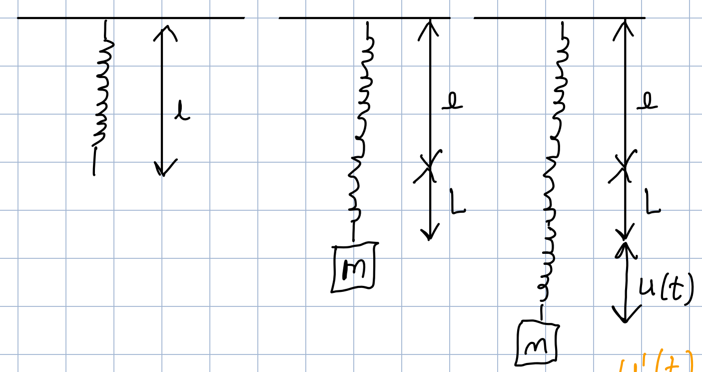
Oscillations/vibrations

Consider a system with a mass on a string

Quiz → 2nd order non homo ODE find gen soln. sum 2 func



Spring has length l, a mass stretches the spring a distance L. if the spring is at equilibrium then we have $mg = kL$

Now consider a displacement $u(t)$ from equilibrium. (down is positive for $u(t)$)

$u'(t)$ would be velocity and $u''(t)$ is acceleration.

$mu''(t)$ = force acting on the mass.

We have (at most) 4 forces acting on the mass:

gravity: mg

Spring: $-k(L+u(t))$

damping: $-\gamma u'(t)$ dampen force depends on velocity

$\gamma > 0$ is damping constant

external force: $g(t)$ can be anything acting on system from external source.

So we get

$$mu''(t) = mg - k(L+u(t)) - \gamma u'(t) + g(t)$$

$$mu''(t) = \underbrace{mg - kL}_{=0} - ku(t) - \gamma u'(t) + g(t)$$

$$mu''(t) + \gamma u'(t) + ku(t) = g(t)$$

2nd order ode with constant coefficients

└ m, γ, k are all positive

initial conditions would be

$u(0) = u_0$ initial position

$u'(0) = v_0$ initial velocity

Likely to be asked 3 things

1) setup ODE/initial value pbm based on info about the system

2) solve the system.

3) determine some other properties/behavior of the system.

Ex. A mass of 10 kg stretches spring 50 cm. mass is acted on by external force of $5\sin(2t)$ N and moves in a medium that imparts a viscous force of 3 N when the speed of the mass is 10 cm/s. If the mass is pulled down 2cm and released, formulate the IVP describing the motion of the mass.

Should look like: $mu'' + \gamma u' + ku = g(t)$

$u(0) = u_0$

$u'(0) = V_0$

m mass

γ damping constant

k spring constant

$g(t)$ external force

u_0 initial displacement

V_0 initial velocity

$$N = \frac{kg \cdot m}{s^2}$$

$m = 10\text{ kg}$

$g(t) = 5\sin(2t)$

"viscous force" = damping $\gamma \cdot u'$

$mg = kL$

$\underset{kg}{10} \cdot \underset{m/s^2}{9.8} = \underset{kg/s^2}{k} \cdot 0.5$ (50 cm in m)

$98 = \frac{k}{2}$

$k = 196$

$\gamma \cdot u' = 3$

$\gamma \cdot 0.1 = 3$

$\gamma = 30$

$u_0 = 2\text{cm} = 0.02$

$V_0 = 0$

(starts stationary $\therefore V_0 = 0$)

$10u'' + 30u' + 196u = 5\sin 2t$

$u(0) = 0.02$

$u'(0) = 0$

Terminology:

if $g(t) = 0$ the oscillation is free (homogeneous)

if $g(t) \neq 0$ the oscillation is forced (non homogeneous)

if $\gamma = 0$ the oscillation is undamped

if $\gamma \neq 0$ the oscillation is damped

if we have undamped free oscillation:

$\hookrightarrow mu'' + ku = 0 \qquad m, k > 0$

soln: $mr^2 + 0r + k = 0$

$$r = \frac{-0 \pm \sqrt{0^2 - 4mk}}{2m} = \frac{\pm\sqrt{-4mk}}{2m} = \frac{\pm 2i\sqrt{mk}}{m} = \frac{\pm i\sqrt{mk}}{m} = 0 \pm i\frac{\sqrt{k}}{\sqrt{m}}$$

$\hookrightarrow u(t) = C_1\cos\left(\sqrt{\frac{k}{m}}t\right) + C_2\sin\left(\sqrt{\frac{k}{m}}t\right)$

$\sqrt{\frac{k}{m}}$ is called the frequency, also denoted ω_0 (Angular freq)

$\frac{1}{\omega_0} = \sqrt{\frac{m}{k}}$ is the period (angular)

Also want amplitude and phase

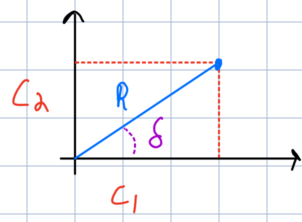
$$u(t) = C_1 \cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + C_2 \sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

can be:

$$= R\cos(\delta)\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + R\sin(\delta)\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

$\cos(\delta) = \frac{C_1}{R}$ $\quad C_1 = R\cos(\delta)$

$\sin(\delta) = \frac{C_2}{R}$ $\quad C_2 = R\sin(\delta)$ $\quad$ because



$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$$

so

$$R\cos(\delta)\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + R\sin(\delta)\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

$$= R\cos\left(\delta - \sqrt{\tfrac{k}{m}}\,t\right) \quad \text{or} \quad \underbrace{R}_{\text{amplitude}}\cos\left(\sqrt{\tfrac{k}{m}}\,t \underbrace{-\,\delta}_{\text{phase}}\right)$$

E.x. Consider a system modeled by

$$u'' + 4u = 0$$

Find: frequency, period, amplitude, phase

$\text{freq} = \sqrt{\frac{k}{m}} = \sqrt{\frac{4}{1}} = 2$

$\text{Period} = \frac{1}{2}$

for amp and phase we need to solve the IVE first

$r^2 + 4 = 0$

$r = \pm 2i$

$u(t) = C_1\cos(2t) + C_2\sin(2t)$

$4 = C_1 \quad C_2 = -3$

$4\cos(2t) - 3\sin(2t)$

$r^2 = 4^2 + -3^2$

$r = 5$

$\delta = \tan^{-1}\left(\frac{-3}{4}\right)$

Undamped forced oscillations

$$mu'' + ku = F\cos(\omega t)$$

$$mr^2 + k = 0$$

$$r = \pm i\sqrt{\tfrac{k}{m}}$$

$$u_H = C_1\cos\left(\sqrt{\tfrac{k}{m}}\,t\right) + C_2\sin\left(\sqrt{\tfrac{k}{m}}\,t\right)$$

the natural freq. or internal freq. of the system is $\omega_0 = \sqrt{\frac{k}{m}}$

the external freq is ω (from the Homogeneous part)

Resonance occurs if we have a system

$mu'' + ku = F\cos(\omega t)$ or $F\sin(\omega t)$

and $\omega = \sqrt{\frac{k}{m}} \leftarrow \omega_0$

Is resonance?

$u'' + 4u = \cos(2t)$ yes

$u'' - 4u = \cos(2t)$ no. needs non-real roots. (isn't even an oscillation)

$u'' + 9u = \cos(2t)$ no (different freqs)

$u'' + 2u' + 5u = \cos(2t)$ no (can't have damping)

Damped Free Oscillations

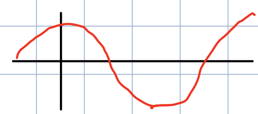
$mu'' + \gamma u' + ku = 0 \qquad m, \gamma, k > 0$

$$mr^2 + \gamma r + k = 0$$

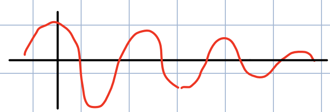
$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

$$\boxed{\begin{array}{l}\text{* 3 cases:}\\ \gamma^2 - 4mk > 0 \text{ over damped}\\ \gamma^2 - 4mk = 0 \text{ crit. damped}\\ \gamma^2 - 4mk < 0 \text{ under damped}\end{array}}$$

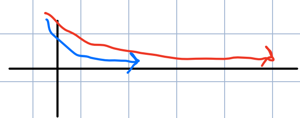
undamped



under damped



over damped (red)



crit. damped (like over damped but goes to 0 faster)

② cases $\boxed{1}$ $\omega \neq \omega_0$

$\boxed{2}$ $\omega = \omega_0$

$\boxed{1}$ $u_p = A\cos(\omega t) + B\sin(\omega t)$

solve for A and B

$$u_p = \frac{F}{K - m\omega^2}\cos(\omega t) + 0$$

then $u_t = C_1\cos(\omega_0 t) + C_2\sin(\omega_0 t) + \frac{F}{K - m\omega^2}\cos(\omega t)$

assuming we have $u(0)=0$ and $u'(0)=0$

solve for $C_1 + C_2$

$$u_t = \frac{-F}{K - m\omega^2}\cos(\omega_0 t) + \frac{F}{K - m\omega^2}\cos(\omega t)$$

$$= \frac{-F}{K - m\omega^2}\left(\cos(\omega_0 t) - \cos(\omega t)\right)$$

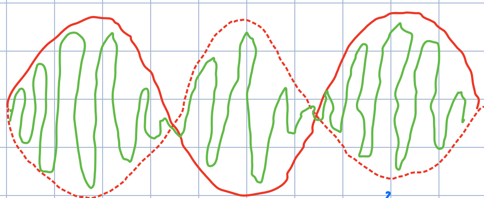
Trig ID

$\cos(\alpha) - \cos(\beta)$

$-2\sin\left(\frac{\alpha-\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$

$$\frac{2F}{K - m\omega^2}\underbrace{\sin\left(\frac{\omega_0 - \omega}{2}t\right)}_{\text{slow}}\underbrace{\sin\left(\frac{\omega_0 + \omega}{2}t\right)}_{\text{quick}}$$

if we have ω_0 and ω close together then we have a sine with a small freq and a sine with a large freq.



green → $u_t(t)$

this gives us an oscillation with an amplitude that changes at the freq $\frac{\omega_0 - \omega}{2}$ (the half difference of the internal and external frequencies). this phenomenon is called beat

EX: does the system $u'' + 4u = \cos(1.9t)$ model a beat. If so, at what frequency does the amplitude vary?

$$\omega_0 = \sqrt{\frac{K}{m}} = \sqrt{\frac{4}{1}} = 2$$

yes it models a beat because $\omega_0 \neq \omega$

Amp varies at the frequency of

$$\frac{\omega_0 - \omega}{2} = \frac{2 - 1.9}{2} = \frac{0.1}{2} = 0.05 \text{ or } \frac{1}{20}$$

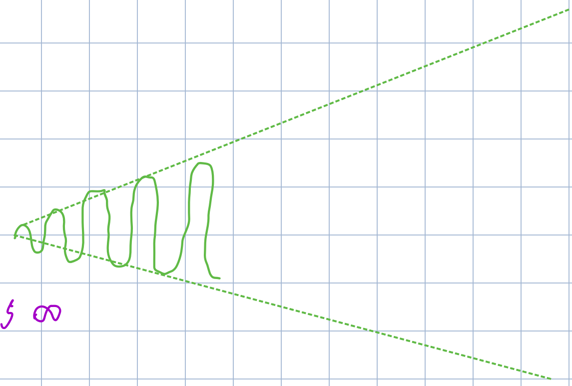
Case $\boxed{2}$ $\omega = \omega_0$ $\quad mu'' + ku = F\cos(\omega t)$

$u_H = C_1\cos(\omega t) + C_2\sin(\omega t) \quad (\omega = \sqrt{\tfrac{k}{m}})$

$u_p = A\cos(\omega t) + B\sin(\omega t)$ wont work

$u_p = At\cos(\omega t) + B\sin(\omega t)$

this gives us the resonance phenomenon.

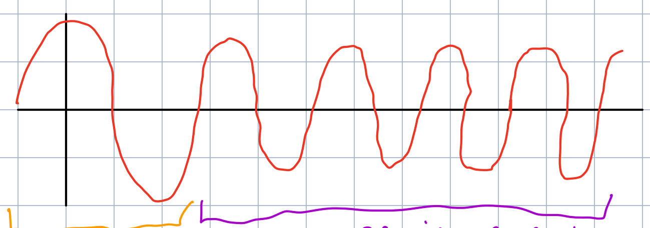


the oscillation has an amplitude that goes ∞ as t increases.

Ex. $u'' + u' + \frac{5}{4}u = 3\cos(t)$

(under damped)

soln. $\underbrace{\frac{22}{17}e^{-t/2}\cos(t) + \frac{14}{17}e^{-t/2}\sin(t)}_{\text{transient soln.}} + \underbrace{\frac{12}{17}\cos(t) + \frac{48}{17}\sin(t)}_{\text{steady-state solution.}}$



starts higher

reduces in magnitude then maintains.

Transient solution decays to 0, while the steady state solution oscilates forever.

Y_H is transient solution

Y_p is the steady state solution.