ColoState   Fall 2023   MATH 340   Exam 2 (Thu.11/02/2023)

Name: ______________________________ CSUID# __________________ Lab Time: ______

Problem	#1	#2	#3	#4	#5	#6	Total
Score							

Bonus (2 points) for submission of your Cheat Sheet (with your name, CSUID#, and Lab Time). It will be returned to you. We just need it to cover the 1st page for your privacy.

Exam Policy

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You may use **one** letter-size 2-sided Cheat Sheet for this exam.

Good luck!

(20 points) *Problem 1.* Determine whether the following statements are correct.
True (T) or False (F). Circle your answer (2 points for each item, no partial credit).

(i) (T) (F) If λ is an eigenvalue of an invertible matrix A, then λ^{-1} is an eigenvalue of A^{-1}.

(ii) (T) (F) If matrix A is nonsingular, then 0 is an eigenvalue.

(iii) (T) (F) An order-3 real matrix has at least one real eigenvalue.

(iv) (T) (F) $\mathbf{y}(t) = e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ is a solution of the ODE system $\mathbf{x}'(t) = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$.

(v) (T) (F) For a dim-2 ODE system $\mathbf{x}'(t) = A\mathbf{x}$, if matrix A is real and $1+2i$ is an eigenvalue, then all trajectories spiral towards the origin.

(vi) (T) (F) For a dim-2 ODE system $\mathbf{x}'(t) = A\mathbf{x}$, if matrix A is real and $\det(A) < 0$, then the origin is a saddle-point.

(vii) (T) (F) For a dim-2 ODE system $\mathbf{x}'(t) = A\mathbf{x}$, if matrix A is real and $\det(A) < 0$, then $\lim\limits_{t\to+\infty} \mathbf{x}(t) = \mathbf{0}$ for any solution $\mathbf{x}(t)$.

(viii) (T) (F) For an ODE system $\mathbf{x}'(t) = A\mathbf{x}$, if matrix A is real and $\mathbf{z}(t)$ is a complex-valued solution, then $\frac{\mathbf{z}(t)-\overline{\mathbf{z}(t)}}{2}$ is a real-valued solution.

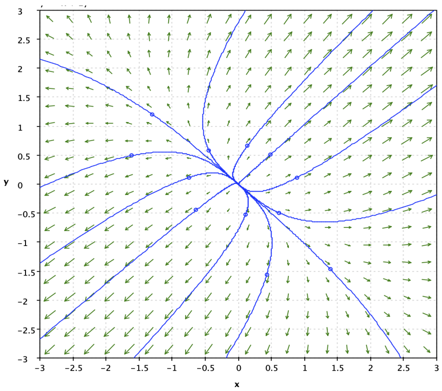
(ix) (T) (F) For a dim-2 ODE system $\mathbf{x}'(t) = A\mathbf{x}$, if $\mathbf{x}_1(t), \mathbf{x}_2(t)$ are solutions and linearly independent, then the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2]$ is never 0.

(x) (T) (F) The exponential matrix e^{tA} is a fundamental matrix of the ODE system $\mathbf{x}'(t) = A\mathbf{x}$.

$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \quad \alpha \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

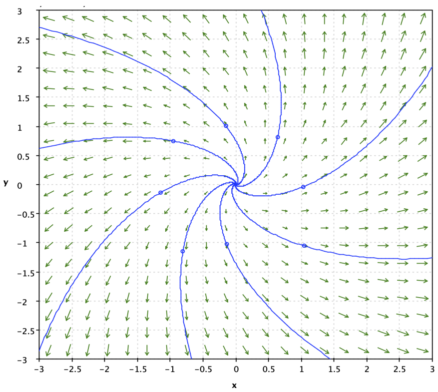
$\begin{bmatrix} 2 & -4 & 2 \\ 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(10 points) *Problem 2.* Examine the following three figures.



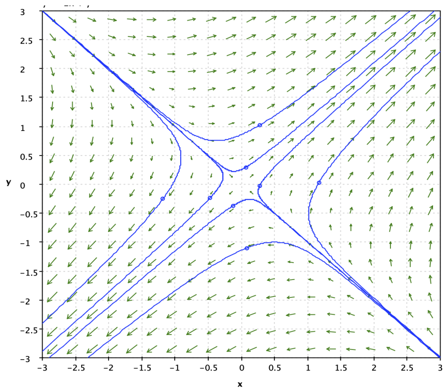
Left

nodal source



Middle

spiral source



Right

saddle point

Examine also these three dim-2 linear ODE systems $\mathbf{x}'(t) = A\,\mathbf{x}$ with

$$(a)\ \ A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \qquad (b)\ \ A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}, \qquad (c)\ \ A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

Now answer the following questions.

(i) Find the ODE system that corresponds to ***the middle panel*** of the above figures.
Circle your choice. (a) (b) (c)

(ii) What is the type of the phase portrait in *the middle panel*?
Circle your choice from the list shown below.

Nodal source *Nodal sink* *Spiral source* *Spiral sink* *Saddle-point*

(iii) For your choice in Part (i), find the eigenvalues.

a) $\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix}$

$(2-\lambda)(2-\lambda)-1$

$\lambda^2-4\lambda+3$

$(\lambda-3)(\lambda-1)$

$\lambda=1,3$

b) $\begin{bmatrix} 2-\lambda & -1 \\ 1 & 2-\lambda \end{bmatrix}$

$\lambda^2-4\lambda+5$

$\dfrac{4\pm\sqrt{16-20}}{2}$

$\lambda=2\pm i$

(15 points) *Problem 3.* Consider the ODE system $\mathbf{x}'(t) = A\mathbf{x}$ with $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

(i) Find the eigenvalues and eigenvectors of matrix A.

(ii) If the eigenvalues are complex, then find the real-valued general solution of the ODE system.

$$\begin{bmatrix} 1-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$$

$1+i$ $\begin{bmatrix} -i & -1 \\ 1 & -i \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -i \\ 0 & 0 \end{bmatrix}$ $\quad v_1 = \alpha i$, $v_2 = \alpha$

$(1-\lambda)(1-\lambda)+1$

$\lambda^2 - 2\lambda + 2$

$\alpha \begin{bmatrix} i \\ 1 \end{bmatrix} \to \underset{a}{\begin{bmatrix} 0 \\ 1 \end{bmatrix}} + \underset{b}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} i$

$\dfrac{2 \pm \sqrt{4-8}}{2}$

$$\vec{x}(t) = C_1 e^t \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \cos t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \right) + C_2 e^t \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} \sin t - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cos t \right)$$

$1 \pm i$

$1-(1+i)$

$0-i$

(20 points) *Problem 4.* Consider the ODE system $\mathbf{x}'(t) = A\mathbf{x}$ with $A = \begin{bmatrix} 3 & -1 \\ 1 & 1 \end{bmatrix}$.

(i) Find the eigenvalues and eigenvectors of matrix A.

(ii) Find the general solution of the ODE system.

(iii) Find the particular solution satisfying the initial condition $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Hint: *You need to find a generalized eigenvector.*

$$\begin{bmatrix} 3-\lambda & -1 \\ 1 & 1-\lambda \end{bmatrix}$$

$$(3-\lambda)(1-\lambda)+1$$

$$\lambda^2-4\lambda+4$$

$$(\lambda-2)(\lambda-2)$$

$$\lambda=2$$

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} v_1 = v_2 \\ v_2 = \alpha \end{matrix} \quad \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{matrix} v_1 = v_2+1 \\ v_2 = \alpha \end{matrix} \quad \alpha\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + c_2 e^{2t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = c_1\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 2 \\ 0 & -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \quad \begin{matrix} c_1 = 1 \\ c_2 = 2 \end{matrix}$$

$$\vec{x}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}e^{2t} + e^{2t}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}t + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

Extra space for Problem 4 or another problem, if needed.

(20 points) *Problem 5.* Consider a dim-3 ODE system $\mathbf{x}'(t) = A\,\mathbf{x}(t)$, where $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(i) Find the eigenvalues of matrix A.

(ii) Find associated eigenvectors for matrix A.

(iii) Find the general solution of the ODE system.

Hint: $\mathbf{p} = [1, -1, 0]^\top$ *is an eigenvector.*

$$\begin{bmatrix} -2-\lambda & 1 & 0 \\ 1 & -2-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{bmatrix}$$

$$(1-\lambda)\left((-2-\lambda)(-2-\lambda)-1\right)$$

$$4+2\lambda+2\lambda+\lambda^2-1$$

$$(1-\lambda)(\lambda^2+4\lambda+3)$$

$$(\lambda+3)(\lambda+1)$$

$$\boxed{\lambda = 1, -1, -3}$$

eigenvectors:

$\lambda = 1$ $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\lambda = -1$ $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\lambda = -3$ $\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$$\vec{x}(t) = C_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} e^{t} + C_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + C_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-3t}$$

$\lambda = 1$ $\begin{bmatrix} -3 & 1 & 0 \\ 1 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 & 0 \\ 0 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$V_1 = 0$ $\quad V_3 = \alpha$ $\quad \alpha \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

$V_2 = 0$

$\lambda = -1$ $\begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$V_1 = V_2$ $\quad \alpha \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$V_2 = \alpha$

$V_3 = 0$

$\lambda = -3$ $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$V_1 = -V_2$ $\quad \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

$V_2 = \alpha$

$V_3 = 0$

Extra space for Problem 5 or another problem, if needed.

(15 points) *Problem 6.* Given an ODE $u''+2u'+5u = \sin(3t)$ and initial conditions $u(0) = 0$, $u'(0) = 1$.

(i) Convert the problem to a 1st-order ODE system with a corresponding initial condition.

(ii) Is the new ODE system linear? Is it homogeneous?

(iii) If the new ODE system is in the form $\mathbf{x}'(t) = A\,\mathbf{x} + \mathbf{b}(t)$, find the eigenvalues of matrix A.

$x_1 = u \qquad x_1' = x_2 \qquad u'' = -2u' - 5u + \sin 3t$

$x_2 = u' \qquad x_2' = u'' \qquad a = 2 \quad b = 5$

$$x' = \begin{bmatrix} 0 & 1 \\ -5 & -2 \end{bmatrix} + \begin{bmatrix} 0 \\ \sin 3t \end{bmatrix} \qquad x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

is linear, not homogeneous

$$\begin{bmatrix} -\lambda & 1 \\ -5 & -2-\lambda \end{bmatrix}$$

$(-\lambda)(-2-\lambda) + 5$

$2\lambda + \lambda^2 + 5$

$\lambda^2 + 2\lambda + 5$

$$\frac{-2 \pm \sqrt{4-20}}{2}$$

$-1 \pm 2i \rightarrow$ eigenvalues for A