DUAL LINEAR PROGRAMMING BOUNDS IN ANY DIMENSIONS USING MODULAR FORMS OF INTEGRAL AND HALF-INTEGRAL WEIGHT

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ABSTRACT. To be completed.

CONTENTS

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1. Introduction

The sphere packing problem concerns the densest possible packing of congruent spheres in $\mathbb{R}^d$. More precisely, it asks for the best configuration of non overlapping spheres. The case $d = 1$ is trivial, the case $d = 2$ was solved by Thue [11], the case $d = 3$ was solved by Hales [7] using a computer-assisted proof, the case $d = 8$ was solved by Viazovska [12], and the case $d = 24$ was solved by Cohn, Kumar, Miller, Radchenko, and Viazovska [4]. The first three cases used extensively the geometry of packings in $\mathbb{R}^d$, while the last two cases used beautifully the connections between exceptional lattices and modular forms to construct magic functions which saturate the linear programming bound of Cohn and Elkies [5].

One may ask if the linear programming bound of Cohn and Elkies is sharp in other dimensions. One way to study such a question was developed by Cohn and Triantafilou [3] using the theory of modular forms. More precisely, they've shown that the linear programming bound cannot be sharp in dimensions $12, 16, 20, 28$ and 32. Their method, developed for dimensions that are multiple of 4, builds on a result of Cohn [2] which shows how to give lower bounds to the linear programming bound.

Proposition 1.1. *Let δ_0 be the delta function at the origin, and μ a tempered distribution on $\mathbb{R}^d$ such that $\mu = \delta_0 + \nu$ with $\nu \geq 0$, supp$(\nu) \subseteq \{x \in \mathbb{R}^d : |x| \geq r\}$ for some $r > 0$, and $\hat{\mu} \geq c\delta_0$ for some $c > 0$. Then the linear programming bound in $\mathbb{R}^d$ is at least*

$$c\left(\frac{r}{2}\right)^d.$$

The main difficulty with this proposition is how to find plenty of distributions μ satisfying the hypotheses. What they observed is that the functional equations of modular forms can be used to extract such distributions. In particular, they've shown the following.

Proposition 1.2. *Let $d = 2k$ with $k \in \mathbb{N}$, let $g \in M_k(\Gamma_1(N))$ be a modular form of weight k for the congruence subgroup $\Gamma_1(N)$, let $W_N = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$,and let*

$$\tilde{g}(\tau) = i^k(g|_k W_N)(\tau) = \frac{i^k}{N^{k/2}z^k} g\left(\frac{-1}{N\tau}\right)$$

be i^k times the image of g under the Fricke involution. Let the q-expansion of g and $\tilde{g}$ be

$$g(\tau) = \sum_{n\geq 0} a_n q^n,\ \tilde{g}(\tau) = \sum_{n\geq 0} b_n q^n,$$

where throughout $q = e^{2\pi i\tau}$. Then, for every radial Schwartz function $f : \mathbb{R}^d \to \mathbb{C}$, we have

$$\sum_{n\geq 0} a_n f(\sqrt{n}) = \left(\frac{2}{\sqrt{N}}\right)^{d/2} \sum_{n\geq 0} b_n \hat{f}\left(\frac{2\sqrt{n}}{\sqrt{N}}\right).$$

The key observation now is that if δ_r denotes a delta function supported on the sphere of radius r centered at the origin in $\mathbb{R}^d$, then this proposition states that the tempered distributions

$$\sum_{n\geq 0} a_n \delta_{\sqrt{n}} \text{ and } \left(\frac{2}{\sqrt{N}}\right)^{d/2} \sum_{n\geq 0} b_n \delta_{2\sqrt{n/N}}$$

are Fourier transforms of each other. What is left to do now is to optimize over such distributions as follows. Let $k = d/2$ be an even integer, and let $M_k(\Gamma_0(N))$ be the space of modular forms of weight k for $\Gamma_0(N)$. We want to find a modular form $g = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma_0(N))$ with the following properties for some $T > 0$.

1. $a_0 = 1, b_0 > 0$,
2. $a_n \geq 0$ and $b_n \geq 0$ for all $n \geq 0$, and
3. $a_n = 0$ for $1 \leq n < T$.

Then, we use the distribution

$$\mu = \sum_{n\geq 0} a_n \delta_{\sqrt{n}}$$

with Proposition 1.1 and Proposition 1.2, by letting $c = (2/\sqrt{N})^{d/2} b_0$ and $r = \sqrt{T}$. The lower bound thus obtained is

$$b_0 \left(\frac{2}{\sqrt{N}}\right)^{d/2} \left(\frac{\sqrt{T}}{2}\right)^{d}$$

for the linear programming bound in $\mathbb{R}^d$, and we wish to maximize b_0 by choosing an optimal g.

This article shows how to generalize the method of Cohn and Triantafillou to arbitrary dimensions using modular forms of integral and half-integral weights. In particular we have the following generalization of Proposition 1.2 to all even dimensions.

Proposition 1.3. *Let $g = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma_0(N), \chi)$ and $\tilde{g} = \sum_{n\geq 0} b_n q^n \in M_k(\Gamma_0(N), \bar{\chi})$ be modular forms of weight $k \in \mathbb{N}$ with characters χ and $\bar{\chi}$, where $\bar{\chi}$ denotes the complex conjuagation. Let $f : \mathbb{R}^{2k} \to \mathbb{R}$ denote any radial Schwartz function. Then we have the following Voronoi type summation formula*

$$\sum_{n\geq 0} a_n f(\sqrt{n}) = \left(\frac{2}{\sqrt{N}}\right)^{k} \sum_{n\geq 0} b_n \hat{f}\left(\frac{2\sqrt{n}}{\sqrt{N}}\right).$$

Furthermore, the linear programming bound in $\mathbb{R}^{2k}$ is at least

$$b_0 \left(\frac{2}{\sqrt{N}}\right)^{k} \left(\frac{\sqrt{T}}{2}\right)^{2k}.$$

In the case of odd dimensions, the objects to consider are half-integral weights modular forms.

Proposition 1.4. *Let k be an odd positive integer and N a positive integer divisible by 4. Let $g = \sum_{n\geq 0} a_n q^n \in M_{k/2}(\Gamma_0(N), \chi)$ and $\tilde{g} = \sum_{n\geq 0} b_n q^n \in M_{k/2}(\Gamma_0(N), \left(\frac{N}{\cdot}\right) \bar{\chi})$ be modular forms of weight $k/2$ with characters χ and $\left(\frac{N}{\cdot}\right) \bar{\chi}$, where $\left(\frac{N}{\cdot}\right)$ denotes the Kronecker symbol. Let $f : \mathbb{R}^k \to \mathbb{R}$ denote any radial Schwartz function. Then we have the following Voronoi type summation formula*

$$\sum_{n\geq 0} a_n f(\sqrt{n}) = \left(\frac{2}{\sqrt{N}}\right)^{k/2} \sum_{n\geq 0} b_n \hat{f}\left(\frac{2\sqrt{n}}{\sqrt{N}}\right).$$

Furthermore, the linear programming bound in $\mathbb{R}^{2k}$ is at least

$$b_0 \left(\frac{2}{\sqrt{N}}\right)^{k/2} \left(\frac{\sqrt{T}}{2}\right)^{k}.$$

Remark 1.5. From the formula we see that if $T > N$ then for k large, the bound should be bigger if the b_0 have more or less the same value across the dimensions and level.

Note that the above formulas apply to the more general spaces $M_k(\Gamma_1(N))$ via the following porpositions.

Proposition 1.6 (Theorem 7.3.4 [1])**.** *We have a canonical direct sum decomposition*

$$M_k(\Gamma_1(N)) = \bigoplus_{\substack{\chi \bmod N \\ \chi(-1)=(-1)^k}} M_k(\Gamma_0(N), \chi),$$

and similarly for cusp forms. The direct sum ranges over all Dirichlet characters χ modulo N such that $\chi(-1) = (-1)^k$.

Proposition 1.7 (Section 5.4 [13])**.** *Let k, N as above. We have the following direct sum decompositions*

$$M_{k/2}(\Gamma_1(N)) = \bigoplus_{\chi} M_{k/2}(\Gamma_0(N), \chi),$$

where the direct sum runs over all even characters modulo N. The same decomposition applies to the space of cusp forms and Eisenstein series.

Hence, it is sufficient to optimize over ...

The coefficients of the above modular forms may be real or complex. However, this is not an obstacle to the generalization of the above algorithm. Indeed, by taking real and imaginary part of the coefficients, we are left with the following algorithm.

Let's fix an integral or half-integral weight k and a level N. Let T and M be positive integers that we will call respectively the *radius* and *imposed positivity threshold*. Using MAGMA and the function ... we can construct a basis of the space $M_k(\Gamma_1(N))$. We then write every basis element as $g^j = \sum_{n\geq 0} a_n^j q^n$ and their involution as $\tilde{g}^j = \sum_{n\geq 0} b_n^j q^n$. Given a form $g \in M_k(\Gamma_1(N))$, we write, for $x_j \in \mathbb{R}$, $g = \sum_j^{\dim(M_k(\Gamma_1(N))} x_j g^j$ and we let $a_n^j = a_{n,\Re}^j + i a_{n,\Im}^j$, and do the same for b_n. Finally, we optimize over the coefficients x_j by solving independently the two linear programs :

We maximize $\sum_j x_j b_{0,\Re}^j$ and $\sum_j x_j b_{0,\Im}^j$, subject to the constraints

- $1 = \sum_j x_j a_{0,\Re}^j$
- $0 = \sum_j x_j a_{n,\Re}^j$ for $1 \leq n < T$,
- $0 \leq \sum_j x_j a_{n,\Re}^j$ for $T \leq n \leq M$, and
- $0 \leq \sum_j x_j b_{n,\Re}^j$ for $1 \leq n \leq M$

and

- $1 = \sum_j x_j a_{0,\Im}^j$
- $0 = \sum_j x_j a_{n,\Im}^j$ for $1 \leq n < T$,
- $0 \leq \sum_j x_j a_{n,\Im}^j$ for $T \leq n \leq M$, and
- $0 \leq \sum_j x_j b_{n,\Im}^j$ for $1 \leq n \leq M$

Then, we are left with choosing the maximal value from $b_{0,\Re}$ and $b_{0,\Im}$.

To be completed with some examples and move certain pieces to other sections.

2. SOME NUMERICAL RESULTS SO FAR

We run all dimensions up to 100 with level up to 100 and precision up to 500.

TABLE 1. Solutions of the LP solver with level $N \leq 100$, $T \leq 25$ and prec$= 500$.

dimension	weight	solution	level	T
1	1/2			
2	1			
3	3/2			
4	2			
5	5/2			
6	3			
7	7/2			
8	4			
9	9/2			
10	5			
11	11/2			
12	6			
13	13/2			
14	7			
15	15/2			
16	8			
17	17/2			
18	9			
19	19/2			
20	10			

Remark 2.1. Must find a feasible density larger than the best upper bound in a given dimension to completely rule out this possibility.

Remark 2.2. Use the theorem we have developed for non negativity of Fourier coefficients for the dual space ! For the whole space of forms, we must study individually the MF and their characters.

3. Nuts and Bolts

This section contains some basic results about the theory of modular forms of integral and half-integral weights. The main references are [1], [6], [10], and [13].

3.1. **Modular Forms of Integral Weight.**

Definition 3.1. Let k, N be positive integers and χ a character modulo N satisfying $\chi(-1) = (-1)^k$. Let $A_k(\Gamma_0(N), \chi)$ be the set of functions on the upper half-plane $\mathbb{H}$ satisfying

1. f is meromorphic on $\mathbb{H}$,
2. $f|_k\gamma = \chi(d)f$ for any $\gamma \in \Gamma_0(N)$,
3. f is meromorphic at each cusp point of $\Gamma_0(N)$.

Then, a functions $f \in A_k(\Gamma_0(N), \chi)$ is called a meromorphic modular form of weight k and Nebentypus (character) χ for $\Gamma_0(N)$. If f is holomorphic in $\mathbb{H}$, then f is called a weakly holomorphic modular form. If f is also holomorphic at the cusps, then f is simply called a (holomorphic) modular form. In this case, if f vanishes at the cusps, then f is called a cusp form. The set of (holomorphic) modular forms will be denoted $M_k(\Gamma_0(N), \chi)$ and the set of cusp forms by $S_k(\Gamma_0(N), \chi)$. From now on, we will refer to holomorphic modular forms simply as modular forms.

Proposition 3.2 (Proposition 10.3.14 [1])**.** *Let $f(\tau) = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma_0(N), \chi)$ be a modular form. Then, we define the Fricke involution*

$$|_k W_N : M_k(\Gamma_0(N), \chi) \to M_k(\Gamma_0(N), \bar{\chi})$$

by

$$f \mapsto (f|_k W_N)(\tau) = N^{-k/2}\tau^{-k} f\left(-1/(N\tau)\right),$$

where W_N denotes the matrix $\begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$.

Remark 3.3. In the case of a real character, the Fricke involution is an endomorphism.

Remark 3.4. In [3], the authors define $\tilde{g} = i^k g|_k W_N$. If instead we rewrite $\tilde{g} = (-i)^{-k} g|_k W_N = (-i\sqrt{N}\tau)^{-k} g\left(\frac{-1}{N\tau}\right)$, then we obtain the same Fricke involution as in the case of half-integral weight modular forms.

Theorem 3.5 (Theorem 5.9 [13])**.** *Let $f(\tau) = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma_0(N), \chi)$ be a modular form, $g(\tau) = (f|_k W_N)(\tau) = \sum_{n\geq 0} b_n q^n$ be its image under the Fricke involution, and $\tilde{g}(\tau) = i^k g(\tau) = (-i\sqrt{N}\tau)^{-k} f\left(\frac{-1}{N\tau}\right)$.*

We define the ζ-function of f by

$$L(f, s) = \sum_{n\geq 1} a_n n^{-s},\ s \in \mathbb{C},$$

and it is absolutely convergent for $\Re(s) > 1 + k$ and converges for $\Re(s) > k$. Furthermore, define the completed L-function

$$\Lambda_n(f, s) = (2\pi)^{-s} N^{s/2} \Gamma(s) L(f, s).$$

Then, $\Lambda_N(f,s)$ can be analytically continued to a meromorphic function on the complex plane with possible simple poles at $s=0$ and $s=k$ of residue a_0 and $i^k b_0$ respectively. Finally, $\Lambda_N(f,s)$ satisfies the functional equation

$$\Lambda_N(f,s) = i^k \Lambda_N(g, k-s) = \Lambda(\tilde{g}, k-s).$$

Proposition 3.6 (Theorem 7.3.4 [1])**.** *We have a canonical direct sum decomposition*

$$M_k(\Gamma_1(N)) = \bigoplus_{\substack{\chi \bmod N \\ \chi(-1)=(-1)^k}} M_k(\Gamma_0(N), \chi),$$

and similarly for cusp forms. The direct sum ranges over all Dirichlet characters χ modulo N such that $\chi(-1) = (-1)^k$.

Theorem 3.7 (Corollary 10.3.15 [1])**.** *If χ is a real Dirichlet character modulo N, we have a natural direct sum decomposition given by*

$$M_k(\Gamma_0(N), \chi) = M_k^+(\Gamma_0(N), \chi) \oplus M_k^-(\Gamma_0(N), \chi),$$

where the spaces $M_k^{\pm}(\Gamma_0(N), \chi)$ are defined by

$$M_k^{\pm}(\Gamma_0(N), \chi) = \left\{ g \in M_k(\Gamma_0(N), \chi) \,:\, g = \pm i^k g|_k W_N \right\}.$$

3.2. **Modular Forms of Half-Integral Weight.** In this section, k will denote an odd integer and N will denote a positive integer always divisible by 4.

Definition 3.8. If d is an odd prime, then let $\left(\frac{c}{d}\right)$ be the usual Legendre symbol. For positive odd d, define $\left(\frac{c}{d}\right)$ by multiplicativity. For negative odd d, we let

$$\left(\frac{c}{d}\right) = \begin{cases} \left(\frac{c}{|d|}\right) & d < 0 \text{ and } c > 0, \\ -\left(\frac{c}{|d|}\right) & d < 0 \text{ and } c < 0. \end{cases}$$

Also, we let $\left(\frac{0}{\pm 1}\right) = 1$.

Definition 3.9. Define ε_d for odd d by

$$\varepsilon_d = \begin{cases} 1 & d = 1 \mod 4, \\ i & d = 3 \mod 4. \end{cases}$$

Remark 3.10. Throughout, we let $\sqrt{z}$ be the branch of the square root having argument is $(-\pi, \pi]$. Hence, $\sqrt{z}$ is a holomorphic function on the complex plane with the negative real axis removed.

Definition 3.11. Let k be an odd positive integer and N a positive integer divisible by 4. Furthermore, suppose that χ is a Dirichlet character modulo N. A meromorphic function $f(\tau)$ on $\mathbb{H}$ is called a meromorphic modular form of half-integral weight with Nebentypus χ and weight $k/2$ if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = \chi(d)\left(\frac{c}{d}\right)^k \varepsilon_d^{-k}(c\tau + d)^{k/2} f(\tau)$$

for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. If $f(\tau)$ is holomorphic on $\mathbb{H}$, then f is called a weakly holomorphic modular form. If in addition it is holomorphic at the cusps, then f is called a holomorphic modular form. In this case, if it is vanishing at the cusps, then f is called a cusp form.

Theorem 3.12 (Section 5.4 [13])**.** *Let k, N as above. We have the following direct sum decompositions*

$$M_{k/2}(\Gamma_1(N)) = \bigoplus_{\chi} M_{k/2}(\Gamma_0(N), \chi),$$

where the direct sum runs over all even characters modulo N. The same decomposition applies to the space of cusp forms and Eisenstein series.

Proposition 3.13. *Let k, N be as above. Let $f \in M_{k/2}(\Gamma_0(N), \chi)$. The Fricke involution W_N is defined by the formula*

$$f(\tau)|_{k/2} W_N = (-i\sqrt{N}z)^{-k/2} f\left(\frac{-1}{Nz}\right),$$

and $f(z)|_{k/2} W_N \in M_{k/2}(\Gamma_0(N), \left(\frac{N}{\cdot}\right)\bar{\chi})$

Remark 3.14. Note that here we define the Fricke involution by adding the i^k term of the definition for integral weight forms.

Theorem 3.15 (Theorem 5.22 [13])**.** *Let $f(\tau) = \sum_{n \geq 0} a_n q^n \in M_{k/2}(\Gamma_0(N), \chi)$, and define*

$$L(f, s) = \sum_{n \geq 1} a_n n^{-s}$$

and

$$\Lambda_N(f,s) = (2\pi)^{-s} N^{s/2} \Gamma(s) L(f,s).$$

Then $L(f,s)$ is absolutely convergent for $\Re(s) > 1+k/2$ and converges for $\Re(s) > k/2$. Furthermore, $\Lambda_N(f,s)$ can be analytically continued to a meromorphic function on the complex plane with possible simple poles at $s = 0$ and $s = k/2$ with residue a_0 and b_0. Furthermore, if $\tilde{f} = f|_{k/2} W_N$, then we have the functional equation

$$\Lambda_N(f,s) = \Lambda_N(\tilde{f}, k/2 - s).$$

Proposition 3.16 (Theorem 6.20 [9])**.** *If f is a newform (a normalized Hecke eigenform that is in the new subspace) in $S_k(\Gamma_0(N),\chi)$, then $a_n = \chi(n)\overline{a_n}$. Hence, a_n is real if $\chi(n) = 1$ while if $\chi(n) \neq 1$, then either a_n is not real or $a_n = 0$. It can happen that a_n is real for all n even if $\chi \neq 1$. For example, one has the newform of weight 3 and level 7 with character $\left(\frac{n}{7}\right)$*

$$q \prod_{n \geq 1} (1-q^n)^3 (1-q^{7n})^3.$$

Such forms must be CM forms and arise from Hecke grossencharacter (see Theorem 12.5 of [8]).

3.3. **Digression on Eisenstein Series of Integral and Half-Intergal Weight.** In this section, we review the general construction of the space of Eisenstein series for arbitrary level and weight. We will closely follow Chapter 7 of [13].

3.3.1. *Integral Weight Eisenstein Series.* Let k, t be positive integers, χ a character modulo N with conductor r, χ_1, χ_2 two primitive characters modulo r_1 and r_2 respectively. Denote by $|A(N,r)|$ the number of triples (t, χ_1, χ_2) satisfying

$$\chi = \chi_1\chi_2,\ tr_1r_2|N.$$

To every such triple, there exists an Eisenstein series $E_k^{\chi_1,\chi_2}(t\tau)$ which is an Eisenstein series of weight k, character χ and level tr_1r_2, thus an Eisenstein series of level N.

Lemma 3.17 (Lemma 7.28 [13])**.** *We have that*

$$|A(N,r)| = \sum\nolimits_{(c,N/c)|N/r}^{c|N} \phi((c, N/c)).$$

Theorem 3.18 (Theorem 7.11, 7.12 [13])**.** *Let $\chi, \chi_1, \chi_2, r_1, r_2$ be as above.*

1. *For $k \geq 3$ or $k = 2$, $\chi \neq \chi_{triv}$, the functions*

$$E_k^{\chi_1,\chi_2}(t\tau) = -L(\chi_1, 0)L(\chi_2, 1-k) + \sum_{n\geq 1}\left(\sum_{m|n}\chi_1(n/m)\chi_2(m)m^{k-1}\right)q^{tn}$$

constitute a basis of the space of Eisenstein series $\mathcal{E}_k(\Gamma_0(n), \chi)$ where (t, χ_1, χ_2) runs over all triples satisfying

$$\chi = \chi_1\chi_2,\ tr_1r_2|N.$$

2. *The functions*

$$E_1^{\chi_1,\chi_2}(t\tau) = -L(\chi_1, 0)L(\chi_2, 0) + \sum_{n\geq 1}\left(\sum_{m|n}\chi_1(n/m)\chi_2(m)m^{k-1}\right)q^{tn}$$

constitute a basis of $\mathcal{E}_1(\Gamma_0(N), \chi)$ where (t, χ_1, χ_2) runs over all triples satisfying $\chi = \chi_1\chi_2$, $tr_1r_2|N$ but only one of (t, χ_1, χ_2) and (t, χ_2, χ_1) can be chosen.

3. *For $k = 2$ and $\chi = \chi_{triv}$, the functions*

$$E_2^{\chi_1,\chi_2}(t\tau) = a_0(t, \chi_1, \chi_2) + \sum_{n\geq 1}\left(\sum_{m|n}\chi_1(n/m)\chi_2(m)m\right)q^{tn}$$

constitute a basis of $\mathcal{E}_2(\Gamma_0(N), \chi_{triv})$, where (t, χ_1, χ_2) runs over all triples satisfying $\chi_1\chi_2 = \chi_{triv}$, $tr_1r_2|N$ and $t \neq 1$ if $r_1 = r_1 = 1$. Here, $a_0(t, \chi_1, \chi_2)$ is defined as

$$a_0(t, \chi_1, \chi_2) = \begin{cases} 0 & \chi_1 \neq \chi_{triv} \\ -\frac{1}{24}\prod_{p|t}(1-p) & \chi_1 = \chi_{triv} \end{cases}$$

3.3.2. *Half-Integral Weight Eisenstein Series.* The explicit formulas are horrible. It may be better to just refer the section and use only what is needed.

3.4. **Bounds on Coefficients.**

3.4.1. *Cusp forms of integral weight.*

Proposition 3.19. *The space of cusp forms has a basis of Hecke eigenforms. Thus, the constant C_g we'll use is $\sum |x_i|$ where the x_i are the linear combination coefficients of the cusp form part. We take absolute value because the cusp form coefficients are not necessarily real so are the x_i.*

Proposition 3.20 (Deligne's bound)**.** *If f is a normalized Hecke eigenform, then $|a_n| \leq \sigma_0(n)n^{(k-1)/2}$.*

3.4.2. *Cusp forms of half-integral weight.* This is a difficult problem. There are no explicit upper bounds known for all n and all cusp forms, only rough estimates in terms of the level and the weight. The various proofs I have studied do not allow for an explicit computations of the big-O constant. This is an interesting fact that arises from odd dimensions for sphere packing, which are already recognized as harder than even dimensions.

Lemma 3.21. *Let $f(\tau) = \sum_{n\geq 1} a_n q^n \in S_{k/2}(\Gamma_0(N), \chi)$ where k is an odd integer and N is a positive integer divisible by 4. Then*

$$|a_n| \leq A_f(2\pi n)^{k/4},$$

where A_f is a constant that depends only on f.

Proof. One can adapt easily the proof on Lemma 5.28 [13]. □

Remark 3.22. We have to run numerical simulations for f and give conjectural results, unless we figure out how to compute it. I have tried many things without success :(

3.4.3. *Eisenstein series of integral weights.* Let $k \geq 3$ be a positive integer and consider a modular form $g = g_{\mathcal{E}} + g_c \in M_k(\Gamma_0(N), \chi)$ where $g_{\mathcal{E}}$ and g_c stand for the Eisenstein and the cusp form part. We can write

$$g_{\mathcal{E}}(\tau) = \sum_{i\in A(N,r)} x_i E_k^{\chi_1,\chi_2}(t\tau),$$

which can be expanded into

$$g_{\mathcal{E}}(\tau) = \sum_{i\in A(N,r)} (-x_i L(\chi_{1,i}, 0)L(\chi_{2,i}, 1-k)) + \sum_{n\geq 1} \sum_{\substack{t|N\\ t|n}} \sum_{\ell m = n/t} \sum_{uv|N/t} x_i \frac{\chi_{1,i}(\ell)}{\ell^{k-1}} \chi_{2,i}(m) \frac{1}{t^{k-1}} n^{k-1} q^n.$$

The main problem in analyzing this sum is that the infimum of

$$a_n/n^{k-1} = \sum_{\substack{t|N\\ t|n}} \sum_{\ell m = n/t} \sum_{uv|N/t} x_i \frac{\chi_{1,i}(\ell)}{\ell^{k-1}} \chi_{2,i}(m) \frac{1}{t^{k-1}}$$

over n may be positive as well as negative. In any case we have the following lower bound.

Lemma 3.23. *Let $n > N^3$ and let $x_{max} = \max_{i\in A(N,r)} |x_i|$. Then*

$$a_n/n^{k-1} = \sum_{\substack{t|N\\ t|n}} \sum_{\ell m = n/t} \sum_{uv|N/t} x_i \frac{\chi_{1,i}(\ell)}{\ell^{k-1}} \chi_{2,i}(m) \frac{1}{t^{k-1}}$$

$$\geq \min_{1\leq n\leq N^3} a_n/n^{k-1} - \left(\zeta(k-1) - \sum_{\ell=1}^{N} \frac{1}{\ell^{k-1}}\right) x_{max} \frac{\sigma_0(N)^2(\sigma_0(N)+1)}{2}.$$

Proof. Let $n > N^3$. Then the character sum given by a_n/n^{k-1} splits into summands with $\ell > N$ and $\ell \leq N$. The sum over $\ell \leq N$ has already appeared as a sum for $n \leq N^3$. Indeed, for each $\ell \leq N$, there exists a unique m such that $\ell m = \frac{n}{t}$. In the sum, the value of m is only important modulo N (or modulo v more precisely). For $t|n, N$ such that $t\ell m = n$, we can in fact view n as $\tilde{n} = n \bmod N^3$, i.e. t, ℓ, m can be taken $\leq N$. It follows that this sum for $\ell \leq N$ has already appeared as a sum for a value of $\tilde{n} \leq N^3$.

Now we would like to give a lower bound for the sum with terms $\ell > N$. We first observe that

$$\sum_{\substack{\ell > N \\ \ell | n/t}} \frac{1}{\ell^{k-1}} \leq \sum_{\ell=N}^{\infty} \frac{1}{\ell^{k-1}} < \zeta(k-1).$$

Also, note that

$$\sum_{\substack{t|N \\ t|n}} \sum_{\substack{\ell m = n/t \\ \ell > N}} \sum_{uv|N/t} x_i \frac{\chi_{1,i}(\ell)}{\ell^{k-1}} \chi_{2,i}(m) \frac{1}{t^{k-1}} \geq \sum_{\substack{t|N \\ t|n}} \sum_{\substack{\ell m = n/t \\ \ell > N}} \sum_{uv|N/t} -\frac{x_{\max}}{\ell^{k-1}}.$$

The most inner sum

$$\sum_{uv|N/t} 1$$

can be bounded by

$$\sigma_0(N)(\sigma_0(N)+1)/2$$

and the outer sum by $\sigma_0(N)$ after replacing

$$\sum_{\substack{\ell > N \\ \ell | n/t}} \frac{1}{\ell^{k-1}}$$

by

$$\zeta(k-1) - \sum_{\ell=N}^{\infty} \frac{1}{\ell^{k-1}}.$$

$\square$

Remark 3.24. add some details for the above proof. Note that it is not useful for g but only for $\tilde{g}$ since the coefficients of index smaller than T all have values 0 except $a_1 = 1$.

3.5. **Digression on Basmaji's Algorithm.** The only known algorithm so far for computing the space of cusp forms of half-integral weight is by an algorithm of Basmaji which assumes the level is divisible by 16. However, Magma computes bases for general level. By analyzing the Magma algorithm of Donnelly and Stein, Purkait was able to write down the algorithm it is relying on. It is in fact a variant of Basmaji's algorithm, obtained by modifying in a very elementary way its construction. We recall the procedure here for completeness.

Let $k > 1$ be an odd integer and $N \in \mathbb{N}$ be an integer divisible by 16. Let χ be a Dirichlet character modulo N. Then, Basmaji in his thesis gave the following algorithm for computing $S_{k/2}(\Gamma_0(N), \chi)$. The main idea is to use theta-series.

Let

$$\Theta(\tau) := 1 + 2\sum_{n\geq 1} q^{n^2}$$

and

$$\Theta_1(\tau) := \sum_{\substack{n\geq 1\\ n=1 \bmod 2}}$$

be the usual theta-series where $q = e^{2\pi i\tau}$. By the celebrated work of Serre and Stark on basis of modular forms of half integral weight, level N, and arbitrary characters, we know that $\Theta \in M_{1/2}(\Gamma_0(4), \chi_{\mathrm{triv}})$ and that $\Theta_1 \in M_{1/2}(\Gamma_0(16), \chi_{\mathrm{triv}})$ (site book ono bringhman etc.).

Let χ_{-1} denote the non trivial Dirichlet character modulo 4 and

$$S := S_{\frac{k+1}{2}}(\Gamma_0(N), \chi\chi_{-1}^{\frac{k+1}{2}}.$$

Basmaji defines the embedding

$$\phi : S_{k/2}(\Gamma_0(N), \chi) \to S \times S, f \mapsto (f\Theta, f\Theta_1).$$

Let U be the subspace of $S \times S$ consisting of pairs (f_1, f_2) such that

$$f_1\Theta_1 = f_2\Theta$$

holds. Then U is isomorphic to $S_{k/2}(\Gamma_0(N), \chi)$ via the map

$$(f_1, f_2) \mapsto f_1/\Theta = f_2/\Theta_1.$$

Since there are known methods for computing bases of modular forms with integral coefficients, one has simply to compute a basis of S, solve the linear system of equations defining U (using eventually the Sturm bound that uniquely defines a form) and compute a basis of U, thus a basis of $S_{k/2}(\Gamma_0(N), \chi)$.

The idea to generalize Basmaji's algorithm is just to realize that the only reason why we impose $16|N$ is to show that $f\Theta_1$ belongs to S. We can circumvent this problem by working with other theta-series and this is what Stein and onnelly have done.

Suppose that $4|N$, $16 \nmid N$, and define

$$\Theta_2(\tau) := \Theta(2\tau) = 1 + 2\sum_{n\geq 1} q^{2n^2} \in M_{1/2}(\Gamma_0(8), \chi_8),$$

where $\chi_8 = \left(\frac{8}{\bullet}\right)$ is the Dirichlet character modulo 8. Let $N' = lcm(N, 8)$ and define $S' = S_{\frac{k+1}{2}}(\Gamma_0(N'), \chi\chi_8\chi_{-1}^{\frac{k+1}{2}})$. Then we have an embedding as before

$$\phi : S_{k/2}(\Gamma_0(N), \chi) \to S \times S',\ f \mapsto (f\Theta, f\Theta_2).$$

Let U' be the subspace of $S \times S'$ consisting of pairs (f_1, f_2) such that

$$f_1\Theta_2 = f_2\Theta.$$

As before, this gives a system of linear equations that we can solve and recover a basis for $S_{k/2}(\Gamma_0(N), \chi)$.

Remark 3.25. Give a reference to the proof.

4. Dual Bounds with Modular Forms

4.1. **Voronoi Type Summation Formulas.** We now present a proof of Proposition 1.3 and 1.4 which is highly based on the proof of Cohn and Triantafillou [3].

Proof. Let k be an integer or a half-integer and N the level. Let $g = \sum_{n\geq 0} a_n q^n \in M_k(\Gamma_0(N), \chi)$, $\tilde{g} = (-i)^{-k} g|_k W_N = \sum_{n\geq 0} b_n q^n$, and let $f : \mathbb{R}^{2k} \to \mathbb{R}$ be a Schwartz function. Define

$$S = \sum_{n\geq 1} a_n f(\sqrt{n}).$$

By the Mellin inversion formula, we can write

$$a_n f(\sqrt{n}) = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} \frac{a_n}{(\sqrt{n})^s}\mathcal{M}f(s)ds,$$

for all $\Re(s) = \sigma > 0$. Replacing in S and exchanging the sum and integral gives

$$S = \frac{1}{2\pi i}\int_{\sigma - i\infty}^{\sigma + i\infty} L(g, s/2)\mathcal{M}f(s)ds,$$

for $\sigma/2 > k$. Let $\varepsilon > 0$ and set $\sigma = 2k + \varepsilon$. Then, we can shift the contour of the integral representation of S to $\sigma = -\varepsilon$ as long as we account for the poles at $s = 0$ and $s = 2k$ by Cauchy's theorem. For the line $\sigma = -\varepsilon$, we obtain

$$S = \frac{1}{2\pi i}\int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} L(g, s/2)\mathcal{M}f(s)ds - a_0 f(0) + 2b_0\left(\frac{2\pi}{\sqrt{N}}\right)^k \frac{1}{\Gamma(k)}\mathcal{M}f(2k).$$

If we define

$$T := \frac{1}{2\pi i}\int_{-\varepsilon - i\infty}^{-\varepsilon + i\infty} L(g, s/2)\mathcal{M}f(s)ds,$$

and use the identity

$$\hat{f}(0) = \frac{2\pi^k}{\Gamma(k)}\mathcal{M}f(2k),$$

which holds for $d = 2k$, then we obtain

$$a_0 f(0) + S = T + \left(\frac{2}{\sqrt{N}}\right)^k b_0 \hat{f}(0).$$

We now make use of the change of variable $s \mapsto 2k - s$ and T becomes

$$T = \frac{1}{2\pi i}\int_{2k+\varepsilon - i\infty}^{2k+\varepsilon + i\infty} L\left(g, \frac{2k - s}{2}\right)\mathcal{M}f(2k - s)ds.$$

Since

$$L\left(g, \frac{2k - s}{2}\right) = (2\pi)^{k-s} N^{\frac{s-k}{2}} \frac{\Gamma(s/2)}{\Gamma(k - s/2)} L(\tilde{g}, s/2),$$

we obtain

$$T = \left(\frac{2}{\sqrt{N}}\right)^k \frac{1}{2\pi i}\int_{2k+\varepsilon - i\infty}^{2k+\varepsilon + i\infty} \left(\frac{4}{N}\right)^{-s/2} L(\tilde{g}, s/2)\mathcal{M}\hat{f}(s)ds,$$

using the fact that

$$\mathcal{M}\hat{f}(s) = \frac{\pi^{k-s}\Gamma(s/2)}{\Gamma(k - s/2)}\mathcal{M}f(2k - s).$$

From the series representation of $L(\tilde{g}, s/2)$ and by switching the order of integration and summation, we find

$$T = \left(\frac{2}{\sqrt{N}}\right)^k \sum_{n\geq 1} \frac{1}{2\pi i} \int_{2k+\varepsilon-i\infty}^{2k+\varepsilon+i\infty} \frac{b_n}{(4n/N)^{s/2}} \mathcal{M}\hat{f}(s)ds = \left(\frac{2}{\sqrt{N}}\right)^k \sum_{n\geq 1} b_n \hat{f}\left(\frac{2\sqrt{n}}{\sqrt{N}}\right).$$

Hence we obtain for $f : \mathbb{R}^{2k} \to \mathbb{R}$ a Schwartz function the Voronoi type summation formula

$$\sum_{n\geq 0} a_n f(\sqrt{n}) = \left(\frac{2}{\sqrt{N}}\right)^k \sum_{n\geq 0} b_n \hat{f}\left(\frac{2\sqrt{n}}{\sqrt{N}}\right).$$

□

Remark 4.1. Note that the same formulas hold by taking the real and imaginary part on both sides.

4.2. **The Cohn-Triantafillou Algorithm.**

Theorem 4.2 (Cohn-Elkies)**.** *state the theorem for the linear programming bound.*

Theorem 4.3 (Dual Bound)**.** *If $\mu = \delta_0 + \nu$ is a tempered distribution on $\mathbb{R}^d$ with $\nu \geq 0$ having support $supp(\nu) \subseteq \{x \in \mathbb{R}^d : |x| \geq r\}$ and $\hat{\mu} \geq c\delta_0$ for some positive c, then the Cohn-Elkies linear programming bound in $\mathbb{R}^d$ is at least*

$$c(\frac{r}{2})^d.$$

The measure μ proposed by Cohn and Triantafillou is $\mu = \sum_{n\geq 0} a_n \delta_{\sqrt{n}}$ where the a_n are Fourier coefficients of modualr forms of even weight and level N. Our formuals generalize this method to all modular forms of all integral weight and half-integral weight, with or without multiplier systems. The Fourier transform of μ is simply the other side of the above formula $\hat{\mu} = (2/\sqrt{N})^k \sum_{n\geq 0} b_n \delta_{2\sqrt{n}/\sqrt{N}}$. The value c in the dual bound theorem that Cohn and Triantafillou picked is the one given by proposition 4.3. We see that it takes into account that b_0 is non zero, excluding automatically cusp forms from the search. Need to find a way to include them. The $\sqrt{T}$ comes from the following considerations. It is the largest radius at which the delta function of μ start being non zero (excluding the term $a_0 f(0)$).

The algorithm of Cohn and Triantafillou where T appears is the following. What they do is actually pick T to be the largest index for which a modular form has non vanishing coefficients.

Proposition 4.4 (Algorithm)**.** *Suppose g is a modular form of weight k for $\Gamma_1(N)$ and Fourier coefficients a_n. Let $\tilde{g}$ be given by Fourier coefficients b_n. Suppose that a_n, b_n obey the following inequalities*

1. $a_n \geq 0, b_n \geq 0,$
2. $a_0 = 1, b_0 > 0,$
3. $a_n = 0$ *for* $1 \leq n \leq T.$

Then, the linear programming bound in $\mathbb{R}^{2k}$ is at least

$$b_0(2/\sqrt{N})^k(\sqrt{T}/2)^{2k}.$$

This is a finite linear programming problem in the sense that the number of variables involved is finite. However, there are infinitely many constraints to check. This is the technical part. Even if we can show condition 2 and 3, we have to develop a method to check for condition 1. We can implement a method to check numerically the constraints for a large index $n \leq M$ but there will still be infinitely many non negativity constraints to check.

See page 9 of Cohn and Triantafillou for an idea of how to implement M in the algorithm. In our case, because the Fricke involution twists the character for half integral weights and conjugates it for integral weights, we may need to optimize over two spaces.

4.3. **Algorithm for integral weight forms.**

- Fix a weight k and level N. Fix a radius T and an M positive which we will call the *imposed positivity threshold.*
- Use function ... to construct $M_k(\Gamma_1(N))$ and a basis for this space.

· Write every basis element as $g^j = \sum_{n\geq 1} a_n^j q^n$ and their involution as $\tilde{g}^j = \sum_{n\geq 1} b_n^j q^n$. Write $g = \sum_j^{\dim(M_k(\Gamma_1(N))} x_j g^j$ and let $a_n^j = a_{n,\Re}^j + i a_{n,\Im}^j$ and do the same for b_n.

· We optimize over the coefficients x_j by solving independently the two linear programs:

maximize : $\sum_j x_j b_{0,\Re}^j$

subject to :
- $1 = \sum_j x_j a_{0,\Re}^j$
- $0 = \sum_j x_j a_{n,\Re}^j$ for $1 \leq n < T$,
- $0 \leq \sum_j x_j a_{n,\Re}^j$ for $T \leq n \leq M$, and
- $0 \leq \sum_j x_j b_{n,\Re}^j$ for $1 \leq n \leq M$

maximize : $\sum_j x_j b_{0,\Im}^j$

subject to :
- $1 = \sum_j x_j a_{0,\Im}^j$
- $0 = \sum_j x_j a_{n,\Im}^j$ for $1 \leq n < T$,
- $0 \leq \sum_j x_j a_{n,\Im}^j$ for $T \leq n \leq M$, and
- $0 \leq \sum_j x_j b_{n,\Im}^j$ for $1 \leq n \leq M$

Basically, we don't really care about what kind of character we use or if the coefficients are complex or real. We can just split the program into real and complex part and optimize over the two independently and see which one offers the best candidate of "b_0". Impose that the x_j are real though.

References

[1] H. Cohen, F. Strömberg, *Modular Forms, A Classical Approach*, Graduate Studies in Mathematics 179, American Mathematical Society, 2017.

[2] H. Cohn, *New upper bounds on sphere packings II*, Geom. Topol. 6 (2002).

[3] H. Cohn, N. Triantafillou, *Dual Linear Programming Bounds for Sphere Packing Using Modular Forms*, https://arxiv.org/pdf/1909.04772.pdf

[4] H. Cohn, A. Kumar, S. D. Miller, D. Radchenko, and M. S. Viazovska, *The sphere packing problem in dimension 24*, Ann. of Math. (2) 185 no. 3 (2017).

[5] H. Cohn, N. Elkies, *New upper bounds on sphere packings I*, Ann. of Math. (2) 157 no. 2 (2013).

[6] F. Diamond, J. Shurman, *A First Course in Modular Forms* , Springer-Verlag New York 2005.

[7] T. C. Hales, *A proof of the Kepler conjecture* , Ann. of Math. (2) 162 (2005), no. 3, 1065–1185. doi:10.4007/annals.2005.162.1065 MR2179728.

[8] H. Iwaniec, *Topics in Classical Automorphic Forms*, Graduate Studies in Mathematics 17, American Mathematical Society, 1997.

[9] H. Iwaniec, E. Kowalski *Analytic Number Theory*, Colloquium Publications 53, American Mathematical Society, 2004.

[10] K. Ono, *The Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*, CBMS 102, American Mathematical Society, 2003.

[11] A. Thue, *Om nogle geometrisk-taltheoretiske Theoremer*, Forhandlingerne ved de Skandinaviske Naturforskeres 14 (1892), 352-353.

[12] M. S. Viazovska, *OThe sphere packing problem in dimension 8*, Ann. of Math. (2) 185, no. 3, 991-1015 (2017).

[13] X. Wang, D. Pei, *Modular Forms with Integral and Half-Integral Weights* , Science Press, Beijing and Springer Berlin Heidelberg 2012.

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