Semantics of Functional Programming

The Scott Model of **PCF**

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Denotational semantics of **PCF**

Instead of specifying *how* a **PCF** program runs, we specify *what* a program is, the *denotation* of a program.

To assign a denotation to a program,

- each type σ is interpreted as a domain D_σ;
- a context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ is interpreted as a product $\prod_{i=1}^{n} D_{\sigma_i}$ of domains;
- in particular, each term of type τ under the empty context is an element of D_τ.

Interpretation of types and contexts

Define the denotation of a type inductively:

Definition 1

Every type σ in **PCF** associates with a domain D_σ as follows:

1. $D_{\mathtt{nat}} := \mathbb{N}_\perp$, and
2. $D_{\tau \to \sigma} := [D_\tau \to D_\sigma]$.

Define the denotation of a context inductively on its length:

Definition 2

For each context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$, the associated domain is defined as

$$D_\Gamma := D_{\sigma_1} \times D_{\sigma_2} \times \cdots \times D_{\sigma_n}$$

and the associated domain of the empty context is $1 = \{*\}$.

Interpretation of judgements

To proceed with the denotational semantics, we further define the denotation for each judgement inductively on its derivation of the following form

- Every judgement $\Gamma \vdash \mathsf{M} : \tau$ is interpreted as a *continuous* function

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!] : D_\Gamma \to D_\tau.$$

- In particular,

$$[\![\, \vdash \mathsf{M} : \tau]\!] : 1 \to D_\tau$$

  is identified with an element $[\![\, \vdash \mathsf{M} : \tau]\!](*) = d$ of D_τ.

Convention

In the following context, $[\![\Gamma \vdash \mathsf{M} : \tau]\!](\vec{d})$ is written as

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!]\ \vec{d}.$$

for any sequence $\vec{d} \in D_\Gamma$ if there is no danger of ambiguity.

(var) Suppose that $\Gamma \vdash \mathsf{M} : \tau$ is of the form

$$x_1 : \sigma_1,\ \ldots,\ x_n : \sigma_n \vdash x_i : \sigma_i$$

derived by the rule (var). It is interpreted as the projection from D_Γ to its i-th component D_{σ_i}

$$[\![x_1 : \sigma_1,\ \ldots,\ x_n : \sigma_n \vdash x_i : \sigma_i]\!]\ \vec{d} := d_i$$

for $i = 1, \ldots, n$ where
$\vec{d} = (d_1, \ldots, d_n) \in D_{\sigma_1} \times \cdots \times D_{\sigma_n}$.

Note that the denotation of this judgement is equal to

$$[\![x_1 : \sigma_1, \ldots, x_n : \sigma_n \vdash x_i : \sigma_i]\!] = \pi_i$$

where $\pi_i \colon D_\Gamma \to D_{\sigma_i}$ is the i-th projection and thus it is a continuous function.

(abs) Let $f := [\![\Gamma, x : \sigma \vdash \mathsf{M} : \tau]\!]$ be the continuous function from $D_\Gamma \times D_\sigma$ to D_τ.

$$[\![\Gamma \vdash \lambda x.\, \mathsf{M} : \sigma \to \tau]\!] := \Lambda f$$

where $\Lambda f : D_\Gamma \to [D_\sigma \to D_\tau]$ is the *curried* f. In other words

$$\left([\![\Gamma \vdash \lambda x.\, \mathsf{M} : \sigma \to \tau]\!]\ \vec{d}\right)\ d = [\![\Gamma, x : \sigma \vdash \mathsf{M} : \tau]\!]\ (\vec{d}, d).$$

(app) Define

$$\begin{aligned} &[\![\Gamma \vdash \mathsf{M}\ \mathsf{N} : \tau]\!]\ \vec{d} \\ := &\ ev\left([\![\Gamma \vdash \mathsf{M} : \sigma \to \tau]\!]\ \vec{d}, [\![\Gamma \vdash \mathsf{N} : \sigma]\!]\ \vec{d}\right) \end{aligned}$$

where $ev\colon [D_1 \to D_2] \times D_1 \to D_2$ is the *evaluation map* which maps a continuous function $f\colon D_1 \to D_2$ with an element $d \in D_1$ to $f(d)$.

The cases for `zero` and `suc` M are rather obvious:

(z) `zero` is a constant, so it does not matter what the context is:

$$[\![\Gamma \vdash \texttt{zero} : \texttt{nat}]\!]\ \vec{d} := 0$$

i.e. a constant function.

(s) The denotation of `suc` is the successor function

$$[\![\Gamma \vdash \texttt{suc}\ \mathsf{M} : \texttt{nat}]\!]\ \vec{d} := (S \circ [\![\Gamma \vdash \mathsf{M} : \texttt{nat}]\!])\ \ \vec{d}$$

where $S \colon \mathbb{N}_\perp \to \mathbb{N}_\perp$ is defined by

$$S(n) := \begin{cases} \perp & \text{if } n = \perp \\ n+1 & \text{if } n \in \mathbb{N}. \end{cases}$$

(Y) The denotation of Y is the fixpoint operation

$$\llbracket \Gamma \vdash \mathtt{Y}x.\,\mathsf{M} : \sigma \rrbracket\ \vec{d} := \mu\left(\llbracket \Gamma \vdash \lambda x.\,\mathsf{M} : \sigma \to \sigma \rrbracket\ \vec{d}\right)$$

where μ is defined previously as $\mu(f) := \bigsqcup_{i \in \mathbb{N}} f^i(\bot)$.

(ifz) The denotation of ifz

$$\begin{aligned} &\llbracket \Gamma \vdash \mathtt{ifz}(\mathsf{M}; \mathsf{M}_0; x.\,\mathsf{M}_1) : \tau \rrbracket\ \vec{d} \\ := {}& \mathit{ifz}_\tau(n, x, f) \end{aligned}$$

where

1. $n := \llbracket \Gamma \vdash \mathsf{M} : \mathtt{nat} \rrbracket\ \vec{d}$,
2. $x := \llbracket \Gamma \vdash \mathsf{M}_0 : \tau \rrbracket\ \vec{d}$,
3. $f := \llbracket \Gamma \vdash \lambda x.\,\mathsf{M}_1 : \sigma \to \tau \rrbracket\ \vec{d}$,

and *ifz* is defined by

$$\mathit{ifz}(n, x, f) := \begin{cases} \bot & \text{if } n = \bot, \\ x & \text{if } n = 0, \\ f(n-1) & \text{otherwise.} \end{cases}$$

Theorem 3

For every judgement $\Gamma \vdash \mathsf{M} : \tau$*, the associated function*

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!] \colon D_\Gamma \to D_\tau$$

is Scott continuous.

Proof sketch.

It is not hard to see that each case of $[\![\Gamma \vdash \mathsf{M} : \tau]\!]$ is a Scott continuous function. $\square$

Example 4

Consider the denotations of the following judgements.

1. $y : \mathtt{nat} \vdash y : \mathtt{nat}$
2. $\vdash \lambda x.\, \underline{0} : \mathtt{nat} \to \mathtt{nat}$
3. $\vdash \mathtt{Y} f.\, \lambda n.\, \mathtt{ifz}(n; \underline{0}\ x.\, f\ x) : \mathtt{nat} \to \mathtt{nat}.$

1. $[\![y : \mathtt{nat} \vdash y : \mathtt{nat}]\!]\ d = d$
2. $[\![\vdash \lambda x.\, \underline{0} : \mathtt{nat} \to \mathtt{nat}]\!] = \Lambda f$ where

$$f := [\![x : \mathtt{nat} \vdash \mathtt{zero} : \mathtt{nat}]\!] = \mathit{const}_0,$$

i.e. the constant function at 0.

3

$$
\begin{aligned}
&[\![\vdash \mathtt{Y} f.\, \lambda n.\, \mathtt{ifz}(n; \underline{0}; x.\, f\ x) : \mathtt{nat} \to \mathtt{nat}]\!] \\
&= \mu(g)
\end{aligned}
$$

where $g : [D_{\mathtt{nat}} \to D_{\mathtt{nat}}] \to [D_{\mathtt{nat}} \to D_{\mathtt{nat}}]$ is defined by

$$
\begin{aligned}
g &:= [\![f : \mathtt{nat} \to \mathtt{nat} \vdash \lambda n.\, \mathtt{ifz}(n; \underline{0}; x.\, f\ x) : \mathtt{nat} \to \mathtt{nat}]\!] \\
&= \Lambda[\![f : \mathtt{nat} \to \mathtt{nat}, n : \mathtt{nat} \vdash \mathtt{ifz}(n; \underline{0}; x.\, f\ x) : \mathtt{nat}]\!]
\end{aligned}
$$

and

$$
\begin{aligned}
&[\![f : \mathtt{nat} \to \mathtt{nat}, n : \mathtt{nat} \vdash \mathtt{ifz}(n; \underline{0}; x.\, f\ x) : \mathtt{nat}]\!]\ (h, d) \\
&= \mathit{ifz}(d, 0, h)
\end{aligned}
$$

Then, what is $\mu(g)$? Let's calculate $g(\perp)$ and $g^2(\perp)$.

$$g(\perp_{D_{\mathtt{nat}} \to D_{\mathtt{nat}}})\ d = ifz(d, 0, \perp_{D_{\mathtt{nat}} \to D_{\mathtt{nat}}}) = \begin{cases} \perp & \text{if } d = \perp \\ 0 & \text{if } d = 0 \\ \perp & \text{otherwise.} \end{cases}$$

$$g(g(\perp))\ d = ifz(d, 0, g(\perp)) = \begin{cases} \perp & \text{if } d = \perp \\ 0 & \text{if } d = 0, 1 \\ \perp & \text{otherwise.} \end{cases}$$

By induction, we can show that

$$g^i(d) = \begin{cases} \perp & \text{if } d = \perp \\ 0 & \text{if } d < i \\ \perp & \text{otherwise,} \end{cases}$$

so $\mu(g)\ d = 0$ if $d \neq \perp$ and $\mu(g)\ d = \perp$ if $d = \perp$.

Exercise

Consider the denotations of the following judgements.

1. $y : \texttt{nat} \vdash (\lambda x.\, \underline{0})\ y : \texttt{nat}$
2. $\vdash \lambda n.\, \texttt{ifz}(n, \underline{0}, x.\, x) : \texttt{nat} \to \texttt{nat}$
3. $\vdash \lambda n.\, \texttt{ifz}(n, \underline{1}, x.\, \underline{0}) : \texttt{nat} \to \texttt{nat}$

Substitution Lemma

Lemma 5

Let $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ *be a context, and* $\Gamma \vdash \mathsf{M} : \tau$ *a judgement. Then the following equation*

$$\begin{aligned} &[\![\Delta \vdash \mathsf{M}[\vec{N}/\vec{x}]]\!]\ \vec{d} \\ = &[\![\Gamma \vdash \mathsf{M}]\!]\left([\![\Delta \vdash \mathsf{N}_1]\!]\ \vec{d}, \ldots, [\![\Delta \vdash \mathsf{N}_n]\!]\ \vec{d}\right) \end{aligned}$$

holds for any context Δ *and judgements* $\Delta \vdash \mathsf{N}_i : \sigma_i$ *for* $i = 1, \ldots, n$.

Corollary 6 (Application)

For every judgement $\Gamma, x : \sigma \vdash \mathsf{M} : \tau$ *and* $\Gamma \vdash \mathsf{N} : \sigma$*, we have*

$$[\![\Gamma \vdash (\lambda x.\, \mathsf{M})\ \mathsf{N} : \tau]\!] = [\![\Gamma \vdash \mathsf{M}[\mathsf{N}/x] : \tau]\!].$$

Observe that

$$\vec{d} = ([\![\Gamma \vdash x_1 : \sigma_1]\!]\ \vec{d}, \ldots, [\![\Gamma \vdash x_n : \sigma_n]\!]\ \vec{d})$$

for any context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$. Then, this corollary is a series of simple facts:

$$\begin{aligned}
&[\![\Gamma \vdash (\lambda x.\, \mathsf{M})\ \mathsf{N} : \tau]\!]\ \vec{d} \\
&= ev\left([\![\Gamma \vdash (\lambda x.\, \mathsf{M}) : \sigma \to \tau]\!]\ \vec{d}, [\![\Gamma \vdash \mathsf{N} : \sigma]\!]\ \vec{d}\right) \\
&= [\![\Gamma, x : \sigma \vdash \mathsf{M} : \tau]\!]\ (\vec{d}, [\![\Gamma \vdash \mathsf{N} : \sigma]\!]\ \vec{d}) \\
&= [\![\Gamma \vdash \mathsf{M}[\vec{x}, \mathsf{N}/\vec{x}, x] : \tau]\!]\ \vec{d} \\
&= [\![\Gamma \vdash \mathsf{M}[\mathsf{N}/x] : \tau]\!]\ \vec{d}
\end{aligned}$$

Example 7

The denotation of

$$\vdash (\lambda n.\, \mathtt{ifz}(n; \underline{1}; x.\, x))\ \underline{1} : \mathtt{nat}$$

and

$$\vdash \mathtt{ifz}(\underline{1}; \underline{1}; x.\, x) : \mathtt{nat}$$

are equal and calculated as follows:

$$\begin{aligned} & [\![\vdash \lambda n.\, \mathtt{ifz}(n; \underline{1}; x.\, x)\ \underline{1}]\!] \\ = {} & [\![\vdash \lambda n.\, \mathtt{ifz}(n; \underline{1}; x.\, x)]\!]([\![\vdash \underline{1} : \mathtt{nat}]\!]) \\ = {} & \mathit{ifz}(1, 1, \mathit{id}) = 0 \end{aligned}$$

Lemma 8 (Weakening)

Let $\Gamma \vdash \mathsf{M} : \tau$ *be a judgement. Then the following*

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!] = [\![\Gamma, x : \sigma \vdash \mathsf{M} : \tau]\!]$$

holds for any variable $x : \sigma$ *not in* Γ*.*

It follows from Substitution Lemma. (*Why?*)

Corollary 9 (η-conversion)

Let $\Gamma \vdash \mathrm{M} : \sigma \to \tau$ *be a judgement. Then,*

$$[\![\Gamma \vdash \lambda x.\, \mathrm{M}\, x : \sigma \to \tau]\!] = [\![\Gamma \vdash \mathrm{M} : \sigma \to \tau]\!]$$

if x *is not a variable in* Γ*.*

For every sequence $\vec{d} \in D_\Gamma$ and $d \in D_\sigma$, we have

$$\begin{aligned}
&\left([\![\Gamma \vdash \lambda x.\, \mathrm{M}\, x : \sigma \to \tau]\!]\ \vec{d}\right)\ d \\
&= [\![\Gamma, x : \sigma \vdash \mathrm{M}\, x : \tau]\!](\vec{d}, d) \\
&= ev\left([\![\Gamma, x : \sigma \vdash \mathrm{M} : \sigma \to \tau]\!](\vec{d}, d), [\![\Gamma, x : \sigma \vdash x : \sigma]\!](\vec{d}, d)\right) \\
&= ev\left([\![\Gamma, x : \sigma \vdash \mathrm{M} : \sigma \to \tau]\!](\vec{d}, d), d\right) \\
&= \left([\![\Gamma, x : \sigma \vdash \mathrm{M} : \sigma \to \tau]\!](\vec{d}, d)\right) d \\
&= \left([\![\Gamma \vdash \mathrm{M} : \sigma \to \tau]\!]\ \vec{d}\right)\ d.
\end{aligned}$$

Compactness

Define $\mathtt{Y}^i x.\, \mathsf{M}$ inductively for each $i \in \mathbb{N}$ by

1. $\mathtt{Y}^0 x.\, \mathsf{M} := \mathtt{Y} x.\, x$ and
2. $\mathtt{Y}^{n+1} x.\, \mathsf{M} := \mathsf{M}[\mathtt{Y}^n x.\, \mathsf{M}/x]$.

Theorem 10

For every judgement $\Gamma, x : \sigma \vdash \mathsf{M} : \sigma$, *we have*

$$[\![\Gamma \vdash \mathtt{Y} x.\, \mathsf{M} : \sigma]\!] = \bigsqcup_{i \in \mathbb{N}} [\![\Gamma \vdash \mathtt{Y}^i x.\, \mathsf{M} : \sigma]\!].$$

To show this theorem, it suffices to show the following

$$[\![\, \vdash \mathtt{Y}^i x.\, \mathsf{M} : \sigma]\!] = [\![x : \sigma \vdash \mathsf{M} : \sigma]\!]^i(\bot)$$

for $i \in \mathbb{N}$. (Why?)

For $n = 0$ we show that $[\![\vdash \mathtt{Y}^0 x.\, \mathsf{M} : \sigma]\!] = \bot_{D_\sigma} \in D_\sigma$.

By definition, $\mathtt{Y}^0 x.\, \mathsf{M} : \sigma$ is equal to $\mathtt{Y}x.\, x$, so

$$[\![\vdash \mathtt{Y}x.\, x : \sigma]\!] = \mu(id) = \bigsqcup_{i \in \mathbb{N}} id^i(\bot)$$
$$= \bigsqcup \bot = \bot$$

For $i = n + 1$ it suffices to show that

$$[\![\vdash \mathtt{Y}^{n+1} x.\, \mathsf{M} : \sigma]\!] = [\![x : \sigma \vdash \mathsf{M} : \sigma]\!] \left([\![\vdash \mathtt{Y}^n x.\, \mathsf{M} : \sigma]\!] \right),$$

so the statement follows by the induction hypothesis.

By definition, $\mathtt{Y}^{n+1} x.\, \mathsf{M}$ is equal to $\mathsf{M}[\mathtt{Y}^n x.\, \mathsf{M}/x]$, and by Substitution Lemma we have

$$[\![\vdash \mathsf{M}[\mathtt{Y}^n x.\, \mathsf{M}/x]]\!] = [\![x : \sigma \vdash \mathsf{M} : \tau]\!] \left([\![\vdash \mathtt{Y}^n x.\, \mathsf{M}]\!] \right).$$

Exercise

Find the denotation of

$$\vdash \mathtt{Y}f.\, \lambda n.\, \mathtt{ifz}(n; \underline{0}; m.\, \mathtt{ifz}((f\ m); \underline{1}; x.\, \underline{0}) : \mathtt{nat} \to \mathtt{nat}$$