Systems of ODEs (linear, homogeneous, constant coefficients)

Have form: $\vec{x}' = A\vec{x}$ ($\vec{x}'$: $n\times 1$ vector; A: $n\times n$ matrix; $\vec{x}$: $n\times 1$ vector) $\quad \vec{x} \to \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ etc

Note: x_1, x_2, etc are dependent vars
t is the independent var.
The derivative $\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \\ \vdots \end{bmatrix}$, the der. of each component w.r.t t

First need guess: (seen this for $n=1$)

$x' = ax$ has gen. soln. $x(t) = ce^{at}$

Expect similar:

Guess: $\vec{v}e^{\lambda t}$ where $\vec{v}$ is a vector and λ is a scalar

Plug guess into ODE:

$\lambda\vec{v}e^{\lambda t} = A\vec{v}e^{\lambda t} \to \lambda\vec{v} = A\vec{v}$ need λ to be an eigenvalue with eigenvector $\vec{v}$

Will work perfectly if A has all real eigenvalues and the maximum number of eigenvectors.

EX:

$x' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix} x$

- first find E.Vals and E.Vecs

$\begin{bmatrix} 1-\lambda & 1 \\ 4 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) - 4$
$\lambda^2 - 2\lambda - 3$
$\lambda - 3, -1$

$\lambda = 3 \begin{bmatrix} -2 & 1 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 4 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad V_1 = \frac{1}{2}\alpha$, $V_2 = \alpha$ $\quad \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

(incorrect $\lambda \to$ wont have row all 0's)

$\lambda = -1 \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad V_1 = -\frac{1}{2}\alpha$, $V_2 = \alpha$ $\quad \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Now have Eigenvalue $\lambda = 3$ with Evector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$
and $\lambda = -1$ with $\begin{bmatrix} -1 \\ 2 \end{bmatrix}$

So solutions are $v_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t}$ and $v_2 e^{\lambda_2 t} = \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}$

The general solution for an $n\times n$ system will be linear combinations of n linearly independent* solutions.

* not multiples of eachother

$\therefore$ the gen. soln. to this example is $x(t) = c_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}$

Ex: only has 1 eigenvalue λ in 2×2

2 possibilities:

1) (easy) λ has 2 eigenvectors
2) λ has 1 eigenvector

① $x' = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} x$ has eigenvalue $\lambda = 3$ and eigenvectors $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

gen soln. $x(t) = c_1 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{3t}$

② If we only have 1 Evalue and 1 Evector then we get a solution but need a 2nd (linearly independent)

reasonable (wrong) guess: $Vte^{\lambda t}$

plug in to get $\underbrace{Ve^{\lambda t} + \lambda Vte^{\lambda t} = \lambda Vte^{\lambda t}}_{\text{not true.}}$

need to add a term to guess.

new guess: $\vec{v}te^{\lambda t} + \vec{w}e^{\lambda t}$ where $\vec{w}$ is some unknown vector.

LHS: $\frac{d}{dt}\left[\vec{v}te^{\lambda t} + \vec{w}e^{\lambda t}\right]$

$= \lambda\vec{v}te^{\lambda t} + \vec{v}e^{\lambda t} + \lambda\vec{w}e^{\lambda t}$

RHS: $A(\vec{v}te^{\lambda t} + \vec{w}e^{\lambda t})$

$= A\vec{v}te^{\lambda t} + Awe^{\lambda t}$

$= \lambda Ate^{\lambda t} + A\vec{w}e^{\lambda t}$

$\lambda vte^{\lambda t} + \vec{V}e^{\lambda t} + \lambda\vec{w}e^{\lambda t} = \lambda\vec{v}te^{\lambda t} + A\vec{w}e^{\lambda t}$

$\vec{V}e^{\lambda t} + \lambda\vec{w}e^{\lambda t} = A\vec{w}e^{\lambda t}$

$\vec{V}e^{\lambda t} = A\vec{w}e^{\lambda t} - \lambda\vec{w}e^{\lambda t}$

$\vec{v} = (A - \lambda I)\underbrace{\vec{w}}$ ← generalized eigenvector

note: $(A - \lambda I)\vec{v} = 0$ ← eigenvector

so if A 2×2 has 1 Evalue and 1 Evector then the general soln. is:

$$\vec{x} = c_1 Ve^{\lambda t} + c_2(\vec{v}te^{\lambda t} + \vec{w}e^{\lambda t})$$

Ex:

$$x' = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix} x$$

eigenvalue 3

eigenvector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

gives

$\begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} \rightarrow$ still need 2nd linearly independent soln.

Need generalized eigenvector

soln. to $(A-\lambda I)\vec{w} = \vec{v}$

$\vec{w} \rightarrow$ gen. eigenvector

$\vec{v} \rightarrow$ eigenvector

set last column of system used to find $\hat{v}$ to $\vec{v}$

$$\left[\begin{array}{cc|c} -2 & -2 & -1 \\ 2 & 2 & 1 \end{array}\right] = \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \omega_1 + \omega_2 = 1/2 \qquad \omega_1 = -\alpha + 1/2$$

$$\omega_2 = \text{free} = \alpha$$

$$\vec{\omega} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -\alpha + \frac{1}{2} \\ \alpha \end{bmatrix} = \alpha \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}^{*}}_{\vec{v}} + \underbrace{\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}^{*}}_{\vec{\omega}}$$

* should always be same as $\vec{v}$ (you got something wrong otherwise)

* can't multiply by something to get better number. Can only change α.

so $\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ is a generalized eigenvector for $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

gives us $\begin{bmatrix} -1 \\ 1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{3t}$

$\therefore$ gen. soln is

$$\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{3t} \right)$$

What if A (2x2) has complex eigenvalues?

recall: • complex eigenvalues come in pairs
• complex eigenvalues have complex eigenvectors
• complex eigenvectors come in pairs $\vec{v} = \vec{a} \pm \vec{b}i$

We expect a solution of the form $\vec{v}e^{\lambda t}$
but we want a real-value solution.

First consider $\lambda = \phi + \mu i$ and $\vec{v} = \vec{a} + \vec{b}i$

$(\vec{a}+\vec{b}i)e^{(\phi+\mu i)t}$. want to get real and imaginary parts. they will be individual solns.

$$(\vec{a}+\vec{b}i)e^{\phi t + \mu i t}$$

$$= (\vec{a}+\vec{b}i)e^{\phi t}e^{\mu i t}$$ use Eulers formula

$$e^{\phi t}(\vec{a}+\vec{b}i)(\cos\mu t + i\sin\mu t)$$ foil

$$e^{\phi t}(\vec{a}\cos\mu t + \vec{a}\sin\mu t i + b\cos\mu t - b\sin\mu t)$$

$$\underbrace{e^{\phi t}(\vec{a}\cos\mu t - b\sin\mu t)}_{\text{real part}} + \underbrace{e^{\phi t}(\vec{a}\sin\mu t + \vec{b}\cos\mu t)i}_{\text{imaginary part}}$$

Note: only need to do to one eigenvector/value.
other will give almost exactly the same thing.
(because $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$)
Ends up with imaginary part multiplied by -1
which can be included in C_2.

$$\vec{x}(t) = C_1 e^{\phi t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t) + C_2 e^{\phi t}(\vec{a}\sin\mu t + \vec{b}\cos\mu t)$$

EX: $x' = \begin{bmatrix} 3 & -2 \\ 4 & -1 \end{bmatrix} x$ find gen. soln.

$|A - \lambda I| = (3-\lambda)(-1-\lambda) + 8$

$= \lambda^2 - 2\lambda + 5$

$\lambda = \dfrac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\lambda = 1+2i$ $\left[\begin{array}{cc|c} 2-2i & -2 & 0 \\ 4 & -2-2i & 0 \end{array}\right] \approx \left[\begin{array}{cc|c} 4 & -2-2i & 0 \\ 2-2i & -2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{1}{2}i & 0 \\ 2-2i & -2 & 0 \end{array}\right]$

add $-(2-2i)$ times R_1 to R_2 $-(2-2i)\left(-\frac{1}{2}-\frac{1}{2}i\right)-2 = 0$

$\left[\begin{array}{cc|c} 1 & -\frac{1}{2}-\frac{1}{2}i & 0 \\ 0 & 0 & 0 \end{array}\right]$

$v_1 + \left(-\frac{1}{2}-\frac{1}{2}i\right) = 0$

$v_1 = \frac{1}{2}+\frac{1}{2}i$

$\alpha \begin{bmatrix} \frac{1}{2}+\frac{1}{2}i \\ 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1+i \\ 2 \end{bmatrix}$

Eigenvector for $\lambda = 1+2i$

Gen. soln.

$\vec{x}(t) = C_1 e^{\phi t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t) + C_2 e^{\phi t}(\vec{a}\sin\mu t + \vec{b}\cos\mu t)$

Plug in $\begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \underset{a}{\begin{bmatrix} 1 \\ 2 \end{bmatrix}} + \underset{b}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} i$

$\phi = 1$

$\mu = 2$

$\vec{x}(t) = C_1 e^{t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\cos 2t - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin 2t\right) + C_2 e^{t}\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\sin 2t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\cos 2t\right)$