from: $\frac{dy}{dt} + P(t)y = g(t)$

$\mu(t)$ is called the integrating factor

$$\left[\begin{array}{l} \mu(t)y = \int \mu(t)g(t)\,dt \\ \mu(t) = e^{\int P(t)\,dt} \end{array}\right]$$

Ex: $\frac{dy}{dt} - 2y = 4 - t$

$P(t) = -2$
$g(t) = 4 - t$

don't need +C

$\mu(t) = e^{\int -2\,dt} = e^{-2t}$

$e^{-2t}y = \int e^{-2t}(4-t)\,dt$

$\int u\,dv = uv - \int v\,du$

$e^{-2t}y = -2e^{-2t} + \frac{1}{2}te^{-2t} - \frac{1}{2}\cdot\left(-\frac{1}{2}\right)e^{-2t} + C$

$u: 4-t$ $\quad dv: e^{-2t}$

$du: -1\,dt$ $\quad v: -\frac{1}{2}e^{-2t}$

$y = -2 + \frac{1}{2}t + \frac{1}{4} + Ce^{2t}$

$y = Ce^{2t} + \frac{1}{2}t - \frac{7}{4}$

$-(4-t)\frac{1}{2}e^{-2t} - \frac{1}{2}\int e^{-2t}\,dt$

$-(4-t)\frac{1}{2}e^{-2t} + e^{-2t}$

$ty' + 2y = 4t^2$ $\quad y(1) = 2$

$\frac{dy}{dt} + \frac{2}{t}y = 4t$

$\mu(t) = e^{\int \frac{2}{t}\,dt} = e^{2\ln|t|} = e^{\ln|t|^2} = t^2$

$t^2 y = \int 4t^3\,dt$

$t^2 y = t^4 + C$

$y = t^2 + \frac{C}{t^2}$

$2 = 1^2 + \frac{C}{1}$

$1 = \frac{C}{1}$

$C = 1$

$y(t) = t^2 + \frac{1}{t^2}$

$[t > 0]$ initial condition $t > 0$
t can't be 0
solution doesn't account for $t < 0$

integrating factors ex:
what if $\mu(t)g(t)$ has no antiderivative?

F.T.C
if $\frac{d}{dt}[F(t)] = f(t)$
then $F(t) = \int_{t_0}^{t} f(s)\,ds$

$2y' + ty = 2 \qquad y(0) = 1$

$y' + \frac{t}{2}y = 1$

$\frac{dy}{dt} + \frac{t}{2}y = 1$

$\mu(t) = e^{\int t/2\,dt} = e^{t^2/4}$

$e^{t^2/4}y = \int e^{t^2/4}\,dt + C$

$e^{t^2}y = \int_0^t e^{s^2/4}\,ds + C$

$y(t) = e^{-t^2/4}\int_0^t e^{s^2/4}\,ds + Ce^{-t^2/4}$ ← gen soln

$1 = e^0 \int_0^0 e^{s^2/4}\,ds + Ce^0$

$1 = C$

$y(t) = e^{-t^2/4}\int_0^t e^{s^24}\,ds + e^{-t^2/4}$ ← specifific soln

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there is no general method that will work to solve all ODEs.
There are only methods that work in specific circumstances.

Exact Differential Equations:

similar concept as integrating factors but using chain rule instead of product rule

Ex:

$2x + y^2 + 2xyy' = 0$

y is dependent
x is independent

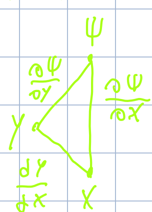
consider $\psi(x,y) = x^2 + xy^2$

partial derivatives $\frac{\partial\psi}{\partial x} = 2x + y^2$

$\frac{\partial\psi}{\partial y} = 2xy$

to check if
$M(x,y) + N(x,y)\frac{dx}{dy} = 0$
is exact: need
$N_x(x,y) = M_y(x,y)$
(same as checking that $\langle M,N\rangle$ is conservative)

multivariable chain rule:



if we think of ψ as just a function of x:

$\frac{d\psi}{dx} = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = 2x + y^2 + 2xyy'$

observe: we have $M(x,y) + N(x,y)\frac{dy}{dx} = 0$
we want ψ such that
$\frac{\partial\psi}{\partial x} = M(x,y)$ and $\frac{\partial\psi}{\partial y} = N(x,y)$

if it is exact we can find ψ as follows:

take the integrals:

$\int M\,dx$ and $\int N\,dy$

$\int 2x+y^2\,dx \quad \int 2xy\,dy$

$$\underbrace{x^2+xy^2+g(y) \leftrightarrow xy^2+h(x)}$$

combine unique terms

$\psi(x,y)=x^2+xy^2$

$\psi(x,y)=$ unique terms of $\int M\,dx$ and $\int N\,dy$

or $\nabla\psi=\langle M(x,y),N(x,y)\rangle$
so ψ is the potential function for $\langle M,N\rangle$

$$\underbrace{2x+y^2+2xyy'=0}$$

this is

$$\frac{d\psi}{dx}=0$$

$$\int\frac{d\psi}{dx}\,dx=\int 0\,dx$$

$\psi=C$

$x^2+xy^2=C$ ← implicit form