

Semantics of Functional Programming

The Scott Model of **PCF**

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Denotational semantics of **PCF**

Instead of specifying *how* a **PCF** program runs, we specify *what* a program is, the *denotation* of a program.

To assign a denotation to a program,

- each type σ is interpreted as a domain D_σ ;
- a context $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$ is interpreted as a product $\prod_{i=1}^n D_{\sigma_i}$ of domains;
- in particular, each term of type τ under the empty context is an element of D_τ .

Interpretation of types and contexts

Define the denotation of a type inductively:

Definition 1

Every type σ in **PCF** associates with a domain D_σ as follows:

- 1 $D_{\text{nat}} := \mathbb{N}_\perp$, and
- 2 $D_{\tau \rightarrow \sigma} := [D_\tau \rightarrow D_\sigma]$.

Define the denotation of a context inductively on its length:

Definition 2

For each context $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$, the associated domain is defined as

$$D_\Gamma := 1 \times D_{\sigma_1} \times D_{\sigma_2} \times \dots \times D_{\sigma_n}$$

Interpretation of judgements

To proceed with the denotational semantics, we further define the denotation for each judgement inductively on its derivation of the following form

- Every judgement $\Gamma \vdash M : \tau$ is interpreted as a *continuous* function

$$\llbracket \Gamma \vdash M : \tau \rrbracket : D_\Gamma \rightarrow D_\tau.$$

- In particular,

$$\llbracket \vdash M : \tau \rrbracket : 1 \rightarrow D_\tau$$

is identified with an element $\llbracket \vdash M : \tau \rrbracket(*) = d$ of D_τ .

Convention

In the following context, $\llbracket \Gamma \vdash M : \tau \rrbracket(\vec{d})$ is written as

$$\llbracket \Gamma \vdash M : \tau \rrbracket \vec{d}.$$

for any sequence $\vec{d} \in D_\Gamma$ if there is no danger of ambiguity.

(var) Suppose that $\Gamma \vdash M : \tau$ is of the form

$$x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash x_i : \sigma_i$$

derived by the rule (var). It is interpreted as the projection from D_Γ to its i -th component D_{σ_i}

$$\llbracket x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash x_i : \sigma_i \rrbracket \vec{d} := d_i$$

for $i = 1, \dots, n$ where

$$\vec{d} = (d_1, \dots, d_n) \in D_{\sigma_1} \times \dots \times D_{\sigma_n}.$$

Note that the denotation of this judgement is equal to

$$\llbracket x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash x_i : \sigma_i \rrbracket = \pi_i$$

where $\pi_i : D_\Gamma \rightarrow D_{\sigma_i}$ is the i -th projection and thus it is a continuous function.

(abs) Let $f := \llbracket \Gamma, x : \sigma \vdash M : \tau \rrbracket$ be the continuous function from $D_\Gamma \times D_\sigma$ to D_τ .

$$\llbracket \Gamma \vdash \lambda x. M : \sigma \rightarrow \tau \rrbracket := \Lambda f$$

where $\Lambda f : D_\Gamma \rightarrow [D_\sigma \rightarrow D_\tau]$ is the *curried* f . In other words

$$\left(\llbracket \Gamma \vdash \lambda x. M : \sigma \rightarrow \tau \rrbracket \vec{d} \right) d = \llbracket \Gamma, x : \sigma \vdash M : \tau \rrbracket (\vec{d}, d).$$

(app) Define

$$\begin{aligned} & \llbracket \Gamma \vdash M N : \tau \rrbracket \vec{d} \\ &:= \text{ev} \left(\llbracket \Gamma \vdash M : \sigma \rightarrow \tau \rrbracket \vec{d}, \llbracket \Gamma \vdash N : \sigma \rrbracket \vec{d} \right) \end{aligned}$$

where $\text{ev} : [D_1 \rightarrow D_2] \times D_1 \rightarrow D_2$ is the *evaluation map* which maps a continuous function $f : D_1 \rightarrow D_2$ with an element $d \in D_1$ to $f(d)$.

The cases for zero and suc M are rather obvious:

- (z) zero is a constant, so it does not matter what the context is:

$$\llbracket \Gamma \vdash \text{zero} : \text{nat} \rrbracket \vec{d} := 0$$

i.e. a constant function.

- (s) The denotation of suc is the successor function

$$\llbracket \Gamma \vdash \text{suc } M : \text{nat} \rrbracket \vec{d} := (S \circ \llbracket \Gamma \vdash M : \text{nat} \rrbracket) \vec{d}$$

where $S: \mathbb{N}_\perp \rightarrow \mathbb{N}_\perp$ is defined by

$$S(n) := \begin{cases} \perp & \text{if } n = \perp \\ n + 1 & \text{if } n \in \mathbb{N}. \end{cases}$$

(Y) The denotation of Y is the fixpoint operation

$$\llbracket \Gamma \vdash Yx. M : \sigma \rrbracket \vec{d} := \mu \left(\llbracket \Gamma \vdash \lambda x. M : \sigma \rightarrow \sigma \rrbracket \vec{d} \right)$$

where μ is defined previously as $\mu(f) := \bigsqcup_{i \in \mathbb{N}} f^i(\perp)$.

(ifz) The denotation of `ifz`

$$\begin{aligned} & \llbracket \Gamma \vdash \text{ifz}(M; M_0; x. M_1) : \tau \rrbracket \vec{d} \\ & := \text{ifz}_\tau(n, e, f) \end{aligned}$$

where

$$1 \quad n := \llbracket \Gamma \vdash M : \text{nat} \rrbracket \vec{d},$$

$$2 \quad e := \llbracket \Gamma \vdash M_0 : \tau \rrbracket \vec{d},$$

$$3 \quad f := \llbracket \Gamma \vdash \lambda x. M_1 : \sigma \rightarrow \tau \rrbracket \vec{d},$$

and ifz is defined by

$$\text{ifz}(n, x, f) := \begin{cases} \perp & \text{if } n = \perp, \\ x & \text{if } n = 0, \\ f(n-1) & \text{otherwise.} \end{cases}$$

Theorem 3

For every judgement $\Gamma \vdash M : \tau$, the associated function

$$\llbracket \Gamma \vdash M : \tau \rrbracket : D_\Gamma \rightarrow D_\tau$$

is Scott continuous.

Proof sketch.

It is not hard to see that each case of $\llbracket \Gamma \vdash M : \tau \rrbracket$ is a Scott continuous function. □

Example 4

Consider the denotations of the following judgements.

- 1 $y : \text{nat} \vdash y : \text{nat}$
- 2 $\vdash \lambda x. \underline{0} : \text{nat} \rightarrow \text{nat}$
- 3 $\vdash \text{Y}f. \lambda n. \text{ifz}(n; \underline{0}; x. f\ x) : \text{nat} \rightarrow \text{nat}.$

1 $\llbracket y : \text{nat} \vdash y : \text{nat} \rrbracket d = d$

2 $\llbracket \vdash \lambda x. \underline{0} : \text{nat} \rightarrow \text{nat} \rrbracket = \Lambda f \text{ where}$

$$f := \llbracket x : \text{nat} \vdash \text{zero} : \text{nat} \rrbracket = \text{const}_0,$$

i.e. the constant function at 0.

3

$$\begin{aligned} & \llbracket \vdash \mathbf{Y}f. \lambda n. \text{ifz}(n; \underline{0}; x. f \ x) : \text{nat} \rightarrow \text{nat} \rrbracket \\ &= \mu(g) \end{aligned}$$

where $g : [D_{\text{nat}} \rightarrow D_{\text{nat}}] \rightarrow [D_{\text{nat}} \rightarrow D_{\text{nat}}]$ is defined by

$$\begin{aligned} g &:= \llbracket f : \text{nat} \rightarrow \text{nat} \vdash \lambda n. \text{ifz}(n; \underline{0}; x. f \ x) : \text{nat} \rightarrow \text{nat} \rrbracket \\ &= \wedge \llbracket f : \text{nat} \rightarrow \text{nat}, n : \text{nat} \vdash \text{ifz}(n; \underline{0}; x. f \ x) : \text{nat} \rrbracket \end{aligned}$$

and

$$\begin{aligned} & \llbracket f : \text{nat} \rightarrow \text{nat}, n : \text{nat} \vdash \text{ifz}(n; \underline{0}; x. f \ x) : \text{nat} \rrbracket (h, d) \\ &= \text{ifz}(d, 0, h) \end{aligned}$$

Then, what is $\mu(g)$? Let's calculate $g(\perp)$ and $g^2(\perp)$.

$$g(\perp_{D_{\text{nat}} \rightarrow D_{\text{nat}}}) \ d = \text{ifz}(d, 0, \perp_{D_{\text{nat}} \rightarrow D_{\text{nat}}}) = \begin{cases} \perp & \text{if } d = \perp \\ 0 & \text{if } d = 0 \\ \perp & \text{otherwise.} \end{cases}$$

$$g(g(\perp)) \ d = \text{ifz}(d, 0, g(\perp)) = \begin{cases} \perp & \text{if } d = \perp \\ 0 & \text{if } d = 0, 1 \\ \perp & \text{otherwise.} \end{cases}$$

By induction, we can show that

$$g^i(d) = \begin{cases} \perp & \text{if } d = \perp \\ 0 & \text{if } d < i \\ \perp & \text{otherwise,} \end{cases}$$

so $\mu(g) \ d = 0$ if $d \neq \perp$ and $\mu(g) \ d = \perp$ if $d = \perp$.

Exercise

Consider the denotations of the following judgements.

- 1 $y : \text{nat} \vdash (\lambda x. \underline{0}) y : \text{nat}$
- 2 $\vdash \lambda n. \text{ifz}(n; \underline{0}; x. x) : \text{nat} \rightarrow \text{nat}$
- 3 $\vdash \lambda n. \text{ifz}(n; \underline{1}; x. \underline{0}) : \text{nat} \rightarrow \text{nat}$

Substitution Lemma

Lemma 5

Let $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$ be a context, and $\Gamma \vdash M : \tau$ a judgement. Then the following equation

$$\begin{aligned} & \llbracket \Delta \vdash M[\vec{N}/\vec{x}] \rrbracket \vec{d} \\ &= \llbracket \Gamma \vdash M \rrbracket \left(\llbracket \Delta \vdash N_1 \rrbracket \vec{d}, \dots, \llbracket \Delta \vdash N_n \rrbracket \vec{d} \right) \end{aligned}$$

holds for any context Δ and judgements $\Delta \vdash N_i : \sigma_i$ for $i = 1, \dots, n$.

Corollary 6 (Application)

For every judgement $\Gamma, x : \sigma \vdash M : \tau$ and $\Gamma \vdash N : \sigma$, we have

$$\llbracket \Gamma \vdash (\lambda x. M) N : \tau \rrbracket = \llbracket \Gamma \vdash M[N/x] : \tau \rrbracket.$$

Observe that

$$\vec{d} = (\llbracket \Gamma \vdash x_1 : \sigma_1 \rrbracket \vec{d}, \dots, \llbracket \Gamma \vdash x_n : \sigma_n \rrbracket \vec{d})$$

for any context $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$. Then, this corollary is a series of simple facts:

$$\begin{aligned} & \llbracket \Gamma \vdash (\lambda x. M) N : \tau \rrbracket \vec{d} \\ &= \text{ev} \left(\llbracket \Gamma \vdash (\lambda x. M) : \sigma \rightarrow \tau \rrbracket \vec{d}, \llbracket \Gamma \vdash N : \sigma \rrbracket \vec{d} \right) \\ &= \llbracket \Gamma, x : \sigma \vdash M : \tau \rrbracket (\vec{d}, \llbracket \Gamma \vdash N : \sigma \rrbracket \vec{d}) \\ &= \llbracket \Gamma \vdash M[\vec{x}, N/\vec{x}, x] : \tau \rrbracket \vec{d} \\ &= \llbracket \Gamma \vdash M[N/x] : \tau \rrbracket \vec{d} \end{aligned}$$

Example 7

The denotation of

$$\vdash (\lambda n. \text{ifz}(n; \underline{1}; x. x)) \underline{1} : \text{nat}$$

and

$$\vdash \text{ifz}(\underline{1}; \underline{1}; x. x) : \text{nat}$$

are equal and calculated as follows:

$$\begin{aligned} & \llbracket \vdash \lambda n. \text{ifz}(n; \underline{1}; x. x) \underline{1} \rrbracket \\ &= \llbracket \vdash \lambda n. \text{ifz}(n; \underline{1}; x. x) \rrbracket (\llbracket \vdash \underline{1} : \text{nat} \rrbracket) \\ &= \text{ifz}(1, 1, id) = 0 \end{aligned}$$

Lemma 8 (Weakening)

Let $\Gamma \vdash M : \tau$ be a judgement. Then the following

$$\llbracket \Gamma \vdash M : \tau \rrbracket = \llbracket \Gamma, x : \sigma \vdash M : \tau \rrbracket$$

holds for any variable $x : \sigma$ not in Γ .

It follows from Substitution Lemma. (*Why?*)

Corollary 9 (η -conversion)

Let $\Gamma \vdash M : \sigma \rightarrow \tau$ be a judgement. Then,

$$\llbracket \Gamma \vdash \lambda x. M \ x : \sigma \rightarrow \tau \rrbracket = \llbracket \Gamma \vdash M : \sigma \rightarrow \tau \rrbracket$$

if x is not a variable in Γ .

For every sequence $\vec{d} \in D_\Gamma$ and $d \in D_\sigma$, we have

$$\begin{aligned} & \left(\llbracket \Gamma \vdash \lambda x. M \ x : \sigma \rightarrow \tau \rrbracket \vec{d} \right) d \\ &= \llbracket \Gamma, x : \sigma \vdash M \ x : \tau \rrbracket (\vec{d}, d) \\ &= \text{ev} \left(\llbracket \Gamma, x : \sigma \vdash M : \sigma \rightarrow \tau \rrbracket (\vec{d}, d), \llbracket \Gamma, x : \sigma \vdash x : \sigma \rrbracket (\vec{d}, d) \right) \\ &= \text{ev} \left(\llbracket \Gamma, x : \sigma \vdash M : \sigma \rightarrow \tau \rrbracket (\vec{d}, d), d \right) \\ &= \left(\llbracket \Gamma, x : \sigma \vdash M : \sigma \rightarrow \tau \rrbracket (\vec{d}, d) \right) d \\ &= \left(\llbracket \Gamma \vdash M : \sigma \rightarrow \tau \rrbracket \vec{d} \right) d. \end{aligned}$$

Compactness

Define $Y^i x. M$ inductively for each $i \in \mathbb{N}$ by

- 1 $Y^0 x. M := Yx. x$ and
- 2 $Y^{n+1} x. M := M[Y^n x. M/x]$.

Theorem 10

For every judgement $\Gamma, x : \sigma \vdash M : \sigma$, we have

$$\llbracket \Gamma \vdash Yx. M : \sigma \rrbracket = \bigsqcup_{i \in \mathbb{N}} \llbracket \Gamma \vdash Y^i x. M : \sigma \rrbracket.$$

To show this theorem, it suffices to show the following

$$\llbracket \vdash Y^i x. M : \sigma \rrbracket = \llbracket x : \sigma \vdash M : \sigma \rrbracket^i(\perp)$$

for $i \in \mathbb{N}$. (Why?)

For $n = 0$ we show that $\llbracket \vdash Y^0 x. M : \sigma \rrbracket = \perp_{D_\sigma} \in D_\sigma$.

By definition, $Y^0 x. M$ is equal to $Y x. x$, so

$$\begin{aligned}\llbracket \vdash Y x. x : \sigma \rrbracket &= \mu(id) = \bigsqcup_{i \in \mathbb{N}} id^i(\perp) \\ &= \bigsqcup \perp = \perp\end{aligned}$$

For $i = n + 1$ it suffices to show that

$$\llbracket \vdash Y^{n+1} x. M : \sigma \rrbracket = \llbracket x : \sigma \vdash M : \sigma \rrbracket (\llbracket \vdash Y^n x. M : \sigma \rrbracket),$$

so the statement follows by the induction hypothesis.

By definition, $Y^{n+1} x. M$ is equal to $M[Y^n x. M/x]$, and by Substitution Lemma we have

$$\llbracket \vdash M[Y^n x. M/x] \rrbracket = \llbracket x : \sigma \vdash M : \tau \rrbracket (\llbracket \vdash Y^n x. M \rrbracket).$$

Exercise

Find the denotation of

$$\vdash \mathsf{Y}f. \lambda n. \mathsf{ifz}(n; \underline{0}; m. \mathsf{ifz}((f\ m); \underline{1}; x. \underline{0})) : \mathsf{nat} \rightarrow \mathsf{nat}$$