ColoState Fall 2024 MATH 340 Final (Wed. Dec.11, 11:50am)

Name ____________________ CSUID# ______________ Class Time ______

Problem	#1	#2	#3	#4	#5	#6	#7	Total
Score								

Notes. The exam is printed single-sided on 7 sheets.
– If you need to use the other side, please make a comment at the top-right corner.
– If you need extra sheets, please put your name and CSUID# on each sheet (at the top-right corner).

Exam Policies

(i) **No** calculator, textbook, homework, or any other references should be used. Please write down all necessary steps, partial credit will be given if deserved.

(ii) You may use **two** letter-size 2-sided Cheat Sheets for this exam.

Good luck!

(20 points) *Problem 1.* Determine whether the following statements are correct.
True (T) or False (F). Circle your answer (2 points for each item, no partial credit).

(i) (T) (F) Multiplying both sides of the equation $(3xy + y^2)dx + (x^2 + xy)dy = 0$ by x, we obtain a new ODE that is exact.

(ii) (T) (F) The equation $y^2 - \cos(x + y) = 0$ implicitly defines a solution to the ODE $\sin(x + y)dx + (2y + \sin(x + y))dy = 0$.

(iii) (T) (F) If $y_p(t)$ is a solution to the ODE $y' = y(20 - y)$ and $y_p(0) = 10$, then $\lim_{t\to+\infty} y_p(t) = 0$.

(iv) (T) (F) The ODE $y''(x) + \cos(y'(x)y(x)) - \sqrt{y(x)} = 2$ is autonomous.

(v) (T) (F) The general solution of the ODE $y'(x) = x^2y^2$ is $y = \dfrac{1}{x^3 + C}$ with C being a constant.

(vi) (T) (F) The ODE $x'' + 4x = \cos(3t)$ models resonance.

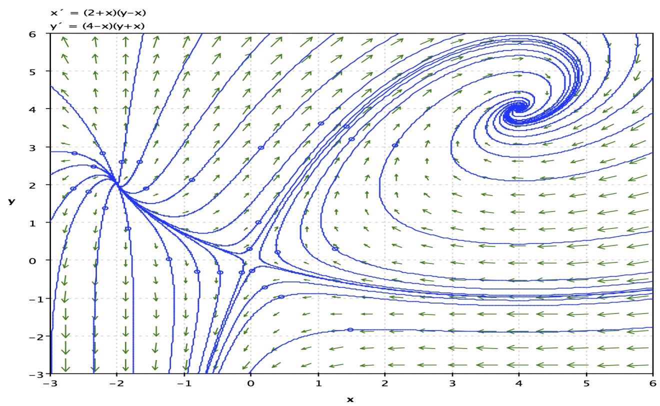
(vii) (T) (F) $\mathbf{y}(t) = e^t \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)$ is a solution of the ODE system $\mathbf{x}'(t) = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix} \mathbf{x}$.

(viii) (T) (F) For a dim-2 ODE system $\mathbf{x}'(t) = A\,\mathbf{x}$, if A is real and $\mathtt{trace}(A) = 5$, $\mathtt{det}(A) = 6$, then the origin is a saddle point.

(ix) (T) (F) If $y(t) = t\,e^{-t}$, then its Laplace transform is $Y(s) = \dfrac{1}{s+1}$.

(x) (T) (F) The nonlinear ODE system $\begin{cases} x'(t) = y \\ y'(t) = -\sin(x) \end{cases}$ has only one saddle point.

(10 points) *Problem 2.* Consider the ODE system $\begin{cases} x'(t) = (2+x)(y-x) \\ y'(t) = (4-x)(y+x) \end{cases}$ and the following figure.



Answer the following questions.

(i) Find the coordinates of the 3 equilibria `(Solving the algebraic system to get the full credit).`

(ii) For each equilibrium, find the Jacobian matrix and its eigenvalues, determine its type:

Nodal source *Nodal sink* *Spiral source* *Spiral sink* *Saddle-point*

(iii) Identify stability for each equilibrium:

Unstable *Asymptotically stable* *Stable but not asymptotically stable*

$(2+x)(y-x)$

$x=-2 \quad x=y$

$6(y-4)$

$6y-24=0$

$6y=24$

$y=4$

$(4-x)(y+x)$

$6(y-2)$

$6y-12=0$

$6y=12$

$y=2$

$x=-y$

$(2-y)(y+y)$

$(2-y)2y=0$

$y=0$

$(-2,2)$

$(4,4)$

$(0,0)$

$x'(t)=(2+x)(y-x)$

$=2y+xy-2x-x^2$

$y'(t)=(4-x)(y+x)$

$=4y-xy+4x-x^2$

$J(x,y)=\begin{bmatrix} y-2x-2 & 2+x \\ -y+4-2x & 4-x \end{bmatrix}$

$J(0,0)\begin{bmatrix} -2 & 2 \\ 4 & 4 \end{bmatrix}$

saddle point

unstable

$(-2-\lambda)(4-\lambda)-8$

$-8-4\lambda+2\lambda+\lambda^2-8$

$\lambda^2-2\lambda-16$

$\frac{2\pm\sqrt{4+64}}{2} \quad \frac{2\pm\sqrt{68}}{2}$

$J(4,4)=\begin{bmatrix} -6 & 6 \\ -8 & 0 \end{bmatrix}$

$(-6-\lambda)(-\lambda)+48$

$\lambda^2+6\lambda+48$

$\frac{-6\pm\sqrt{36-4(48)}}{2}$

spiral sink

Asym. stable

$J(-2,2)\begin{bmatrix} 4 & 0 \\ 6 & 6 \end{bmatrix}$

$(4-\lambda)(6-\lambda)$

$\lambda=4,6$

nodal source

unstable

$\sqrt{64}\leq\sqrt{68}\leq\sqrt{81}$

$8\leq\sqrt{68}\leq 4$

$\frac{2-\sqrt{68}}{2}<0$

(10 points) *Problem 3.* Consider the ODE $x'(t) = 2te^t\sqrt{x}$.

(i) Find the general solution.

(ii) Find the particular solution $x_p(t)$ satisfying the condition $x_p(0) = 1$.

(iii) Is the constant function $x(t) = 0$ a solution of the ODE?
If yes, is it included in the general solution? → no

$$\frac{1}{\sqrt{x}}\frac{dx}{dt} = 2te^t$$

$$\int x^{-1/2} = 2\int te^t$$

$$\frac{x^{1/2}}{1/2} = 2(t-1)e^t + C$$

$$2\sqrt{x} = 2(t-1)e^t + C$$

$$\sqrt{x} = e^t(t-1) + \frac{C}{2}$$

$$x = (e^t(t-1)+C)^2$$

$$x = te^{2t} - e^{2t} + C$$

$$1 = 0 - 1 + C$$

$$2 = C$$

$$x = te^{2t} - e^{2t} + 2$$

(15 points) *Problem 4.* Consider the ODE system $\begin{bmatrix} x'(t) \\ v'(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x(t) \\ v(t) \end{bmatrix}$ with $\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$.

(i) Find the general solution of the ODE system.
Note: It is also possible to convert the ODE system into a 2nd order ODE and solve that.

(ii) Find the particular solution $\begin{bmatrix} x_p(t) \\ v_p(t) \end{bmatrix}$ of the ODE system satisfying $\begin{cases} x_p(0) = 1, \\ v_p(0) = 0. \end{cases}$

(iii) Compute the energy $\int_0^T (v_p(t))^2 dt$ for any $T > 0$.

$\begin{bmatrix} 0-\lambda & 1 \\ -1 & -2-\lambda \end{bmatrix}$ $(-\lambda)(-2-\lambda)+1$

$\lambda^2 + 2\lambda + 1$

$(\lambda+1)^2$

$\lambda = -1$

$\lambda = -1$ $\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ $V_1 = -V_2$, $V_2 = \alpha$ $\alpha \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$

$x(t) = C_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + C_2 e^{-t} \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$

$\begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ -1 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$ $C_1 = 0$, $C_2 = -1$

$x(t) = -e^{-t} \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} -1 \\ 0 \end{bmatrix} \right)$

(15 points) *Problem 5.* Consider the ODE system $\mathbf{x}'(t) = \mathbf{A}\,\mathbf{x}(t)$, where $\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{bmatrix}$.

Find the ***real-valued*** general solution.

$$\begin{bmatrix} 1-\lambda & 0 & 0 \\ 2 & 1-\lambda & -2 \\ 3 & 2 & 1-\lambda \end{bmatrix} \quad 1-\lambda \begin{vmatrix} 1-\lambda & -2 \\ 2 & 1-\lambda \end{vmatrix}$$

$$(1-\lambda)\left((1-\lambda)(1-\lambda)+4\right) \qquad \frac{2\pm\sqrt{4-20}}{2}$$

$$(1-\lambda)(\lambda^2-2\lambda+5) \qquad \frac{2\pm 4i}{2} \qquad 1\pm 2i$$

$$\lambda=1$$

$$\lambda=1 \quad \begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & -2 \\ 3 & 2 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3/2 \\ 0 & 0 & 0 \end{bmatrix}$$

$V_1 = V_3$

$V_2 = -3/2\, V_3$ $\quad \alpha \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix}$

$V_3 = \alpha$

$$\lambda = 1-2i \quad \begin{bmatrix} 2i & 0 & 0 \\ 2 & 2i & -2 \\ 3 & 2 & 2i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2i & -2 \\ 3 & 2 & 2i \end{bmatrix}$$

$1-(1-2i)$

$2i$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2i & -2 \\ 0 & 2 & 2i \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & i \end{bmatrix}$$

$V_1 = 0$

$V_2 = \alpha$ $\quad \alpha \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

$V_3 = -i$ $\qquad a \qquad b$

$1-2i$

$\cancel{0}\ \mu$

$$x(t) = C_1 \begin{bmatrix} 2 \\ -3 \\ 2 \end{bmatrix} e^{t} + C_2 e^{t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\cos(-2t) - \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\sin(-2t)\right) + C_3 e^{t}\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\sin(-2t) + \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}\cos(-2t)\right)$$

(15 points) *Problem 6.* We know the ODE $my'' + by' + ky = f(t)$ with constant coefficients $m > 0, b, k$ could be used to model oscillations.

(i) Consider the unforced motion, i.e., $f(t) = 0$. Find a condition (an inequality) about m, b, k so that the ODE indeed models an oscillatory motion. *Hint: Consider complex eigenvalues.*

(ii) Now consider a concrete case: $m = 1, b = -2, k = 2, f(t) = e^{-t}$. Find the general solution.

(15 points) *Problem 7.* Apply both Laplace and inverse Laplace transforms to solve the ODE IVP: $y'' - 2y' + 2y = e^{-t}, y(0) = 0, y'(0) = 1$.

$$s^2 Y(s) - s\,y(0) - y'(0) - 2\left(sY(s) - y(0)\right) + 2Y(s) = \frac{1}{s+1}$$

$$(s^2 - 2s + 1) + 1$$

$$(s-1)^2 + 1$$

$$s^2 Y(s) - 1 - 2sY(s) + 2Y(s) = \frac{1}{s+1}$$

$$Y(s)\left(s^2 - 2s + 2\right) = \frac{1}{s+1} + 1$$

$$\frac{s+2}{(s+1)(s^2-2s+2)} = \frac{A}{(s+1)} + \frac{Bs+C}{s^2-2s+2}$$

$$Y(s) = \frac{s+2}{(s+1)(s^2-2s+2)}$$

$$s+2 = A(s^2-2s+2) + (Bs+C)(s+1)$$

$$s+2 = As^2 - 2As + 2A + Bs^2 + Cs + Bs + C$$

$$Y(s) = \frac{1}{5}\cdot\frac{1}{(s+1)} - \frac{1}{5}\,\frac{(s-1)+1}{(s-1)^2+1} + \frac{8}{5}\,\frac{1}{(s-1)^2+1}$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos t + \frac{7}{5}e^{t}\sin t$$

$$1 = -2A + C + B \qquad 1 = 3B + C$$

$$2 = 2A + C \qquad 2 - 2A = C$$

$$0 = A + B \qquad 1 = 3B + 2 + 2B$$

$$A = -B \qquad 1 = 5B + 2$$

$$-1 = 5B \qquad B = -\frac{1}{5}$$

$$1 = -\frac{2}{5} + C - \frac{1}{5}$$

$$A = \frac{1}{5}$$

$$1 = -\frac{3}{5} + C$$

$$\frac{8}{5} = C$$

$$-\frac{1}{5}s + \frac{8}{5}$$

$$Y(s) = \frac{s+2}{(s+1)(s^2-2s+2)} \qquad \frac{s+2}{(s+1)(s^2-2s+2)} = \frac{A}{(s+1)} + \frac{Bs+C}{s^2-2s+2}$$

$$B = -\frac{1}{5}$$

$$A = \frac{1}{5}$$

$$\frac{8}{5} = C$$

$$Y(s) = \frac{1}{5} \cdot \frac{1}{(s+1)} + \frac{-\frac{1}{5}s + \frac{8}{5}}{(s-1)^2+1}$$

$$= \frac{1}{5}\frac{1}{(s+1)} - \frac{1}{5}\frac{(s-1)+1}{(s-1)^2+1} + \frac{8}{5}\frac{1}{(s-1)^2+1}$$

$$= \frac{1}{5}\frac{1}{(s+1)} - \frac{1}{5}\frac{(s-1)}{(s-1)^2+1} - \frac{1}{5}\frac{1}{(s-1)^2+1} + \frac{8}{5}\frac{1}{(s-1)^2+1}$$

$$= \frac{1}{5}e^{-t} - \frac{1}{5}e^{t}\cos t + \frac{7}{5}e^{t}\sin t$$