Semantics of Functional Programming

The Scott Model of **PCF**

Chuang, Tyng-Ruey
`trc.iis.sinica.edu.tw`

Institute of Information Science, Academia Sinica

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Interpretation of types and contexts

Every type is interpreted as a domain and a context is a product of domains:

Definition 1

Every type σ in **PCF** associates with a domain D_σ as follows:

1. $D_{\mathtt{nat}} := \mathbb{N}_\perp$, and
2. $D_{\tau \to \sigma} := [D_\sigma \to D_\tau]$.

Definition 2

For each context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, the associated domain is defined as

$$D_\Gamma := D_{\sigma_1} \times D_{\sigma_2} \times \cdots \times D_{\sigma_n}$$

and the associated domain of the empty context is $1 = \{*\}$.

Interpretation of simply type lambda calculus and natural numbers

Every judgement $\Gamma \vdash \mathrm{M} : \tau$ is interpreted as a Scott-continuous function

$$[\![\Gamma \vdash \mathrm{M} : \tau]\!] : D_\Gamma \to D_\tau$$

and defined inductively on the derivation of $\Gamma \vdash \mathrm{M} : \tau$ as follows.

1. For (var), we interpret the judgement as the projection from the product to the component:

   $$[\![x_1 : \sigma_1, \ldots, x_i : \sigma_i, ldots, x_n : \sigma_n \vdash x_i : \sigma_i]\!] := \pi$$

   or pointwise

   $$[\![\Gamma, x : \tau, \Delta \vdash x : \tau]\!](\vec{x}) = x$$

   $$[\![\Gamma \vdash \lambda x.\, \mathrm{M} : \sigma \to \tau]\!] := \Lambda[\![\Gamma, x : \sigma \vdash \mathrm{M} : \tau]\!]$$

   $$[\![\Gamma \vdash \mathrm{M}\ \mathrm{N} : \tau]\!] = ev \circ \langle [\![\Gamma \vdash \mathrm{M} : \sigma \to \tau]\!], [\![\Gamma \vdash \mathrm{N} : \sigma]\!] \rangle$$

   $$[\![\Gamma \vdash \mathtt{zero} : \mathtt{nat}]\!](\vec{d}) := 0$$

   $$[\![\Gamma \vdash \mathtt{suc}\ \mathrm{M} : \mathtt{nat}]\!] := S \circ [\![\Gamma \vdash \mathrm{M} : \mathtt{nat}]\!]$$

Interpretation of general recursion and if-zero test

$$[\![\Gamma \vdash \mathtt{Y}x.\, \mathsf{M} : \sigma]\!] := \mu \circ \Lambda[\![\Gamma, x : \sigma \vdash \mathsf{M} : \sigma]\!]$$

$$\begin{aligned} &[\![\Gamma \vdash \mathtt{ifz}(\mathsf{M}; \mathsf{M}_0; \mathsf{M}_1) : \tau]\!] \\ := \ &\mathit{ifz}_\tau \circ \langle\, [\![\Gamma \vdash \mathsf{M} : \mathtt{nat}]\!], [\![\Gamma \vdash \mathsf{M}_0 : \tau]\!], \Lambda[\![\Gamma, x : \mathtt{nat} \vdash \mathsf{M}_1 : \tau]\!] \,\rangle \end{aligned}$$

where S and ifz_τ are defined by

$$S(n) := \begin{cases} \perp & \text{if } d = \perp \\ n+1 & \text{otherwise.} \end{cases} \qquad \mathit{ifz}_\tau(n, x, f) := \begin{cases} \perp & \text{if } n = \perp, \\ x & \text{if } n = 0, \\ f(m) & \text{if } n = m+1. \end{cases}$$

In particular, every program $\mathsf{M} : \tau$ associates with a function from 1 to $[\![\tau]\!]$, and thus determines a unique element $[\![\mathsf{M}]\!](*)$ of $[\![\tau]\!]$.

Convention

For brevity, we write $[\![\mathsf{M}]\!]$ instead of $[\![\vdash \mathsf{M} : \tau]\!](*)$ for every program M of type τ

Theorem 3

For every judgement $\Gamma \vdash \mathsf{M} : \tau$*, the associated function*

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!] : [\![\Gamma]\!] \to [\![\tau]\!]$$

is Scott continuous.