Semantics of Functional Programming: Sample Solution

Liang-Ting Chen and Tyng-Ruey Chuang

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1. (20%) Which of the following posets are cpos? Why?

   (a) The set $(\mathbb{N}, \leq)$ of natural numbers with the natural ordering.

   Proof. No. The set $\mathbb{N}$ of all natural numbers is directed, because for every two elements x and y, either $x \leq y$ or $y \leq x$. However, there is no upper bound of $\mathbb{N}$ in $\mathbb{N}$. $\square$

   (b) $(\mathbb{N} \cup \{\infty\}, \leq')$ with the natural ordering i.e. $n \leq' \infty$ for every $n \in \mathbb{N} \cup \{\infty\}$ and $n \leq' m$ if $n \leq m$ for $m \in \mathbb{N}$.

   Proof. Yes.

   i. 0 is the element satisfying $0 \leq n$ with $n \in \mathbb{N} \cup \{\infty\}$, i.e. the bottom element.
   ii. We claim that every directed subset has the least upper bound. Let D be a directed subset. If D is finite, then by induction there is an upper bound z of D *in* D so that z is the least upper bound. If D is infinite, then every element $x \in \mathbb{N}$ is not an upper bound. Otherwise D is finite, since there are only finitely many elements below x. By definition, ∞ is the upper bound of D, and any element x with $\infty \leq x$ must be ∞.

   $\square$

2. (30%) Evaluate the following terms using $\rightsquigarrow$ step by step *without* formal derivations.

   (a) $(\lambda x.\, \lambda y.\, \lambda z.\, z)\ (\mathtt{Y} n.\, n)\ \underline{3}\ \underline{1}$

   Answer. $(\lambda x.\, \lambda y.\, \lambda z.\, z)\ (\mathtt{Y} n.\, n)\ \underline{3}\ \underline{1} \rightsquigarrow (\lambda y.\, \lambda z.\, z)\ \underline{3}\ \underline{1} \rightsquigarrow (\lambda z.\, z)\ \underline{1} \rightsquigarrow \underline{1}$ $\square$

   (b) $(\lambda z.\, \lambda n.\, \mathtt{ifz}(n; z; x.\, x))\ \ \mathtt{ifz}(\underline{0}; \underline{1}; m.\, \mathtt{suc}\ m)$

   Answer.

$$
\begin{aligned}
&(\lambda z.\, \lambda n.\, \mathtt{ifz}(n; z; x.\, x))\ \ \mathtt{ifz}(\underline{0}; \underline{1}; m.\, \mathtt{suc}\ m) \\
\rightsquigarrow &(\lambda n.\, \mathtt{ifz}(n; \mathtt{ifz}(\underline{0}; \underline{1}; m.\, \mathtt{suc}\ m); x.x)
\end{aligned}
$$

   $\square$

3. (30%) Find the denotation of the following terms.

(a) $z : \sigma \vdash \lambda x.\, \lambda y.\, x : \sigma_1 \to \sigma_2 \to \sigma_1$

Answer.

$$\begin{aligned} & [\![z : \sigma \vdash \lambda x.\, \lambda y.\, x : \sigma_1 \to \sigma_2 \to \sigma_1]\!] \\ = {} & \Lambda[\![z : \sigma, x : \sigma_1 \vdash \lambda y.\, x : \sigma_2 \to \sigma_1]\!] \\ = {} & \Lambda(\Lambda[\![z : \sigma, x : \sigma_1, y : \sigma_2 \vdash x : \sigma_1]\!]) = \Lambda(\Lambda \pi_2) \end{aligned}$$

where $\pi_2 : 1 \times D_\sigma \times D_{\sigma_1} \times D_{\sigma_2}$ maps $(*, d, d_1, d_2) \in 1 \times D_\sigma \times D_{\sigma_1} \times D_{\sigma_2}$ to d_1. □

(b) $x : \mathtt{nat}, y : \sigma, z : \sigma \vdash \mathtt{ifz}(x; y; m.\, z) : \sigma$

Answer. For every $(*, d_1, d_2, d_3) \in 1 \times D_{\mathtt{nat}} \times D_\sigma \times D_\sigma$, we have

$$\begin{aligned} & [\![x : \mathtt{nat}, y : \sigma, z : \sigma \vdash \mathtt{ifz}(x; y; m.\, z) : \sigma]\!]\ (*, d_1, d_2, d_3) \\ = {} & \mathit{ifz}(d_1, d_2, \mathit{const}_{d_3}) \end{aligned}$$

or equivalently

$$[\![x : \mathtt{nat}, y : \sigma, z : \sigma \vdash \mathtt{ifz}(x; y; m.\, z) : \sigma]\!]\ (*, d_1, d_2, d_3) = \begin{cases} \bot & \text{if } d_1 = \bot \\ d_2 & \text{if } d_1 = 0 \\ d_3 & \text{if } d_1 > 0 \end{cases}$$

□

4. (10%) Show that there is no term V satisfying $\mathtt{Y}x.\, x \Downarrow \mathsf{V}$. *Hint.* Use the agreement of the one-step reduction and the big-step reduction, and the fact that $\rightsquigarrow$ is deterministic, i.e. if $\mathsf{M} \rightsquigarrow \mathsf{M}_1$ and $\mathsf{M} \rightsquigarrow \mathsf{M}_2$ then $\mathsf{M}_1 = \mathsf{M}_2$.

   Proof. Suppose that there is a term V with $\mathtt{Y}x.\, x \Downarrow V$. By the agreement, $\mathtt{Y}x.\, x \rightsquigarrow^* V$ and V is a value. By ($\rightsquigarrow$-fix), $\mathtt{Y}x.\, x \rightsquigarrow \mathtt{Y}x.\, x$ and by determinacy every term V with $\mathtt{Y}x.\, x \rightsquigarrow V$ must be $\mathtt{Y}x.\, x$ By induction, every term V with $\mathtt{Y}x.\, x \rightsquigarrow^* \mathsf{V}$ must be $\mathtt{Y}x.\, x$.

   However, $\mathtt{Y}x.\, x$ is not a value, so we reach a contradiction. □

   Proof. Alternatively, we do induction on the derivation of $\mathtt{Y}x.\, x \Downarrow \mathsf{V}$, and the only case is ($\Downarrow$-fix), i.e. to have $\mathtt{Y}x.\, x \Downarrow \mathsf{V}$ we must have $x[\mathtt{Y}x.\, x/x] \Downarrow \mathsf{V}$ as a premise satisfying the induction hypothesis. That is, there is no term V satisfying $\mathtt{Y}x.\, x \Downarrow \mathsf{V}$. □

5. (10%) Find a monotonic function $f : \mathbb{N} \cup \{\infty\} \to \mathbb{N} \cup \{\infty\}$ which is *not* continuous. You have to give an example such that $f(\bigsqcup S) \neq \bigsqcup_{x \in S} f(x)$.

   Answer. Let $f(n) = 0$ for $n < \infty$ and $f(\infty) = \infty$. It is clear that f is a monotonic function. Since ∞ is the greatest element of $\mathbb{N}$, $\bigsqcup \mathbb{N} = \infty$. Thus, $f(\bigsqcup \mathbb{N}) = \infty \neq 0 = \bigsqcup_{n \in \mathbb{N}} f(n)$. □