Semantics of Functional Programming

The Scott Model of **PCF**

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1 Scott domain model

Denotational semantics of PCF

Instead of specifying *how* a **PCF** program runs, we specify *what* a program is, the *denotation* of a program. To assign a denotation to a program,

- each type σ is interpreted as a domain D_σ;
- a context $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$ is interpreted as a product $\prod_{i=1}^{n} D_{\sigma_i}$ of domains;
- in particular, each term of type τ under the empty context is an element of D_τ.

Interpretation of types and contexts

Define the denotation of a type inductively:

Definition 1. Every type σ in **PCF** associates with a domain D_σ as follows:

1. $D_{\texttt{nat}} := \mathbb{N}_\perp$, and
2. $D_{\tau \to \sigma} := [D_\tau \to D_\sigma]$.

Define the denotation of a context inductively on its length:

Definition 2. For each context $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$, the associated domain is defined as

$$D_\Gamma := 1 \times D_{\sigma_1} \times D_{\sigma_2} \times \cdots \times D_{\sigma_n}$$

Interpretation of judgements

To proceed with the denotational semantics, we further define the denotation for each judgement inductively on its derivation of the following form

- Every judgement $\Gamma \vdash \mathsf{M} : \tau$ is interpreted as a *continuous* function
$$[\![\Gamma \vdash \mathsf{M} : \tau]\!] : D_\Gamma \to D_\tau.$$
- In particular,
$$[\![\vdash \mathsf{M} : \tau]\!] : 1 \to D_\tau$$
is identified with an element $[\![\vdash \mathsf{M} : \tau]\!](*) = d$ of D_τ.

Convention

In the following context, $[\![\Gamma \vdash \mathsf{M} : \tau]\!](\vec{d})$ is written as

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!]\ \vec{d}.$$

for any sequence $\vec{d} \in D_\Gamma$ if there is no danger of ambiguity.

(var) Suppose that $\Gamma \vdash \mathsf{M} : \tau$ is of the form

$$x_1 : \sigma_1,\ \dots,\ x_n : \sigma_n \vdash x_i : \sigma_i$$

derived by the rule (var). It is interpreted as the projection from D_Γ to its i-th component D_{σ_i}

$$[\![x_1 : \sigma_1,\ \dots,\ x_n : \sigma_n \vdash x_i : \sigma_i]\!]\ \vec{d} := d_i$$

for $i = 1, \dots, n$ where $\vec{d} = (*, d_1, \dots, d_n) \in 1 \times D_{\sigma_1} \times \cdots \times D_{\sigma_n}$.

Note that the denotation of this judgement is equal to

$$[\![x_1 : \sigma_1, \dots, x_n : \sigma_n \vdash x_i : \sigma_i]\!] = \pi_i$$

where $\pi_i \colon D_\Gamma \to D_{\sigma_i}$ is the i-th projection and thus it is a continuous function.

(abs) Let $f := [\![\Gamma, x : \sigma \vdash \mathsf{M} : \tau]\!]$ be the continuous function from $D_\Gamma \times D_\sigma$ to D_τ.

$$[\![\Gamma \vdash \lambda x.\, \mathsf{M} : \sigma \to \tau]\!] := \Lambda f$$

where $\Lambda f : D_\Gamma \to [D_\sigma \to D_\tau]$ is the *curried* f. In other words

$$\left([\![\Gamma \vdash \lambda x.\, \mathsf{M} : \sigma \to \tau]\!]\ \vec{d}\right)\ d = [\![\Gamma, x : \sigma \vdash \mathsf{M} : \tau]\!]\ (\vec{d}, d).$$

(app) Define

$$[\![\Gamma \vdash \mathsf{M}\ \mathsf{N} : \tau]\!]\ \vec{d}$$
$$:= ev\left([\![\Gamma \vdash \mathsf{M} : \sigma \to \tau]\!]\ \vec{d}, [\![\Gamma \vdash \mathsf{N} : \sigma]\!]\ \vec{d}\right)$$

where $ev \colon [D_1 \to D_2] \times D_1 \to D_2$ is the *evaluation map* which maps a continuous function $f \colon D_1 \to D_2$ with an element $d \in D_1$ to $f(d)$.

The cases for zero and suc M are rather obvious:

(z) $\mathtt{zero}$ is a constant, so it does not matter what the context is:

$$[\![\Gamma \vdash \mathtt{zero} : \mathtt{nat}]\!]\ \vec{d} := 0$$

i.e. a constant function.

(s) The denotation of $\mathtt{suc}$ is the successor function

$$[\![\Gamma \vdash \mathtt{suc}\ \mathsf{M} : \mathtt{nat}]\!]\ \vec{d} := (S \circ [\![\Gamma \vdash \mathsf{M} : \mathtt{nat}]\!])\ \vec{d}$$

where $S \colon \mathbb{N}_\perp \to \mathbb{N}_\perp$ is defined by

$$S(n) := \begin{cases} \perp & \text{if } n = \perp \\ n+1 & \text{if } n \in \mathbb{N}. \end{cases}$$

(Y) The denotation of $\mathtt{Y}$ is the fixpoint operation

$$[\![\Gamma \vdash \mathtt{Y}x.\,\mathsf{M} : \sigma]\!]\ \vec{d} := \mu\left([\![\Gamma \vdash \lambda x.\,\mathsf{M} : \sigma \to \sigma]\!]\ \vec{d}\right)$$

where μ is defined previously as $\mu(f) := \bigsqcup_{i\in\mathbb{N}} f^i(\perp)$.

(ifz) The denotation of $\mathtt{ifz}$

$$\begin{aligned}&[\![\Gamma \vdash \mathtt{ifz}(\mathsf{M}; \mathsf{M}_0; x.\,\mathsf{M}_1) : \tau]\!]\ \vec{d} \\ := {}& \mathit{ifz}_\tau(n, e, f)\end{aligned}$$

where

1. $n := [\![\Gamma \vdash \mathsf{M} : \mathtt{nat}]\!]\ \vec{d}$,
2. $e := [\![\Gamma \vdash \mathsf{M}_0 : \tau]\!]\ \vec{d}$,
3. $f := [\![\Gamma \vdash \lambda x.\,\mathsf{M}_1 : \mathtt{nat} \to \tau]\!]\ \vec{d}$,

and ifz is defined by

$$\mathit{ifz}(n, x, f) := \begin{cases} \perp & \text{if } n = \perp, \\ x & \text{if } n = 0, \\ f(n-1) & \text{otherwise.} \end{cases}$$

Theorem 3. *For every judgement $\Gamma \vdash \mathsf{M} : \tau$, the associated function*

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!] \colon D_\Gamma \to D_\tau$$

is Scott continuous.

Proof sketch. It is not hard to see that each case of $[\![\Gamma \vdash \mathsf{M} : \tau]\!]$ is a Scott continuous function. □

Example 4. Consider the denotations of the following judgements.

1. $y : \mathtt{nat} \vdash y : \mathtt{nat}$
2. $\vdash \lambda x.\,\underline{0} : \mathtt{nat} \to \mathtt{nat}$
3. $\vdash \mathtt{Y}f.\,\lambda n.\,\mathtt{ifz}(n; \underline{0}; x.\,f\ x) : \mathtt{nat} \to \mathtt{nat}$.

1. $[\![y : \mathtt{nat} \vdash y : \mathtt{nat}]\!]\ d = d$
2. $[\![\vdash \lambda x.\,\underline{0} : \mathtt{nat} \to \mathtt{nat}]\!] = \Lambda f$ where

$$f := [\![x : \mathtt{nat} \vdash \mathtt{zero} : \mathtt{nat}]\!] = \mathit{const}_0,$$

i.e. the constant function at 0.

3.

$$\begin{aligned}&[\![\vdash \mathtt{Y}f.\,\lambda n.\,\mathtt{ifz}(n; \underline{0}; x.\,f\ x) : \mathtt{nat} \to \mathtt{nat}]\!] \\ &= \mu(g)\end{aligned}$$

where $g : [D_{\mathtt{nat}} \to D_{\mathtt{nat}}] \to [D_{\mathtt{nat}} \to D_{\mathtt{nat}}]$ is defined by

$$\begin{aligned}g &:= [\![f : \mathtt{nat} \to \mathtt{nat} \vdash \lambda n.\,\mathtt{ifz}(n; \underline{0}; x.\,f\ x) : \mathtt{nat} \to \mathtt{nat}]\!] \\ &= \Lambda[\![f : \mathtt{nat} \to \mathtt{nat}, n : \mathtt{nat} \vdash \mathtt{ifz}(n; \underline{0}; x.\,f\ x) : \mathtt{nat}]\!]\end{aligned}$$

and

$$\begin{aligned}&[\![f : \mathtt{nat} \to \mathtt{nat}, n : \mathtt{nat} \vdash \mathtt{ifz}(n; \underline{0}; x.\,f\ x) : \mathtt{nat}]\!]\ (h, d) \\ &= \mathit{ifz}(d, 0, h)\end{aligned}$$

Then, what is $\mu(g)$? Let's calculate $g(\perp)$ and $g^2(\perp)$.

$$g(\perp_{D_{\mathtt{nat}} \to D_{\mathtt{nat}}})\ d = \mathit{ifz}(d, 0, \perp_{D_{\mathtt{nat}} \to D_{\mathtt{nat}}}) = \begin{cases} \perp & \text{if } d = \perp \\ 0 & \text{if } d = 0 \\ \perp & \text{otherwise.} \end{cases}$$

$$g(g(\perp))\ d = \mathit{ifz}(d, 0, g(\perp)) = \begin{cases} \perp & \text{if } d = \perp \\ 0 & \text{if } d = 0, 1 \\ \perp & \text{otherwise.} \end{cases}$$

By induction, we can show that

$$g^i(d) = \begin{cases} \perp & \text{if } d = \perp \\ 0 & \text{if } d < i \\ \perp & \text{otherwise,} \end{cases}$$

so $\mu(g)\ d = 0$ if $d \neq \perp$ and $\mu(g)\ d = \perp$ if $d = \perp$.

Exercise
Consider the denotations of the following judgements.

1. $y : \mathtt{nat} \vdash (\lambda x.\,\underline{0})\ y : \mathtt{nat}$
2. $\vdash \lambda n.\,\mathtt{ifz}(n; \underline{0}; x.\,x) : \mathtt{nat} \to \mathtt{nat}$
3. $\vdash \lambda n.\,\mathtt{ifz}(n; \underline{1}; x.\,\underline{0}) : \mathtt{nat} \to \mathtt{nat}$

2 Substitution and Compactness

Substitution Lemma

Lemma 5. *Let $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$ be a context, and $\Gamma \vdash \mathsf{M} : \tau$ a judgement. Then the following equation*

$$\begin{aligned} &[\![\Delta \vdash \mathsf{M}[\vec{N}/\vec{x}]]\!]\ \vec{d} \\ &= [\![\Gamma \vdash \mathsf{M}]\!] \left([\![\Delta \vdash \mathsf{N}_1]\!]\ \vec{d}, \ldots, [\![\Delta \vdash \mathsf{N}_n]\!]\ \vec{d}\right) \end{aligned}$$

holds for any context Δ and judgements $\Delta \vdash \mathsf{N}_i : \sigma_i$ for $i = 1, \ldots, n$.

Corollary 6 (Application)**.** *For every judgement $\Gamma, x : \sigma \vdash \mathsf{M} : \tau$ and $\Gamma \vdash \mathsf{N} : \sigma$, we have*

$$[\![\Gamma \vdash (\lambda x. \mathsf{M})\ \mathsf{N} : \tau]\!] = [\![\Gamma \vdash \mathsf{M}[\mathsf{N}/x] : \tau]\!].$$

Observe that

$$\vec{d} = ([\![\Gamma \vdash x_1 : \sigma_1]\!]\ \vec{d}, \ldots, [\![\Gamma \vdash x_n : \sigma_n]\!]\ \vec{d})$$

for any context $\Gamma = x_1 : \sigma_1, \ldots, x_n : \sigma_n$. Then, this corollary is a series of simple facts:

$$\begin{aligned} &[\![\Gamma \vdash (\lambda x. \mathsf{M})\ \mathsf{N} : \tau]\!]\ \vec{d} \\ &= ev\left([\![\Gamma \vdash (\lambda x. \mathsf{M}) : \sigma \to \tau]\!]\ \vec{d}, [\![\Gamma \vdash \mathsf{N} : \sigma]\!]\ \vec{d}\right) \\ &= [\![\Gamma, x : \sigma \vdash \mathsf{M} : \tau]\!]\ (\vec{d}, [\![\Gamma \vdash \mathsf{N} : \sigma]\!]\ \vec{d}) \\ &= [\![\Gamma \vdash \mathsf{M}[\vec{x}, \mathsf{N}/\vec{x}, x] : \tau]\!]\ \vec{d} \\ &= [\![\Gamma \vdash \mathsf{M}[\mathsf{N}/x] : \tau]\!]\ \vec{d} \end{aligned}$$

Example 7. The denotation of

$$\vdash (\lambda n. \mathtt{ifz}(n; \underline{1}; x. x))\ \underline{1} : \mathtt{nat}$$

and

$$\vdash \mathtt{ifz}(\underline{1}; \underline{1}; x. x) : \mathtt{nat}$$

are equal and calculated as follows:

$$\begin{aligned} &[\![\vdash \lambda n. \mathtt{ifz}(n; \underline{1}; x. x)\ \underline{1}]\!] \\ &= [\![\vdash \lambda n. \mathtt{ifz}(n; \underline{1}; x. x)]\!]([\![\vdash \underline{1} : \mathtt{nat}]\!]) \\ &= \mathit{ifz}(1, 1, \mathit{id}) = 0 \end{aligned}$$

Lemma 8 (Weakening)**.** *Let $\Gamma \vdash \mathsf{M} : \tau$ be a judgement. Then the following*

$$[\![\Gamma \vdash \mathsf{M} : \tau]\!]\ \vec{d} = [\![\Gamma, x : \sigma \vdash \mathsf{M} : \tau]\!]\ (\vec{d}, d)$$

holds for any variable $x : \sigma$ not in Γ, $\vec{d} \in D_\Gamma$ and $d \in D_\sigma$.

It follows from Substitution Lemma. (*Why?*)

Corollary 9 (η-conversion)**.** *Let $\Gamma \vdash \mathsf{M} : \sigma \to \tau$ be a judgement. Then,*

$$[\![\Gamma \vdash \lambda x. \mathsf{M}\ x : \sigma \to \tau]\!] = [\![\Gamma \vdash \mathsf{M} : \sigma \to \tau]\!]$$

if x is not a variable in Γ.

For every sequence $\vec{d} \in D_\Gamma$ and $d \in D_\sigma$, we have

$$\begin{aligned} &\left([\![\Gamma \vdash \lambda x. \mathsf{M}\ x : \sigma \to \tau]\!]\ \vec{d}\right)\ d \\ &= [\![\Gamma, x : \sigma \vdash \mathsf{M}\ x : \tau]\!](\vec{d}, d) \\ &= ev\left([\![\Gamma, x : \sigma \vdash \mathsf{M} : \sigma \to \tau]\!](\vec{d}, d), [\![\Gamma, x : \sigma \vdash x : \sigma]\!](\vec{d}, d)\right) \\ &= ev\left([\![\Gamma, x : \sigma \vdash \mathsf{M} : \sigma \to \tau]\!](\vec{d}, d), d\right) \\ &= \left([\![\Gamma, x : \sigma \vdash \mathsf{M} : \sigma \to \tau]\!](\vec{d}, d)\right)\ d \\ &= \left([\![\Gamma \vdash \mathsf{M} : \sigma \to \tau]\!]\ \vec{d}\right)\ d. \end{aligned}$$

Compactness

Define $\mathtt{Y}^i x. \mathsf{M}$ inductively for each $i \in \mathbb{N}$ by

1. $\mathtt{Y}^0 x. \mathsf{M} := \mathtt{Y} x. x$ and
2. $\mathtt{Y}^{n+1} x. \mathsf{M} := \mathsf{M}[\mathtt{Y}^n x. \mathsf{M}/x]$.

Theorem 10. *For every judgement $\Gamma, x : \sigma \vdash \mathsf{M} : \sigma$, we have*

$$[\![\Gamma \vdash \mathtt{Y} x. \mathsf{M} : \sigma]\!] = \bigsqcup_{i \in \mathbb{N}} [\![\Gamma \vdash \mathtt{Y}^i x. \mathsf{M} : \sigma]\!].$$

To show this theorem, it suffices to show the following

$$[\![\vdash \mathtt{Y}^i x. \mathsf{M} : \sigma]\!] = [\![x : \sigma \vdash \mathsf{M} : \sigma]\!]^i(\bot)$$

for $i \in \mathbb{N}$. (Why?)

For $n = 0$ we show that $[\![\vdash \mathtt{Y}^0 x. \mathsf{M} : \sigma]\!] = \bot_{D_\sigma} \in D_\sigma$.

By definition, $\mathtt{Y}^0 x. \mathsf{M}$ is equal to $\mathtt{Y} x. x$, so

$$\begin{aligned} [\![\vdash \mathtt{Y} x. x : \sigma]\!] &= \mu(\mathit{id}) = \bigsqcup_{i \in \mathbb{N}} \mathit{id}^i(\bot) \\ &= \bigsqcup \bot = \bot \end{aligned}$$

For $i = n + 1$ it suffices to show that

$$[\![\vdash \mathtt{Y}^{n+1} x. \mathsf{M} : \sigma]\!] = [\![x : \sigma \vdash \mathsf{M} : \sigma]\!]\left([\![\vdash \mathtt{Y}^n x. \mathsf{M} : \sigma]\!]\right),$$

so the statement follows by the induction hypothesis.

By definition, $\mathtt{Y}^{n+1} x. \mathsf{M}$ is equal to $\mathsf{M}[\mathtt{Y}^n x. \mathsf{M}/x]$, and by Substitution Lemma we have

$$[\![\vdash \mathsf{M}[\mathtt{Y}^n x. \mathsf{M}/x]]\!] = [\![x : \sigma \vdash \mathsf{M} : \tau]\!]\left([\![\vdash \mathtt{Y}^n x. \mathsf{M}]\!]\right).$$

Exercise
Find the denotation of

$\vdash \mathtt{Y} f.\, \lambda n.\, \mathtt{ifz}(n; \underline{0}; m.\, \mathtt{ifz}((f\ m); \underline{1}; x.\, \underline{0})) : \mathtt{nat} \to \mathtt{nat}$