Nonlinear Programming Assignment – Version 2

SC42056 Optimization for Systems and Control 2025/2026

Note: Changes w.r.t. the original version are marked in blue.

E_1, E_2, and E_3 are parameters ranging from 0 to 18 for each group according to the sum of the last three numbers of the student IDs:

$$E_1 = D_{a,1} + D_{b,1}, \qquad E_2 = D_{a,2} + D_{b,2}, \qquad E_3 = D_{a,3} + D_{b,3}$$

with $D_{a,3}$ the right-most digit of the student ID of the first student, $D_{b,3}$ the right-most digit of the student ID of the other student, $D_{a,2}$ the one but last digit of the student ID of the first student, and so on.

Important: Please note that all questions regarding this assignment should preferably be asked via the Brightspace Discussion forum.

Important: Please ensure your work is your own. A plagiarism check will be performed on all submitted material.

Important: Please note that for the coding component of the assignment, the use of AI assistance is permitted only to the extent that it explains the working of functions. Using AI to generate your code is not allowed. Checks will be run upon submitted files.

In recent years, the number of vehicles on our roads has been increasing consistently, and, despite the existence of large and robustly designed freeways and urban traffic networks, traffic congestion remains a persistent issue. Traffic jams occur frequently and have a severe impact when many people need to use the traffic infrastructure with limited capacity simultaneously, especially during rush hours. Traffic congestion can lead to transportation delays, economic losses, pollution, and other problems.

There are two main solutions to address the traffic congestion problem. The first is to improve the existing transportation infrastructure by building new roads or expanding the existing ones. For this, considerable investments in time and financial resources are required, which highlights the urgent need for alternative and immediate approaches. In this context, the second solution consists of the deployment of effective traffic control methods. These strategies are the most efficient and immediate solutions to address traffic congestion problems.

One of the most widely adopted control measures in freeway traffic management is ramp metering, involving a device positioned at freeway on-ramps to regulate the entering traffic, and typically implemented as a basic traffic signal system, as shown in Figure 1a. The metering rate determines the fraction of ramp traffic allowed to merge onto the freeway in each time interval. Ramp metering systems have been proven to alleviate traffic congestion and to enhance driver safety. Another promising control approach is the use of Variable Speed Limits (VSLs), which are overhead signs showing a variable speed limit that changes according to the current road conditions, as shown in Figure 1b.

The objective of this assignment is to compute the optimal ramp metering rates and the values of the variable speed limits that minimize congestion on a specific freeway stretch.

(a) Ramp metering device at the entrance of a freeway.

(b) Variable speed limit signals on a freeway stretch.

Figure 1: Examples of freeway traffic control devices.

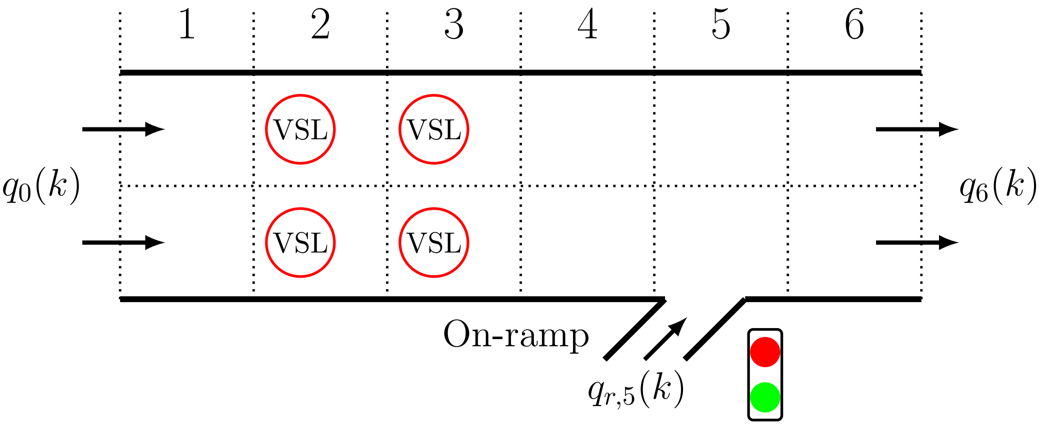

Figure 2: Schematic representation of the freeway stretch considered.

Let us assume that we want to model and control the behavior of the freeway stretch shown in Figure 2. The stretch considered has 6 segments, each of which has a length of $L_i = 1000$ [m] and $\lambda_i = 2$ lanes. The ramp metering installation is located on the on-ramp of segment 5, and VSLs are located at the start of segments 2 and 3.

The discrete-time macroscopic model METANET [1, 2] will be used to model the behavior of the freeway. METANET represents the traffic network as a graph where the segments correspond to edges. They are indexed by an index i and have a length L_i with λ_i lanes. The freeway is dynamically characterized at time step k and for each segment i by the traffic flow $q_i(k)$, the traffic density $\rho_i(k)$, the mean speed $v_i(k)$, and by the number of vehicles $w_\mathrm{r}(k)$ waiting in the queue on the on-ramp. Here, the index k is the time step corresponding to time instant $t = kT$, where T is the simulation time step (in our case, T =10 s). The traffic density ρ is defined as the number of vehicles per unit length and per lane, and is expressed in the units [veh/(km lane)], where 'veh' stands for vehicles. The traffic flow q is defined as the number of vehicles passing a given location per time unit, and is expressed in [veh/h]. The mean speed v and the queue length w are expressed in [km/h] and [veh], respectively.

Two main equations describe the system dynamics of the METANET model. The first one expresses the conservation of vehicles:

$$\rho_i(k+1) = \rho_i(k) + \frac{T}{\lambda_i L_i}(q_{i-1}(k) - q_i(k) + q_{\mathrm{r},i}(k)), \tag{1}$$

where $q_{\mathrm{r},i}(k)$ is the traffic flow that enters the freeway segment i from the connected on-ramp (if any) for time step k. The traffic flow $q_i(k)$ in segment i can be computed for each time step as $q_i(k) = \lambda_i \rho_i(k) v_i(k)$.

The second equation expresses the mean speed as a sum of the previous mean speed, a relaxation term, a convection term, and an anticipation term:

$$v_i(k+1) = v_i(k) + \frac{T}{\tau}(V_i(k) - v_i(k)) + \frac{T}{L_i} v_i(k)(v_{i-1}(k) - v_i(k)) - \frac{\mu T(\rho_{i+1}(k) - \rho_i(k))}{\tau L_i(\rho_i(k) + K)}, \quad (2)$$

where τ, μ, and K are model parameters that for this assignment are assumed to be constant for all segments, and $V_i(k)$ is the desired speed for the drivers for the given density, and it is modeled by the following equation:

$$V_i(k) = \min\left((1+\alpha)V_{\mathrm{SL},i}(k), v_{\mathrm{f}} \exp\left(-\frac{1}{a}\left(\frac{\rho_i(k)}{\rho_{\mathrm{c}}}\right)^a\right)\right), \quad (3)$$

where a is a model parameter; v_{f} is the free-flow speed that the cars reach in steady state on a freeway with a very low density, and ρ_{c} is the critical density (i.e., the density corresponding to the maximum flow); $V_{\mathrm{SL},i}(k)$ is the variable speed limit applied on segment i at time step k; and α reflects the non-compliance of the drivers to the speed limit shown on the VSL panels.

If a VSL is not active for a given segment, then the desired speed in (3) is equal to the 2nd argument of the min operator. For the given freeway stretch, the variable speed limit V_{SL} affects both segments 2 and 3 at once (i.e. $V_{\mathrm{SL},2}(k) = V_{\mathrm{SL},3}(k) = V_{\mathrm{SL}}(k)$). The minimum and maximum values for the VSL are respectively 60 and 120 [km/h].

In order to complete the model, the following equation defines the flow that enters the controlled on-ramp connected to segment 5:

$$q_{\mathrm{r},5}(k) = \min\left(r(k)C_{\mathrm{r}}, D_{\mathrm{r}}(k) + \frac{w_{\mathrm{r}}(k)}{T}, C_{\mathrm{r}}\frac{\rho_{\mathrm{m}} - \rho_5(k)}{\rho_{\mathrm{m}} - \rho_{\mathrm{c}}}\right), \quad (4)$$

where $r(k) \in [0,1]$ is the ramp metering rate applied at time step k; ρ_{m} is the maximum density; C_{r} is the on-ramp capacity; $D_{\mathrm{r}}(k)$ is the demand of the on-ramp in [veh/h]; and $w_{\mathrm{r}}(k)$ is the queue length on the on-ramp of segment 5, which is updated as follows:

$$w_{\mathrm{r}}(k+1) = w_{\mathrm{r}}(k) + T(D_{\mathrm{r}}(k) - q_{\mathrm{r},5}(k)). \quad (5)$$

The Total Time Spent (TTS) by the drivers during each interval $[kT, (k+1)T]$, and expressed as [veh·h] is taken as the output of the system:

$$y(k) = Tw_{\mathrm{r}}(k) + T\sum_{i=1}^{6} L_i \lambda_i \rho_i(k) \ . \quad (6)$$

For the freeway considered, the values of the parameters of the METANET model are listed in Table 1.

Moreover, assume that, at $k = 0$, the densities of all the segments are equal to 25 [veh/(km lane)], and the speeds are equal to 80 [km/h], and the initial ramp queue length $w_{\mathrm{r}}(0)$ is equal to 0 [veh]. Also, assume that the ramp demand is constant and equal to $D_{\mathrm{r}}(k) = 1500$ [veh/h] $\forall k$, and that the flow entering the mainline is defined as follows:

τ	μ	C_{r}	ρ_{m}	α	K	a	v_{f}	ρ_{c}
19 [s]	60 $\left[\frac{\mathrm{km}^2}{\mathrm{h}}\right]$	2000$\left[\frac{\mathrm{veh}}{\mathrm{h}}\right]$	120 $\left[\frac{\mathrm{veh}}{\mathrm{km \cdot lane}}\right]$	0.1	40 $\left[\frac{\mathrm{veh}}{\mathrm{km \cdot lane}}\right]$	1.867	120 $\left[\frac{\mathrm{km}}{\mathrm{h}}\right]$	$33 + \frac{1}{3}E_1\left[\frac{\mathrm{veh}}{\mathrm{km \cdot lane}}\right]$

Table 1: Parameters for the METANET model.

$$q_0(k) = \begin{cases} 3000 + 50 \cdot E_2 \text{ [veh/h]} & \text{if } \; k < 60 \\ 1000 + 50 \cdot E_2 \text{ [veh/h]} & \text{if } \; k \geq 60 \end{cases} \tag{7}$$

Finally, some boundary conditions are defined:

- The downstream density of the final segment is considered to be equal to the density of the last segment: $\rho_7(k) = \rho_6(k), \forall L$.
- The speed upstream the first segment is considered to be equal to the speed of the first segment: $v_0(k) = v_1(k), \forall L$.

Note 1: When reporting your results for the tasks below, be sure to include the resulting values of the cost functions, as well as the total computation time of your code (for each case).

Note 2: Comment your MATLAB or Python code such that it will be understandable for the evaluation.

Tasks

1. Formulate the discrete-time state space model that predicts the speed and density in each segment, and the number of vehicles in the on-ramp queue for the next simulation cycle $k+1$ as follows:

   $$x(k+1) = f(x(k), u(k))$$
   $$y(k) = g(x(k))$$

   where f and g are vector-valued, nonlinear functions.

   Hints: Choose the states properly so that the output can be expressed as a function of the states. All states, inputs, and output variables must be expressed with the same units: [h], [km], [veh], etc.

   Write a MATLAB or Python function to simulate the system for given control signals $\{r(k)\}_{k=1}^{N}$ and $\{V_{\mathrm{SL}}(k)\}_{k=1}^{N}$. Take into account that we have an actual control time update step of 1 minute. So for a signal $r(k)$ defined with a sample step of $T = 10$ s, we thus have $r(6l+1) = r(6l+2) = \cdots = r(6l+6) = r_{\mathrm{control}}(l)$ for $l = 0, 1, 2,$ ... where r_{control} is the actual control signal with sample step $T_{\mathrm{c}} = 1$ min. A similar explanation holds for the variable speed limit control signal.

   Hint: To transform r_{control} into r in MATLAB, the function `repelem` might be useful; in Python, the function `numpy.repeat` might be useful.

2. Assume that both the ramp metering installation and the variable speed limits can be used to alter the traffic flow in the freeway stretch. Formulate an optimization problem to find the values of the ramp metering rates and of the VSLs that minimize the TTS by the drivers for the time interval $[0T,\ 120T]$.

3. (a) Select an appropriate optimization algorithm to solve the problem of Task 2, and motivate your choice. Then, write the corresponding MATLAB or Python code for running the selected algorithm. Run the optimization algorithm using two different starting points:
   (1) $r_{\text{control}}(l) = 0,\ V_{\text{SL,control}}(l) = 60,\ \forall l;$
   (2) $r_{\text{control}}(l) = 1,\ V_{\text{SL,control}}(l) = 120,\ \forall l;$ [i.e., the no-control case]

   Is there a substantial difference between the solutions obtained for the two different start points? Why? Can you prove that the solution obtained is the global optimum? How can the solution be improved (if possible)?

   (b) Repeat all of the above with $q_0(k)$ values that 50% higher than the ones of (7).

4. Plot the states and inputs of your simulations of Task 3 (a) and (b), and compare them with the ones obtained for the no-control case (i.e. for the control signal defined in Task 3(a)(2) above). Analyze the traffic situation for the considered simulation period. Moreover, provide the instantaneous and cumulative TTS values over the entire simulation period.

5. Plot $(1+\alpha)V_{\text{SL},i}(k)$ versus $V_i(k)$ (desired speed) in segments 2 and 3, and check if they are equal? If not, is there a way to bring $(1+\alpha)V_{\text{SL},i}(k)$ closer to $V_i(k)$ without affecting the optimal TTS?
   Plot $r(k)C_{\text{r}}$ versus the on-ramp flow $q_{\text{r},5}(k)$, and check whether the first of these signals is always above the second one, and if so, what could be done to bring them closer without affecting the optimal TTS?
   Determine the optimal ramp metering rates and the optimal variable speed limits that minimize the TTS and the differences mentioned above.

6. Starting from the setting of Tasks 3(b) and 4, consider the additional hard constraint $w_{\text{r}}(k) \leqslant 23 + \frac{1}{6}E_1,\ \forall k$. Extend your code to take this hard constraint into account and determine optimal ramp metering rates and the optimal variable speed limits that minimize the TTS subject to the given constraint.
   Plot your the states and inputs and verify whether the constraint is satisfied.
   Discuss and explain the effect of adding the constraint on the optimal TTS.

   Note:

   - If the optimization problem is infeasible, then you are allowed to gradually increase the upper bound for $w_{\text{r}}(k)$ with steps of 10% to get a feasible solution.
   - If your initial problem is feasible, but you notice that the upper bound for $w_{\text{r}}(k)$ is not reached at all, you should gradually decrease it with steps of 10% until it becomes active.

7. Starting from the setting of Tasks 3(b) and 4, consider that the ramp metering rate $r_{\text{control}}(l)$ and the VSLs $V_{\text{SL,contro}}(l)$ are now limited to assume values in the discrete sets $\{0.2, 0.4, 0.6, 0.8\}$ and $\{60, 80, 100, 120\}$ respectively for all l.
   Select an appropriate optimization algorithm to determine the values of the ramp metering rates and of the VSLs that minimize the TTS by the drivers for the time interval $[0T, 120T]$, and motivate your choice. Then, write the corresponding MATLAB or Python code and determine the optimal control signals. Can you prove that

the solution obtained is the global optimum? How can the solution be improved (if possible)?

8. Plot the states and inputs of your simulations in Task 7, and compare them with the ones obtained for the no-control case (defined in Task 3(a)(2) above). Analyze the traffic situation for the considered simulation period. Moreover, provide the instantaneous and cumulative TTS values over the entire simulation period.

9. Add a final discussion to analyze and compare all the different approaches for traffic control implemented in the assignment, and the corresponding algorithms. Draw your conclusions, and explain the results.

Practical Information

The solutions of the assignment should be uploaded to Brightspace before Monday, October 27, 2025, at 17:00 as:

1. One .pdf file (no other formats allowed) containing a **digitally** written report on the practical exercise, addressing the required tasks. Include an appendix in the report containing the MATLAB or Python code you used[1].

   After uploading, please verify the file to ensure that it is correct and not broken.

2. File(s) with the MATLAB or Python code you used. Please make sure your code is error-free.

Please also note that you will lose 0.5 point from your grade for this assignment for each (started) day of delay in case you exceed the deadline.

References

[1] A. Ferrara, S. Sacone, and S. Siri. *Freeway Traffic Modelling and Control*. Springer, 2018. 324 pp.

[2] A. Kotsialos, M. Papageorgiou, C. Diakaki, Y. Pavlis, and F. Middelham. "Traffic Flow Modeling of Large-Scale Motorway Networks Using the Macroscopic Modeling Tool METANET". In: *IEEE Transactions on Intelligent Transportation Systems* 3.4 (Dec. 2002), pp. 282–292.

[1] In LaTeX, you can use the package `listings` to create the appendix with the MATLAB or Python code.