Moving into working with problems that have 1 independent var. and multiple dependent vars. (2 to 3)

Systems of 1st order linear ODE's

- covering Linear Algebra basics

- Matricies:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 3 & 5 \\ 4 & \frac{1}{2} & 2 \end{bmatrix}$$

3x3 matrix

row
column

notation:
A - capitol for matrix iD
i, j, k - lowercase for individual enteries

$A = (a_{ij})$ $\quad a_{11} = 1 \quad a_{12} = 2$
$a_{21} = -1 \quad a_{22} = 3$

entry of A in ith row jth column

- will mostly see 2 kinds of matrices:
  - Square - rows = columns
  - Vectors - column vectors: n x 1 matrix ← mainly looking at
    - row vectors: 1 x n matrix
    - (calc 3 vector knowledge still applies)

Ex. dot product $\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 2 + 12 + 30 = 44$ ← scalar (not vector)

- Properties of matricies
  - only equal if every entry is same

$$\begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} \neq \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}$$

  - the zero matrix is the n x m matrix of all zeroes

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(is bold)

  - the identity matrix has 1s on diagonal 0s otherwise

"diagonal" → top left to bottom right

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Sometimes written I_n to mean n x n identity.

  - Addition and Subtraction

$$A + B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 5 \\ 1 & 5 \end{bmatrix}$$

Note: • A and B must be same size
• Addition easy, multiplication hard.

↳ Matrix Multiplication

$$AB = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} (2)(-1)+(1)(0) & (2)(4)+(1)(2) \\ (1)(-1)+(3)(0) & (1)(4)+(3)(2) \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ -1 & 10 \end{bmatrix}$$

to multiply A and B to get : the C_{ij} entry
take dot product of i_{th} row of A and j_{th} column of B.

Note: - don't need A and B to be same size.
Num rows in A needs to equal num columns in B.
Ex: A 3x2 B 2x3

- If A and B are square, they must be the same size.

- Matrix multiplication is not commutative.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 & 1 \\ 0 & 4 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 4 \end{bmatrix} = \begin{bmatrix} (2)(-1)+(3)(0) & \\ & \end{bmatrix}$$

- if you multiply row vector by column vector, you get same thing as dot product

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix} = 1\cdot 3 + 2\cdot 4 + 3\cdot(-2) = 5$$

- if you multiply $m \times n$ matrix with $n \times q$ matrix the product will be an $m \times q$ matrix
3x2 times 2x4 gives 3x4

- you can multiply a Matrix by a Scalar

$$\alpha A = \alpha \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} \alpha 3 & \alpha 2 \\ \alpha 1 & \alpha 4 \end{bmatrix}$$

- multiplying a matrix by a column vector is really just multiplying 2 matrices but the second has 1 column.

$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & 0 & -1 \\ 4 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1\cdot 2 + 3\cdot 2 + 1\cdot 1 \\ 2\cdot 2 + 0\cdot 2 + (-1)\cdot 1 \\ 4\cdot 2 + 1\cdot 2 + 1\cdot 1 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ 11 \end{bmatrix}$$

Row Reduction

used for:
- solving linear systems
- determining whether vectors are linearly independent.
- finding inverse matricies

How to row reduce a matrix

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

[3 operations]

① swap any 2 rows $\begin{bmatrix} 4 & 5 & 6 \\ 1 & 2 & 3 \\ 7 & 8 & 9 \end{bmatrix}$

② Multiply any row by a scalar. $\begin{bmatrix} 4 & 5 & 6 \\ -3 & -6 & -9 \\ 7 & 8 & 9 \end{bmatrix}$

③ add a [scalar] multiple of a row to another row. *can be 1

$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 7 & 9 \\ 7 & 8 & 9 \end{bmatrix}$ row 1 added to row 2.

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$ -5 times r1 added to r2.

When we row reduce a matrix we use the 3 row operations to convert the matrix to reduced row echelon form. [RREF]

[RREF]

- All rows with 0 entries are at the bottom
- The leading entry (left most) of every row, called the pivot, is to the right of every row above.
- The leading entry in every non-zero row is 1.
- Each column containing a leading 1 has zero's in all it's other enteries.

E.X. $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 5 \\ -1 & 2 & 1 & -1 \end{bmatrix}$ convert to RREF

① Get a 1 in top left entry.

② Get a 0 in every entry below that. [using third elementary row operation]

-2 times R1 to R2

(2a) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ -1 & 2 & 1 & -1 \end{bmatrix}$

add R_1 to R_3

(2b) $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 3 & 2 & -1 \end{bmatrix}$

③ Get a 1 on diagonal of next column. (if possible). [Multiply a row by a scalar]

$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & 0 & 1 \\ 0 & 3 & 2 & -1 \end{bmatrix}$ multiply R_2 by -1 $\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & 2 & -1 \end{bmatrix}$

④ Get zero's below the new leading 1. (use 3rd elem. row op. to add row with leading 1 to row 1 below) add -3 times $R2$ to $R3$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 3 & 2 & -1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{bmatrix}$$

5. repeat steps 3 and 4 for next row.

multiply $R3$ by $1/2$

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 2 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

6. after every row has leading 1 (or is all zeroes)
get zeroes above all leading 1s.
(doesn't always matter but start right to left)

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{add } -1 \text{ times } R_3 \text{ to } R1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \xrightarrow{\text{add } -1 \text{ times } R_2 \text{ to } R_1} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

Result is in RREF: $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

Other Examples of RREF:

$$\begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Interpreting the RREF

$\begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 5 \\ -1 & 2 & 1 & -1 \end{bmatrix} \rightarrow$ Can interpret this matrix representing a system of equations, where the last column is the right hand side of the eqns. and the other columns are the coefficients of the vars. on the left hand side.

"Augmented matrix"

$$\begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 + x_2 + 2x_3 = 5 \\ -x_1 + 2x_2 + x_3 = -1 \end{cases} \equiv \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 2 & 5 \\ -1 & 2 & 1 & -1 \end{array}\right] \qquad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{bmatrix} \rightarrow \text{says soln. is } \begin{matrix} x_1 = 1 \\ x_2 = -1 \\ x_3 = 2 \end{matrix}$$

A Matrix equation (or system) is often written

$A\vec{x} = \vec{b}$ ⟵ $n \times 1$ vector

$\vec{x}$: $n \times 1$ vector

A: $n \times n$ matrix

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 2 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \end{bmatrix} \equiv \begin{cases} x_1 + x_2 + x_3 = 2 \\ 2x_1 + x_2 + 2x_3 = 5 \\ -x_1 + 2x_2 + x_3 = -1 \end{cases}$$