Semantics of Functional Programming

The Scott Model of **PCF**

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Formosan Summer School on Logic, Language, and Computation 2014

1 Scott domain model of PCF

Interpretation of types and contexts

Every type is interpreted as a domain and a context is a product of domains:

Definition 1. Every type σ in **PCF** associates with a domain D_σ as follows:

1. $D_{\mathtt{nat}} \coloneqq \mathbb{N}_\perp$, and
2. $D_{\tau\to\sigma} \coloneqq [D_\sigma \to D_\tau]$.

Definition 2. For each context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, the associated domain is defined as

$$D_\Gamma \coloneqq D_{\sigma_1} \times D_{\sigma_2} \times \cdots \times D_{\sigma_n}$$

and the associated domain of the empty context is $1 = \{*\}$.

Interpretation of simply type lambda calculus and natural numbers

Every judgement $\Gamma \vdash \mathsf{M} : \tau$ is interpreted as a Scott-continuous function

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!] : D_\Gamma \to D_\tau$$

and defined inductively on the derivation of $\Gamma \vdash \mathsf{M} : \tau$ as follows.

1. For (var), we interpret the judgement as the projection from the product to the component:

   $$[\![x_1 : \sigma_1, \ldots, x_i : \sigma_i, ldots, x_n : \sigma_n \vdash x_i : \sigma_i]\!] \coloneqq \pi$$

   or pointwise

   $$[\![\Gamma, x : \tau, \Delta \vdash x : \tau]\!](\vec{x}) = x$$

$$[\![\Gamma \vdash \lambda x.\,\mathsf{M} : \sigma \to \tau]\!] \coloneqq \Lambda[\![\Gamma, x : \sigma \vdash \mathsf{M} : \tau]\!]$$

$$[\![\Gamma \vdash \mathsf{M}\ \mathsf{N} : \tau]\!] = ev \circ \langle [\![\Gamma \vdash \mathsf{M} : \sigma \to \tau]\!], [\![\Gamma \vdash \mathsf{N} : \sigma]\!] \rangle$$

$$[\![\Gamma \vdash \mathtt{zero} : \mathtt{nat}]\!](\vec{d}) \coloneqq 0$$

$$[\![\Gamma \vdash \mathtt{suc}\ \mathsf{M} : \mathtt{nat}]\!] \coloneqq S \circ [\![\Gamma \vdash \mathsf{M} : \mathtt{nat}]\!]$$

Interpretation of general recursion and if-zero test

$$[\![\Gamma \vdash \mathtt{Y}x.\,\mathsf{M} : \sigma]\!] \coloneqq \mu \circ \Lambda[\![\Gamma, x : \sigma \vdash \mathsf{M} : \sigma]\!]$$

$$[\![\Gamma \vdash \mathtt{ifz}(\mathsf{M}; \mathsf{M}_0; \mathsf{M}_1) : \tau]\!]$$
$$\coloneqq \mathit{ifz}_\tau \circ \langle\, [\![\Gamma \vdash \mathsf{M} : \mathtt{nat}]\!], [\![\Gamma \vdash \mathsf{M}_0 : \tau]\!], \Lambda[\![\Gamma, x : \mathtt{nat} \vdash \mathsf{M}_1 : \tau]\!] \,\rangle$$

where S and ifz_τ are defined by

$$S(n) \coloneqq \begin{cases} \perp & \text{if } d = \perp \\ n+1 & \text{otherwise.} \end{cases}$$

$$\mathit{ifz}_\tau(n, x, f) \coloneqq \begin{cases} \perp & \text{if } n = \perp, \\ x & \text{if } n = 0, \\ f(m) & \text{if } n = m+1. \end{cases}$$

In particular, every program $\mathsf{M} : \tau$ associates with a function from 1 to $[\![\tau]\!]$, and thus determines a unique element $[\![\mathsf{M}]\!](*)$ of $[\![\tau]\!]$.

Convention
For brevity, we write $[\![\mathsf{M}]\!]$ instead of $[\![\ \vdash \mathsf{M} : \tau]\!](*)$ for every program M of type τ

Theorem 3. *For every judgement $\Gamma \vdash \mathsf{M} : \tau$, the associated function*

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!] \colon [\![\Gamma]\!] \to [\![\tau]\!]$$

is Scott continuous.