Chapter 1

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July 15, 2020

Problem 1.A

1.

Suppose a and b are real numbers, not both 0 . Find real numbers c and d such that

$$\frac{1}{a+bi} = c + di.$$

Solution:

$$\frac{1}{a+bi} = \frac{a-bi}{(a+bi)(a-bi)} = \frac{a-bi}{a^2+b^2} = c+di \Rightarrow \begin{cases} c = \frac{a}{a^2+b^2} \\ d = -\frac{b}{a^2+b^2} \end{cases}$$

2.

Show that

$$\frac{-1+\sqrt{3}i}{2}$$

is a cube root of 1(meaning that its cube equals 1).

Solution:

Simply, we have

$$\begin{aligned} (\frac{-1+\sqrt{3}i}{2})^3 &= \frac{(-1+\sqrt{3}i)^3}{8} \\ &= \frac{(-2-2\sqrt{3}i)(-1+\sqrt{3}i)}{8} \\ &= \frac{2(1+\sqrt{3}i)(1-\sqrt{3}i)}{8} \\ &= \frac{2\times(1+3)}{8} \\ &= 1 \end{aligned}$$

More tricky solution is as below which depends on the computation methods of n-th root of a complex.

$$
\begin{aligned}
s &= \sqrt[3]{1} \\
&= \sqrt[3]{1+0\cdot i} \\
&= \sqrt[3]{\cos 0 + i\sin 0} \\
&= \cos\phi + i\sin\phi \\
&(\phi = \frac{1}{3}(0+2k\pi) \quad k=0,1,2)
\end{aligned}
\qquad \Rightarrow s = 1, \frac{-1+\sqrt{3}i}{2}, \frac{-1-\sqrt{3}i}{2}
$$

3.

Find two distinct square roots of i.

Solution:

Use the same tricky method shown above.

$$
\begin{aligned}
s &= \sqrt{i} \\
&= \sqrt{0+1\cdot i} \\
&= \sqrt{\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}} \\
&= \cos\phi + i\sin\phi \\
&(\phi = \frac{1}{2}(\frac{\pi}{2}+2k\pi) \quad k=0,1)
\end{aligned}
\qquad \Rightarrow s = \frac{\sqrt{2}(1+i)}{2}, \frac{-\sqrt{2}(1+i)}{2}
$$

For another solution, let $i = (a+bi)^2$, we have

$$
\begin{aligned}
i &= (a+bi)^2 \\
&= a^2 - b^2 + 2abi
\end{aligned}
\Rightarrow \begin{cases} a^2 - b^2 = 0 \\ 2ab = 1 \end{cases} \quad \Rightarrow a = b = \pm\frac{\sqrt{2}}{2}
$$

And we will get the same answer.

4.

Show that $\alpha + \beta = \beta + \alpha$ for all $\alpha, \beta \in \boldsymbol{C}$.

Solution:

Let $\alpha = a + bi, \beta = c + di$ for all $a, b, c, d \in \boldsymbol{R}$, we have

$$
\begin{aligned}
\alpha + \beta &= (a+bi) + (c+di) \\
&= (a+c) + (b+d)i \\
\beta + \alpha &= (c+di) + (a+bi) \\
&= (c+a) + (d+b)i \\
&= (a+c) + (b+d)i \\
\Rightarrow \quad & \alpha + \beta = \beta + \alpha
\end{aligned}
$$

5.

Show that $(\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)$ for all $\alpha, \beta, \lambda \in \boldsymbol{C}$.

Solution:

Let $\alpha = a+bi, \beta = c+di, \lambda = e+fi$ for all $a, b, c, d, e, f \in \boldsymbol{R}$, we have

$$
\begin{aligned}
(\alpha+\beta)+\lambda &= ((a+bi)+(c+di))+(e+fi) \\
&= ((a+c)+(b+d)i)+(e+fi) \\
&= (a+c+e)+(b+d+f)i \\
\alpha+(\beta+\lambda) &= (a+bi)+((c+di)+(e+fi)) \\
&= (a+bi)+((c+e)+(d+f)i) \\
&= (a+c+e)+(b+d+f)i \\
\Rightarrow \quad & (\alpha+\beta)+\lambda = \alpha+(\beta+\lambda)
\end{aligned}
$$

6.

Show that $(\alpha\beta)\lambda = \alpha(\beta\lambda)$for all $\alpha, \beta, \lambda \in \boldsymbol{C}$.

Solution:

Let $\alpha = a+bi, \beta = c+di, \lambda = e+fi$ for all $a, b, c, d, e, f \in \boldsymbol{R}$, we have

$$
\begin{aligned}
(\alpha\beta)\lambda &= ((a+bi)(c+di))(e+fi) \\
&= ((ac-bd)+(ad+bc)i)(e+fi) \\
&= (ace-bde-adf-bcf)+(ade+bce+acf-bdf)i \\
\alpha(\beta\lambda) &= (a+bi)((c+di)(e+fi)) \\
&= (a+bi)((ce-fd)+(cf+ed)i) \\
&= (ace-bde-adf-bcf)+(ade+bce+acf-bdf)i \\
\Rightarrow \quad & (\alpha\beta)\lambda = \alpha(\beta\lambda)
\end{aligned}
$$

7.

Show that for every $\alpha \in \boldsymbol{C}$, there exists a unique $\beta \in \boldsymbol{C}$ such that $\alpha+\beta = 0$.

Solution:

We know for every $a \in \boldsymbol{R}$, there exists a unique $b \in \boldsymbol{R}$ such that $a+b=0$, more specifically, $b=-a$, so we let $\alpha = a+bi, \beta = c+di, a, b, c, d \in \boldsymbol{R}$. Let $c=-a, d=-b$, there will be $\alpha+\beta = (a+c)+(b+d)i = 0$, so the unique one is $\beta = -a-bi = -\alpha$.

8.

Show that for every $\alpha \in \boldsymbol{C}$ with $\alpha \neq 0$, there exists a unique $\beta \in \boldsymbol{C}$ such that $\alpha\beta = 1$.

Solution:

From the Problem 1, we know the unique one of $\alpha = a + bi, a, b \in \boldsymbol{R}$ is

$$\beta = \frac{a}{a^2 + b^2} - \frac{b}{a^2 + b^2} i$$

9.

Show that $\lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta$for all $\lambda, \alpha, \beta \in \boldsymbol{C}$.

Solution:

Let $\alpha = a + bi, \beta = c + di, \lambda = e + fi$ for all $a, b, c, d, e, f \in \boldsymbol{R}$, we have

$$\begin{aligned}
\lambda(\alpha + \beta) &= (e + fi)(a + bi + c + di) \\
&= (e + fi)((a + c) + (b + d)i) \\
&= (ae + ce - bf - df) + (af + cf + be + de)i \\
\lambda\alpha + \lambda\beta &= (e + fi)(a + bi) + (e + fi)(c + di) \\
&= (ae - bf) + (af + be)i + (ce - df) + (cf + de)i \\
&= (ae + ce - bf - df) + (af + cf + be + de)i \\
\Rightarrow \quad & \lambda(\alpha + \beta) = \lambda\alpha + \lambda\beta
\end{aligned}$$

10.

Find $x \in \boldsymbol{R}^4$such that

$$(4, -3, 1, 7) + 2x = (5, 9, -6, 8).$$

Solution:

Let $x = (a, b, c, d), a, b, c, d \in \boldsymbol{R}$, and according to the equation above, we have

$$\begin{cases} 4 + 2a = 5 \\ -3 + 2b = 9 \\ 1 + 2c = -6 \\ 7 + 2d = 8 \end{cases} \Rightarrow \begin{cases} a = 5 \\ b = 6 \\ c = -5 \\ d = 5 \end{cases} \Rightarrow x = (5, 6, -5, 5)$$

11.

Explain why there does not exist $\lambda \in \mathbf{C}$ such that

$$\lambda(2 - 3i, 5 + 4i, -6 + 7i) = (12 - 5i, 7 + 22i, -32 - 9i).$$

Solution:

From the equation above, we have

$$\begin{cases} \lambda(2 - 3i) = 12 - 5i \\ \lambda(5 + 4i) = 7 + 22i \\ \lambda(-6 + 7i) = -32 - 9i \end{cases}$$

For the first equation, we have $\lambda = 3 + 2i$. Also for the second equation, we need $\lambda = 3 + 2i$. But for the third equation, when $\lambda = 3 + 2i$, the left side will be $(3 + 2i)(-6 + 7i) = -32 + 9i \neq -32 - 9i$. So here exists conflicts.

12.

Show that $(x + y) + z = x + (y + z)$ for all $x, y, z \in \mathbf{F}^n$.

Solution:

$$\begin{aligned}
(x + y) + z &= ((x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)) + (z_1, z_2, \ldots, z_n) \\
&= (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) + (z_1, z_2, \ldots, z_n) \\
&= (x_1 + y_1 + z_1, x_2 + y_2 + z_2, \ldots, x_n + y_n + z_n) \\
&= (x_1 + (y_1 + z_1), x_2 + (y_2 + z_2), \ldots, x_n + (y_n + z_n)) \\
&= (x_1, x_2, \ldots, x_n) + (y_1 + z_1, y_2 + z_2, \ldots, y_n + z_n) \\
&= (x_1, x_2, \ldots, x_n) + ((y_1, y_2, \ldots, y_n) + (z_1, z_2, \ldots, z_n)) \\
&= x + (y + z)
\end{aligned}$$

13.

Show that $(ab)x = a(bx)$ for all $x \in \mathbf{F}^n$ and all $a, b \in \mathbf{F}$.

Solution:

$$\begin{aligned}
(ab)x &= (ab)(x_1, x_2, \ldots, x_n) \\
&= (abx_1, abx_2, \ldots, abx_n) \\
&= (a(bx_1), a(b_x 2), \ldots, a(bx_n)) \\
&= a(bx_1, bx_2, \ldots, bx_n) \\
&= a(bx)
\end{aligned}$$

14.

Show that $1x = x$ for all $x \in \mathbf{F}^n$.

Solution:

$$1x = 1(x_1, x_2, \ldots, x_n) = (1x_1, 1x_2, \ldots, 1x_n) = (x_1, x_2, \ldots, x_n) = x$$

15.

Show that $\lambda(x + y) = \lambda x + \lambda y$ for all $\lambda \in \mathbf{F}$ and all $x, y \in \mathbf{F}^n$.

Solution:

$$\begin{aligned}
\lambda(x+y) &= \lambda((x_1, x_2, \ldots, x_n) + (y_1, y_2, \ldots, y_n)) \\
&= \lambda(x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n) \\
&= (\lambda(x_1 + y_1), \lambda(x_2 + y_2), \ldots, \lambda(x_n + y_n)) \\
&= (\lambda x_1 + \lambda y_1, \lambda x_2 + \lambda y_2, \ldots, \lambda x_n + \lambda y_n) \\
&= (\lambda x_1, \lambda x_2, \ldots, \lambda x_n) + (\lambda y_1, \lambda y_2, \ldots, \lambda y_n) \\
&= \lambda x + \lambda y
\end{aligned}$$

16.

Show that $(a+b)x = ax + bx$ for all $a, b \in \mathbf{F}$ and all $x \in \mathbf{F}^n$.

Solution:

$$\begin{aligned}
(a+b)x &= (a+b)(x_1, x_2, \ldots, x_n) \\
&= ((a+b)x_1, (a+b)x_2, \ldots, (a+b)x_n) \\
&= (ax_1 + bx_1, ax_2 + bx_2, \ldots, ax_n + bx_n) \\
&= (ax_1, ax_2, \ldots, ax_n) + (bx_1, bx_2, \ldots, bx_n) \\
&= ax + bx
\end{aligned}$$

Problem 1.B

1.

Prove that $-(-v) = v$ for every $v \in V$.

Solution:

By the definition of additive inverse, we have $w + (-w) = 0$. The unique additive inverse of v is $-v$ and for $-v$ the unique one is $-(-v)$. So we get $-(-v) = v$ for every $v \in V$.

2.

Suppose $a \in \mathbf{F}, v \in V$, and $av = 0$.Prove that $a = 0$ or $v = 0$.

Solution:

If $a = 0$, then we are done. If $a \neq 0$, we have $\frac{1}{a}(av) = \frac{1}{a}(0) \Rightarrow v = 0$. What should be pointed is that the 0 on the right side of the equation is a vector, so we do depend on the rule that a number times the vector 0 will be vector 0.

3.

Suppose $v, w \in V$. Explain why there exists a unique $x \in V$ such that $v + 3x = w$.

Solution:

Adding $-w$ and then multiplying $\frac{1}{3}$ to the both sides of the equation, we have

$$x + \frac{1}{3}(v - w) = 0$$

Suppose $y = \frac{1}{3}(v - w)$ and we have $x + y = 0$. We know $-y$ is the unique additive inverse of y, so we get $x = -y$. And that means for every y, x is unique. So we are Done.

4.

The empty set is not a vector space. The empty set fails to satisfy only one of the requirements listed in 1.19. Which one?

Solution:

Every one, I think. Because the empty set has no element.

5.

Show that in the definition of a vector space (1.19), the additive inverse condition can be replaced with the condition that

$$0v = 0 \text{ for all } v \in V$$

Here the 0 on the left side is the number 0 , and the 0 on the right side is the additive identity of V. (The phrase "a condition can be replaced" in a definition means that the collection of objects satisfying the definition is unchanged if the original condition is replaced with the new condition.)

Solution:

We can easily get the equation below.

$$0v = (1 + (-1))v = 1v + (-1v) = v + (-v)$$

So from $0v = 0$, we have $v + (-v) = 0$. This means the additive inverse condition can be replace with this condition.

6.

Let ∞ and $-\infty$ denote two distinct objects, neither of which is in $\mathbf{R}$. Define an addition and scalar multiplication on $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ as you could guess from the notation. Specifically, the sum and product of two real numbers is as usual, and for $t \in \mathbf{R}$ define

$$t\infty = \begin{cases} -\infty & \text{if } t < 0, \\ 0 & \text{if } t = 0, \\ \infty & \text{if } t > 0, \end{cases} \quad t(-\infty) = \begin{cases} \infty & \text{if } t < 0 \\ 0 & \text{if } t = 0 \\ -\infty & \text{if } t > 0 \end{cases}$$

$$\begin{aligned} t + \infty = \infty + t &= \infty, \\ t + (-\infty) = (-\infty) + t &= -\infty \\ \infty + \infty &= \infty, \\ (-\infty) + (-\infty) &= -\infty, \\ \infty + (-\infty) &= 0 \end{aligned}$$

Is $\mathbf{R} \cup \{\infty\} \cup \{-\infty\}$ a vector space over $\mathbf{R}$? Explain.

Solution:

No. If it is a vector space over $\mathbf{R}$, for example, we have

$$\infty = (2-1)\infty = 2\infty + (-1)\infty = \infty + (-\infty) = 0$$

But this is impossible!

Problem 1.C

1.

For each of the following subsets of $\mathbf{F}^3$, determine whether it is a subspace of $\mathbf{F}^3$

(a) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$
(b) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$
(c) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$
(d) $\{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$

Solution:

(a) Let $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 0\}$.

1) Obviously, $(0,0,0) \in U$.

2) Suppose $u = (x_1, x_2, x_3), v = (y_1, y_2, y_3) \in U$ and we have

$$\begin{aligned} &\begin{cases} x_1 + 2x_2 + 3x_3 = 0 \\ y_1 + 2y_2 + 3y_3 = 0 \end{cases} \\ \Rightarrow \quad & (x_1 + y_1) + 2(x_2 + y_2) + 3(x_3 + y_3) = 0 \\ \Rightarrow \quad & u + v = (x_1 + y_1, x_2 + y_2, x_3 + y_3) \in U \end{aligned}$$

3) Suppose $a \in F, u = (x_1, x_2, x_3) \in U$ and we have

$$\begin{aligned} & x_1 + 2x_2 + 3x_3 = 0 \\ \Rightarrow \quad & a(x_1 + 2x_2 + 3x_3) = ax_1 + 2ax_2 + 3ax_3 = 0 \\ \Rightarrow \quad & au = (ax_1, ax_2, ax_3) \in U \end{aligned}$$

Because U is a nonempty subset of F^3 that is closed under addition and scalar multiplication, U is a subspace of F^3.

(b) Let $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 + 2x_2 + 3x_3 = 4\}$. Simply, we know $(0,0,0) \notin U$, so U is not a subspace of F^3.

(c) Let $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 x_2 x_3 = 0\}$. Obviously, $(0,0,0) \in U$. But U is not closed under addition and scalar multiplication, so U is not a subspace of F^3.

(d) Let $U = \{(x_1, x_2, x_3) \in \mathbf{F}^3 : x_1 = 5x_3\}$. This format is the same with (a). So it is a subspace of F^3, too.

2.

Verify all the assertions in Example 1.35.

Solution:

(a) If this set is a subspace of $\mathbf{F}^4$, then $(0,0,0,0) \in \mathbf{F}^4$, then $0 = 5 \cdot 0 + b$. Hence $b = 0$. (b) (c) and (d) is similar to Problem 3 and 4. Now let us consider (e).

Denote the set of all sequences of complex numbers with limit 0 by A.

1) Additive identity: it is clear that $(0, 0, \cdots) \in A$.

2) Closed under addition: if $(a_1, a_2, \cdots), (b_1, b_2, \cdots) \in A$, then

$$\lim_{n\to\infty} a_n = 0 \quad \text{and} \quad \lim_{n\to\infty} b_n = 0.$$

It is easy to see

$$\lim_{n\to\infty} (a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n = 0 + 0 = 0.$$

This means $(a_1 + b_1, a_2 + b_2, \cdots) = (a_1, a_2, \cdots) + (b_1, b_2, \cdots) \in A$.

3) Closed under scalar multiplication: if $(a_1, a_2, \cdots) \in A$, then

$$\lim_{n\to\infty} a_n = 0.$$

For any $\lambda \in \mathbf{C}$, it is easy to see

$$\lim_{n\to\infty} (\lambda a_n) = \lambda \lim_{n\to\infty} a_n = \lambda 0 = 0.$$

This means $\lambda(a_1, a_2, \cdots) = (\lambda a_1, \lambda a_2, \cdots) \in A$.

3.

Show that the set of differentiable real-valued functions f on the interval (-4,4) such that $f'(-1) = 3f(2)$ is a subspace of $\mathbf{R}^{(-4,4)}$

Solution:

Denote the set of differentiable real-valued functions f on the interval $(-4, 4)$ such that $f'(-1) = 3f(2)$ by V.

Additive identity: it is clear that the constant function $f \equiv 0$ is contained in V.

Closed under addition: if $f, g \in V$, then f and g are differentiable real-valued functions. So is $f + g$. Moreover,

$$(f+g)'(-1) = f'(-1) + g'(-1) = 3f(2) + 3g(2) = 3(f(2) + g(2)) = 3(f+g)(2).$$

This concludes V is closed under addition.

Closed under scalar multiplication: if $f \in V$, for any $\lambda \in \mathbf{R}$, then f is differentiable real-valued functions. So is λf. Moreover,

$$(\lambda f)'(-1) = \lambda f'(-1) = \lambda(3f)(2) = 3(\lambda f)(2).$$

This deduces V is closed under scalar multiplication.

4.

Suppose $b \in \mathbf{R}$. Show that the set of continuous real-valued functions f on the interval [0,1] such that $\int_0^1 f = b$ is a subspace of $\mathbf{R}^{[0,1]}$ if and only if $b = 0$

Solution:

Denote the set of continuous real-valued functions f on the interval $[0, 1]$ such that $\int_0^1 f = b$ by V_b.

If V_b is a subspace of $\mathbf{R}^{[0,1]}$, then for any $f \in V_b$, we have $\int_0^1 f = b$. And we also have $kf \in V_b$ for any $k \in \mathbf{R}$. Hence

$$b = \int_0^1 (kf) = k \int_0^1 f = kb$$

for all $k \in \mathbf{R}$, and this happens if and only if $b = 0$.

Now if $b = 0$, then for any $f, g \in V_0$ and $\lambda \in \mathbf{R}$. We have

$$\int_0^1 (f+g) = \int_0^1 f + \int_0^1 g = 0 + 0 = 0$$

and $f + g$ is continuous real-valued functions since f and g are. So V_0 is closed under addition.

Similarly,

$$\int_0^1 (\lambda f) = \lambda \int_0^1 f = k0 = 0$$

and λf is continuous real-valued functions since f is. So V_0 is closed under scalar multiplication.

At last, the constant function $f \equiv 0 \in V_0$, which is also the additive identity in $\mathbf{R}^{[0,1]}$.

Hence V_0 is a subspace of $\mathbf{R}^n$.

5.

Is $\mathbf{R}^2$ a subspace of the complex vector space $\mathbf{C}^2$?

Solution:

No. $i \in C$ and $(1,1) \in \mathbf{R}^2$, but $i(1,1) = (i,i) \notin \mathbf{R}^2$.

6.

(a) Is $\{(a,b,c) \in \mathbf{R}^3 : a^3 = b^3\}$ a subspace of $\mathbf{R}^3$?
(b) Is $\{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\}$ a subspace of $\mathbf{C}^3$?

Solution:

(a) $\{(a,b,c) \in \mathbf{R}^3 : a^3 = b^3\} = \{(a,b,c) \in \mathbf{R}^3 : a = b\}$ It's obviously a subspace of $\mathbf{R}^3$.

(b) $\{(a,b,c) \in \mathbf{C}^3 : a^3 = b^3\} = \left\{(a,b,c) \in \mathbf{C}^3 : a = b \text{ or } a = \frac{1 \pm \sqrt{3}i}{2} b\right\} = A$ So we have $x = (\frac{1+\sqrt{3}i}{2}, 1, 0) \in A$ and $y = (\frac{1-\sqrt{3}i}{2}, 1, 0) \in A$ but $x + y = (1, 2, 0) \notin A$.

7.

Give an example of a nonempty **subset** U of $\mathbf{R}^2$ such that U is closed under addition and under taking additive inverses (meaning $-u \in U$ whenever $u \in U$), but U is not a **subspace** of $\mathbf{R}^2$

Solution:

Suppose $U = \{(x,y) \in \mathbf{R}^2 : x, y \in \mathbf{Z}\}$. It's easy to proof that U is closed under addition and under taking additive inverses. The most important is that $\frac{1}{2} \in R$ and $(1,1) \in U$ but $\frac{1}{2}(1,1) = (\frac{1}{2}, \frac{1}{2}) \notin U$

8.

Give an example of a nonempty subset U of $\mathbf{R}^2$ such that U is closed under scalar multiplication, but U is not a subspace of $\mathbf{R}^2$

Solution:

Suppose $U = \{(x,y) \in \mathbf{R}^2 : x = 0 \text{ or } y = 0\}$, then U is nonempty. $(0,y), (x,0) \in U$, then for $\lambda \in \mathbf{R}$, we have $\lambda(0,y) = (0, \lambda y) \in U$. Similarly, we have $\lambda(x,0) = (\lambda x, 0) \in U$, hence U is under scalar multiplication. However, $(1,0), (0,1) \in U$ while $(0,1) + (1,0) = (1,1) \notin U$. That means U is not a subspace of $\mathbf{R}^2$.

9.

A function $f : \mathbf{R} \to \mathbf{R}$ is called periodic if there exists a positive number p such that $f(x) = f(x+p)$ for all $x \in \mathbf{R}$. Is the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ a subspace of $\mathbf{R}^{\mathbf{R}}$? Explain.

Solution:

Denote the set of periodic functions from $\mathbf{R}$ to $\mathbf{R}$ by S. We have $f(x) = \cos(x) \in S$ and $g(x) = \sin\sqrt{2}(x) \in S$, then $h(x) = f(x)+g(x) = \cos x+\sin\sqrt{2}x$. Assume that there exists a positive number p such that $h(x) = h(x+p)$ for all $x \in R$, then we have

$$1 = h(0) = h(p) = h(-p) \Rightarrow \begin{cases} \cos p + \sin\sqrt{2}p = \cos(-p) + \sin\sqrt{2}(-p) \\ \cos p + \sin\sqrt{2}p = 1 \end{cases}$$

$$\Rightarrow \begin{cases} \sin\sqrt{2}p = 0 \\ \cos p = 1 \end{cases} \Rightarrow \begin{cases} p = 2k\pi \\ \sqrt{2}p = l\pi \end{cases} \quad (k, l \in \mathbf{Z})$$

Hence,

$$\sqrt{2} = \frac{l\pi}{2k\pi} = \frac{l}{2k} \in \mathbf{Q}$$

which is impossible. Therefore we get the conclusion.

10.

Suppose U_1 and U_2 are subspaces of V. Prove that the intersection $U_1 \cap U_2$ is a subspace of V

Solution:

1. Additive identity: By definition, we know $0 \in U_1$ and $0 \in U_2$, hence $0 \in U_1 \cap U_2$.
2. Closed under addition: If $x, y \in U_1 \cap U_2$ then $x, y \in U_1$ and $x, y \in U_2$, hence $x + y \in U_1$ and also $x + y \in U_2$, then $x + y \in U_1 \cap U_2$.
3. Closed under scalar multiplication: If $x \in U_1 \cap U_2$, then $x \in U_1$ and $x \in U_2$. Then for any $\lambda in \mathbf{F}$, we have $\lambda x \in U_1$ and $\lambda x \in U_2$ since U_1, U_2 is closed under scalar multiplication. Therefore $\lambda x \in U_1 \cap U_2$.

11.

Prove that the intersection of every collection of subspaces of V is a subspace of V

Solution:

Assume U_i are subspaces of V, where $i \in I$. Now we will show $\cap_{i\in I} U_i$ is a subspace of V.

Additive identity: by definition $0 \in U_i$ for every $i \in I$, hence $0 \in \cap_{i\in I} U_i$.

Closed under addition: if $x \in \cap_{i\in I} U_i$ and $y \in \cap_{i\in I} U_i$, then for any given $i \in I$, we have $x \in U_i$ and $y \in U_i$, hence $x + y \in U_i$ for U_i is closed under addition. Therefore $x + y \in \cap_{i\in I} U_i$.

Closed under scalar multiplication: if $x \in \cap_{i\in I} U_i$, then $x \in U_i$ for any given $i \in I$. Then for any $\lambda \in \mathbf{F}$, we have $\lambda x \in U_i$ since U_i is closed under scalar multiplication. Therefore $\lambda x \in \cap_{i\in I} U_i$.

12.

Prove that the union of two subspaces of V is a subspace of V if and only if one of the subspaces is contained in the other.

Solution:

Suppose U, W are both subspaces of V.

If $U \cup W$ is a subspace of V, moreover $U \not\subset W$ and $W \not\subset U$. Consider $u \in U - W$ and $w \in W - U$, then $u + w \in U \cup W$. Hence $u + w \in U$ or $\in W$. If $u + w \in U$, then $w = (u + w) - u \in U$. We get a contradiction. If $u + w \in W$, then $u = (u + w) - w \in W$. We get another contradiction. Hence if $U \cup W$is a subspace of V, we must have $U \subset W$ or $W \subset U$.

If $U \subset W$ or $W \subset U$, without loss of generality, we can assume $U \subset W$, then $U \sup W = W$ is obviously a subspace of V.

13.

Prove that the union of three subspaces of V is a subspace of V if and only if one of the subspaces contains the other two. [This exercise is surprisingly harder than the previous exercise, possibly because this exercise is not true if we replace $\mathbf{F}$ with a field containing only two elements.]

Solution:

TODO

14.

Verify the assertion in Example 1.38

Solution:

It is clear that U and W are subspaces of $\mathbf{F}^4$.

Now assume that $(x_1, x_1, y_1, y_1) \in U$ and $(x_2, x_2, x_2, y_2) \in W$, then

$$\begin{aligned} &(x_1, x_1, y_1, y_1) + (x_2, x_2, x_2, y_2) \\ =&(x_1 + x_2, x_1 + x_2, y_1 + x_2, y_1 + y_2) \in \{(x, x, y, z) : x, y, z \in \mathbf{F}\}. \end{aligned}$$

Hence $U + W \subset \{(x, x, y, z) : x, y, z \in \mathbf{F}^4\}$. For any $x, y, z \in \mathbf{F}$, we have $(0, 0, y - x, y - x) \in U$ and $(x, x, x, z + x - y) \in W$.

However,

$$(x, x, y, z) = (0, 0, y - x, y - x) + (x, x, x, z + x - y) \in U + W$$

Hence $\{(x, x, y, z) : x, y, z \in \mathbf{F}^4\} \subset U + W$.

Combining this with previous argument, it follows that $U+W = \{(x, x, y, z) : x, y, z \in \mathbf{F}\}$.

15.

Suppose U is a subspace of V. What is $U + U$?

Solution:

It will still be U. Because U is a subspace of V, hence closed under addition. Therefore for any $x, y \in U$, we have $x + y \in U$, i.e. $U + U \subset U$. Note that if $x \in U$, then $x = x + 0 \in U + U$, hence $U \subset U + U$. So $U + U = U$

16.

Is the operation of addition on the subspaces of V commutative? In other words, if U and W are subspaces of V, is $U + W = W + U$?

Solution:

For $x \in U$ and $y \in W$, because U is a subspace of V, hence closed under addition in V is commutative, we have $x + y = y + x \in W + U$. This implies $U + W \subset W + U$. Similarly, we have $W + U \subset U + W$. Hence $U + W = W + U$.

17.

Is the operation of addition on the subspaces of V associative? In other words, if U_1, U_2, U_3 are subspaces of V, is

$$(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)?$$

Solution:

Note that in V, we have $(x + y) + z = x + (y + z)$. Hence this similar to Problem 16. Let $x_i \in U_i, i = 1, 2, 3$, then

$$(x_1 + x_2) + x_3 = x_1 + (x_2 + x_3) \in U_1 + (U_2 + U_3)$$

Since every element in $(U_1+U_2)+U_3$ can be expressed as the form $(x_1+x_2)+x_3$. It follows that $(U_1 + U_2) + U_3 \subset U_1 + (U_2 + U_3)$. Similarly, we also have $U_1 + (U_2 + U_3) \subset (U_1 + U_2) + U_3$. Hence $(U_1 + U_2) + U_3 = U_1 + (U_2 + U_3)$.

18.

Does the operation of addition on the subspaces of V have an additive identity? Which subspaces have additive inverses?

Solution:

Denote the additive identity by U. Suppose W is a subspace of V. By definition, we have $U + W = W$. This means $U \subset W$ by the similar arguments in Problem 15 and 16. Hence the only possibility is $U = 0$.

Suppose subspace W of V has additive inverses, then there exists a subspace S of V such that $S + W = 0$. This can only happen when $W = 0$ since $W \subset W + S$.

19.

Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$U_1 + W = U_2 + W$$

then $U_1 = U_2$

Solution:

It's easy to give. Suppose $U_1 = V, U_2 \subseteq V, W = V - U_2 \subset V$. Then we have

$$U_2 + W = V - U_2 + U_2 = V = U_1 = U_1 + W$$

But $U1 \neq U_2$.

20.

Suppose

$$U = \{(x, x, y, y) \in \mathbf{F}^4 : x, y \in \mathbf{F}\}$$

Find a subspace W of $\mathbf{F}^4$ such that $\mathbf{F}^4 = U \oplus W$

Solution:

$W = \{(0, z, 0, w) \in \mathbf{F}^4 : z, w \in \mathbf{F}\}$. For any $(x, y, z, w) \in \mathbf{F}^4$, we have

$$(x, y, z, w) = (x, x, z, z) + (0, y - x, 0, w - z) \in U + W$$

since $(x, x, z, z) \in U$ and $(0, y - x, 0, w - z) \in W$. We have $\mathbf{F}^4 = U + W$.

Moreover, if $(x, y, z, w) \in U \cap W$, then we must have $x = y$ and $z = w$ since $(x, y, z, w) \in U$.

Similarly, since $(x, y, z, w) \in W$, we have $x = 0$ and $z = 0$. Therefore, $x = y = 0$ and $z = w = 0$, hence $(x, y, z, w) = (0, 0, 0, 0)$. It follows that $U \cap W = 0$. Hence $F = U \oplus W$.

21.

Suppose

$$U = \{(x, y, x + y, x - y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$$

Find a subspace W of $\mathbf{F}^5$ such that $\mathbf{F}^5 = U \oplus W$

Solution:

$W = \{(0, 0, a, b, c) \in \mathbf{F}^5 : a, b, c \in \mathbf{F}\}$. For any $(x, y, a, b, c) \in \mathbf{F}^5$, we have

$$(x, y, a, b, c) = (x, y, x + y, x - y, 2x) + (0, 0, a - x - y, b - x + y, c - 2x) \in U + W$$

since $(x, y, x + y, x - y, 2x) \in U$ and $(0, 0, a - x - y, b - x + y, c - 2x) \in W$. Therefore $\mathbf{F}^5 = U + W$.

And we will have the same procedure to proof $U \cap W = 0$ like Problem 20. So $\mathbf{F}^5 = U \oplus W$.

22.

Suppose
$$U = \{(x, y, x+y, x-y, 2x) \in \mathbf{F}^5 : x, y \in \mathbf{F}\}$$
Find three subspaces W_1, W_2, W_3 of $\mathbf{F}^5$, none of which equals $\{0\}$, such that $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$

Solution:

It's easy to find.

$$\begin{aligned} W_1 &= \{(0,0,a,0,0) \in \mathbf{F}^5 : a \in \mathbf{F}\} \\ W_2 &= \{(0,0,0,b,0) \in \mathbf{F}^5 : b \in \mathbf{F}\} \\ W_3 &= \{(0,0,0,0,c) \in \mathbf{F}^5 : c \in \mathbf{F}\} \end{aligned}$$

By the same argument as in Problem 20&21, we have $\mathbf{F}^5 = U \oplus W_1 \oplus W_2 \oplus W_3$.

23.

Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \quad \text{and} \quad V = U_2 \oplus W$$

then $U_1 = U_2$

Solution:

Let $V = \mathbf{R}^2, U_1 = \{(x,0) \in \mathbf{R}^2 : x \in \mathbf{R}\}, U_2 = \{(0,y) \in \mathbf{R}^2 : y \in \mathbf{R}\}, W = \{(a,a) \in \mathbf{R}^2 : a \in \mathbf{R}\}$. We have $V = U_1 \oplus W$ and $V = U_2 \oplus W$, but $U_1 \neq U_2$.

24.

A function $f : \mathbf{R} \to \mathbf{R}$ is called even if

$$f(-x) = f(x)$$

for all $x \in \mathbf{R}$.
A function $f : \mathbf{R} \to \mathbf{R}$ is called odd if

$$f(-x) = -f(x)$$

for all $x \in \mathbf{R}$.
Let U_e denote the set of real-valued even functions on $\mathbf{R}$ and let U_o denote the set of real-valued odd functions on $\mathbf{R}$. Show that

$$\mathbf{R}^{\mathbf{R}} = U_\mathrm{e} \oplus U_\mathrm{o}.$$

Solution:

Given $f \in \mathbf{R}^{\mathbf{R}}$ and two other function made by f:

$$f_e = \frac{f(x) + f(-x)}{2} \quad f_o = \frac{f(x) - f(-x)}{2}$$

Of course, $f_e, f_o \in \mathbf{R}^{\mathbf{R}}$.

Apparently, $f = f_e + f_o \in U_e + U_o$, so we get $\mathbf{R}^{\mathbf{R}} = U_e + U_o$.

Let $f \in U_e \cap U_o$, then $f(x) = f(-x)$ since $f \in U_e$ and $f(x) = -f(x)$ since $f \in U_o$ for all $x \in \mathbf{R}$. Sum up the two equations, we get $f(x) = 0$ for all $x \in \mathbf{R}$.

Hence $f = 0$, which implies $U_e \cap U_o = \{0\}$ and $\mathbf{R}^{\mathbf{R}} = U_e \oplus U_o$.