A $x' = Ax$ with a 4x4 will always be relatively nice matrix.

Almost certainly has 4 eigenvectors, maybe 3 eigenvectors with a generalized one.

New case

4 complex eigenvalues, or complex values with multiplicity 2.

4x4 in this class will always be block diagonal.

$$x' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

$$|A - \lambda I| = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} \begin{vmatrix} -1-\lambda & 2 \\ -1 & 1-\lambda \end{vmatrix}$$

$$= (\lambda^2+1)(\lambda^2+1)$$

$\lambda = \pm i$ both with multiplicity 2

eigenvectors for $\lambda = i$ $\begin{bmatrix} -i \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1-i \\ 1 \end{bmatrix}$ $\lambda = -i$ $\begin{bmatrix} i \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1+i \\ 1 \end{bmatrix}$

for gen. soln. consider each eigenvector of $\lambda = i$ separately

$$\underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}} \qquad \underbrace{\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}}$$

$$\vec{x}(t) = c_1\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\cos t - \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\sin t\right) + c_2\left(\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}\sin t + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 0 \end{bmatrix}\cos t\right)$$

$$+ c_3\left(\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\cos t - \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}\sin t\right) + c_4\left(\begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}\sin t + \begin{bmatrix} 0 \\ 0 \\ -1 \\ 0 \end{bmatrix}\cos t\right)$$

Initial Value Problems for $x' = Ax$

initial value will be a vector $\vec{x}(0) = [\ ]$

Ex $x' = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$ $x(0) = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$

gen. soln. $\vec{x}(t) = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3t} + c_2\begin{bmatrix} -1 \\ 2 \end{bmatrix}e^{-t}$

Plug in $x(0)$

$$\begin{bmatrix} -7 \\ 4 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

can write as

$$\begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 2 & -1 & -7 \\ 1 & 2 & 4 \end{array}\right]$$

$2c_1 - c_2 = -7$

$c_1 + 2c_2 = 4$

solve for c_1, c_2

$c_1 = -2$ $c_2 = 3$

$$x(t) = -2\begin{bmatrix} 2 \\ 1 \end{bmatrix}e^{3t} + 3\begin{bmatrix} -1 \\ 2 \end{bmatrix}e^{-t}$$