Matrix:

- ↳ swap rows
- ↳ multiply rows by scalar
- ↳ add rows to eachother

$\begin{bmatrix}1&0&3\\0&1&2\\0&0&0\end{bmatrix}$ ✓ $\begin{bmatrix}1&0&2\\0&1&0\\0&0&1\end{bmatrix}$ ✗

vectors linearly independent?

$\begin{bmatrix}1\\2\\-1\end{bmatrix},\begin{bmatrix}2\\1\\3\end{bmatrix},\begin{bmatrix}-4\\1\\11\end{bmatrix} \rightarrow \begin{bmatrix}1&2&-4&0\\2&1&1&0\\-1&3&11&0\end{bmatrix}$

$\begin{bmatrix}1&0&2&0\\0&1&-3&0\\0&0&0&0\end{bmatrix} \neq \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\end{bmatrix}$ No. not = I.D. matrix

A^{-1} (inverse of A) of 2x2

$A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ $A^{-1}=\frac{1}{|A|}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$

$=\frac{1}{ad-bc}\begin{bmatrix}d&-b\\-c&a\end{bmatrix}$ ($|A|$ = determinant)

Trace: sum of diagonal

Invertable matrix theorem

(All must be true or false)

For square $n \times n$ matrix

- A is invertable
- A has RREF = I.D. matrix
- $A\vec{x}=\vec{b}$ has exactly 1 soln. (for any b)
- the columns are linearly independent.
- Determinant of $A \neq 0$
- 0 is not an eigenval of A

Quadratic formula

$$0=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

$|A|$ = product of the eigenvals

trace(A) = sum of eigenvals

∴ if A is 2x2 we can find eigenvals if given $|A|$ and tr(A)

Ex: tr(A) = 5 $|A|=6$ $\lambda=2,3$

$\lambda^2-5\lambda+6$

(2 nums multiply to $|A|$ add to tr(A))

Eigenvalues and Eigenvectors

$x'=\begin{bmatrix}a&b&c\\d&e&f\\g&h&k\end{bmatrix}x$ (this is A)

- subtract λ from diag. entries
- find determinant of A
- find 0s of polynomial
- solve $|A-\lambda I|$ for all λ

☆ if $\lambda=\varnothing-\mu i$ is an eigenvalue then $\lambda=\varnothing+\mu i$ also is.

☆ if $\vec{v}=\vec{a}+\vec{b}i$ is eigenvector then $\vec{v}=\vec{a}-\vec{b}i$ also is.

RREF will always have atleast 1 row of all zeroes.

0 can't be an eigenvector.

☆ if $|A-\lambda I|$ is not invertable then the system has ∞ solutions.

Gen soln:

* $\vec{x}(t)=C_1\alpha_1e^{\lambda_1 t}+C_2\alpha_2e^{\lambda_2 t}+C_n\ldots$

* Will have 1 independent soln. per row/column in A.

If A only has 1 eigenval and 1 eigenvector we find a generalized eigenvector.

Ex: $x'=\begin{bmatrix}1&-2\\2&5\end{bmatrix}$ evector: $\begin{bmatrix}-1\\1\end{bmatrix}$ eval: 3

$\lambda=3 \rightarrow \begin{bmatrix}-2&-2\\2&2\end{bmatrix}$ bring evector in

$\begin{bmatrix}-2&-2&-1\\2&2&1\end{bmatrix}=\begin{bmatrix}1&1&1/2\\0&0&0\end{bmatrix}$

$V_1+V_2=\frac{1}{2}$ $V_1=-V_2+\frac{1}{2}$

$V_2=\alpha$ $\alpha\begin{bmatrix}-1\\1\end{bmatrix}+w\begin{bmatrix}1/2\\0\end{bmatrix}$

Gen. Soln:

$\vec{x}(t)=C_1\begin{bmatrix}-1\\1\end{bmatrix}e^{3t}+C_2\left(\begin{bmatrix}-1\\1\end{bmatrix}te^{3t}+\begin{bmatrix}1/2\\0\end{bmatrix}e^{3t}\right)$

☆ Complex eigenvalues and complex eigenvectors always come in pairs.

Eval: $\lambda=\varnothing+\mu i$ Evec: $\vec{V}=\vec{a}+\vec{b}i$

$\vec{x}(t)=C_1e^{\varnothing t}(\vec{a}\cos\mu t-\vec{b}\sin\mu t)+C_2e^{\varnothing t}(\vec{a}\sin\mu t+\vec{b}\cos\mu t)$

$\alpha\begin{bmatrix}1+i\\2\end{bmatrix}=\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{a}+\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{b}i$

$\cos(-x)=\cos(x)$

$\sin(-x)=-\sin(x)$

Synthetic Division

Ex: $1\lambda^3-3\lambda^2-10\lambda+24$

- Factor independent term

[1, 2, 3, 4, 6, 12, 24] ±

$$\begin{array}{r|rrrr} 2 & 1 & -3 & -10 & 24 \\ & & +2 & +-2 & +-24 \\ \hline & 1 & -1 & -12 & 0 \end{array}$$

- Test factors until last term = 0
- here 2 is a root so we have

$(\lambda-2)(1\lambda^2-1\lambda-12)=0$

$(\lambda-2)(\lambda-4)(\lambda+3)$

☆ This will not find complex roots.

Initial Value Problems:

- Put eigenvector solns. into matrix with condit. vector

Ex: evectors: $\begin{bmatrix}1\\0\\0\end{bmatrix}\begin{bmatrix}0\\1\\1\end{bmatrix}\begin{bmatrix}0\\-1\\1\end{bmatrix}$

$x(0)=\begin{bmatrix}1\\2\\3\end{bmatrix}$

$$\begin{array}{cccc} C_1 & C_2 & C_3 & \end{array}$$
$$\begin{bmatrix}1&0&0&1\\0&1&-1&2\\0&1&1&3\end{bmatrix}$$

- find RREF. $V_1=C_1$ etc.

Wronskian for linear sys.

- If y_1, y_2 etc are solutions to $x'=Ax$ they are linearly independent only if $W[y_1,y_2]\neq 0$
- find W by forming matrix with y_1, y_2, etc as the columns and take the determinant.

Ex: $C_1\begin{bmatrix}2\\1\end{bmatrix}e^{3t}+C_2\begin{bmatrix}-1\\2\end{bmatrix}e^{-t}$

$\begin{vmatrix}2e^{3t}&-e^{-t}\\e^{3t}&2e^{-t}\end{vmatrix}\neq 0$ ∴ are independent

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Algebraic multiplicity:

root is repeated when factoring for λ.

(doesn't always mean 2 eigenvectors)

Geometric multiplicity:

eigenvalue has multiple eigenvectors.

Phase Planes: 2x2 systems

6 cases

real:
- Nodal source → Both positive
- Nodal sink → Both negative
- saddle-point → 1 pos. 1 neg

complex:
- spiral source → real part pos.
- spiral sink → real part neg
- center → real part 0

Real eigenvalues:
- Nodal source: "source" means everything goes away from (0,0)
- A node will always have an asymptote (usually 2).
  - this is a solution curve that follows a straight line.
  - are actually the eigenvectors.
- nodal 'sink' means all solution curves go toward (0,0)
- saddle-point will also have asymptotes, 1 with solutions approaching and 1 with solutions going away. othe solutions approach (0,0) before turning away.

complex eigenvalues:
- spiral source does not have asymptotes, solution curves are never straight lines. solutions spiral away from source (0,0)
- spiral sink no asymptotes. soln. curves spiral toward center.
- center no asymptotes. soln. curves form circles/oval loops around center.

Limit behavior:
- Nodal sink – both vals neg as $t \to \infty$ approaches $\begin{bmatrix}0\\0\end{bmatrix}$
- Nodal source – both vals pos as $t \to \infty$ diverges
- saddle-point – 1 pos 1 neg val as $t \to \infty$ if $c_1 = 0 \to$ approaches $\begin{bmatrix}0\\0\end{bmatrix}$; $c_1 \neq 0 \to$ diverges.
- spiral sink – $\lambda = \alpha \pm ui$, $\alpha < 0$ as $t \to \infty$ $e^{\alpha t} \to \begin{bmatrix}0\\0\end{bmatrix}$ while sin/cos oscillate. soln $\to \begin{bmatrix}0\\0\end{bmatrix}$
- spiral source – $\alpha > 0$ as $t \to \infty$ diverges
- center $\alpha = 0$ as $t \to \infty$ soln. loops around origin

☆ $x' = Ax$ will always have an equilibrium point at (0,0). can have more if 0 is an eigenvalue.

Relax, you can do it. You have more time left. Don't rush.

Converting a 2nd order ODE to system of 1st order ODE's (linear)

Idea: If we have an ODE $u'' + au' + bu = 0$

Define: $x_1 = u$, $x_2 = u'$ then $x_1' = u' = x_2$ and $x_2' = u''$
$= 0x_1 + 1 \cdot x_2$

so $x' = Ax$ will be

$$x' = \begin{bmatrix}0 & 1\\ -b & -a\end{bmatrix} x$$

E.x. $2u'' - 3u' + 4u = 0$ $u(0) = -1$ $u'(0) = 2$

Convert IVP to a system of 1st order ODE's

First find correct form: notice $a = -\frac{3}{2}$ $-a = \frac{3}{2}$; $b = 2$ $-b = -2$

$\underbrace{u'' - \tfrac{3}{2}u'}_{a} + \underbrace{2u}_{b} = 0$

u shouldn't have prefix

$x_1 = u$ $x_1' = x_2$
$x_2 = u'$ $x_2' = u''$

$u'' = \frac{3}{2}u' - 2u$

$u'' = -2x_1 + \frac{3}{2}x_2 \to x' = \begin{bmatrix}0 & 1\\ -2 & 3/2\end{bmatrix} x$

$$x(0) = \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix} = \begin{bmatrix}u(0)\\ u'(0)\end{bmatrix} = \begin{bmatrix}-1\\ 2\end{bmatrix}$$

What if 2nd order ODE isn't homogeneous?

$u'' + u' + u = \sin t$

note:
$x' - A\vec{x} = \vec{b}$ nonhomogeneous
$x' - A\vec{x} = 0$ homogeneous

$x_1 = u$ $x_1' = x_2$
$x_2 = u'$ $x_2' = u''$

$u'' = -x_1 - x_2 + \sin t$
$u'' = -u - u' + \sin t$

so system is $x' = \begin{bmatrix}0 & 1\\ -1 & 1\end{bmatrix} + \begin{bmatrix}0\\ \sin t\end{bmatrix}$ (always 0)

Oscillations: Terminology:

if $g(t) = 0$ oscillation is free
if $g(t) \neq 0$ oscillation is forced
if $\gamma = 0$ oscillation is undamped
if $\gamma \neq 0$ oscillation is damped

soln: $u(t) = C_1 \cos\left(\sqrt{\frac{K}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{K}{m}}t\right)$

$\sqrt{\frac{K}{m}}$ = frequency ω_0 (Angular) frequency = $\frac{1}{T}$

$\frac{1}{\omega_0} = \sqrt{\frac{m}{K}}$ = period (angular) T = period

Damped Free Oscillations

$mu'' + \gamma u' + Ku = 0$ $m, \gamma, K > 0$

$mr^2 + \gamma r + K = 0$

$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mK}}{2m}$

3 cases:
- $\gamma^2 - 4mK > 0$ over damped
- $\gamma^2 - 4mK = 0$ crit. damped
- $\gamma^2 - 4mK < 0$ under damped

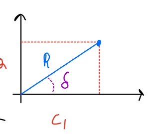
Also want amplitude and phase

$u(t) = C_1 \cos\left(\sqrt{\frac{K}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{K}{m}}t\right)$

can be:

$= R\cos(\delta)\cos\left(\sqrt{\frac{K}{m}}t\right) + R\sin(\delta)\sin\left(\sqrt{\frac{K}{m}}t\right)$

$C_1 = R\cos(\delta)$
$C_2 = R\sin(\delta)$



$\cos(\delta) = \frac{C_1}{R}$

$\sin(\delta) = \frac{C_2}{R}$

$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

so

$R\cos(\delta)\cos\left(\sqrt{\frac{K}{m}}t\right) + R\sin(\delta)\sin\left(\sqrt{\frac{K}{m}}t\right)$

$= R\cos\left(\delta - \sqrt{\frac{K}{m}}t\right)$ or $\underbrace{R}_{\text{amplitude}}\cos\left(\sqrt{\frac{K}{m}}t \underbrace{- \delta}_{\text{phase}}\right)$

Ex:

$u'' + 4u = 0$

find: freq, period, amp, phase

freq = $\sqrt{\frac{K}{m}} = \sqrt{\frac{4}{1}} = 2$

Period = $\frac{1}{2}$ for amp and phase we need to solve the IVE first

$r^2 + 4 = 0$
$r = \pm 2i$

$u(t) = C_1\cos(2t) + C_2\sin(2t)$

$4 = C_1$ $C_2 = -3$

$4\cos(2t) - 3\sin(2t)$

$r^2 = 4^2 + -3^2$

$r = 5$ $\delta = \tan^{-1}\left(\frac{-3}{4}\right)$

integrals

$\int \sec^2 x = \tan x$ $\int \ln x = x\ln|x| - x$

$\int \sec x \tan x = \sec x$ $\int \frac{1}{x} = \ln|x|$

$\int \csc x \cot x = -\csc x$

$\int \csc^2 x = -\cot x$ $\int \frac{1}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right)$

$\int \tan x = \ln|\sec x|$

$\int \sec x = \ln|\sec x + \tan x$

$\int \frac{1}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right)$

$\int \frac{10}{x(1 - \frac{x}{c})} dx = 10(\ln|x| - \ln|x - c|)$

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if A is invertable then 0 is an eigenvalue? F

if 2 is an eigenvalue of an invertable matrix, then $\frac{1}{2}$ is an eigenvalue of A^{-1} T

if A is real and $z(t)$ is complex-val soln then $\frac{z(t) - \overline{z(t)}}{2}$ is a real val soln? F

if Matrix A is non-singular then 0 is an eigenvalue? F

if A is real and $\det(A) < 0$ then $\lim_{t \to +\infty} x(t) = 0$ for any soln. $x(t)$? F

The exponential matrix e^{tA} is a fundamental matrix of the ODE system $x'(t) = Ax$? T

if λ is an eigenvalue of an invertable matrix A, then λ^{-1} is an eigenvalue of A^{-1}. T

If the general soln. of a 2-dimensional ODE system $x'(t) = Ax(t)$ is $C_1V + C_2(Vt + W)$ then A has 0 as an eigenvalue with algebraic multiplicity of 2. T

If A is an order 2 real matrix with eigenvalues 1 and −1 then every solution to $x'(t) = Ax(t)$ that does not start at the origin will diverge. F

Ex: Oscillations

$u'' + 4u = 0$

find: freq, period, amp, phase

freq = $\sqrt{\frac{K}{m}} = \sqrt{\frac{4}{1}} = 2$

Period = $\frac{1}{2}$ for amp and phase we need to solve the IVE first

$r^2 + 4 = 0$
$r = \pm 2i$

$u(t) = C_1\cos(2t) + C_2\sin(2t)$

$4 = C_1$ $C_2 = -3$

$4\cos(2t) - 3\sin(2t)$

$r^2 = 4^2 + -3^2$

$r = 5$ $\delta = \tan^{-1}\left(\frac{-3}{4}\right)$

Log/exp rules:

$\ln(ab) = \ln a + \ln b$
$\ln(a/b) = \ln a - \ln b$
$\ln(a^b) = b\ln a$
$\ln(1) = 0$
$\ln(e) = 1$
$\ln(e^b) = b$
$a^x \cdot a^y = a^{x+y}$
$(a^x)^y = a^{xy}$