ColoState Fall 2024 MATH 340 Exam 1 (Thu.09/26, 5-6:50pm)

Name ______________________ CSUID# ________________ Class Time ________

Problem	#1	#2	#3	#4	#5	#6	#7	Total
Score								

Bonus (2 points) for submission of your Cheat Sheet (with your name, CSUID#, and class time on it). It will be returned to you. We just need it to cover the 1st page for your privacy.

Exam Policies

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You may use **one** letter-size 2-sided Cheat Sheet for this exam.

Good luck!

(20 points) *Problem 1.* Determine whether the following statements are correct. True (T) or False (F). Circle your answer (2 points for each item, no partial credit).

(i) (T) (F) The constant function $y(x) = \pi$ is a solution of the ODE $y'(x) = x\,\sin^2 y$.

(ii) (T) (F) The ODE $(6y^2 - x^2 + 3) + (3x^2 - 2xy + 2)y' = 0$ is exact. $M_y = N_x$? no

$12y$ $6x - 2y$

(iii) (T) (F) For the initial value problem (IVP) $x'(t) = 2x + e^t$, $x(0) = 0$, the solution is $x_{\text{IVP}}(t) = e^{2t} - e^t$.

(iv) (T) (F) The ODE $x'(t) + \sin(x(t)) = 0$ is linear.

(v) (T) (F) The autonomous ODE $y'(t) = y^2(1-y)^2$ has only two equilibria: $y = 0$, $y = 1$.

(vi) (T) (F) For the 2nd order linear homogeneous ODE $y'' + 4y' + 13y = 0$, the two functions $\{y_1 = e^{-3t}\cos(2t), y_2 = e^{-3t}\sin(2t)\}$ form a fundamental set of solutions. $\frac{-4 \pm \sqrt{16-52}}{2}$ $\frac{-4 \pm 6i}{2}$ $-2 \pm 3i$

(vii) (T) (F) For the 2nd order linear homogeneous ODE $y'' - 5y' + 6y = 0$, the two functions $y_1(t) = e^{2t}, y_2(t) = e^{3t}$ are solutions and their Wronskian $W[y_1, y_2]$ is never zero. $\frac{5 \pm \sqrt{25-24}}{2}$ $\frac{5 \pm 1}{2} < \begin{matrix} 3 \\ 2 \end{matrix}$

(viii) (T) (F) The ODE $y'' + y = \cos t$ models a forced oscillation that incurs the *resonance* phenomenon.

(ix) (T) (F) For the 3rd order linear nonhomogeneous ODE $y''' - 4y'' + 3y = e^t$, we can find a particular solution in the form $y_p(t) = Ae^t$. $Ae^t - 4Ae^t + 3Ae^t = e^t$ ×

(x) (T) (F) For the 4th order linear homogeneous ODE $y^{(4)} + 18y'' + 81y = 0$, the general solution is $y(t) = (A_0 + A_1 t)\cos(3t) + (B_0 + B_1 t)\sin(3t)$.

$r^4 + 18r^2 + 81 = 0$

$s^2 + 18s + 81 = 0$

$\frac{-18 \pm \sqrt{18^2 - 324}}{2}$ $\frac{-18 \pm \sqrt{324-324}}{2}$

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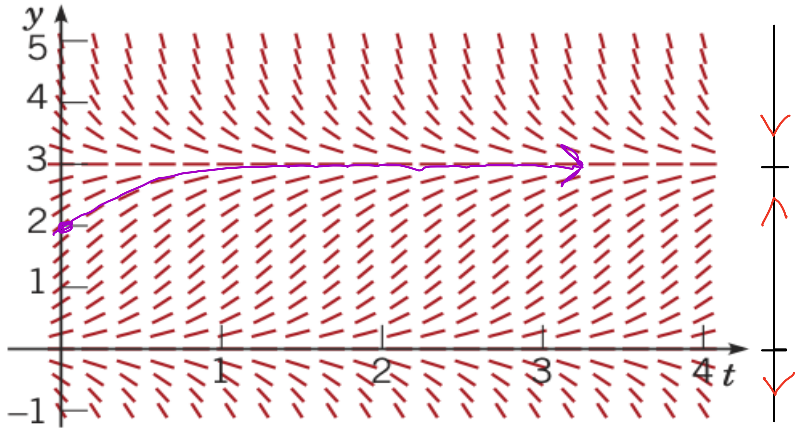
$\begin{bmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{bmatrix} = (e^{2t})(3e^{3t}) - (e^{3t})(2e^{2t})$

$3e^{5t} - 2e^{5t} = e^{5t} \neq 0$

(15 points) *Problem 2.* The direction field of an autonomous ODE $y'(t) = f(y)$ is shown below.

$y=3$

$y=0$



$3-y=0$

$3=y$

$y-3=0$

$y=0$

(i) Identify the differential equation that corresponds to the direction field. Briefly explain why.
(a) $y' = y(y-3)$; [(b) $y' = y(3-y)$;] (c) $y' = y - 3$; (d) $y' = y$.

(ii) Based on the direction field, identify two equilibrium points (equilibria) of the ODE. $y=3,0$

(iii) Draw arrows on ***the phase line*** and indicate stability of the two equilibrium points.

$y=0$ unstable $\qquad$ $y=3$ stable

−1	0	2	3	4	
−		+		+	y
−		−		+	$y-3$
+		−		+	

−1	0	2	3	4	
−		+		+	y
+		+		−	$3-y$
−		+		−	

(15 points) *Problem 3.*

(i) A 1st order ODE is written as $P(x, y)\, dx + Q(x, y)\, dy = 0$. Write down the condition for the ODE to be exact.

(ii) Consider the ODE $(\overset{M}{e^x y^2 + \sin y})\, dx + (\overset{N}{2e^x y + x \cos y})\, dy = 0$. Determine whether it is exact.

(iii) For the ODE in Part (ii), if it is exact, find the general solution.

Ode is exact iff $M_y = N_x$

$M = e^x y^2 + \sin y$

$M_y = 2e^x y + \cos y$

$N = 2e^x y + x\cos y$

$N_x = 2e^x y + \cos y$

$M_y = N_x$

$\int (e^x y^2 + \sin y)\, dx$
$= e^x y^2 + x\sin y$

$\int 2e^x y + x\cos y \;\, dy$
$= e^x y^2 + x\sin y$

$$\boxed{e^x y^2 + x\sin y = C}$$

(15 points) *Problem 4*. Consider the population model $y'(t) = \frac{y}{10}\left(1 - \frac{y}{K}\right)$ with capacity $K > 0$.

(i) Find the general solution of the ODE.

(ii) Assume the initial population is one-third of the capacity. Find the time at which the population has doubled.

(iii) Sketch the solution in Part (ii) on the ty-plane. Find the limit $\lim_{t\to+\infty} y(t)$.

Extra space (for Problem 4 or other problems).

(15 points) *Problem 5.*

(i) Find the general solution of the homogeneous ODE $x'' - 4x' + 13x = 0$.

(ii) Find the general solution of the homogeneous ODE $x'' - 4x' + 3x = 0$.

(iii) Find a particular solution of the nonhomogeneous ODE $x'' - 4x' + 3x = e^t$.

$r^2 - 4r + 13 = 0$ $\quad \dfrac{4 \pm \sqrt{16-52}}{2}$ $\quad \dfrac{4 \pm \sqrt{-36}}{2}$ $\quad \dfrac{4 \pm 6i}{2}$ $\quad 2 \pm 3i$

1) $y_H(t) = C_1 e^{2t} \sin 3t + C_2 e^{2t} \cos 3t$

2) $r^2 - 4r + 3 = 0$

$(r-1)(r-3) = 0$

$r = 1, 3$

$y_H(t) = C_1 e^t + C_2 e^{3t}$

$y(t) = C_1 e^t + C_2 e^{3t} - \frac{1}{2} t e^t$

$y_P(t) = Ate^t$

$y'_P(t) = Ae^t + Ate^t$

$y''_P(t) = Ae^t + Ae^t + Ate^t$

$2Ae^t + Ate^t$

$\underbrace{2Ae^t} + Ate^t - 4Ate^t \underbrace{- 4Ae^t} + 3Ate^t = e^t$

$-2Ae^t = e^t$

$-2A = 1$

$A = -\frac{1}{2}$

$y_P(t) = -\frac{1}{2} t e^t$

(10 points) *Problem 6.* Consider an initial value problem (IVP) $\begin{cases} Q'(t) + \dfrac{r}{100}Q(t) = \dfrac{r}{4} \\ Q(0) = Q_0 \end{cases}$

with two positive parameters r, Q_0.

(i) Find the solution $Q_{\text{IVP}}(t)$ to the IVP. (ii) Find the limit $\lim\limits_{t\to+\infty} Q_{\text{IVP}}(t)$.

$$\frac{dQ}{dt} + \frac{r}{100}Q = \frac{r}{4}$$

$$p(t) = \frac{r}{100} \qquad g(t) = \frac{r}{4}$$

$$\mu(t) = e^{\frac{rt}{100}}$$

$$e^{\frac{rt}{100}} Q(t) = \int \frac{r}{4} e^{\frac{rt}{100}}$$

$$e^{rt/100} Q(t) = 25 e^{rt/100} + C$$

$$Q(t) = 25 + Ce^{-rt/100}$$

$$Q(0) = Q_0$$

$$Q_0 = 25 + C$$

$$C = Q_0 - 25$$

$$Q(t) = 25 + (Q_0 - 25)e^{-rt/100}$$

$$\lim_{t\to+\infty} Q(t) = 25$$

(10 points) *Problem 7.* Consider the 2nd order ODE $x'' + 4x = \sin(t)$.

(i) Does the ODE model a free oscillation? Is the oscillation over-damped?

(ii) Find a particular solution of the above nonhomogeneous ODE.

(iii) Find the solution of the ODE that satisfies the initial conditions $x(0) = 1,\ x'(0) = 0$.

1) since $g(t) \neq 0$ the oscillation is not free
not damped at all, no $\gamma u'$ term

2) $r^2+4=0$ $\quad C_1\cos 2t + C_2\sin 2t = Y_H(t)$ $\quad Y_p(t) = A\sin t + B\cos t$

$r^2 = -2$ $\quad y'(t) = A\cos t - B\sin t$

$r = \pm 2i$ $\quad y''(t) = -A\sin t - B\cos t$

$-A\sin t - B\cos t + 4A\sin t + 4B\cos t = \sin t$

$Y_p(t) = \frac{1}{3}\sin t$ $\quad 3A\sin t + 3B\cos t = \sin t + 0\cos t$

$y(t) = C_1\cos 2t + C_2\sin 2t + \frac{1}{3}\sin t$

$3A = 1 \quad A = \frac{1}{3}$

$1 = C_1\cos 0 + C_2\sin 0 + \frac{1}{3}\sin 0$ $\quad 3B = 0 \quad B = 0$

$1 = C_1$

$y'(t) = -2C_1\sin 2t + 2C_2\cos 2t + \frac{1}{3}\cos t$

$0 = -C_1\sin 0 + C_2\cos 0 + \frac{1}{3}\cos 0$

$0 = C_2 + \frac{1}{3} \quad 2C_2 = -\frac{1}{3} \quad C_2 = -\frac{1}{6}$

$y(t) = \cos 2t - \frac{1}{6}\sin 2t + \frac{1}{3}\sin t$