

# Syntax and Semantics of Cedille

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## 1 Introduction

The type theory of Cedille is called the Calculus of Dependent Lambda Eliminations (CDLE). This document presents the version of CDLE as of December 1, 2020. We have made many changes from the first paper on CDLE [14], mostly in the form of dropping constructs we discovered (to our surprise) could be derived [15]. I have also omitted *lifting* – a technique for large eliminations with lambda encodings – in this document’s version of CDLE. Some uses of lifting can be simulated other ways within the system, though the limits of this are still under investigation. We also include a construct  $\delta$ , for deriving a contradiction from a proof that lambda-encoded true equals lambda-encoded false. This also compensates somewhat for the lack of lifting.

At a high level, CDLE is an extrinsic (i.e., Curry-style) type theory extending the Calculus of Constructions with three additional constructs, which allow deriving induction principles within the theory, for inductive datatypes. The goal is to support usual idioms of dependently typed programming and proving as in Agda or similar tools, but using pure lambda encodings for all data, and requiring a much smaller core theory.

The current Cedille implementation of CDLE extends the system described below with a number of features intended to make programming in the system more convenient and with less redundancy. These features all compile away to a slightly simplified version of the theory presented in this document, called Cedille Core, described here: <https://github.com/astump/cedille-core-spec>.

## 2 Classification Rules

The classification rules are given in Figures 1, 2, and 3. For brevity, we take these figures as implicitly specifying the syntax of kinds  $\kappa$ , types  $T$ , and annotated terms  $t$ ; these may use term variables  $x$  and type variables  $X$ , which we assume come from distinct sets. So terms and types are syntactically distinguished. The typing rules (Figure 3) are bidirectional [13], while the kinding and superkinding rules (Figure 2 and 1) are only synthesizing. We write  $\Leftrightarrow$  to range over  $\{\Leftarrow, \Rightarrow\}$ , and when this symbol occurs multiple times in a rule, it is intended that such occurrences be read the same way (i.e., either all as  $\Leftarrow$  or all as  $\Rightarrow$ ). We follow the syntax of our implementation Cedille, which distinguishes application of a term or type  $e$  to a type ( $e \cdot T$ ), from application to a term ( $e \ t$ ), and application to an erased term argument ( $e \ -t$ , in which case  $e$  must be a term). The rules are intended, with a few points of nondeterminism, to be read bottom-up (in a standard way; c.f. [11, 12]) as an algorithm for synthesizing a classifier from a context and an expression ( $\Rightarrow$ ) (*type synthesis*) or checking an expression against a classifier in context ( $\Leftarrow$ ) (*type checking*). Specifically, nondeterminism arises in the two rules allowing both synthesized and checked types to be reduced, written  $T \rightsquigarrow_{\beta}^* T'$ , so that type constructors required by other rules may be revealed. The reduction relation  $\rightsquigarrow_{\beta}$  is call-by-name normalization, as this is sufficient to reduce types to weak head normal form.

The classification rules refer to an erasure function, defined in Figure 4. The type theory is *extrinsic* (aka,

$$\frac{}{\Gamma \vdash \star} \quad \frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash \kappa}{\Gamma \vdash \Pi x : T. \kappa} \quad \frac{\Gamma \vdash \kappa' \quad \Gamma, X : \kappa' \vdash \kappa}{\Gamma \vdash \Pi X : \kappa'. \kappa}$$

Figure 1: Rules for checking that a kind is well-formed ( $\Gamma \vdash \kappa$ )

$$\begin{array}{c} \frac{(X : \kappa) \in \Gamma}{\Gamma \vdash X \Rightarrow \kappa} \quad \frac{\Gamma \vdash \kappa \quad \Gamma, X : \kappa \vdash T \Rightarrow \star}{\Gamma \vdash \forall X : \kappa. T \Rightarrow \star} \\[10pt] \frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash T' \Rightarrow \star}{\Gamma \vdash \forall x : T. T' \Rightarrow \star} \quad \frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash T' \Rightarrow \star}{\Gamma \vdash \Pi x : T. T' \Rightarrow \star} \\[10pt] \frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash T' \Rightarrow \kappa}{\Gamma \vdash \lambda x : T. T' \Rightarrow \Pi x : T. \kappa} \quad \frac{\Gamma \vdash \kappa \quad \Gamma, X : \kappa \vdash T' \Rightarrow \kappa'}{\Gamma \vdash \lambda X : \kappa. T' \Rightarrow \Pi X : \kappa. \kappa'} \\[10pt] \frac{\Gamma \vdash T \Rightarrow \Pi x : T'. \kappa \quad \Gamma \vdash t \Leftarrow T'}{\Gamma \vdash T t \Rightarrow [t/x]\kappa} \quad \frac{\Gamma \vdash T_1 \Rightarrow \Pi X : \kappa_2. \kappa_1 \quad \Gamma \vdash T_2 \Rightarrow \kappa'_2 \quad \kappa_2 \cong \kappa'_2}{\Gamma \vdash T_1 \cdot T_2 \Rightarrow [T_2/X]\kappa_1} \\[10pt] \frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash T' \Rightarrow \star}{\Gamma \vdash \iota x : T. T' \Rightarrow \star} \quad \frac{FV(t \ t') \subseteq \text{dom}(\Gamma)}{\Gamma \vdash \{t \simeq t'\} : \star} \end{array}$$

Figure 2: Rules for synthesizing a kind for a type ( $\Gamma \vdash T \Rightarrow \kappa$ )

Curry-style), and hence we only consider erasures  $|t|$  of terms when testing for  $\beta\eta$ -equivalence. This applies also to the conversion relation  $T \cong T'$  on types and  $\kappa \cong \kappa'$  on kinds, whose central rules are given in Figure 5. That figure omits the various congruence rules needed to equate bigger expressions by equating subexpressions. The main ideas of conversion shown in the figure are to use  $\beta$ -equivalence at the type level, and  $\beta\eta$ -equivalence of erased terms at the term level. Note in particular that for conversion we check full  $\beta$  and  $\beta\eta$ -equivalence, not just equivalence with respect to call-by-name normalization.

## 2.1 Overview of the constructs

CDLE has as a subsystem the extrinsic Calculus of Constructions (CC). We have dependent types  $\Pi x : T. T'$  and kinds  $\Pi x : T. \kappa$ , as well as term- and type-level quantification over (possibly higher-kinded) types  $\forall X : \kappa. T$  and  $\Pi X : \kappa. \kappa'$ . We use  $\forall$  when the corresponding argument will be erased, and  $\Pi$  when it will be retained. Since we do not erase term or type arguments from type-level applications, we thus write  $\Pi X : \kappa. \kappa'$  instead of  $\forall X : \kappa. \kappa'$ . We write  $\lambda$  to correspond to  $\Pi$  and  $\Lambda$  to correspond to  $\forall$ . As noted above, application to a type is denoted with center dot  $(\cdot)$ .

To Curry-style CC, CDLE adds: implicit products, introduced originally by Miquel [9]; a primitive equality type  $\{t \simeq t'\}$ ; and dependent intersection types  $\iota x : T. T'$ , introduced by Kopylov [8]. Implicit products are used for erased arguments to functions, found also in systems like Agda (cf. [10]). Dependent intersections are a rather exotic construct allowing us to assign type  $\iota x : T'. T$  to erased term  $t$  when we can assign  $T'$  to  $t$ , and also assign  $[t/x]T$  to  $t$ . For an annotated introduction form, we write  $[t, t']$ , where  $t$  checks against type  $T'$ ,  $t'$  checks against  $[t/x]T$ , and  $t$  and  $t'$  have  $\alpha\beta\eta$ -equivalent erasures. Dependent intersections thus enable a controlled form of self-reference in the type. Previous work showed how to use this to derive induction for Church-encoded natural numbers [15]. We will see below further uses of this construct.

The typing rules include conversion checks in a few places; e.g., as standardly, when switching from checking to synthesizing mode. As mentioned earlier, two rules near the top of Figure 3 state that one may (nondeterministically)  $\beta$ -reduce the type one is synthesizing or checking, before proceeding. This allows reduction to weak head normal form, to match the form of type required by other rules. Finally, we include the construct

$$\begin{array}{c}
\frac{(x:t) \in \Gamma}{\Gamma \vdash x \Rightarrow T} \qquad \frac{\Gamma \vdash t \Leftarrow T \quad T' \rightsquigarrow_{\beta}^* T}{\Gamma \vdash t \Leftarrow T'} \\
\\
\frac{\Gamma \vdash t \Rightarrow T \quad T \rightsquigarrow_{\beta}^* T'}{\Gamma \vdash t \Rightarrow T'} \qquad \frac{\Gamma \vdash t \Rightarrow T' \quad T' \cong T}{\Gamma \vdash t \Leftarrow T} \\
\\
\frac{\Gamma, x:T \vdash t \Leftarrow T'}{\Gamma \vdash \lambda x. t \Leftarrow \Pi x:T. T'} \qquad \frac{\Gamma \vdash t \Rightarrow \Pi x:T'. T \quad \Gamma \vdash t' \Leftarrow T'}{\Gamma \vdash t \cdot t' \Rightarrow [t'/x]T} \\
\\
\frac{\Gamma, X:\kappa \vdash t \Leftarrow T}{\Gamma \vdash \Lambda X. t \Leftarrow \forall X:\kappa. T} \qquad \frac{\Gamma \vdash t \Rightarrow \forall X:\kappa. T \quad \Gamma \vdash T' \Leftarrow \kappa}{\Gamma \vdash t \cdot T' \Rightarrow [T'/X]T} \\
\\
\frac{\Gamma, x:T' \vdash t \Leftarrow T \quad x \notin FV(|t|)}{\Gamma \vdash \Lambda x. t \Leftarrow \forall x:T'. T} \qquad \frac{\Gamma \vdash t \Rightarrow \forall x:T'. T \quad \Gamma \vdash t' \Leftarrow T'}{\Gamma \vdash t \cdot t' \Rightarrow [t'/x]T} \\
\\
\frac{\Gamma \vdash t \Leftarrow T \quad \Gamma \vdash t' \Leftarrow [t/x]T' \quad |t| =_{\beta\eta} |t'|}{\Gamma \vdash [t, t'] \Leftarrow \iota x:T. T'} \qquad \frac{\Gamma \vdash t \Rightarrow \iota x:T. T'}{\Gamma \vdash t.1 \Rightarrow T} \\
\\
\frac{\Gamma \vdash t \Rightarrow \iota x:T. T'}{\Gamma \vdash t.2 \Rightarrow [t.1/x]T'} \qquad \frac{\Gamma \vdash FV(t' \ t_1 \ t_2) \subseteq \text{dom}(\Gamma) \quad |t_1| =_{\beta\eta} |t_2|}{\Gamma \vdash \beta\{t'\} \Leftarrow \{t_1 \simeq t_2\}} \\
\\
\frac{\Gamma \vdash t \Leftarrow \{\lambda x. \lambda y. x \simeq \lambda x. \lambda y. y\}}{\Gamma \vdash \delta \cdot t \Leftarrow T} \qquad \frac{\Gamma \vdash t' \Rightarrow \{t_1 \simeq t_2\} \quad \Gamma \vdash [t_2/x]T \Rightarrow \star \quad \Gamma \vdash t \Rightarrow T' \quad T' \cong [t_1/x]T}{\Gamma \vdash \rho \ t' @x. T \cdot t \Rightarrow [t_2/x]T} \\
\\
\frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma \vdash t \Leftarrow T}{\Gamma \vdash \chi \ T \cdot t \Rightarrow T} \qquad \frac{\Gamma \vdash t \Rightarrow \{t' \simeq t''\} \quad \Gamma \vdash t' \Leftarrow T}{\Gamma \vdash \varphi \ t \cdot t' \{t''\} \Leftarrow T}
\end{array}$$

Figure 3: Rules for checking a term against a well-kinded type ( $\Gamma \vdash t \Leftarrow T$ ) and synthesizing a type for a term ( $\Gamma \vdash t \Rightarrow T$ )

$$\begin{array}{ll}
|x| & = x & |\lambda x:t'. t| & = \lambda x. |t| \\
|t \ t'| & = |t| \ |t'| & |t \cdot T| & = |t| \\
|\Lambda x:t'. t| & = |t| & |t \cdot t'| & = |t| \\
|[t, t']| & = |t| & |t.1| & = |t| \\
|t.2| & = |t| & |\beta\{t\}| & = |t| \\
|\delta \cdot t| & = \lambda x. x & |\rho \ t @x. T \cdot t'| & = |t'| \\
|\varphi \ t \cdot t' \{t''\}| & = |t''| & |\chi \ T \cdot t'| & = |t'|
\end{array}$$

Figure 4: Erasure for annotated terms

$$\begin{array}{c}
\frac{T \rightsquigarrow_{\beta}^* T_1 \quad T' \rightsquigarrow_{\beta}^* T_2 \quad T_1 \cong^t T_2}{T \cong T'} \qquad \frac{T \cong^t T'}{T \cong T'} \\
\\
\frac{T \cong^t T' \quad |t| =_{\beta\eta} |t'|}{T \cdot t \cong^t T \cdot t'} \qquad \frac{|t_1| =_{\beta\eta} |t'_1| \quad |t_2| =_{\beta\eta} |t'_2|}{\{t_1 \simeq t_2\} \cong^t \{t'_1 \simeq t'_2\}}
\end{array}$$

Figure 5: Non-congruence rules for conversion

$\chi T - t$  which allows a term whose type can only be checked to be ascribed a user-provided type so that the whole expression may occur where type synthesis is expected.

We have modified the rules for equality types  $\{t \simeq t'\}$  so that we require nothing of  $t$  and  $t'$  except that the set  $\text{dom}(\Gamma)$  of variables declared by  $\Gamma$  includes their free variables  $FV(t\ t')$ . Further modifications over the version of CDLE in [15] are:

- To prove  $\{t_1 \simeq t_2\}$  for definitionally equal terms (that is, terms that are  $\beta\eta$ -equivalent modulo erasure), one now writes  $\beta\{t'\}$ , with the critical idea that  $|\beta\{t'\}|$  erases to (a possibly unrelated)  $|t'|$ . We call this the **Kleene trick**, because it goes back to Kleene’s numeric realizability [7], which accepts any number  $n$  as a realizer of a true equation. Here, we accept any term  $t'$  as a realizer of  $\{t_1 \simeq t_2\}$ , provided the free variables of  $t'$  are declared in the context. This means that in Cedille, any such term — even otherwise untypable terms, non-normalizing terms, etc. — prove trivially true equations.
- The  $\rho$  construct allows one to rewrite occurrences of  $t_1$  to  $t_2$  in the synthesized type of the second subexpression. The version presented here requires a “guide”  $@x.T$ , where  $x$  indicates the occurrences of  $t_1$  to substitute.  $[t_2/x]T$  must be a well-kinded type, but it is not required that  $[t_1/x]T$  also be, only that  $T'$  and  $[t_1/x]T$  are convertible modulo erasure — it is not guaranteed that  $t_1$  and  $t_2$  will have the same type, or will even be typeable. In Cedille, the guide is optional and the construct may be used to rewrite a contextually given type; a heuristic is used to produce a resulting type that is well-kinded whose details are beyond the scope of this document.
- We adopt a strong form of Nuprl’s **direct computation rules** [2]: If we have a term  $t'$  of type  $T$  and a proof  $t$  that  $\{t' \simeq t''\}$ , then we may conclude that  $t''$  has type  $T$  by writing the annotated term  $\varphi\ t - t' \{t''\}$ , which erases to  $t''$ .
- Where the previous version of CDLE uses  $\beta$ -equivalence for (erased) terms, we here adopt  $\beta\eta$ -equivalence. This allows us to observe in many cases that retyping functions are actually  $\beta\eta$ -equivalent to  $\lambda x.x$ . While  $\beta\eta$ -equivalence takes more work to incorporate into intrinsic type theory [5], it raises no difficulties for our extrinsic one.
- In this version, we add an explicit axiom  $\delta$  saying that Church-encoded boolean *true* is different from *false*. In the implementation of Cedille, this is generalized to the rule where the proof  $t$  synthesizes an equation  $\{t_1 \simeq t_2\}$  in which  $|t_1|$  and  $|t_2|$  are separable using the *Böhm-out algorithm* [1].

In the first version of CDLE, such an axiom was derivable from *lifting*, a construct allowing simply typable terms to be lifted to the type level [14]. We omit lifting in this new version of CDLE, because while sound, lifting as defined in that previous work is complicated and appears to be incomplete. Developing a new form of lifting remains to future work.

The equality type remains **intensional**: we equate terms iff they are  $\beta\eta$ -equal.

## 2.2 Semantics and metatheory

Figure 6 gives a realizability semantics for types and kinds, following the semantics given in the previous papers on CDLE [15, 14]. Details of this semantics are presented further in Section 2.3 below. Using the semantics and the definition in Figure 7 of  $\llbracket \Gamma \rrbracket$ , we can prove the following theorem:

**Theorem 1** (Soundness). *Suppose  $(\sigma, \rho) \in \llbracket \Gamma \rrbracket$ . Then we have:*

1. *If  $\Gamma \vdash \kappa$ , then  $\llbracket \kappa \rrbracket_{\sigma, \rho}$  is defined.*
2. *If  $\Gamma \vdash T \Rightarrow \kappa$ , then  $\llbracket T \rrbracket_{\sigma, \rho} \in \llbracket \kappa \rrbracket_{\sigma, \rho}$ .*
3. *If  $\Gamma \vdash t \Rightarrow T$  then  $[\sigma|t]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ .*
4. *If  $\Gamma \vdash t \Leftarrow T$  and  $\llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ , then  $[\sigma|t]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ .*
5. *If  $T \cong T'$  or  $T \cong^t T'$  and  $\llbracket T \rrbracket_{\sigma, \rho}$  and  $\llbracket T' \rrbracket_{\sigma, \rho}$  are both defined, then they are equal.*

6. If  $\kappa \cong \kappa'$  and  $\llbracket \kappa \rrbracket_{\sigma, \rho}$  and  $\llbracket \kappa' \rrbracket_{\sigma, \rho}$  are both defined, then they are equal.

An easy corollary, by the semantics of  $\forall$ -types, is then:

**Theorem 2** (Logical consistency). *There is no term  $t$  such that  $\vdash t : \forall X : \star. X$ .*

It may worry some readers that we have:

**Observation 3.** *There are typable terms  $t$  which fail to normalize.*

Defining **Top** to be  $\{\lambda x. x \simeq \lambda x. x\}$ , we may assign **Top** to any closed term  $\mathbf{t}$ , including non-normalizing ones. In our annotated syntax, we write  $\beta\{\mathbf{t}\}$ . Even without this, the presence of  $\delta$  in combination with  $\varphi$  allows us to type non-normalizing terms assuming an erased argument  $x$  of type  $\{tt \simeq ff\}$  for Church-encoded booleans  $tt$  and  $ff$ . For example,  $\delta -x$  has type  $\{\lambda x. x \simeq \lambda x. x\}$ , and with  $\varphi$  we can use this to type  $\Omega$  by changing the typed term  $\text{id} \cdot \text{True id}$ , where **True** is  $\forall X : \star. X \rightarrow X$ . But failure of normalization does not impinge on Theorem 2. Extensional Martin-Löf type theory (MLTT) is also non-normalizing, for a very similar reason, but this fact does not contradict its logical soundness [4]. In CDLE, the guarantees one gets about the behavior of terms are expressed almost entirely in their types. If the types are weak, then not much is guaranteed; but stronger types can guarantee properties like normalization, as demonstrated by the following theorem:

**Theorem 4** (Call-by-name normalization of functions). *Suppose  $\Gamma \vdash t : T$ ,  $t$  is closed, and there exists a closed term  $t'$  which erases to  $\lambda x. x$  and whose type is  $T \rightarrow \Pi x : T_1. T_2$  for some  $T_1, T_2$ . Then  $|t|$  is call-by-name normalizing.*

Given the lack of normalization in general, several checks in the typing rules – for things like  $t =_{\beta\eta} t'$  – are formally undecidable. In practice, we simply impose a bound on the number of steps of reduction, and thus restore formal decidability (we are checking “typable within a given budget”). In practice, the same is done for Coq and Agda, where type checking is decidable but, in general, infeasible (since one may write astronomically slow terminating functions).

Finally, in line with ideas recently advocated by Dreyer, we are less concerned with syntactic type preservation as we are with *semantic* type preservation [3]. Note that by construction, semantic types  $\llbracket T \rrbracket_{\sigma, \rho}$  are preserved by  $\beta\eta$ -reduction:

**Theorem 5** (Semantic type preservation). *If  $t \rightsquigarrow_{\beta\eta} t'$  and  $t \in \llbracket T \rrbracket_{\sigma, \rho}$ , then  $t' \in \llbracket T \rrbracket_{\sigma, \rho}$ .*

Confluence of  $\beta\eta$ -reduction for (erased) terms is nothing other than confluence of untyped lambda calculus. This is because, as easily verified by inspecting Figure 4, the erasure function maps annotated terms  $t$  to terms  $|t|$  of pure untyped lambda calculus.

We make a concession to syntactic classifier preservation in the case of types and kinds. During type inference, types may be reduced using a call-by-name operational semantics to reveal type constructors. With the removal of lifting from CDLE, terms cannot compute types and so no terms need to be reduced during this process.

**Theorem 6** (Syntactic kind preservation). *If  $\Gamma \vdash T \Rightarrow \kappa$  and  $T \rightsquigarrow_{\beta} T'$  then  $\Gamma \vdash T' \Rightarrow \kappa'$  for some  $\kappa'$  such that  $\kappa \cong \kappa'$ .*

## 2.3 Some details about the semantics and the proof of Theorem 1

Following the development in [14], we work with set-theoretic partial functions for the semantics of higher-kinded types. Types are interpreted as  $\beta\eta$ -closed sets of closed terms. Let  $\mathcal{L}$  be the set of closed terms of pure lambda calculus (differently from [14], we include all terms at this point, even non-normalizing ones). We write  $=_{c\beta\eta}$  for standard  $\beta\eta$ -equivalence of pure lambda calculus, restricted to closed terms; and  $[t]_{c\beta\eta}$  for  $\{t' \mid t =_{c\beta\eta} t'\}$ . This is extended to sets  $S$  of terms by writing  $[S]_{c\beta\eta}$  for  $\{[t]_{c\beta\eta} \mid t \in S\}$ . In a few places we write  $nf(t)$  for the (unique)  $\beta\eta$ -normal form of term  $t$ , if it has one. If (in our meta-language) we affirm

$$\begin{aligned}
\llbracket X \rrbracket_{\sigma, \rho} &= \rho(X) \\
\llbracket \Pi x : T_1. T_2 \rrbracket_{\sigma, \rho} &= [\{\lambda x. t \mid \forall E \in \llbracket T_1 \rrbracket_{\sigma, \rho}. \\
&\quad [[\zeta(E)/x]t]_{c\beta\eta} \in \llbracket T_2 \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho} \wedge t = |t|\}]_{c\beta\eta} \\
\llbracket \forall X : \kappa. T \rrbracket_{\sigma, \rho} &= \cap \{ \llbracket T \rrbracket_{\sigma, \rho[X \mapsto S]} \mid S \in \llbracket \kappa \rrbracket_{\sigma, \rho} \} \\
\llbracket \forall x : T. T' \rrbracket_{\sigma, \rho} &= \cap_{\star} \{ \llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho} \mid E \in \llbracket T \rrbracket_{\sigma, \rho} \} \\
\llbracket \iota x : T. T' \rrbracket_{\sigma, \rho} &= \{ E \in \llbracket T \rrbracket_{\sigma, \rho} \mid E \in \llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho} \} \\
\llbracket \lambda X : \kappa. T \rrbracket_{\sigma, \rho} &= (S \in \llbracket \kappa \rrbracket_{\sigma, \rho} \mapsto \llbracket T \rrbracket_{\sigma, \rho[X \mapsto S]}) \\
\llbracket \lambda x : T. T' \rrbracket_{\sigma, \rho} &= (E \in \llbracket T \rrbracket_{\sigma, \rho} \mapsto \llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}) \\
\llbracket T \ T' \rrbracket_{\sigma, \rho} &= \llbracket T \rrbracket_{\sigma, \rho} (\llbracket T' \rrbracket_{\sigma, \rho}) \\
\llbracket T \ t \rrbracket_{\sigma, \rho} &= \llbracket T \rrbracket_{\sigma, \rho} ([\sigma|t]_{c\beta\eta}) \\
\llbracket \{t \simeq t'\} \rrbracket_{\sigma, \rho} &= [\{t'' \mid \sigma|t| =_{\beta\eta} \sigma|t'| \wedge t'' = |t''|\}]_{c\beta\eta} \\
&\quad \text{if } FV(t \ t') \subseteq \text{dom}(\sigma) \\
\llbracket \star \rrbracket_{\sigma, \rho} &= \mathcal{R} \\
\llbracket \Pi x : T. \kappa \rrbracket_{\sigma, \rho} &= (E \in \llbracket T \rrbracket_{\sigma, \rho} \rightarrow \llbracket \kappa \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}), \\
&\quad \text{if } \llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R} \\
\llbracket \Pi x : \kappa. \kappa' \rrbracket_{\sigma, \rho} &= (S \in \llbracket \kappa \rrbracket_{\sigma, \rho} \rightarrow \llbracket \kappa' \rrbracket_{\sigma, \rho[X \mapsto S]}) \\
\cap_{\star} X &= \begin{cases} \cap X, & \text{if } X \neq \emptyset \\ [\mathcal{L}]_{c\beta\eta}, & \text{otherwise} \end{cases}
\end{aligned}$$

Figure 6: Semantics for types and kinds

$$\begin{aligned}
(\sigma \uplus [x \mapsto t], \rho) \in \llbracket \Gamma, x : T \rrbracket &\Leftrightarrow (\sigma, \rho) \in \llbracket \Gamma \rrbracket \wedge [t]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R} \wedge t = |t| \\
(\sigma, \rho \uplus [X \mapsto S]) \in \llbracket \Gamma, X : \kappa \rrbracket &\Leftrightarrow (\sigma, \rho) \in \llbracket \Gamma \rrbracket \wedge S \in \llbracket \kappa \rrbracket_{\sigma, \rho} \\
(\emptyset, \emptyset) \in \llbracket \cdot \rrbracket &
\end{aligned}$$

Figure 7: Semantics of typing contexts  $\Gamma$

a statement involving application of a partial function, then it is to be understood that that application is defined.

**Definition 7** (Reducibility candidates).  $\mathcal{R} := \{[S]_{c\beta\eta} \mid S \subseteq \mathcal{L}\}$ .

Throughout the development we find it convenient to use a **choice function**  $\zeta$ . Given any nonempty set  $E$  of terms,  $\zeta$  returns some element of  $E$ . Note that if  $a \in A \in \mathcal{R}$ , then  $a$  is a nonempty set of terms of pure lambda calculus; it can also happen that  $A \in \mathcal{R}$  is empty. The proof of Theorem 1 (see appendix) is then a straightforward adaptation of [14].

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## A Proof of Theorem 1

First a few lemmas (easy proofs omitted):

**Lemma 8.**  $\llbracket \kappa \rrbracket_{\sigma, \rho}$  is nonempty if defined.

**Lemma 9.** If  $E$  is nonempty, then  $[\zeta(E)]_{c\beta\eta} = E$

**Lemma 10.** The set  $\mathcal{R}$  ordered by subset forms a complete lattice, with greatest element  $[\mathcal{L}]_{c\beta\eta}$  and greatest lower bound of a nonempty set of elements given by intersection. Also,  $\emptyset$  is the least element.

**Lemma 11** (Term substitution and interpretation). If  $t' =_{c\beta\eta} \sigma|t|$ , then:

- $\llbracket T \rrbracket_{\sigma[x \mapsto t'], \rho} = \llbracket [t/x]T \rrbracket_{\sigma, \rho}$
- $\llbracket \kappa \rrbracket_{\sigma[x \mapsto t'], \rho} = \llbracket [t/x]\kappa \rrbracket_{\sigma, \rho}$

**Lemma 12** (Type substitution and interpretation). •  $\llbracket T \rrbracket_{\sigma, \rho[X \mapsto \llbracket T' \rrbracket_{\sigma, \rho}]} = \llbracket [T'/X]T \rrbracket_{\sigma, \rho}$

- $\llbracket \kappa \rrbracket_{\sigma, \rho[X \mapsto \llbracket T' \rrbracket_{\sigma, \rho}]} = \llbracket [T'/X]\kappa \rrbracket_{\sigma, \rho}$

**Lemma 13.** If  $T \rightsquigarrow_{\beta}^* T'$  and  $\llbracket T \rrbracket_{\sigma, \rho}$  is defined, then  $\llbracket T' \rrbracket_{\sigma, \rho}$  is also defined and equals  $\llbracket T \rrbracket_{\sigma, \rho}$ .

*Proof.* This follows by induction on the reduction derivation, making use of the previous substitution lemmas.  $\square$

*Soundness (Theorem 1).* The following proof is adapted from [14]. It proceeds by mutual induction on the assumed typing, kinding, or superkinding derivation, for each part of the lemma. We prove the parts successively.

### A.1 Proof of part (1)

Case:

$$\overline{\Gamma \vdash \star}$$

$\llbracket \star \rrbracket_{\sigma, \rho}$  is just  $\mathcal{R}$ , which is defined.

Case:

$$\frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash \kappa}{\Gamma \vdash \Pi x : T. \kappa}$$

By the IH,  $\llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ , and so  $\llbracket \Pi x : T. \kappa \rrbracket_{\sigma, \rho}$  is  $(E \in \llbracket T \rrbracket_{\sigma, \rho} \rightarrow \llbracket \kappa \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho})$ . The latter quantity is defined if for all  $E \in \llbracket T \rrbracket_{\sigma, \rho}$ ,  $\llbracket \kappa \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$  is, too. Since  $\llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ , every element  $E$  of  $\llbracket T \rrbracket_{\sigma, \rho}$  is nonempty, as noted above, so  $\zeta(E)$  is defined. We may apply the IH to the second premise, since  $(\sigma[x \mapsto \zeta(E)], \rho) \in \llbracket \Gamma, x : T \rrbracket$ , because  $E \in \llbracket T \rrbracket_{\sigma, \rho}$  (by assumption) and  $[\zeta(E)]_{c\beta\eta} = E$ . This gives definedness of the semantics of the  $\Pi$ -kind.

Case:

$$\frac{\Gamma \vdash \kappa' \quad \Gamma, X : \kappa' \vdash \kappa}{\Gamma \vdash \Pi X : \kappa'. \kappa}$$



We must show  $(S \in \llbracket \kappa \rrbracket_{\sigma, \rho} \rightarrow \llbracket \kappa \rrbracket_{\sigma, \rho[X \mapsto S]})$  is defined. This is true if  $\llbracket \kappa \rrbracket_{\sigma, \rho}$  is defined, which is the case by the IH applied to the first premise; and if for all  $S \in \llbracket \kappa \rrbracket_{\sigma, \rho}$ ,  $\llbracket \kappa \rrbracket_{\sigma, \rho[X \mapsto S]}$  is defined. The latter is true by the IH applied to the second premise.

## A.2 Proof of part (2)

Case:

$$\frac{(X : \kappa) \in \Gamma}{\Gamma \vdash X \Rightarrow \kappa}$$

From the definition of  $\llbracket \Gamma \rrbracket$ , we obtain  $\rho(x) \in \llbracket \kappa \rrbracket_{\sigma, \rho}$ .

Case:

$$\frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash T' \Rightarrow \star}{\Gamma \vdash \Pi x : T. T' \Rightarrow \star}$$

We must show  $\llbracket \Pi x : T. T' \rrbracket_{\sigma, \rho} \in \mathcal{R}$ . The semantics defines  $\llbracket \Pi x : T. T' \rrbracket_{\sigma, \rho}$  to be  $[A]_{c\beta\eta}$  for a certain  $A$ , where if  $A$  is defined, then  $A \subseteq \mathcal{L}$ . So it suffices to show definedness. By the IH for the first premise,  $\llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ . This means that if  $E \in \llbracket T \rrbracket_{\sigma, \rho}$ ,  $\zeta(E)$  is defined. We can then apply the IH to the second premise, since  $\sigma[x \mapsto \zeta(E)] \in \llbracket \Gamma, x : T \rrbracket$ , to obtain definedness of  $\llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$ .

Case:

$$\frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash T' \Rightarrow \star}{\Gamma \vdash \forall x : T. T' \Rightarrow \star}$$

By the IH for the second premise,  $\llbracket T_2 \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho} \in \mathcal{R}$ , for every  $E \in \llbracket T_1 \rrbracket_{\sigma, \rho}$  where  $\llbracket T_1 \rrbracket_{\sigma, \rho} \in \mathcal{R}$ . By the IH for the first premise, we indeed have  $\llbracket T_1 \rrbracket_{\sigma, \rho} \in \mathcal{R}$ . So if  $\llbracket T_1 \rrbracket_{\sigma, \rho}$  is non-empty, then the intersection of all the sets  $\llbracket T_2 \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$  where  $E \in \llbracket T_1 \rrbracket_{\sigma, \rho}$  is a reducibility candidate, since each of those sets is. By the semantics of  $\forall$ -types quantifying over terms, this is sufficient. If  $\llbracket T_1 \rrbracket_{\sigma, \rho}$  is empty, then the interpretation of the  $\forall$ -type is  $[\mathcal{L}]_{c\beta\eta}$  by the definition of  $\cap_\star$ , and this is in  $\mathcal{R}$ .

Case:

$$\frac{\Gamma \vdash \kappa \quad \Gamma, X : \kappa \vdash T \Rightarrow \star}{\Gamma \vdash \forall X : \kappa. T \Rightarrow \star}$$

Similarly to the previous case: by the IH for the second premise,  $\llbracket T_2 \rrbracket_{\sigma, \rho[X \mapsto S]} \in \mathcal{R}$ , for every  $S \in \llbracket \kappa \rrbracket_{\sigma, \rho}$ . By the IH part for the first premise,  $\llbracket \kappa \rrbracket_{\sigma, \rho}$  is defined. So the intersection of all the sets  $\llbracket T_2 \rrbracket_{\sigma, \rho[X \mapsto S]}$  where  $S \in \llbracket \kappa \rrbracket_{\sigma, \rho}$  is a reducibility candidate, since each of those sets is. The intersection is nonempty, since  $\llbracket \kappa \rrbracket_{\sigma, \rho}$  is (as stated in a lemma above). By the semantics of  $\forall$ -types quantifying over types, this is sufficient.

Case:

$$\frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash T' \Rightarrow \star}{\Gamma \vdash \iota x : T. T' \Rightarrow \star}$$

The set  $\llbracket \iota x : T. T' \rrbracket_{\sigma, \rho}$  is explicitly defined to be a subset of  $\llbracket T \rrbracket_{\sigma, \rho}$ , which is in  $\mathcal{R}$ , by the IH applied to the first premise. Since for any  $A \subseteq \mathcal{L}$ ,  $[A]_{c\beta\eta}$  is in  $\mathcal{R}$ , to show that  $\llbracket \iota x : T. T' \rrbracket_{\sigma, \rho}$  is also in  $\mathcal{R}$  it suffices to show definedness of  $\llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$  (which is used in the predicate picking out the particular subset of  $\llbracket T \rrbracket_{\sigma, \rho}$ ), for  $E \in \llbracket T \rrbracket_{\sigma, \rho}$ . For such  $E$ ,  $\zeta(E)$  is defined (since  $\llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$  and hence  $E \in \llbracket T \rrbracket_{\sigma, \rho}$  is nonempty) and in  $E$ , so  $\sigma[x \mapsto \zeta(E)] \in \llbracket \Gamma, x : T \rrbracket$ . So by the IH for the second premise,  $\llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$  is defined.

Case:

$$\frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma, x : T \vdash T' \Rightarrow \kappa}{\Gamma \vdash \lambda x : T. T' \Rightarrow \Pi x : T. \kappa}$$

By the semantics,  $\llbracket \lambda x : T.T' \rrbracket_{\sigma, \rho}$  is  $(E \in \llbracket T \rrbracket_{\sigma, \rho} \mapsto \llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho})$ . We must show that this (meta-level) function is in  $\llbracket \Pi x : T.\kappa \rrbracket_{\sigma, \rho}$ . By the semantics of kinds, the latter quantity, if defined, is  $(E \in \llbracket T \rrbracket_{\sigma, \rho} \rightarrow_{c\beta\eta} \llbracket \kappa \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho})$ . By the IH for the first premise,  $\llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ . So we must just show that for any  $E \in \llbracket T \rrbracket_{\sigma, \rho}$ ,  $\llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho} \in \llbracket \kappa \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$ . But this follows by the IH for the second premise.

**Case:**

$$\frac{\Gamma \vdash \kappa \quad \Gamma, X : \kappa \vdash T' \Rightarrow \kappa'}{\Gamma \vdash \lambda X : \kappa. T' \Rightarrow \Pi X : \kappa. \kappa'}$$

This case is an easier version of the previous one. It suffices to assume an arbitrary  $S \in \llbracket \kappa \rrbracket_{\sigma, \rho}$  and show  $\llbracket T' \rrbracket_{\sigma, \rho[X \mapsto S]} \in \llbracket \kappa' \rrbracket_{\sigma, \rho[X \mapsto S]}$ . But this follows by the IH applied to the second premise. And we have definedness of  $\llbracket \kappa \rrbracket_{\sigma, \rho}$  by the IH for the first premise.

**Case:**

$$\frac{\Gamma \vdash T \Rightarrow \Pi x : T'. \kappa \quad \Gamma \vdash t \Leftarrow T'}{\Gamma \vdash T \ t \Rightarrow [t/x]\kappa}$$

By the IH for the first premise,  $\llbracket T \rrbracket_{\sigma, \rho} \in \llbracket \Pi x : T'. \kappa \rrbracket_{\sigma, \rho}$ . By the semantics of  $\Pi$ -kinds, this means that  $\llbracket T \rrbracket_{\sigma, \rho}$  is a function which given any  $E \in \llbracket T' \rrbracket_{\sigma, \rho}$ , will produce a result in  $\llbracket \kappa \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$ . By the semantics of type applications,  $\llbracket T \ t \rrbracket_{\sigma, \rho}$  is equal to  $\llbracket T \rrbracket_{\sigma, \rho}(\llbracket \sigma[t] \rrbracket_{c\beta\eta})$ . This is defined, since  $\llbracket \sigma[t] \rrbracket_{c\beta\eta} \in \llbracket T' \rrbracket_{\sigma, \rho}$ , by the IH for the second premise; note that  $\llbracket T' \rrbracket_{\sigma, \rho}$  is defined since otherwise  $\llbracket \Pi x : T'. \kappa \rrbracket_{\sigma, \rho}$  would not be defined. The result of applying the function is thus indeed in  $\llbracket [t/x]\kappa \rrbracket_{\sigma, \rho}$ , since by Lemma 11, this equals  $\llbracket \kappa \rrbracket_{\sigma[x \mapsto \zeta(\llbracket \sigma[t] \rrbracket_{c\beta\eta})], \rho}$  (the codomain of the function being applied).

**Case:**

$$\frac{\Gamma \vdash T_1 \Rightarrow \Pi X : \kappa_2. \kappa_1 \quad \Gamma \vdash T_2 \Rightarrow \kappa'_2 \quad \kappa_2 \cong \kappa'_2}{\Gamma \vdash T_1 \cdot T_2 \Rightarrow [T_2/X]\kappa_1}$$

By the IH applied to the first premise,  $\llbracket T_1 \rrbracket_{\sigma, \rho} \in \llbracket \Pi X : \kappa_2. \kappa_1 \rrbracket_{\sigma, \rho}$ . By the semantics of  $\Pi$ -kinds, this means that for any  $S \in \llbracket \kappa_2 \rrbracket_{\sigma, \rho}$ ,  $\llbracket T_1 \rrbracket_{\sigma, \rho} S$  is in  $\llbracket \kappa_1 \rrbracket_{\sigma, \rho[X \mapsto S]}$ . By the IH for the second premise, we have  $\llbracket T_2 \rrbracket_{\sigma, \rho} \in \llbracket \kappa'_2 \rrbracket_{\sigma, \rho}$ , and by the IH for the third premise, we have  $\llbracket \kappa_2 \rrbracket_{\sigma, \rho} = \llbracket \kappa'_2 \rrbracket_{\sigma, \rho}$ . So we get  $\llbracket T_1 \rrbracket_{\sigma, \rho}(\llbracket T_2 \rrbracket_{\sigma, \rho}) \in \llbracket \kappa_1 \rrbracket_{\sigma, \rho[X \mapsto \llbracket T_2 \rrbracket_{\sigma, \rho}]}$ , which suffices by Lemma 12.

**Case:**

$$\frac{FV(t \ t') \subseteq \text{dom}(\Gamma)}{\Gamma \vdash \{t \simeq t'\} : \star}$$

Either  $\sigma[t] =_{c\beta\eta} \sigma[t']$  or not. Either way, the interpretation is defined and in  $\mathcal{R}$ , since  $FV(t \ t') \subseteq \text{dom}(\sigma)$  (as an easy consequence of  $(\sigma, \rho) \in \llbracket \Gamma \rrbracket$ ).

### A.3 Proof of parts (3) and (4)

**Case:**

$$\frac{(x : t) \in \Gamma}{\Gamma \vdash x \Rightarrow T}$$

This follows from the definition of  $\llbracket \Gamma \rrbracket$ .

**Case:**

$$\frac{\Gamma \vdash t \Leftarrow T \quad T' \rightsquigarrow_{\beta}^* T}{\Gamma \vdash t \Leftarrow T'}$$

We are assuming  $\llbracket T' \rrbracket_{\sigma, \rho}$  is defined, since this is a checking judgment. The desired result then follows from Lemma 13.

Case:

$$\frac{\Gamma \vdash t \Rightarrow T \quad T \rightsquigarrow_{\beta}^* T'}{\Gamma \vdash t \Rightarrow T'}$$

This also follows from Lemma 13 and the induction hypothesis for the first premise, which implies  $\llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$  (and hence defined).

Case:

$$\frac{\Gamma \vdash t \Rightarrow T' \quad T' \cong T}{\Gamma \vdash t \Leftarrow T}$$

By the IH applied to the first premise, we have  $[\sigma t]_{c\beta\eta} \in \llbracket T' \rrbracket_{\sigma, \rho} \in \mathcal{R}$ . By assumption,  $\llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ , and so by the IH applied to the second premise, we have  $[\sigma t]_{c\beta\eta} \in \llbracket T' \rrbracket_{\sigma, \rho} = \llbracket T \rrbracket_{\sigma, \rho}$ .

Case:

$$\frac{\Gamma, x : T \vdash t \Leftarrow T'}{\Gamma \vdash \lambda x. t \Leftarrow \Pi x : T. T'}$$

To show  $[\sigma \lambda x. t]_{c\beta\eta} \in \llbracket \Pi x : T. T' \rrbracket_{\sigma, \rho}$  (noting that the latter is defined and in  $\mathcal{R}$  by assumption), it suffices to assume an arbitrary  $E \in \llbracket T \rrbracket_{\sigma, \rho}$ , and show  $[[\zeta(E)/x] \sigma t]_{c\beta\eta} \in \llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$ . By the IH, we have  $[\sigma[x \mapsto \zeta(E)] t]_{c\beta\eta} \in \llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$ . But  $[\sigma[x \mapsto \zeta(E)] t]_{c\beta\eta} = [[\zeta(E)/x] \sigma t]_{c\beta\eta}$ , so this is sufficient.

Case:

$$\frac{\Gamma \vdash t \Rightarrow \Pi x : T'. T \quad \Gamma \vdash t' \Leftarrow T'}{\Gamma \vdash t t' \Rightarrow [t'/x] T}$$

By the IH applied to the first premise,  $[\sigma t]_{c\beta\eta} \in \llbracket \Pi x : T'. T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ . This means that there exists a  $\lambda$ -abstraction  $\lambda x. \hat{t}$  such that  $\lambda x. \hat{t} =_{c\beta\eta} \sigma t$ , by the semantics of  $\Pi$ -types. Furthermore, for any  $E \in \llbracket T' \rrbracket_{\sigma, \rho}$ ,  $[[\zeta(E)/x] \hat{t}]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$ . By the IH applied to the second premise,  $[\sigma t']_{c\beta\eta} \in \llbracket T' \rrbracket_{\sigma, \rho}$ , so we can instantiate the quantifier in the previous formula to obtain

$$[[\zeta([\sigma t']_{c\beta\eta})/x] \hat{t}]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma[x \mapsto \zeta([\sigma t']_{c\beta\eta})], \rho}$$

By Lemma 11, this is equivalent to

$$[[\zeta([\sigma t']_{c\beta\eta})/x] \hat{t}]_{c\beta\eta} \in \llbracket [t'/x] T \rrbracket_{\sigma, \rho}$$

Since  $\sigma(t t') =_{c\beta\eta} (\lambda x. \hat{t}) \sigma t' =_{c\beta\eta} [[\zeta([\sigma t']_{c\beta\eta})/x] \hat{t}]_{c\beta\eta}$ , this is sufficient.

Case:

$$\frac{\Gamma, X : \kappa \vdash t \Leftarrow T}{\Gamma \vdash \Lambda X. t \Leftarrow \forall X : \kappa. T}$$

By the IH,  $[\sigma t]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma, \rho[X \mapsto S]}$ , for all  $S \in \llbracket \kappa \rrbracket_{\sigma, \rho}$ . This is sufficient to prove  $[\sigma \Lambda X. t]_{c\beta\eta} \in \llbracket \forall X : \kappa. T \rrbracket_{\sigma, \rho}$ , by the semantics of  $\forall$ -types and definition of erasure.

Case:

$$\frac{\Gamma \vdash t \Rightarrow \forall X : \kappa. T \quad \Gamma \vdash T' \Leftarrow \kappa}{\Gamma \vdash t \cdot T' \Rightarrow [T'/X] T}$$

By the semantics of  $\forall$ -types and the IH applied to the first premise, we have  $[\sigma t]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma, \rho[X \mapsto S]}$ , for all  $S \in \llbracket \kappa \rrbracket_{\sigma, \rho}$ . Since  $\llbracket T' \rrbracket_{\sigma, \rho} \in \llbracket \kappa \rrbracket_{\sigma, \rho}$  by the IH applied to the second premise, we can derive  $[\sigma t]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma, \rho[X \mapsto \llbracket T' \rrbracket_{\sigma, \rho}]}$ . By Lemma 12, this is equivalent to the required  $[\sigma t]_{c\beta\eta} \in \llbracket [T'/X] T \rrbracket_{\sigma, \rho}$ , using also the definition of erasure.

Case:

$$\frac{\Gamma, x : T' \vdash t \Leftarrow T \quad x \notin FV(|t|)}{\Gamma \vdash \Lambda x. t \Leftarrow \forall x : T'. T}$$

By the IH applied to the first premise, we have  $[\sigma[x \mapsto \zeta(E)]t]_{c\beta\eta} \in \llbracket T' \rrbracket_{\sigma[x \mapsto \zeta(E)], \rho}$ , for any  $E \in \llbracket T \rrbracket_{\sigma, \rho}$ . This is because  $\llbracket T \rrbracket_{\sigma, \rho} \in \mathcal{R}$ , since  $\llbracket \forall x : T'. T \rrbracket_{\sigma, \rho}$  is in  $\mathcal{R}$  and hence defined, by assumption. Since  $x \notin FV(t)$ , we know  $[[\sigma[x \mapsto \zeta(E)]t]_{c\beta\eta} = [\sigma t]_{c\beta\eta}$ . By the semantics of  $\forall$ -types and definition of erasure, this suffices to show the desired conclusion.

Case:

$$\frac{\Gamma \vdash t \Rightarrow \forall x : T'. T \quad \Gamma \vdash t' \Leftarrow T'}{\Gamma \vdash t - t' \Rightarrow [t'/x]T}$$

The result follows easily by the IH applied to the premises, the semantics of  $\forall$ -types, definition of erasure, and Lemma 11.

Case:

$$\frac{\Gamma \vdash t \Leftarrow T \quad \Gamma \vdash t' \Leftarrow [t/x]T' \quad |t| =_{\beta\eta} |t'|}{\Gamma \vdash [t, t'] \Leftarrow \iota x : T. T'}$$

By the IH, we have  $[\sigma|t|]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma, \rho}$  and  $[\sigma|t|]_{c\beta\eta} \in \llbracket [t/x]T' \rrbracket_{\sigma, \rho}$ . By Lemma 11, the latter is equivalent to  $[\sigma t]_{c\beta\eta} \in \llbracket T' \rrbracket_{\sigma[x \mapsto \zeta([\sigma t]_{c\beta\eta})], \rho}$ . These two facts about  $[\sigma t]_{c\beta\eta}$  are sufficient, by the semantics of  $\iota$ -types, for the desired conclusion, using also the fact (from the third premise) that  $\sigma|t| =_{c\beta\eta} \sigma|t'|$ .

Case:

$$\frac{\Gamma \vdash t \Rightarrow \iota x : T. T'}{\Gamma \vdash t.1 \Rightarrow T}$$

The desired conclusion follows easily from the IH and the semantics of  $\iota$ -types.

Case:

$$\frac{\Gamma \vdash t \Rightarrow \iota x : T. T'}{\Gamma \vdash t.2 \Rightarrow [t.1/x]T'}$$

Similar to the previous case, using Lemma 11.

Case:

$$\frac{\Gamma \vdash FV(t' \ t_1 \ t_2) \subseteq \text{dom}(\Gamma) \quad |t_1| =_{\beta\eta} |t_2|}{\Gamma \vdash \beta\{t'\} \Leftarrow \{t_1 \simeq t_2\}}$$

Since by assumption  $|t_1| =_{\beta\eta} |t_2|$  it follows that  $\sigma|t_1| =_{\beta\eta} \sigma|t_2|$ , and also by assumption it follows that  $FV(t_1 \ t_2) \subseteq \text{dom}(\sigma)$ . Also, we see  $\sigma|t'|$  is a closed term. So,  $[\sigma|t'|]_{c\beta\eta} \in \llbracket \{t_1 \simeq t_2\} \rrbracket_{\sigma, \rho}$  follows directly from the semantics of equality types.

Case:

$$\frac{\Gamma \vdash t \Rightarrow \{\lambda x. \lambda y. x \simeq \lambda x. \lambda y. y\}}{\Gamma \vdash \delta - t \Leftarrow T}$$

By the semantics of equality types,  $[\sigma|t|]_{c\beta\eta}$  cannot be in the interpretation of the equation in the premise, since the two terms in question are closed and not  $\beta\eta$ -equal. By the IH applied to the first premise, however,  $[\sigma|t|]_{c\beta\eta}$  is in the interpretation of that equation. This is a contradiction.

Case:

$$\frac{\begin{array}{cc} \Gamma \vdash t' \Rightarrow \{t_1 \simeq t_2\} & \Gamma \vdash [t_2/x]T \Rightarrow \star \\ \Gamma \vdash t \Rightarrow T' & T' \cong [t_1/x]T \end{array}}{\Gamma \vdash \rho \ t' \ @x.T - t \Rightarrow [t_2/x]T}$$

By the IH applied to the first premise in the first row,  $\sigma|t_1| =_{\beta\eta} \sigma|t_2|$ . By the IH applied to the first premise in the second row,  $[\sigma|t|]_{c\beta\eta} \in \llbracket T' \rrbracket_{\sigma,\rho}$ . By the IH applied to the second premise in the second row,  $\llbracket T' \rrbracket_{\sigma,\rho} = \llbracket [t_1/x]T \rrbracket_{\sigma,\rho}$ . The result then follows by applying Lemma 11.

**Case:**

$$\frac{\Gamma \vdash T \Rightarrow \star \quad \Gamma \vdash t \Leftarrow T}{\Gamma \vdash_{\chi} T - t \Rightarrow T}$$

We apply the IH for the first premise to get that  $\llbracket T \rrbracket_{\sigma,\rho}$  is in  $\mathcal{R}$  and hence defined. Using this, we apply the IH on the second premise to get  $[\sigma|t|]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma,\rho}$ , which by the definition of erasure is what we must show.

**Case:**

$$\frac{\Gamma \vdash t \Rightarrow \{t' \simeq t''\} \quad \Gamma \vdash t' \Leftrightarrow T}{\Gamma \vdash \varphi \ t - t' \{t''\} \Leftrightarrow T}$$

By the IH for the first premise,  $\sigma|t'| =_{c\beta\eta} \sigma|t''|$ . By the IH for the second premise,  $[\sigma|t'|]_{c\beta\eta} \in \llbracket T \rrbracket_{\sigma,\rho}$ . This suffices for the desired conclusion, using also the definition of erasure ( $|\varphi \ t - t' \{t''\}| = |t''|$ ).

## Proof of part (5)

**Case:**

$$\frac{T \rightsquigarrow_{\beta}^* T_1 \quad T' \rightsquigarrow_{\beta}^* T_2 \quad T_1 \cong^t T_2}{T \cong T'}$$

By Lemma 13, we have

$$\begin{aligned} \llbracket T \rrbracket_{\sigma,\rho} &= \llbracket T_1 \rrbracket_{\sigma,\rho} \\ \llbracket T' \rrbracket_{\sigma,\rho} &= \llbracket T_2 \rrbracket_{\sigma,\rho} \end{aligned}$$

By the IH for the third premise, we have  $\llbracket T_1 \rrbracket_{\sigma,\rho} = \llbracket T_2 \rrbracket_{\sigma,\rho}$ , which suffices.

**Case:**

$$\frac{T \cong^t T'}{T \cong T'}$$

By the IH.

**Case:**

$$\frac{T \cong^t T' \quad |t| =_{\beta\eta} |t'|}{T \ t \cong^t T' \ t'}$$

By the semantics,  $\llbracket T \ t \rrbracket_{\sigma,\rho} = \llbracket T \rrbracket_{\sigma,\rho}([\sigma|t|]_{c\beta\eta})$ . By the second premise and the IH for the first premise, this equals  $\llbracket T' \rrbracket_{\sigma,\rho}([\sigma|t'|]_{c\beta\eta})$ , as required.

**Case:**

$$\frac{|t_1| =_{\beta\eta} |t'_1| \quad |t_2| =_{\beta\eta} |t'_2|}{\{t_1 \simeq t_2\} \cong^t \{t'_1 \simeq t'_2\}}$$

This follows easily from the premises and the semantics of equality types.

## Proof of Part (6)

The convertibility relation for kinds consists entirely of congruential rules for quantification over types and kinds.

□

## B Proof of Theorem 4

*Proof.* Theorem 1 implies that since  $t' t$  is closed and of type  $\Pi x : T_1. T_2$ , we have  $[[t' t]]_{c\beta\eta} \in [[\Pi x : T_1. T_2]]_{\cdot, \rho}$ , where  $[[t' t]]_{c\beta\eta}$  is the set of closed terms which are  $\beta\eta$ -equivalent to  $|t' t|$ ; and  $(\cdot, \rho) \in [[\Gamma]]$  gives interpretations  $\cdot$  for term- and  $\rho$  for type-variables in  $\Gamma$ . By the semantics of types defined in Figure 6, the interpretation of a  $\Pi$ -type consists of sets of the form  $[\lambda x. t']_{c\beta\eta}$ . So we have that  $|t' t|$  is  $\beta\eta$ -equivalent to  $\lambda x. t'$  for some  $x, t'$ . Since  $|t' t|$  is  $\beta$ -equivalent to  $t$ , we know  $t =_{\beta\eta} \lambda x. t'$ . It is then an easy consequence of the standardization theorem for untyped lambda calculus that  $|t|$  is call-by-name normalizing (cf. [6]). □

## C Proof of Theorem 6

First a few lemmas (easy proofs omitted):

**Lemma 14.**

- If  $\Gamma \vdash T \Rightarrow \kappa$  then  $\Gamma \vdash \kappa$
- If  $\Gamma \vdash t T$  then  $\Gamma \vdash T \Rightarrow \kappa$  for some  $\kappa$

**Lemma 15.**

- If  $\Gamma, x : T \vdash T \Rightarrow \kappa$  and  $\Gamma \vdash t \Leftarrow T$  then  $\Gamma \vdash [t/x]T \Rightarrow [t/x]\kappa$
- If  $\Gamma, X : \kappa_2 \vdash T_1 \Rightarrow \kappa_1$  and  $\Gamma \vdash T_2 \Rightarrow \kappa_2$  then  $\Gamma \vdash [T_2/X]T_1 \Rightarrow [T_2/X]\kappa_1$

**Lemma 16.** If  $\Gamma \vdash t \Leftarrow T_1$  where  $\Gamma \vdash T_1 \Rightarrow \kappa_1$ , and we have  $\Gamma \vdash T_2 \Rightarrow \kappa_2$  with  $T_1 \cong T_2$  and  $\kappa_1 \cong \kappa_2$ , then  $\Gamma \vdash t \Leftarrow T_2$ .

*Proof (of Theorem 6).* We may rule out the cases where  $T$  is a variable or formed by a type constructor, as this would contradict the assumption that  $T \rightsquigarrow_\beta T'$  for some  $T'$  (recall that  $\rightsquigarrow_\beta$  is call-by-name reduction for types).

Case:

$$\frac{\Gamma \vdash T \Rightarrow \Pi x : T_1. \kappa_1 \quad \Gamma \vdash t \Leftarrow T_1}{\Gamma \vdash T t \Rightarrow [t/x]\kappa_1}$$

There are two subcases to consider for the derivation of  $T t \rightsquigarrow_\beta T'$ . In the first case, we have  $T \rightsquigarrow_\beta T''$  for some  $T''$  (so  $T' = T'' t$ ). By the IH, we have  $\Gamma \vdash T'' \Rightarrow \Pi x : T'_1. \kappa'_1$  for some  $T'_1 \cong T_1$  and  $\kappa'_1 \cong \kappa_1$  (we have by assumption the kind of  $T''$  must be convertible with  $\Pi x : T_1. \kappa_1$ ). By Lemma 16,  $\Gamma \vdash t \Leftarrow T'_1$ . By the rule,  $\Gamma \vdash T'' t \Rightarrow [t/x]\kappa'_1$ , and the synthesized kind is clearly convertible with  $[t/x]\kappa_1$ .

In the other subcase, the subject of kinding is of the form  $(\lambda x : T_1. T'') t$  and our assumed reduction is to  $[t/x]T''$ . By inversion of the kinding derivation, we have  $\Gamma, x : T_1 \vdash T'' \Rightarrow \kappa_1$ . By Lemma 15, we have  $\Gamma \vdash [t/x]T'' \Rightarrow [t/x]\kappa_1$ .

Case:

$$\frac{\Gamma \vdash T_1 \Rightarrow \Pi X : \kappa_2. \kappa_1 \quad \Gamma \vdash T_2 \Rightarrow \kappa'_2 \quad \kappa_2 \cong \kappa'_2}{\Gamma \vdash T_1 \cdot T_2 \Rightarrow [T_2/X]\kappa_1}$$

There are two subcases to consider for the derivation of  $T_1 \cdot T_2 \rightsquigarrow_\beta T'$ . In the first subcase, we have  $T_1 \rightsquigarrow_\beta T''_1$  for some  $T''_1$  (so  $T' = T''_1 \cdot T_2$ ). By the IH, we have  $\Gamma \vdash T''_1 \Rightarrow \Pi X : \kappa'_2. \kappa'_1$  for some  $\kappa'_2 \cong \kappa_2$  and  $\kappa'_1 \cong \kappa_1$  (we have by assumption the kind of  $T''_1$  must be convertible with  $\Pi X : \kappa_2. \kappa_1$ ). By transitivity we can conclude

$\kappa_2'' \cong \kappa_2$ . By the rule, we have  $\Gamma \vdash T_1'' \cdot T_2 \Rightarrow [T_2/X]\kappa_1''$ , and the synthesized kind is clearly convertible with  $[T_2/X]\kappa_1$ .

In the other subcase, the subject of kinding is of the form  $(\lambda X : \kappa_2. T_1'') \cdot T_2$  and our assumed reduction is to  $[T_2/X]T_1''$ . By inversion of the kinding derivation, we have  $\Gamma, X : \kappa_2 \vdash T_1'' \Rightarrow \kappa_1$ . By Lemma 15, we have  $\Gamma \vdash [T_2/X]T_1'' \Rightarrow [T_2/X]\kappa_1$ .

□