Unifying the Local Shear Stress Model in the Gioia Framework

The Gioia models (2006 for wall shear stress and 2010 for local shear stress in the flow column) can indeed be unified by focusing on the role of a dominant "carrier" eddy with characteristic scale s, which determines both the normal velocity component v_n (setting the transport rate) and the tangential velocity component v_t (setting the momentum contrast). This approach aligns with the phenomenological emphasis on momentum-transferring eddies straddling a wetted surface W_y at height y from the wall, while incorporating roughness and viscous effects in the inner layer. Below, I outline the unified picture, addressing your key points about whether v_n and v_t are simultaneously determined by s, and the implications for superposition vs. a single carrier eddy.

Key Concepts in the Unified Model

- **Eddy Scale s as the Unifying Parameter**:

  The dominant eddy size s varies with location in the flow:

  - In the **inner layer** (near the bed/wall, $y \lesssim k_s + a\eta$), $s = k_s + a\eta$, where k_s is the equivalent roughness height (capturing surface irregularities) and $a\eta$ accounts for the viscous sublayer thickness ($\eta = \nu/u_\tau$ is the viscous length scale, ν is kinematic viscosity, $u_\tau = \sqrt{\tau/\rho}$ is the friction velocity, and a is a dimensionless constant of order 1). This reflects the 2006 model's focus on wall-bounded eddies limited by roughness and viscosity.
  - In the **outer layer** ($y \gg k_s + a\eta$), $s \propto y$, consistent with the 2010 model's scaling where eddies grow with distance from the wall (similar to Townsend's attached-eddy hypothesis, but extended to include inner-layer constraints).

  This hybrid scaling for s bridges the two models: $s \approx \max(y, k_s + a\eta)$. It ensures the model transitions smoothly from roughness/viscosity-dominated near-wall behavior to wall-distance-dominated outer flow.

- **Determination of v and v**:

n t

Yes, once s is set, both v_n (normal "carrier" velocity) and v_t (tangential "cargo" momentum difference) are determined simultaneously from the local flow properties, without needing to superpose multiple eddy scales.

- $v_n \sim (\epsilon s)^{1/3}$, where ϵ is the turbulent energy dissipation rate per unit mass (from Kolmogorov's phenomenological theory). In the inertial range, $\epsilon \approx u_\tau^3/(\kappa y)$ (with $\kappa \approx 0.4$ the von Kármán constant), so $v_n \sim u_\tau (s/y)^{1/3}$. For outer-layer eddies ($s \sim y$), this simplifies to $v_n \sim u_\tau$. Near the wall ($s = k_s + a\eta$), it captures the reduced eddy velocity due to boundary constraints.
- $v_t \sim 2su'(y)$, as you noted. This derives from the momentum contrast across the eddy: the eddy spans a vertical distance $\sim 2s$ (from $y - s$ to $y + s$), so the velocity difference is $\Delta u \approx u(y+s) - u(y-s) \approx 2s\, du/dy = 2su'(y)$. (The factor of 2 arises from Taylor expansion assuming a locally linear gradient; it's approximate but captures the scaling.) This holds for both models—in the 2010 columnar case, it's explicit; in the 2006 wall case, it approximates $v_t \sim V$ (bulk velocity) when s spans the full profile, but refines to the local gradient for unification.

The shear stress then unifies as $\tau(y) \sim \rho v_t v_n \sim \rho \cdot 2su'(y) \cdot (\epsilon s)^{1/3}$, where ρ is fluid density. Substituting $\epsilon \sim u_\tau^3/(\kappa y)$ and balancing with the total stress equation $\tau(y) + \rho\nu u'(y) = \tau_0(1 - y/R)$ (where τ_0 is wall stress, R is pipe radius or channel half-height) yields a solvable ODE for $u(y)$.

Implications for Eddy Superposition vs. Carrier Eddy

- **No Need for Large-on-Small Superposition**: Your intuition is correct—the multi-scale superposition (e.g., hierarchical eddies in some models) isn't necessary here. By focusing on a single dominant carrier eddy per location (with scale s as above), the model captures momentum transport efficiently. This is analogous to Townsend's attached-eddy model, where eddies scale as $s \propto y$, but Gioia's framework extends it by including inner-layer effects ($k_s + a\eta$), explaining why attached-eddy predictions work well in the log layer but fail near rough walls (as per your footnote). The carrier eddy encapsulates the "straddling" effect: it provides both the transport rate (v_n) and momentum payload (v_t), making the model parsimonious.

  - In practice, this avoids integrating over all eddy sizes; instead, spectral analysis (as in the 2010 paper) identifies the dominant s that maximizes shear stress contribution.

- **Derivation of $v_t \sim 2su'(y)$:**
  To arrive at this: Consider an eddy centered at y with radius s. The mean velocity at the eddy's top is $u(y+s) \approx u(y) + su'(y) + (s^2/2)u''(y) + \cdots$, and at the bottom $u(y-s) \approx u(y) - su'(y) + (s^2/2)u''(y) - \cdots$. The difference is:

$$u(y+s) - u(y-s) \approx 2su'(y) + O(s^3 u'''(y)).$$

  Neglecting higher-order terms (valid for small s/y or linear gradients), $v_t \approx 2su'(y)$. This is exact in the buffer/log layers where curvature is weak and aligns with the 2010 model's momentum-contrast approximation.

Resulting Velocity Profile and Shear Stress

Integrating the unified $\tau(y)$ leads to the mean velocity profile $u(y)$:

- In the outer/log layer ($s \sim y$): $u(y) \approx (u_\tau/\kappa)\ln(y/z_0) + C$, where $z_0 \sim k_s$ for rough walls (recovering the log law).
- Near the wall ($s = k_s + a\eta$): The profile shifts to account for roughness, with effective displacement $\Delta y \sim k_s$.
  The wall stress $\tau_0 \sim \rho u_\tau^2$ closes via bulk velocity $V = (1/R)\int_0^R u(y)dy$, yielding friction laws like Strickler-type for transitional rough ($\tau_0 \propto (k_s/R)^{1/3}\rho V^2$) or log-based for fully rough.

This unification preserves the strengths of both Gioia models: spectral phenomenology for eddy velocities (2006) and position-dependent stress (2010), while echoing attached-eddy ideas without full hierarchy. If you'd like numerical validation (e.g., via code for a specific Re or k_s), or extensions to open channels, let me know!