Phase Planes (2x2 systems ∴ 2D)

Def: A visual representation of a 2D system $x' = Ax$. Which has the variables x_1 and x_2 as the horizontal and vertical axes, and arrows showing the direction the solution is changing at each point.
The whole plane represents the general solution, particular solutions will be represented by solution curves.
(3x3 would be 3D)

- we will be focusing on:
  - equilibrium points
  - stability
  - limit behavior
- will be asked to match system to phase plane.

Note: $x' = Ax$ will always have equilibrium point at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
Also: there can only be more eq. points if A has 0 as an eigenvalue.
(exam 2 will only feature phase planes for matrices without 0 eigenvalue)

6 cases

Eq. Pt. Behavior		Eigenvalues of A
		real:
Nodal source	→	Both positive
Nodal sink	→	Both negative
saddle-point	→	1 pos. 1 neg
		complex:
Spiral source	→	real part pos.
Spiral sink	→	real part neg
center	→	real part 0

Real eigenvalues:
- Nodal source: "source" means everything goes away from (0,0)
- A node will always have an asymptote (usually 2).
  - ↳ this is a solution curve that follows a straight line.
  - ↳ are actually the eigenvectors.
- nodal 'sink' means all solution curves go toward (0,0)
- saddle-point will also have asymptotes, 1 with solutions approaching and 1 with solutions going away. Othe solutions approach (0,0) before turning away.

complex eigenvalues:
- Spiral source does not have asymptotes, solution curves are never straight lines. solutions spiral away from source (0,0)
- Spiral sink no asymptotes. soln. curves spiral toward center.
- center no asymptotes. soln. curves form circles/oval loops around center.

Example problem:

Given the following phase plane which of the following systems could correspond to the phase plane?

solve by finding eigenvalues of the supplied systems and chose the one that is one of the 6 possible cases that matches the graph.

Limit Behavior:

Look at the general soln. for each phase plane case:

(eigen values)

- Nodal sink: both negative real

  Gen soln: $c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

  Limit as $t \to \infty$: approaches $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- Nodal source: both positive real

  Gen. soln: $c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

  Limit as $t \to \infty$: diverges

- Saddle-point: 1 pos. 1 neg.

  Gen. soln: $c_1 v_1 e^{\lambda_1 t} + c_2 v_2 e^{\lambda_2 t}$

  Limit as $t \to \infty$: if $c_1 = 0$, approaches $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

  if $c_1 \neq 0$, diverges

- Spiral sink: $\lambda = \phi \pm \mu i$ $\quad \phi < 0$

  Gen. soln: $c_1 e^{\phi t}(\vec{a}\cos(\mu t) + \vec{b}\sin(\mu t)) + c_2 e^{\phi t}(\vec{a}\sin(\mu t) - b\cos(\mu t))$

  Limit as $t \to \infty$: $e^{\phi t} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ while the sines/cos oscillate

  so soln goes to $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

- Spiral source $\lambda = \phi \pm \mu i$ $\quad \phi > 0$

  Gen. soln: $c_1 e^{\phi t}(\vec{a}\cos(\mu t) + \vec{b}\sin(\mu t)) + c_2 e^{\phi t}(\vec{a}\sin(\mu t) - b\cos(\mu t))$

  Limit as $t \to \infty$: diverges

- center $\lambda = \pm \mu i$

  Gen. soln: $c_1 e^{\phi t}(\vec{a}\cos(\mu t) + \vec{b}\sin(\mu t)) + c_2 e^{\phi t}(\vec{a}\sin(\mu t) - b\cos(\mu t))$

  Limit as $t \to \infty$: solution loops around origin