Autonomous Eqn.s and Equilibrium Analysis

Def: An ODE is Autonomous if it can be written in the form $\frac{dy}{dx} = f(y)$
i.e. the independent variable does not appear explicitly.

An equilibrium solution of an Autonomous ODE is a soln where the derivative is 0.
(so value of y does not change)

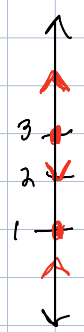
In equilibrium problems we'll have 2 diagrams we draw:
- Phase line
- solution curves

E.x. $\frac{dy}{dt} = (1-y)(3-y)$

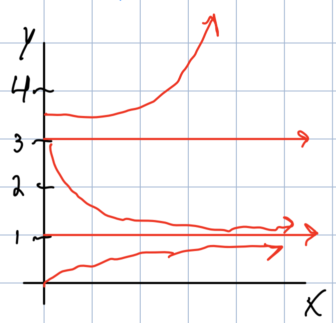
- find all equilibrium solutions.

$\hookrightarrow y = 1, 3$

Phase line



solution curves



equilibrium solns are horizontal lines.
inbetween equilibrium draw 1 solution curve in correct direction

Next need to determine if $\frac{dy}{dt}$ is positive or negative between equilibrium points.

	$y<1$	$1<y<3$	$y>3$
$(1-y)$	+	−	−
$(3-y)$	+	+	−
	+	−	+

Equilibrium solutions will be classified as:
- stable $\longrightarrow$ is the asymptote. solns approach equilibrium on both sides
- unstable $\longrightarrow$ solutions go away on both sides.
- semi stable $\rightarrow$ solutions approach on 1 side.

Concavity: can be found by looking at 2nd derivative

CCU: 2nd derivative is >0
CCd: 2nd derivative is <0

We have $\frac{dy}{dt} = f(y)$, want to look at $\frac{d^2y}{dt^2}$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{dy}{dt}\right] = \frac{d}{dt}\left[f(y)\right] \rightarrow \text{chain rule } f'(y)\frac{dy}{dt}$$

$$\frac{d^2y}{dt^2} = f'(y) \cdot f(y)$$

$$\frac{dy}{dt} = (1-y)(3-y)$$

$3 - 4y + y^2$

want to find concavity

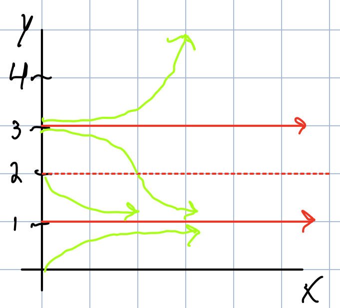
and derivative is 0 at inflection pts and equilibrium pts.

$$f'(y) = -4+2y \qquad \frac{d^2y}{dt^2} = (-4+2y)(1-y)(3-y)$$

$\frac{d^2y}{dt^2} = 2, 1, 3$

inflection pt

solution curves



determine +/- between equilibrium/inflection pts.

	1	2	3	
+	−	−	−	$(1-y)$
+	+	+	−	$(3-y)$
−	−	+	+	$(-4+2y)$
−	+	−	+	

[Logistic Equation → population dynamics.

Basic population model: exponential growth.

$$\frac{dy}{dt} = ry$$

population grows exponentially with no other factors.

solution is $y(t) = Ce^{rt}$

[more realistic model

$$\frac{dy}{dt} = (r - ay)y$$

as y increases, growth rate decreases

usually

$$\underbrace{\frac{dy}{dt} = r\left(1 - \frac{y}{K}\right)y}_{\text{logistic eqn}}$$

renamed $K = \frac{r}{a}$ → carrying capacity

Equilibrium analysis $\quad y(0) = y_0$ where $r > 0$ is the "intrinsic growth rate"

$$0 = r\left(1 - \frac{y}{K}\right)y$$

equilibrium: $y = 0$, $y = K$

look for pos/neg growth.

	0		K	
$1-\frac{y}{K}$	+	+	−	
y	−	+	+	
	−	+	−	
	$y\downarrow$	$\uparrow$	$\downarrow$	

concavity:

$$\frac{d^2y}{dt^2} = f'(y)f(y)$$

$$f(y) = ry - \frac{r}{K}y^2$$

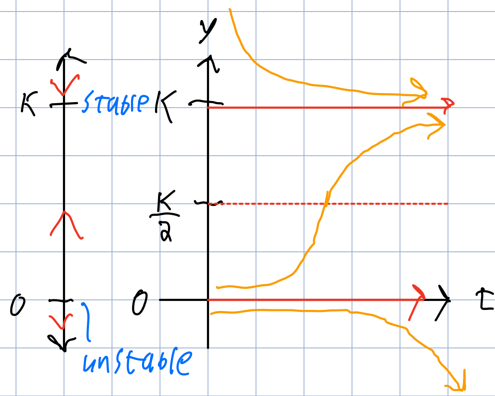
$$f' = r - \frac{2ry}{K}$$

$$\frac{d^2y}{dt^2} = r\left(1-\frac{y}{K}\right)y \cdot \left(1-\frac{2}{K}y\right)r$$

$$0 = \quad '' \qquad\qquad ''$$

$$y = \underbrace{0, K}_{E.Q.}, \underbrace{\frac{K}{2}}_{\text{inflection}}$$

	0	$\frac{K}{2}$	K	
+	+	−	−	$1-\frac{2}{K}y$
+	+	+	−	$1-\frac{y}{K}$
−	+	+	+	y
−	+	−	+	
CCD	CCU	CCD	CCU	



At what value of y is the population increasing the fastest?

at $y = \frac{K}{2}$

that gives max value of $f(y)$ (which $= \frac{dP}{t}$)

what is the solution?

$$y' = r\left(1-\frac{y}{K}\right)y$$

$$\frac{1}{\left(1-\frac{y}{K}\right)y}\frac{dy}{dt} = r$$

$$\underbrace{\int \frac{1}{\left(1-\frac{y}{K}\right)y}}_{PFD} dy = \int r\,dt$$

$$\frac{1}{\left(1-\frac{y}{K}\right)y} = \frac{A}{1-\frac{y}{K}} , \frac{B}{y}$$

$$0y + 1 = Ay + B\left(1-\frac{y}{K}\right)$$

$$1 = B$$

$$0 = A - \frac{B}{K} \qquad 0 = A - \frac{1}{K}$$

$$A = \frac{1}{K}$$

$$\int \frac{\frac{1}{K}}{1-y/K}dy + \int \frac{1}{y}dy = \int r\,dt$$

$$u = 1-\frac{y}{K}$$

$$du = -\frac{1}{K}dy$$

$$-\int \frac{1}{u}du$$

$$-\ln|u| + \ln|y| - rt = C$$

$$-\ln\left|1-\frac{y}{K}\right| + \ln|y| - rt = C$$

$$e^{-\ln|1-\frac{y}{K}|}\, e^{\ln|y|}\, e^{-rt} = C$$

$$\frac{y}{\left(1-\frac{y}{K}\right)e^{rt}} = C$$

$$y = Ce^{rt}\left(1-\frac{y}{K}\right) = Ce^{rt} - \frac{y}{K}Ce^{rt}$$

$$y + \frac{y}{K}\cdot Ce^{rt} = Ce^{rt}$$

$$y\left(1+\frac{Ce^{rt}}{K}\right) = Ce^{rt} \qquad y = \frac{Ce^{rt}}{1+\frac{Ce^{rt}}{K}}$$

Logistic Eqn:

$$y' = r\left(1-\frac{y}{K}\right)y$$

LE with threshold

$$y' = r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y \qquad r>0,\ 0<T<K$$

Equilibriums:

$$0 = -r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y$$

$$y = 0,\ T,\ K)$$

← difficult to find y'' and 0s. Probably wont be asked.

check where y' is -/+

	$y<0$	$0<y<T$	$T<y<K$	$y>K$
y	−	+	+	+
$1-\frac{y}{T}$	+	+	+	−
$1-y/K$	+	+	−	−
$-r$	−	−	−	−
	+	−	+	−