Laplace transforms

Def. $\mathcal{L}\{f(t)\} = F(s) = \underbrace{\int_0^\infty e^{-st} f(t)\,dt}$ note: after computing integral we get a func of s and not t.

Note: Laplace transform takes a func. as input and outputs a different func.

note: since the laplace transform is defined using an improper integral, in general it will only be defined for certain values of s.

Ex: $\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,dt = \lim_{A\to\infty} \int_0^A e^{-st}\,dt = \lim_{A\to\infty} -\frac{1}{s} e^{-st}\Big|_0^A$

constant function $f(t) \equiv 1$

$\lim_{A\to\infty} \left(\underbrace{-\frac{1}{s} e^{-sA}}_{\substack{0 \\ \text{if } s>0}} + \underbrace{\frac{1}{s} e^0}_{1/s}\right) \quad \therefore \mathcal{L}\{1\} = \frac{1}{s},\ s>0$

Ex: $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \lim_{A\to\infty} \int_0^A e^{(A-s)t}\,dt = \lim_{A\to\infty} \underbrace{\frac{1}{a-s} \cdot e^{(a-s)A}}_{\substack{0 \text{ if } a-s<0 \\ s>a}} - \frac{1}{a-s} e^0$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a},\ s>a$

Ex: $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt \qquad \mathcal{L}\{f(t)\} = F(s)$

$= \lim_{A\to\infty} \int_0^A e^{-st} f'(t)\,dt \qquad u = e^{-st} \quad v = f(t)$

$\qquad du = -se^{-st} \quad dv = f'(t)$

$= \lim_{A\to\infty} \left(\int_0^A e^{-st} f(t)\Big|_0^A + \int_0^A e^{-st} f(t)\,dt \right)$

$= \lim_{A\to\infty} \left(\underbrace{e^{-sA} f(A)}_{\substack{\to 0 \text{ if} \\ s>0}} - \underbrace{e^0 f(0)}_{f(0)}\right) + \underbrace{s \lim_{A\to\infty} \int_0^A e^{-st} f(t)\,dt}_{\mathcal{L}\{f(t)\} = F(s)}$

$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \quad s>0$

$\mathcal{L}\{f''(t)\} = \mathcal{L}\left\{\frac{d}{dt}[f'(t)]\right\}$ we know what $\mathcal{L}\{f'(t)\}$ is.

$= s^2 F(s) - sf(0) - f'(0)$

generalizes to

$\mathcal{L}\{f^n(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - f^{n-1}(0)$

Fact: If f is piecewise continuous on interval $0 \le t \le A$ for any positive A and there exists real constants K, a, M with K and M positive such that $|f(t)| \le Ke^{at}$ when $t > M$, then $L\{f(t)\} = F(s)$ exists for