Finding Eigenvalues and Eigenvectors

Quiz → compute evals/evecs for 2x2 with real nums.

We want: $A\vec{v} = \lambda\vec{v}$

$$A\vec{v} - \lambda\vec{v} = 0$$

$I\vec{v} = \vec{v}$

$$A\vec{v} - \lambda I\vec{v} = 0$$

$$(A-\lambda I)\vec{v} = 0$$

Is a matrix and a scalar times a vector.

this is a system, it has $\vec{v}=0$ as a soln. If $(A-\lambda I)$ is invertable then $\vec{v}=0$ is the only soln. (Eigenvectors cant be 0)

Why not $(A-\lambda)\vec{v}=0$? A is a matrix and λ is a scalar. Can't have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \lambda = 0$

If $(A-\lambda I)$ is singular (not invertable) then the system has ∞ solutions (that are all eigenvectors)

The easiest way to check this is to look for where $|A-\lambda I|=0$. So we solve for λ and that will give all eigenvalues of A.

Ex: $A = \begin{bmatrix} 2 & 2 \\ -2 & 3 \end{bmatrix}$

1. $A-\lambda I = \begin{bmatrix} 2 & 2 \\ -2 & -3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A-\lambda I = \begin{bmatrix} 2 & 2 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 2-\lambda & 2 \\ -2 & -3-\lambda \end{bmatrix}$

1. find eigenvalues of A. $|A-\lambda I|=0$

2. $\begin{bmatrix} 2-\lambda & 2 \\ -2 & -3-\lambda \end{bmatrix} = (2-\lambda)(-3-\lambda)-(2)(-2)$
$= \lambda^2 - 2\lambda + 3\lambda - 6 + 4$
$= \lambda^2 + \lambda - 2$ → Characteristic polynomial of A

2. find determinant

3. $\lambda^2 + \lambda - 2 = (\lambda+2)(\lambda-1)$
$(\lambda+2)(\lambda-1) = 0$
$\lambda = -2, 1$

3. find roots

4. find eigenvectors
$(A-\lambda I)\vec{v}=0$ solve for all λ

4. $(A-\lambda I)\vec{v}=0$

$\lambda = -2$ $\quad \left[\begin{array}{cc|c} 2-(-2) & 2 & 0 \\ -2 & -3-(-2) & 0 \end{array}\right]$

$$\begin{bmatrix}4&2&0\\-2&-1&0\end{bmatrix}\to\begin{bmatrix}1&\frac{1}{2}&0\\-2&-1&0\end{bmatrix}\to\begin{bmatrix}1&\frac{1}{2}&0\\0&0&0\end{bmatrix}$$

$v_1+\frac{1}{2}v_2=0$

$v_2=\text{free}=\alpha$

$v_1=-\frac{1}{2}\alpha$

$$\begin{bmatrix}v_1\\v_2\end{bmatrix}=\alpha\begin{bmatrix}-\frac{1}{2}\\1\end{bmatrix}$$

represents all possible ways of writing eigenvector. (besides $\alpha=0$) so we pick most convenient value of α so that there are no fractions.

so $\begin{bmatrix}-1\\2\end{bmatrix}$ is eigenvector for $\lambda=-2$.

RREF will always have a row of all zeroes if you didn't make a mistake finding your eigenvalues.

$\lambda=1$ $\left[\begin{array}{cc|c}2-(1)&2&0\\-2&-3-(1)&0\end{array}\right]$

$$\left[\begin{array}{cc|c}1&2&0\\-2&-4&0\end{array}\right]\to\left[\begin{array}{cc|c}1&2&0\\0&0&0\end{array}\right]$$

$v_1+2v_2=0$

$v_2=\alpha$

$v_1=-2\alpha$

$$\begin{bmatrix}v_1\\v_2\end{bmatrix}=\alpha\begin{bmatrix}-2\\1\end{bmatrix}$$

no fractions so use $\alpha=1$

so $\begin{bmatrix}-2\\1\end{bmatrix}$ is eigenvector for $\lambda=1$

Complex EigenValues

- will always have complex eigenvectors

2 helpful facts:

1. If $\lambda=\phi-\mu i$ is an eigenvalue of A then $\bar{\lambda}=\phi+\mu i$ is also an eigenvalue of A
2. If $\vec{v}=\vec{a}+\vec{b}i$ is an eigenvector of $\phi+\mu i$ then $\bar{\vec{v}}=\vec{a}-\vec{b}i$ is also an eigenvector of $\phi+\mu i$

$\therefore$ once you solve for one complex eigenvector you immediately get the other complex eigenvector as well.

Ex. $A=\begin{bmatrix}-1&5\\-1&3\end{bmatrix}$ $|A-\lambda I|=\begin{bmatrix}-1-\lambda&5\\-1&3-\lambda\end{bmatrix}$

$(-1-\lambda)(3-\lambda)-(5)(-1)$

$=\lambda^2+\lambda-3\lambda-3+5$

$=\lambda^2-2\lambda+2$

$\lambda=\dfrac{2\pm\sqrt{4-8}}{2}$ $\lambda=1\pm i$

$$\begin{bmatrix}-1-(1+i)&5&0\\-1&3-(1+i&0\end{bmatrix}\to\begin{bmatrix}-2-i&5&0\\-1&2-i&0\end{bmatrix}$$

$$\begin{bmatrix}-1&2-i&0\\-2-i&5&0\end{bmatrix}\to\begin{bmatrix}1&-2+i&0\\-2-i&5&0\end{bmatrix}$$

add $(2+i)$ row 1 to row 2.

$(2+i)(-2+i)+5$

$-4+2i-2i+i^2+5$

$-5+5=0$

$$\begin{bmatrix}1&-2+i&0\\0&0&0\end{bmatrix}$$

$v_1+(-2+i)v_2=0$

$v_2=\alpha$

so $\begin{bmatrix}2-i\\1\end{bmatrix}$ is eigenvector for $1+i$

$\therefore$ $\begin{bmatrix}2+i\\1\end{bmatrix}$ is eigenvector for $1-i$

Eigenvalues/vectors examples:

$$A=\begin{bmatrix}1&2&0\\-1&-2&0\\0&0&1\end{bmatrix}$$

$$\begin{bmatrix}1-\lambda&2&0\\-1&-2-\lambda&0\\0&0&-1-\lambda\end{bmatrix}=0-0+(-1-\lambda)\begin{vmatrix}1-\lambda&2\\-1&-2-\lambda\end{vmatrix}$$

$$=(-1-\lambda)\big((1-\lambda)(-2-\lambda)-(2)(-1)\big)$$

$$=(-1-\lambda)(\lambda^2-\lambda+2\lambda-2+2)$$

$$=(-1-\lambda)(\lambda^2+\lambda)$$ ← Always want Ch. Poly. in factored form. (don't distribute)

$$=(-1-\lambda)\lambda(\lambda+1)$$

$\lambda=-1$ $\quad\lambda=0$ $\quad\lambda=-1$

- 0 can be an eigenvalue but $\vec{0}$ can't be an eigenvector.
- an eigenvalue can be a repeated root. the # of times it is repeated is called Algebraic multiplicity.

$\lambda=-1$ has Algebraic multiplicity of 2.

$\lambda=0$ has 1

$\sum$ A.M. = size of matrix

$\lambda=0$: $\begin{bmatrix}1&2&0&0\\-1&-2&0&0\\0&0&-1&0\end{bmatrix}$

$$\begin{bmatrix}1&2&0&0\\0&0&0&0\\0&0&-1&0\end{bmatrix}\begin{bmatrix}1&2&0&0\\0&0&1&0\\0&0&0&0\end{bmatrix}$$

$V_1+2V_2=0$

$V_3=0$

$V_2=\text{free}$

$V_2=\alpha$

$\alpha\begin{bmatrix}-2\\1\\0\end{bmatrix}$ $\qquad\begin{bmatrix}-2\\1\\0\end{bmatrix}$ is eigenvector for $\lambda=0$

When an eigenvalue has Algebraic multiplicity of 2 there are 2 possibilities for the eigenvector.

1. there are 2 linearly independent eigenvectors. Geometric Multiplicity 2
2. there is only 1 e.v. Geometric multiplicity 1

$\lambda=-1$ $\begin{bmatrix}2&2&0&0\\-1&-1&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}1&1&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$

$V_1+V_2=0$ $\qquad V_1=-\alpha$

$V_2=\text{free}$ $\qquad V_2=\alpha$

$V_3=\text{free}$ $\qquad V_3=\beta$

$$\begin{bmatrix}V_1\\V_2\\V_3\end{bmatrix}=\alpha\begin{bmatrix}-1\\1\\0\end{bmatrix}+\beta\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$\begin{bmatrix}-1\\1\\0\end{bmatrix},\begin{bmatrix}0\\0\\1\end{bmatrix}$ are eigenvectors for $\lambda=-1$

4x4

any 4x4 in this class will be block diagonal

$$\begin{bmatrix} 5 & -4 & 0 & 0 \\ 4 & -5 & 0 & 0 \\ 0 & 0 & -1 & -3 \\ 0 & 0 & 2 & 4 \end{bmatrix} \quad \begin{bmatrix} 5-\lambda & -4 & 0 & 0 \\ 4 & -5-\lambda & 0 & 0 \\ 0 & 0 & -1-\lambda & -3 \\ 0 & 0 & 2 & 4-\lambda \end{bmatrix}$$

* take determinant of each block and multifly

$$\begin{vmatrix} 5-\lambda & -4 \\ 4 & -5-\lambda \end{vmatrix} \cdot \begin{vmatrix} -1-\lambda & -3 \\ 2 & 4-\lambda \end{vmatrix}$$

$$((5-\lambda)(-5-\lambda)-(-4)(4))((-1-\lambda)(4-\lambda)-(-3)(2))$$

$$(\lambda^2-9)(\lambda^2-3\lambda+2)$$

$$=(\lambda+3)(\lambda-3)(\lambda-2)(\lambda-1) \qquad \lambda=-3,3,2,1$$