HW1: Convex sets

Matrix

January 5, 2024

Homework 1, due Friday 7/1/22: 2.9, 2.12a-e, 2.15, 2.4, A1.4, A2.7, 2.13.

Solution: [2.9(a)]

Voronoi region definition yields

$$\begin{aligned}
\|x - x_0\|_2 \le \|x - x_i\|_2 &\iff (x - x_0)^T(x - x_0) \le (x - x_i)^T(x - x_i) \\
&\iff x^Tx - 2x_0^Tx + x_0^Tx_0 \le x^Tx - 2x_i^Tx + x_i^Tx_i \\
&\iff (x_i - x_0)^Tx \le \frac{1}{2}(x_i - x_0)^T(x_i + x_0).
\end{aligned}$$

The result above defines a halfspace for each i. Thus, we can express V in the form $V = \{x \mid Ax \preceq b\}$ with

$$A = \begin{bmatrix} (x_1 - x_0)^T \\ \vdots \\ (x_K - x_0)^T \end{bmatrix}, \ b = \begin{bmatrix} \frac{1}{2}(x_1 - x_0)^T(x_1 + x_0) \\ \vdots \\ \frac{1}{2}(x_K - x_0)^T(x_K - x_0) \end{bmatrix}.$$

Solution: [2.9(b)]

Since polyhedron P has nonempty interior, we can express P in the form $P = \{x \mid Ax \preceq b\}$[1] with $A \in \mathbf{R}^{K\times n}$ and $b \in \mathbf{R}^K$. We can choose any point x_0 from P's interior, then take a mirror image of x_0 with respect to a hyperplane $\{a_i^Tx = b_i\}$ with $a_i = (x_i - x_0)$ and $b_i = \frac{1}{2}(x_i - x_0)^T(x_i - x_0)$ to get x_i. Thus, any point $x \in P$ has shorter (or equal, when on hyperplane) distance to x_0 than x_i.

The mirror image x_i of x_0 with respect to hyperplane $\{a_i^Tx = b_i\}$ satisfies

$$\begin{cases} \frac{\|a_i^Tx_0 - b_i\|}{\|a_i\|} = \frac{\|a_i^Tx_i - b_i\|}{\|a_i\|} \iff a_i^Tx_0 - b_i = -1 \cdot (a_i^Tx_i - b_i), \\ x_i = x_0 + \lambda a_i. \end{cases}$$

Solving λ yields:

$$\lambda = \frac{2(b_i - a_i^Tx_0)}{\|a_i\|^2}.$$

[1] If P only contains hyperplane, then P has empty interior, contradicting the assumption. This form contains both hyperplanes and halfspaces.

Thus, we can choose

$$x_i = x_0 + \frac{2(b_i - a_i^T x_0)}{\|a_i\|^2} a_i, \;\; i = 1, \ldots, K$$

so that the polyhedron P is the *Voronoi region* of x_0 with respect to $x_1, \ldots, x_K$.
Solution: [2.9(c)]

A polyhedron decomposition of $\mathbf{R}^n$ can not always be described as the *Voronoi regions.*

There is a counterexample in $\mathbf{R}^2$ plane, shown in figure 1. Consider a polyhedron decomposition with respect to two hyperplanes H_1 and H_2, which yields four polyhedrons P_1, P_2, P_3 and P_4. Suppose P_1 is a *Voronoi region*, then choose any of point $x_1 \in P_1$, its mirror image x_3 with respect to H_2 must be in $\tilde{P}_1 \subset P_3$. Now we consider P_2 as another *Voronoi region*, its mirror image x_3 with respect to H_1 must be in $\tilde{P}_2 \subset P_3$. There is not such point $x_3 \in P_3$ satisfying the two conditions.

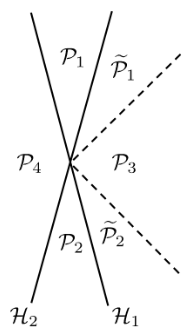

Figure 1: Polyhedral decomposition/Voronoi region partition counterexample in $\mathbf{R}^2$

Solution: [2.12(a)-(e)]

(a) *salb*: $\{x \in \mathbf{R}^n \mid \alpha \leq a^T x \leq \beta\}$. Choose any two point x_1 and x_2 in *slab*. For any $\theta \in [0, 1]$ we have convex combination

$$\alpha \leq \theta a^T x_1 + (1-\theta) a^T x_2 \leq \beta.$$

Thus, *slab* is convex set.

(b) *rectangle*: $\{x \in \mathbf{R}^n \mid \alpha_i \leq x_i \leq \beta_i, \; i = 1, \ldots, n\}$. For any two point x and y in *rectangle* and $\theta \in [0, 1]$, we have convex combination

$$\alpha_i \leq \theta x_i + (1-\theta) y_i \leq \beta_i, \;\; i = 1, \ldots, n.$$

Thus, *rectangle* is convex set.

(c) *wedge*: $\{x \in \mathbf{R}^n \mid a_1^T x \leq b_1, \; a_2^T x \leq b_2\}$. For any two point x and y in *wedge* and $\theta \in [0, 1]$, we have convex combination

$$\theta a_1^T x + (1-\theta) a_1^T y \leq b_1, \;\; \theta a_2^T x + (1-\theta) a_2^T y \leq b_2.$$

Thus, *wedge* is convex set.

(d) Set: $\{x \mid \|x - x_0\|_2 \leq \|x - y\|_2 \text{ for all } y \in S\}$ where $S \subseteq \mathbf{R}^n$. From the definiton, we have the following:

$$\begin{aligned}
\|x - x_0\|_2 \leq \|x - y\|_2 &\iff (x - x_0)^T(x - x_0) \leq (x - y)^T(x - y) \\
&\iff x^Tx - 2x_0^Tx + x_0^Tx_0 \leq x^Tx - 2y^Tx + y^Ty \\
&\iff (y - x_0)^Tx \leq \frac{1}{2}(y - x_0)^T(y + x_0).
\end{aligned}$$

Choose any two point x_1 and x_2 in set. For any $\theta \in [0, 1]$ we have convex combination

$$\begin{aligned}
\theta(y - x_0)^Tx_1 + (1 - \theta)(y - x_0)^Tx_2 &\leq \theta \cdot \frac{1}{2}(y - x_0)^T(y + x_0) \\
&\quad + (1 - \theta) \cdot \frac{1}{2}(y - x_0)^T(y + x_0) \\
&= \frac{1}{2}(y - x_0)^T(y + x_0).
\end{aligned}$$

Thus, the set is convex.

(e) Set: $\{x \mid \mathbf{dist}(x, S) \leq \mathbf{dist}(x, T)\}$ where $S, T \subseteq \mathbf{R}^n$, and $\mathbf{dist}(x, S) = \inf\{\|x - z\|_2 \mid z \in S\}$. For any norm, we have the following for any two points x_1 and x_2 in $\mathbf{R}^n$ and $\theta \in [0, 1]$:

$$\begin{aligned}
\|\theta x_1 + (1 - \theta)x_2 - z\|_2 &= \|\theta(x_1 - z) + (1 - \theta)(x_2 - z)\|_2 \\
&\leq \theta\|x_1 - z\|_2 + (1 - \theta)\|x_2 - z\|_2, \ \forall z \in \mathbf{R}^n.
\end{aligned}$$

Consider x_1 and x_2 in the set, we have convex combination

$$\begin{aligned}
\mathbf{dist}(\theta x_1 + (1 - \theta)x_2, S) &\leq \theta\,\mathbf{dist}(x_1, S) + (1 - \theta)\,\mathbf{dist}(x_2, S) \\
&\leq \theta\,\mathbf{dist}(x_1, T) + (1 - \theta)\,\mathbf{dist}(x_2, T).
\end{aligned}$$

We can not conclude that $\mathbf{dist}(\theta x_1 + (1 - \theta)x_2, S) \leq \mathbf{dist}(\theta x_1 + (1 - \theta)x_2, T)$ from above results.

The set is not convex, with a counterexample in $\mathbf{R}^2$ shown in figure 2. With S (dark area) and T (green area) both are not convex set, the set (pink area) is not convex.

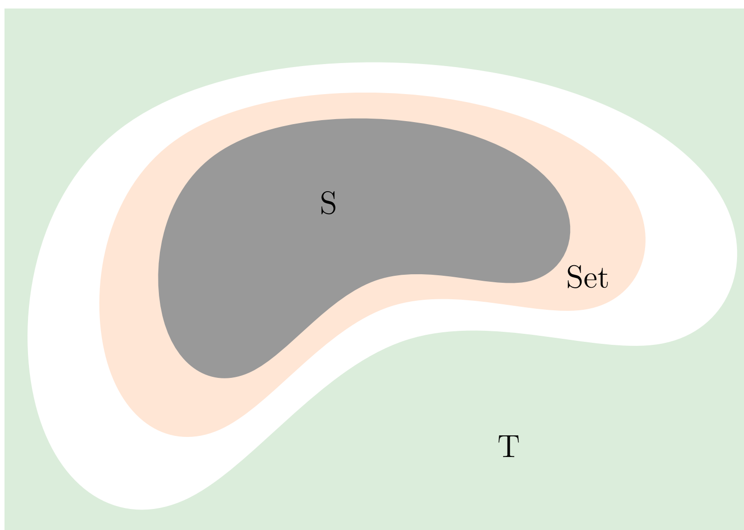

Figure 2: 2.12(e) counterexample with non-convex set in $\mathbf{R}^2$