Populations: The group of individuals under study is called population or universe.

Sample: A finite subset of statistical individuals in a population is called samle.

Sample Size: The number of individuals in a sample is called the sample size.

Sampling distribution: The probability of the statistic that would be obtained if the number of samples each of the same size, were very large is called the sampling distribution of the statistic.

If we draw a sample of ~~s~~ size n from a given finite population of size N, then total number of possible samples is

$$^{N}C_n = \frac{N!}{n!\,(N-n)!} = k\ (\text{say})$$

* ~~Sample mean~~

Standard Error:

The standard deviation of the sampling distribution of a static (s.d.) is known as Standard Error and represent by S.E.

Let n is the sample size, σ^2 is the population variance and P the population proportion and $Q = 1-P$. n_1 and n_2 represent the sizes of two independent random sample respectively drawn from the given population.

S.E. of some of the well known statistics, for large samples, are given below. (2)

Statistic — S. Error

$t = \{x_1, x_2 - x_n\}$

t is function of random variable and t is called statistic

1. Sample mean : $\bar{x} = \sigma/\sqrt{n}$

2. Sample proportion 'p' $= \sqrt{\frac{PQ}{n}}$ with Replacement

3. Sample S.D: $s = \sqrt{\sigma^2/2n}$

4. ~~Sample variance~~ s^2 Sample variance $= \frac{\sigma^2}{...}$

$\sigma^2 \to$ population variance

Sample moment:

$$\mu_3 = \sigma^3\sqrt{\frac{96}{n}} \quad , \quad \mu_4 = \sigma^4\sqrt{\frac{96}{n}}$$

5. Difference of two sample means: $(\bar{x}_1 - \bar{x}_2) = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$

6. Difference of two sample S.D.: $(S_1 - S_2) = \sqrt{\frac{\sigma_1^2}{2n_1} + \frac{\sigma_2^2}{2n_2}}$

7. Difference of two sample proportion:

$$P_1 - P_2 = \sqrt{\frac{P_1Q_1}{n_1} + \frac{P_2Q_2}{n_2}}$$

8. Sample proportion WOR

$$= \sqrt{\frac{PQ}{n}}\sqrt{\frac{N-n}{N-1}}$$

Utility of Standard Error: If 't' is any statistic, then for large samples:

$$Z = \frac{t - E(t)}{\sqrt{V(t)}} \sim N(0,1)$$

$$\Rightarrow \quad Z = \frac{t - E(t)}{S.E.(t)} \sim N(0,1), \text{ for large samples.}$$

Prob. A population consists of 5 numbers 2, 3, 6, 8, 11. Find the mean and variance of the population. By drawing all possible simple samples of size 2 find the sampling distribution of means and the standard deviation of the distribution, with replacement

Q. A population consists of four numbers 2, 3, 4, 5. Consider all possible distinct samples of size two with replacement. Find (a) the population mean (b) the population standard deviation (s.d.) (c) the sampling distribution of means (d) the mean of the sampling distribution (S.D.) of means (S.D.M.) (e) s.d. of S.D. of mean. Verify (d) and (e) directly from (a) and (b) by use of suitable formulae. (Also solve this problem without replacement)

Soln. For the 4 members of the population

$$\Sigma x = 2+3+4+5 = 14$$

$$\Sigma x^2 = 2^2+3^2+4^2+5^2 = 4+9+16+25 = 54$$

(a) Population mean $(\mu) = \frac{14}{4} = 3.5$ or $\mu = \frac{2+3+4+5}{4} = 3.5$

(b) Population variance $(\sigma^2) = \frac{\Sigma x^2}{N} - \left(\frac{\Sigma x}{N}\right)^2 \quad (N=4)$

$$= \frac{54}{4} - \left(\frac{14}{4}\right)^2$$

$$= 13.5 - 12.25 = 1.25$$

Variance $= \frac{\Sigma(x-\bar{x})^2}{N}$

$= \frac{\Sigma(x^2+\bar{x}^2-2x\bar{x})}{N} = \frac{\Sigma x^2}{N} + \frac{\bar{x}^2 \Sigma 1}{N} - \frac{2\bar{x}\Sigma x}{N}$ $\boxed{\text{or}}$

$= \frac{\Sigma x^2}{N} - \bar{x}^2 = \frac{\Sigma x^2}{N} - \left(\frac{\Sigma x}{N}\right)^2$

Population variance $\cancel{\sigma^2 = }$

$$\sigma^2 = \frac{(2-3.5)^2+(3-3.5)^2+(4-3.5)^2+(5-3.5)^2}{4}$$

$$= 1.25$$

∴ Population standard deviation $(\sigma) = \sqrt{1.25} = 1.118033$.

Case I Simple Random Sampling with replacement (SRSWR) or Sampling with replacement (infinite population):

The total number of samples with replacement is $N^n = 4^2 = 16$. Here N = population size and n = sample size.

The possible random sample of size 2 with replacement and the sample means are shown below: (2,3,4,5) x (2,3,4,5)

S.No.	Sample values	Total	Sample mean ($\bar{x}$)	S.No.	Sample values	Total	Sample mean ($\bar{x}$)
1	(2, 2)	4	2	9	(4, 2)	6	3
2	(2, 3)	5	2.5	10	(4, 3)	7	3.5
3	(2, 4)	6	3	11	(4, 4)	8	4
4	(2, 5)	7	3.5	12	(4, 5)	9	4.5
5	(3, 2)	5	2.5	13	(5, 2)	7	3.5
6	(3, 3)	6	3	14	(5, 3)	8	4
7	(3, 4)	7	3.5	15	(5, 4)	9	4.5
8	(3, 5)	8	4	16	(5, 5)	10	5

(sample mean is obtained by total values of sample values / n (sample size))

(c) Sampling distribution of mean (S.D.M.)

Sample mean $\bar{x}$	2	2.5	3	3.5	4	4.5	5	
frequency	1	2	3	4	3	2	1	
probability	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{3}{16}$	$\frac{4}{16}$	$\frac{3}{16}$	$\frac{2}{16}$	$\frac{1}{16}$	Total = 1

(d) Mean of the sampling distribution of means:

mean of these 16 means is known as mean of the sampling distribution of means

$$E(\bar{x}) = \mu_{\bar{x}} = 2\times\frac{1}{16} + 2.5\times\frac{2}{16} + 3\times\frac{3}{16} + 3.5\times\frac{4}{16}$$
$$+ 4\times\frac{3}{16} + 4.5\times\frac{2}{16} + 5\times\frac{1}{16}$$
$$= \frac{1}{16}(2+5+9+14+12+9+5)$$
$$= \frac{1}{16}(56) = 3.5$$

OR add all $\bar{x}$ and divide by n

$$E(\bar{x}) = \frac{\Sigma\bar{x}}{n}$$

$Var(\bar{x}) = (\text{or } \sigma^2_{\bar{x}}) = E(\bar{x}^2) - E(\bar{x})^2$

$= \left[2^2 \times \frac{1}{16} + (2.5)^2 \times \frac{1}{16} + \ldots + 5^2 \times \frac{1}{16}\right] - (3.5)^2$

$= \frac{1}{16}[4 + \overset{12.5}{\cancel{10}} + 27 + 28 + 48 + 18 + 25) - \ldots$

$= \frac{10}{16} = 0.625$

s.d. of the sampling distribution of means

~~S.D. ($\bar{x}$~~ $S.D.(\bar{x}) = \sigma_{\bar{x}} = \sqrt{Var(\bar{x})} = \sqrt{0.625}$

$= 0.7905694$

verification

(i) $E(\bar{x}) = 3.5 = \mu$ (population mean μ = Sample mean $E(\bar{x})$ or $\mu_{\bar{x}}$)

(ii) $S.E(\bar{x}) = \frac{\sigma}{\sqrt{n}} = \frac{\sqrt{1.25}}{\sqrt{2}} = \frac{1.118033}{\sqrt{2}} = 0.79057 = \sigma_{\bar{x}}$ (with replacement $S.E = \frac{\sigma}{\sqrt{2}}$, WOR $S.E = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}$)

Also we know that Standard deviation of sampling distribution of means are Standard Error

So $S.E. = \frac{\sigma}{\sqrt{n}} = 0.79057$ ($\sigma \rightarrow$ Population variance)

Note ① ~~Population mean and Sample mean are equal~~

② S.E and S.D. are also equal.

Case II Simple Random Sampling without Replacement (SRSWOR)

or Sampling without Replacement (finite population).

The total number of samples without replacement is $^{N}C_{n} = {}^{4}C_{2} = 6$. Therefor 6 Random samples of size 2 without replacement and the sample means are shown below:

S.No	Sample values	total	Sample mean ($\bar{x}$)
1	(2,3)	5	2.5
2	(2,4)	6	3
3	(2,5)	7	3.5
4	(3,4)	7	3.5
5	(3,5)	8	4
6	(4,5)	9	4.5

(c) Sampling distribution of mean

Sample mean ($\bar{x}$)	2.5	3	3.5	4	4.5	
frequency	1	1	2	1	1	
probability	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	Total 1

(d) mean of the sampling distribution of means

$$E(\bar{x}) = \mu_{\bar{x}} = \frac{1}{6}(2.5 + 3 + 3.5 \times 2 + 4 + 4.5)$$

$$= \frac{21}{6} = 3.5$$

(e) $Var(\bar{x}) = \sigma^2_{\bar{x}} = E(\bar{x}^2) - \{E(\bar{x})\}^2$

$= \frac{2.5}{6} = 0.4166$

So $S.D.(\bar{x}) = \sigma_{\bar{x}} = \sqrt{0.4166} = 0.645497$

and so $S.E(\bar{x}) = \frac{\sigma}{\sqrt{n}} = \frac{1.118083}{\sqrt{2}}$

$= \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = \frac{1.11803}{\sqrt{2}}\sqrt{\frac{4-2}{4-1}}$

$= \cancel{0.4166}\ 0.6454$

verification -

(i) $\therefore E(\bar{x}) = 3.5 = \mu$

(ii) $S.D.(\bar{x}) = 0.645497 = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}}\ \frac{1.118083}{\sqrt{2}}\sqrt{\frac{4-2}{4-1}}$

$= 0.645497 = \sigma_{\bar{x}}$

6. A population consists of the four members 3, 7, 11, 15. Consider all possible samples of size 2 which can be drawn with replacement from this population. Find

(a) Population mean

(b) Population standard deviation

(c) mean of the sampling distribution of means

(d) the standard deviation of the samply distribution of means

Verify (c) and (d) from (a) and (b) by use of suitable formulae. (Also solve for WOR)

Case I SRSWR

(a) 9, (b) $\sqrt{20}$

(c) $E(\bar{x}) = 9$ (d) $Var(\bar{x}) = 10$

$S.D. = \sqrt{10}$

Case II (a) 9 (b) $\sqrt{20}$

(c) $E(\bar{x}) = 9$ (d) $S.D. = \sqrt{\frac{20}{3}}$

Q. A population consist of the following elements 3, 4, 5, 6, 7. Draw all possible samples of size 3 drawn without replacement. Find the sampling distribution of sample mean. Hence find the sample mean and standard error. (8)

Soln. SRSWOR

Total no. of samples $^5C_3 = \frac{\lfloor 5}{\lfloor 3 \lfloor 2} = \frac{5 \times 4^2 \times \lfloor 3}{\lfloor 3 \times \lfloor 2} = 10$

S.No.	Sample value	Total	Sample mean($\bar{x}$)
1	3,4,5		
2	3,5,6		
3	3,6,7		
4	4,5,6		
5	4,6,7		
6	5,6,7		
7	3,4,6		
8	3,4,7		
9			
10			

S.No.	Sample value	Total	Sample mean ($\bar{x}$)
1	3,4,5		4
2	3,4,6		4.33
3	3,4,7		4.67
4	3,5,6		4.67
5	3,5,7		5
6	3,6,7		5.33
7	4,5,6		5
8	4,5,7		5.33
9	4,6,7		5.67
10	5,6,7		6

Sampling distribution of mean

Sample mean($\bar{x}$)	4	4.33	4.67	5	5.33	5.67	6	
frequency	1	1	2	2	2	1	1	
probability	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{2}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	Total 1

mean of S.D.M $E(\bar{x}) = \frac{1}{10}(4 + 4.33 + 4.67 \times 2 + 5 \times 2 + 5.33 \times 2 + 5.67 \times 1 + 6 \times 1)$

$= \frac{50}{10} = 5$

Variance of sampling distribution of mean $Var(\bar{x}) = E(\bar{x}^2) - \{E(\bar{x})\}^2$

$$= \frac{1}{10}\{16 + 18.7489 + 43.6178 + 50 + 56.81787$$ (9)

$$32.1489 + 36\times 1\} - 5^2$$

$$= \frac{1}{10}\{253.3334\} - 25$$

$$= 25.33334 - 25 = 0.33334$$

$$S.D. = \sqrt{0.33334} = 0.577356$$

(S.D. of sample mean)

$$S.E. = \frac{\sigma}{\sqrt{n}}\sqrt{\frac{N-n}{N-1}} = \frac{\sigma}{\sqrt{3}}\sqrt{\frac{5-3}{5-1}} = \frac{\sigma}{\sqrt{3}}\sqrt{\frac{2}{4}}$$

S.D. of population

$$= \frac{\sigma}{\sqrt{3}}\sqrt{\frac{1}{2}}$$

$$= \frac{\sigma}{\sqrt{3}}\sqrt{0.5}$$

$$\sigma = \frac{(3-5)^2 + (4-5)^2 + (5-5)^2 + (6-5)^2 + (7-5)^2}{5}$$

S.D. of population

$$= \frac{4+1+0+1+4}{5} = \frac{10}{5} = 2$$

$$\therefore S.E. = \frac{2\sqrt{0.5}}{\sqrt{3}} = \frac{2\times 0.70710}{1.7320} = 0.8165$$

Problems. 1 A population Consists of 4 members 0, 4, 6 and 6. Draw all possible Samples of size 2 drawn with replacement. Find the sampling distribution of sample mean. Hence find the mean and variance of the sample mean.

Soln Since the samples of size 2 are drawn with replacement there are $4 \times 4 = 16$ samples which are given by cross product $\{0, 4, 6, 6\} \times \{0, 4, 6, 6\}$

S.No

Sample Values	Sample total	Sample mean
0, 0	0	0
0, 4	4	2
0, 6	6	3
0, 6	6	3
4, 0	4	2
4, 4	8	4
4, 6	10	5
4, 6	10	5
6, 0	6	3
6, 4	10	5
6, 6	12	6
6, 6	12	6
6, 0	6	~~6~~ 3
6, 4	10	5
6, 6	12	6
6, 6	12	6

The sampling distribution of sample mean $(\bar{x})$ is given below

value of sample mean $(\bar{x})$	0	2	3	4	5	6	Total
Frequency (f)	1	2	4	1	4	4	$\Sigma f = 16$
Probability	$\frac{1}{16}$	$\frac{2}{16}$	$\frac{4}{16}$	$\frac{1}{16}$	$\frac{4}{16}$	$\frac{4}{16}$	1

$$E(\bar{x}) = \sum \bar{x}\, p(\bar{x})$$

$$= 0 + 2\times\frac{2}{16} + 3\times\frac{4}{16} + 4\times\frac{1}{16} + 5\times\frac{4}{16} + 6\times\frac{4}{16}$$

$$= \frac{1}{16}\left[4+12+4+20+24\right) = \frac{64}{16} = 4.$$

$$E(\bar{x}^2) = \sum \bar{x}^2 p(\bar{x}) = 0 + 4\times\frac{2}{16} + 9\times\frac{4}{16} + 16\times\frac{1}{16} + 25\times\frac{4}{16} + 36\times\frac{4}{16}$$

$$= \frac{1}{16}\left(8+36+16+100+144\right)$$

$$= \frac{304}{16} = 19.$$

$$\mathrm{Var}(\bar{x}) = E(\bar{x}^2) - \left(E(\bar{x})\right)^2$$

$$= 19 - (4)^2 = 3.$$

Ex. A population consists of five numbers 2, 3, 6, 8, 11. Consider all positive samples of size two which can be drawn with replacement from this population. Calculation the standard error of sample mean.

Soln Given that five number 2, 3, 6, 8, 11 drawn in the first draw can be associated with any one of these five number drawn in the second draw and hence the total number of possible sample of size 2 is $5 \times 5 = 25$.

$(2, 3, 6, 8, 11) \times (2, 3, 6, 8, 11)$

(2, 2)	2
(3, 2)	2.5
(6, 2)	4.0
(8, 2)	5.0
(11, 2)	6.5
(2, 3)	2.5
(3, 3)	3
(6, 3)	4.5
(8, 3)	5.5
(11, 3)	7.0
(2, 6)	4.0
(3, 6)	4.5
(6, 6)	6.0
(8, 6)	7.0
(11, 6)	8.5
(2, 8)	5.0
(3, 8)	5.5
(6, 8)	7.0
(11, 8)	8.0
(2, 11)	9.5
(3, 11)	6.5
(6, 11)	7.0
(8, 11)	8.5
(11, 11)	9.5
	Total 150

$$E(\bar{x}) = \frac{\Sigma \bar{x}}{25} = \frac{150}{25} = 6$$

The variance of sampling distribution of the mean is given by.

$$Var(\bar{x}) = \frac{1}{25} \Sigma \left[\bar{x} - E(\bar{x})\right]^2$$

$$= \frac{1}{25} \left[(2-6)^2 + (2.5-6)^2 + \cdots + (11-6)^2\right]$$

$$= \frac{135}{25} = 5.40$$

$$S.E.(\bar{x}) = \sqrt{Var(x)} = \sqrt{5.4} = 2.32$$

Second method:

We know that S.E. of sample mean in random sampling from an infinite population or in random sampling with replacement is given by

$Var(\bar{x}) = \frac{\sigma^2}{n} \Rightarrow S.E.(\bar{x}) = \frac{\sigma}{\sqrt{n}}$, where σ is the population S.D.

The population values are 2, 3, 6, 8, 11

$$\mu = \frac{1}{5}(2+3+6+8+11) = \frac{30}{5} = 6$$

$$\sigma^2 = \frac{1}{5}\left[(2-6)^2 + (3-6)^2 + (6-6)^2 + (8-6)^2 + (11-6)^2\right]$$

$$= \frac{1}{5}\left[16+9+0+4+25\right]$$

$$= \frac{54}{5}$$

$$Var(\bar{x}) = \frac{\sigma^2}{n} = \frac{54}{5\times2} = \frac{54}{10} = 5.4.$$

$$S.E.(\bar{x}) = \sqrt{5.4}$$

$$= \underline{2.32}$$