eqType} {alpha : Rel A C} {beta beta' : Rel B C}:
 Rel_sum alpha (beta ∩ beta') ⊆ (Rel_sum alpha beta ∩ Rel_sum alpha beta').
Proof.
rewrite -cup_cap_distr_l.
apply cup_inc_compat_l.
apply comp_cap_distr_l.
Qed.
```

Lemma 286 (sum_cap_distr_r) *Let $\alpha, \alpha' : A \rightharpoondown C$ and $\beta : B \rightharpoondown C$. Then,*

$$(\alpha \sqcap \alpha') \perp \beta \sqsubseteq (\alpha \perp \beta) \sqcap (\alpha' \perp \beta).$$

```
Lemma sum_cap_distr_r
 {A B C : eqType} {alpha alpha' : Rel A C} {beta : Rel B C}:
 Rel_sum (alpha ∩ alpha') beta ⊆ (Rel_sum alpha beta ∩ Rel_sum alpha' beta).
Proof.
rewrite -cup_cap_distr_r.
apply cup_inc_compat_r.
apply comp_cap_distr_l.
Qed.
```

Lemma 287 (sum_cup_distr_l) *Let $\alpha : A \rightharpoondown C$ and $\beta, \beta' : B \rightharpoondown C$. Then,*

$$\alpha \perp (\beta \sqcup \beta') = (\alpha \perp \beta) \sqcup (\alpha \perp \beta').$$

```
Lemma sum_cup_distr_l
 {A B C : eqType} {alpha : Rel A C} {beta beta' : Rel B C}:
 Rel_sum alpha (beta ∪ beta') = Rel_sum alpha beta ∪ Rel_sum alpha beta'.
Proof.
rewrite -cup_assoc (@cup_comm _ _ (Rel_sum alpha beta)) -cup_assoc.
by [rewrite cup_idem cup_assoc -comp_cup_distr_l].
Qed.
```

Lemma 288 (sum_cup_distr_r) *Let $\alpha, \alpha' : A \rightharpoondown C$ and $\beta : B \rightharpoondown C$. Then,*

$$(\alpha \sqcup \alpha')\perp\beta = (\alpha\perp\beta) \sqcup (\alpha'\perp\beta).$$

```
Lemma sum_cup_distr_r
 {A B C : eqType} {alpha alpha' : Rel A C} {beta : Rel B C}:
 Rel_sum (alpha ∪ alpha') beta = (Rel_sum alpha beta ∪ Rel_sum alpha' beta).
Proof.
rewrite cup_assoc (@cup_comm _ _ (inr_r A B # • beta)) cup_assoc.
by [rewrite cup_idem -cup_assoc -comp_cup_distr_l].
Qed.
```

Lemma 289 (comp_sum_distr_r) *Let $\alpha : A \rightharpoondown C$, $\beta : B \rightharpoondown C$ and $\gamma : C \rightharpoondown D$. Then,*

$$(\alpha\perp\beta) \cdot \gamma = \alpha \cdot \gamma\perp\beta \cdot \gamma.$$

```
Lemma comp_sum_distr_r
 {A B C D : eqType} {alpha : Rel A C} {beta : Rel B C} {gamma : Rel C D}:
 (Rel_sum alpha beta) • gamma = Rel_sum (alpha • gamma) (beta • gamma).
Proof.
by [rewrite comp_cup_distr_r comp_assoc comp_assoc].
Qed.
```

13.2 関係の直積

13.2.1 射影対, 関係直積の定義

射影対の存在公理 (Axiom 24) で射影対が存在することまでは仮定済みなので, 実際に射影対 $p : A \times B \rightharpoondown A, k : A \times B \rightharpoondown B$ を定義する関数を定義する.

```
Definition prod_r (A B : eqType):
 {x : (Rel (prod_eqType A B) A) × (Rel (prod_eqType A B) B) |
 (fst x) # • (snd x) = ∇ A B ∧
 ((fst x) • (fst x) #) ∩ ((snd x) • (snd x) #) = Id (prod_eqType A B) ∧
 univalent_r (fst x) ∧ univalent_r (snd x)}.
apply constructive_indefinite_description.
elim (@pair_of_projections A B) ⇒ p.
elim ⇒ q H.
∃ (p,q).
simpl.
apply H.
```

```
Defined.
Definition fst_r (A B : eqType):= fst (sval (prod_r A B)).
Definition snd_r (A B : eqType):= snd (sval (prod_r A B)).
```

またこの定義による射影対が, 射影対としての性質 (Axiom 24) $+\alpha$ を満たしていることも事前に証明しておく.

```
Lemma fst_snd_universal {A B : eqType}: fst_r A B # • snd_r A B = ∇ A B.
Proof.
apply (proj2_sig (prod_r A B)).
Qed.
Lemma snd_fst_universal {A B : eqType}: snd_r A B # • fst_r A B = ∇ B A.
Proof.
apply inv_invol2.
rewrite comp_inv inv_invol inv_universal.
apply fst_snd_universal.
Qed.
Lemma fst_snd_cap_id {A B : eqType}:
 (fst_r A B • fst_r A B #) ∩ (snd_r A B • snd_r A B #) = Id (prod_eqType A B).
Proof.
apply (proj2_sig (prod_r A B)).
Qed.
Lemma fst_function {A B : eqType}: function_r (fst_r A B).
Proof.
move : (proj2_sig (prod_r A B)).
elim ⇒ H.
elim ⇒ H0 H1.
split.
rewrite /total_r.
rewrite -H0.
apply cap_l.
apply H1.
Qed.
Lemma snd_function {A B : eqType}: function_r (snd_r A B).
Proof.
move : (proj2_sig (prod_r A B)).
elim ⇒ H.
elim ⇒ H0 H1.
split.
rewrite /total_r.
rewrite -H0.
apply cap_r.
```

```
apply H1.
Qed.
```

さらに $\alpha : A \rightharpoondown B$ と $\beta : A \rightharpoondown C$ の関係直積 $\alpha \top \beta : A \rightharpoondown B \times C$ を, $\alpha \top \beta := \alpha \cdot p^\sharp \sqcap \beta \cdot q^\sharp$ で定義する.

```
Definition Rel_prod {A B C : eqType} (alpha : Rel A B) (beta : Rel A C):=
 (alpha · fst_r B C #) ∩ (beta · snd_r B C #).
```

13.2.2 関係直積の性質

Lemma 290 (prod_inc_compat) *Let $\alpha, \alpha' : A \rightharpoondown B$ and $\beta, \beta' : A \rightharpoondown C$. Then,*

$$\alpha \sqsubseteq \alpha' \wedge \beta \sqsubseteq \beta' \Rightarrow \alpha \top \beta \sqsubseteq \alpha' \top \beta'.$$

```
Lemma prod_inc_compat
 {A B C : eqType} {alpha alpha' : Rel A B} {beta beta' : Rel A C}:
 alpha ⊆ alpha' → beta ⊆ beta' → Rel_prod alpha beta ⊆ Rel_prod alpha' beta'.
Proof.
move ⇒ H H0.
apply cap_inc_compat.
apply (comp_inc_compat_ab_a'b H).
apply (comp_inc_compat_ab_a'b H0).
Qed.
```

Lemma 291 (prod_inc_compat_l) *Let $\alpha : A \rightharpoondown B$ and $\beta, \beta' : A \rightharpoondown C$. Then,*

$$\beta \sqsubseteq \beta' \Rightarrow \alpha \top \beta \sqsubseteq \alpha \top \beta'.$$

```
Lemma prod_inc_compat_l
 {A B C : eqType} {alpha : Rel A B} {beta beta' : Rel A C}:
 beta ⊆ beta' → Rel_prod alpha beta ⊆ Rel_prod alpha beta'.
Proof.
move ⇒ H.
apply (prod_inc_compat (@inc_refl _ _ alpha) H).
Qed.
```

Lemma 292 (prod_inc_compat_r) *Let $\alpha, \alpha' : A \rightharpoondown B$ and $\beta : A \rightharpoondown C$. Then,*

$$\alpha \sqsubseteq \alpha' \Rightarrow \alpha \top \beta \sqsubseteq \alpha' \top \beta.$$

```
Lemma prod_inc_compat_r
 {A B C : eqType} {alpha alpha' : Rel A B} {beta : Rel A C}:
```

```
 alpha ⊆ alpha' → Rel_prod alpha beta ⊆ Rel_prod alpha' beta.
Proof.
move ⇒ H.
apply (prod_inc_compat H (@inc_refl _ _ beta)).
Qed.
```

Lemma 293 (total_prod) *Let $\alpha : A \rightharpoondown B$ and $\beta : A \rightharpoondown C$ are total relations, then $\alpha \top \beta$ is also a total relation.*

```
Lemma total_prod {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 total_r alpha → total_r beta → total_r (Rel_prod alpha beta).
Proof.
move ⇒ H H0.
rewrite domain_total cap_domain cap_comm.
apply Logic.eq_sym.
apply inc_def1.
apply (@inc_trans _ _ _ _ _ H).
rewrite comp_inv inv_invol comp_assoc.
apply comp_inc_compat_ab_ab'.
apply (@inc_trans _ _ _ (alpha # • (beta • beta #))).
apply (comp_inc_compat_a_ab H0).
rewrite -comp_assoc -comp_assoc fst_snd_universal.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
Qed.
```

Lemma 294 (univalent_prod) *Let $\alpha : A \rightharpoondown B$ and $\beta : A \rightharpoondown C$ are univalent relations, then $\alpha \top \beta$ is also a univalent relation.*

```
Lemma univalent_prod {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 univalent_r alpha → univalent_r beta → univalent_r (Rel_prod alpha beta).
Proof.
move ⇒ H H0.
rewrite /univalent_r/Rel_prod.
rewrite inv_cap_distr comp_inv inv_invol comp_inv inv_invol.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_l)).
rewrite -fst_snd_cap_id.
apply cap_inc_compat.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_l)).
rewrite comp_assoc -(@comp_assoc _ _ _ _ _ alpha).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H).
```

```
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_r)).
rewrite comp_assoc -(@comp_assoc _ _ _ _ _ beta).
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_ab_b H0).
Qed.
```

Lemma 295 (function_prod) *Let $\alpha : A \to B$ and $\beta : A \to C$ are functions, then $\alpha \top \beta$ is also a function.*

```
Lemma function_prod {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 function_r alpha → function_r beta → function_r (Rel_prod alpha beta).
Proof.
elim ⇒ H H0.
elim ⇒ H1 H2.
split.
apply (total_prod H H1).
apply (univalent_prod H0 H2).
Qed.
```

Lemma 296 (prod_fst_surjection) *Let $p : B \times C \to B$ be a projection. Then,*

$$\text{“}p \text{ is a surjection”} \Leftrightarrow \forall D, \nabla_{BD} = \nabla_{BC} \cdot \nabla_{CD}.$$

```
Lemma prod_fst_surjection {B C : eqType}:
 surjection_r (fst_r B C) ↔ ∀ D : eqType, ∇ B D = ∇ B C • ∇ C D.
Proof.
split; move ⇒ H.
move ⇒ D.
elim H ⇒ H0 H1.
apply inc_antisym.
apply (@inc_trans _ _ _ ((fst_r B C # • (fst_r B C #) #) • ∇ B D)).
apply (comp_inc_compat_b_ab H1).
rewrite inv_invol.
apply (@inc_trans _ _ _ (((fst_r B C # • snd_r B C) • (snd_r B C # • fst_r B C)) •
∇ B D)).
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -(@comp_assoc _ _ _ _ (snd_r B C)).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply snd_function.
rewrite (@comp_assoc _ _ _ _ _ _ _ (∇ B D)).
apply comp_inc_compat.
```

```
apply inc_alpha_universal.
apply inc_alpha_universal.
apply inc_alpha_universal.
split.
apply fst_function.
rewrite /total_r.
rewrite -(@cap_universal _ _ (Id B)) (H B) -(@fst_snd_universal B C) cap_comm comp_assoc.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite comp_id_r.
apply cap_r.
Qed.
```

Lemma 297 (prod_snd_surjection) *Let $q : B \times C \to C$ be a projection. Then,*

$$\text{“}q \textit{ is a surjection”} \Leftrightarrow \forall D, \nabla_{CD} = \nabla_{CB} \cdot \nabla_{BD}.$$

```
Lemma prod_snd_surjection {B C : eqType}:
 surjection_r (snd_r B C) ↔ ∀ D : eqType, ∇ C D = ∇ C B • ∇ B D.
Proof.
split; move ⇒ H.
move ⇒ D.
elim H ⇒ H0 H1.
apply inc_antisym.
apply (@inc_trans _ _ _ ((snd_r B C # • (snd_r B C #) #) • ∇ C D)).
apply (comp_inc_compat_b_ab H1).
rewrite inv_invol.
apply (@inc_trans _ _ _ (((snd_r B C # • fst_r B C) • (fst_r B C # • snd_r B C)) •
∇ C D)).
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -(@comp_assoc _ _ _ _ (fst_r B C)).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_b_ab.
apply fst_function.
rewrite (@comp_assoc _ _ _ _ _ _ _ (∇ C D)).
apply comp_inc_compat.
apply inc_alpha_universal.
apply inc_alpha_universal.
apply inc_alpha_universal.
split.
apply snd_function.
rewrite /total_r.
rewrite -(@cap_universal _ _ (Id C)) (H C) -(@snd_fst_universal B C) cap_comm comp_assoc.
```

```
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite comp_id_r.
apply cap_r.
Qed.
```

Lemma 298 (prod_fst_domain1) *Let $p : B \times C \to B$ be a projection, $\alpha : A \rightharpoondown B$ and $\beta : A \rightharpoondown C$. Then,*

$$(\alpha \top \beta) \cdot p = \lfloor \beta \rfloor \cdot \alpha.$$

```
Lemma prod_fst_domain1 {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 (Rel_prod alpha beta) • fst_r B C = domain beta • alpha.
Proof.
rewrite (@comp_inv_inv A A) domain_inv.
rewrite domain_universal2 inv_cap_distr comp_inv inv_invol inv_invol inv_universal.
rewrite -snd_fst_universal.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
rewrite comp_assoc comp_assoc.
apply cap_inc_compat_r.
apply comp_inc_compat_ab_a.
apply fst_function.
rewrite cap_comm -comp_assoc.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
rewrite cap_comm.
apply inc_refl.
Qed.
```

Lemma 299 (prod_fst_domain2) *Let $p : B \times C \to B$ be a projection, $\alpha : A \rightharpoondown B$ and $\beta : A \rightharpoondown C$. Then,*

$$(\alpha \top \beta) \cdot p = \alpha \Leftrightarrow \lfloor \alpha \rfloor \sqsubseteq \lfloor \beta \rfloor.$$

```
Lemma prod_fst_domain2 {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 (Rel_prod alpha beta) • fst_r B C = alpha ↔ domain alpha ⊆ domain beta.
Proof.
rewrite prod_fst_domain1.
split; move ⇒ H.
apply domain_lemma2b.
assert ((domain beta • alpha) ⊆ ((beta • beta #) • alpha)).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite H in H0.
apply H0.
```

```
apply inc_antisym.
apply comp_inc_compat_ab_b.
apply cap_r.
apply (@inc_trans _ _ _ (domain alpha • alpha)).
rewrite domain_comp_alpha1.
apply inc_refl.
apply (comp_inc_compat_ab_a'b H).
Qed.
```

Lemma 300 (prod_snd_domain1) *Let $q : B \times C \to C$ be a projection, $\alpha : A \rightharpoondown B$ and $\beta : A \rightharpoondown C$. Then,*

$$(\alpha \top \beta) \cdot q = \lfloor \alpha \rfloor \cdot \beta.$$

```
Lemma prod_snd_domain1 {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 (Rel_prod alpha beta) • snd_r B C = domain alpha • beta.
Proof.
rewrite (@comp_inv_inv A A) domain_inv.
rewrite domain_universal2 inv_cap_distr comp_inv inv_invol inv_invol inv_universal.
rewrite -fst_snd_universal.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
rewrite comp_assoc comp_assoc cap_comm.
apply cap_inc_compat_r.
apply comp_inc_compat_ab_a.
apply snd_function.
rewrite cap_comm -comp_assoc.
apply dedekind2.
Qed.
```

Lemma 301 (prod_snd_domain2) *Let $q : B \times C \to C$ be a projection, $\alpha : A \rightharpoondown B$ and $\beta : A \rightharpoondown C$. Then,*

$$(\alpha \top \beta) \cdot q = \beta \Leftrightarrow \lfloor \beta \rfloor \sqsubseteq \lfloor \alpha \rfloor.$$

```
Lemma prod_snd_domain2 {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 (Rel_prod alpha beta) • snd_r B C = beta ↔ domain beta ⊆ domain alpha.
Proof.
rewrite prod_snd_domain1.
split; move ⇒ H.
apply domain_lemma2b.
assert ((domain alpha • beta) ⊆ ((alpha • alpha #) • beta)).
apply comp_inc_compat_ab_a'b.
apply cap_l.
rewrite H in H0.
```

```
apply H0.
apply inc_antisym.
apply comp_inc_compat_ab_b.
apply cap_r.
apply (@inc_trans _ _ _ (domain beta • beta)).
rewrite domain_comp_alpha1.
apply inc_refl.
apply (comp_inc_compat_ab_a'b H).
Qed.
```

Lemma 302 (prod_to_cap) *Let $\alpha : A \rightharpoondown B$ and $\beta : A \rightharpoondown C$. Then,*

$$\lfloor \alpha \top \beta \rfloor = \lfloor \alpha \rfloor \sqcap \lfloor \beta \rfloor.$$

```
Lemma prod_to_cap {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 domain (Rel_prod alpha beta) = domain alpha ∩ domain beta.
Proof.
replace (domain (Rel_prod alpha beta)) with (domain (Rel_prod alpha beta • snd_r B
C)).
rewrite prod_snd_domain1 comp_domain8.
apply dedekind_id3.
apply cap_r.
apply cap_r.
apply cap_r.
apply comp_domain3.
apply snd_function.
Qed.
```

Lemma 303 (prod_conjugate1) *Let $\alpha : A \to B$ and $\beta : A \to C$ be functions, $p : B \times C \to B$ and $q : B \times C \to C$ be projections. Then,*

$$(\alpha \top \beta) \cdot p = \alpha \wedge (\alpha \top \beta) \cdot q = \beta.$$

```
Lemma prod_conjugate1 {A B C : eqType} {alpha : Rel A B} {beta : Rel A C}:
 function_r alpha → function_r beta →
 Rel_prod alpha beta • fst_r B C = alpha ∧ Rel_prod alpha beta • snd_r B C = beta.
Proof.
move ⇒ H H0.
split.
rewrite prod_fst_domain1.
elim H0 ⇒ H1 H2.
apply inc_def1 in H1.
rewrite /domain.
```

```
by [rewrite cap_comm -H1 comp_id_l].
rewrite prod_snd_domain1.
elim H => H1 H2.
apply inc_def1 in H1.
rewrite /domain.
by [rewrite cap_comm -H1 comp_id_l].
Qed.
```

Lemma 304 (prod_conjugate2) *Let $\gamma : A \to B \times C$ be a function, $p : B \times C \to B$ and $q : B \times C \to C$ be projections. Then,*

$$(\gamma \cdot p) \top (\gamma \cdot q) = \gamma.$$

```
Lemma prod_conjugate2 {A B C : eqType} {gamma : Rel A (prod_eqType B C)}:
 function_r gamma -> Rel_prod (gamma • fst_r B C) (gamma • snd_r B C) = gamma.
Proof.
move => H.
rewrite /Rel_prod.
rewrite comp_assoc comp_assoc -(function_cap_distr_l H).
by [rewrite fst_snd_cap_id comp_id_r].
Qed.
```

Lemma 305 (diagonal_conjugate) *Let $p : B \times C \to B$ and $q : B \times C \to C$ be projections. Then,*

$$\frac{\alpha : A \rightharpoondown B}{u \sqsubseteq id_{A\times B}} \ \frac{\alpha = p^\sharp \cdot u \cdot q}{u = \lfloor p \cdot \alpha \sqcap q \rfloor}.$$

```
Lemma diagonal_conjugate {A B : eqType} {alpha : Rel A B}:
 conjugate A B (prod_eqType A B) (prod_eqType A B)
 True_r (fun u => u ⊆ Id (prod_eqType A B))
 (fun u => (fst_r A B # • u) • snd_r A B)
 (fun alpha => domain ((fst_r A B • alpha) ∩ snd_r A B)).
Proof.
split.
move => alpha0 H.
split.
apply cap_r.
rewrite cap_domain.
apply inc_antisym.
apply (@inc_trans _ _ _ ((fst_r A B # • ((fst_r A B • alpha0) • snd_r A B #)) • snd_r
A B)).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_ab'.
```

```
apply cap_l.
rewrite comp_assoc comp_assoc -comp_assoc -(@comp_assoc _ _ _ _ (fst_r A B #)).
apply (@inc_trans _ _ _ ((fst_r A B # • fst_r A B) • alpha0)).
apply comp_inc_compat_ab_a.
apply snd_function.
apply comp_inc_compat_ab_b.
apply fst_function.
apply (@inc_trans _ _ _ (alpha0 ∩ ((fst_r A B # • Id (prod_eqType A B)) • snd_r A
B))).
rewrite comp_id_r fst_snd_universal cap_universal.
apply inc_refl.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm inv_invol comp_assoc.
apply inc_refl.
move ⇒ u H.
split.
by [].
replace ((fst_r A B • ((fst_r A B # • u) • snd_r A B)) ∩ snd_r A B) with (u • snd_r
A B).
apply domain_inc_id in H.
move : (@snd_function A B) ⇒ H0.
elim H0 ⇒ H1 H2.
by [rewrite (comp_domain3 H1) H].
rewrite comp_assoc -comp_assoc.
apply inc_antisym.
apply (@inc_trans _ _ _ ((u • snd_r A B) ∩ snd_r A B)).
apply inc_cap.
split.
apply inc_refl.
apply (comp_inc_compat_ab_b H).
apply cap_inc_compat_r.
apply comp_inc_compat_b_ab.
apply fst_function.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_b.
rewrite -fst_snd_cap_id.
apply cap_inc_compat_l.
apply comp_inc_compat_ab_ab'.
```

```
apply inc_inv.
apply (comp_inc_compat_ab_b H).
Qed.
```

13.2.3 鋭敏性

この節の補題は以下の 1 つのみだが, 証明が異様に長いため単独の節を設ける.

Lemma 306 (sharpness) *In below figure,*

$$\alpha \cdot \gamma^\sharp \sqcap \beta \cdot \delta^\sharp = (\alpha \cdot p^\sharp \sqcap \beta \cdot q^\sharp) \cdot (p \cdot \gamma^\sharp \sqcap q \cdot \delta^\sharp).$$

$$\begin{tikzcd}
 & Y & \\
X \arrow[ur, "\alpha"] \arrow[dr, "\beta"'] & Y \times Z \arrow[u, "p"'] \arrow[d, "q"'] & W \arrow[ul, "\gamma"'] \arrow[dl, "\delta"] \\
 & Z &
\end{tikzcd}$$

```
Lemma sharpness {W X Y Z : eqType}
 {alpha : Rel X Y} {beta : Rel X Z} {gamma : Rel W Y} {delta : Rel W Z}:
 (alpha · gamma #) ∩ (beta · delta #) =
 ((alpha · fst_r Y Z #) ∩ (beta · snd_r Y Z #))
  · ((fst_r Y Z · gamma #) ∩ (snd_r Y Z · delta #)).
Proof.
apply inc_antisym.
move : (rationality _ _ alpha) ⇒ H.
move : (rationality _ _ beta) ⇒ H0.
move : (rationality _ _ (gamma #)) ⇒ H1.
move : (rationality _ _ (delta #)) ⇒ H2.
elim H ⇒ R.
elim ⇒ f0.
elim ⇒ g0 H3.
elim H0 ⇒ R0.
elim ⇒ f1.
elim ⇒ g1 H4.
elim H1 ⇒ R1.
elim ⇒ h0.
elim ⇒ k0 H5.
elim H2 ⇒ R2.
elim ⇒ h1.
elim ⇒ k1 H6.
```

```
move : (rationality _ _ (g0 ⋅ h0 #)) => H7.
move : (rationality _ _ (g1 ⋅ h1 #)) => H8.
move : (rationality _ _ ((alpha ⋅ gamma #) ∩ (beta ⋅ delta #))) => H9.
elim H7 => R3.
elim => s0.
elim => t0 H10.
elim H8 => R4.
elim => s1.
elim => t1 H11.
elim H9 => R5.
elim => x.
elim => z H12.
assert (alpha ⋅ gamma # = (f0 # ⋅ (s0 # ⋅ t0)) ⋅ k0).
replace alpha with (f0 # ⋅ g0).
replace (gamma #) with (h0 # ⋅ k0).
rewrite -comp_assoc (@comp_assoc _ _ _ _ (f0 #)).
apply f_equal2.
apply f_equal.
apply H10.
by [].
apply Logic.eq_sym.
apply H5.
apply Logic.eq_sym.
apply H3.
assert (beta ⋅ delta # = (f1 # ⋅ (s1 # ⋅ t1)) ⋅ k1).
replace beta with (f1 # ⋅ g1).
replace (delta #) with (h1 # ⋅ k1).
rewrite -comp_assoc (@comp_assoc _ _ _ _ (f1 #)).
apply f_equal2.
apply f_equal.
apply H11.
by [].
apply Logic.eq_sym.
apply H6.
apply Logic.eq_sym.
apply H4.
assert (t0 ⋅ h0 = s0 ⋅ g0).
apply function_inc.
apply function_comp.
apply H10.
apply H5.
apply function_comp.
```

```
apply H10.
apply H3.
apply (@inc_trans _ _ _ (s0 · ((s0 # · t0) · h0))).
rewrite comp_assoc -comp_assoc.
apply comp_inc_compat_b_ab.
apply H10.
apply comp_inc_compat_ab_ab'.
replace (s0 # · t0) with (g0 · h0 #).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H5.
apply H10.
assert (t1 · h1 = s1 · g1).
apply function_inc.
apply function_comp.
apply H11.
apply H6.
apply function_comp.
apply H11.
apply H4.
apply (@inc_trans _ _ _ (s1 · ((s1 # · t1) · h1))).
rewrite comp_assoc -comp_assoc.
apply comp_inc_compat_b_ab.
apply H11.
apply comp_inc_compat_ab_ab'.
replace (s1 # · t1) with (g1 · h1 #).
rewrite comp_assoc.
apply comp_inc_compat_ab_a.
apply H6.
apply H11.
remember ((x · (s0 · f0) #) ∩ (z · (t0 · k0) #)) as m0.
remember ((x · (s1 · f1) #) ∩ (z · (t1 · k1) #)) as m1.
assert (total_r m0).
rewrite Heqm0.
apply domain_corollary1.
apply H12.
apply H12.
replace (x # · z) with ((alpha · gamma #) ∩ (beta · delta #)).
apply (@inc_trans _ _ _ _ _ (cap_l)).
rewrite comp_inv H13 -comp_assoc comp_assoc.
apply inc_refl.
apply H12.
```

```
assert (total_r m1).
rewrite Heqm1.
apply domain_corollary1.
apply H12.
apply H12.
replace (x # · z) with ((alpha · gamma #) ∩ (beta · delta #)).
apply (@inc_trans _ _ _ _ _ (cap_r)).
rewrite comp_inv H14 -comp_assoc comp_assoc.
apply inc_refl.
apply H12.
remember (m0 · (s0 · g0)) as n0.
remember (m1 · (s1 · g1)) as n1.
assert (total_r n0).
rewrite Heqn0.
apply (total_comp H17).
apply total_comp.
apply H10.
apply H3.
assert (total_r n1).
rewrite Heqn1.
apply (total_comp H18).
apply total_comp.
apply H11.
apply H4.
assert (total_r ((n0 · fst_r Y Z #) ∩ (n1 · snd_r Y Z #))).
apply (domain_corollary1 H19 H20).
rewrite fst_snd_universal.
apply inc_alpha_universal.
assert ((x # · n0) ⊆ alpha).
replace alpha with (f0 # · g0).
rewrite Heqn0 Heqm0.
apply (@inc_trans _ _ _ (((x # · x) · f0 #) · ((s0 # · s0) · g0))).
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -comp_inv.
apply cap_l.
apply comp_inc_compat.
apply comp_inc_compat_ab_b.
apply H12.
```

```
apply comp_inc_compat_ab_b.
apply H10.
apply Logic.eq_sym.
apply H3.
assert ((x # · n1) ⊆ beta).
replace beta with (f1 # · g1).
rewrite Heqn1 Heqm1.
apply (@inc_trans _ _ _ (((x # · x) · f1 #) · ((s1 # · s1) · g1))).
rewrite comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc -comp_inv.
apply cap_l.
apply comp_inc_compat.
apply comp_inc_compat_ab_b.
apply H12.
apply comp_inc_compat_ab_b.
apply H11.
apply Logic.eq_sym.
apply H4.
assert ((n0 # · z) ⊆ gamma #).
replace (gamma #) with (h0 # · k0).
rewrite Heqn0 Heqm0 -H15 comp_inv comp_inv inv_cap_distr.
apply (@inc_trans _ _ _ ((h0 # · (t0 # · t0)) · (k0 · (z # · z)))).
rewrite -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc comp_assoc comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc (@comp_inv _ _ _ z) inv_invol.
apply cap_r.
apply comp_inc_compat.
apply comp_inc_compat_ab_a.
apply H10.
apply comp_inc_compat_ab_a.
apply H12.
apply Logic.eq_sym.
apply H5.
assert ((n1 # · z) ⊆ delta #).
replace (delta #) with (h1 # · k1).
```

```
rewrite Heqn1 Heqm1 -H16 comp_inv comp_inv inv_cap_distr.
apply (@inc_trans _ _ _ ((h1 # · (t1 # · t1)) · (k1 · (z # · z)))).
rewrite -comp_assoc -comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
rewrite comp_assoc comp_assoc comp_assoc comp_assoc.
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc (@comp_inv _ _ _ z) inv_invol.
apply cap_r.
apply comp_inc_compat.
apply comp_inc_compat_ab_a.
apply H11.
apply comp_inc_compat_ab_a.
apply H12.
apply Logic.eq_sym.
apply H6.
replace ((alpha · gamma #) ∩ (beta · delta #)) with (x # · z).
apply (@inc_trans _ _ _ ((x # · (((n0 · fst_r Y Z #) ∩ (n1 · snd_r Y Z #)) · (((n0
· fst_r Y Z #) ∩ (n1 · snd_r Y Z #))) #)) · z)).
apply comp_inc_compat_ab_a'b.
apply (comp_inc_compat_a_ab H21).
rewrite -comp_assoc comp_assoc.
apply comp_inc_compat.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_l)).
apply cap_inc_compat.
rewrite -comp_assoc.
apply (comp_inc_compat_ab_a'b H22).
rewrite -comp_assoc.
apply (comp_inc_compat_ab_a'b H23).
rewrite inv_cap_distr comp_inv comp_inv inv_invol inv_invol.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply cap_inc_compat.
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H24).
rewrite comp_assoc.
apply (comp_inc_compat_ab_ab' H25).
apply Logic.eq_sym.
apply H12.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_l)).
apply cap_inc_compat.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_l)).
```

```
rewrite -comp_assoc (@comp_assoc _ _ _ _ alpha).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a.
apply fst_function.
apply (@inc_trans _ _ _ _ _ (comp_cap_distr_r)).
apply (@inc_trans _ _ _ _ _ (cap_r)).
rewrite -comp_assoc (@comp_assoc _ _ _ _ beta).
apply comp_inc_compat_ab_a'b.
apply comp_inc_compat_ab_a.
apply snd_function.
Qed.
```

13.2.4 分配法則

Lemma 307 (prod_cap_distr_l) *Let $\alpha : A \rightharpoondown B$ and $\beta, \beta' : A \rightharpoondown C$. Then,*

$$\alpha \top (\beta \sqcap \beta') = (\alpha \top \beta) \sqcap (\alpha \top \beta').$$

```
Lemma prod_cap_distr_l {A B C : eqType} {alpha : Rel A B} {beta beta' : Rel A C}:
  Rel_prod alpha (beta ∩ beta') = Rel_prod alpha beta ∩ Rel_prod alpha beta'.
Proof.
rewrite /Rel_prod.
rewrite -cap_assoc (@cap_comm _ _ _ (alpha • fst_r B C #)) -cap_assoc cap_idem
cap_assoc.
apply f_equal.
apply function_cap_distr_r.
apply snd_function.
Qed.
```

Lemma 308 (prod_cap_distr_r) *Let $\alpha, \alpha' : A \rightharpoondown B$ and $\beta : A \rightharpoondown C$. Then,*

$$(\alpha \sqcap \alpha') \top \beta = (\alpha \top \beta) \sqcap (\alpha' \top \beta).$$

```
Lemma prod_cap_distr_r {A B C : eqType} {alpha alpha' : Rel A B} {beta : Rel A C}:
  Rel_prod (alpha ∩ alpha') beta = Rel_prod alpha beta ∩ Rel_prod alpha' beta.
Proof.
rewrite /Rel_prod.
rewrite cap_assoc (@cap_comm _ _ (beta • snd_r B C #)) cap_assoc cap_idem -cap_assoc.
apply (@f_equal _ _ (fun x => @cap _ _ x (beta • snd_r B C #))).
apply function_cap_distr_r.
apply fst_function.
Qed.
```

Lemma 309 (prod_cup_distr_l) *Let $\alpha : A \rightharpoonup B$ and $\beta, \beta' : A \rightharpoonup C$. Then,*

$$\alpha \top (\beta \sqcup \beta') = (\alpha \top \beta) \sqcup (\alpha \top \beta').$$

```
Lemma prod_cup_distr_l {A B C : eqType} {alpha : Rel A B} {beta beta' : Rel A C}:
 Rel_prod alpha (beta ∪ beta') = Rel_prod alpha beta ∪ Rel_prod alpha beta'.
Proof.
by [rewrite -cap_cup_distr_l -comp_cup_distr_r].
Qed.
```

Lemma 310 (prod_cup_distr_r) *Let $\alpha, \alpha' : A \rightharpoonup B$ and $\beta : A \rightharpoonup C$. Then,*

$$(\alpha \sqcup \alpha') \top \beta = (\alpha \top \beta) \sqcup (\alpha' \top \beta).$$

```
Lemma prod_cup_distr_r {A B C : eqType} {alpha alpha' : Rel A B} {beta : Rel A C}:
 Rel_prod (alpha ∪ alpha') beta = Rel_prod alpha beta ∪ Rel_prod alpha' beta.
Proof.
by [rewrite -cap_cup_distr_r -comp_cup_distr_r].
Qed.
```

Lemma 311 (comp_prod_distr_l) *Let $\alpha : A \rightharpoonup B$, $\beta : B \rightharpoonup C$ and $\gamma : B \rightharpoonup D$. Then,*

$$\alpha \cdot (\beta \top \gamma) \sqsubseteq \alpha \cdot \beta \top \alpha \cdot \gamma.$$

```
Lemma comp_prod_distr_l
 {A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel B D}:
 alpha • Rel_prod beta gamma ⊆ Rel_prod (alpha • beta) (alpha • gamma).
Proof.
rewrite /Rel_prod.
rewrite comp_assoc comp_assoc.
apply comp_cap_distr_l.
Qed.
```

Lemma 312 (function_prod_distr_l) *Let $\alpha : A \rightharpoonup B$ be a function, $\beta : B \rightharpoonup C$ and $\gamma : B \rightharpoonup D$. Then,*

$$\alpha \cdot (\beta \top \gamma) = \alpha \cdot \beta \top \alpha \cdot \gamma.$$

```
Lemma function_prod_distr_l
 {A B C D : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel B D}:
 function_r alpha → alpha • Rel_prod beta gamma = Rel_prod (alpha • beta) (alpha •
gamma).
Proof.
```

```
move => H.
rewrite /Rel_prod.
rewrite comp_assoc comp_assoc.
apply (function_cap_distr_l H).
Qed.
```

Lemma 313 (comp_prod_universal) *Let $\alpha : A \rightharpoonup B$, $\beta : B \rightharpoonup C$ and $\gamma : D \rightharpoonup E$. Then,*

$$\alpha \cdot (\beta \top \nabla_{BD} \cdot \gamma) = \alpha \cdot \beta \top \nabla_{AD} \cdot \gamma.$$

```
Lemma comp_prod_universal
 {A B C D E : eqType} {alpha : Rel A B} {beta : Rel B C} {gamma : Rel D E}:
 alpha • Rel_prod beta (∇ B D • gamma) = Rel_prod (alpha • beta) (∇ A D • gamma).
Proof.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (comp_prod_distr_l)).
apply prod_inc_compat_l.
rewrite -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
rewrite /Rel_prod.
rewrite comp_assoc.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
apply comp_inc_compat_ab_ab'.
apply cap_inc_compat_l.
rewrite comp_assoc comp_assoc -comp_assoc.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
Qed.
```

Lemma 314 (fst_cap_snd_distr) *Let $u, v : A \times B \rightharpoonup A \times B$ and $u, v \sqsubseteq id_{A\times B}$, $p : B \times C \to B$ and $q : B \times C \to C$ be projections. Then,*

$$p^\sharp \cdot (u \sqcap v) \cdot q = p^\sharp \cdot u \cdot q \sqcap p^\sharp \cdot v \cdot q.$$

```
Lemma fst_cap_snd_distr
 {A B : eqType} {u v : Rel (prod_eqType A B) (prod_eqType A B)}:
 u ⊆ Id (prod_eqType A B) → v ⊆ Id (prod_eqType A B) →
 fst_r A B # • (u ∩ v) • snd_r A B =
 ((fst_r A B # • u) • snd_r A B) ∩ ((fst_r A B # • v) • snd_r A B).
Proof.
move => H H0.
apply inc_antisym.
```

```
apply (fun H' ⇒ @inc_trans _ _ _ _ _ H' (comp_cap_distr_r)).
apply comp_inc_compat_ab_a'b.
apply comp_cap_distr_l.
apply (@inc_trans _ _ _ _ _ (dedekind1)).
rewrite -(dedekind_id3 H H0) -(@comp_assoc _ _ _ _ _ u) (@comp_assoc _ _ _ _ (fst_r A
B # • u) v).
apply comp_inc_compat_ab_ab'.
rewrite cap_comm comp_assoc -comp_assoc.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
apply comp_inc_compat_ab_b.
rewrite comp_inv comp_inv inv_invol -fst_snd_cap_id.
apply cap_inc_compat.
rewrite comp_assoc (dedekind_id1 H).
apply (comp_inc_compat_ab_b H).
rewrite -comp_assoc (dedekind_id1 H0).
apply (comp_inc_compat_ab_a H0).
Qed.
```

Chapter 14

Library Point_Axiom

```
Require Import Basic_Notations.
Require Import Basic_Lemmas.
Require Import Relation_Properties.
Require Import Functions_Mappings.
Require Import Dedekind.
Require Import Rationality.
```

14.1 I-点

14.1.1 I-点の定義

Dedekind 圏における域 X の I-点 x とは, 関数 $x : I \rightharpoonup X$ のことであり, 記号 $x\dot{\in}X$ によって表される. また関係 $\rho : I \rightharpoonup X$ と I-点 $x : I \rightharpoonup X$ に対して, 記号 $x\dot{\in}\rho$ で $x \sqsubseteq \rho$ を表すものとする.
ちなみに I-点の定義 $x\dot{\in}X$ は $x\dot{\in}\nabla_{IX}$ と言い換えることも可能である.

```
Definition point_inc {X : eqType} (x rho : Rel i X) := function_r x ∧ x ⊆ rho.
Definition point {X : eqType} (x : Rel i X) := point_inc x (∇ i X).
```

14.1.2 I-点の性質

Lemma 315 (point_property1) *Let* $x, y\dot{\in}X$. *Then,*

$$x = y \Leftrightarrow x \cdot y^\sharp = id_I.$$

```
Lemma point_property1 {X : eqType} {x y : Rel i X}:
 point x → point y → (x = y ↔ x • y # = Id i).
Proof.
```

```
move => H H0.
split; move => H1.
apply inc_antisym.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
rewrite H1.
apply H0.
apply Logic.eq_sym.
apply function_inc.
apply H0.
apply H.
rewrite -(@comp_id_l _ _ y) -H1 comp_assoc.
apply comp_inc_compat_ab_a.
apply H0.
Qed.
```

Lemma 316 (point_property2a, point_property2b) *Let $\rho : I \rightharpoondown X$ be a total relation. Then,*

$$\rho \cdot \rho^\sharp = \rho \cdot \nabla_{XI} = id_I.$$

```
Lemma point_property2a {X : eqType} {rho : Rel i X}:
 total_r rho → rho • rho # = Id i.
Proof.
move => H.
apply inc_antisym.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
apply H.
Qed.
```

```
Lemma point_property2b {X : eqType} {rho : Rel i X}:
 total_r rho → rho • rho # = rho • ∇ X i.
Proof.
move => H.
apply inc_antisym.
apply comp_inc_compat_ab_ab'.
apply inc_alpha_universal.
rewrite (point_property2a H) unit_identity_is_universal.
apply inc_alpha_universal.
Qed.
```

Lemma 317 (point_property3) *Let $\rho : I \rightharpoonup X$. Then,*

$$\exists x \dot{\in} \rho \Rightarrow \text{“}\rho \textit{ is total”} \wedge \rho \neq \phi_{IX}.$$

```
Lemma point_property3 {X : eqType} {rho : Rel i X}:
 (∃ x : Rel i X, point_inc x rho) → total_r rho ∧ rho ≠ ϕ i X.
Proof.
elim ⇒ x H.
assert (total_r rho).
elim H ⇒ H0 H1.
elim H0 ⇒ H2 H3.
apply (@inc_trans _ _ _ _ _ H2).
apply comp_inc_compat.
apply H1.
apply (@inc_inv _ _ _ _ H1).
split.
apply H0.
move ⇒ H1.
rewrite /total_r in H0.
rewrite H1 comp_empty_l in H0.
apply unit_identity_not_empty.
apply inc_antisym.
apply H0.
apply inc_empty_alpha.
Qed.
```

Lemma 318 (point_property4)

$$\exists x \dot{\in} X \Rightarrow \text{“}\nabla_{IX} \textit{ is total”} \wedge \nabla_{IX} \neq \phi_{IX}.$$

```
Lemma point_property4 {X : eqType}:
 (∃ x : Rel i X, point x) → total_r (∇ i X) ∧ (∇ i X) ≠ ϕ i X.
Proof.
move ⇒ H.
apply (@point_property3 _ (∇ i X) H).
Qed.
```

14.2 I-点に関する諸公理

14.2.1 点公理

この "点公理" を使えば, I-点に関する様々な定理や補題が導出できる.

Lemma 319 (point_axiom) *Let $\rho : I \rightharpoondown X$. Then,*

$$\rho = \sqcup_{x \dot{\in} \rho} x.$$

```
Lemma lemma_for_PA {X : eqType} {rho : Rel i X}:
 (((rho = φ i X) → False) → False) → rho = φ i X.
Proof.
move ⇒ H.
case (@unit_empty_or_universal (rho • rho #)) ⇒ H0.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (relation_rel_inv_rel)).
rewrite H0 comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
apply False_ind.
apply H.
move ⇒ H1.
rewrite H1 comp_empty_l in H0.
apply (unit_empty_not_universal H0).
Qed.
```

```
Lemma point_axiom {X : eqType} {rho : Rel i X}:
 rho = ∪_{fun x : Rel i X ⇒ point_inc x rho} id.
Proof.
apply inc_antisym.
apply bool_lemma2.
assert ((∃ x : Rel i X, point_inc x ((∪_{fun x : Rel i X ⇒ point_inc x rho} id) ∩
(∪_{fun x : Rel i X ⇒ point_inc x rho} id) ^)) → False).
move ⇒ H.
move : (point_property3 H) ⇒ H0.
apply H0.
apply cap_complement_empty.
assert ((∃ x : Rel i X, point_inc x (rho ∩ (∪_{fun x : Rel i X ⇒ point_inc x rho} id)
^)) → False).
move ⇒ H0.
apply H.
elim H0 ⇒ x H1.
```

```
∃ x.
split.
apply H1.
apply inc_cap.
split.
assert (point_inc x rho).
split.
apply H1.
elim H1 ⇒ H2 H3.
apply inc_cap in H3.
apply H3.
clear H1.
move : x H2.
apply inc_cupP.
apply inc_refl.
elim H1 ⇒ H2 H3.
apply inc_cap in H3.
apply H3.
apply lemma_for_PA.
move ⇒ H1.
apply H0.
apply weak_axiom_of_choice.
rewrite /total_r.
remember (rho ∩ (∪_{fun x : Rel i X ⇒ point_inc x rho} id) ^) as rho'.
case (@unit_empty_or_universal (rho' • rho' #)) ⇒ H2.
apply False_ind.
apply H1.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (relation_rel_inv_rel)).
rewrite H2 comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
rewrite H2.
apply inc_alpha_universal.
apply inc_cupP.
move ⇒ beta H.
apply H.
Qed.
```

Lemma 320 (PA_corollary1)

$$\nabla_{IX} = \sqcup_{x \dot{\in} X} x.$$

```
Lemma PA_corollary1 {X : eqType}: ∇ i X = ∪_{fun x : Rel i X ⇒ point x} id.
Proof.
apply point_axiom.
Qed.
```

Lemma 321 (PA_corollary2)

$$id_X = \sqcup_{x \dot{\in} X} x^\sharp \cdot x.$$

```
Lemma PA_corollary2 {X : eqType}:
 Id X = ∪_{fun x : Rel i X ⇒ point x} (fun x : Rel i X ⇒ x # • x).
Proof.
rewrite -(@cap_universal _ _ (Id X)) -lemma_for_tarski2 PA_corollary1.
rewrite comp_cupP_distr_l cap_cupP_distr_l.
apply cupP_eq.
move ⇒ alpha H.
apply inc_antisym.
rewrite cap_comm.
apply (@inc_trans _ _ _ _ _ (dedekind2)).
rewrite comp_id_l cap_comm cap_universal.
apply inc_refl.
apply inc_cap.
split.
apply H.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
Qed.
```

Lemma 322 (PA_corollary3) *Let $\alpha, \beta : X \rightharpoondown Y$. Then,*

$$(\forall x \dot{\in} X, x \cdot \alpha = x \cdot \beta) \Rightarrow \alpha = \beta.$$

```
Lemma PA_corollary3 {X Y : eqType} {alpha beta : Rel X Y}:
 (∀ x : Rel i X, point x → x • alpha = x • beta) → alpha = beta.
Proof.
move ⇒ H.
rewrite -(@comp_id_l _ _ alpha) -(@comp_id_l _ _ beta) PA_corollary2.
rewrite comp_cupP_distr_r comp_cupP_distr_r.
apply cupP_eq.
move ⇒ gamma H0.
by [rewrite comp_assoc comp_assoc (H gamma H0)].
Qed.
```

14.2.2 その他の公理

Lemma 323 (total_axiom) *Let $\rho : I \rightharpoonup X$. Then,*

$$\rho \neq \phi_{IX} \Rightarrow id_I = \rho \cdot \rho^\sharp.$$

```
Lemma total_axiom {X : eqType} {rho : Rel i X}:
 rho ≠ φ i X → Id i = rho • rho #.
Proof.
move ⇒ H.
case (@unit_empty_or_universal (rho • rho #)) ⇒ H0.
apply False_ind.
apply H.
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (relation_rel_inv_rel)).
rewrite H0 comp_empty_l.
apply inc_refl.
apply inc_empty_alpha.
by [rewrite H0 unit_identity_is_universal].
Qed.
```

Lemma 324 (nonempty_axiom) *Let $\rho : I \rightharpoonup X$. Then,*

$$\rho \neq \phi_{IX} \Rightarrow \exists x \dot{\in} \rho.$$

```
Lemma nonempty_axiom {X : eqType} {rho : Rel i X}:
 rho ≠ φ i X → ∃ x : Rel i X, point_inc x rho.
Proof.
move : (@weak_axiom_of_choice _ rho) ⇒ H.
move ⇒ H0.
apply H.
rewrite /total_r.
rewrite (total_axiom H0).
apply inc_refl.
Qed.
```

Lemma 325 (axiom_of_subobjects) *Let $\rho : I \rightharpoonup X$. Then,*

$$\exists S, \exists j : S \rightharpoonup X, \rho = \nabla_{IS} \cdot j \wedge j \cdot j^\sharp = id_S.$$

```
Lemma axiom_of_subobjects {X : eqType} {rho : Rel i X}:
 ∃ (S : eqType)(j : Rel S X), rho = ∇ i S • j ∧ j • j # = Id S.
Proof.
```

```
elim (@rationality _ _ rho) ⇒ R.
elim ⇒ f.
elim ⇒ g.
elim ⇒ H.
elim ⇒ H0.
elim ⇒ H1 H2.
∃ R.
∃ g.
split.
rewrite H1.
apply inc_antisym.
apply comp_inc_compat_ab_a'b.
apply inc_alpha_universal.
apply comp_inc_compat_ab_a'b.
apply (@inc_trans _ _ _ (∇ i R • (f • f #))).
apply comp_inc_compat_a_ab.
apply H.
rewrite -comp_assoc.
apply comp_inc_compat_ab_b.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
rewrite -H2 cap_comm inc_def1.
assert ((f # • g) ⊆ rho).
rewrite H1.
apply inc_refl.
apply (function_move1 H) in H3.
apply (@inc_trans _ _ _ ((f • rho) • (f • rho) #)).
apply comp_inc_compat.
apply H3.
apply (@inc_inv _ _ _ _ H3).
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ rho).
apply comp_inc_compat_ab_ab'.
apply comp_inc_compat_ab_b.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
Qed.
```

Lemma 326 (axiom_of_choice) *Let* $\alpha : X \rightharpoonup Y$ *be a total relation. Then,*

$$\exists f : X \to Y, f \sqsubseteq \alpha.$$

```
Lemma axiom_of_choice {X Y : eqType} {alpha : Rel X Y}:
 total_r alpha → ∃ (f : Rel X Y), function_r f ∧ f ⊆ alpha.
```

```
Proof.
move ⇒ H.
case (@unit_empty_or_universal (∇ i X • ∇ X i)) ⇒ H0.
∃ (φ X Y).
repeat split.
assert (Id X = φ X X).
apply inc_antisym.
apply (@inc_trans _ _ _ _ _ (@inc_alpha_universal _ _ _)).
apply (@inc_trans _ _ _ (∇ X X • ∇ X X)).
apply comp_inc_compat_a_ab.
apply inc_alpha_universal.
rewrite -unit_universal comp_assoc -(@comp_assoc _ _ _ _ (∇ i X)).
rewrite H0 comp_empty_l comp_empty_r.
apply inc_refl.
apply inc_empty_alpha.
rewrite /total_r.
rewrite H1 comp_empty_l.
apply inc_refl.
rewrite /univalent_r.
rewrite comp_empty_r.
apply inc_empty_alpha.
apply inc_empty_alpha.
assert (∀ x : Rel i X, point x → ∃ g : Rel i Y, function_r g ∧ g ⊆ (x • alpha)).
move ⇒ x H1.
apply weak_axiom_of_choice.
apply total_comp.
apply H1.
apply H.
assert (∀ x : Rel i X, point x → ∃ g' : Rel X Y, univalent_r g' ∧ g' ⊆ alpha).
move ⇒ x H2.
elim (H1 x H2) ⇒ g H3.
∃ (x # • g).
split.
rewrite /univalent_r.
rewrite comp_inv inv_invol.
apply (@inc_trans _ _ _ (g # • g)).
rewrite comp_assoc.
apply comp_inc_compat_ab_ab'.
rewrite -comp_assoc.
apply comp_inc_compat_ab_b.
rewrite unit_identity_is_universal.
apply inc_alpha_universal.
```

```
apply H3.
apply function_move1.
apply H2.
apply H3.
assert (∃ f' : Rel i X → Rel X Y, function_r (∪_{fun x ⇒ point x} f') ∧ (∪_{fun x
⇒ point x} f') ⊆ alpha).
move : (@weak_axiom_of_choice _ (∇ i X • alpha)) ⇒ H1.
assert (total_r (∇ i X • alpha)).
rewrite /total_r.
rewrite comp_inv comp_assoc -(@comp_assoc _ _ _ _ alpha).
rewrite unit_identity_is_universal -H0 inv_universal.
apply comp_inc_compat_ab_ab'.
apply (comp_inc_compat_b_ab H).
apply H1 in H2.
elim H2 ⇒ g H3.
Qed.
```

Bibliography

[1] R. Affeldt and M. Hagiwara. Formalization of Shannon 's Theorems in SSReflect-Coq. In 3rd Conference on Interactive Theorem Proving, LNCS 7406, 233–249, 2012.