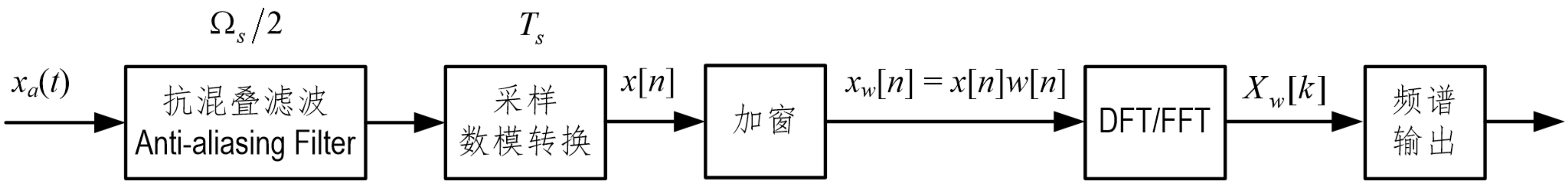
Digital Frequency Spectrum Analysis Based on DFT

Spectrum Analysis

- Acquire the physical characteristics of signals
- Widely used in signal processing and analysis
- Light spectrum (different colors)
- Frequency spectrum
  - Based on Fourier transform
  - Strength of each frequency component in the signal

Diagram block of spectrum analysis



- Interested in frequency band $[-\Omega_H, \Omega_H]$
  - Set up sampling frequency $\Omega_s \geq 2\Omega_H$
  - Anti-aliasing filter, $\Omega_s/2$
  - ADC, sampling interval $T_s = 2\pi/\Omega_s$,
  - Windowing, $x_w[n] = x[n]w[n]$
  - DFT, $X_w[k] = \mathrm{DFT}\{x_w[n]\}$

Key steps in spectrum analysis

- For $x_a(t)$ ←CTFT→ $X_a(j\Omega)$
- After sampling, $x[n]$ ←DTFT→ $X(e^{j\omega})$
  - Following Nyquist rule, $X(e^{j\omega})$ and $X_a(j\Omega)$ contain the same information
- Windowing, i.e., fetch finite (M) samples of $x[n]$, $x_w[n] = x[n]w[n]$ ←DTFT→ $X_w(e^{j\omega})$
  - Usually, $X_w(e^{j\omega}) \neq X(e^{j\omega})$
- N-point DFT, get discrete values of $X_w(e^{j\omega})$ with sampling interval $2\pi/N$

How windowing affects the analysis?

- Time domain multiplication, frequency domain convolution

$$X_w(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})e^{j\theta}d\theta$$

  - Computational capability constraint
  - Memory storage constraint
  - Real time requirement

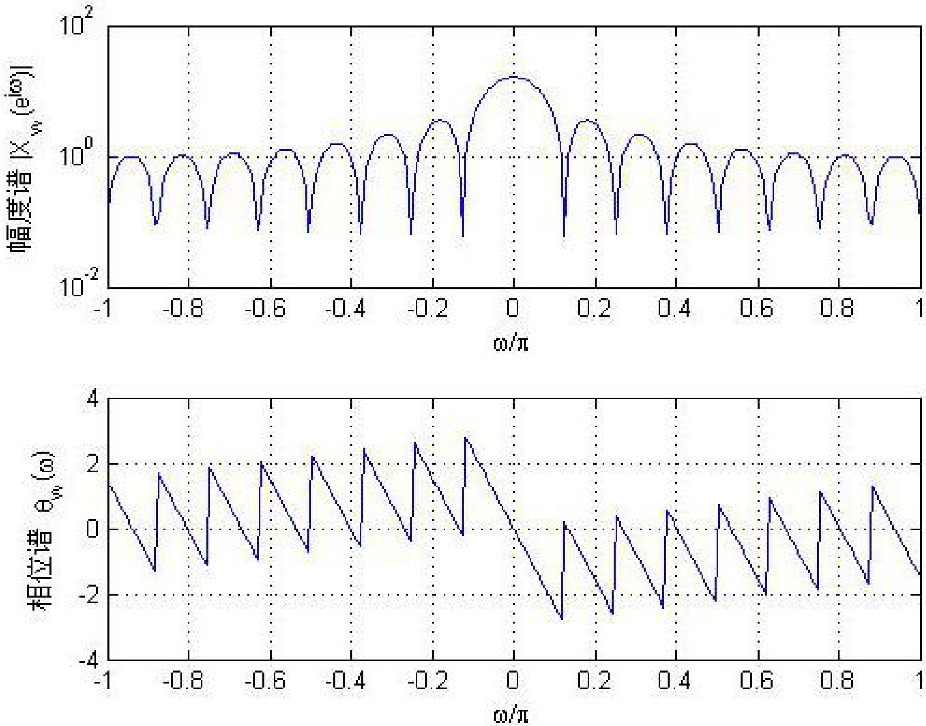
How windowing affects the analysis?

- Rectangular Window

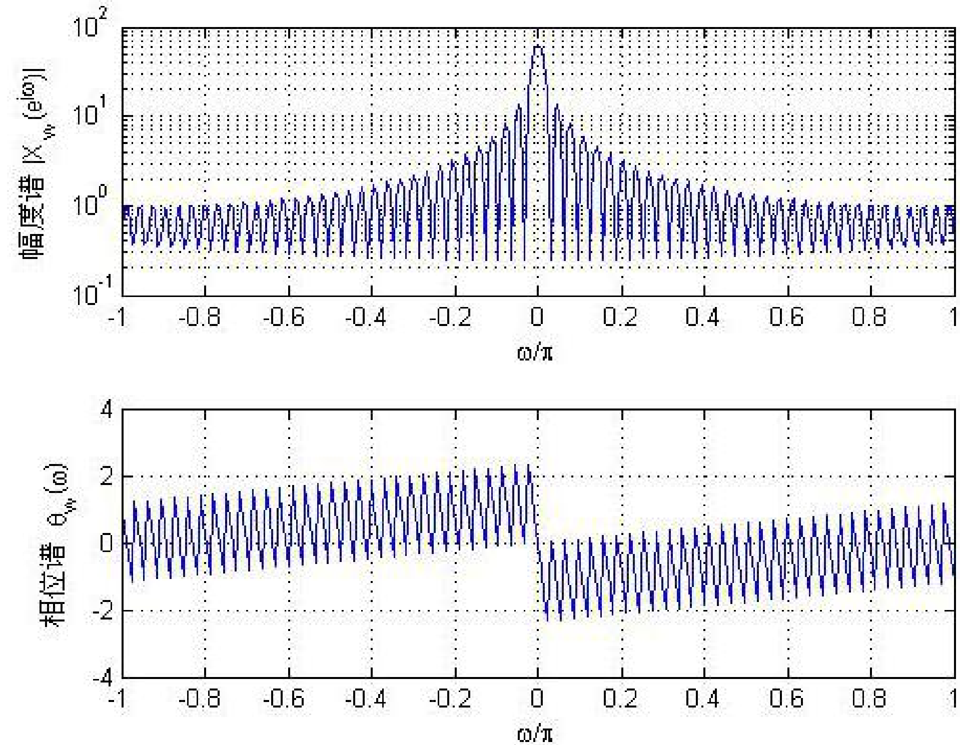
$$w_R[n] = \begin{cases} 1, & 0 \le n \le M-1 \\ 0, & \text{otherwise} \end{cases}$$

$$W_R(e^{j\omega}) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{\sin\left(M\omega/2\right)}{\sin(\omega/2)} e^{-j\frac{(M-1)\omega}{2}}$$

How windowing affects the analysis?



$M = 16$



$M = 32$

Main lobe width: $4\pi/M$, side lobe

Zero points: $2\pi m/M$, $m = \pm 1, \pm 2, \ldots$

How windowing affects the analysis?

- Sinusoidal sequence $x[n] = \cos(\omega_0 n)$
  - Spectrum: two impulses

$$\cos\left(\omega_0 n\right) \xleftrightarrow{\text{DTFT}} \pi\delta\left(\omega+\omega_0\right)+\pi\delta\left(\omega-\omega_0\right), \quad |\omega| \le \pi$$

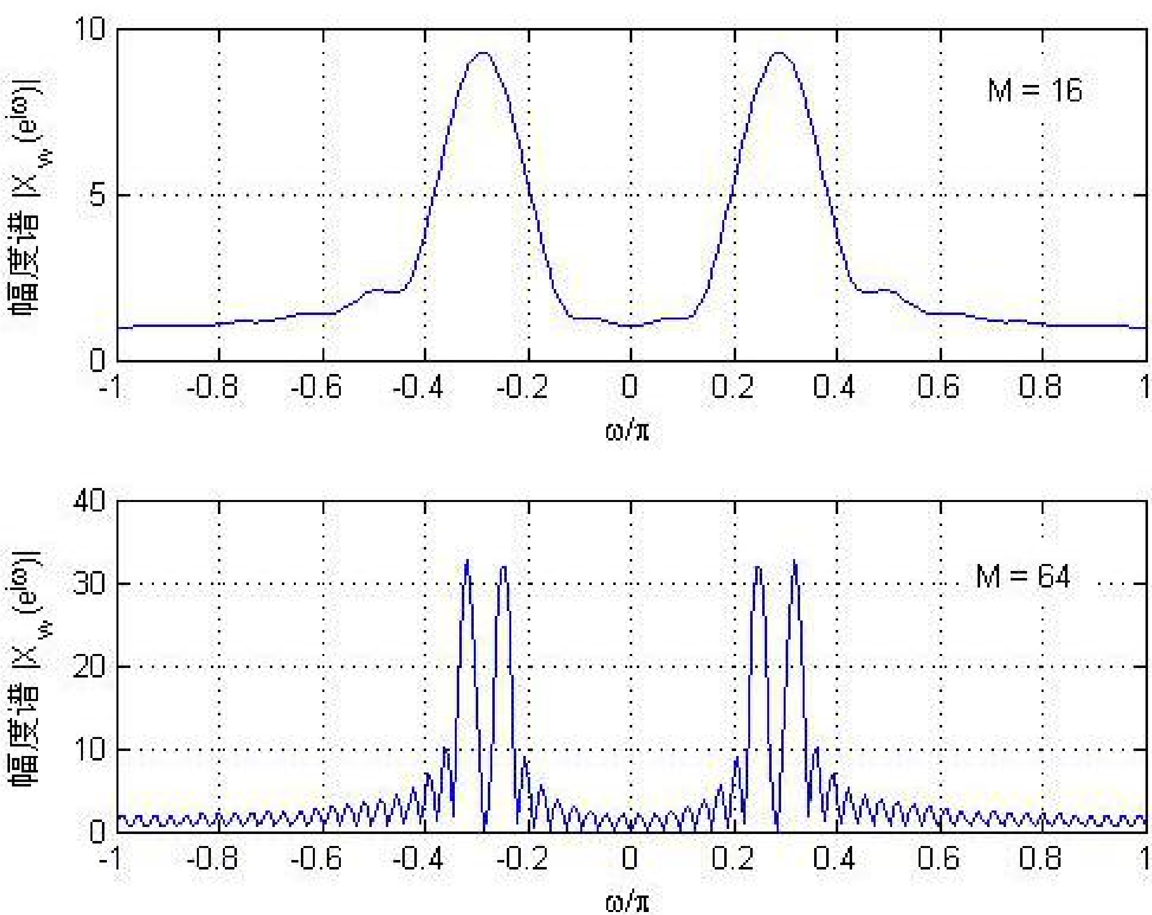
  - After windowing

$$X_w(e^{j\omega}) = \pi W_R\left(e^{j\left(\omega+\omega_0\right)}\right)+\pi W_R\left(e^{j\left(\omega-\omega_0\right)}\right), \quad |\omega| \le \pi$$

  - Multiple frequency components, resolution?

How windowing affects the analysis?

- Sinusoidal sequence: $x[n] = \cos\left(\frac{\pi n}{4}\right) + \cos\left(\frac{5\pi n}{16}\right)$



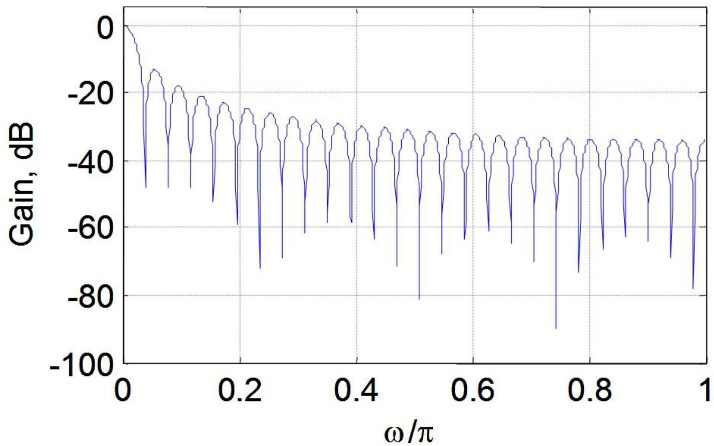
How windowing affects the analysis?

- Different windows have different mainlobe width and different side lobe levels (find the details in the textbook)
- Side lobe may conceal some “weak” freq. components

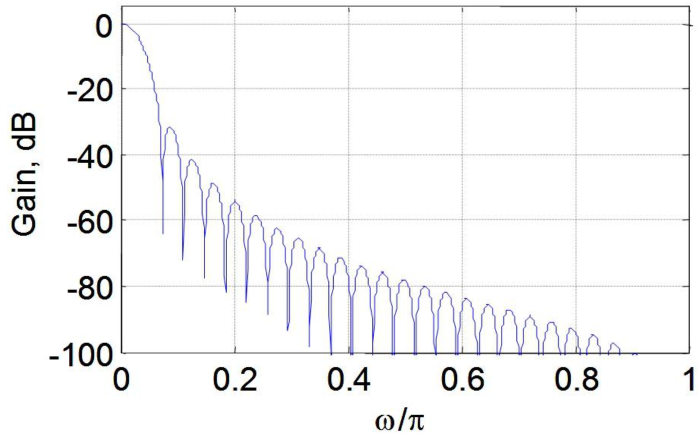
表 11.9-1 常用窗函数及其参数（主瓣峰值归一化，长度为 M）

窗函数	主瓣宽度	3dB 带宽	最大旁瓣电平
矩形（Rectangular）窗	$4\pi/M$	$0.89\times 2\pi/M$	-13dB
汉宁（Hann）窗	$8\pi/M$	$1.44\times 2\pi/M$	-31dB
汉明（Hamming）窗	$8\pi/M$	$1.30\times 2\pi/M$	-41dB
布莱克曼（Blackman）窗	$12\pi/M$	$1.68\times 2\pi/M$	-57dB

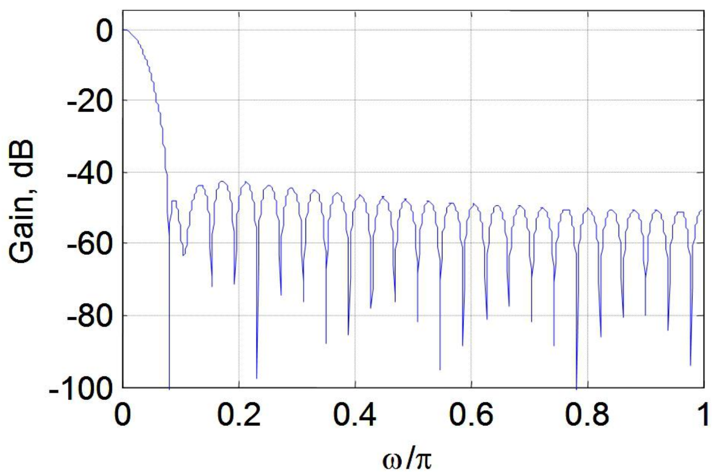
How windowing affects the analysis?



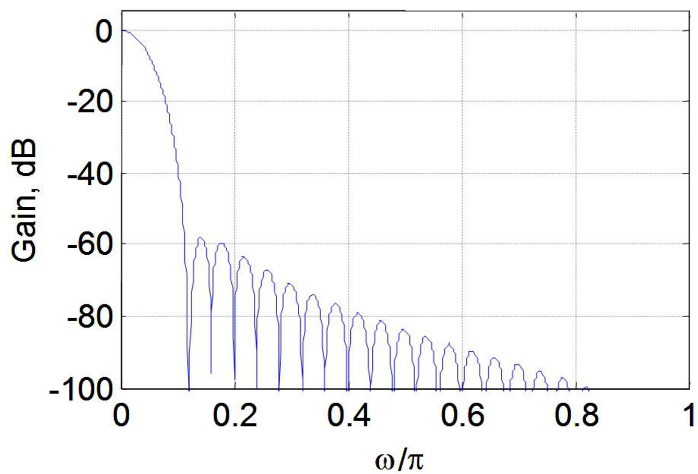
(a) 矩形窗Rectangular



(b) 汉宁窗Hanning



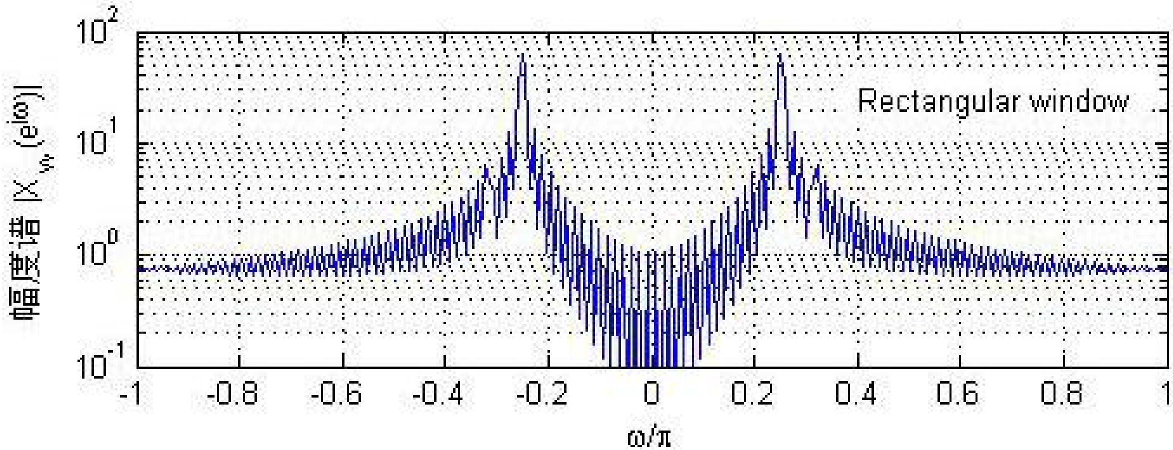
(c) 汉明窗Hamming

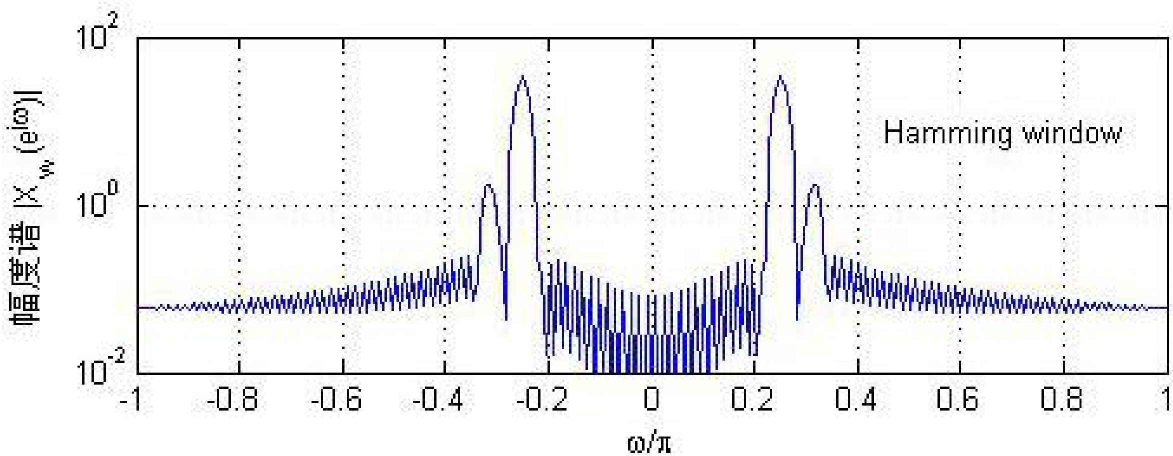


(d) 布莱克曼窗Blackman

How windowing affects the analysis?

- Consider the signal: $x[n] = \cos\left(\frac{\pi n}{4}\right) + 0.05\cos\left(\frac{5\pi n}{16}\right)$





How DFT length affects the analysis?

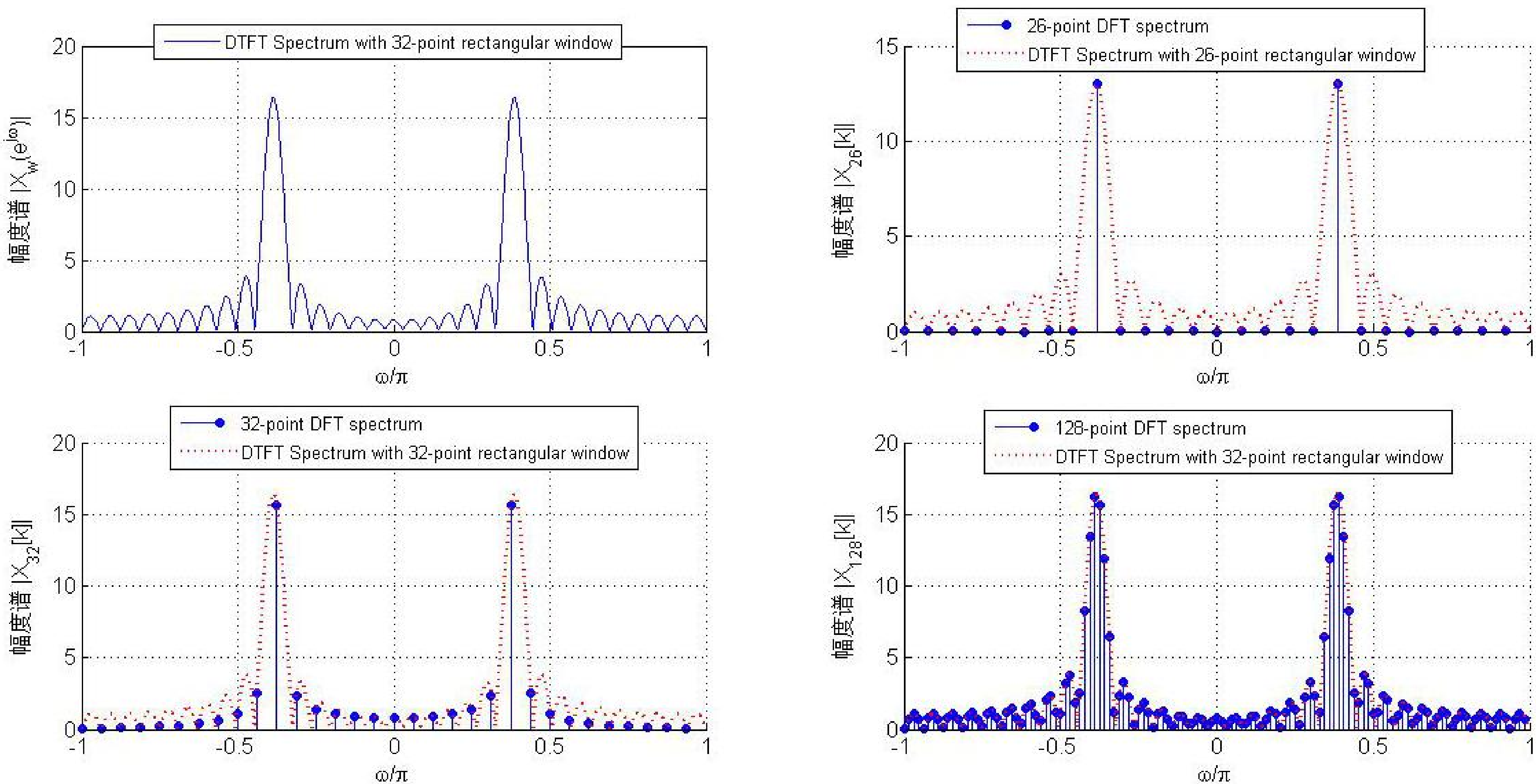
- N-point DFT means sampling $X_w(e^{j\omega})$ with the interval of $2\pi/N$
- The larger N, the better performance?
- Consider the signal:

$$x_w[n] = \cos\left(\frac{5\pi n}{13}\right) w_R[n]$$

$w_R[n]$, 32-point rectangular window

Perform 26-point, 32-point, 128-poing DFT

How DFT length affects the analysis?



How DFT length affects the analysis?

- 26-point DFT has the best result, why?
  - Two clean spectrum lines
  - Corresponding to the actual frequency
- 32-point and 128-point DFT
  - Spectrum leakage
    - Many nonzero spectrum lines
  - Picket fence effect
    - Cannot sample the mainlobe peak

How DFT length affects the analysis?

- **Explanation I**: DFT v.s. DTFT
  - For 26-point DFT, just sample the mainlobe peak and zeros of $X_{w\text{-}26}(e^{j\omega})$

$$X_{w-26}(e^{j\omega}) = \pi W_{R-26}\left(e^{j(\omega+5\pi/13)}\right) + \pi W_{R-26}\left(e^{j(\omega-5\pi/13)}\right)$$

$$W_{R-26}(e^{j\omega}) = \frac{\sin(13\omega)}{\sin(\omega/2)} e^{-j25\omega/2}$$

How DFT length affects the analysis?

- **Explanation I**: DFT v.s. DTFT
  - For 32/128-point DFT, cannot sample the main lobe peak and zeros

$$X_{w-32}(e^{j\omega}) = \pi W_{R-32}\left(e^{j(\omega+5\pi/13)}\right) + \pi W_{R-32}\left(e^{j(\omega-5\pi/13)}\right)$$

$$W_{R-32}(e^{j\omega}) = \frac{\sin(16\omega)}{\sin(\omega/2)} e^{-j31\omega/2}$$

  - Increasing N alleviates Picket fence effect

How DFT length affects the analysis?

- **Explanation II**: DFT v.s. DFS
  - N-point DFT actually calculates the DFS of another periodical sequence with the period of N

$$\tilde{x}[n] = x[n], \quad n = 0,\ 1,\ \ldots,\ N-1$$

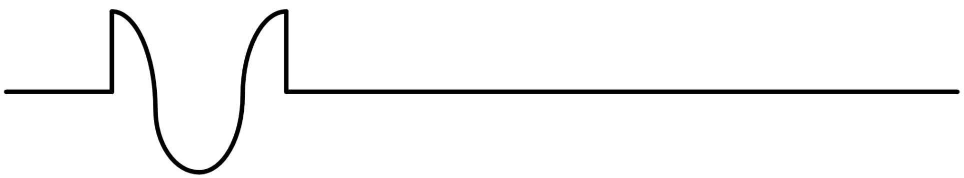
  - For $x[n] = \cos\left(\dfrac{5\pi n}{13}\right)$

    the period is actually 26.

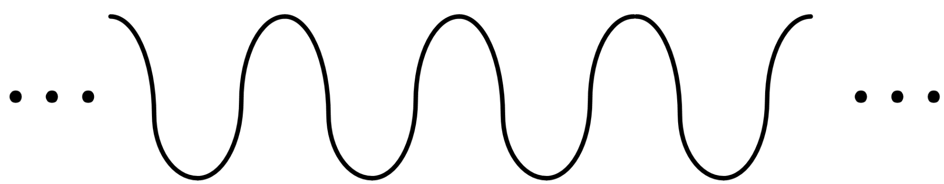
How DFT length affects the analysis?

- **Explanation II**: DFT v.s. DFS
  - For 26-point DFT

$x_{w-26}[n]$



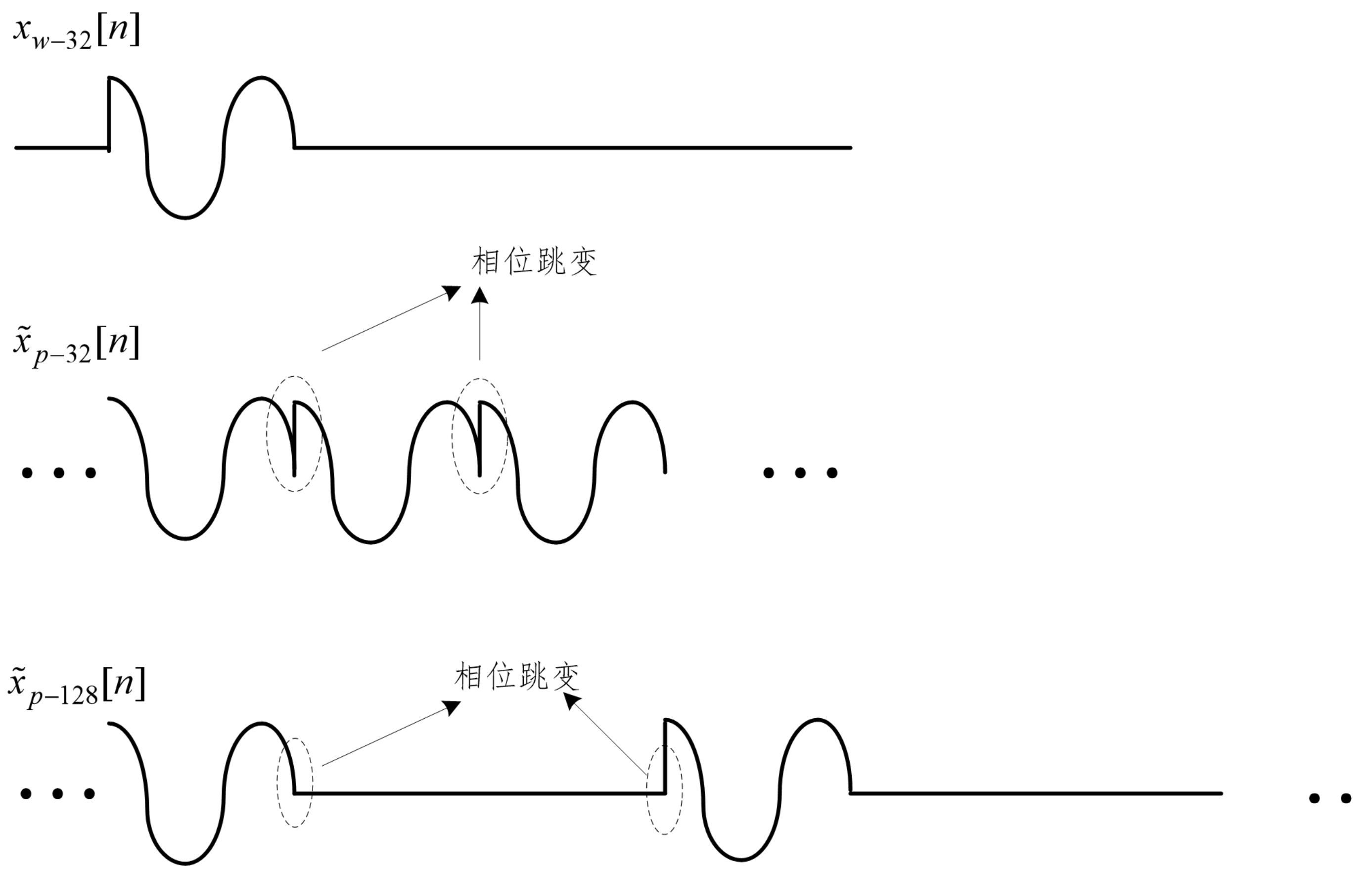
$\tilde{x}_{p-26}[n]$



$$\tilde{x}_{p-26}[n] = x[n] = \cos(5\pi n/13)$$

How DFT length affects the analysis?

- **Explanation II**: DFT v.s. DFS
  - For 32/128-point DFT



Project

- Consider the following signal

$$x(t) = 10\sin\left(2\pi\times 64t\right) + \sin\left(2\pi\times 250t\right) + 20\sin\left(2\pi\times 256t\right)$$
$$+ 3\sin\left(2\pi\times 260t\right) + 10\sin\left(2\pi\times 512t\right)$$

- Design the digital spectrum analysis scheme
  - Determine the sampling rate
  - Choose the window (type and length)
  - Design the length of DFT
- Use as less data as possible
- Complexity issue