2nd Order ODEs

- For 1st order we want a general solution which contains an arbitrary constant. Then can use initial condition to solve for C to find specific solution.
- For 2nd order we want to find 2 "fundamental solutions" which will combine to get a general solution with 2 arbitrary constants. need 2 initial conditions to find a specific solution.
- How to tell if you have 2 fundamental solutions:
  Wronskian: If the wronskian is not 0 you have fundamental solutions.

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 \cdot y_2' - y_2 \cdot y_1'$$

Exam "Prove the solutions are fundamental" = compute wronskian "W"

If we take 2 solutions that are multiples of eachother, $(2e^t, e^t)$ they are not fundamental and W will be 0.

- to get general solution we take a linear combination of the fundamental solutions
  - multiply each soln by unique arbitrary constant (C_1, C_2) and add them together

Simple Ex:

$y'' = y \qquad \left(\frac{d^2y}{dt^2} = y\right)$

solutions: $y_1(t) = e^t$
$y_2(t) = e^{-t}$

these are the 2 fundamental solutions.

$$W[e^t, e^{-t}] = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix} =$$

$$e^t(-e^{-t}) - e^t(e^{-t}) = -1 - 1 = -2$$

$W \neq 0$ not zero so y_1 and y_2 are fundamental solutions

$y = C_1 e^t + C_2 e^{-t}$ (linear combination) we will look at 2nd order linear ODEs with constant coefficients.

$ay''+by'+cy=g(t)$ $a,b,c \in \mathbb{R}$

will start with problems where $g(t)=0$

Looking for solutions of form:

$y=e^{rt}$ (r constant)

$y'=re^{rt}$ $y''=r^2e^{rt}$

↳ plug into ODE $ay''+by'+cy=0$

$ar^2e^{rt}+bre^{rt}+ce^{rt}=0$

$\underbrace{e^{rt}}(ar^2+br+c)=0$

does not $=0$ for any value of t.

so we need to solve $ar^2+br+c=0$ which is our characteristic polynomial

2.11)

3 possibilities for the solutions of $ar^2+br+c=0$

- 2 real roots
- 2 complex roots
- 1 repeated root (real)

$y''+5y'+6y=0$

characteristic polynomial:

$r^2+5r+6=0$

$(r+2)(r+3)$ $r=-2,-3$ ← 2 real roots of ch eq. so we get 2 fundamental solutions to the ODE:

$y_1(t)=e^{-2t}$ $y_2(t)=e^{-3t}$

so gen soln is $y(t)=C_1e^{-2t}+C_2e^{-3t}$

If we have initial conditions $y(0)=2$, $y'(0)=3$ find specific solution

$2=C_1+C_2$

to plug in 2nd initial condition need to find $y'(t)$

$y'(t)=-2C_1e^{-2t}-3C_2e^{-3t}$

$3=-2C_1-3C_2$ $C_1=2-C_2$

$3=-2(2-C_2)-3C_2$

$3=4+2C_2-3C_2$ $-7=C_2$ $C_1=9$

$y(t)=9e^{-2t}-7e^{-3t}$

what if roots are complex?

↳ If they are complex, they will always be complex conjugates.

$r = \underbrace{\lambda}_{\text{real}} + \underbrace{\mu i}_{\text{imaginary}}$

the imaginary part "3-3i" is the real number that is multiplied by i.

based on our guess our solutions to the ODE would be $e^{(\lambda \pm \mu i)t}$

but we want real-valued solutions

How to get real-valued solutions

Eulers formula: $e^{i\theta} = \cos\theta + i\sin\theta$

this splits complex exponential into real and imaginary parts.

$e^{(\lambda+\mu i)t}$

$= e^{(\lambda t + \mu i t)}$

$= e^{\lambda t} e^{\mu i t}$

$= e^{\lambda t}(\cos\mu t + i\sin\mu t)$

$\underbrace{e^{\lambda t}\cos\mu t}_{\text{real part}} + \underbrace{i e^{\lambda t}\sin\mu t}_{\text{imaginary part}}$

$e^{(\lambda-\mu i)t}$

$= e^{\lambda t} e^{-i\mu t}$

$= e^{\lambda t}(\cos -\mu t + i\sin -\mu t)$

$= e^{\lambda t}\cos(-\mu t) + i e^{\lambda t}\sin(-\mu t)$

$\cos(-x) = \cos(x)$

$= \underbrace{e^{\lambda t}\cos(\mu t)}_{\text{real}} - \underbrace{i e^{\lambda t}\sin(\mu t)}_{\text{imaginary}}$

Fact; If $y(t)$ is a solution, then the real and imaginary parts of $y(t)$ are also solutions individually.

ie: if ie^t is soln then e^t is also soln

So our fundamental set of solutions is $e^{\lambda t}\cos(\mu t)$ and $e^{\lambda t}\sin\mu t$

and our general solution is $\boxed{y(t) = C_1 e^{\lambda t}\cos(\mu t) + C_2 e^{\lambda t}\sin(\mu t)}$

in a problem once you have roots you can go straight to ⟶

$y'' + 6y' + 13y = 0 \quad y(0) = 1 \quad y'(0) = 5$

ch eq: $r^2 + 6r + 13 = 0$

$r = \frac{-6 \pm \sqrt{36-52}}{2} = \frac{-6 \pm \sqrt{-16}}{2} = -3 \pm 2i$

→ real part $\lambda = -3$

→ imag part $\mu = 2$

$y(t) = C_1 e^{-3t}\cos(2t) + C_2 e^{-3t}\sin(2t)$ → gen soln

Plug in $y(0)=1$

$1 = C_1 e^0 \cos 0 + C_2 e^0 \sin 0$

$1 = C_1$

for $y'(0)=5$

find $y'(t)$ (need product rule)

$y'(t) = -3e^{-3t}\cos(2t) - 2e^{-3t}\sin(2t) - 3C_2 e^{-3t}\sin(2t) + 2C_2 e^{-3t}\cos(2t)$

$5 = -3e^0\cos(0) - 2e^0\sin(0) - 3C_2 e^0 \sin 0 + 2C_2 e^0 \cos 0$

$5 = -3 + 2C_2 \qquad 4 = C_2$

$y(t) = e^{-3t}\cos(2t) + 4e^{-3t}\sin(2t)$

Double root 2nd order ODE

Ex:

$y'' + 9y = 0$

Ch eq: $r^2 + 9 = 0$

$r = \frac{-0 \pm \sqrt{0-36}}{2} = \pm 3i$

$0 \pm 3i \qquad e^{\lambda t} = e^0 = 1$

general solution is:

$y(t) = C_1 \cos(3t) + C_2 \sin(3t)$

repeated roots: (only 1 root) $\quad r^2+2r+1 = (r+1)(r+1) \quad r=-1$

one fundamental solution easily: $y_1(t) = e^{rt}$

To get second: $y_2(t) = te^{rt}$

$y_2''(t) = r^2 te^{rt} + 2re^{rt}$

When plugged back into ODE $ay'' + by' + cy$

$ar^2te^{rt} + 2are^{rt} + brte^{rt} + be^{rt} + cte^{rt}$

$= r(ar^2e^{rt} + bre^{rt} + ce^{rt}) + 2are^{rt} + be^{rt}$

gen solution

$y(t) = C_1 e^{-t} + C_2 t e^{-t}$

summary where RHS=0

$ay'' + by' + cy = 0$

Ch. eq.: $ar^2 + br + c = 0$

3 cases

① 2 real roots: $y(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$

② 2 complex roots: $y(t) = C_1 e^{\lambda t}\cos(\mu t) + C_2 e^{\lambda t}\sin(\mu t)$ $\quad \lambda$ = real part, μ = imaginary part

③ 1 repeated root: $y(t) = C_1 e^{rt} + C_2 t e^{rt}$