Exact Eqns.

$M+(N)y'=0$

$N_x(x,y)=M_y(x,y)$

$\psi(x,y)=$ unique terms of $\int M\,dx$ and $\int N\,dy$

add $=C$

Can be made Exact if: $\frac{M_y-N_x}{N}$ is only a function of x.

$\frac{M_y-N_x}{N}=\frac{1}{\mu}\frac{d\mu}{dx}$

or if $\frac{N_x-M_y}{M}$ depends only on y. then

$\frac{d\mu}{dy}=\mu(y)\frac{N_x-M_y}{M}$

$\frac{d\mu}{dx}=\mu(y)\cdot\frac{1}{y}\rightarrow\mu(y)=y$

2nd Order ODE's

$ay''+by'+cy=g(t)$

Ch. eqn $ar^2+br+c=0$

$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

roots: $y_H(t)=$

2 real: $C_1e^{r_1t}+C_2e^{r_2t}$

1 real: $C_1e^{rt}+C_2te^{rt}$

unreal: $C_1e^{\lambda t}\cos(\mu t)+C_2e^{\lambda t}\sin(\mu t)$

$\lambda\rightarrow$ real $\mu\rightarrow$ unreal

$g(t)\rightarrow y_p(t)$

$e^{nt}\rightarrow Ae^{nt}$

$\sin/\cos(nt)\searrow$

$A\sin(nt)+B\cos(nt)$

$t^n\rightarrow At^2+Bt+C$

if $y_p(t)$ is soln of Homo multiply by t

Are solutions fundamental?

find Wronskian

$W[y_1,y_2]=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

fund if determinant is not 0

answer $=C$ implicit

$y(x)=$ answer explicit

Integrating factor

from: $\frac{dy}{dt}+P(t)y=g(t)$

$\mu(t)y=\int\mu(t)g(t)\,dt$

$\mu(t)=e^{\int P(t)dt}$ $(+C!!!)$

PFD

$\frac{1-v}{v^2-4}=\frac{1-v}{(v-2)(v+2)}$

$\frac{A}{v+2}+\frac{B}{v-2}=\frac{1-v}{(v-2)(v+2)}$

$A(v-2)+B(v+2)=1-v$

$Av-2A+vB+2B=1-v$

$\boxed{1=-2A+2B}$ int terms

$\boxed{-1=A+B}$ v terms

3rd and 4th order

Very similar to 2nd order

- find roots of ch. eqn.
- real, complex, repeated roots handled same way

~y_p is found the same way

$y^{(4)}-4y'''+4y''=0$

$r^4-4r^3+4r^2=0$

$r^2(r^2-4r+4)=0$

$r^2(r-2)(r-2)=0$

$r=0,0,2,2$

$y(t)=C_1+C_2t+C_3e^{2t}+C_4te^{2t}$

Need 4 initial conditions

$r^4+2r^2+1=0$

replace $r^2=s$

$s^2+2s+1=0$

$(r^2+1)(r^2+1)=0$

$r^2+1=0$ $r^2+1=0$

$r=\pm i$ $r=\pm i$

$y^{(4)}-y=0$

$r^{(4)}-1=0$

$(r^2-1)(r^2+1)=0$

$(r-1)(r+1)(r^2+1)$

Separable ODE'S

need: $f(y)\frac{dy}{dx}=G(x)$

E.x: $\frac{dy}{dx}=\frac{2x}{y-1}$ integrate both sides

$(y-1)\frac{dy}{dx}=2x$

IBP

$\int u\,dv=uv-\int v\,du$

$\int \ln x\,dx$ $u=\ln x$ $dv=dx$ $du=\frac{1}{x}$ $v=x$

Trig sub: $\tan^2\theta+1=\sec^2\theta$

$x^2+1\rightarrow\tan\theta$

$x^2-1\rightarrow\sec\theta$

$1-x^2\rightarrow\sin\theta$

Newtons law of cooling:

T: temp of obj

M: temp surroundings

t: time

K: proportionality

$\frac{dT}{dt}=K(M-T),\ K>0$

$\int\frac{dT}{M-T}=\int K\,dt$

$-\ln|M-T|=Kt+C$

cooling

$M-T=e^{-Kt-C}$

$T=M+Ae^{-Kt}$

Heating

$T=M-Ae^{-Kt}$

$y'(t)=\frac{y}{10}\left(1-\frac{y}{K}\right)$

1) find gen soln.

2) init pop $=\frac{1}{3}$ cap

find 2x pop

3) sketch soln

find lim $y(t)$

$\frac{dy}{dt}=\frac{y}{10}\left(1-\frac{y}{K}\right)$

$\frac{1}{y\left(1-\frac{y}{K}\right)}=\frac{1}{10}$

PFD

$\left(\frac{1}{y}+\frac{1/K}{1-y/K}\right)\frac{dy}{dt}=\frac{1}{10}$

$\int\frac{1}{y}+\frac{1/K}{1-y/K}\,dy=\int\frac{1}{10}$

$\ln|y|-\ln\left|1-\frac{y}{K}\right|=\frac{t}{10}+C$

$y(0)=K/3$ sol. 4 C

$\ln\left|\frac{K}{3}\right|-\ln|1-1/3|=C$

$\ln\left(\frac{K/3}{2/3}\right)=\ln\left(\frac{K}{2}\right)=C$

find t_1 $y(t_1)=\frac{2K}{3}$

If $\frac{dy}{dx}=f(x,y)$ and $f(x,y)$ can be a func of $\frac{y}{x}$ or $\frac{x}{y}$ it is Homogeneous

$v=\frac{y}{x}$ and $\left[v+x\frac{dv}{dx}=\frac{dy}{dx}\right]$

Autonomous Eqns and Equilibrium Analysis:

Form: $\frac{dy}{dx}=f(y)$

Equilibrium soln the derivative is 0

- find $y=0$ eq pts.
- find signs between
- draw Phase line
- soln curves
- unstable, stable, semi

$\frac{d^2y}{dx^2}=0$ for inflection

- CCU $\frac{d^2y}{dx}>0$
- CCD <0

$\frac{d^2y}{dt^2}=f'(y)\cdot f(y)$

logistic Eqn's

$\frac{dy}{dt}=(r-ay)y$ $K=\frac{r}{a}$

$\frac{dy}{dt}=r\left(1-\frac{y}{K}\right)y$ $y(0)=y_0$

With threshold

$y'=r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y$

$r>0,\ 0<T<K$

$y=0,T,K$

initial pop is $\frac{1}{3}$ cap:

$y(0)=\frac{K}{3}$

Integrals

$\int\sec^2x=\tan x$ $\int\ln x=x\ln|x|-x$

$\int\sec x\tan x=\sec x$ $\int\frac{1}{x}=\ln|x|$

$\int\csc x\cot x=-\csc x$

$\int\csc^2x=-\cot x$ $\int\frac{1}{\sqrt{a^2-u^2}}=\sin^{-1}\left(\frac{u}{a}\right)$

$\int\tan x=\ln|\sec x|$

$\int\sec x=\ln|\sec x+\tan x|$

$\int\frac{1}{a^2+u^2}=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$

$\int\frac{10}{x(1-\frac{x}{C})}dx=10(\ln|x|-\ln|x-C|)$

$\ln|y|-\ln\left|1-\frac{y}{K}\right|=\frac{t}{10}+\ln\left(\frac{K}{2}\right)$

$y(t_1)=\frac{2K}{3}$

$\ln\left(\frac{2K}{3}\right)-\ln\left(1-\frac{2}{3}\right)=\frac{1}{10}t_1+\ln\left(\frac{K}{2}\right)$

$\ln\left(\frac{2K}{3}\right)-\ln\left(\frac{1}{3}\right)-\ln\left(\frac{K}{2}\right)=\frac{1}{10}t_1$

$\ln\left(\frac{2K/3}{\frac{1}{3}\cdot\frac{K}{2}}\right)=\frac{1}{10}t_1$

$\ln(4)=\frac{1}{10}t_1$

$10\ln(4)=t_1$

$\lim_{t\to\infty}y(t)=K$

A mass of 10 kg stretches spring 50cm. mass is acted on by external force of $5\sin(2t)$ N and moves in a medium that imparts a viscous force of 3 N when the speed of the mass is 10 cm/s. If the mass is pulled down 2cm and released, formulate the IVP describing the motion of the mass.

$m = 10\,kg$ $\quad N = \frac{kg \cdot m}{s^2}$ $\quad$ "viscous force" = damping

$g(t) = 5\sin(2t)$ $\quad \gamma \cdot u' \quad \gamma \cdot u' = 3 \quad \gamma = 30$

$\gamma \cdot 0.1 = 3$

$mg = kL$

$u_0 = 2cm = 0.02$

$\underbrace{10}_{kg} \cdot \underbrace{9.8}_{m/s^2} = \underbrace{k}_{kg/s^2} \cdot 0.5$

$V_0 = 0$

$10u'' + 30u' + 196u = 5\sin 2t$

$98 = \frac{k}{2}$

$k = 196$

$u(0) = 0.02$

$u'(0) = 0$ (starts stationary $\therefore V_0 = 0$)

$mu''(t) + \gamma u'(t) + ku(t) = g(t)$

$\hookrightarrow m, \gamma, k$ are all positive

initial conditions would be

$u(0) = u_0$ initial position

$u'(0) = V_0$ initial velocity

m mass

γ damping constant

k spring constant

$g(t)$ external force

Resonance if we have

$mu'' + ku = F\cos(\omega t)$ or $F\sin(\omega t)$

and $\omega = \sqrt{\frac{k}{m}} \leftarrow \omega_0$

Is resonance?

$u'' + 4u = \cos(2t)$ yes

$u'' - 4u = \cos(2t)$ no.

needs non-real roots.

Terminology:

if $g(t) = 0$ oscillation is free (homogeneous)

if $g(t) \neq 0$ oscillation is forced (non homogeneous)

if $\gamma = 0$ oscillation is undamped

if $\gamma \neq 0$ oscillation is damped

if we have undamped free oscillation:

$\hookrightarrow mu'' + ku = 0 \qquad m, k > 0$

soln: $mr^2 + 0r + k = 0$

$$r = \frac{-0 \pm \sqrt{0^2 - 4mk}}{2m} \qquad 0 \pm i\frac{\sqrt{k}}{\sqrt{m}}$$

$$\hookrightarrow u(t) = C_1\cos\left(\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(\sqrt{\frac{k}{m}}t\right)$$

$\sqrt{\frac{k}{m}}$ = frequency ω_0 (Angular)

$\frac{1}{\omega_0} = \sqrt{\frac{m}{k}}$ = period (angular)

Also want amplitude and phase

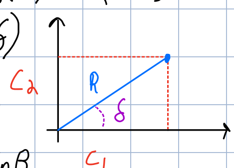
$$u(t) = C_1\cos\left(\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(\sqrt{\frac{k}{m}}t\right)$$

can be:

$$= R\cos(\delta)\cos\left(\sqrt{\frac{k}{m}}t\right) + R\sin(\delta)\sin\left(\sqrt{\frac{k}{m}}t\right)$$

$\cos(\delta) = \frac{C_1}{R} \qquad C_1 = R\cos(\delta)$

$\sin(\delta) = \frac{C_2}{R} \qquad C_2 = R\sin(\delta)$



$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

so

$$R\cos(\delta)\cos\left(\sqrt{\frac{k}{m}}t\right) + R\sin(\delta)\sin\left(\sqrt{\frac{k}{m}}t\right)$$

$$= R\cos\left(\delta - \sqrt{\frac{k}{m}}t\right) \text{ or } \underbrace{R}_{\text{amplitude}}\cos\left(\sqrt{\frac{k}{m}}t - \underbrace{\delta}_{\text{phase}}\right)$$

Damped Free Oscillations

$mu'' + \gamma u' + ku = 0 \qquad m, \gamma, k > 0$

$mr^2 + \gamma r + k = 0$

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

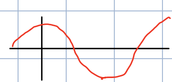
$\divideontimes$ 3 cases:

$\gamma^2 - 4mk > 0$ over damped

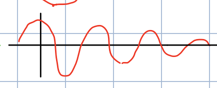
$\gamma^2 - 4mk = 0$ crit. damped

$\gamma^2 - 4mk < 0$ under damped

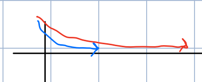
undamped



under damped



over damped (red)



crit. damped (like over damped but goes to 0 faster)

Log/exp rules:

$\ln(ab) = \ln a + \ln b$

$\ln(a/b) = \ln a - \ln b$

$\ln(a^b) = b\ln a$

$\ln(1) = 0$

$\ln(e) = 1$

$\ln(e^b) = b$

$a^x \cdot a^y = a^{x+y}$

$(a^x)^y = a^{xy}$

Ex:

$u'' + 4u = 0$

find: freq, period, amp, phase

freq $= \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2$

Period $= \frac{1}{2}$ $\quad$ for amp and phase we need to solve the IVE first

$r^2 + 4 = 0$

$r = \pm 2i$

$u(t) = C_1\cos(2t) + C_2\sin(2t)$

$4 = C_1 \quad C_2 = -3$

$4\cos(2t) - 3\sin(2t)$

$r^2 = 4^2 + -3^2$

$r = 5 \qquad \delta = \tan^{-1}\left(\frac{-3}{4}\right)$

$g(t)$	$Y_p(t)$
$g(t) = 2t^2e^{-t} + e^{-t} = (2t^2+1)e^{-t}$	$Y_p(t) = At^2e^{-t} + Bte^{-t} + Ce^{t}$
$g(t) = \underbrace{\sin(2t)}_{g_1} + \underbrace{e^{3t}}_{g_2}$	$Y_{p1}(t) = A\sin 2t + B\cos 2t$ $Y_{p2}(t) = Ce^{3t}$
$g(t) = te^{t/2}\cos 3t$	$Y_p(t) = Ate^{t/2}\sin 3t + Bte^{t/2}\cos 3t + Ce^{t/2}\sin 3t + De^{t/2}\cos 3t$ $(t+1)e^{t/2}(\sin 3t + \cos 3t)$ w/o constants
$g(t) = 5\sin(2t) - 3\cos(2)$	$Y_p(t) = A\sin 2t + B\cos 2t$
$g(t) = \underbrace{4\sin 2t}_{g_1} + \underbrace{2\cos 3t}_{g_2}$	must deal with them separately