ColoState Fall 2023 MATH 340 Final Exam (Mon.12/11/2023)

Name: ______________________________ CSUID# __________________ Lab Time: ______

Problem	#1	#2	#3	#4	#5	#6	Total
Score							

Bonus (2 points) for submission of your Cheat Sheet (with your name and CSUID# on it).

Exam Policy

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You may use **two** letter-size 2-sided Cheat Sheets for this exam.

Good luck!

(20 points) *Problem 1.* Determine whether the following statements are correct.
True (T) or False (F). Circle your answer (2 points for each item, no partial credit).

(i) (T) (F) For the 1st order ODE $\dfrac{dy}{dx} = \dfrac{-x + \frac{x^3}{6}}{y}$, the general solution is $x^2 - \dfrac{x^4}{12} + y^2 = C$.

(ii) (T) (F) The ODE $\theta''(t) + 0.1\,\theta'(t) + 9.8\,\sin(\theta(t)) = 0$ is 2nd order and linear.

(iii) (T) (F) The ODE $y'' + 2y' + 5y = \cos(t)$ models a free oscillation.

(iv) (T) (F) For a dim-3 ODE system $\mathbf{x}'(t) = A\,\mathbf{x}$, if matrix A is real, then it has at least one real eigenvalue.

(v) (T) (F) $\mathbf{y}(t) = e^{-t}\left(\begin{bmatrix} -1 \\ 0 \end{bmatrix} + t\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right)$ is a solution of the ODE system $\mathbf{x}'(t) = \begin{bmatrix} -3 & 4 \\ -1 & 1 \end{bmatrix}\mathbf{x}$.

(vi) (T) (F) For a dim-2 ODE system $\mathbf{x}'(t) = A\,\mathbf{x}$, if matrix A is real and $(1-2i)$ is an eigenvalue, then all trajectories spiral towards the origin.

(vii) (T) (F) For a dim-2 ODE system $\mathbf{x}'(t) = A\,\mathbf{x}$, if matrix A is real and $(-2i)$ is an eigenvalue, then all trajectories circle around the origin.

(viii) (T) (F) For the ODE system $\mathbf{x}'(t) = A\,\mathbf{x}$, the exponential matrix e^{tA} is invertible for any t.

(ix) (T) (F) If $\mathcal{L}\{y(t)\}(s) = Y(s)$, then $\mathcal{L}\{e^{ct}y(t)\}(s) = Y(s+c)$.

(x) (T) (F) The ODE system $\begin{cases} x'(t) = y \\ y'(t) = -\sin(x) \end{cases}$ has infinitely many equilibria.

(10 points) *Problem 2.* Consider a planar nonlinear ODE system $\begin{cases} x'(t) = x + x^2 + y^2, \\ y'(t) = y - xy. \end{cases}$

(i) Find its **two equilibria** (by either solving algebraic equations or examining the given graphics).

(ii) For each equilibrium, calculate its Jacobian matrix and the eigenvalues. Then determine its type (saddle-point, nodal source, center, etc.) and stability.

$x'(t) = x(1+x) + y^2$

$y'(t) = y(1-x)$

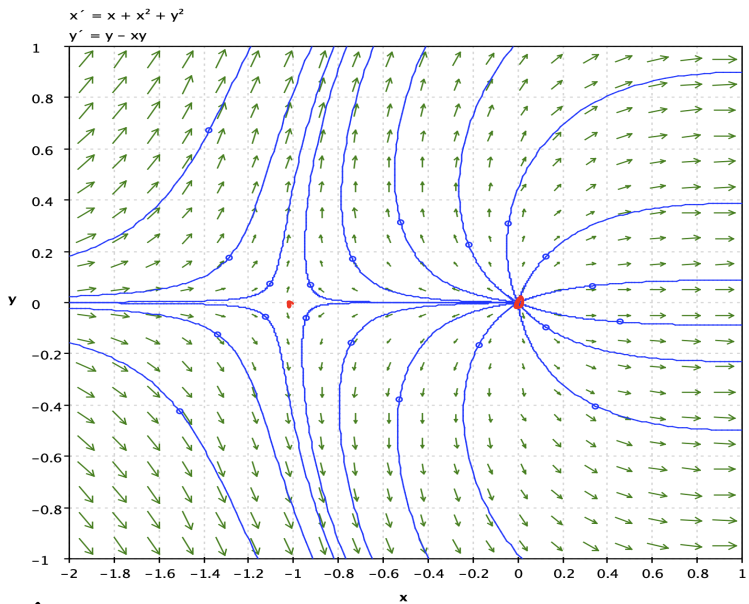
$(0,0)$

$-1(1-1) + y^2 = 0$

$y^2 = 0$

$y = 0$

$(-1,0)$



$$J(x,y) = \begin{bmatrix} 1+2x & 2y \\ -y & 1-x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$(1-\lambda)(1-\lambda)$

$\lambda = 1$

proper node source at $(0,0)$ (unstable)

$$J(-1,0) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$$

$(-1-\lambda)(2-\lambda)$

$\lambda = -1, 2$

saddle pt a $(-1,0)$ (unstable)

(15 points) *Problem 3.* A tank initially contains 100 litres (L) of pure water. A mixture containing a concentration of γ (g/L) of salt enters the tank at a rate of 2 (L/min). The well-stirred mixture leaves the tank at the same rate.

(i) Establish an ODE and an initial condition for the amount of salt $Q(t)$ in the tank.

(ii) Find an expression in terms of γ for $Q(t)$.

(iii) Find the limiting amount of salt in the tank as $t \to \infty$.

Hints: *Sketching a tank with info for the entry, exit, and within-the-tank will be helpful.*

$Q(t)$ $\quad \frac{Q}{100} \cdot 2$

$Q'(t) = 2\gamma - \frac{Q}{50}$

(20 points) *Problem 4.* Consider an ODE system $\mathbf{x}'(t) = A\,\mathbf{x}$ with $A = \begin{bmatrix} -3 & 1 \\ -1 & -1 \end{bmatrix}$.

(i) Find the eigenvalues and eigenvectors of matrix A, and generalized eigenvectors if needed.

(ii) Find the general solution of the ODE system.

(iii) Find the particular solution satisfying the initial condition $\mathbf{x}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$(-3-\lambda)(-1-\lambda)+1$

$\lambda^2+4\lambda+4$

$(\lambda+2)^2$

$\lambda=-2$

$\lambda=-2 \quad \begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad m \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$$x(t) = C_1 e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + C_2 e^{-2t} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$C_1 = 1 \quad C_2 = 2$

$$x(t) = e^{-2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 2 e^{-2t} \left(\begin{bmatrix} 1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$$

Extra space for Problem 4, if needed.

(20 points) *Problem 5.* Given an ODE system $\mathbf{x}'(t) = A\,\mathbf{x}(t) + \mathbf{b}(t)$, $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, $\mathbf{b}(t) = \begin{bmatrix} e^{-t} \\ t \end{bmatrix}$.

(i) Find the eigenvalues and eigenvectors of matrix A.

(ii) It is known matrix A is diagonalizable. Find an order-2 diagonal matrix D and an order-2 matrix P such that $P^{-1}AP = D$. Write down P^{-1} explicitly.

(iii) Perform change of variables $\mathbf{x}(t) = P\,\mathbf{y}(t)$ to rewrite the original ODE system as $\mathbf{y}'(t) = D\,\mathbf{y}(t) + \mathbf{c}(t)$, where D is the diagonal matrix in (ii), $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$. Find $\mathbf{c}(t)$.

(iv) Solve the two single ODEs obtained in Part (iii).

Bonus 5 points. Find the general solution of the original ODE system.

$(-2-\lambda)(-2-\lambda)-1$

$\lambda^2+4\lambda+3$

$(\lambda+1)(\lambda+3)$

$\lambda=-1, -3$

$\lambda=-1 \quad \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} \quad \propto \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\lambda=-3 \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad \propto \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Extra space for Problem 5.

(15 points) *Problem 6.* Apply both Laplace and inverse Laplace transforms to solve the IVP: $y'' + 2y' + 5y = 8e^t,\ y(0) = 0,\ y'(0) = 0.$

$$s^2Y(s) - \underbrace{sY(0)}_{0} - \underbrace{Y'(0)}_{0} + 2sY(s) - \underbrace{2Y(0)}_{0} + 5Y(s) = \frac{8}{s-1}$$

$$s^2Y(s) + 2sY(s) + 5Y(s) = \frac{8}{s-1} \qquad (s+1)^2 + 4$$

$$Y(s)(s^2 + 2s + 5) = \frac{8}{s-1}$$

$$\frac{8}{(s-1)(s^2+2s+5)} = \frac{A}{(s-1)} + \frac{Bs+C}{(s^2+2s+5)} \qquad (Bs+c)(s-1)$$

$$Bs^2 + Cs - Bs - C$$

$$8 = As^2 + A2s + A5 + Bs^2 + Cs - Bs - C$$

$8 = A5 - C$

$0 = A2 + C - B$

$0 = A + B$

$A = -B$

$0 = A3 + C$

$C = -A3$

$8 = A5 + A3$

$8 = A8$

$A = 1$

$B = -1$

$C = -3$

$$Y(s) = \frac{1}{(s-1)} + \frac{-s-3}{(s+1)^2+4} = \frac{1}{(s-1)} - \frac{(s+3-2)+2}{(s+1)^2+4} = \frac{1}{(s-1)} - \left(\frac{s+1}{(s+1)^2+4} + \frac{2}{(s+1)^2+4}\right)$$

$$= \frac{1}{(s-1)} - \frac{s+1}{(s+1)^2+4} - \frac{2}{(s+1)^2+4}$$

$$= e^t - e^{-t}\cos(2t) - e^{-t}\sin(2t)$$