Fun With The Fundamental Group

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April 14, 2020

First Definitions:

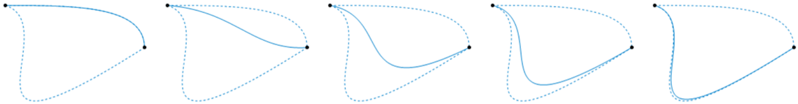
A **path** in a space X is a continuous map $f : I \to X$ where I is the unit interval.

A **homotopy** of paths in X is a family of maps $f_t : I \to X$, $0 \leqslant t \leqslant 1$, such that:

1. The endpoints $f_t(0) = x_0$ and $f_t(1) = x_1$ are independent of t.
2. The associated map $F : I \times I \to X$ defined by $F(s, t) = f_t(s)$ is continuous.

When two paths f_0 and f_1 are connected in this way by a homotopy f_t, they are said to be homotopic.

Example:



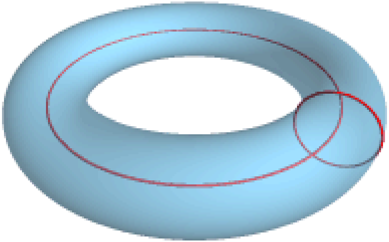
Example: Linear Homotopies:

Any two paths in $\mathbb{R}^n$ with the same endpoints are homotopic, by the homotopy $f_t(s) = (1 - t)\, f_0(s) + t\, f_1(s)$

More on Homotopy

Proposition: The relation of homotopy on paths with fixed endpoints in any space is an equivalence relation!

Example of a topological space with more than one homotopy equivalence class:



Homeomorphism

Definition: A map $f : X \to Y$ is called a **homotopy equivalence** if there is a map $g : Y \to X$ such that $fg \simeq gf \simeq \mathbb{1}$ (the identity map). We then say that X and Y are **homotopically equivalent**.

Definition: A map $f : X \to Y$ is called a **homeomorphism** if f is bijective, continuous and f^{-1} is continuous.

Remark: Every homeomorphism is a homotopy equivalence, but the converse is not true! Example, $\mathbb{R}^n$ and point!

Introducing the **Fun**damental Group

Theorem: Let $\pi_1(X, x_0)$ be the set of all homotopy equivalence classes of loops with basepoint $x_0 \in X$. $\pi_1(X, x_0)$ forms a group with operation $[f][g] = [f \cdot g]$, where "$\cdot$" denotes path-product.

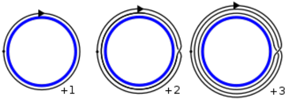
If X is path-connected, the group $\pi_1(X, x_0)$ is independent of the choice of basepoint x_0, up to isomorphism. In this case, we may write $\pi_1(X, x_0)$ as $\pi_1(X)$.

Example: $\mathbb{R}^n$: By our construction of the linear homotopy sending any path to any path, there is only one equivalence class! Thus:

$$\pi_1(\mathbb{R}^n) = 0.$$

Computing *Fun*damental Group of Circle!

Theorem: $\pi_1(S^1) \cong \mathbb{Z}$, generated by the homotopy class of the loop $w(s) = (\cos(2\pi s), \sin(2\pi s))$ based at $x_0 := (1, 0)$.



Definition: Given a topological space X, a **covering space** of X is a topological space $\widetilde{X}$ and a map $p : \widetilde{X} \to X$ such that the following condition is satisfied:

For each point $x \in X$, there is an open neighbourhood U of X such that $p^{-1}(U)$ is a union of disjoint open sets, each of which is mapped homeomorphically onto U by p.

Computing *Fun*damental Group of Circle!

Definition: Given a path $f : I \to X$ and a covering space $\widetilde{X}$ with the associated map $p : \widetilde{X} \to X$, we say that the path $\widetilde{f} : I \to \widetilde{X}$ is a **lift** of the path f if they satisfy $p\widetilde{f} = f$.

Lemma 1: For each path $f : I \to X$ starting at $x \in X$ and each $\widetilde{x} \in p^{-1}(x)$, there exists a unique lift $\widetilde{f} : I \to \widetilde{X}$ starting at $\widetilde{x}$.

Lemma 2: For each homotopy of paths $f_t : I \to X$ starting at $x \in X$ and each $\widetilde{x} \in p^{-1}(x)$, there exists a unique lifted homotopy $\widetilde{f}_t : I \to \widetilde{X}$ of paths starting at $\widetilde{x}$.

Computing *Fun*damental Group of Circle!

Proof: We use $\mathbb{R}$ as a covering space of S^1, and $p : \mathbb{R} \to S^1$ given by $p(s) = (\cos(2\pi s), \sin(2\pi s))$.

Define $w_n(s) = (\cos(2\pi ns), \sin(2\pi ns))$ for $n \in \mathbb{Z}$.

Note that $[w_1]^n = [w_n]$, thus the theorem is equivalent to the fact that every loop in S^1 based at $x_0 = (1, 0)$ is homotopic to w_n for some unique $n \in \mathbb{Z}$.

Let $f : I \to S^1$ be a loop with basepoint x_0. Lemma 1 $\Rightarrow$ there is a unique lift $\widetilde{f} : I \to R$ starting at 0. This path $\widetilde{f}$ ends at some integer $n \in \mathbb{R}$ since $p\widetilde{f}(1) = f(1) = x_0$, and since $p^{-1}(x_0) = \mathbb{Z} \subset \mathbb{R}$.

Computing *Fun*damental Group of Circle!

Another path from 0 to n in $\mathbb{R}$ is $\widetilde{w}_n$.

$\widetilde{f} \simeq \widetilde{w}_n$ by the linear homotopy $(1-t)\widetilde{f} + t\widetilde{w}_n$.

Composing this homotopy with p gives a homotopy from f to w_n

$$\Rightarrow f \simeq w_n$$

Remains to show n is uniquely determined by $[f]$. Suppose that $w_m \simeq f \simeq w_n$. Let f_t be a homotopy from $w_m = f_0$ to $w_n = f_1$. Lemma 2 $\Rightarrow$ this homotopy lifts to a homotopy $\widetilde{f}_t$ of paths starting at 0. By uniqueness in Lemma 1, $\widetilde{f}_0 = \widetilde{w}_m$ and $\widetilde{f}_1 = \widetilde{w}_n$.

Endpoints of $\widetilde{f}_t(1)$ independent of t.

$t = 0 \Rightarrow \widetilde{f}_0(1) = \widetilde{w}_m(1) = m$
$t = 1 \Rightarrow \widetilde{f}_1(1) = \widetilde{w}_n(1) = n$

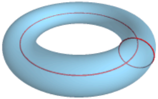
$$\Longrightarrow m = n \qquad \square$$

Applications of $\pi_1(S^1)$

Proposition: If X and Y are two path-connected topological spaces, then $\pi_1(X \times Y) \cong \pi_1(X) \times \pi_1(Y)$.

Example: The 2-torus $\mathbb{T}^2 = S^1 \times S^1$, so:

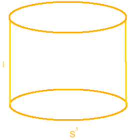
$$\pi_1(S^1 \times S^1) \cong \pi_1(S^1) \times \pi_1(S^1) \cong \mathbb{Z} \times \mathbb{Z}$$



Abelian!

Example: The unit cylinder $S^1 \times I$, so:

$$\pi_1(S^1 \times I) \cong \pi_1(S^1) \times \pi_1(I) \cong \mathbb{Z} \times 1 \cong \mathbb{Z}$$



Fundamental Theorem of Algebra!

Proof: Let $p(z) = z^n + a_1 z^{n-1} + ... + a_n$

If $p(z)$ has no roots in $\mathbb{C}$, then $f_r(s) = \frac{p(re^{2\pi i s})/p(r)}{|p(re^{2\pi i s})/p(r)|}$ defines a loop in the unit circle $S^1 \subset \mathbb{C}$ based at 1, for all $r \geqslant 0$, $r \in \mathbb{R}$.

As r varies, f_r is a homotopy of loops based at 1. f_0 is the trivial loop, and $f_0 \simeq f_r$ via this homotopy, so $[f_r] = [0] \in \pi_1(S^1)$.

Now fix r large such that $r > \max\{1, |a_1| + ... + |a_n|\}$, then for $|z| = r$, we have that

$$\begin{aligned} |z^n| &> (|a_1| + ... + |a_n|)\,|z^{n-1}| \\ &> |a_1 z^{n-1}| + ... + |a_n| \\ &\geqslant |a_1 z^{n-1} + ... + a_n| \end{aligned}$$

Fundamental Theorem of Algebra!

It follows that $p_t(z) = z^n + t\left(a_1 z^{n-1} + \dots + a_n\right)$ has no roots on circle $|z| = r$ when $0 \leqslant t \leqslant 1$.

Replacing p by p_t in the formula for f_r:

$$\tilde{f}_r(s) = \frac{p_t(re^{2\pi is})/p_t(r)}{|p_t(re^{2\pi is})/p_t(r)|}$$

and letting t go from 1 to 0, we obtain a homotopy $\tilde{f}_r(s)$ from f_r to $w_n(s) = e^{2\pi ins}$, so $[w_n] = [f_r] = [0]$.

But by previous computation of $\pi_1(S^1)$:

$$[w_n] = [0] \Rightarrow n = 0 \qquad \square$$

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Thanks! :)