Semantics of Functional Programming
The Scott Model of **PCF**

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Denotational semantics of **PCF**

Instead of specifying *how* a **PCF** program runs, we specify *what* a program is, the *denotation* of a program.

To assign a denotation to a program,

- each type σ is interpreted as a domain D_σ;
- a context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ is interpreted as a product $\prod_{i=1}^{n} D_{\sigma_i}$ of domains;
- in particular, each term of type τ under the empty context is an element of D_τ.

Interpretation of types and contexts

Define the denotation of a type inductively:

Definition 1

Every type σ in **PCF** associates with a domain D_σ as follows:

1. $D_{\mathtt{nat}} := \mathbb{N}_\perp$, and
2. $D_{\tau \to \sigma} := [D_\tau \to D_\sigma]$.

Define the denotation of a context inductively on its length:

Definition 2

For each context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, the associated domain is defined as

$$D_\Gamma := 1 \times D_{\sigma_1} \times D_{\sigma_2} \times \cdots \times D_{\sigma_n}$$

Interpretation of judgements

To proceed with the denotational semantics, we further define the denotation for each judgement inductively on its derivation of the following form

- Every judgement $\Gamma \vdash M : \tau$ is interpreted as a *continuous* function

$$[\![\Gamma \vdash M : \tau]\!] : D_\Gamma \to D_\tau.$$

- In particular,

$$[\![\vdash M : \tau]\!] : 1 \to D_\tau$$

is identified with an element $[\![\vdash M : \tau]\!](*) = d$ of D_τ.

Convention

In the following context, $[\![\Gamma \vdash M : \tau]\!](\vec{d})$ is written as

$$[\![\Gamma \vdash M : \tau]\!]\ \vec{d}.$$

for any sequence $\vec{d} \in D_\Gamma$ if there is no danger of ambiguity.

(var) Suppose that $\Gamma \vdash M : \tau$ is of the form

$$x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash x_i : \sigma_i$$

derived by the rule (var). It is interpreted as the projection from D_Γ to its i-th component D_{σ_i}

$$[\![x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash x_i : \sigma_i]\!] \, \vec{d} := d_i$$

for $i = 1, \ldots, n$ where
$\vec{d} = (*, d_1, \ldots, d_n) \in 1 \times D_{\sigma_1} \times \cdots \times D_{\sigma_n}$.

Note that the denotation of this judgement is equal to

$$[\![x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash x_i : \sigma_i]\!] = \pi_i$$

where $\pi_i \colon D_\Gamma \to D_{\sigma_i}$ is the i-th projection and thus it is a continuous function.

(abs) Let $f := [\![\Gamma, x : \sigma \vdash M : \tau]\!]$ be the continuous function from $D_\Gamma \times D_\sigma$ to D_τ.

$$[\![\Gamma \vdash \lambda x. M : \sigma \to \tau]\!] := \Lambda f$$

where $\Lambda f : D_\Gamma \to [D_\sigma \to D_\tau]$ is the *curried* f. In other words

$$\left([\![\Gamma \vdash \lambda x. M : \sigma \to \tau]\!] \, \vec{d} \right) \, d = [\![\Gamma, x : \sigma \vdash M : \tau]\!] \, (\vec{d}, d).$$

(app) Define

$$[\![\Gamma \vdash M \, N : \tau]\!] \, \vec{d}$$
$$:= ev \left([\![\Gamma \vdash M : \sigma \to \tau]\!] \, \vec{d}, [\![\Gamma \vdash N : \sigma]\!] \, \vec{d} \right)$$

where $ev : [D_1 \to D_2] \times D_1 \to D_2$ is the *evaluation map* which maps a continuous function $f : D_1 \to D_2$ with an element $d \in D_1$ to $f(d)$.

The cases for zero and suc M are rather obvious:

(z) zero is a constant, so it does not matter what the context is:

$$\llbracket \Gamma \vdash \texttt{zero} : \texttt{nat} \rrbracket \ \vec{d} := 0$$

i.e. a constant function.

(s) The denotation of suc is the successor function

$$\llbracket \Gamma \vdash \texttt{suc M} : \texttt{nat} \rrbracket \ \vec{d} := (S \circ \llbracket \Gamma \vdash \texttt{M} : \texttt{nat} \rrbracket) \ \vec{d}$$

where $S \colon \mathbb{N}_\perp \to \mathbb{N}_\perp$ is defined by

$$S(n) := \begin{cases} \perp & \text{if } n = \perp \\ n+1 & \text{if } n \in \mathbb{N}. \end{cases}$$

(Y) The denotation of `Y` is the fixpoint operation

$$\llbracket \Gamma \vdash \mathtt{Y}x.\, \mathsf{M} : \sigma \rrbracket \, \vec{d} := \mu \left(\llbracket \Gamma \vdash \lambda x.\, \mathsf{M} : \sigma \to \sigma \rrbracket \, \vec{d} \right)$$

where μ is defined previously as $\mu(f) := \bigsqcup_{i \in \mathbb{N}} f^i(\bot)$.

(ifz) The denotation of `ifz`

$$\llbracket \Gamma \vdash \mathtt{ifz}(\mathsf{M}; \mathsf{M}_0; x.\, \mathsf{M}_1) : \tau \rrbracket \, \vec{d}$$
$$:= \mathit{ifz}_\tau(n, e, f)$$

where

1. $n := \llbracket \Gamma \vdash \mathsf{M} : \mathtt{nat} \rrbracket \, \vec{d}$,
2. $e := \llbracket \Gamma \vdash \mathsf{M}_0 : \tau \rrbracket \, \vec{d}$,
3. $f := \llbracket \Gamma \vdash \lambda x.\, \mathsf{M}_1 : \sigma \to \tau \rrbracket \, \vec{d}$,

and ifz is defined by

$$\mathit{ifz}(n, x, f) := \begin{cases} \bot & \text{if } n = \bot, \\ x & \text{if } n = 0, \\ f(n-1) & \text{otherwise.} \end{cases}$$

For every judgement $\Gamma \vdash M : \tau$, the associated function

$$\llbracket \Gamma \vdash M : \tau \rrbracket \colon D_\Gamma \to D_\tau$$

is Scott continuous.

Proof sketch.

It is not hard to see that each case of $\llbracket \Gamma \vdash M : \tau \rrbracket$ is a Scott continuous function. $\qquad\square$

Example 4

Consider the denotations of the following judgements.

1. $y : \mathtt{nat} \vdash y : \mathtt{nat}$
2. $\vdash \lambda x. \underline{0} : \mathtt{nat} \rightarrow \mathtt{nat}$
3. $\vdash \mathtt{Y}f. \lambda n. \mathtt{ifz}(n; \underline{0}; x.\, f\, x) : \mathtt{nat} \rightarrow \mathtt{nat}$.

1. $[\![y : \mathtt{nat} \vdash y : \mathtt{nat}]\!]\, d = d$

2. $[\![\vdash \lambda x. \underline{0} : \mathtt{nat} \rightarrow \mathtt{nat}]\!] = \Lambda f$ where

$$f := [\![x : \mathtt{nat} \vdash \mathtt{zero} : \mathtt{nat}]\!] = const_0,$$

   i.e. the constant function at 0.

$$\llbracket \,\vdash \mathtt{Y}f.\,\lambda n.\,\mathtt{ifz}(n;\underline{0};x.\,f\ x) : \mathtt{nat} \to \mathtt{nat}\rrbracket$$
$$= \mu(g)$$

where $g : [D_{\mathtt{nat}} \to D_{\mathtt{nat}}] \to [D_{\mathtt{nat}} \to D_{\mathtt{nat}}]$ is defined by

$$g := \llbracket f : \mathtt{nat} \to \mathtt{nat} \vdash \lambda n.\,\mathtt{ifz}(n;\underline{0};x.\,f\ x) : \mathtt{nat} \to \mathtt{nat}\rrbracket$$
$$= \Lambda \llbracket f : \mathtt{nat} \to \mathtt{nat}, n : \mathtt{nat} \vdash \mathtt{ifz}(n;\underline{0};x.\,f\ x) : \mathtt{nat}\rrbracket$$

and

$$\llbracket f : \mathtt{nat} \to \mathtt{nat}, n : \mathtt{nat} \vdash \mathtt{ifz}(n;\underline{0};x.\,f\ x) : \mathtt{nat}\rrbracket\,(h, d)$$
$$= \mathit{ifz}(d, 0, h)$$

Then, what is $\mu(g)$? Let's calculate $g(\bot)$ and $g^2(\bot)$.

$$g(\bot_{D_{\mathrm{nat}} \to D_{\mathrm{nat}}}) \, d = \mathit{ifz}(d, 0, \bot_{D_{\mathrm{nat}} \to D_{\mathrm{nat}}}) = \begin{cases} \bot & \text{if } d = \bot \\ 0 & \text{if } d = 0 \\ \bot & \text{otherwise.} \end{cases}$$

$$g(g(\bot)) \, d = \mathit{ifz}(d, 0, g(\bot)) = \begin{cases} \bot & \text{if } d = \bot \\ 0 & \text{if } d = 0, 1 \\ \bot & \text{otherwise.} \end{cases}$$

By induction, we can show that

$$g^i(d) = \begin{cases} \bot & \text{if } d = \bot \\ 0 & \text{if } d < i \\ \bot & \text{otherwise,} \end{cases}$$

so $\mu(g) \, d = 0$ if $d \neq \bot$ and $\mu(g) \, d = \bot$ if $d = \bot$.

Exercise

Consider the denotations of the following judgements.

1. $y : \mathtt{nat} \vdash (\lambda x.\, \underline{0})\, y : \mathtt{nat}$

2. $\vdash \lambda n.\, \mathtt{ifz}(n; \underline{0}; x.\, x) : \mathtt{nat} \to \mathtt{nat}$

3. $\vdash \lambda n.\, \mathtt{ifz}(n; \underline{1}; x.\, \underline{0}) : \mathtt{nat} \to \mathtt{nat}$

Substitution Lemma

Lemma 5

Let $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ be a context, and $\Gamma \vdash M : \tau$ a judgement. Then the following equation

$$\llbracket \Delta \vdash M[\vec{N}/\vec{x}] \rrbracket \ \vec{d}$$
$$= \llbracket \Gamma \vdash M \rrbracket \left(\llbracket \Delta \vdash N_1 \rrbracket \ \vec{d}, \ldots, \llbracket \Delta \vdash N_n \rrbracket \ \vec{d} \right)$$

holds for any context Δ and judgements $\Delta \vdash N_i : \sigma_i$ for $i = 1, \ldots, n$.

Corollary 6 (Application)

For every judgement $\Gamma, x : \sigma \vdash M : \tau$ and $\Gamma \vdash N : \sigma$, we have

$$[\![\Gamma \vdash (\lambda x. M)\, N : \tau]\!] = [\![\Gamma \vdash M[N/x] : \tau]\!].$$

Observe that

$$\vec{d} = ([\![\Gamma \vdash x_1 : \sigma_1]\!]\ \vec{d}, \ldots, [\![\Gamma \vdash x_n : \sigma_n]\!]\ \vec{d})$$

for any context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$. Then, this corollary is a series of simple facts:

$$
\begin{aligned}
&[\![\Gamma \vdash (\lambda x. M)\, N : \tau]\!]\ \vec{d} \\
&= ev\left([\![\Gamma \vdash (\lambda x. M) : \sigma \to \tau]\!]\ \vec{d}, [\![\Gamma \vdash N : \sigma]\!]\ \vec{d}\right) \\
&= [\![\Gamma, x : \sigma \vdash M : \tau]\!]\ (\vec{d}, [\![\Gamma \vdash N : \sigma]\!]\ \vec{d}) \\
&= [\![\Gamma \vdash M[\vec{x}, N/\vec{x}, x] : \tau]\!]\ \vec{d} \\
&= [\![\Gamma \vdash M[N/x] : \tau]\!]\ \vec{d}
\end{aligned}
$$

Example 7

The denotation of

$$\vdash (\lambda n. \mathtt{ifz}(n; \underline{1}; x. x)) \, \underline{1} : \mathtt{nat}$$

and

$$\vdash \mathtt{ifz}(\underline{1}; \underline{1}; x. x) : \mathtt{nat}$$

are equal and calculated as follows:

$$\begin{aligned}
&\llbracket \vdash \lambda n. \mathtt{ifz}(n; \underline{1}; x. x) \, \underline{1} \rrbracket \\
&= \llbracket \vdash \lambda n. \mathtt{ifz}(n; \underline{1}; x. x) \rrbracket (\llbracket \vdash \underline{1} : \mathtt{nat} \rrbracket) \\
&= \mathit{ifz}(1, 1, \mathit{id}) = 0
\end{aligned}$$

Let $\Gamma \vdash M : \tau$ be a judgement. Then the following

$$[\![\Gamma \vdash M : \tau]\!] = [\![\Gamma, x : \sigma \vdash M : \tau]\!]$$

holds for any variable $x : \sigma$ not in Γ.

It follows from Substitution Lemma. (*Why?*)

Let $\Gamma \vdash M : \sigma \to \tau$ be a judgement. Then,

$$\llbracket \Gamma \vdash \lambda x.\, M\, x : \sigma \to \tau \rrbracket = \llbracket \Gamma \vdash M : \sigma \to \tau \rrbracket$$

if x is not a variable in Γ.

For every sequence $\vec{d} \in D_\Gamma$ and $d \in D_\sigma$, we have

$$
\begin{aligned}
&\left(\llbracket \Gamma \vdash \lambda x.\, M\, x : \sigma \to \tau \rrbracket\, \vec{d} \right)\, d \\
&= \llbracket \Gamma, x : \sigma \vdash M\, x : \tau \rrbracket (\vec{d}, d) \\
&= ev \left(\llbracket \Gamma, x : \sigma \vdash M : \sigma \to \tau \rrbracket (\vec{d}, d), \llbracket \Gamma, x : \sigma \vdash x : \sigma \rrbracket (\vec{d}, d) \right) \\
&= ev \left(\llbracket \Gamma, x : \sigma \vdash M : \sigma \to \tau \rrbracket (\vec{d}, d), d \right) \\
&= \left(\llbracket \Gamma, x : \sigma \vdash M : \sigma \to \tau \rrbracket (\vec{d}, d) \right) d \\
&= \left(\llbracket \Gamma \vdash M : \sigma \to \tau \rrbracket\, \vec{d} \right)\, d.
\end{aligned}
$$

Compactness

Define $Y^i x.\, M$ inductively for each $i \in \mathbb{N}$ by

1. $Y^0 x.\, M := Y x.\, x$ and
2. $Y^{n+1} x.\, M := M[Y^n x.\, M / x]$.

Theorem 10

For every judgement $\Gamma, x : \sigma \vdash M : \sigma$, we have

$$\llbracket \Gamma \vdash Y x.\, M : \sigma \rrbracket = \bigsqcup_{i \in \mathbb{N}} \llbracket \Gamma \vdash Y^i x.\, M : \sigma \rrbracket.$$

To show this theorem, it suffices to show the following

$$\llbracket\, \vdash Y^i x.\, M : \sigma \rrbracket = \llbracket x : \sigma \vdash M : \sigma \rrbracket^i(\bot)$$

for $i \in \mathbb{N}$. (Why?)

For $n = 0$ we show that $\llbracket \vdash Y^0 x.\, M : \sigma \rrbracket = \bot_{D_\sigma} \in D_\sigma$.

By definition, $Y^0 x.\, M$ is equal to $Yx.\, x$, so

$$\llbracket \vdash Yx.\, x : \sigma \rrbracket = \mu(id) = \bigsqcup_{i \in \mathbb{N}} id^i(\bot)$$
$$= \bigsqcup \bot = \bot$$

For $i = n + 1$ it suffices to show that

$$\llbracket \vdash Y^{n+1} x.\, M : \sigma \rrbracket = \llbracket x : \sigma \vdash M : \sigma \rrbracket (\llbracket \vdash Y^n x.\, M : \sigma \rrbracket),$$

so the statement follows by the induction hypothesis.

By definition, $Y^{n+1} x.\, M$ is equal to $M[Y^n x.\, M/x]$, and by Substitution Lemma we have

$$\llbracket \vdash M[Y^n x.\, M/x] \rrbracket = \llbracket x : \sigma \vdash M : \tau \rrbracket (\llbracket \vdash Y^n x.\, M \rrbracket).$$

Exercise

Find the denotation of

$$\vdash \mathtt{Y}f.\, \lambda n.\, \mathtt{ifz}(n; \underline{0}; m.\, \mathtt{ifz}((f\ m); \underline{1}; x.\, \underline{0})) : \mathtt{nat} \to \mathtt{nat}$$