Name Answer Key CSUID# ______________ Class Time ______

Problem	#1	#2	#3	#4	#5	#6	#7	Total
Score								

Bonus (2 points) for submission of your Cheat Sheet (with your name, CSUID#, and class time on it). It will be returned to you. We just need it to cover the 1st page for your privacy.

Exam Policies

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You may use **one** letter-size 2-sided Cheat Sheet for this exam.

Good luck!

(20 points) *Problem 1.* Determine whether the following statements are correct.
True (T) or False (F). Circle your answer (2 points for each item, no partial credit).

(i) (T) (F) The constant function $y(x) = \pi$ is a solution of the ODE $y'(x) = x \sin^2 y$.

(ii) (T) (F) The ODE $(6y^2 - x^2 + 3) + (3x^2 - 2xy + 2)y' = 0$ is exact.

(iii) (T) (F) For the initial value problem (IVP) $x'(t) = 2x + e^t$, $x(0) = 0$, the solution is $x_{\text{IVP}}(t) = e^{2t} - e^t$.

(iv) (T) (F) The ODE $x'(t) + \sin(x(t)) = 0$ is linear.

(v) (T) (F) The autonomous ODE $y'(t) = y^2(1-y)^2$ has only two equilibria: $y = 0$, $y = 1$.

(vi) (T) (F) For the 2nd order linear homogeneous ODE $y'' + 4y' + 13y = 0$, the two functions $\{y_1 = e^{-3t}\cos(2t), y_2 = e^{-3t}\sin(2t)\}$ form a fundamental set of solutions.

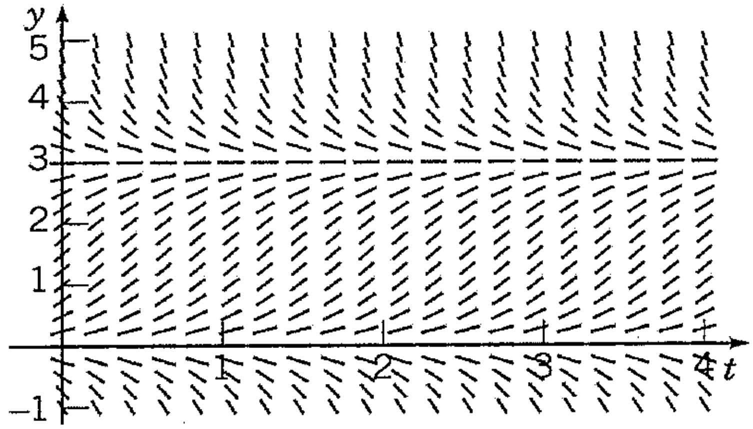
(vii) (T) (F) For the 2nd order linear homogeneous ODE $y'' - 5y' + 6y = 0$, the two functions $y_1(t) = e^{2t}, y_2(t) = e^{3t}$ are solutions and their Wronskian $W[y_1, y_2]$ is never zero.

(viii) (T) (F) The ODE $y'' + y = \cos t$ models a forced oscillation that incurs the *resonance* phenomenon.

(ix) (T) (F) For the 3rd order linear nonhomogeneous ODE $y''' - 4y'' + 3y = e^t$, we can find a particular solution in the form $y_p(t) = Ae^t$.

(x) (T) (F) For the 4th order linear homogeneous ODE $y^{(4)} + 18y'' + 81y = 0$, the general solution is $y(t) = (A_0 + A_1 t)\cos(3t) + (B_0 + B_1 t)\sin(3t)$.

(15 points) *Problem 2.* The direction field of an autonomous ODE $y'(t) = f(y)$ is shown below.



(i) Identify the differential equation that corresponds to the direction field. Briefly explain why.

(a) $y' = y(y-3)$; (b) $y' = y(3-y)$; (c) $y' = y - 3$; (d) $y' = y$.

(ii) Based on the direction field, identify two equilibrium points (equilibria) of the ODE.

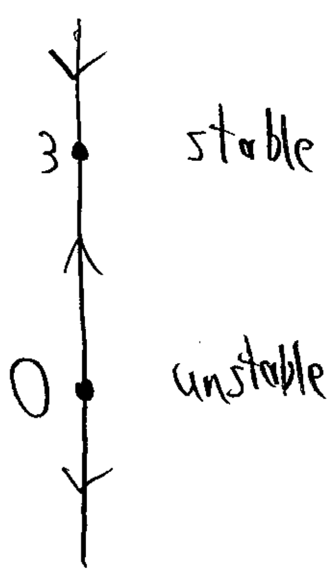
(iii) Draw arrows on ***the phase line*** and indicate stability of the two equilibrium points.

(i) equilibrium points at 0 and 3

function is increasing between 0 and 3 (option a is decreasing between 0 and 3)

(ii) $y=0$ and $y=3$

(iii)



(15 points) *Problem 3.*

(i) A 1st order ODE is written as $P(x, y)\,dx + Q(x, y)\,dy = 0$. Write down the condition for the ODE to be exact.

(ii) Consider the ODE $(e^x y^2 + \sin y)\,dx + (2e^x y + x\cos y)\,dy = 0$. Determine whether it is exact.

(iii) For the ODE in Part (ii), if it is exact, find the general solution.

(i) $P_y = Q_x$

(ii) $P_y = 2e^x y + \cos(y)$

$Q_x = 2e^x y + \cos(y)$

yes it is exact

(iii) $\int e^x y^2 + \sin(y)\,dx$

$= e^x y^2 + x\sin(y) + g(y)$

$\int 2e^x y + x\cos(y)\,dy$

$= e^x y^2 + x\sin(y) + h(x)$

$$\boxed{e^x y^2 + x\sin(y) = C}$$

(15 points) *Problem 4.* Consider the population model $y'(t) = \frac{y}{10}\left(1 - \frac{y}{K}\right)$ with capacity $K > 0$.

(i) Find the general solution of the ODE.

(ii) Assume the initial population is one-third of the capacity. Find the time at which the population has doubled.

(iii) Sketch the solution in Part (ii) on the ty-plane. Find the limit $\lim_{t\to+\infty} y(t)$.

(i) $\frac{dy}{dt} = \frac{y}{10}\left(1-\frac{y}{K}\right)$

$\frac{1}{y\left(1-\frac{y}{K}\right)}\frac{dy}{dt} = \frac{1}{10}$

partial fractions: $\frac{1}{y\left(1-\frac{y}{K}\right)} = \frac{A}{y} + \frac{B}{1-\frac{y}{K}}$

$1 = A\left(1-\frac{y}{K}\right) + By$

$1 = A - \frac{A}{K}y + By$

$\begin{cases} 1 = A \\ 0 = -\frac{A}{K} + B \end{cases} \to B = \frac{1}{K}$

$\left(\frac{1}{y} + \frac{1/K}{1-y/K}\right)\frac{dy}{dt} = \frac{1}{10}$

$\int \frac{1}{y} + \frac{1/K}{1-y/K}\, dy = \int \frac{1}{10}\, dt$

$\ln|y| - \ln\left|1-\frac{y}{K}\right| = \frac{1}{10}t + C$

(ii) $y(0) = \frac{K}{3}$ solve for C

$\ln\left|\frac{K}{3}\right| - \ln\left|1-\frac{1}{3}\right| = \frac{1}{10}(0) + C$

$\ln\left(\frac{K}{3}\right) - \ln\left(\frac{2}{3}\right) = C$

$\ln\left(\frac{K/3}{2/3}\right) = \ln\left(\frac{K}{2}\right) = C$

Now solve for t_1 where $y(t_1) = \frac{2K}{3}$

$\frac{dy}{dt} = \frac{y}{10}\left(1-\frac{y}{K}\right)$

$\frac{1}{y\left(1-\frac{y}{K}\right)} = \frac{1}{10}$

PFD

$\left(\frac{1}{y} + \frac{1/K}{1-y/K}\right)\frac{dy}{dt} = \frac{1}{10}$

$\int \frac{1}{y} + \frac{1/K}{1-y/K}\, dy = \int \frac{1}{10}$

$\ln|y| - \ln\left|1-\frac{y}{K}\right| = \frac{t}{10} + C$

$y(0) = K/3$ sol. 4 C

$\ln\left|\frac{K}{3}\right| - \ln|1-1/3| = C$

$\ln\left(\frac{K/3}{2/3}\right) = \ln\left(\frac{K}{2}\right) = C$

find t_1 $y(t_1) = \frac{2K}{3}$

continued next page:

Extra space (for Problem 4 or other problems).

$$\ln|y| - \ln\left|1-\frac{y}{K}\right| = \frac{1}{10}t + \ln\left(\frac{K}{2}\right)$$

plug in $y(t_1) = \frac{2K}{3}$

$$\ln\left(\frac{2K}{3}\right) - \ln\left(1-\frac{2}{3}\right) = \frac{1}{10}t_1 + \ln\left(\frac{K}{2}\right)$$

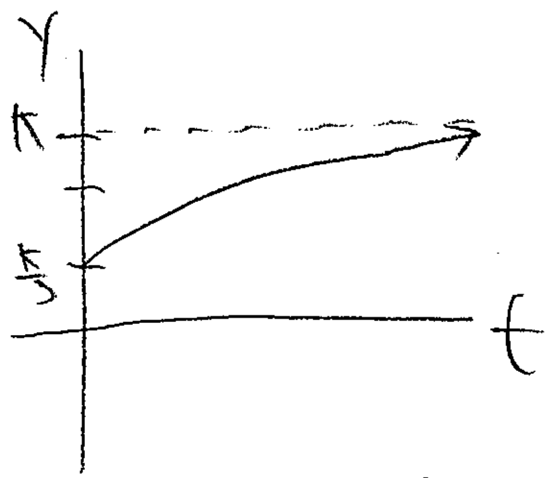
$$\ln\left(\frac{2K}{3}\right) - \ln\left(\frac{1}{3}\right) - \ln\left(\frac{K}{2}\right) = \frac{1}{10}t_1$$

$$\ln\left(\frac{2K/3}{\frac{1}{3}\cdot\frac{K}{2}}\right) = \frac{1}{10}t_1$$

$$\ln(4) = \frac{1}{10}t_1$$

$$\boxed{10\ln(4) = t_1}$$

(iii)



$$\lim_{t\to\infty} y(t) = K$$

$\ln|y| - \ln\left|1-\frac{y}{K}\right| = \frac{t}{10} + \ln\left(\frac{K}{2}\right)$

$y(t_1) = \frac{2K}{3}$

$\ln\left(\frac{2K}{3}\right) - \ln\left(1-\frac{2}{3}\right) = \frac{1}{10}t_1 + \ln\left(\frac{K}{2}\right)$

$\ln\left(\frac{2K}{3}\right) - \ln\left(\frac{1}{3}\right) - \ln\left(\frac{K}{2}\right) = \frac{1}{10}t_1$

$\ln\left(\frac{2t/3}{\frac{1}{3}\cdot\frac{K}{2}}\right) = \frac{1}{10}t_1$

$\ln(4) = \frac{1}{10}t_1$

$10\ln(4) = t_1$

$\lim_{t\to\infty} y(t) = K$

$y'(t) = \frac{y}{10}\left(1-\frac{y}{K}\right)$

1) find gen soln.
2) init pop $= \frac{1}{3}$ cap find 2x pop
3) sketch soln find $\lim y(t)$

(15 points) *Problem 5.*

(i) Find the general solution of the homogeneous ODE $x'' - 4x' + 13x = 0$.

(ii) Find the general solution of the homogeneous ODE $x'' - 4x' + 3x = 0$.

(iii) Find a particular solution of the nonhomogeneous ODE $x'' - 4x' + 3x = e^t$.

(i) $r^2 - 4r + 13 = 0$

$$r = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$$\boxed{x(t) = c_1 e^{2t}\cos(3t) + c_2 e^{2t}\sin(3t)}$$

(ii) $r^2 - 4r + 3 = 0$

$(r-3)(r-1) = 0$ $\quad r = 3, 1$

$$\boxed{x(t) = c_1 e^{3t} + c_2 e^{t}}$$

(iii) $\cancel{Y_p(t) = Ae^t}$ won't work

$Y_p(t) = Ate^t$

$Y_p'(t) = Ate^t + Ae^t$

$Y_p''(t) = Ate^t + 2Ae^t$

plug in: $Ate^t + 2Ae^t - 4Ate^t - 4Ae^t + 3Ate^t = e^t$

$-2Ae^t = e^t$

$A = -\frac{1}{2}$

$$\boxed{Y_p(t) = -\frac{1}{2}te^t}$$

(10 points) *Problem 6.* Consider an initial value problem (IVP) $\begin{cases} Q'(t) + \dfrac{r}{100}Q(t) = \dfrac{r}{4} \\ Q(0) = Q_0 \end{cases}$

with two positive parameters r, Q_0.

(i) Find the solution $Q_{\text{IVP}}(t)$ to the IVP. (ii) Find the limit $\lim_{t \to +\infty} Q_{\text{IVP}}(t)$.

integrating factor:

$$\mu(t) = e^{\int \frac{r}{100} dt} = e^{\frac{r}{100}t}$$

$$\mu(t) Q(t) = \int \mu(t) g(t) dt$$

$$e^{\frac{r}{100}t} Q = \int e^{\frac{r}{100}t} \cdot \frac{r}{4}$$

$$u = \frac{r}{100}t$$

$$du = \frac{r}{100}dt \rightarrow 25 du = \frac{r}{4} dt$$

$$e^{\frac{rt}{100}} \cdot Q = 25 \int e^{u} du$$

$$Q e^{\frac{rt}{100}} = 25 e^{\frac{rt}{100}} + C$$

$$Q(t) = 25 + Ce^{-\frac{rt}{100}}$$

$$Q(0) = Q_0 \rightarrow Q_0 = 25 + C \rightarrow C = Q_0 - 25$$

$$\boxed{Q_{\text{IVP}}(t) = 25 + (Q_0 - 25)e^{-\frac{rt}{100}}}$$

(ii) $\boxed{\lim_{t \to \infty} Q_{\text{IVP}}(t) = 25}$

(10 points) *Problem 7.* Consider the 2nd order ODE $x'' + 4x = \sin(t)$.

(i) Does the ODE model a free oscillation? Is the oscillation over-damped?

(ii) Find a particular solution of the above nonhomogeneous ODE.

(iii) Find the solution of the ODE that satisfies the initial conditions $x(0) = 1,\ x'(0) = 0$.

(i) No, forced oscillation

No, not overdamped (in fact, not damped at all)

(ii) $Y_p(t) = A\cos(t) + B\sin(t)$

$Y_p'(t) = -A\sin(t) + B\cos(t)$

$Y_p''(t) = -A\cos(t) - B\sin(t)$

plug in:

$-A\cos(t) - B\sin(t) + 4A\cos(t) + 4B\sin(t) = \sin(t)$

$$\begin{cases} -A + 4A = 0 & \to A = 0 \\ -B + 4B = 1 & \to 3B = 1 \to B = 1/3 \end{cases}$$

$$\boxed{Y_p(t) = \frac{1}{3}\sin(t)}$$

(iii) Need $Y_H(t)$: $r^2 + 4 = 0 \to r = \pm 2i$

$Y_H = C_1\cos(2t) + C_2\sin(2t)$

$Y(t) = C_1\cos(2t) + C_2\sin(2t) + \frac{1}{3}\sin(t) \to Y(0) = C_1 = 1$

$Y'(t) = -2C_1\sin(2t) + 2C_2\cos(2t) + \frac{1}{3}\cos(t)$

$\to Y'(0) = 2C_2 + \frac{1}{3} = 0 \to 2C_2 = -1/3$

$C_2 = -1/6$

$$\boxed{Y(t) = \cos(2t) - \frac{1}{6}\sin(2t) + \frac{1}{3}\sin(t)}$$