Name ____________________ CSUID# ______________ Class Time _______

Problem	#1	#2	#3	#4	#5	#6	#7	Total
Score								

Bonus (2 points) for submission of your Cheat Sheet (with your name, CSUID#, and class time on it). It will be returned to you. We just need it to cover the 1st page for your privacy.

Exam Policies

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You may use **one** letter-size 2-sided Cheat Sheet for this exam.

Good luck!

(20 points) *Problem 1.* Determine whether the following statements are correct.
True (T) or False (F). Circle your answer (2 points for each item, no partial credit).

(i) (T) (F) If matrix A is invertible, then 0 is an eigenvalue.

(ii) (T) (F) If 2 is an eigenvalue of an invertible matrix A, then $\frac{1}{2}$ is an eigenvalue of A^{-1}.

(iii) (T) $\boxed{\text{(F)}}$ If an eigenvalue has algebraic multiplicity 2, then it has surely two linearly independent eigenvectors.

(iv) (T) (F) If an order-2 real matrix A has one complex eigenvalue, then it must have a second complex eigenvalue.

(v) (T) (F) If an order-3 real matrix already has a pair of conjugate complex eigenvalues, then the 3rd eigenvalue must be real.

(vi) (T) (F) For an ODE system $\mathbf{x}'(t) = A\mathbf{x}$, if matrix A is real and $\mathbf{z}(t)$ is a complex-valued solution, then $\frac{\mathbf{z}(t)-\overline{\mathbf{z}(t)}}{2}$ is a real-valued solution.

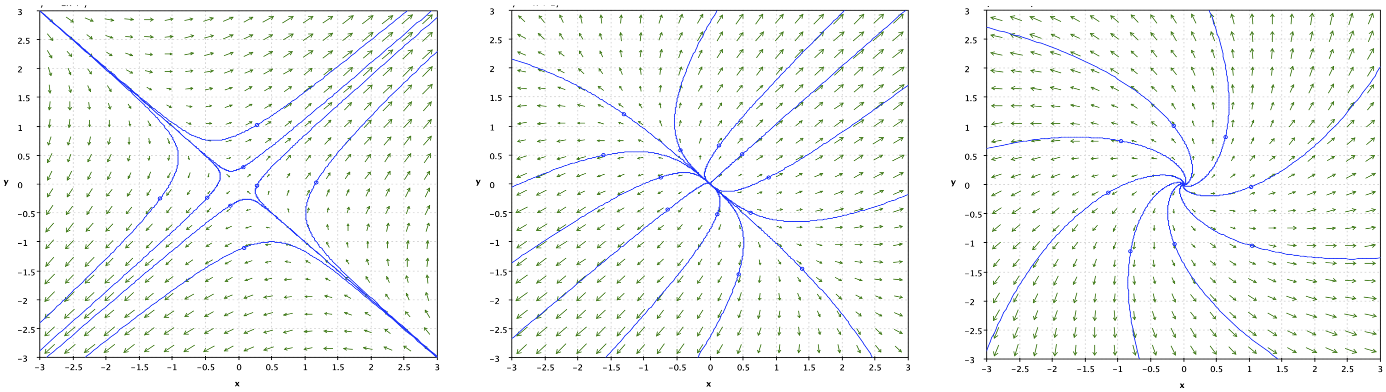
(vii) (T) (F) For a dim-2 ODE system $\mathbf{x}'(t) = A\mathbf{x}$, if $-2, 3$ are the eigenvalues of A, then $\lim_{t\to+\infty} \mathbf{x}(t) = \mathbf{0}$ for any solution $\mathbf{x}(t)$.

(viii) (T) (F) For a dim-2 ODE system $\mathbf{x}'(t) = A\mathbf{x}$, if $-2, 3$ are the eigenvalues of A, then the origin is a saddle point.

(ix) (T) (F) For a dim-3 ODE system $\mathbf{x}'(t) = A\mathbf{x}$, if $\mathbf{u} = [1, 0, -1]^T$ (a dim-3 constant vector), then $\mathbf{x}(t) = \mathtt{exp}(tA)\,\mathbf{u}$ is a solution of the ODE system.

(x) (T) (F) If $\mathbf{x}_1(t), \mathbf{x}_2(t)$ are two linearly independent solutions of a dim-2 ODE system $\mathbf{x}'(t) = A\mathbf{x}$, then the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2]$ is never 0.

(10 points) *Problem 2.* Examine the following figure.



Left | Middle | Right

saddle pt | nodal source | spiral source

Examine also these three dim-2 linear ODE systems $\mathbf{x}'(t) = A\,\mathbf{x}$ with

$$(a)\ \ A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad (b)\ \ A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}, \qquad (c)\ \ A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Answer the following questions.

(i) What is the type of the phase portrait in ***the left panel***?
Circle your choice from the list shown below.

Nodal source   *Nodal sink*   *Spiral source*   *Spiral sink*   *Saddle-point*

(ii) Find the eigenvalues of all three matrices.

(iii) Find the ODE system that corresponds to ***the left panel*** in the above figure.
Circle your choice.   (a)   (b)   (c)

A) $\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

$(2-\lambda)(2-\lambda)-1$

$\lambda^2-4\lambda+3$

$(\lambda-3)(\lambda-1)$

$\lambda=3,1$

B) $\begin{bmatrix} -2-\lambda & -1 \\ 1 & -2-\lambda \end{bmatrix}$

$(-2-\lambda)(-2-\lambda)+1$

$\lambda^2+4\lambda+5$

$\dfrac{-4\pm\sqrt{16-20}}{2}$

$\lambda=-2\pm i$

C) $\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix}$

$(1-\lambda)(1-\lambda)-4$

$\lambda^2-2\lambda-3$

$(\lambda-3)(\lambda+1)$

$\lambda=3,-1$

(15 points) *Problem 3.* Consider the ODE system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, with $A = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

(i) Find the eigenvalues and eigenvectors of A.

(ii) Write down the general solution of the ODE system.

(iii) Find the particular solution $\mathbf{x}_p(t)$ satisfying the initial condition $\mathbf{x}_p(0) = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$.

$$\begin{bmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{bmatrix}$$

$(1-\lambda)(1-\lambda)-4$

$\lambda^2-2\lambda-3$

$(\lambda-3)(\lambda+1)$

$\lambda = 3, -1$

$\lambda = 3 \quad \begin{bmatrix} -2 & 4 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$

$V_1 = 2V_2 \qquad \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$V_2 = \alpha$

$\lambda = -1 \quad \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

$V_1 = -2V_2 \qquad \alpha \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

$V_2 = \alpha$

$$\vec{x}(t) = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}$$

$$\begin{bmatrix} 4 \\ 0 \end{bmatrix} = C_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + C_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & -2 & 4 \\ 1 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \qquad C_1 = 1, \quad C_2 = -1$$

$$\vec{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} e^{-t}$$

(15 points) *Problem 4.* Consider the ODE system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$.

(i) Find the eigenvalues of A and the algebraic multiplicities.

(ii) Find the associated eigenvector(s) and (if necessary) generalized eigenvectors.

(iii) Write down the general solution of the ODE system.

$$\begin{bmatrix} 3-\lambda & -4 \\ 1 & -1-\lambda \end{bmatrix}$$

$$(3-\lambda)(-1-\lambda)+4$$

$$\lambda^2-2\lambda+1$$

$$(\lambda-1)(\lambda-1)$$

$$\lambda=1$$

Algebraic multiplicity 2

$$\lambda=1 \quad \begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

$$v_1 = 2\alpha \qquad \alpha\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$v_2 = \alpha$$

$$\begin{bmatrix} 2 & -4 & 2 \\ 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$v_1 = 2v_2 + 1 \qquad \alpha\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$
$$v_2 = \alpha$$

$$\vec{x}(t) = C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^t + C_2e^t\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

Extra space (for Problem 4 or other problems).

(15 points) *Problem 5.* Consider the ODE system $\mathbf{x}'(t) = A\mathbf{x}$ with $A = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix}$.

Hints: The eigenvalues of A are complex.

(i) Find the eigenvalues and eigenvectors of A.

(ii) Write down a ***real-valued general solution*** for the ODE system.

$$\begin{bmatrix} -2-\lambda & -1 \\ 1 & -2-\lambda \end{bmatrix}$$

$$(-2-\lambda)(-2-\lambda)+1$$

$$\lambda^2+4\lambda+5$$

$$\frac{-4 \pm \sqrt{16-20}}{2}$$

$$\lambda = -2 \pm i$$

$$-2-(-2-i)$$

$$0+i$$

$$\lambda = -2-i$$

$$\begin{bmatrix} i & -1 \\ 1 & i \end{bmatrix}$$

$$\begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \to \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

$$V_1 = -iV_2$$

$$V_2 = \alpha$$

$$\alpha \begin{bmatrix} -i \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} i$$

$$\vec{x}(t) = C_1 e^{-2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} -1 \\ 0 \end{bmatrix}\sin t\right) + C_2 e^{-2t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\sin t + \begin{bmatrix} -1 \\ 0 \end{bmatrix}\cos t\right)$$

(15 points) *Problem 6.* Consider the ODE system $\mathbf{x}'(t) = A\mathbf{x}(t)$, where $A = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$. It is known that A has one single eigenvalue and one double eigenvalue.

(i) Find the single eigenvalue, an associated eigenvector, and then a solution for the ODE system.

(ii) Find the double eigenvalue, an associated eigenvector, and then a solution for the ODE system.

(iii) For the double eigenvalue, find a ***generalized eigenvector***, then use it to get another solution for the ODE system.

$$|A-\lambda I| = \begin{bmatrix} 2-\lambda & 0 & 0 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{bmatrix}$$

$$(-1-\lambda)(2-\lambda)(2-\lambda)$$

$$\lambda = -1, 2$$

$$\lambda = -1 \quad \begin{bmatrix} 3 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$V_1 = 0$

$V_2 = 0$

$V_3 = \alpha$

$$\alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad c_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{-t}$$

$$\lambda = 2 \quad \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$V_1 = 0$

$V_2 = \alpha$

$V_3 = 0$

$$\alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^{2t}$$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & -3 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$V_1 = 1$

$V_2 = \alpha$

$V_3 = 0$

$$\alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$c_2 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} t e^{2t} \right)$$

(10 points) *Problem 7.* Consider an ODE $u'' + 0.25u' + 4u = 2\cos(3t)$ along with initial conditions $u(0) = 1$, $u'(0) = -2$.

(i) Convert the problem to a 1st-order ODE system with a corresponding initial condition.

(ii) Is the new ODE system linear? Is it homogeneous?

(iii) If the new ODE system is in the form $\mathbf{x}'(t) = A\mathbf{x} + \mathbf{b}(t)$, find the eigenvalues of matrix A.

$x_1 = u \qquad x_1' = u' = x_2$

$x_2 = u' \qquad x_2' = u''$

$$x' = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} x + \begin{bmatrix} 0 \\ 2\cos 3t \end{bmatrix}$$

$$x' = \begin{bmatrix} 0 & 1 \\ -4 & -\frac{1}{4} \end{bmatrix} x + \begin{bmatrix} 0 \\ 2\cos 3t \end{bmatrix}$$

$$x(0) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

is linear

is not homogeneous

$$\begin{bmatrix} -\lambda & 1 \\ -4 & -\frac{1}{4}-\lambda \end{bmatrix}$$

$(-\lambda)(-\frac{1}{4}-\lambda)+4$

$\lambda^2 + \frac{\lambda}{4} + 4$

$$\frac{-\frac{1}{4} \pm \sqrt{\frac{1}{16} - 16}}{2}$$

$$-\frac{1}{8} \pm \frac{\sqrt{\frac{1}{16} - 16}}{2}$$