Networking EE4C06 Student group project graph theory

September 8, 2016

Topic: Spectral analysis of networks
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Summary: The Laplacian matrix Q is a matrix representation of a network G. Dynamic processes on networks, such as epidemic spreading, and the robustness of networks against cascading failures are characterized by the spectrum of their Laplacian matrix. This project will numerically and theoretically analyze the Laplacian eigenvalues and how the spectrum reflects the topological properties of the network.

Research questions: What is the relation between the distribution of the Laplacian eigenvalues and the distribution of the nodal degrees?

Task: Your task will start with simulations on random graph models. For this purpose, we suggest that you generate the following networks:

1. Erdős-Rényi graph $G_p(N)$ (ER): this graph is generated from a set of N nodes by randomly and independently assigning a link between each node pair with probability p.

2. Barabási-Albert graph (BA): this graph is generated by starting with m nodes. At every time step, a new node with m links is connected to the m existing nodes in the network. A new node connects to a node i in step t with probability $p = d_i/2L_t$, where d_i is the degree of node i and L_t is the total number of links at time t.

3. Watts-Strogatz small-world network (WS): This graph is generated from a ring lattice (see Figure 1 as an example) of N nodes and k links per node. The end node of a link that connects a node n to its nearest neighbours is replaced with probability p by another randomly chosen node that is not the direct neighbours of that node n.

To evaluate the simulation results, all parameters for generating random graphs are fixed:

(i) for an ER graph, $N = 500$ and $p = \frac{d_{av}}{N-1}$ and $d_{av} = 12$;

(ii) for a BA graph, $N = 500$ and $m = 6$;

(iii) for a WS graph, $N = 500$, $k = 12$ and $p = 0.1$.

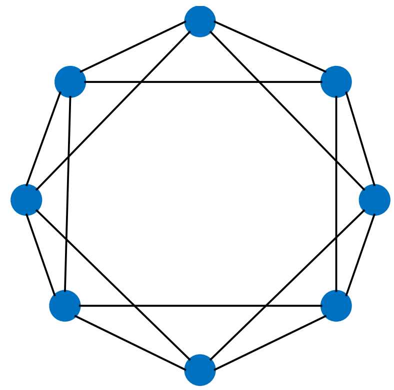

Figure 1: Ring Lattice on $N = 8$ nodes, each connected to its $k = 4$ nearest neighbours.

The code to generate random graphs is provided in the part of Resources and literature.

As a next step, analyze the relation between the Laplacian eigenvalues and the nodal degrees. The analysis includes two parts: simulations and theoretical analysis.

Part I: Simulations

The eigenvalues of the Laplacian matrix Q of a graph on N nodes and L links are denoted as $0 = \mu_N \leq \mu_{N-1} \leq \cdots \leq \mu_1$. The degree is ordered as $d_{(N)} \leq d_{(N-1)} \leq \cdots \leq d_{(1)}$, where $d_{(k)}$ denotes the k-th largest degree. (Please follow the notations on the Lecture slides.)

1. Compute the degree vector and the Laplacian eigenvalues and plot the degree vector and the Laplacian eigenvalues in the same figure. Draw at least two conclusions from the results (In order to find a proper relation, make sure that the degree and the Laplacian eigenvalues are ordered from the smallest to the largest). See Figure 2 as an example.

2. Compute the degree distribution and the distribution of the Laplacian eigenvalues and draw at least two conclusions from the results. Fit the distribution of the Laplacian eigenvalues for BA graphs. (First generate one random graph and compute the degree vector and the Laplacian eigenvalues. Then 10^5 random graphs need to be generated. Finally, compute the probability distribution of the nodal degree and the Laplacian eigenvalues. For BA graphs, use log-log plot. The two distributions should be plotted in one figure.) Analyze how the relation between distributions of the degree and the Laplacian eigenvalues is influenced by the topology of a graph. In other words, compare results obtained from ER, BA and WS graphs.) See Figure 3 as an example.

3. Let the set $S = \{k : \mu_k \geq d_{(k)}\}$ be the set of the Laplacian eigenvalues that are larger than the degree. Let the number $s = |S|$ be the size of the set S. Simulate the distribution of the number s and fit the distribution for ER graphs. When fitting the distribution of the number s, y-axis should be in log-scale. See Figure 4 as an example.

Part II: Theoretical analysis

1. Show that $\mu_N = 0$ for all undirected graphs.

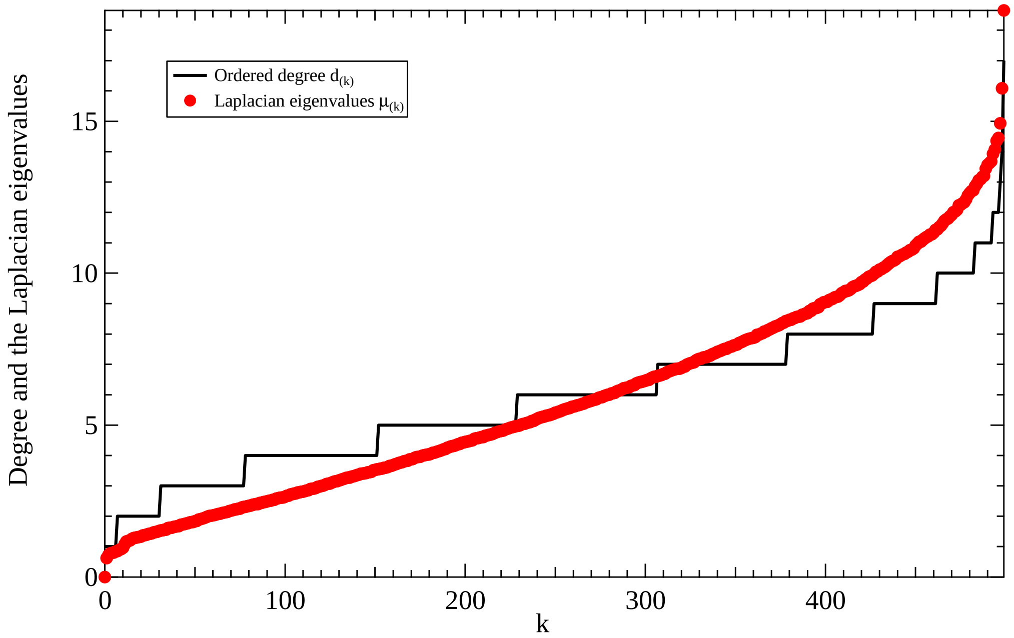

Figure 2: The degree vector and the Laplacian eigenvalues of a graph.

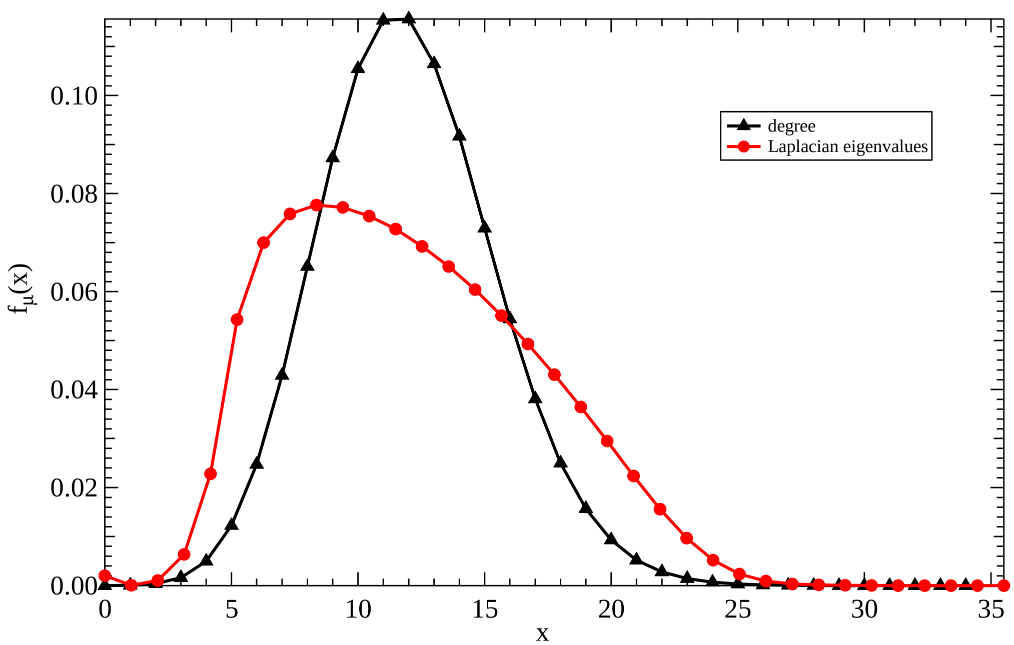

Figure 3: The degree distribution and the distribution of the Laplacian eigenvalues.

2. Derive the equation for the average $E[\mu]$ of the Laplacian eigenvalues in terms of the number of nodes N and the number of links L.

3. Can you find a linear relation between the average eigenvalue $E[\mu]$ and the average degree $E[D]$?

4. Show that the trace of the second power of the adjacency matrix, i.e., trace $\left(A^2\right)$ equals to the sum of nodal degrees, namely trace $\left(A^2\right) = \sum_{j=1}^{N} d_j$.

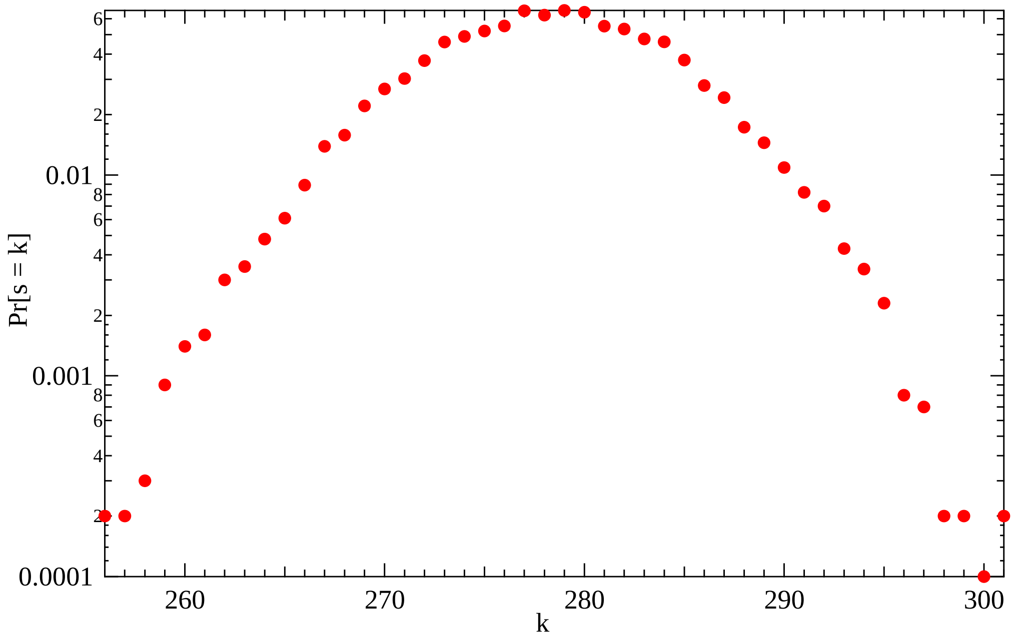

Figure 4: The distribution of the number m.

5. By using the definition of the Laplacian matrix $Q = \Delta - A$, where Δ is the diagonal matrix of nodal degrees, can you derive a functional relation between the trace of the second power of the Laplacian matrix, i.e., trace $\left(Q^2\right)$, and the trace of the second power of the adjacency matrix trace $\left(A^2\right)$?

6. Can you find a functional relation between the variance $\text{Var}[\mu]$ of the Laplacian eigenvalues and the variance $\text{Var}[D]$ of the degree? Use this result to understand the simulation results of the simulation task 2. (Hint: employ the results of 4 and 5 in Part II.)

7. Based on results 1 and 3 in Part II, can you show that there exists at least one integer k such that $\mu_k \geq d_{(k)}$?

8. Derive an upper and lower bound for the number s. Derive the formula for the mean $E[s]$ of the number of s.

To pass this project, hand in a report in .pdf-format via eMail containing the name of all students working on the project. The report should include all the intermediary steps. For figures, the labels for the x-y axis should be clear and the type of random graphs with graph parameters should be clearly documented. A good grade can be obtained by providing required figures in Part I and the solutions in tasks 1-6 in Part II. An exceptional grade can be obtained by demonstrating a deep understanding of the relations between the Laplacian eigenvalues and the degrees including

(i) clear conclusions from each figure;

(ii) fitting for the distribution of the Laplacian eigenvalues for ER, BA and WS graphs;

(iii) fitting for the distribution of the number s for ER graphs;

(iv) based on the fitting result, try to explain why the distributions follow the fitting functions;

(v) solutions for tasks 7,8 in part II;

(vi) try to explain why the Laplacian eigenvalues and the nodal degrees are close yet different;

(vii) how the difference between the Laplacian eigenvalues and the degrees can be influenced by the topology of a graph.

Resources and literature

This project requires programming skills, knowledge on matrix theory and basics of graph theory. A few recommendations for literatures and tools:

Network analysis libraries:
MATLAB code for generating random graphs is as follows:

1. Generate a random Erdős-Rényi graph: erdos_reyni(N,p)
`http://nl.mathworks.com/matlabcentral/fileexchange/10922-matlabbgl/content/matlab_bgl/erdos_reyni.m`

2. Generate a Barabási-Albert graph: scalefree(N, m, m)
`https://github.com/maybmdz/NetworkBased-Modeling-for-the-Spread-of-Scientific-Ideas/blob/master/code/scalefree.m`

3. Generate a Watts-Strogatz graph: small_world(N, k, p)
`https://github.com/msssm/lecture_files/blob/master/simulation_networks/small_world.m`

Literature on Graph spectra:

1. P. Van Mieghem. "*Graph spectra for complex networks.*" Cambridge University Press, Oxford, U.K., 2012.

2. Mohar, Bojan, et al. "The Laplacian spectrum of graphs." Graph theory, combinatorics, and applications 2.871-898 (1991): 12.

3. Merris, Russell. "Laplacian matrices of graphs: a survey." Linear algebra and its applications 197 (1994): 143-176.