

Semantics of Functional Programming

Lecture I: **PCF** and its Operational Semantics

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1 Introduction

The meaning of programs

How can we tell if a program is correct?

1. The meaning of a program is ideally independent of its actual implementation.
2. A rigorous specification of language is essential. Everything must be defined without any ambiguities. No undefined behaviour.
3. A structural approach to semantics. The meaning of a program is built from its parts, so verification is possible.
4. The precise definitions of notions, such as strict and lazy evaluation strategies.

Two approaches to be taught

1. **Operational approach:** How values and functions are computed? E.g., for $\text{add} : \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$ and numerals $\underline{2}, \underline{4}$

$$\begin{aligned} \text{add } \underline{2} \ \underline{4} &\rightsquigarrow \text{succ}(\text{add } \underline{1} \ \underline{4}) \\ &\rightsquigarrow \text{succ succ}(\text{add } \underline{0} \ \underline{4}) \rightsquigarrow \text{succ succ } \underline{4} \equiv \underline{6} \end{aligned}$$

2. **Denotational approach:** What the values and functions are? The set \mathbb{N}_\perp of natural numbers with *divergence* \perp is the denotation of the type **nat**, e.g.,

$$\begin{aligned} \llbracket \text{add } \underline{2} \ \underline{4} \rrbracket &= \llbracket \text{add } \underline{2} \rrbracket \llbracket \underline{4} \rrbracket = (\llbracket \text{add} \rrbracket \llbracket \underline{2} \rrbracket) \llbracket \underline{4} \rrbracket \\ &= (x \mapsto 2 + x) \llbracket \underline{4} \rrbracket = 2 + 4 = 6 \end{aligned}$$

where $\llbracket \text{add} \rrbracket : \mathbb{N}_\perp \rightarrow (\mathbb{N}_\perp \rightarrow \mathbb{N}_\perp)$.

2 Programming in PCF

2.1 Syntax and typing rules for PCF

What is PCF?

PCF stands for **P**rogramming **C**omputable **F**unctionals,

1. an extension of simply typed lambda calculus with **nat** and general recursion;
2. extremely simple compared to modern programming languages;
3. Turing complete, i.e. every computable function on natural numbers can be defined in **PCF**.

Syntax of PCF

Definition 1. Types in **PCF** are defined by the inference rules

$$\frac{}{\text{nat set}} \quad \frac{\tau_1 \text{ set} \quad \tau_2 \text{ set}}{\tau_1 \rightarrow \tau_2 \text{ set}}$$

or equivalently by the grammar $\tau ::= \text{nat} \mid \tau \rightarrow \tau$.

Definition 2. The collection of terms in **PCF** is defined inductively:

$$\begin{aligned} M ::= & x \mid \lambda x. M \mid M \ N \mid \text{zero} \mid \text{succ } M \\ & \mid \text{ifz}(M; M; x. M) \mid Yx. M \end{aligned}$$

where x is a variable.

The operator Y is called the **fixpoint operator**, or **general recursion**.

Typing Rules for PCF

A **judgement** $\Gamma \vdash M : \tau$ denotes that M has type τ under the context Γ . **PCF** consists of

- simply typed lambda calculus (with \rightarrow only):

$$\frac{}{\Gamma, x : \sigma, \Delta \vdash x : \sigma} \text{ (var)}$$

$$\frac{\Gamma, x : \sigma \vdash M : \tau}{\Gamma \vdash \lambda x. M : \sigma \rightarrow \tau} \text{ (abs)}$$

$$\frac{\Gamma \vdash M : \sigma \rightarrow \tau \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M \ N : \tau} \text{ (app)}$$

- the type of natural numbers:

$$\frac{}{\Gamma \vdash \text{zero} : \text{nat}} \text{ (z)}$$

$$\frac{\Gamma \vdash M : \text{nat}}{\Gamma \vdash \text{succ } M : \text{nat}} \text{ (s)}$$

- **if zero** test: it is meant to be the *case analysis* on natural numbers:

$$\frac{\Gamma \vdash M : \text{nat} \quad \Gamma \vdash M_0 : \tau \quad \Gamma, x : \text{nat} \vdash M_1 : \tau}{\Gamma \vdash \text{ifz}(M; M_0; x. M_1) : \tau} \text{ (ifz)}$$

- **general recursion** (to be explained):

$$\frac{\Gamma, x : \sigma \vdash M : \sigma}{\Gamma \vdash Yx. M : \sigma} \text{ (Y)}$$

Definition 3. A term M of type τ is called a **program** of type τ in **PCF** if it is derivable under an empty context, i.e. the judgement

$$() \vdash M : \tau$$

is derivable where $()$ denotes the empty context for emphasis.

E.g. $Yx. \text{succ } x$ and $\text{ifz}(\text{zero}; \lambda x. \text{zero}; y. \lambda z. y)$ are programs, but $\lambda y. x y$ or $\text{succ } (\lambda x. \text{succ } x)$ are not.

Example: Predecessor

The predecessor of natural numbers can be defined as

$$\text{pred} := \lambda x. \text{ifz}(x; \text{zero}; y. y) : \text{nat} \rightarrow \text{nat}$$

with the following typing derivation:

$$\frac{\frac{\frac{\Gamma \vdash x : \text{nat} \quad \Gamma \vdash \text{zero} : \text{nat} \quad \Gamma, y : \text{nat} \vdash y : \text{nat}}{\Gamma \vdash \text{ifz}(x; \text{zero}; y. y) : \text{nat}}}{\vdash \lambda x. \text{ifz}(x; \text{zero}; y. y) : \text{nat} \rightarrow \text{nat}}}$$

where $\Gamma := x : \text{nat}$.

2.2 Operational semantics

One-step reduction

Definition 4. A **closed value** denoted **val** is one of the following:

$$\begin{array}{c} \frac{}{\text{zero val}} \\ \frac{M \text{ val}}{\text{succ } M \text{ val}} \\ \frac{}{\lambda x. M \text{ val}} \end{array}$$

A value is meant to be the final result of computation. For example, natural numbers **zero**, **succ zero** and lambda functions $\lambda x. x$ etc. This formulation also includes ill-typed terms such as $\text{succ } (\lambda x. M)$.

Notation

The notation \rightsquigarrow is a relation between terms, denoted

$$M \rightsquigarrow M'$$

which means that the term M reduces to M' in *one step*.

Reduction of general recursion and natural numbers

For general recursion, each occurrence of $Y.M$ reduces to an substitution of x in M by itself:

$$\frac{}{Yx. M \rightsquigarrow M[Yx. M/x]} \text{ (}\rightsquigarrow\text{-fix)}$$

For the *eager evaluation*, $\text{succ } M$ reduces to $\text{succ } M'$ if M reduces to M'

$$\frac{M \rightsquigarrow M'}{\text{succ } M \rightsquigarrow \text{succ } M'} \text{ (}\rightsquigarrow\text{-succ)}$$

On the other hand, it is possible to defer the evaluation of natural numbers, and this evaluation is known as the *lazy evaluation*. To do so, we simply remove this reduction rule $\rightsquigarrow\text{-succ}$ and modify the definition of closed values for **succ** to

$$\frac{}{\text{succ } M \text{ val}}$$

without any assumptions.

Reduction of ifz and Y

For the if-zero test, the first argument must be reduced to a closed value before branching, but branching can be done before the evaluation on branches:

$$\frac{M \rightsquigarrow M'}{\text{ifz}(M; M_0; x. M_1) \rightsquigarrow \text{ifz}(M'; M_0; x. M_1)} \text{ (}\rightsquigarrow\text{-ifz)}$$

$$\frac{}{\text{ifz}(\text{zero}; M_0; x. M_1) \rightsquigarrow M_0} \text{ (}\rightsquigarrow\text{-ifz}_0\text{)}$$

$$\frac{\text{succ } M \text{ val}}{\text{ifz}(\text{succ } M; M_0; x. M_1) \rightsquigarrow M_1[M/x]} \text{ (}\rightsquigarrow\text{-ifz}_1\text{)}$$

Reduction for application: call-by-name and call-by-value

In call-by-name evaluation, arguments are substituted directly into the function body. It is a *non-strict* evaluation strategy, because application with non-terminating arguments can be terminating.

$$\frac{M \rightsquigarrow M'}{M N \rightsquigarrow M' N} (\rightsquigarrow\text{-lapp})$$

$$\frac{}{(\lambda x. M) N \rightsquigarrow M[N/x]} (\rightsquigarrow\text{-by-name})$$

In call-by-value evaluation, each argument is evaluated before application, so we replace (\rightsquigarrow -by-name) by the following two rules. It is a *strict* evaluation strategy, as non-terminating arguments always lead to non-terminating terms.

$$\frac{M \text{ val} \quad N \rightsquigarrow N'}{M N \rightsquigarrow M' N} (\rightsquigarrow\text{-by-value-1})$$

$$\frac{N \text{ val}}{(\lambda x. M) N \rightsquigarrow M[N/x]} (\rightsquigarrow\text{-by-value-2})$$

In the following context, we adopt the *call-by-name* interpretation.

Many-step reduction

Definition 5. The relation \rightsquigarrow^* between terms is defined inductively:

$$\frac{}{M \rightsquigarrow^* M}$$

$$\frac{M_1 \rightsquigarrow M_2 \quad M_2 \rightsquigarrow^* M_3}{M_1 \rightsquigarrow^* M_3}$$

Note that $M \rightsquigarrow^* M'$ if M' is reachable from M after finitely many steps of reduction, i.e. $M = M_0 \rightsquigarrow M_1 \rightsquigarrow \dots M_k = M'$.

Proposition 6. *The relation \rightsquigarrow^* is reflexive and transitive.*

Proof. Easy exercise. \square

Example: Calculating the factorials

To define the factorials, we are seeking for a function **fact** satisfying

$$\text{fact}: n \mapsto \begin{cases} 0 & \text{if } n = 1 \\ n \times \text{fact}(n') & \text{if } n = n' + 1 \end{cases}$$

and this can be understood as a fixpoint of the functional F mapping $f: \mathbb{N} \rightarrow \mathbb{N}$ to $f': \mathbb{N} \rightarrow \mathbb{N}$ defined by

$$f': n \mapsto \begin{cases} 1 & \text{if } n = 0 \\ n \times f(n') & \text{if } n = n' + 1 \end{cases}$$

where f' does not depend on itself.

Under the context of $\Gamma := f: \text{nat} \rightarrow \text{nat}$, we define a function depending on f as follows:

$$\Gamma \vdash \lambda n. \text{ifz}(n; \text{suc zero}; m. n \times (f m)) : \text{nat} \rightarrow \text{nat}$$

and thus we derive its fixpoint by Y :

$$\text{fact} := Yf. \lambda n. \text{ifz}(n; \text{suc zero}; m. n \times (f m))$$

where the term **suc zero** represents the natural number 1.

Example 7. Let $\underline{0} := \text{zero}$ and $\underline{n+1} := \text{suc } \underline{n}$. We calculate **fact** $\underline{2}$:

$$\begin{aligned} \text{fact } \underline{2} &\rightsquigarrow (\lambda n. \text{ifz}(n; \underline{1}; m. n \times (\text{fact } m))) \underline{2} \\ &\rightsquigarrow \text{ifz}(\underline{2}; \underline{1}; m. \underline{2} \times (\text{fact } m)) \\ &\rightsquigarrow \underline{2} \times (\text{fact } \underline{1}) \\ &\rightsquigarrow \underline{2} \times (\lambda n. \text{ifz}(n; \underline{1}; m. n \times (\text{fact } m)) \underline{1}) \\ &\rightsquigarrow \dots \rightsquigarrow \underline{2} \times (\underline{1} \times \underline{1}) \rightsquigarrow^* \underline{2} \end{aligned}$$

where the definition of $\times: \text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$ is left as an exercise.

In-class exercise

Try to be familiar with **ifz**.

1. Calculate **pred** M for $M \text{ val}$ to closed values with their derivations: For the base case **zero**:

$$\text{pred zero} \rightsquigarrow^* ?$$

For the inductive case $M = \text{suc } N$:

$$\frac{\text{suc } N \text{ val}}{\text{pred } (\text{suc } N) \rightsquigarrow^* ?}$$

2. Define **flip**: $\text{nat} \rightarrow \text{nat}$ such that **flip zero** \rightsquigarrow^* **suc zero** and **flip (suc M)** \rightsquigarrow^* **zero**.

In-class exercise: fold on natural numbers

fold on natural numbers is defined in Haskell as follows:

$$\begin{aligned} \text{fold} &:: (a \rightarrow a) \rightarrow a \rightarrow \text{Integer} \rightarrow a \\ \text{fold } f \ e \ 0 &= e \\ \text{fold } f \ e \ n &= f (\text{fold } f \ e \ (n - 1)) \end{aligned}$$

By modifying the definition of **fact**, give the corresponding term of **fold** in **PCF**.

3 Type safety

Progress Theorem

Every well-typed is either a closed value or a reducible term.

Theorem 8. *If $\vdash M : \tau$ then either M is a closed value or there exists M' such that $M \rightsquigarrow M'$.*

Proof. By induction on the derivation of $\vdash M : \tau$. For the case that

$$\frac{M : \text{nat}}{\vdash \text{succ } M : \text{nat}}$$

M is either a closed value or a reducible term by induction hypothesis:

1. If M is a closed value, then $\text{succ } M$ is also a closed value by definition.
2. Suppose that $M \rightsquigarrow M'$ for some M' . Then, by the rule $(\rightsquigarrow\text{-succ})$, we also have $\text{succ } M \rightsquigarrow \text{succ } M'$.

Proofs are programs

Remark 3.1. Note that given a program $M : \tau$, the previous proof produces either a closed value or a **PCF** term M' with a proof that M reduces to M' . Forgetting the proof, the proof itself is indeed a program which asks a **PCF** term, performs a single reduction and return a term tagged either **done** or **not yet**.

Substitution Lemma

If a variable $x : \tau$ in a term M is substituted by another term N of the same type, then the type of the resulting term remains.

Lemma 9. *If $\Gamma, x : \sigma \vdash M : \tau$ and $\Gamma \vdash N : \sigma$, then $\Gamma \vdash M[N/x] : \tau$.*

Proof. Induction on the derivation of $\Gamma, x : \sigma \vdash M : \tau$. Suppose that $\Gamma, x : \sigma \vdash M : \tau$ is derived from

$$\frac{}{\Delta, y : \tau, x : \sigma \vdash y : \tau} (\text{var})$$

that is, $\Gamma = \Delta, y : \tau$ and $M = y$ for some variable y . Then, we need to show that $\Delta, y : \tau \vdash y[N/x] : \tau$.

1. If $x = y$, then $y[N/x] = N : \sigma$ and $\sigma = \tau$.
2. Otherwise, $x \neq y$, then $y[N/x] = y : \tau$.

Other cases follow similarly. \square

Preservation Theorem

The one-step evaluation preserves types. This property is also called **Subject Reduction**.

Theorem 10. *If $\vdash M : \tau$ and $M \rightsquigarrow N$ then $\vdash N : \tau$.*

Proof. We prove it by induction on the derivation of $\vdash M : \tau$ and $M \rightsquigarrow M'$. For the case that

$$\frac{x : \sigma \vdash M : \sigma}{\vdash Yx. M : \sigma}$$

we do induction on \rightsquigarrow , but there is exactly one rule applicable:

$$\frac{}{Yx. M \rightsquigarrow M[Yx. M/x]} (\rightsquigarrow\text{-fix})$$

By Substitution Lemma, it follows that $\vdash M[Yx. M/x] : \sigma$, and other cases follow similarly. \square

4 Big-step semantics

Call-by-name big-step semantics

Instead of the one-step reduction relation \rightsquigarrow , we turn to the **big-step** reduction relation \Downarrow , formulating the notion that a term M reduce to its final value V .

$$\frac{}{\text{zero} \Downarrow \text{zero}} (\Downarrow\text{-zero})$$

$$\frac{M \Downarrow \underline{n}}{\text{succ } M \Downarrow \text{succ } \underline{n}} (\Downarrow\text{-succ})$$

$$\frac{}{\lambda x. M \Downarrow \lambda x. M} (\Downarrow\text{-lam})$$

$$\frac{M \Downarrow \lambda x. E \quad E[N/x] \Downarrow V}{M N \Downarrow V} (\Downarrow\text{-app})$$

$$\frac{M \Downarrow \text{zero} \quad M_0 \Downarrow V}{\text{ifz}(M; M_0; x. M_1) \Downarrow V} (\Downarrow\text{-ifz}_0)$$

$$\frac{M \Downarrow \text{succ } \underline{n} \quad M_1[\underline{n}/x] \Downarrow V}{\text{ifz}(M; M_0; x. M_1) \Downarrow V} (\Downarrow\text{-ifz}_1)$$

$$\frac{M[Yx. M/x] \Downarrow V}{Yx. M \Downarrow V} (\Downarrow\text{-fix})$$

Closed values

We shall justify the intended meaning: whenever $M \Downarrow V$, the term V is always a closed value:

Lemma 11. *For every terms M and V , the term V is a closed value if $M \Downarrow V$.*

Proof. By induction on the formulation of $M \Downarrow V$. \square

Moreover, a closed value reduces to itself:

Lemma 12. *If V is a closed value, then $V \Downarrow V$.*

Proof. By structural induction on V **val**. That is, it is sufficient to check that **zero** \Downarrow **zero**, $\lambda x.M \Downarrow \lambda x.M$; **suc** $M \Downarrow$ **suc** M if $M \Downarrow M$ by induction hypothesis. \square

Agreement of big-step and one-step semantics

Indeed, the big-step reduction can be characterised with respect to the one-step step reduction as follows:

Theorem 13. *For every term M and V , $M \Downarrow V$ if and only if $M \rightsquigarrow^* V$ with V **val**.*

Proof Sketch.

1. First we show that if $M \Downarrow V$ then $M \rightsquigarrow^* V$ by induction on \Downarrow and \rightsquigarrow^* .
2. Second, by induction on \rightsquigarrow and \Downarrow we show that

$$\frac{M \rightsquigarrow N \Downarrow V}{M \Downarrow V}$$

3. Finally, by induction on \rightsquigarrow^* we show that

$$\frac{M \rightsquigarrow^* N \Downarrow V}{M \Downarrow V}$$

In particular, for every $M \rightsquigarrow^* V$ with V **val**, we always have $V \Downarrow V$, so it follows $M \Downarrow V$.

Proof. 1. We show the case (\Downarrow -fix), which is similar to other cases:

$$\frac{\frac{Yx.M \rightsquigarrow M[Yx.M/x]}{Yx.M \rightsquigarrow^* V} \quad \frac{M[Yx.M/x] \rightsquigarrow^* V}{Yx.M \rightsquigarrow^* V}}{Yx.M \rightsquigarrow^* V}$$

and by assumption V has no further reduction.

2. We show the case (\rightsquigarrow -fix), which is similar to other cases. By hypothesis, we have $Yx.M \rightsquigarrow M[Yx.M/x]$. If $M[Yx.M/x] \Downarrow V$, then by (\Downarrow -fix) it follows that $Yx.M \Downarrow V$.

3. Induction on \rightsquigarrow^* . \square

By the agreement of big-step and one-step semantics, we easily conclude that the Subject Reduction also holds for big-step semantics:

Corollary 14. *If $\vdash M : \tau$ and $M \Downarrow V$ then $\vdash V : \tau$.*

Exercises

Basic

1. Define the following programs in **PCF**.
 - (a) Multiplication of natural numbers *Hint*. Define addition first;
 - (b) Fibonacci numbers;
 - (c) Parity test, i.e. a function determines whether the given argument is an odd or even number. Return **zero** if even, **suc zero** otherwise.
2. Let **bool** be a type with two constructors:

$$\frac{}{\mathbf{true} : \mathbf{bool}}$$

$$\frac{}{\mathbf{false} : \mathbf{bool}}$$

- (a) Provide the typing rule for the conditional construct **if**:

$$\frac{?}{\Gamma \vdash \mathbf{if}(M_0; M_1; M_2) : \tau}$$

- (b) Provide its one-step semantics.

Advanced

3. Define primitive recursion in **PCF**

$$\mathbf{rec} : \tau \rightarrow (\mathbf{nat} \rightarrow \tau \rightarrow \tau) \rightarrow \mathbf{nat} \rightarrow \tau$$

such that it reduces to

$$\begin{aligned} \mathbf{rec} \ e_0 \ f \ \mathbf{zero} &\rightsquigarrow^* e_0 \\ \mathbf{rec} \ e_0 \ f \ (\mathbf{suc} \ M) &\rightsquigarrow^* f \ M \ (\mathbf{rec} \ e_0 \ f \ M) \end{aligned}$$

respectively

4. Show that $\mathbf{rec} \ e_0 \ f \ \underline{n} \Downarrow V$ for a closed value V if f terminating and \underline{n} a closed value.

Consider Gödel's **T**, simply typed lambda calculus with natural numbers and *primitive recursion*:

$$\frac{\Gamma \vdash e_0 : \tau \quad \Gamma \vdash M : \mathbf{nat} \quad \Gamma, x : \mathbf{nat}, y : \tau \vdash e_1 : \tau}{\Gamma \vdash \mathbf{rec}(e_0; x.y.e_1; M) : \tau}$$

5. Provide the one-step and big-step reductions for **rec**.
6. Show that for every program M in \mathbf{T} , there is a closed value V with $M \Downarrow V$. This property is called *totality*. *Hint.* Use the structural induction principle on the typing rules for \mathbf{T} .

Reference

Denotational Semantics and this lecture are based on the following two books:

1. Thomas Streicher, *Domain-Theoretic Foundations of Functional Programming*, World Scientific, 2006
2. Robert Harper, *Practical Foundations for Programming Languages*, Cambridge University Press, 2012

Their preprints are available on the Internet.