HW1: Convex sets

Matrix

January 3, 2024

Homework 1, due Friday 7/1/22: 2.9, 2.12a-e, 2.15, 2.4, A1.4, A2.7, 2.13.

Solution: [2.9(a)]

Voronoi region definition yields

$$\begin{aligned}\|x - x_0\|_2 \le \|x - x_i\| &\iff (x - x_0)^T (x - x_0) \le (x - x_i)^T (x - x_i)\\ &\iff x^T x - 2xx_0^T + x_0^T x_0 \le x^T x - 2xx_i^T + x_i^T x_i\\ &\iff (x_i - x_0)^T x \le \frac{1}{2}(x_i - x_0)^T (x_i + x_0).\end{aligned}$$

Note that the result above defines a halfspace for each i. Thus, we can express V in the form $V = \{x \mid Ax \preceq b\}$ with

$$A = \begin{bmatrix} (x_1 - x_0)^T \\ \vdots \\ (x_K - x_0)^T \end{bmatrix},\ b = \begin{bmatrix} \frac{1}{2}(x_1 - x_0)^T (x_1 + x_0) \\ \vdots \\ \frac{1}{2}(x_K - x_0)^T (x_K - x_0) \end{bmatrix}.$$

Solution: [2.9(b)]

Since polyhedron P has nonempty interior, we can express P in the form $P = \{x \mid Ax \preceq b\}$[1] with $A \in \mathbf{R}^{K \times n}$ and $b \in \mathbf{R}^K$. We can choose any point x_0 from P's interior, then take a mirro image of x_0 with respect to a hyperplane $\{a_i^T x = b_i\}$ with $a_i = (x_i - x_0)$ and $b_i = \frac{1}{2}(x_i - x_0)^T (x_i - x_0)$ to get x_i. Thus, any point $x \in P$ has shorter (or equal, when on hyperplane) distance to x_0 than x_i.

The mirro image x_i of x_0 with respect to hyperplane $\{a_i^T x = b_i\}$ satisfies

$$\begin{cases} \frac{\|a_i^T x_0 - b_i\|}{\|a_i\|} = \frac{\|a_i^T x_i - b_i\|}{\|a_i\|} \iff a_i^T x_0 - b_i = -1 \cdot (a_i^T x_i - b_i), \\ x_i = x_0 + \lambda a_i. \end{cases}$$

Solving λ yields:

$$\lambda = \frac{2(b_i - a_i^T x_0)}{\|a_i\|^2}.$$

[1] If P only contains hyperplane, then P has empty interior, contradicting the assumption. This form contains both hyperplanes and halfspaces.

Thus, we can choose

$$x_i = x_0 + \frac{2(b_i - a_i^T x_0)}{\|a_i\|^2} a_i, \ i = 1, \ldots, K$$

so that the polyhedron P is the *Voronoi region* of x_0 with respect to $x_1, \ldots, x_K$.
Solution: [2.9(c)]