# Semantics of Functional Programming

Lecture IV: Computational Adequacy and Further Topics

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So far we have given two kinds of semantics for **PCF**. For a program $\mathsf{M}$ of type $\sigma$,

- one gives how the program $\mathsf{M}$ is evaluated to a closed value $\mathsf{V}$ via the reduction relation $\mathsf{M} \Downarrow \mathsf{V}$;
- the other defines what the denotation $[\![\mathsf{M}]\!]$ of $\mathsf{M}$ is.

In this lecture, we will compare these two approaches and discuss some issues arising from them:

**Correctness**

**Completeness**

**Computational adequacy**

**Full abstraction**

## 1 Computational Adequacy

**Closed values of `nat` do not diverge**

The bottom element $\bot$ in a domain models the divergence of computation, and a closed value $\mathsf{V}$ of `nat` is meant to be some natural number. Let's justify this idea.

**Lemma 1.** *For every closed value* $\mathsf{V}$ *of type* `nat`*, the denotation* $[\![\mathsf{V}]\!]$ *is an element of* $\mathbb{N}$*. In particular,* $[\![\mathsf{V}]\!] \neq \bot$*.*

*Proof.* By structural induction on closed values. For the following cases

$$\frac{}{\mathtt{zero}\ \mathbf{val}}$$

$$\frac{\mathsf{M}\ \mathbf{val}}{\mathsf{suc}\ \mathsf{M}\ \mathbf{val}}$$

$$\frac{}{\lambda x.\,\mathsf{M}\ \mathbf{val}}$$

it is easy to see that $[\![\mathtt{zero}]\!]$ and $[\![\mathtt{suc}\ \mathsf{M}]\!]$, if $[\![\mathsf{M}]\!] \in \mathbb{N}$, are elements of $\mathbb{N}$ by the definition of $[\![-]\!]$. On the other hand, $\lambda x.\,\mathsf{M}$ cannot be of type `nat`, so this case holds vacuously. ☐

By inspection of the above proof, we conclude that $[\![\underline{n}]\!] = n$ — what a numeral $\underline{n}$ of $n$ should mean.

**Correctness**

Now we show that denotational semantics is correct with respect to denotational semantics:

**Theorem 2.** *For every two programs* $\mathsf{M}$ *and* $\mathsf{V}$*,* $\mathsf{M} \Downarrow \mathsf{V}$ *implies* $[\![\mathsf{M}]\!] = [\![\mathsf{V}]\!]$*.*

A sanity check: By Preservation Theorem, it is known that $[\![\vdash \mathsf{M} : \tau]\!]$ and $[\![\vdash \mathsf{V} : \sigma]\!]$ are of the same type if $\mathsf{M} \Downarrow \mathsf{V}$, so their range $[\![\tau]\!]$ and $[\![\sigma]\!]$ are the same.

*Proof Sketch.* Prove $[\![\mathsf{M}]\!] = [\![\mathsf{V}]\!] \in [\![\tau]\!]$ by structural induction on the derivation of $\mathsf{M} \Downarrow \mathsf{V}$. ☐

**Proof of correctness**

We show the case ($\Downarrow$-suc) first and the cases ($\Downarrow$-zero) and ($\Downarrow$-lam) are similar and easy.

- For ($\Downarrow$-suc), we show that $[\![\mathtt{suc}\ \mathsf{M}]\!] = [\![\mathtt{suc}\ \mathsf{V}]\!]$ if $[\![\mathsf{M}]\!] = [\![\mathsf{V}]\!]$. By definition, we simply calculate its denotation directly:

  $$[\![\mathtt{suc}\ \mathsf{M}]\!] = S([\![\mathsf{M}]\!]) = S([\![\mathsf{V}]\!]) = [\![\mathtt{suc}\ \mathsf{V}]\!]$$

  where the middle equality follows from the induction hypothesis.

Try to do the cases ($\Downarrow$-zero), ($\Downarrow$-lam), and ($\Downarrow$-$\mathtt{ifz}_0$).

The case ($\Downarrow$-app) is slightly complicated, as we have to address the binding structure using Substitution Lemma.

- For ($\Downarrow$-app), we show that $[\![\mathsf{M}\ \mathsf{N}]\!] = [\![V]\!]$ if $[\![\mathsf{M}]\!] = [\![\lambda x.\,\mathsf{E}]\!]$ and $[\![E[\mathsf{N}/x]]\!] = [\![V]\!]$. We calculate the denotation as follows

  $$\begin{aligned}[\![\mathsf{M}\ \mathsf{N}]\!] &= ev([\![\mathsf{M}]\!], [\![\mathsf{N}]\!]) \\ &= ev([\![\lambda x.\,\mathsf{E}]\!], [\![\mathsf{N}]\!]) \\ &= ev([\![x : \sigma \vdash \mathsf{E} : \tau]\!], [\![\mathsf{N}]\!]) \\ &= [\![x : \sigma \vdash \mathsf{E} : \tau]\!]([\![\mathsf{N}]\!]) = [\![\mathsf{E}[\mathsf{N}/x]]\!] = [\![\mathsf{V}]\!]\end{aligned}$$

where the last but one equation follows from Substitution Lemma.

- Complete the remaining two (interesting) cases ($\Downarrow$-$\texttt{ifz}_1$) and ($\Downarrow$-fix). *Hint.* Consider Substitution Lemma and the properties of the fixpoint operator $\mu$.