Decidability of Bidirectional Type Checking and Synthesis

See our paper for the algorithm/proof

- For a mode-correct bidirectional type system specified by (Σ, Ω),
  - if $|\Gamma| \vdash_{\Sigma,\Omega} t^{\Rightarrow}$, then it is decidable whether $\Gamma \vdash_{\Sigma,\Omega} t :^{\Rightarrow} A$ for some A;
  - if $|\Gamma| \vdash_{\Sigma,\Omega} t^{\Leftarrow}$, then it is decidable for any A whether $\Gamma \vdash_{\Sigma,\Omega} t :^{\Leftarrow} A$;

Proof (of Theorem 5.7). We prove this statement by induction on the mode derivation $|\Gamma| \vdash_{\Sigma,\Omega} t^{d}$. The two cases $\text{VAR}^{\Rightarrow}$ and $\text{ANNO}^{\Rightarrow}$ are straightforward and independent of mode-correctness. The case $\text{SUB}^{\Leftarrow}$ invokes the uniqueness of synthesised types to refute the case that $\Gamma \vdash_{\Sigma,\Omega} t \Rightarrow B$ but $A \neq B$ for a given type A. The first three cases follow essentially the same reasoning provided by Wadler et al. [30], so we only detail the last case OP, which is new (but has been discussed informally above). For brevity we omit the subscript (Σ, Ω).

For a mode derivation of $|\Gamma| \vdash \mathsf{op}_o(\vec{x}_1.\,t_1;\ldots;\vec{x}_n.\,t_n)^{d}$, we first claim:

Claim. For an argument list $[\Delta_1]A_1^{d_1},\ldots,[\Delta_n]A_n^{d_n}$ and any partial substitution ρ from Ξ to $\emptyset$, either

1. there is a minimal extension $\bar{\rho}$ of ρ such that

$$dom(\bar{\rho}) \supseteq fv^{\Rightarrow}([\Delta_1]A_1^{d_1},\ldots,[\Delta_n]A_n^{d_n}) \text{ and } \Gamma, \vec{x}_i : \Delta_i\langle\bar{\rho}\rangle \vdash t_i \colon A_i\langle\bar{\rho}\rangle^{d_i} \tag{2}$$

   for $i = 1,\ldots,n$, or
2. there is no extension σ of ρ such that (2) holds.

Then, we proceed with a case analysis on d in the mode derivation:

- d is $\Rightarrow$: We apply our claim with the partial substitution ρ_0 defined nowhere.
  1. If there is no $\sigma \geq \rho$ such that (2) holds but $\Gamma \vdash \mathsf{op}_o(\vec{x}_1.\,t_1;\ldots;\vec{x}_n.\,t_n) \Rightarrow A$ for some A, then by inversion we have $\rho\colon \mathsf{Sub}_{\Sigma}(\Xi,\emptyset)$ such that
  $$\Gamma, \vec{x}_i : \Delta_i\langle\rho\rangle \vdash t_i \colon A_i\langle\rho\rangle^{d_i}$$
  for every i. Obviously, $\rho \geq \rho_0$ and $\Gamma, \vec{x}_i : \Delta_i\langle\rho\rangle \vdash t_i \colon A_i\langle\rho\rangle^{d_i}$ for every i, which contradict the assumption that no such extension exists.
  2. If there exists a minimal $\bar{\rho} \geq \rho_0$ defined on $fv^{\Rightarrow}([\Delta_1]A_1^{d_1},\ldots,[\Delta_n]A_n^{d_n})$ such that (2) holds, then by mode-correctness $\bar{\rho}$ is total, and thus
  $$\Gamma \vdash \mathsf{op}_o(\vec{x}_1.\,t_1;\ldots;\vec{x}_n.\,t_n) \Rightarrow A_0\langle\bar{\rho}\rangle\,.$$
- d is $\Leftarrow$: Let A be a type and apply Lemma 5.10 with ρ_0 defined nowhere.
  1. If there is no $\sigma \geq \rho_0$ s.t. $A_0\langle\sigma\rangle = A$ but $\Gamma \vdash \mathsf{op}_o(\vec{x}_1.\,t_1;\ldots;\vec{x}_n.\,t_n) \Leftarrow A$, then inversion gives us a substitution ρ s.t. $A = A_0\langle\rho\rangle$—a contradiction.
  2. If there is a minimal $\bar{\rho} \geq \rho_0$ s.t. $A_0\langle\bar{\rho}\rangle = A$, then apply our claim with $\bar{\rho}$:
     (a) If no $\sigma \geq \bar{\rho}$ satisfies (2) but $\Gamma \vdash \mathsf{op}_o(\vec{x}_1.\,t_1;\ldots;\vec{x}_n.\,t_n) \Leftarrow A$, then by inversion there is γ s.t. $A_0\langle\gamma\rangle = A$ and $\Gamma, \vec{x}_i : \Delta_i\langle\gamma\rangle \vdash t_i \colon A_i\langle\gamma\rangle^{d_i}$ for every i. Given that $\bar{\rho} \geq \rho$ is minimal s.t. $A_0\langle\bar{\rho}\rangle = A$, it follows that γ is an extension of $\bar{\rho}$, but by assumption no such an extension satisfying $\Gamma, \vec{x}_i : \Delta_i\langle\gamma\rangle \vdash t_i \colon A_i\langle\gamma\rangle^{d_i}$ exists, thus a contradiction.
     (b) If there is a minimal $\bar{\bar{\rho}} \geq \bar{\rho}$ s.t. (2), then by mode-correctness $\bar{\bar{\rho}}$ is total and
     $$\Gamma \vdash \mathsf{op}_o(\vec{x}_1.\,t_1;\ldots;\vec{x}_n.\,t_n) \Leftarrow A_0\langle\bar{\bar{\rho}}\rangle$$
     where $A_0\langle\bar{\bar{\rho}}\rangle = A_0\langle\bar{\rho}\rangle = A$ since $\bar{\bar{\rho}}(x) = \bar{\rho}$ for every x in $dom(\bar{\rho})$.

We have proved the decidability by induction on the derivation of $|\Gamma| \vdash_{\Sigma,\Omega} t^{d}$, assuming the claim.

Proof (of Claim). We prove it by induction on the list $[\Delta_1]A_1^{d_1},\ldots,[\Delta_n]A_n^{d_n}$:

1. For the empty list, ρ is the minimal extension of ρ itself satisfying (2) trivially.
2. For $\overrightarrow{[\Delta_i]A_i^{d_i}}, [\Delta_{m+1}]A_{m+1}^{d_{m+1}}$, by induction hypothesis on the list, we have two cases:
   (a) If there is no $\sigma \geq \rho$ s.t. (2) holds for all $1 \leq i \leq m$ but a minimal $\gamma \geq \rho$ such that (2) holds for all $1 \leq i \leq m+1$, then we have a contradiction.
   (b) There is a minimal $\bar{\rho} \geq \rho$ s.t. (2) holds for $1 \leq i \leq m$. By case analysis on d_{m+1}:
   - d_{m+1} is $\Leftarrow$: By mode-correctness, $\Delta_{m+1}\langle\bar{\rho}\rangle$ and $A_{m+1}\langle\bar{\rho}\rangle$ are defined. By the ind. hyp. $\Gamma, \vec{x}_{m+1} : \Delta_{m+1}\langle\bar{\rho}\rangle \vdash t_{m+1} \Leftarrow A_{m+1}\langle\bar{\rho}\rangle$ is decidable. Clearly, if $\Gamma, \vec{x}_{m+1} : \Delta_{m+1}\langle\bar{\rho}\rangle \vdash t_{m+1} \Leftarrow A_{m+1}\langle\bar{\rho}\rangle$ then the desired statement is proved; otherwise we easily derive a contradiction.
   - d_{m+1} is $\Rightarrow$: By mode-correctness, $\Delta_{m+1}\langle\bar{\rho}\rangle$ is defined. By the ind. hyp., '$\Gamma, \vec{x}_{m+1} : \Delta_{m+1}\langle\bar{\rho}\rangle \vdash t_{m+1} \Rightarrow A$ for some A' is decidable:
     i. If $\Gamma, \vec{x}_{m+1} : \Delta_{m+1}\langle\bar{\rho}\rangle \nvdash t_{m+1} \Rightarrow A$ for any A but there is $\gamma \geq \rho$ s.t. (2) holds for $1 \leq i \leq m+1$, then $\gamma \geq \bar{\rho}$. Therefore $\Delta_{m+1}\langle\bar{\rho}\rangle = \Delta_{m+1}\langle\gamma\rangle$, and we derive a contradiction because $\Gamma, \vec{x}_{m+1} : \Delta_{m+1}\langle\bar{\rho}\rangle \vdash t_{m+1} \Rightarrow A_{m+1}\langle\gamma\rangle$.
     ii. If $\Gamma, \vec{x}_{m+1} : \Delta_{m+1}\langle\bar{\rho}\rangle \vdash t_{m+1} \Rightarrow A$ for some A, then Lemma 5.10 gives the following two cases:
       - Suppose no $\sigma \geq \bar{\rho}$ s.t. $A_{m+1}\langle\sigma\rangle = A$ but an extension $\gamma \geq \rho$ s.t. (2) holds for $1 \leq i \leq m+1$. Then, $\gamma \geq \bar{\rho}$ by the minimality of $\bar{\rho}$ and thus $\Gamma, \vec{x}_{m+1} : \Delta_{m+1}\langle\bar{\rho}\rangle \vdash t_{m+1} \Rightarrow A_{m+1}\langle\gamma\rangle$. However, by Lemma 5.6, the synthesised type $A_{m+1}\langle\gamma\rangle$ must be unique, so γ is an extension of $\bar{\rho}$ s.t. $A_{m+1}\langle\gamma\rangle = A$, i.e. a contradiction.
       - If there is a minimal $\bar{\bar{\rho}} \geq \bar{\rho}$ such that $A_{m+1}\langle\bar{\bar{\rho}}\rangle = A$, then it is not hard to show that $\bar{\bar{\rho}}$ is also the minimal extension of ρ such that (2) holds for all $1 \leq i \leq m+1$.

We have proved our claim for any argument list by induction. ■

We have completed the proof of Theorem 5.7. □

$$\Gamma \vdash_{\Sigma,\Omega} t :^{d} A$$

$$\Gamma \vdash_{\Sigma,\Omega} t : A$$

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Soundness

Completeness

$$|\Gamma| \vdash_{\Sigma,\Omega} t^{d}$$

Decoration

Mode

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$$(\Sigma, \Omega)$$

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