Exact Eqns.

$M+(N)y'=0$

$N_x(x,y)=M_y(x,y)$

$\psi(x,y)=$ unique terms of $\int M dx$ and $\int N dy$

add $=C$

Can be made Exact if: $\frac{M_y-N_x}{N}$ is only a function of x.

$\frac{M_y-N_x}{N}=\frac{1}{\mu}\frac{d\mu}{dx}$

or if $\frac{N_x-M_y}{M}$ depends only on y. then

$\frac{d\mu}{dy}=\mu(y)\frac{N_x-M_y}{M}$

$\frac{d\mu}{dx}=\mu(y)\cdot\frac{1}{y}\rightarrow\mu(y)=y$

2nd Order ODE's

$ay''+by'+cy=g(t)$

Ch. eqn $ar^2+br+c=0$

$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

roots: $y_H(t)=$

2 real: $C_1e^{r_1t}+C_2e^{r_2t}$

1 real: $C_1e^{rt}+C_2te^{rt}$

unreal:

$C_1e^{\lambda t}\cos(\mu t)+C_2e^{\lambda t}\sin(\mu t)$

$\lambda\rightarrow$ real $\mu\rightarrow$ unreal

$g(t)\rightarrow y_P(t)$

$e^{nt}\rightarrow Ae^{nt}$

$\sin/\cos(nt)\searrow$

$A\sin(nt)+B\cos(nt)$

$t^n\rightarrow At^2+Bt+C$

if $y_P(t)$ is soln of Homo multiply by t

Are solutions fundamental?

find Wronskian

$W[y_1,y_2]=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$

fund if determinant is not 0

answer $=C$ implicit

$y(x)=$ answer explicit

Integrating factor

from: $\frac{dy}{dt}+P(t)y=g(t)$

$\mu(t)y=\int\mu(t)g(t)dt$

$\mu(t)=e^{\int P(t)dt}$ (+C!!!)

must be linear

Separable ODE's

need: $f(y)\frac{dy}{dx}=G(x)$

E.x: $\frac{dy}{dx}=\frac{2x}{y-1}$ integrate both sides

$(y-1)\frac{dy}{dx}=2x$

If $\frac{dy}{dx}=f(x,y)$ and $f(x,y)$ can be a func of $\frac{y}{x}$ or $\frac{x}{y}$ it is Homogeneous

$V=\frac{y}{x}$ and $\boxed{V+x\frac{dV}{dx}=\frac{dy}{dx}}$

PFD

$\frac{1-V}{V^2-4}=\frac{1-V}{(V-2)(V+2)}$

$\frac{A}{V+2}+\frac{B}{V-2}=\frac{1-V}{(V-2)(V+2)}$

$A(V-2)+B(V+2)=1-V$

$AV-2A+VB+2B=1-V$

$\boxed{1=-2A+2B}$ int terms

$\boxed{-1=A+B}$ V terms

IBP

$\int u\,dv=uv-\int v\,du$

$\int \ln x\,dx$ $u=\ln x$ $dv=dx$ $du=\frac{1}{x}$ $v=x$

Trig sub: $\tan^2\theta+1=\sec^2\theta$

$x^2+1\rightarrow\tan\theta$

$x^2-1\rightarrow\sec\theta$

$1-x^2\rightarrow\sin\theta$

Newtons law of cooling:

T: temp of obj

M: temp surroundings

t: time

K: proportionality

$\frac{dT}{dt}=K(M-T),\ K>0$

$\int\frac{dT}{M-T}=\int K dt$

$-\ln|M-T|=Kt+C$

cooling

$M-T=e^{-Kt-C}$

$T=M+Ae^{-Kt}$

Heating

$T=M-Ae^{-Kt}$

Autonomous Eqns and Equilibrium Analysis:

Form: $\frac{dy}{dx}=f(y)$

Equilibrium soln the derivative is 0

- find $y=0$ eq pts.
- find signs between
- draw Phase line
- soln curves
- unstable, stable, semi

$\frac{d^2y}{dx^2}=0$ for inflection

.ccu $\frac{d^2y}{dx}>0$

.ccd <0

$\frac{d^2y}{dt^2}=f'(y)\cdot f(y)$

logistic Eqn's

$\frac{dy}{dt}=(r-ay)y$ $K=\frac{r}{a}$

$\frac{dy}{dt}=r\left(1-\frac{y}{K}\right)y$ $y(0)=y_0$

With threshold

$y'=r\left(1-\frac{y}{T}\right)\left(1-\frac{y}{K}\right)y$

$r>0,\ 0<T<K$

$y=0,T,K$

initial pop is $\frac{1}{3}$ cap:

$y(0)=\frac{K}{3}$

3rd and 4th order

Very similar to 2nd order

- find roots of ch. eqn.
- real, complex, repeated roots handled same way

~Y_P is found the same way

$y^{(4)}-4y'''+4y''=0$

$r^4-4r^3+4r^2=0$

$r^2(r^2-4r+4)=0$

$r^2(r-2)(r-2)=0$

$r=0,0,2,2$

$y(t)=C_1+C_2t+C_3e^{2t}+C_4te^{2t}$

Need 4 initial conditions

$r^4+2r^2+1=0$

replace $r^2=s$

$s^2+2s+1=0$

$(r^2+1)(r^2+1)=0$

$r^2+1=0$ $r^2+1=0$

$r=\pm i$ $r=\pm i$

$y^{(4)}-y=0$

$r^{(4)}-1=0$

$(r^2-1)(r^2+1)=0$

$(r-1)(r+1)(r^2+1)$

$y'(t)=\frac{y}{10}\left(1-\frac{y}{K}\right)$

1) find gen soln.

2) init pop $=\frac{1}{3}$ cap find 2x pop

3) sketch soln find lim $y(t)$

$\frac{dy}{dt}=\frac{y}{10}\left(1-\frac{y}{K}\right)$

$\frac{1}{y\left(1-\frac{y}{K}\right)}\frac{dy}{dt}=\frac{1}{10}$

PFD

$\left(\frac{1}{y}+\frac{1/K}{1-y/K}\right)\frac{dy}{dt}=\frac{1}{10}$

$\int\frac{1}{y}+\frac{1/K}{1-y/K}\,dy=\int\frac{1}{10}$

$\ln|y|-\ln\left|1-\frac{y}{K}\right|=\frac{t}{10}+C$

$y(0)=K/3$ sol. 4 C

$\ln\left|\frac{K}{3}\right|-\ln|1-1/3|=C$

$\ln\left(\frac{K/3}{2/3}\right)=\ln\left(\frac{K}{2}\right)=C$

find t_1 $y(t_1)=\frac{2K}{3}$

$\ln|y|-\ln\left|1-\frac{y}{K}\right|=\frac{t}{10}+\ln\left(\frac{K}{2}\right)$

$y(t_1)=\frac{2K}{3}$

$\ln\left(\frac{2K}{3}\right)-\ln\left(1-\frac{2}{3}\right)=\frac{1}{10}t_1+\ln\left(\frac{K}{2}\right)$

$\ln\left(\frac{2K}{3}\right)-\ln\left(\frac{1}{3}\right)-\ln\left(\frac{K}{2}\right)=\frac{1}{10}t_1$

$\ln\left(\frac{2K/3}{\frac{1}{3}\cdot\frac{K}{2}}\right)=\frac{1}{10}t_1$

$\ln(4)=\frac{1}{10}t_1$

$10\ln(4)=t_1$

$\lim_{t\to\infty}y(t)=K$

A mass of 10 kg stretches spring 50cm. mass is acted on by external force of $5\sin(2t)$ N and moves in a medium that imparts a viscous force of 3 N when the speed of the mass is 10 cm/s. If the mass is pulled down 2cm and released, formulate the IVP describing the motion of the mass.

$m = 10\,kg$ $\quad N = \frac{kg \cdot m}{s^2}$

$g(t) = 5\sin(2t)$

"viscous force" = damping

$\gamma \cdot u'$ $\quad \gamma \cdot u' = 3$ $\quad \gamma = 30$

$\gamma \cdot 0.1 = 3$

$mg = kL$

$\underset{kg}{10} \cdot \underset{m/s^2}{9.8} = \underset{kg/s^2}{k} \cdot 0.5$

$98 = \frac{k}{2}$

$k = 196$

$u_0 = 2cm = 0.02$

$V_0 = 0$

$10u'' + 30u' + 196u = 5\sin 2t$

$u(0) = 0.02$

$u'(0) = 0$ (starts stationary $\therefore V_0 = 0$)

$mu''(t) + \gamma u'(t) + ku(t) = g(t)$

$\hookrightarrow m, \gamma, k$ are all positive

initial conditions would be

$u(0) = u_0$ initial position

$u'(0) = V_0$ initial velocity

m mass

γ damping constant

k spring constant

$g(t)$ external force

Resonance if we have

$mu'' + ku = F\cos(\omega t)$ or $F\sin(\omega t)$

and $\omega = \sqrt{\frac{k}{m}} \leftarrow \omega_0$

Is resonance?

$u'' + 4u = \cos(2t)$ yes

$u'' - 4u = \cos(2t)$ no.

needs non-real roots.

and frequency matches

Terminology:

if $g(t) = 0$ oscillation is free (homogeneous)

if $g(t) \neq 0$ oscillation is forced (non homogeneous)

if $\gamma = 0$ oscillation is undamped

if $\gamma \neq 0$ oscillation is damped

if we have undamped free oscillation:

$\hookrightarrow mu'' + ku = 0$ $\quad m, k > 0$

soln: $mr^2 + 0r + k = 0$

$r = \frac{-0 \pm \sqrt{0^2 - 4mk}}{2m}$ $\quad 0 \pm i\frac{\sqrt{k}}{\sqrt{m}}$

$\hookrightarrow u(t) = C_1\cos\left(\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(\sqrt{\frac{k}{m}}t\right)$

$\sqrt{\frac{k}{m}}$ = frequency ω_0 (Angular)

$\frac{1}{\omega_0} = \sqrt{\frac{m}{k}}$ = period (angular)

Also want amplitude and phase

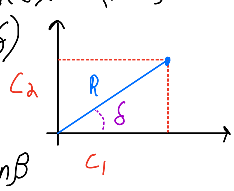
$u(t) = C_1\cos\left(\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(\sqrt{\frac{k}{m}}t\right)$

can be:

$= R\cos(\delta)\cos\left(\sqrt{\frac{k}{m}}t\right) + R\sin(\delta)\sin\left(\sqrt{\frac{k}{m}}t\right)$

$\cos(\delta) = \frac{C_1}{R}$ $\quad C_1 = R\cos(\delta)$

$\sin(\delta) = \frac{C_2}{R}$ $\quad C_2 = R\sin(\delta)$



$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

so

$R\cos(\delta)\cos\left(\sqrt{\frac{k}{m}}t\right) + R\sin(\delta)\sin\left(\sqrt{\frac{k}{m}}t\right)$

$= R\cos\left(\delta - \sqrt{\frac{k}{m}}t\right)$ or $\underbrace{R}_{\text{amplitude}}\cos\left(\sqrt{\frac{k}{m}}t - \underbrace{\delta}_{\text{phase}}\right)$

Damped Free Oscillations

$mu'' + \gamma u' + ku = 0$ $\quad m, \gamma, k > 0$

$mr^2 + \gamma r + k = 0$

$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$

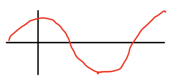
3 cases:

$\gamma^2 - 4mk > 0$ over damped

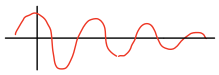
$\gamma^2 - 4mk = 0$ crit. damped

$\gamma^2 - 4mk < 0$ under damped

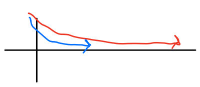
undamped



under damped



over damped (red)



crit. damped (like over damped but goes to 0 faster)

Ex:

$u'' + 4u = 0$

find: freq, period, amp, phase

freq $= \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2$

Period $= \frac{1}{2}$

for amp and phase we need to solve the IVE first

$r^2 + 4 = 0$

$r = \pm 2i$

$u(t) = C_1\cos(2t) + C_2\sin(2t)$

$4 = C_1$ $\quad C_2 = -3$

$4\cos(2t) - 3\sin(2t)$

$r^2 = 4^2 + -3^2$

$r = 5$ $\quad \delta = \tan^{-1}\left(\frac{-3}{4}\right)$

$g(t)$	$Y_p(t)$
$g(t) = 2t^2e^{-t} + e^{-t}$ $= (2t^2+1)e^{-t}$	$Y_p(t) = At^2e^{-t} + Bte^{-t} + Ce^{t}$
$g(t) = \underbrace{\sin(2t)}_{g_1} + \underbrace{e^{3t}}_{g_2}$	$Y_{p1}(t) = A\sin 2t + B\cos 2t$ $Y_{p2}(t) = Ce^{3t}$
$g(t) = te^{t/2}\cos 3t$	$Y_p(t) = Ate^{t/2}\sin 3t + Bte^{t/2}\cos 3t + Ce^{t/2}\sin 3t + De^{t/2}\cos 3t$ $(t+1)e^{t/2}(\sin 3t + \cos 3t)$ w/o constants
$g(t) = 5\sin(2t) - 3\cos(2)$	$Y_p(t) = A\sin 2t + B\cos 2t$
$g(t) = \underbrace{4\sin 2t}_{g_1} + \underbrace{2\cos 3t}_{g_2}$	must deal with them separately

Matrix:

↳ swap rows
↳ Multiply rows by scalar
↳ add rows to eachother

$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$ ✓ $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ ✗

vectors linearly independent?

$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} -4 \\ 1 \\ 11 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & -4 & 0 \\ 2 & 1 & 1 & 0 \\ -1 & 3 & 11 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ NO. not = I.D. matrix $\neq \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

A^{-1} (inverse of A) of 2x2

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$= \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ ($|A|$ = determinant)

Trace: sum of diagonal

Invertable matrix theorem

(All must be true or false)

For square nxn matrix

- A is invertable
- A has RREF = I.D. matrix
- $A\vec{x} = \vec{b}$ has exactly 1 soln. (for any b)
- the columns are linearly independent.
- Determinant of $A \neq 0$
- 0 is not an eigenval of A

Quadratic formula

$$0 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$|A|$ = product of the eigenvals

trace(A) = sum of eigenvals

∴ if A is 2x2 we can find eigenvals if given $|A|$ and tr(A)

Ex: tr(A) = 5 $|A| = 6$ $\lambda = 2, 3$

$\lambda^2 - 5\lambda + 6$

(2 nums multiply to $|A|$ add to tr(A))

Eigenvalues and Eigenvectors

$x' = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & k \end{bmatrix} x$ (this is A)

- subtract λ from diag. entries
- find determinant of A
- find 0s of polynomial
- solve $|A - \lambda I|$ for all λ

☆ if $\lambda = \emptyset - \mu i$ is an eigenvalue then $\lambda = \emptyset + \mu i$ also is.

☆ if $\vec{v} = \vec{a} + \vec{b}i$ is eigenvector then $\vec{v} = \vec{a} - \vec{b}i$ also is.

RREF will always have atleast 1 row of all zeroes.

0 can't be an eigenvector.

☆ if $|A - \lambda I|$ is not invertable then the system has ∞ solutions.

Gen soln:

* $\vec{x}(t) = C_1 \alpha_1 e^{\lambda_1 t} + C_2 \alpha_2 e^{\lambda_2 t} + C_n \ldots$

* Will have 1 independent soln. per row/column in A.

If A only has 1 eigenval and 1 eigenvector we find a generalized eigenvector.

Ex: $x' = \begin{bmatrix} 1 & -2 \\ 2 & 5 \end{bmatrix}$ evector: $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ eval: 3

$\lambda = 3 \rightarrow \begin{bmatrix} -2 & -2 \\ 2 & 2 \end{bmatrix}$ bring evector in

$\begin{bmatrix} -2 & -2 & -1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 0 & 0 \end{bmatrix}$

$V_1 + V_2 = \frac{1}{2}$ $V_1 = -V_2 + \frac{1}{2}$

$V_2 = \alpha$ $\alpha\begin{bmatrix} -1 \\ 1 \end{bmatrix} + w\begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$

Gen. Soln:

$\vec{x}(t) = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{3t} + C_2\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t e^{3t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{3t}\right)$

☆ Complex eigenvalues and complex eigenvectors always come in pairs.

Eval: $\lambda = \emptyset + \mu i$ Evec: $\vec{V} = \vec{a} + \vec{b}i$

$\vec{x}(t) = C_1 e^{\emptyset t}(\vec{a}\cos\mu t - \vec{b}\sin\mu t) + C_2 e^{\emptyset t}(\vec{a}\sin\mu t + \vec{b}\cos\mu t)$

$\alpha\begin{bmatrix} 1+i \\ 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{a} + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{b} i$

$\cos(-x) = \cos(x)$

$\sin(-x) = -\sin(x)$

Synthetic Division

Ex: $1\lambda^3 - 3\lambda^2 - 10\lambda + 24$

- Factor independent term

$[1, 2, 3, 4, 6, 12, 24] \pm$

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -10 & 24 \\ & \downarrow & +2 & +-2 & +-24 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

- Test factors until last term = 0
- here 2 is a root so we have

$(\lambda - 2)(1\lambda^2 - 1\lambda - 12) = 0$

$(\lambda - 2)(\lambda - 4)(\lambda + 3)$

☆ This will not find complex roots.

Initial Value Problems:

- Put eigenvector solns. into matrix with condit. vector

Ex: evectors: $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$

$x(0) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\begin{array}{cccc} C_1 & C_2 & C_3 & \\ \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 2 \\ 0 & 1 & 1 & 3 \end{bmatrix}$$

- find RREF. $V_1 = C_1$ etc.

Wronskian for linear sys.

- If y_1, y_2 etc are solutions to $x' = Ax$ they are linearly independent only if $W[y_1, y_2] \neq 0$
- find W by forming matrix with y_1, y_2, etc as the columns and take the determinant.

Ex: $C_1\begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{3t} + C_2\begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-t}$

$\begin{vmatrix} 2e^{3t} & -e^{-t} \\ e^{3t} & 2e^{-t} \end{vmatrix} \neq 0$ ∴ are independent

Algebraic multiplicity:

root is repeated when factoring for λ.
(doesn't always mean 2 eigenvectors)

Geometric multiplicity:

eigenvalue has multiple eigenvectors.

Phase Planes: 2x2 systems

6 cases

real:
- Nodal source → both positive
- Nodal sink → both negative
- saddle-point → 1 pos. 1 neg

complex:
- spiral source → real part pos.
- spiral sink → real part neg
- center → real part 0

Real eigenvalues:
- Nodal source: "source" means everything goes away from (0,0)
- A node will always have an asymptote (usually 2).
  this is a solution curve that follows a straight line.
  are actually the eigenvectors.
- nodal 'sink' means all solution curves go toward (0,0)
- saddle-point will also have asymptotes, 1 with solutions approaching and 1 with solutions going away. other solutions approach (0,0) before turning away.

complex eigenvalues:
- spiral source does not have asymptotes, solution curves are never straight lines. solutions spiral away from source (0,0)
- spiral sink no asymptotes. soln. curves spiral toward center.
- center no asymptotes. soln. curves form circles/oval loops around center.

Limit behavior:
- Nodal sink – both vals neg as $t\to\infty$ approaches $\begin{bmatrix}0\\0\end{bmatrix}$
- Nodal source – both vals pos as $t\to\infty$ diverges
- saddle-point – 1 pos 1 neg val as $t\to\infty$ if $c_1=0$ → approaches $\begin{bmatrix}0\\0\end{bmatrix}$; $c_1\neq0$ → diverges.
- spiral sink – $\lambda=\emptyset\pm\mu i$ $\emptyset<0$ as $t\to\infty$ $e^{\emptyset t}\to\begin{bmatrix}0\\0\end{bmatrix}$ while sin/cos oscillate. soln $\to\begin{bmatrix}0\\0\end{bmatrix}$
- spiral source – $\emptyset>0$ as $t\to\infty$ diverges
- center $\emptyset=0$ as $t\to\infty$ soln. loops around origin

☆ $x'=Ax$ will always have an equilibrium point at (0,0). can have more if 0 is an eigenvalue.

Relax, you can do it. You have more time left. Don't rush.

Converting a 2nd order ODE to system of 1st order ODE's (linear)

Idea: If we have an ODE $u''+au'+bu=0$

Define: $x_1=u$, $x_2=u'$ then $x_1'=u'=x_2$ and $x_2'=u''$
$=0x_1+1\cdot x_2$

so $x'=Ax$ will be

$$x'=\begin{bmatrix}0&1\\-b&-a\end{bmatrix}x$$

u1 shouldn't have prefix

E.x. $2u''-3u'+4u=0$ $u(0)=-1$ $u'(0)=2$

Convert IVP to a system of 1st order ODE's

First find correct form: $\underbrace{u''-\tfrac{3}{2}u'}_{a}+\underbrace{2u}_{b}=0$ notice $a=-\frac{3}{2}$ $-a=\frac{3}{2}$, $b=2$ $-b=-2$

$u''=\frac{3}{2}u'-2u$

$x_1=u$ $x_1'=x_2$

$x_2=u'$ $x_2'=u''$

$u''=-2x_1+\frac{3}{2}x_2 \to x'=\begin{bmatrix}0&1\\-2&3/2\end{bmatrix}x$

$$x(0)=\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}=\begin{bmatrix}u(0)\\u'(0)\end{bmatrix}=\begin{bmatrix}-1\\2\end{bmatrix}$$

What if 2nd order ODE isn't homogeneous?

$u''+u'+u=\sin t$

note: $x'-A\vec{x}=\vec{b}$ nonhomogeneous; $x'-A\vec{x}=0$ homogeneous

$x_1=u$ $x_1'=x_2$

$x_2=u'$ $x_2'=u''$

$u''=-x_1-x_2+\sin t$

$u''=-u-u'+\sin t$

so system is $x'=\begin{bmatrix}0&1\\-1&1\end{bmatrix}+\begin{bmatrix}0\\\sin t\end{bmatrix}$ (always 0)

Laplace transforms

1	$\frac{1}{s}$, $s>0$
e^{at}	$\frac{1}{s-a}$, $s>a$
t^n	$\frac{n!}{s^{n+1}}$, $s>0$
$\sin(bt)$	$\frac{b}{s^2+b^2}$, $s>0$
$\cos(bt)$	$\frac{s}{s^2+b^2}$, $s>0$
$\cosh(bt)$	$\frac{s}{s^2-b^2}$
$\sinh(bt)$	$\frac{b}{s^2-b^2}$
$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$, $s>a$
$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$, $s>a$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s>a$
$e^{ct}f(t)$	$F(s-c)$
$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-\dots-f^{(n-1)}(0)$

Example:

$y''-2y'+2y=e^{-t}$, $y(0)=0$, $y'(0)=1$

$s^2Y(s)-sy(0)-y'(0)-2(sY(s)-y(0))+2Y(s)=\frac{1}{s+1}$

Non-linear systems:
- find eq points
- find Jacobian
- For each EQ. pt. plug into J, get matrix, then find eigenvalues.

$$J(x,y)=\begin{bmatrix}\frac{\partial f}{\partial x}&\frac{\partial f}{\partial y}\\\frac{\partial g}{\partial x}&\frac{\partial g}{\partial y}\end{bmatrix}$$

Asymptotically stable: nodal sink, spiral sink

unstable: nodal source, spiral source, saddle

stable: center

Level Curves: $\frac{dy}{dt}=\frac{dy}{dx}\frac{dx}{dt}$

Ex: $x'(t)=4-2y$ $\quad y'(t)=\frac{dy}{dx}x'(t)$

$y'(t)=12-3x^2$

$12-3x^2=\frac{dy}{dx}(4-2y)$

If A has 1 eigenvalue with algebraic multiplicity of 2 (2 must be real)
A will have a special kind of node.
2 cases:
1) λ has geo. mult. 1: improper node
2) λ has geo. mult. 2: proper node

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if A is invertable then 0 is an eigenvalue? F

if 2 is an eigenvalue of an invertable matrix, then $\frac{1}{2}$ is an eigenvalue of A^{-1} T

if A is real and $z(t)$ is complex-val soln then $\frac{z(t)-\bar{z}(t)}{2}$ is a real val soln? F

if Matrix A is non-singular then 0 is an eigenvalue? F

if A is real and $\det(A)<0$ then $\lim_{t\to+\infty}x(t)=0$ for any soln. $x(t)$? F

The exponential matrix e^{tA} is a fundamental matrix of the ODE system $x'(t)=Ax$? T

if λ is an eigenvalue of an invertable matrix A, then λ^{-1} is an eigenvalue of A^{-1}. T

If the general soln. of a 2-dimensional ODE system $x'(t)=Ax(t)$ is $c_1V+c_2(Vt+W)$ then A has 0 as an eigenvalue with algebraic multiplicity of 2. T

If A is an order 2 real matrix with eigenvalues 1 and −1 then every solution to $x'(t)=Ax(t)$ that does not start at the origin will diverge. F

Log/exp rules:
$\ln(ab)=\ln a+\ln b$
$\ln(a/b)=\ln a-\ln b$
$\ln(a^b)=b\ln a$
$\ln(1)=0$
$\ln(e)=1$
$\ln(e^b)=b$
$a^x\cdot a^y=a^{x+y}$
$(a^x)^y=a^{xy}$

Integrals
$\int\sec^2x=\tan x$ $\quad\int\ln x=x\ln|x|-x$
$\int\sec x\tan x=\sec x$ $\quad\int\frac{1}{x}=\ln|x|$
$\int\csc x\cot x=-\csc x$
$\int\csc^2x=-\cot x$ $\quad\int\frac{1}{\sqrt{a^2-u^2}}=\sin^{-1}\left(\frac{u}{a}\right)$
$\int\tan x=\ln|\sec x|$
$\int\sec x=\ln|\sec x+\tan x$
$\int\frac{1}{a^2+u^2}=\frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$
$\int\frac{10}{x(1-\frac{x}{c})}dx=10(\ln|x|-\ln|x-c|)$
$\int xe^xdx=(x-1)e^x+C$

Logistic Model for Population of one species:

$x'(t) = x(r - ax)$ or $x'(t) = r(1 - \frac{x}{K})x$

y/x = population
r = growth rate
b/K = carrying cap
y'/x' = rate change

$x'(t) = x(r_1 - a_1 x - b_1 y)$

$y'(t) = y(r_2 - a_2 y - b_2 x)$