3 possible cases $x' = Ax$ for a 2×2 Need 2 linearly independent solutions

(1) A has 2 real eigenvectors
gen. soln is of the form
$\vec{x}(t) = c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$ v_n is eigenvector for λ_n

(2) A has 1 real eigenvector
Gen. soln.
$\vec{x}(t) = c_1 \vec{v} e^{\lambda t} + c_2(\vec{v} t e^{\lambda t} + \vec{w} e^{\lambda t})$ $\vec{w}$ is generalized eigenvector for $\vec{v}$

(3) A has 2 complex eigenvectors and values
Gen. soln.
$\vec{x}(t) = c_1 e^{\phi t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t) + c_2 e^{\phi t}(\vec{a}\sin\mu t + \vec{b}\cos\mu t)$
$\vec{v}_1 = \vec{a} + \vec{b}i$ is eigenvector for $\lambda_1 = \phi + \mu i$

3x3 is basically the same need 3 linearly dependent solutions

Best case (easiest)
A has 3 eigenvector → Ex. eigenvalues $-2, 3, -3$
eigenvectors $\begin{bmatrix}-2\\-1\\3\end{bmatrix}, \begin{bmatrix}-3\\-4\\2\end{bmatrix}, \begin{bmatrix}0\\-1\\2\end{bmatrix}$

soln: $\vec{x}(t) = c_1\begin{bmatrix}-2\\-1\\3\end{bmatrix}e^{-2t} + c_2\begin{bmatrix}-3\\-4\\2\end{bmatrix}e^{3t} + c_3\begin{bmatrix}0\\-1\\2\end{bmatrix}e^{-3t}$

Less nice:
A has 2 eigenvectors

Bad case (wont cover)
A has 1 eigenvector

so easy we won't cover
1 eigenvalue 3 eigenvectors

Ex: $x' = \begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}x$ has eigenvalue $\lambda = 3$

$\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} \rightarrow$ $v_1 = $ free, $v_2 = $ free, $v_3 = $ free $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$

3x3 Always has at least 1 real eigenvalue

3x3 EX:

$\vec{x} = Ax$ $\quad A = \begin{bmatrix}1&0&2\\0&-1&0\\-2&0&-3\end{bmatrix}$ $\quad |A-\lambda I| = \begin{bmatrix}1-\lambda&0&2\\0&-1-\lambda&0\\-2&0&-3-\lambda\end{bmatrix}$

$$= -(1-\lambda)\begin{vmatrix}1-\lambda & 2\\ -2 & -3-\lambda\end{vmatrix} = (-1-\lambda)\big((1-\lambda)(-3-\lambda)+4\big)$$

$$=(-1-\lambda)(\lambda^2-\lambda+3\lambda-3+4$$

$$=(-1-\lambda)(\lambda+1)(\lambda+1)$$

$\lambda=-1$ Algebraic multiplicity 3

$\lambda=-1$ $\quad \left[\begin{array}{ccc|c}2&0&2&0\\0&0&0&0\\-2&0&-2&0\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&1&0\\0&0&0&0\\0&0&0&0\end{array}\right]$ $\quad v_1+v_3=0 \quad v_1=-\beta$

$v_2=\alpha$

$v_3=\beta$

$$\begin{bmatrix}-\beta\\ \alpha\\ \beta\end{bmatrix} = \alpha\begin{bmatrix}0\\1\\0\end{bmatrix} + \beta\begin{bmatrix}-1\\0\\1\end{bmatrix}$$ 2 eigenvectors $\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\0\\1\end{bmatrix}$

$\lambda=-1$ has geometric multiplicity 2

have 2 of 3 solutions

$\begin{bmatrix}0\\1\\0\end{bmatrix}e^{-t}$ and $\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-t}$

need a third

we want to find a generalized eigenvector $\vec{w}$

need to check both eigenvectors to find gen.

(there will only be 1)

try $\vec{v}_1 = \begin{bmatrix}0\\1\\0\end{bmatrix}$ $\quad \left[\begin{array}{ccc|c}2&0&2&0\\0&0&0&\underbrace{1}\\-2&0&-2&0\end{array}\right] \to$ no solution $\quad v_1+v_3=-\frac{1}{2} \quad v_1=-\frac{1}{2}-\beta$

$v_2=\alpha$

$v_3=\beta$

$\vec{v}_2 = \begin{bmatrix}-1\\0\\1\end{bmatrix}$ $\quad \left[\begin{array}{ccc|c}2&0&2&-1\\0&0&0&0\\-2&0&-2&1\end{array}\right] \to \left[\begin{array}{ccc|c}1&0&1&-\frac{1}{2}\\0&0&0&0\\0&0&0&0\end{array}\right]$ $\quad \alpha\begin{bmatrix}0\\1\\0\end{bmatrix} + \beta\begin{bmatrix}-1\\0\\1\end{bmatrix} + \begin{bmatrix}-\frac{1}{2}\\0\\0\end{bmatrix}$

$w = \begin{bmatrix}-1/2\\0\\0\end{bmatrix}$

General solution:

$$\vec{x}(t) = C_1\begin{bmatrix}0\\1\\0\end{bmatrix}e^{-t} + C_2\begin{bmatrix}-1\\0\\1\end{bmatrix}e^{-t} + C_3\left(\underbrace{\begin{bmatrix}-1\\0\\1\end{bmatrix}}_{\text{Eigenvector used to find } \vec{w}.} te^{-t} + \begin{bmatrix}-1/2\\0\\0\end{bmatrix}e^{-t}\right)$$

Ex. case 2

$$x' = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & 3 \end{bmatrix} x \qquad |A-\lambda I| = (3-\lambda)\begin{vmatrix} 1-\lambda & -1 \\ 1 & 3-\lambda \end{vmatrix} = (3-\lambda)(\lambda^2-4\lambda+4)$$

$$= (3-\lambda)(2-\lambda)(2-\lambda)$$

$$\lambda = 3, 2$$

eigenvectors $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ for $\lambda=3$ $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ for $\lambda=2$

need generalized eigenvector, will come from an eigenvalue with algebraic multiplicity > 1.

(so $\lambda=2$)

finding gen eigenvector gives $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{x}(t) = C_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{3t} + C_2\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}e^{2t} + C_3\left(\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}te^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}e^{2t}\right)$$

3x3 2 complex eigenvalues 1 real

Eigenvalues: $2, 1+2i, 1-2i$ $\quad \phi = 1 \quad \mu = 2$

Eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1-i \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1+i \\ 1 \end{bmatrix}$ $\quad a = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad b = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$

$$\vec{x}(t) = C_1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}e^{2t} + C_2 e^t\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\cos 2t - \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\sin 2t\right) + C_3 e^t\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\sin 2t + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\cos 2t\right)$$