More Phase Planes:

For linear systems $x' = Ax$, we covered most of the cases based on the eigenvalues of A.

We skipped: A has repeated eigenvalue
A has 0 eigenvalue

If A has 1 eigenvalue with algebraic multiplicity of 2 (λ must be real)
A will have a special kind of node.

2 cases:

1) λ has geometric multiplicity 1: improper node
2) λ has geometric multiplicity 2: proper node

Ex: $A = \begin{bmatrix} 3 & 4 \\ -1 & -1 \end{bmatrix}$ $\lambda = 1$ $\vec{v} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ (Alg mult: 2 geo. mult: 1)

We get an improper node which looks like a node but with exactly 1 asymptote.

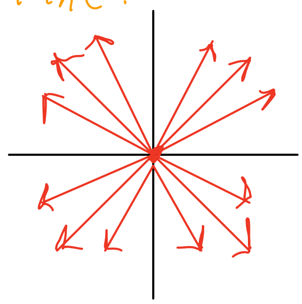
If $\lambda > 0$ we get a source unstable.
If $\lambda < 0$ we get a sink asymptotically stable.
If $\lambda = 0$ we get something else.

Ex: $x' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x$

eigenvalue: $\lambda = 1$
eigenvectors: $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

proper node: has straight lines in every direction.

Phase plane:



What if $\lambda=0$ is an eigenvalue?

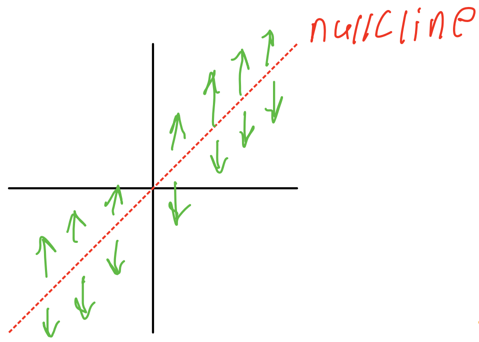
Ex:

$$x' = \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} x \qquad \text{eigenvalues: } \lambda = 1, 0 \qquad \text{eigenvectors: } \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

General soln: $\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} e^t$

notice: if we have $c_2 = 0$ then the soln. does not depend on t. this means we have infinitely many equilibrium points.

These ∞ equilibria will lie on a nullcline which corresponds to the eigenvector for $\lambda=0$.



in this case the second eigenvalue is positive so solutions move away from the nullcline.
if the second eval. is negative the solns. would move twords it.

What if $\lambda=0$ is the only eigenvalue?

Ex:

$$x' = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \qquad \lambda = 0 \qquad \vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

algebraic multiplicity: 2
geometric multiplicity: 1

Gen. soln: $\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$

solutions are parallel to the nullcline going in opposite directions accross the nullcline.

What if we have: $\lambda=0$ algebraic multiplicity: 2
geometric multiplicity: 2
(only happens with 0 matrix)

$$x' = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} x \qquad \lambda = 0 \qquad \vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\vec{x}' = \vec{0}$ for every point $\therefore$ every point is an equilibrium point.