

# Semantics of Functional Programming

## Computational Adequacy

Chen, Liang-Ting  
`lxc@iis.sinica.edu.tw`

Institute of Information Science, Academia Sinica

Formosan Summer School on Logic, Language, and  
Computation 2014

# Overview

So far we have given two kinds of semantics for **PCF**. For a well-typed closed terms  $M$  of type  $\sigma$ ,

- one gives how the well-typed closed term  $M$  is evaluated to a value  $V$  via the reduction relation  $M \Downarrow V$ ;
- the other defines what the denotation  $\llbracket M \rrbracket$  of  $M$  is in a domain  $D_\sigma$ .

In this lecture, we will compare these two approaches and discuss some issues arising from them:

**Correctness**  $M \Downarrow V$  implies  $\llbracket M \rrbracket = \llbracket V \rrbracket$ .

**Completeness**  $\llbracket M \rrbracket = n$  implies  $M \Downarrow \underline{n}$

**Computational adequacy** Both of correctness and completeness hold.

## **nat** values always converges

The bottom element  $\perp$  models the divergence of computation. A value of **nat** is meant to be some natural number, so it shouldn't diverge.

### Lemma 1

*For every value  $V$  of type **nat**, the denotation  $\llbracket V \rrbracket$  is an element of  $\mathbb{N}$ . In particular,  $\llbracket V \rrbracket \neq \perp$ .*

### Proof.

By structural induction on values:

$$\frac{}{\text{zero } \mathbf{val}}$$

$$\frac{M \ \mathbf{val}}{\text{suc } M \ \mathbf{val}}$$

$$\frac{M \ \mathbf{term}}{\lambda x. M \ \mathbf{val}}$$



## Theorem 2

*For every two well-typed closed terms  $M$  and  $V$ ,  $M \Downarrow V$  implies  $\llbracket M \rrbracket = \llbracket V \rrbracket$ .*

### Proof sketch.

Prove  $\llbracket M \rrbracket = \llbracket V \rrbracket \in \llbracket \tau \rrbracket$  by structural induction on the derivation of  $M \Downarrow V$ . □

We show the case ( $\Downarrow$ -suc) first and the cases ( $\Downarrow$ -zero) and ( $\Downarrow$ -lam) are similar and straightforward.

- For ( $\Downarrow$ -suc), we show that  $\llbracket \text{suc } M \rrbracket = \llbracket \text{suc } V \rrbracket$  if  $\llbracket M \rrbracket = \llbracket V \rrbracket$ .  
By definition, we simply calculate its denotation directly:

$$\llbracket \text{suc } M \rrbracket = S(\llbracket M \rrbracket) = S(\llbracket V \rrbracket) = \llbracket \text{suc } V \rrbracket$$

where the middle equality follows from the induction hypothesis.

The case ( $\Downarrow$ -app) is interesting, because there is the binding structure.

- For ( $\Downarrow$ -app), we show that  $\llbracket M N \rrbracket = \llbracket V \rrbracket$  if  $\llbracket M \rrbracket = \llbracket \lambda x. E \rrbracket$  and  $\llbracket E[N/x] \rrbracket = \llbracket V \rrbracket$ . We calculate the denotation as follows

$$\begin{aligned}\llbracket M N \rrbracket &= \text{ev}(\llbracket M \rrbracket, \llbracket N \rrbracket) \\ &= \text{ev}(\llbracket \lambda x. E \rrbracket, \llbracket N \rrbracket) \\ &= \text{ev}(\llbracket x : \sigma \vdash E : \tau \rrbracket, \llbracket N \rrbracket) \\ &= \llbracket x : \sigma \vdash E : \tau \rrbracket(\llbracket N \rrbracket) = \llbracket E[N/x] \rrbracket = \llbracket V \rrbracket\end{aligned}$$

where  $\llbracket x : \sigma \vdash E : \tau \rrbracket(\llbracket N \rrbracket) = \llbracket E[N/x] \rrbracket$  follows from Substitution Lemma.

- Complete the cases ( $\Downarrow$ -ifz<sub>1</sub>) and ( $\Downarrow$ -fix). *Hint.* Consider Substitution Lemma and the properties of the fixpoint operator  $\mu$ .

# Logical Equivalence

## Definition 3 (Applicative approximation)

For each type  $\sigma$ , we define a relation  $\preceq_\sigma$  between well-typed closed terms  $\vdash M : \sigma$ .

- 1 For  $\text{nat}$ , define

$$M \preceq_{\text{nat}} N$$

if for all  $n \in \mathbb{N}$ ,  $M \Downarrow \underline{n}$  implies  $N \Downarrow \underline{n}$

- 2 For  $\sigma \rightarrow \tau$ , define

$$M \preceq_{\sigma \rightarrow \tau} N$$

if  $M P \preceq_\tau N P$ , for every well-typed closed term  $P$ .

Two well-typed closed terms  $M$  and  $N$  of the same type  $\sigma$  are **logically equivalent** denoted  $M \simeq_\sigma N$  if  $M \preceq_\sigma N$  and  $N \preceq_\sigma M$ .

The relation  $\lesssim_\sigma$  is a preorder, so  $\simeq_\sigma$  is indeed an equivalence.<sup>2</sup>

#### Proposition 4

*The logical equivalence  $\simeq_\sigma$  is an equivalence relation.*

A well-typed closed term  $M$  can be replaced by another well-typed closed term  $N$  without changing its result if  $M \simeq_\sigma N$ .

#### Example 5

The following two well-typed closed terms are logically equivalent:

$$\lambda x. x : \text{nat} \rightarrow \text{nat} \quad \text{and} \quad \lambda x. \text{pred} (\text{suc } x) : \text{nat} \rightarrow \text{nat}$$

---

<sup>2</sup>Why? Prove it. Note that an equivalence relation is defined to be a reflexive, symmetric, and transitive relation.

# Reduction respects logical equivalence

Recall that from  $M \rightsquigarrow^* M'$  and  $M' \Downarrow V$  it follows that  $M \Downarrow V$  in the agreement between  $\rightsquigarrow$  and  $\Downarrow$ .

## Proposition 6

*Let  $M$  and  $M'$  be well-typed closed terms of type  $\sigma$ . If  $M \rightsquigarrow^* M'$ , then  $M \lesssim_\sigma M'$ .*

The other direction follows from the determinacy and values cannot be reduced further:

## Proposition 7

*For every  $M \Downarrow V$  and  $M \rightsquigarrow^* M'$ , we have  $M' \Downarrow V$ .*

Therefore, if  $M \rightsquigarrow^* M'$ , then  $M \simeq_\sigma M'$ .



However, logical equivalence goes beyond reduction. Consider the following two well-typed closed terms of type  $\text{nat} \rightarrow \text{nat} \rightarrow \text{nat}$ :

$$\lambda x. \lambda y. x + y$$

and

$$\lambda x. \lambda y. y + x$$

Surely the addition of natural numbers are commutative, but *why*?

By definition they are already values, so they cannot be reduced to each other.

### Remark 2.1

We can show directly that these two well-typed closed terms are logically equivalent in dependent type theory. Yet, we will present an external approach using denotational semantics in the absense of the identity type.

In the following, we will show that for every  $\vdash M : \text{nat}$  if  $\llbracket M \rrbracket = n$  then  $M$  reduces to the numeral  $\underline{n}$ .

- Define a relation  $R_\sigma$  for each type  $\sigma$  between the domain  $\llbracket \sigma \rrbracket = D_\sigma$  and the collection of well-typed closed terms of type  $\sigma$ :

$$R_\sigma \subseteq D_\sigma \times \text{Prg}_\sigma$$

for every type  $\sigma$  where  $\text{Prg}_\sigma = \{ M \mid \vdash M : \sigma \}$ .

- Then show that  $\llbracket M \rrbracket R_\sigma M$  for every well-typed closed term  $M$  of type  $\sigma$ , and by construction  $\llbracket M \rrbracket R_{\text{nat}} M$  is equivalent to that  $\llbracket M \rrbracket = n$  implies  $M \Downarrow \underline{n}$ .

With this property, we can conclude that denotational equivalence entails logical equivalence.<sup>3</sup>

---

<sup>3</sup>But, the converse may fail.

# Logical relation between semantics and syntax

## Definition 8 (Logical relation)

For every type  $\sigma$ , define a relation  $R_\sigma \subseteq D_\sigma \times \text{Prg}_\sigma$  inductively as follows:

- $d R_{\text{nat}} M$  if  $M$  reduces to  $\underline{n}$  whenever  $d$  is a natural number:

$$d R_{\text{nat}} M \quad \text{if} \quad \forall n \in \mathbb{N}. d = n \implies M \Downarrow \underline{n}$$

- for every function type,  $f R_{\sigma \rightarrow \tau} M$  if the outcome is always related whenever the input is related:

$$f R_{\sigma \rightarrow \tau} M \quad \text{if}$$

$$\forall d, N. d R_\sigma N \implies f(d) R_\tau M N$$

For example,  $0 R_{\text{nat}} \text{zero}$ , and  $n + 1 R_{\text{nat}} \text{suc } M$  wherever  $n R_{\text{nat}} M$  for  $n \in \mathbb{N}$ .

# Properties of $R_\sigma$

## Lemma 9

For every type  $\sigma$ , the following statements are true:

- 1 If  $d' \sqsubseteq d$  and  $d \textcolor{red}{R}_\sigma M$ , then  $d' \textcolor{red}{R}_\sigma M$ ;
- 2 For every  $M \in \text{Prg}_\sigma$ , the set

$$R_\sigma M := \{ d \in D_\sigma \mid d \textcolor{red}{R}_\sigma M \}$$

contains  $\perp$  and is closed under directed sups;<sup>4</sup>

- 3 If  $d \textcolor{red}{R}_\sigma M$  and  $M \preceq_\sigma M'$ , then  $d \textcolor{red}{R}_\sigma M'$ .

## Proof.

By induction on  $\sigma$ . □

---

<sup>4</sup>Let  $S$  be an arbitrary directed subset of  $D_\sigma$ , if  $d \textcolor{red}{R}_\sigma M$  for every  $d \in S$ , then  $\bigsqcup S \textcolor{red}{R}_\sigma M$ .

## Lemma 10 (General recursion)

If we have  $f \text{ } R_{\sigma \rightarrow \sigma} (\lambda x. M)$ , then  $\mu(f) \text{ } R_{\sigma} (\Upsilon x. M)$ .

### Proof sketch.

By definition  $\mu(f)$  is the directed supremum of the following directed sequence

$$\perp \sqsubseteq f(\perp) \sqsubseteq f^2(\perp) \sqsubseteq \cdots \sqsubseteq f^i(\perp) \sqsubseteq \cdots,$$

so it suffices to show that

$$f^i(\perp) \text{ } R_{\sigma} (\Upsilon x. M)$$

for every  $i \in \mathbb{N}$ , because  $R_{\sigma}(\Upsilon x. M)$  is closed under directed sups. We prove it by induction on  $i$  and properties of  $R_{\sigma}$ . □

The complete proof is listed below.

For  $i = 0$ : By definition  $f^0(\perp) = \perp$ , so  $\perp R_\sigma (Yx.M)$  follows.

For  $i = n + 1$ : By the assumption  $f R_{\sigma \rightarrow \sigma} (\lambda x.M)$ , it follows that

$$f^{n+1}(\perp) R_\sigma (\lambda x.M) (Yx.M)$$

by the induction hypothesis  $f^n(\perp) R_\sigma (Yx.M)$ .

The RHS reduces to  $M[Yx.M/x]$  and

$Yx.M \rightsquigarrow M[Yx.M/x]$ , so the RHS is logically equivalent to  $Yx.M$ . Hence, it follows that

$$f^{n+1}(\perp) R_\sigma (Yx.M).$$

Therefore, it follows that  $\bigsqcup_{i \in \mathbb{N}} f^i(\perp) R_\sigma (Yx.M)$ .

# Substitution Lemm and completeness

## Lemma 11 (Substitution)

Let  $\Gamma = x_1 : \sigma_1, \dots, x_n : \sigma_n$  be a context and  $d_i \textcolor{red}{R}_{\sigma_i} N_i$  for all  $i = 1, \dots, n$ . For every well-typed term  $M$  we have

$$\llbracket \Gamma \vdash M : \tau \rrbracket (\vec{d}) \textcolor{red}{R}_{\tau} M[\vec{N}/\vec{x}]$$

where  $\vec{d}$  stands for  $(d_1, \dots, d_n)$  and  $\vec{N}$  stands for  $(N_1, \dots, N_n)$ .

## Theorem 12 (Completeness)

For every  $\vdash M : \text{nat}$ , we have  $M \Downarrow \underline{n}$  if  $\llbracket M \rrbracket = n$ .

## Proof.

A special case of the previous lemma:

$$\llbracket \vdash M : \tau \rrbracket (*) \textcolor{red}{R}_{\sigma} M$$

where the LHS is  $\llbracket M \rrbracket$



# Proof of Substitution Lemma

To prove the lemma, do induction on the typing rules for **PCF**. For convenience, we write

$$\vec{d} \text{ } R \vec{N} \quad \text{for} \quad d_i \text{ } R_{\sigma_i} N_i \quad \text{indexed by } i = 1, \dots, n$$

where  $\vec{d}$  stands for  $(d_1, \dots, d_n)$  and  $\vec{N}$  stands for  $(N_1, \dots, N_n)$ .

(z), (s) These two cases follow from  $0 \text{ } R_{\text{nat}} \text{ zero}$  and  $n + 1 \text{ } R_{\text{nat}} \text{ suc } M$  whenever  $n \text{ } R_{\text{nat}} M$ .



(var) To show that

$$\llbracket \dots, x_i : \sigma_i, \dots \vdash x_i : \sigma_i \rrbracket \textcolor{red}{R}_{\sigma_i} x_i[\vec{N}/\vec{x}]$$

we check both sides separately. By definition, we have

$$\llbracket \dots, x_i : \sigma_i, \dots \vdash x_i : \sigma_i \rrbracket(\vec{d}) = d_i \quad \text{and} \quad [\vec{N}/\vec{x}] = N_i.$$

Therefore, from the assumption it follows that  $d_i \textcolor{red}{R}_{\sigma} N_i$  for every  $i$ .

(abs) We need to show that

$$\llbracket \Gamma \vdash \lambda x. M : \tau \rrbracket(\vec{d}) \mathrel{R_{\sigma \rightarrow \tau}} (\lambda x. M)[\vec{N}/\vec{x}] \quad (1)$$

under the induction hypothesis

$$\llbracket \Gamma, x : \sigma \vdash M : \tau \rrbracket(\vec{d}, d) \mathrel{R_{\tau}} M[\vec{N}, N / \vec{x}, x].$$

- For the LHS, we have by definition

$$\begin{aligned} & \llbracket \Gamma \vdash \lambda x. M : \tau \rrbracket(\vec{d})(d) \\ &= \llbracket \Gamma, x : \sigma \vdash M : \tau \rrbracket(\vec{d}, d). \end{aligned}$$

- For the RHS, we have

$$\begin{aligned} & (\lambda x. M)[\vec{N}/\vec{x}] N \\ & \rightsquigarrow (\lambda x. M)[\vec{N}/\vec{x}][N/x] \\ &= (\lambda x. M)[\vec{N}, N / \vec{x}, x] \end{aligned}$$

and it follows that these two terms are logically equivalent. Thus, (1) follows by the definition of  $R_{\sigma \rightarrow \tau}$ .

(Y) We show that  $\llbracket \Gamma \vdash Yx. M : \sigma \rrbracket(\vec{d}) \mathrel{R_\sigma} (Yx. M)[\vec{N}/\vec{x}]$   
under the assumption that

$$\llbracket \Gamma, x : \sigma \vdash M : \sigma \rrbracket(\vec{d}, d) \mathrel{R_\sigma} M[\vec{N}, N/\vec{x}, x] \quad (2)$$

Recall the lemma for general recursion. It suffices to show  $\wedge \llbracket \Gamma, x : \sigma \vdash M : \sigma \rrbracket(\vec{d}) \mathrel{R_{\sigma \rightarrow \sigma}} \lambda x. M[\vec{N}/\vec{x}]$  or, equivalently

$$\llbracket \Gamma, x : \sigma \vdash M : \sigma \rrbracket(\vec{d}, d) \mathrel{R_\sigma} (\lambda x. M[\vec{N}/\vec{x}]) N \quad (3)$$

for every  $d \mathrel{R_\sigma} N$ . The RHS can be reduced to

$$M[\vec{N}/\vec{x}][N/x] = M[\vec{N}, N / \vec{x}, x],$$

so (2) implies (3) by logical equivalence.

(app), (ifz) Exercises.

# Applicative approximation coincides with logical relation

## Lemma 13

For every  $\vdash M : \sigma$  and  $\vdash N : \sigma$ ,

$$M \lesssim_{\sigma} N \quad \text{if and only if} \quad \llbracket M \rrbracket R_{\sigma} N.$$

## Proof.

$M \lesssim_{\sigma} N$ . By adequacy, we have  $\llbracket M \rrbracket R_{\sigma} M$ , so  $\llbracket M \rrbracket R_{\sigma} N$ .

$\llbracket M \rrbracket R_{\sigma} N$ . Prove it by induction on  $\sigma$ .

**nat:** If  $\llbracket M \rrbracket R_{\text{nat}} N$ , then  $N \Downarrow \underline{n}$  whenever  $\llbracket M \rrbracket = n$ .

$\sigma \rightarrow \tau$ : For  $\sigma \rightarrow \tau$ , by adequacy, we have  $\llbracket P \rrbracket R_{\sigma} P$  for every  $P$ , so by assumption and  $\llbracket M P \rrbracket = \llbracket M \rrbracket (\llbracket P \rrbracket) R_{\tau} N P$ . By induction hypothesis,  $M P \lesssim_{\tau} N P$  for every  $P$ , so  $M \lesssim_{\sigma \rightarrow \tau} N$  by definition.



## Corollary 14

*Given two  $\vdash M : \sigma$  and  $\vdash N : \sigma$ , if  $\llbracket M \rrbracket = \llbracket N \rrbracket$ , then  $M$  and  $N$  are logically equivalent.*

### Proof.

- 1 By adequacy  $\llbracket M \rrbracket \textcolor{red}{R} M$  and by assumption  $\llbracket N \rrbracket = \llbracket M \rrbracket \textcolor{red}{R} M$ , it follows that  $N \lesssim M$ .
- 2 Similarly,  $\llbracket M \rrbracket \textcolor{red}{R} N$ , so  $M \lesssim N$ .

Hence,  $M$  and  $N$  are logically equivalent. □

From this property, techniques and results in denotational semantics can be used to argue logical equivalence and reductions.

# Compactness

Recall that the semantics of general recursion is the least upper bound of its finite unfoldings

$$\llbracket Yx. M \rrbracket = \bigsqcup_{i \in \mathbb{N}} \llbracket Y^i x. M \rrbracket$$

where  $Y^i x. M$  is defined inductively by

1  $Y^0 x. M := Yx. x$  and

2  $Y^{n+1} x. M := M[Y^n x. M/x]$

and  $\llbracket Y^i x. M \rrbracket = \llbracket \lambda x. M \rrbracket^i(\perp)$ .

## Theorem 15

*Suppose that  $x \neq y$ ,*

$$y : \sigma \vdash E : \text{nat} \quad \text{and} \quad \vdash Yx. M : \sigma.$$

*If  $E[Yx. M/y] \Downarrow \underline{n}$  then  $E[Y^m x. M/y] \Downarrow \underline{n}$  for some  $m$ .*

## Proof.

By the Substitution Lemma, we have

$$\llbracket E[Yx.M/y] \rrbracket = \llbracket y : \sigma \vdash E : \text{nat} \rrbracket (\llbracket Yx.M \rrbracket).$$

Let  $g := \llbracket y : \sigma \vdash E : \text{nat} \rrbracket$  and  $f := \llbracket x : \sigma \vdash M : \sigma \rrbracket$ .

$$\begin{aligned} \llbracket y : \sigma \vdash E : \text{nat} \rrbracket (\llbracket Yx.M \rrbracket) &= g(\mu f) \\ &= g\left(\bigsqcup_{i \in \mathbb{N}} f^i(\perp)\right) \\ &= \bigsqcup_{i \in \mathbb{N}} (g \circ f^i)(\perp) = n \end{aligned}$$

Therefore there exists some  $m \in \mathbb{N}$  such that  $(g \circ f^m)(\perp) = n$ . By adequacy, it follows that  $E[Y^m x.M/y] \Downarrow \underline{n}$ .  $\square$

# Finite unfoldings approximate general recursion

## Lemma 16

*Suppose that  $x : \sigma \vdash M : \sigma$ . Then for every  $i \in \mathbb{N}$ , we have*

$$Y^i x. M \preceq_{\sigma} Y x. M.$$

The proof is left as an exercise.

## Theorem 17 (Fixed Point Induction)

*Suppose that  $x : \sigma \vdash M : \sigma$ ,  $x : \sigma \vdash N : \sigma$  and*

$$Y^i x. M \simeq_{\sigma} Y^i x. N$$

*for every  $i \in \mathbb{N}$ . Then, we also have*

$$Y x. M \simeq_{\sigma} Y x. N$$



## Proof.

We show that  $Y_X.M \lesssim_\sigma Y_X.N$ , or equivalently  $\llbracket Y_X.M \rrbracket R_\sigma Y_X.N$ , and the other direction follows similarly.

Let  $f := \llbracket x : \sigma \vdash M : \sigma \rrbracket$  and  $g := \llbracket x : \sigma \vdash N : \sigma \rrbracket$ . Since the set

$$R_\sigma(Y_X.N) = \{ d \in D_\sigma \mid d R_\sigma Y_X.N \}$$

is closed under directed supremum, it suffices to show that

$$\llbracket Y^i_X.M \rrbracket R_\sigma Y_X.N$$

for every  $i$ .

By assumption, we have  $\llbracket Y^i_X.M \rrbracket R_\sigma Y^i_X.N$ , so it suffices to show that  $Y^i_X.N \lesssim_\sigma Y_X.N$ . By the previous lemma the statement follows. □