Predator Prey (lotka-Volterra)

Idea: Show the dynamics for a prey species given by $x(t)$ and a predator species $y(t)$.

Assume:

- With no predator the prey population grows exponentially.
  $x'(t) = ax$ for $a > 0$ (when $y(t) = 0$)
- With no prey the predator pop. goes extinct.
  $y'(t) = -cy$ for $c > 0$ (when $x'(t) = 0$)

- Gives system: ($x(t)$ decreases as $y(t)$ increases)

$$x'(t) = ax - \alpha xy = x(a - \alpha y)$$
$$y'(t) = -cy + \gamma xy = y(-c + \gamma x)$$

$(a, c, \alpha, \gamma > 0)$

Ex:

$$x'(t) = x\left(1 - \tfrac{1}{2}y\right)$$
$$y'(t) = y\left(-\tfrac{3}{4} + \tfrac{1}{4}x\right)$$

Note: systems of this form will always have 2 equilibria.

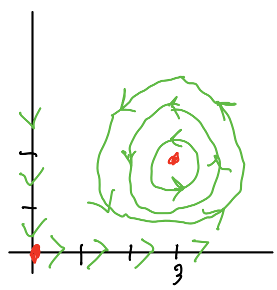
$x = 0$

$\swarrow$ $y = 0$ $\quad\searrow$ $-\frac{3}{4} + \frac{1}{4}x = 0$ $\downarrow$ no soln

$1 - \frac{1}{2}y = 0$

$y = 2$

$\swarrow$ $2 = 0$ $\quad\searrow$ $-\frac{3}{4} + \frac{1}{4}x = 0$ $\downarrow$ $(3, 2)$



$$J(x,y) = \begin{bmatrix} 1 - \frac{1}{2}y & -\frac{1}{2}x \\ \frac{1}{4}y & -\frac{3}{4} + \frac{1}{4}x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -3/4 \end{bmatrix} \quad \lambda = 1, -\tfrac{3}{4}$$ saddle point. unstable

$$J(3,2) = \begin{bmatrix} 0 & -\frac{3}{2} \\ 1/2 & 0 \end{bmatrix} \quad \lambda = \pm \frac{\sqrt{3}}{2} i$$ center. stable

Note: we assumed that the prey grows exponentially without predators. What if the prey had carying capacity.

$$x'(t) = x(a - \alpha y - \sigma x) \quad \text{for } \sigma > 0$$
$$y'(t) = y(-c + \gamma x)$$

the carying cap will depend on σ.

Ex: $x'(t) = x(1 - \frac{1}{2}y - \sigma x)$ $\sigma > 0$
$y'(t) = y(-3/4 + 1/4 x)$

(modified predator/prey with carying capacity on prey)

1. find equilibria

$x = 0 \to y = 0$, no soln; $(0,0)$

$1 - \frac{1}{2}y - \sigma x = 0 \to y = 0$, $x = \frac{1}{\sigma}$, $(\frac{1}{\sigma}, 0)$

$-\frac{3}{4} + \frac{1}{4}x = 0$
$x\frac{1}{4} = \frac{3}{4}$
$x = 3$
$1 - \frac{1}{2}y - 3\sigma = 0$
$y = 2 - 6\sigma$

note: with $\sigma = 0$ from previous example we got 2 equilibria. $(0,0)$ and $(3,2)$.
Now at the stable equilibrium $(3, 2-6\sigma)$ the prey pop. is the same but pred. pop. is lower.

$x - \frac{1}{2}xy - \sigma x^2$

$-\frac{3}{4}y + \frac{1}{4}xy$

$$J(x,y) = \begin{bmatrix} 1 - \frac{1}{2}y - 2\sigma x & -\frac{1}{2}x \\ \frac{1}{4}y & -\frac{3}{4} + \frac{1}{4}x \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -3/4 \end{bmatrix} \quad \lambda = 1, -3/4$$ Saddle at $(0,0)$

$$J(\tfrac{1}{\sigma}, 0) = \begin{bmatrix} -1 & -\frac{1}{2\sigma} \\ 0 & -\frac{3}{4} + \frac{1}{4}(\frac{1}{\sigma}) \end{bmatrix} \quad \lambda = -1, -\frac{3}{4} + \frac{1}{4\sigma}$$

$-\frac{3}{4} + \frac{1}{4\sigma} > 0$

$\frac{1}{4\sigma} > \frac{3}{4}$
$\frac{1}{\sigma} > 3$
$1 > 3\sigma$
$\frac{1}{3} > \sigma$

saddle point if $\sigma < \frac{1}{3}$

nodal sink if $\sigma > \frac{1}{3}$

$$J(3, 2-6\sigma) \begin{bmatrix} 1 - 1/2(2-6\sigma) & -3/2 \\ \frac{1}{2}(2-6\sigma) & -\frac{3}{4} + \frac{3}{4} \end{bmatrix} = \begin{bmatrix} -3\sigma & -3/2 \\ \frac{1}{2} - \frac{3}{2}\sigma & 0 \end{bmatrix}$$

$$(-3\sigma - \lambda)(-\lambda) - (-3/2)(1/2 - 3/2\sigma) = 0$$
$$\lambda^2 + 3\sigma\lambda + 3/4 - \frac{9}{4}\sigma = 0$$

$$\lambda = \frac{-3\sigma \pm \sqrt{9\sigma^2 - 3 + 9\sigma}}{2}$$

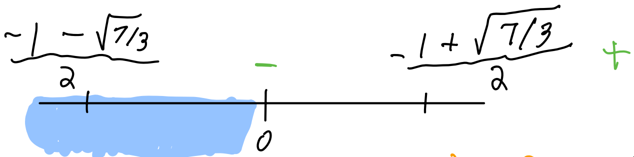
$$\lambda = \frac{-3\sigma \pm 3\sqrt{\sigma^2 + \sigma - 1/3}}{2}$$

real or complex?

$\sigma^2 + \sigma - 1/3$ ← if positive, λ is real; if negative, λ is complex

$$\sigma^2 + \sigma - \frac{1}{3} = 0$$

$$\sigma = \frac{-1 \pm \sqrt{1 - 4(7/3)}}{2} = \frac{-1 \pm \sqrt{7/3}}{2}$$



(negative need $\sigma > 0$)

Plug in 0: $-\frac{1}{3} < 0$

Plug in $\sigma = 10$: $100 + 10 - 1/3 > 0$

so if we have $\sigma < \frac{-1+\sqrt{7/3}}{2}$ we have complex eigenvalues so $(2, 2-6\sigma)$ is a spiral sink.

If $\sigma > \frac{-1+\sqrt{7/3}}{2}$ we have real eigenvalues but need to test for node or saddle (λ is real) by determining signs of λ.

$$\frac{-3\sigma - 3\sqrt{\sigma^2 + \sigma - 1/3}}{2} \leftarrow \text{negative}$$

$$\frac{-3\sigma + 3\sqrt{\sigma^2 + \sigma - 1/3}}{2} > 0$$

$$-3\sigma + 3\sqrt{\sigma^2 + \sigma - 1/3} > 0$$

$$\sqrt{\sigma^2 + \sigma - 1/3} > \sigma$$
$$\sigma^2 + \sigma - 1/3 > \sigma^2$$
$$\sigma - 1/3 > 0$$
$$\sigma > 1/3$$

so if $\sigma > 1/3$ nodal sink $(3, 2-6\sigma)$

if $\sigma < 1/3$ nodal source at $(3, 2-6\sigma)$

Now have 3 cases for $(3, 2-6\sigma)$ based off of $1/3$ and $\frac{-1+\sqrt{7/3}}{2}$. Which is bigger? $\frac{1}{3} > \frac{-1+\sqrt{7/3}}{2}$

case 1: if $\sigma > 1/3$ saddle

2: $\frac{-1+\sqrt{7/3}}{2} < \sigma < 1/3$ nodal sink

3: $0 < \sigma < \frac{-1+\sqrt{7/3}}{2}$ spiral sink

4: $\sigma = 0$ center