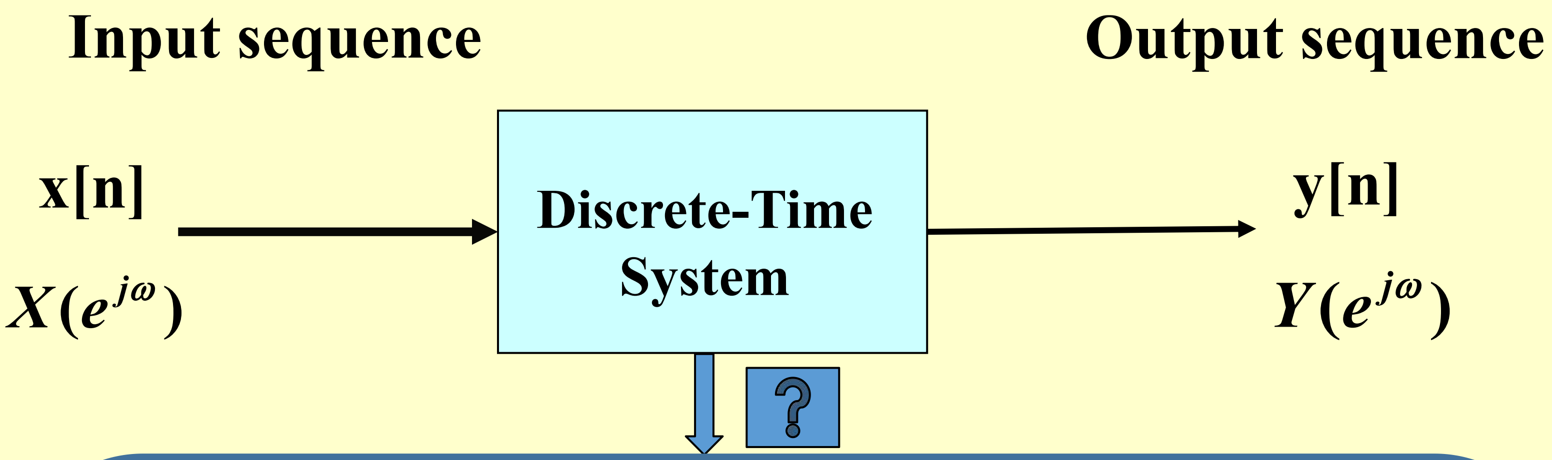
Review



频域: $h[n] \rightarrow H(e^{j\omega})$

$$H(e^{j\omega}) = Y(e^{j\omega}) / X(e^{j\omega})$$

$$H(e^{j\omega}) = \frac{\sum_{k=0}^{M} p_k e^{-j\omega k}}{\sum_{k=0}^{N} d_k e^{-j\omega k}} = \frac{p_0}{d_0} \cdot \frac{\prod_{k=1}^{M}(1-\xi_k e^{-j\omega})}{\prod_{k=1}^{N}(1-\lambda_k e^{-j\omega})}$$

$$= \frac{p_0}{d_0} e^{j\omega(N-M)} \cdot \frac{\prod_{k=1}^{M}(e^{j\omega}-\xi_k)}{\prod_{k=1}^{N}(e^{j\omega}-\lambda_k)}$$

Review

4.8节重要结论：

$$e^{j\omega_0 n} \xrightarrow{H} H(e^{j\omega_0})e^{j\omega_0 n} = \left|H(e^{j\omega_0})\right| e^{j(\omega_0 n+\theta(\omega_0))} \qquad (4.69)$$

$$\cos(\omega_0 n) \xrightarrow{H} \left|H(e^{j\omega_0})\right| \cos(\omega_0 n+\theta(\omega_0)) \qquad (4.86)$$

$$\sin(\omega_0 n) \xrightarrow{H} \left|H(e^{j\omega_0})\right| \sin(\omega_0 n+\theta(\omega_0)) \qquad (4.86)$$

切记：输入输出信号都是时域信号

Review

系统频响对系统分析的意义

$$x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$$

信号分解为基信号的线性叠加

+

$$e^{j\omega_0 n}\xrightarrow{H}H(e^{j\omega_0})e^{j\omega_0 n}$$

基信号通过系统的响应

$$y[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}H(e^{j\omega})X(e^{j\omega})e^{j\omega n}d\omega$$

线性系统：输出为各个基信号响应的线性叠加

- Thus, by appropriately choosing the values of the magnitude function $|H(e^{j\omega})|$ of the LTI digital filter at frequencies corresponding to the frequencies of the sinusoidal components of the input, some of these components can be selectively heavily attenuated or filtered with respect to the others

Chapter 5
Finite-Length Discrete Transforms

5.2 The Discrete Fourier Transform (5.2.2小节和5.2.4小节不看）

5.3 Relation Between the DTFT and the DFT and Their Inverses

5.2 The Discrete Fourier Transform

5.2.1 Definition

- **<u>Definition</u>** - The **Discrete Fourier Transform (DFT)** of the length-N time-domain sequence $x[n]$ $(0 \le n \le N-1)$ is

$$\boxed{X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n} \qquad k = 0 \cdots N-1}$$

IDFT:

$$\boxed{x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k] e^{j\frac{2\pi k}{N}n} = \frac{1}{N}\sum_{k=0}^{N-1} X[k] e^{j\omega_k n} \qquad \omega_k = \frac{2\pi k}{N} \qquad k = 0 \cdots N-1}$$

§ 5.2 The Discrete Fourier Transform

5.2.1 Definition

DFT作为一种傅里叶变换，它把信号做分解的基信号是什么？

$$x[n]=\frac{1}{N}\sum_{n=0}^{N-1}X[k]e^{j\frac{2\pi k}{N}n}=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\omega_k n}\qquad \omega_k=\frac{2\pi k}{N}\qquad k=0\cdots N-1$$

基信号：时域离散频率也离散的复正弦信号

★ 对比 $x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\omega_k n}$ 与 $x[n]=\frac{1}{2\pi}\int_{-\pi}^{\pi}X(e^{j\omega})e^{j\omega n}d\omega$

发现用 $0\le\omega<2\pi$ 范围内N个频率（$\omega_k=\frac{2\pi k}{N}$ $k=0\cdots N-1$）的复正弦信号而不是所有频率的复正弦信号就可以恢复有限长的时域信号。

§ 5.2 The Discrete Fourier Transform

5.2.1 Definition

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi k}{N}n} \qquad k=0\cdots N-1$$

在DFT定义式中 $X[k]$里的k代表的实际频率位置为多少?

k代表频率$\omega_k=\dfrac{2\pi k}{N} \qquad k=0\text{L} \;\; N-1$

$x[n]$ 的序号n代表的实际时间位置是什么（采样间隔为T）？

n代表时间nT

在DFT的定义式里时域信号是什么类型的信号(连续或离散)，长度是多少？通过定义式的哪个参数可以看出来?

时域信号$x[n]$是离散时间信号，长度为N,可通过定义式中参数n看出

计算出的 $X[k]$是什么类型的信号(连续或离散)，长度是多少?通过定义式的哪个参数可以看出来?

频域信号$X[k]$是离散时间信号，长度为N,可通过定义式中参数k看出

5.2.1 Definition

X[k]是实函数还是复函数，是否是周期函数，周期是多少？

Since $e^{-j2\pi kn/N}$ is a periodic sequence in k with a period N, it follows from Eq(5.7) that $X[k]$ can also be considered as a periodic sequence with a period N in k in the range $-\infty < k < \infty$,

$$X[k+N] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi(k+N)}{N}n} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n} \cdot e^{-j2\pi n} = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

$$= X[k] \quad k = 0 \ldots N-1$$

However, we shall consider the DFT $X[k]$ *to be* a length-N sequence in the transform domain by restricting the frequency domain integer variable k to be in the range $0 \le k \le N-1$, Often, the length-N DFT sequence is referred to as the *N-point* DFT

5.2.1 Definition

DFT:

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi k}{N}n} \qquad k=0 \cdots N-1$$

$$X(e^{j\omega_k})=\sum_{n=0}^{N-1}x[n]e^{-j\omega_k n} \qquad k=0 \cdots N-1$$

问题： DFT存在的 条件？

Since the computation of the DFT samples involves a finite sum, for a finite-length time-domain sequence with finite sample values, **the DFT always exists** .

5.2.1 Definition

引入变量 W_N 后，DFT的表示形式，W_N的表达式。

- Using the notation $W_N = e^{-j2\pi/N}$ the DFT is usually expressed as:

记住

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn},\ 0 \le k \le N-1$$

$$X[k] = \sum_{n=0}^{N-1} x[n] e^{-j\frac{2\pi k}{N}n}$$

- The inverse discrete Fourier transform (IDFT) is given by

$$x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn},\ 0 \le n \le N-1$$

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] e^{j\frac{2\pi k}{N}n}$$

5.2.1 Definition (N-point DFT of a length-N sequence)

DFT的三种形式

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi k}{N}n}\qquad 0\le k\le N-1$$

$$X(e^{j\omega_k})=\sum_{n=0}^{N-1}x[n]e^{-j\omega_k n}\qquad \omega_k=\frac{2\pi k}{N}$$

$$0\le k\le N-1$$

$$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn}\qquad 0\le k\le N-1$$

IDFT的三种形式

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi k}{N}n}\qquad 0\le n\le N-1$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\omega_k n}\qquad 0\le n\le N-1$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn}\qquad 0\le n\le N-1$$

5.2.1 Definition

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$x[n] = \begin{cases} 1, & n = 0 \\ 0, & 1 \le n \le N-1 \end{cases}$$

- Its N-point DFT is given by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} = x[0] W_N^0 = 1$$

$$0 \le k \le N-1$$

5.2.1 Definition

Discrete Fourier Transform

- Example - Consider the length-N sequence

$$y[n] = \begin{cases} 1, & n = m \\ 0, & 0 \le n \le m-1,\ m+1 \le n \le N-1 \end{cases}$$

- Its N-point DFT is given by

$$Y[k] = \sum_{n=0}^{N-1} y[n] W_N^{kn} = y[m] W_N^{km} = W_N^{km}$$

$$0 \le k \le N-1$$

5.2.1 Definition

Discrete Fourier Transform

- Example - Consider the length-N sequence defined for $0 \le n \le N-1$

$$g[n] = \cos(2\pi rn/N), \quad 0 \le r \le N-1$$

- Using a trigonometric identity we can write

$$g[n] = \frac{1}{2}\left(e^{j2\pi rn/N} + e^{-j2\pi rn/N}\right)$$

$$= \frac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

5.2.1 Definition

Discrete Fourier Transform

- The N-point DFT of $g[n]$ is thus given by

$$G[k] = \sum_{n=0}^{N-1} g[n] W_N^{kn}$$

$$g[n] = \tfrac{1}{2}\left(W_N^{-rn} + W_N^{rn}\right)$$

$$= \frac{1}{2}\left(\sum_{n=0}^{N-1} W_N^{-(r-k)n} + \sum_{n=0}^{N-1} W_N^{(r+k)n}\right),$$

$$0 \le r \le N-1 \qquad 0 \le k \le N-1$$

5.2.1 Definition

$$\sum_{n=0}^{N-1} W_N^{-mn} = \frac{1-W_N^{-Nm}}{1-W_N^{-m}} = \begin{cases} N, & for \quad m = lN \\ 0, & otherwise \end{cases}$$

- Making use of the identity

$$\sum_{n=0}^{N-1} W_N^{-(k-r)n} = \begin{cases} N, & for \quad k-r = \ell N, \quad \ell \quad an \quad integer \\ 0, & otherwise \end{cases}$$

$$\sum_{n=0}^{N-1} W_N^{(k+r)n} = \begin{cases} N, & for \quad k+r = lN, \quad l \quad an \quad integer \\ 0, & otherwise \end{cases}$$

we get

$$G[k] = \begin{cases} N/2, & \text{for } k = r \\ N/2, & \text{for } k = N-r \\ 0, & \text{otherwise} \end{cases}$$

$$0 \le k \le N-1 \qquad 0 \le r \le N-1$$

5.2.2 Computational Complexity Issue

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi k}{N}n}\qquad 0\le k\le N-1$$

复数乘法次数（the number of complex multiplication）：N^2

复数加法次数（the number of complex addition）：$N\cdot(N-1)$

5.2.3 Matrix Relations

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j2\pi kn/N} \qquad 0\le k\le N-1$$

$$\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix} \xrightarrow{DFT} \begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix}$$

5.2.3 Matrix Relations

$$X(e^{j\omega_k}) = X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n} = \sum_{n=0}^{N-1} x[n]W_N^{kn}$$

$$X[k] = \left[\exp(-j\frac{2\pi k}{N}\cdot 0) \quad \exp(-j\frac{2\pi k}{N}\cdot 1) \quad \cdots \quad \exp(-j\frac{2\pi k}{N}\cdot (N-1))\right]\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$X[k] = \left[W_N^{k\cdot 0} \quad W_N^{k\cdot 1} \quad \cdots \quad W_N^{k\cdot (N-1)})\right]\begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

5.2.3 Matrix Relations

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W_N^{0\cdot 0} & W_N^{0\cdot 1} & \cdots & W_N^{0\cdot(N-1)} \\ W_N^{1\cdot 0} & W_N^{1\cdot 1} & \cdots & W_N^{1\cdot(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ W_N^{(N-1)\cdot 0} & W_N^{(N-1)\cdot 1} & \cdots & W_N^{(N-1)\cdot(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

$$\mathbf{X} = \mathbf{D}_N \mathbf{x}$$

$$\mathbf{X} = \begin{bmatrix} X[0] & X[1] & \cdots\cdots & X[N-1] \end{bmatrix}^T$$

$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & \cdots\cdots & x[N-1] \end{bmatrix}^T$$

5.2.3 Matrix Relations

- The DFT samples defined by

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn}, \quad 0 \le k \le N-1$$

can be expressed in matrix form as

$$\boxed{\mathbf{X} = \mathbf{D}_N \mathbf{x}}$$

where

$$\mathbf{X} = \begin{bmatrix} X[0] & X[1] & \cdots & X[N-1] \end{bmatrix}^T$$

$$\mathbf{x} = \begin{bmatrix} x[0] & x[1] & \cdots & x[N-1] \end{bmatrix}^T$$

5.2.3 Matrix Relations

1. 矩阵维数和什么有关？
2. 矩阵元素是什么，有什么规律？

矩阵行数是DFT的点数，矩阵列数是时域序列的长度

W_N^{kn}：k是行-1，n是列-1

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W_N^{0\cdot 0} & W_N^{0\cdot 1} & \cdots & W_N^{0\cdot(N-1)} \\ W_N^{1\cdot 0} & W_N^{1\cdot 1} & \cdots & W_N^{1\cdot(N-1)} \\ \vdots & \vdots & \vdots & \vdots \\ W_N^{(N-1)\cdot 0} & W_N^{(N-1)\cdot 1} & \cdots & W_N^{(N-1)\cdot(N-1)} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

and $\mathbf{D}_N$ is the $N \times N$ **DFT matrix** given by

$$\mathbf{D}_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{(N-1)} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{(N-1)} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)^2} \end{bmatrix}$$

5.2.3 Matrix Relations

- Likewise, the IDFT relation given by

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k] W_N^{-kn}, \; 0 \le n \le N-1$$

can be expressed in matrix form as

$$\mathbf{x} = \mathbf{D}_N^{-1}\mathbf{X}$$

where $\mathbf{D}_N^{-1}$ is the $N \times N$ **IDFT matrix**

5.2.3 Matrix Relations

$$\boxed{\frac{1}{N}\mathbf{D}_N\mathbf{D}_N^* = \mathbf{I}} \Rightarrow \boxed{\mathbf{D}_N^{-1} = \frac{1}{N}\mathbf{D}_N^*}$$

$$\mathbf{D}_N^{-1} = \frac{1}{N}\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N^{-1} & W_N^{-2} & \cdots & W_N^{-(N-1)} \\ 1 & W_N^{-2} & W_N^{-4} & \cdots & W_N^{-2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & W_N^{-(N-1)} & W_N^{-2(N-1)} & \cdots & W_N^{-(N-1)^2} \end{bmatrix}$$

$$\mathbf{x} = \frac{1}{N}\mathbf{D}_N^*\mathbf{X}$$

5.2.3 Matrix Relations

- **Example:** **Compute 4 points DFT of sequences $x[n]=\{\underline{5},0,-3,4\}$**

$$D_4=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}$$

说说为什么此处值是 –j？

$$\Rightarrow W_4^{(行-1)\cdot(列-1)}=W_4^1=e^{-j\frac{2\pi}{4}}=-j$$

$$\mathbf{X}=D_4\mathbf{x}=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}\begin{bmatrix}5\\ 0\\ -3\\ 4\end{bmatrix}=\begin{bmatrix}6\\ 8+j4\\ -2\\ 8-j4\end{bmatrix}$$

§ 5.3 Relation Between the DTFT and the DFT and Their Inverses

5.3.1 Relation with DTFT

(用数学表达式说说DTFT与DFT之间的关系)

$$X[k]=\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n} \qquad k=0 \ldots N-1$$

$$X(e^{j\omega})=\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

- From the definition of the DTFT we thus have

$$X[k]=X(e^{j\omega})\Big|_{\omega=2\pi k/N}=\sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N},$$

$$0\le k\le N-1$$

§ 5.3 Relation Between the DTFT and the DFT and Their Inverses

5.3.1 Relation with DTFT

(用语言说说DTFT与DFT之间的关系)

- From the definition of the DTFT we thus have

$$X[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/N}, \quad 0 \le k \le N-1$$

The N-point DFT sequence $X[k]$ is precisely the set of frequency samples of the Fourier transform $X(e^{j\omega})$ of the length-N sequence $x[n]$ at N equally spaced frequencies $\omega = \omega_k = 2\pi k/N, \quad 0 \le k \le N-1$

5.3.2 Numerical Computation of the DTFT Using the DFT

这一节的标题中的数值计算（Numerical Computation）是什么意思？

数值计算 指有效使用数字计算机求数学问题近似解的方法与过程，以及由相关理论构成的学科。数值计算主要研究如何利用计算机更好的解决各种数学问题。

是否可以在数字设备上计算存储DTFT所有频率点上的频谱值？

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

5.3.2 Numerical Computation of the DTFT Using the DFT

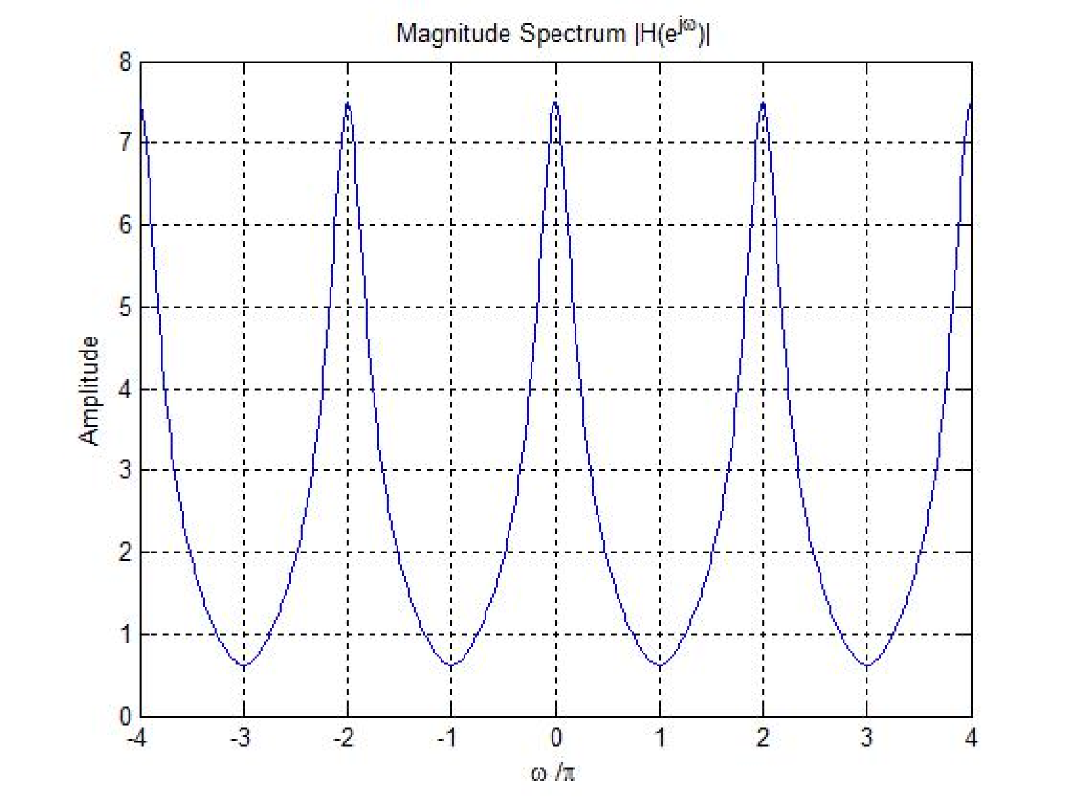
了解：计算并画出 $H(e^{j\omega}) = \dfrac{p_0 + p_1 e^{-j\omega} + \cdots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \cdots + d_N e^{-j\omega N}}$ 给出的离散时间傅里叶变换

```
% Program P3_1
% Evaluation of the DTFT
clf;
% Compute the frequency samples of the DTFT
w = -4*pi:8*pi/511:4*pi;
num = [2 1];den = [1 -0.6];
h = freqz(num, den, w);
plot(w/pi,abs(h))（或者stem(w/pi,abs(h))）
```

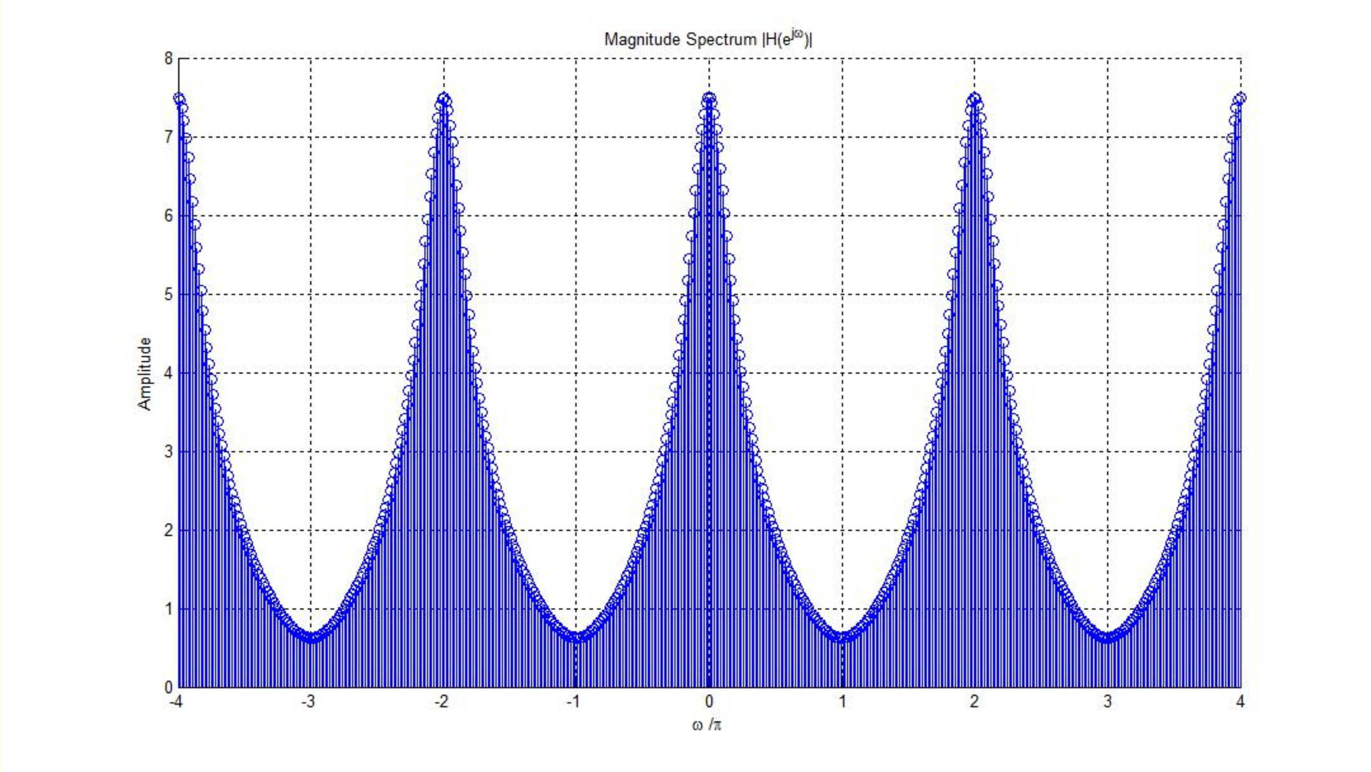
5.3.2 Numerical Computation of the DTFT Using the DFT

了解：计算并画出 $H(e^{j\omega}) = \dfrac{p_0 + p_1 e^{-j\omega} + \cdots + p_M e^{-j\omega M}}{d_0 + d_1 e^{-j\omega} + \cdots + d_N e^{-j\omega N}}$ 给出的离散时间傅里叶变换

plot(w/pi,abs(H))



stem(w/pi,abs(H))



5.3.2 Numerical Computation of the DTFT Using DFT

如果要用DFT完成DTFT的数值计算,对DFT定义式中的参数有何要求?

- A practical approach to the numerical computation of the DTFT of a finite-length sequence

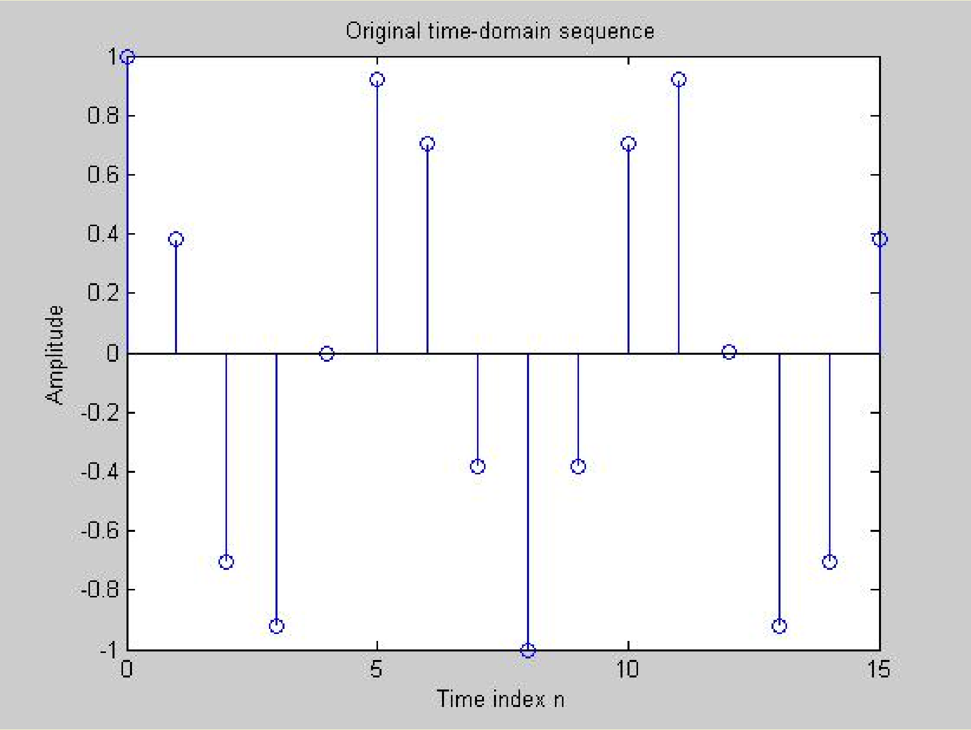
- We wish to evaluate $X(e^{j\omega})$ at a dense grid of frequencies $\omega_k = 2\pi k / M$, $0 \le k \le M-1$, where $M >> N$:

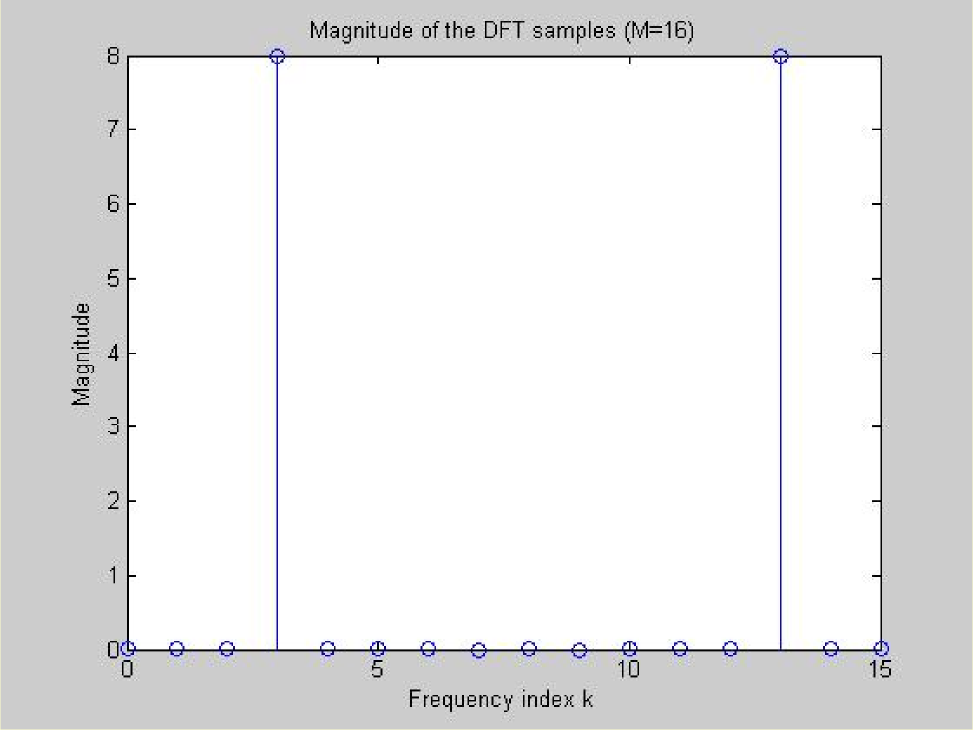
$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} = \sum_{n=0}^{\boxed{N}-1} x[n] e^{-j2\pi kn/\boxed{M}}$$

（时域序列长度和频域采样点数可以不相等）

5.3.2 Numerical Computation of the DTFT Using the DFT

- **Example**

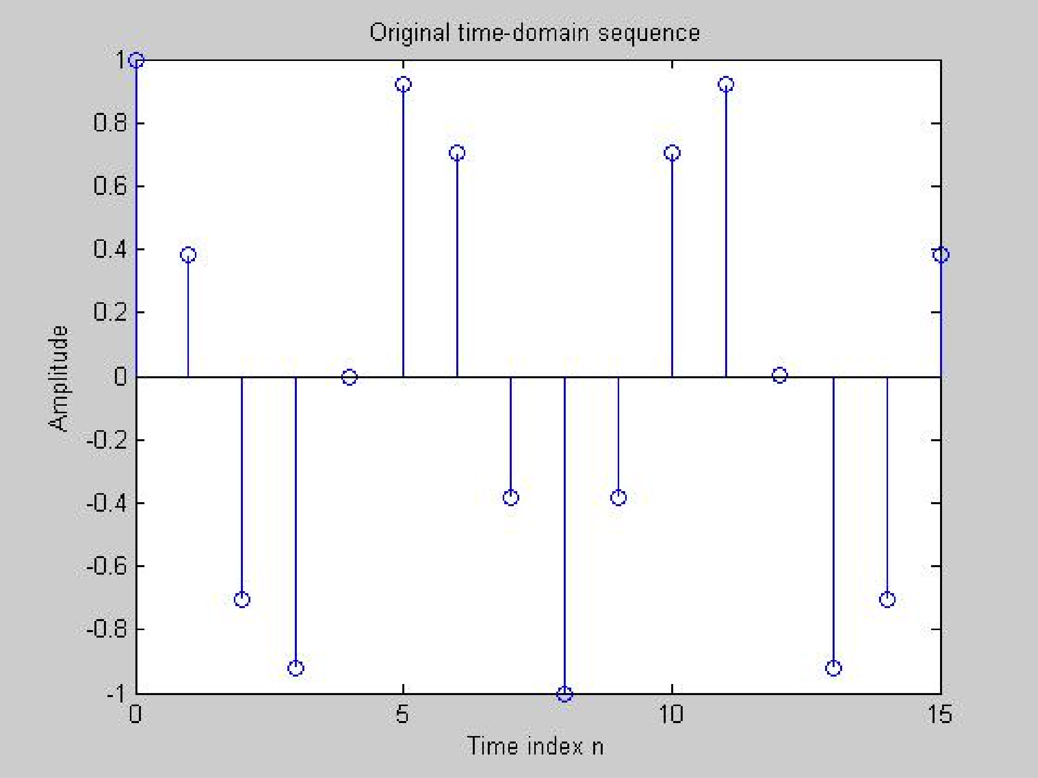


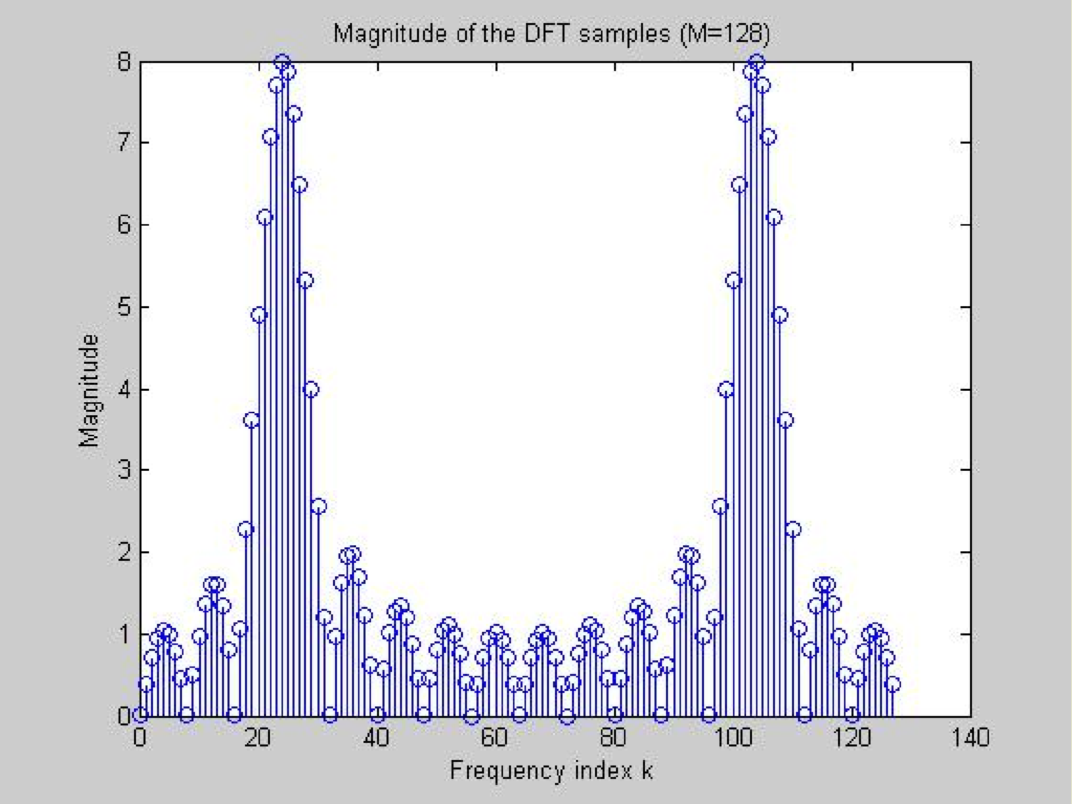


$$x[n] = \cos(3\pi n/8) \qquad n = 0 \ldots 15$$

$$X[k] = \sum_{n=0}^{15} x[n] e^{-j2\pi kn/16} \qquad k = 0 \ldots 15$$

5.3.2 Numerical Computation of the DTFT Using the DFT

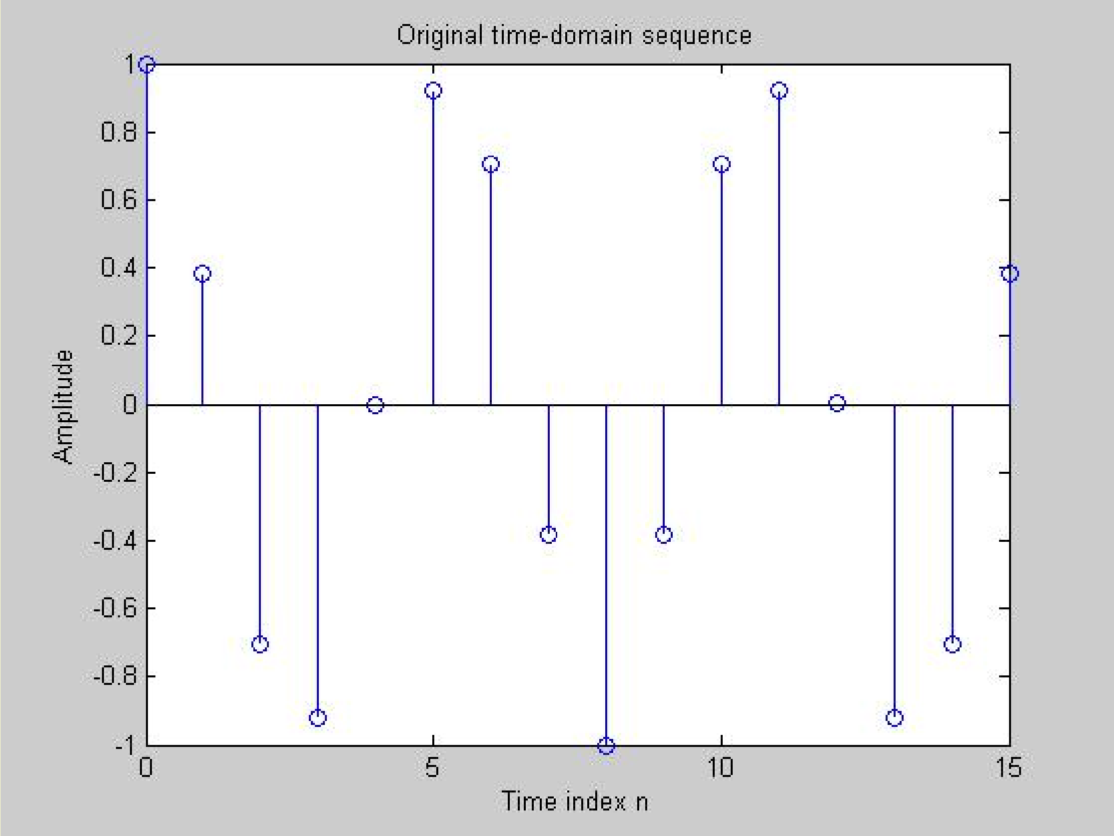


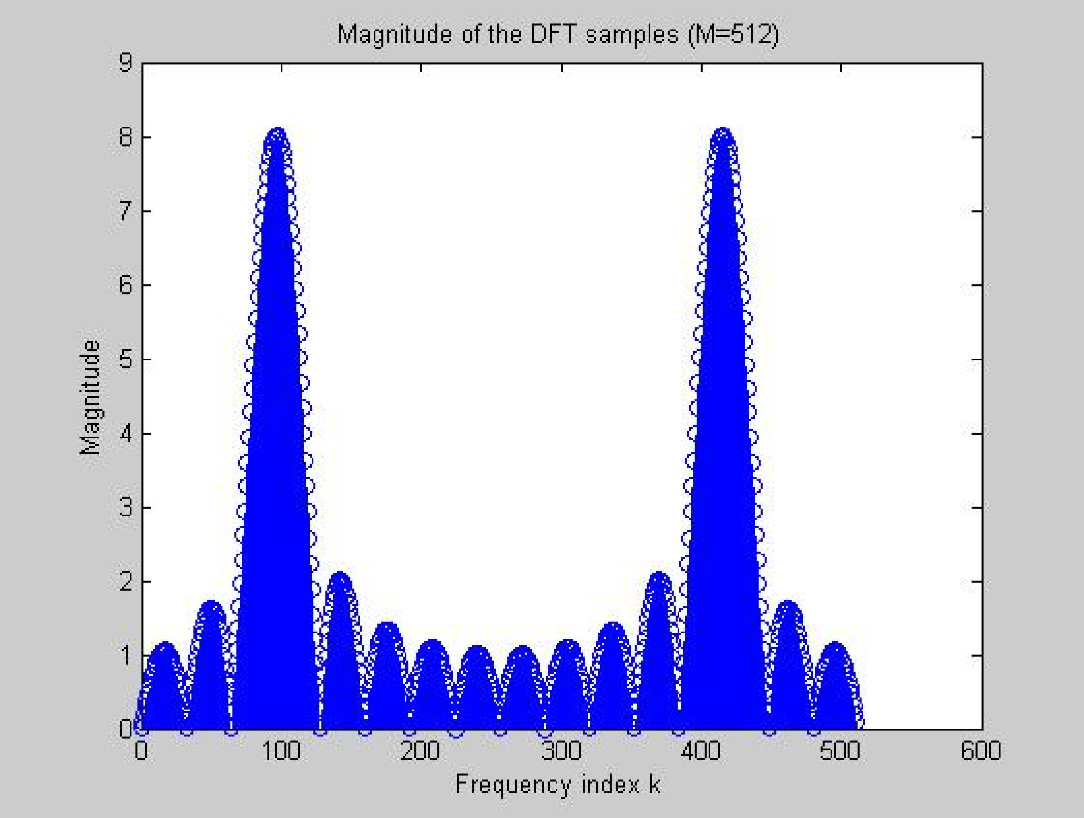


$$x[n] = \cos(3\pi n / 8) \qquad n = 0 \ldots 15$$

$$X[k] = \sum_{n=0}^{15} x[n] e^{-j2\pi kn/128} \qquad k = 0 \ldots 127$$

5.3.2 Numerical Computation of the DTFT Using the DFT

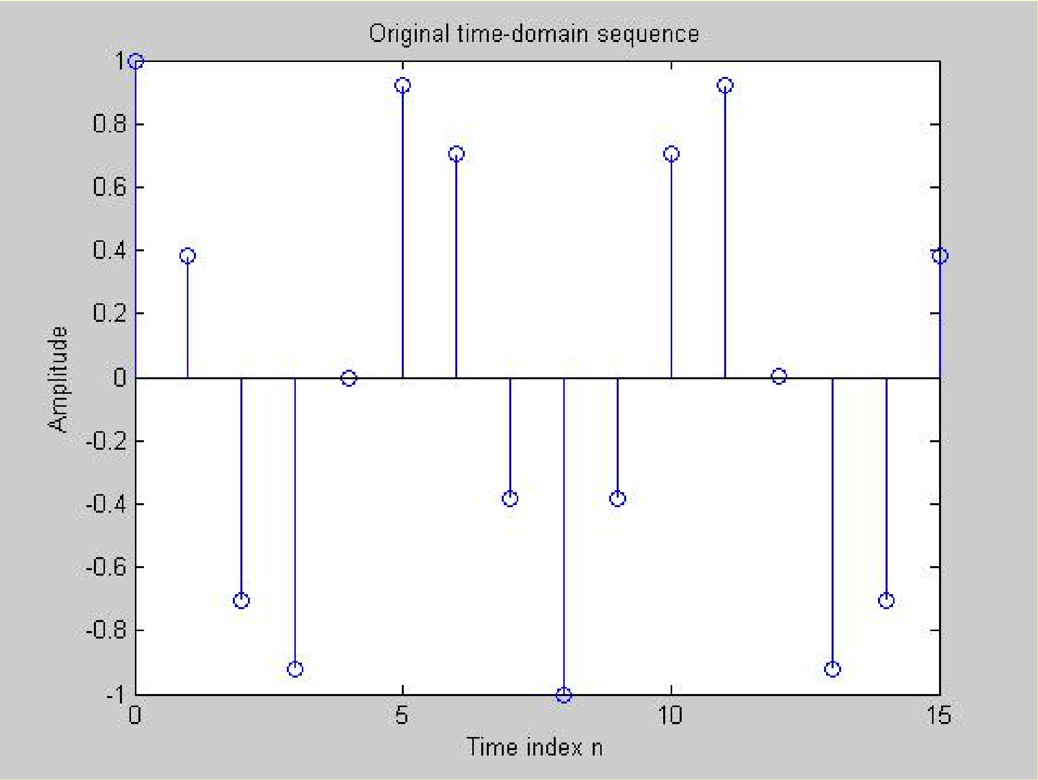


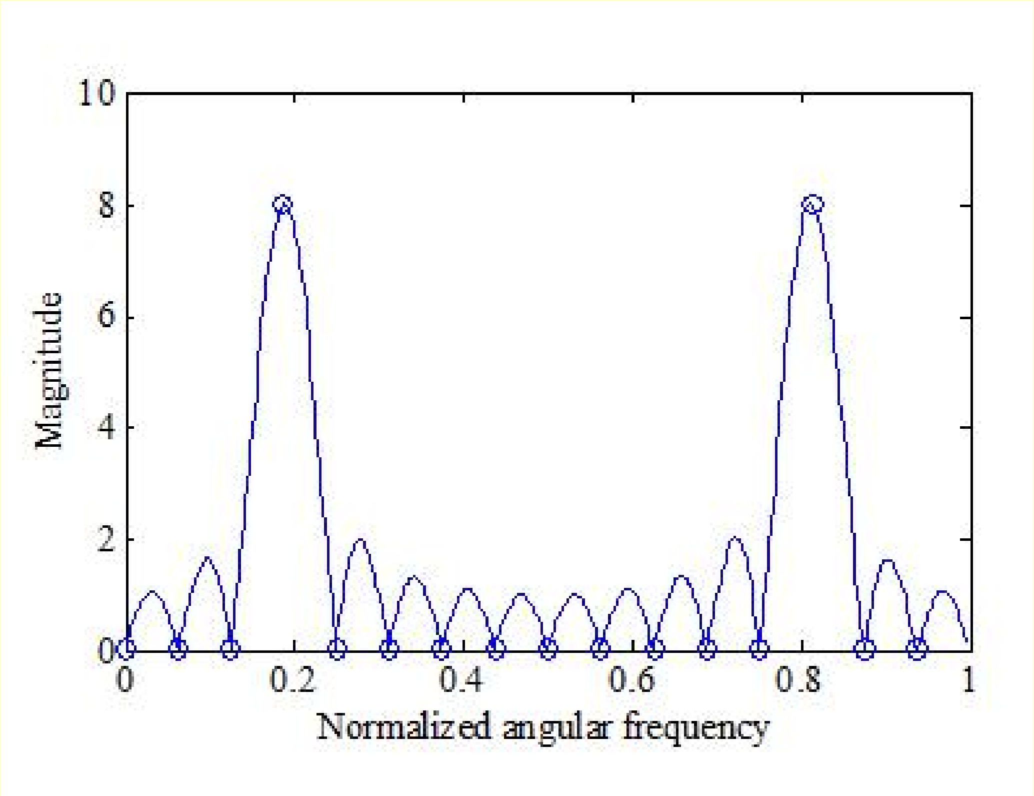


$$x[n] = \cos(3\pi n/8) \qquad n = 0 \ldots 15$$

$$X[k] = \sum_{n=0}^{15} x[n] e^{-j2\pi kn/512} \qquad k = 0 \ldots 511$$

5.3.2 Numerical Computation of the DTFT Using the DFT





$$x[n] = \cos(3\pi n / 8) \qquad n = 0 \ldots 15$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

5.3.2 Numerical Computation of the DTFT Using the DFT

$$X(e^{j\omega_k}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega_k n} = \sum_{n=0}^{N-1} x[n] e^{-j2\pi kn/M}$$

- Define a new sequence

$$x_e[n] = \begin{cases} x[n], & 0 \le n \le N-1 \\ 0, & N \le n \le M-1 \end{cases}$$

通过补零时域序列长度和频域采样点数相等

- Then

$$X(e^{j\omega_k}) = \sum_{n=0}^{M-1} x_e[n] e^{-j2\pi kn/M}$$

- Thus $X(e^{j\omega_k})$ is essentially an M-point DFT $X_e[k]$ of the length-M sequence $x_e[n]$

5.3.2 Numerical Computation of the DTFT Using the DFT

DFT在数字信号处理中的作用是什么?

- **DTFT** $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$ **is the Fourier Transform of discrete-time sequence. The spectrum is continuous. Thus it can not be processed by digital system because the digital system could only process discrete sequence.**

- **The DTFT can only be evaluated by the digital system at discrete frequency point.**

$$X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi k}{N}n} \qquad k = 0 \ldots N-1$$

5.3.3 DTFT from DFT by Interpolation

$$X[k] = X(e^{j\omega})\Big|_{\omega=2\pi k/N}$$

$$X[k] \xrightarrow{?} X(e^{j\omega})$$

- The N-point DFT $X[k]$ of a length-N sequence $x[n]$ is simply the frequency samples of its DTFT $X(e^{j\omega})$ evaluated at N uniformly spaced frequency points

$$\omega = \omega_k = 2\pi k / N, \quad 0 \le k \le N-1$$

- Given the N-point DFT $X[k]$ of a length-N sequence $x[n]$, its DTFT $X(e^{j\omega})$ can be uniquely determined from $X[k]$

5.3.3 DTFT from DFT by Interpolation

$$X[k] \xrightarrow{?} X(e^{j\omega})$$

- Thus

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n}$$

$$= \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn} \right] e^{-j\omega n}$$

$$= \frac{1}{N} \sum_{k=0}^{N-1} X[k] \underbrace{\sum_{n=0}^{N-1} e^{-j(\omega - 2\pi k/N)n}}_{S}$$

5.3.3 DTFT from DFT by Interpolation

- Therefore

$$X(e^{j\omega})$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X[k] \boxed{\frac{\sin\left(\frac{\omega N - 2\pi k}{2}\right)}{\sin\left(\frac{\omega N - 2\pi k}{2N}\right)} \cdot e^{-j[(\omega - 2\pi k/N)][(N-1)/2]}} \rightarrow \Phi(\omega - \frac{2k\pi}{N})$$

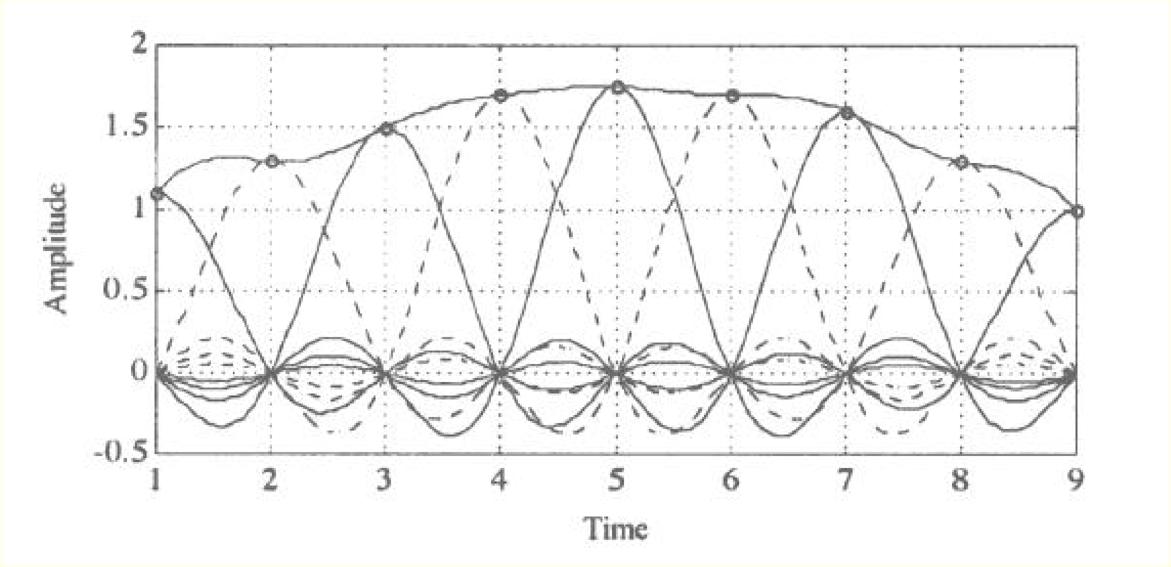
$$\Phi(\omega)\big|_{\omega = 2\pi l/N} = \begin{cases} 1 & l = 0 \\ 0 & 1 \le l \le N-1 \end{cases}$$

$$\Rightarrow \boxed{X(e^{j\omega})\big|_{\omega = 2\pi l/N} = X[l] \quad 0 \le l \le N-1}$$

5.3.3 DTFT from DFT by Interpolation

把这一小节的内容和公式（3.82）的含义联系在一起对照理解。

- The ideal bandlimited interpolation process is illustrated below 回顾：时域采样恢复



$$\hat{g}_a(t) = \sum_{n=-\infty}^{\infty} g[n] \frac{\sin[\pi(t-nT)/T]}{\pi(t-nT)/T}$$ 公式（3.82）

5.3.4 Sampling the Fourier Transform

- Now $$X(e^{j\omega}) = \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j\omega\ell}$$

- Thus $$Y[k] = X(e^{j\omega_k}) = X(e^{j2\pi k/N})$$
$$= \sum_{\ell=-\infty}^{\infty} x[\ell] e^{-j2\pi k\ell/N} = \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell}$$

- An IDFT of $Y[k]$ yields
$$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} Y[k] W_N^{-kn}$$

5.3.4 Sampling the Fourier Transform

- i.e. $$y[n] = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{\ell=-\infty}^{\infty} x[\ell] W_N^{k\ell} W_N^{-kn}$$

$$= \sum_{\ell=-\infty}^{\infty} x[\ell] \left[\frac{1}{N} \sum_{k=0}^{N-1} W_N^{-k(n-\ell)} \right]$$

- Making use of the identity

$$\frac{1}{N} \sum_{k=0}^{N-1} W_N^{-k(n-1)} = \begin{cases} 1 & for \quad 1 = n + mN \ \text{或} n - 1 = mN \\ 0 & otherwise \end{cases}$$

5.3.4 Sampling the Fourier Transform

we arrive at the desired relation

$$\boxed{y[n]=\sum_{m=-\infty}^{\infty} x[n+m\underbrace{N}_{\text{频域采样点数}}],\quad 0\le n\le N-1}$$

- Thus $y[n]$ is obtained from $x[n]$ by adding an infinite number of shifted replicas of $x[n]$, with each replica shifted by an integer multiple of N sampling instants, and observing the sum only for the interval $0\le n\le N-1$

$$x[n] \xrightarrow{\text{DTFT}} X(e^{j\omega}) \xrightarrow{\text{一个周期 } N\text{点采样}} Y[k]=X(e^{j\omega_k}) \xrightarrow{\text{IDFT}} y[n]$$

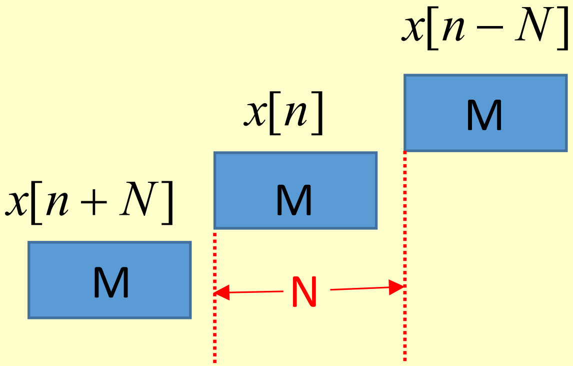
结论：频域采样带来时域序列的周期复制

5.3.4 Sampling the Fourier Transform

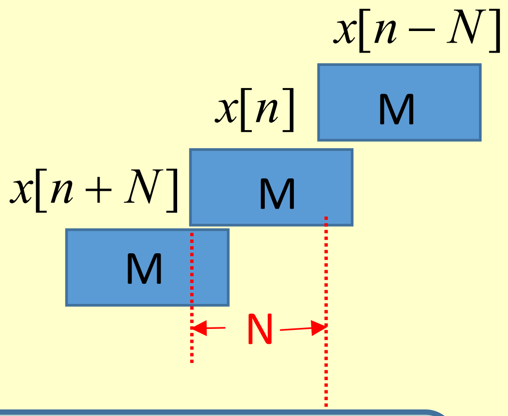
周期信号

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \quad 0 \le n \le N-1$$

- Thus if $x[n]$ is a length-M sequence with $M \le N$, then $y[n] = x[n]$ for $0 \le n \le N-1$



- If $M > N$, there is a time-domain aliasing of samples of $x[n]$ in generating $y[n]$, and $x[n]$ cannot be recovered from $y[n]$



这一小节要讲的内容就是频域采样带来的时域信号的周期复制，根据傅里叶变换的时频域对偶性，大家可以和3.8.1这一小节联系对照理解。

5.3.4 Sampling the Fourier Transform

- Example - Let $\{x[n]\} = \{\underset{\uparrow}{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$
- By sampling its DTFT $X(e^{j\omega})$ at $\omega_k = 2\pi k/4$, $0 \le k \le 3$ and then applying a 4-point IDFT to these samples, we arrive at the sequence $y[n]$ given by

$$y[n] = \mathrm{L} \; + x[n+4] + x[n] + x[n-4] + \mathrm{L} \qquad 0 \le n \le 3$$

$$\boxed{\{0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}}$$

$$\boxed{\{\underset{\uparrow}{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}}$$

$$x[n+4] = \boxed{\{0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}}$$

5.3.4 Sampling the Fourier Transform

$$y[n] = \mathrm{L} \;\; + x[n+4] + x[n] + x[n-4] + \mathrm{L} \qquad 0 \le n \le 3$$

$$\{0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$$

$$\{\underset{\uparrow}{0} \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$$

$$\{0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5\}$$

$$\mathrm{L} \quad 2 \quad 3 \quad 4 \quad 6 \quad 2 \quad 3 \quad 4 \quad 6 \quad 2 \quad 3 \quad \mathrm{L}$$

- i.e.

$$\{y[n]\} = \{\underset{\uparrow}{4} \quad 6 \quad 2 \quad 3\}$$

$\{x[n]\}$ cannot be recovered from $\{y[n]\}$

补充：

Periodicity in one domain corresponds to discreteness in another domain

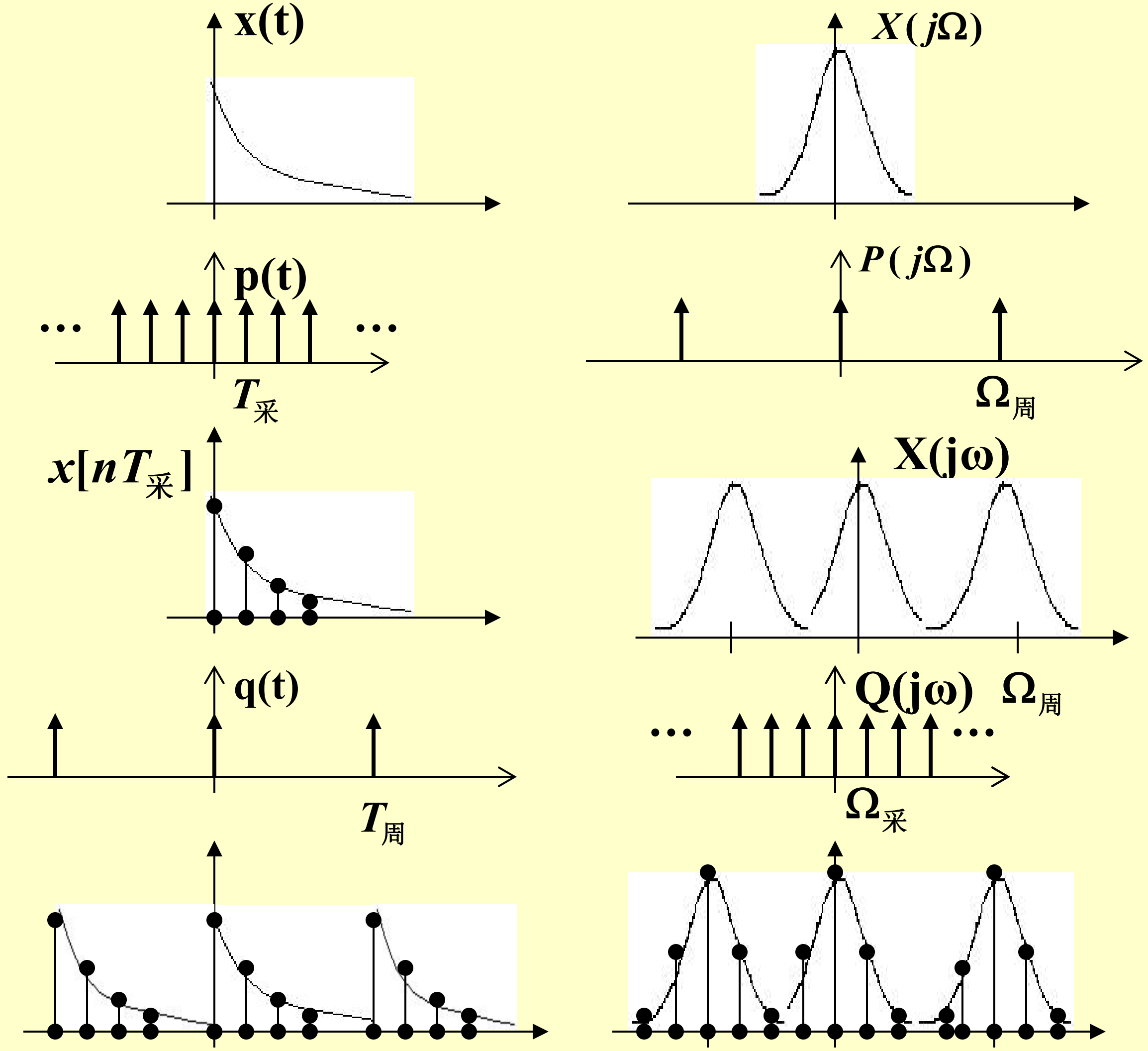
Time domain			Frequency domain	
Continuous	aperiodic	$\leftarrow$ CTFT $\rightarrow$	Continuous	aperiodic
Discrete	aperiodic	$\leftarrow$ DTFT $\rightarrow$	periodic	continuous
Discrete	periodic	$\leftarrow$ DFT $\rightarrow$	periodic	discrete

(DFT隐含时域的周期性)

$$y[n] = \sum_{m=-\infty}^{\infty} x[n+mN], \ \ 0 \le n \le N-1$$

补充：

Make a signal discrete and periodic



CTFT

$$\Omega_{周} = 2\pi / T_{采}$$

DTFT $\omega = \Omega \cdot T_{采}$

$$\Omega_{采} = 2\pi / T_{周}$$

DFT

$$\Omega_{周} = N \cdot \Omega_{采} \Rightarrow$$

$$\frac{2\pi}{T_{采}} = N \cdot \frac{2\pi}{T_{周}} \Rightarrow T_{周} = N \cdot T_{采}$$

(DFT隐含时域的周期性)

问题：DFT 时域序列长度和频域采样点数是否必须相等？

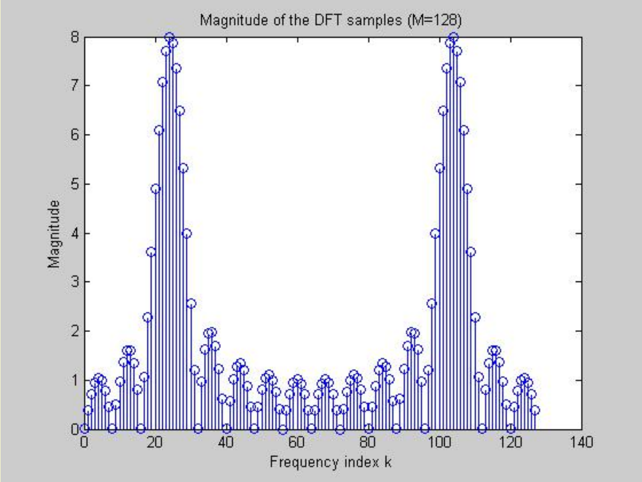
DFT: $$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi k}{N}n}\qquad k=0\cdots N-1$$

DFT: $$X(e^{j\omega_k})=\sum_{n=0}^{N-1}x[n]e^{-j\omega_k n}\qquad \omega_k=\frac{2\pi k}{N}\qquad k=0\cdots N-1\qquad \omega_k\in[0,\ 2\pi)$$

★不一定相等，DFT的表达式也可以写成

$$X[k]=\sum_{n=0}^{L-1}x[n]e^{-j\frac{2\pi k}{N}n}\qquad k=0\cdots N-1$$

（时域序列长度L不一定等于DFT点数N）



$$X[k]=\sum_{n=0}^{15}x[n]e^{-j2\pi kn/128}\qquad k=0\cdots 127$$

问题： 在DFT定义式中时域序列长度和频域采样点数相等的意义是什么？

$$X[k]=\sum_{n=0}^{L-1}x[n]e^{-j\frac{2\pi k}{N}n}\qquad k=0\ldots N-1$$

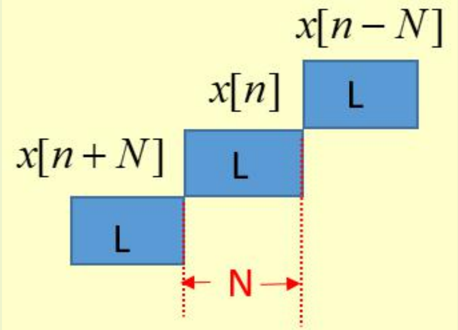
$$y[n]=\frac{1}{N}\sum_{n=0}^{N-1}X[k]e^{j\frac{2\pi k}{N}n}\qquad n=0\ldots L-1$$

$$y[n]=\sum_{m=-\infty}^{\infty}x[n+mN]$$

频域离散性带来时域周期性

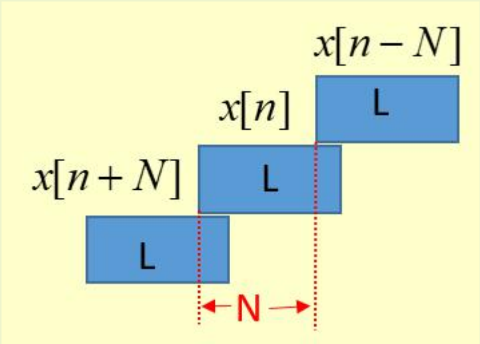
$$L=N\quad\Rightarrow y[n]=x[n]\qquad n=0\ldots N-1$$

$$L>N\quad\Rightarrow y[n]\neq x[n]\qquad n=0\ldots N-1$$



L=N

$y[n]=x[n]\qquad 0\le n\le N-1$



$L>N$

$y[n]\neq x[n]\qquad 0\le n\le N-1$

★ **当时域序列长度与DFT频域采样点数相等时，IDFT定义式等号左边才能为$x[n]$**

$$x[n]=\frac{1}{N}\sum_{n=0}^{N-1}X[k]e^{j\frac{2\pi k}{N}n}\qquad n=0\ldots N-1$$

第八次课 要点

1. DFT定义式的全面理解。
2. DFT的存在问题。
3.DFT和IDFT的三种表示形式

$$X[k]=\sum_{n=0}^{N-1}x[n]e^{-j\frac{2\pi k}{N}n} \qquad 0\le k\le N-1$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\frac{2\pi k}{N}n} \qquad 0\le n\le N-1$$

$$X(e^{j\omega_k})=\sum_{n=0}^{N-1}x[n]e^{-j\omega_k n} \qquad \omega_k=\frac{2\pi k}{N} \qquad 0\le k\le N-1$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]e^{j\omega_k n} \qquad 0\le n\le N-1$$

$$X[k]=\sum_{n=0}^{N-1}x[n]W_N^{kn} \qquad 0\le k\le N-1$$

$$x[n]=\frac{1}{N}\sum_{k=0}^{N-1}X[k]W_N^{-kn} \qquad 0\le n\le N-1$$

4. 按定义式计算时域序列的DFT
5. DFT的复数乘法运算次数和复数加法运算次数
6. DFT和IDFT的矩阵形式（矩阵元素的规律），会用矩阵形式计算DFT
7. DFT、DTFT、Z变换之间的关系
8. 5.3 小节所讲内容画一下DFT与DTFT之间关系的思维导图。

第八次课第五章(1)DFT作业：

5.9(a)

5.9 Determine the N-point DFTs of the following length-N sequences defined for $0 \le n \le N-1$:

(a) $y_a[n] = \alpha^n$,

5.25

5.25 Let $X(e^{j\omega})$ denote the DTFT of the length-9 sequence $x[n] = \{1,\ -3,\ 4,\ -5,\ 7,\ -5,\ 4,\ -3,\ 1\}$.

(a) For the DFT sequence $X_1[k]$, obtained by sampling $X(e^{j\omega})$ at uniform intervals of $\pi/6$ starting from $\omega = 0$, determine the IDFT $x_1[n]$ of $X_1[k]$ without computing $X(e^{j\omega})$ and $X_1[k]$. Can you recover $x[n]$ from $x_1[n]$?

(b) For the DFT sequence $X_2[k]$, obtained by sampling $X(e^{j\omega})$ at uniform intervals of $\pi/4$ starting from $\omega = 0$, determine the IDFT $x_2[n]$ of $X_2[k]$ without computing $X(e^{j\omega})$ and $X_2[k]$. Can you recover $x[n]$ from $x_2[n]$?

第八次课第五章(1)DFT作业：

读书笔记：

根据5.3 小节所讲内容画一下DFT与DTFT之间关系的思维导图。