Nonlinear ODE systems:

↳ linear systems $X' = Ax$ where A is some constant matrix

$$\vec{x}(t) = \begin{bmatrix} x_1' t \\ x_2' t \end{bmatrix} = \begin{bmatrix} a_1 x_1 + a_2 x_2 \\ b_1 x_1 + b_2 x_2 \end{bmatrix}$$

(note this is a linear ODEs)

Nonlinear (but locally linear)

(we will only see 2D x, y)

- Autonomous (independent var. t does not appear explicitly)
- This is also called a planar system.

Equilibrium points of nonlinear systems.

for linear systems $x' = Ax$ there is always EQ point at $(0,0)$.
For nonlinear there could be multiple EQ points at any location.

Ex:

$$x'(t) = xy$$
$$y'(t) = x + y + 2$$

nonlinear bc multiplication

set $x'(t)$ and $y'(t)$ equal to 0 and solve for x and y.

1 $0 = xy$
2 $0 = x + y + 2$

1) $xy = 0$

$x = 0 \quad y = 0$

2)

$x = -2 \quad (0, -2)$
$y = -2 \quad (-2, 0)$

2 EQ Points

Ex: $x'(t) = (x-y)(x+y)$
$y'(t) = x(y+1)$

1. $0 = (x-y)(x+y)$ → $x - y = 0 \quad x + y = 0$
$x = y \quad x = -y$
2. $0 = x(y+1)$ → $y(y+1) = 0$

$y = 0 \quad y = -1$
$x = -1$

$(0,0) \quad (-1,-1)$

$(0,0) \quad (1,-1)$

3 EQ Points

Classifying Equilibrium points

- For $x' = Ax$ we classified eq at $(0,0)$ based on the eigenvalues of A.*
  * Can't do this for nonlinear system.
- For nonlinear systems we can get an approximation, using a taylor series, of our system around each equilibrium that is linear.

2 Variable taylor series for the system
$$x'(t) = f(x,y)$$
$$y'(t) = g(x,y)$$
with the series centered at the eq pt.

$$\begin{cases} f(x,y) = f(x_0,y_0) + \frac{\partial f}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)\cdot(y-y_0) + \dots \text{(higher order terms)} \\ g(x,y) = g(x_0,y_0) + \frac{\partial g}{\partial x}(x_0,y_0)\cdot(x-x_0) + \frac{\partial g}{\partial y}(x_0,y_0)\cdot(y-y_0) + \dots \end{cases}$$

Note:
- since (x_0,y_0) is an e.q point $f(x_0,y_0)=0$, $g(x_0,y_0)=0$
  ∴ we can ignore the first term in the Taylor series.
- since we only need approximation close to (x_0,y_0) the (x_0,y_0) will be small, so the higher order terms will be much smaller than the 1st order terms. ∴ we only need the first order terms.

$$f(x,y) = \frac{\partial f}{\partial y}(x_0,y_0)(x-x_0) + \frac{\partial f}{\partial y}(x_0,y_0)(y-y_0)$$

$$g(x,y) = \frac{\partial g}{\partial y}(x_0,y_0)(x-x_0) + \frac{\partial g}{\partial y}(x_0,y_0)(y-y_0)$$

now use change of variables:

$$u = x - x_0 \qquad u(t) = x(t) - x_0 \qquad u'(t) = x'(t) = f(x,y)$$
$$v = y - y_0 \qquad v(t) = y(t) - y_0 \qquad v'(t) = y'(t) = g(x,y)$$

$$\therefore\ u'(t) = \frac{\partial f}{\partial x}(x_0,y_0)\,u + \frac{\partial f}{\partial y}(x_0,y_0)\,v$$

$$v'(t) = \frac{\partial g}{\partial x}(x_0,y_0)\,u + \frac{\partial g}{\partial y}(x_0,y_0)\,v$$

this is a linear system

$$\begin{bmatrix} u' \\ v' \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial f}{\partial x}(x_0,y_0) & \frac{\partial f}{\partial y}(x_0,y_0) \\ \frac{\partial g}{\partial x}(x_0,y_0) & \frac{\partial g}{\partial y}(x_0,y_0) \end{bmatrix}}_{} \begin{bmatrix} u \\ v \end{bmatrix}$$

this is the jacobian usually $J(x_0,y_0)$

so if we want to classify eq points of a system $x'(t) = f(x,y)$
$y'(t) = g(x,y)$

- find eq points
- find jacobian $J(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$ matrix with functions as the entries.
- For each EQ. Pt. plug into J, get matrix, then find eigenvalues.

Classify nonlinear system eq pts.

Still have the ones from linear systems
↳ nodal, source, saddle, center.

we will define 3 types of stability:

1. Unstable → some solutions that start close to eq. pt. move away.

2. Stable → solutions that start close to eq. pt. stay close.

3. Asymptotically stable. → solutions that start close to eq. pt. approach it.

<u>unstable:</u>	<u>stable:</u>	<u>Asymptotically stable:</u>
nodal source	center	nodal sink
spiral source		spiral sink
saddle		

EX: $x'(t) = xy$

$y'(t) = x + y + 2$

1. Find eq. pts. $(0,-2)$ $(-2,0)$
2. solve jacobian $\begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix} = \begin{bmatrix} y & x \\ 1 & 1 \end{bmatrix}$
3. Plug in equilibria and compute eigenvalues

$J(0,-2) = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix}$ locally linear system around $(0,-2)$

$\begin{bmatrix} -2-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix} = (-2-\lambda)(1-\lambda)$

$\lambda = -2, 1$ real and opposite sign so $(-2,0)$ is saddle pt.

$J(-2,0) = \begin{bmatrix} 0 & -2 \\ 1 & 1 \end{bmatrix}$ $(-\lambda)(1-\lambda)+2$

$\lambda^2 - \lambda + 2$

$\lambda = \frac{1 \pm \sqrt{1-8}}{2} = \frac{1 \pm i\sqrt{7}}{2}$ spiral source at $(-2,0)$

Ex: $x'(t) = (1-y)(2x-y)$
$y'(t) = (2+x)(x-2y)$

Classify eq pts and determine their stability.

① for finding equilibria, better to have f and g factored

$f=0$

$1-y=0$ → $y=1$

- $2+x=0$ → $x=-2$ → $(-2,1)$
- $x-2y=0$ → $x=2$ → $(2,1)$

$2x-y=0$ → $y=2x$

- $y=2(-2)$ → $y=-4$ → $(-2,-4)$
- $x-2(2x)=0$ → $x-4x=0$ → $-3x=0$ → $x=0$ → $(0,0)$

② jacobian

$f(x,y) = (1-y)(2x-y) = 2x - y - 2xy + y^2$

$g(x,y) = (2+x)(x-2y) = 2x - 4y + x^2 - 2xy$

$$J(x,y) = \begin{bmatrix} 2-y & -1-2x+2y \\ 2+2x-2y & -4-2x \end{bmatrix}$$

③ $J(-2,1) = \begin{bmatrix} 0 & 5 \\ -4 & 0 \end{bmatrix}$ $\lambda = \pm\sqrt{20}\, i$

center, stable

$J(2,1) = \begin{bmatrix} 0 & -3 \\ 4 & -8 \end{bmatrix}$ $\lambda = -6, -2$

nodal sink, asymptotically stable

$J(-2,-4) = \begin{bmatrix} 10 & -5 \\ 6 & 0 \end{bmatrix}$ $\lambda = 5 \pm \sqrt{5}\, i$

spiral source, unstable

$J(0,0) = \begin{bmatrix} 2 & -1 \\ 2 & -4 \end{bmatrix}$ $\lambda = \dfrac{-2 \pm \sqrt{28}}{2} = \dfrac{-2 \pm 2\sqrt{7}}{2} = -1 \pm \sqrt{7}$

need to know signs

saddle, unstable

- $-1-\sqrt{7}$ negative
- $-1+\sqrt{7}$: $\sqrt{4} \le \sqrt{7} \le \sqrt{9}$, $2 \le \sqrt{7} \le 3$ positive