Terminology:

$ay'' + by' + cy = 0$ is called "homogeneous"
(unrelated to other homogeneous ODE)

$ay'' + by' + cy = g(t)$ non Homogeneous

How to solve non-Homogeneous

- First find solution to homogeneous eqn.
  $ay'' + by' + cy = 0$
- Then we'll find 1 particular solution to non homogeneous version. (solution is "random" and does not correspond to initial conditions.
- Add the homogeneous solution to particular solution of non homogeneous soln to get general soln of non homogeneous eqn.

How to do step 2 depends on what $g(t)$ is.

$y'' + 2y' - 3y = 2e^{t/2}$ Here $g(t)$ is an exponential

$r^2 + 2r - 3 = 0$

$(r+3)(r-1)$

$y_H(t) = C_1 e^{-3t} + C_2 e^t$

for step 2 we make a guess for particular func
if $g(t)$ is exponential our guess is that the particular solution $y_p(t)$ is also exponential with same exponent.

Guess $y_p(t) = Ae^{t/2}$
find A so that $y_p(t)$ solves the ODE

$y_p'(t) = \frac{A}{2}e^{t/2}$ $y_p''(t) = \frac{A}{4}e^{t/2}$

$\frac{A}{4}e^{t/2} + 2\frac{A}{2}e^{t/2} - 3Ae^{t/2} = 2e^{t/2}$

$e^{t/2}\left(\frac{A}{4} + A - 3A\right)$

$\frac{A}{4} + A - 3A = 2$

$\frac{A}{4} - 2A = 2$

$-\frac{2A}{4} = 2$ $A = -\frac{8}{7}$

$y_p(t) = -\frac{8}{7}e^{t/2}$

$y_H(t) = C_1 e^{-3t} + C_2 e^t$

$y_p(t) + y_H(t) =$

$y(t) = C_1 e^{-3t} + C_2 e^t - \frac{8}{7}e^{t/2}$

- If $g(t)$ is exponential $\to Y_p(t)$ is exponential
- If $g(t)$ is sin or cos $\to Y_p(t)$ is sum of a sin and a cos
- If $g(t)$ is a polynomial $\to Y_p(t)$ is polynomial of same degree

If $g(t)$ is sum of functions we can deal with them seperately.

Ex. $y'' + 2y' - 3y = \underbrace{2e^{t/2}} + \underbrace{4\sin(2t)} + \underbrace{t^2 - 1}$

$$Y_{p1}(t) = Ae^{t/2}$$

$$Y_{p2}(t) = A\sin(2t) + B\cos(2t)$$

$$Y_{p3}(t) = At^2 + Bt + C$$

then solve for each $Y_p(t)$ ignoring other terms so solve for $Y_{p1}(t)$, calculate $Y_{p1}'(t), Y_{p1}''(t)$ and plug into $ay'' + by' + cy = d_1(t)$

so $y(t) = \underbrace{C_1e^t + C_2e^{-3t}}_{Y_H(t)} \underbrace{- \frac{8}{7}e^{t/2}}_{Y_{p1}(t)} \underbrace{- \frac{28}{65}\sin(2t) - \frac{16}{65}\cos(2t)}_{Y_{p2}(t)} \underbrace{- \frac{1}{3}t^2 - \frac{4}{9}t - \frac{5}{27}}_{Y_{p3}(t)}$

If $g(t)$ is the product of functions we can't split it up, our guess for $Y_p(t)$ will look like the product of our guesses for the factors

Sums can be split up because derivatives of sums can be split up. derivatives of products can't be easily split up.

Ex. $y'' + 2y' - 3y = 3e^t \cos(t)$

$Y_p(t)$ should be the product of an exponential for e^t and the sum of a sine and a cos for $\cos(t)$

so $Y_p(t) = e^t(\sin(t) + \cos(t))$ but with constants multiplied

$$Y_p(t) = Ae^t \sin(t) + Be^t \cos(t) = y_p(t)$$

$$y_p'(t) = Ae^t \sin(t) + Ae^t \cos(t) + Be^t \cos(t) - Be^t \sin(t)$$
$$= (A-B)e^t \sin(t) + (A+B)e^t \cos(t)$$

$$y_p''(t) = (A-B)e^t \sin(t) + (A-B)e^t \cos(t) + (A+B)e^t \cos(t) - (A+B)e^t \sin(t)$$
$$= (A-B-A-B)e^t \sin(t) + (A-B+A+B)e^t \cos t$$
$$= 2Be^t \sin(t) + 2Ae^t \cos(t)$$

Plug into ODE

$$-2Be^t \sin(t) + 2Ae^t \cos(t) + 2(A-B)e^t \sin(t) + 2(A+B)e^t \cos(t) - 3Ae^t \sin(t)$$
$$-3Be^t \cos(t) = 3e^t \cos(t)$$

linear system

$$\begin{cases} 2A + 2A + 2B - 3B = 3 & \text{cos terms} \\ -2B + 2A - 2B - 3A = 0 & \text{sin terms} \end{cases}$$

$4A - B = 3$ $\qquad$ $4(-4B) - B = 3$

$-A - 4B = 0$ $\qquad$ $-17B = 3$

$\qquad\qquad\qquad\qquad$ $B = -\frac{3}{17}$

$-4B = A$

$\qquad\qquad\qquad\qquad$ $A = \frac{12}{17}$

$$y(t) = \underbrace{C_1 e^t + C_2 e^t}_{Y_H} + \underbrace{\frac{12}{17} e^t \sin(t) - \frac{3}{17} e^t \cos(t)}_{Y_p(t)}$$

Examples of $Y_p(t)$ based on $g(t)$

$g(t)$	$Y_p(t)$
$g(t) = 2t^2e^{-t} + e^{-t}$ $= (2t^2+1)e^{-t}$	$Y_p(t) = At^2e^{-t} + Bte^{-t} + Ce^{t}$
$g(t) = \underbrace{\sin(2t)}_{g_1} + \underbrace{e^{3t}}_{g_2}$	$Y_{p_1}(t) = A\sin 2t + B\cos 2t$ $Y_{p_2}(t) = Ce^{3t}$
$g(t) = te^{t/2}\cos 3t$	$Y_p(t) = Ate^{t/2}\sin 3t + Bte^{t/2}\cos 3t$ $+ Ce^{t/2}\sin 3t + De^{t/2}\cos 3t$ $(t+1)e^{t/2}(\sin 3t + \cos 3t)$ w/ constants missing
$g(t) =$ $5\sin(2t) - 3\cos(2)$	$Y_p(t) = A\sin 2t + B\cos 2t$
$g(t) =$ $\underbrace{4\sin 2t}_{g_1} + \underbrace{2\cos 3t}_{g_2}$	must deal with them separately

Ex. $y'' + 2y' - 3y = 2e^{-3t}$

$$Y_p(t) = Ce^{-3t}$$
$$y_p'(t) = -3Ce^{-3t}$$
$$y_p''(t) = 9Ce^{-3t}$$

$$9Ce^{-3t} - 6Ce^{-3t} - 3Ce^{-3t} = 2e^{-3t}$$
$$0 = 2e^{-3t}$$

our guess was wrong because Ce^{-3t} is a solution to the homogeneous equation.

If this happens we need a different "guess"

multiply previous (incorrect) guess by t.

new guess is $Y_p(t) = Cte^{-3t}$ $\quad y_p'(t) = -3Cte^{-3t} + Ce^{-3t}$ $\quad y_p''(t) = 9Cte^{-3t} - 6Ce^{-3t}$

plug in $9Cte^{-3t} - 6Ce^{-3t} - 6Cte^{-3t} + 2Ce^{-3t} - 3Ate^{-3t} = 2e^{-3t}$

$$0 - 4Ce^{-3t} = 2e^{-3t}$$
$$-4A = 2$$
$$A = -\frac{1}{2}$$

Ex. find solution of initial value problem

$y'' - 4y = 6e^{2t}$ $\quad y(0)=1 \quad y'(0)=0$

find $Y_H(t)$ to the homogeneous equation:

$r^2 - 4r = 0$

$(r+2)(r-2)$ $\quad Y_H(t) = C_1 e^{-2t} + C_2 e^{2t}$

$r = -2, 2$

$Y_p(t) = Ae^{2t}$ wont work b/c it is part of homogeneous solution

$Y_p(t) = Ate^{2t}$

$Y_p'(t) = 2Ate^{2t} + Ae^{2t}$

$Y''_p(t) = 4Ate^{2t} + 2Ae^{2t} + 2Ae^{2t}$

$4Ate^{2t} + 4Ae^{2t} - 4Ate^{2t} = 6e^{2t}$

$4Ae^{2t} = 6e^{2t}$

$4A = 6$

$A = \cancel{6}^{3}/\cancel{4}_{2}$

gen soln:

$y(t) = C_1 e^{-2t} + C_2 e^{2t} + \frac{3}{2} te^{2t}$

$y(0)=1$

$1 = C_1 + C_2$

$C_1 = 7/8$

$y'(0)=0$

$y'(t) = -2C_1 e^{-2t} + 2C_2 e^{2t} + 3te^{2t} + \frac{3}{2} e^{2t}$

$0 = -2C_1 + 2C_2 + \frac{3}{2}$

$0 = -2(1-C_2) + 2C_2 + \frac{3}{2}$

$C_2 = \frac{1}{8}$

$y(t) = \frac{7}{8} e^{-2t} + \frac{1}{8} e^{2t} + \frac{3}{2} te^{2t}$