Spherical Harmonics and Representations of Lie Groups

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First Definitions:

Definition: A **Lie Group** is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps. These two requirements can be combined to the single requirement that the mapping

$$\mu : G \times G \to G \quad \mu(x, y) = x^{-1}y$$

be a smooth mapping of the product manifold into G.

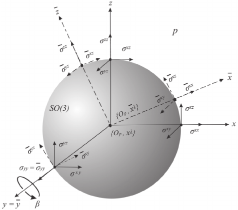
Definition: Given a Lie group G, and a vector space V, of dimension n, **a linear representation** of G of order n is a group homomorphism, $U : G \to GL(V)$, such that the map,$g \to U(g)(u)$, is continuous $\forall u \in V$. The space V, called the representation space.

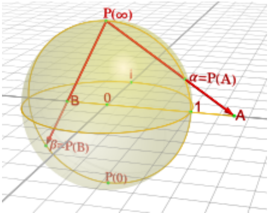
First Definitions:

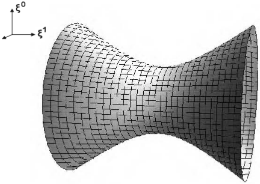
Definition: Let V be a G-module. A subspace $U \subset V$ which is G-invariant, meaning $gu \in U$ for $g \in G, u \in U$ is called a submodule of V or a subrepresentation.

Definition: A representation is called **irreducible** if it does not contain any non-trivial submodule.

Examples of Lie Groups:







Examples:

$SO(n) = \{Q \in GL(n,\mathbb{R}) : Q^\top Q = 1, det(Q) = 1\}$

$\mathrm{R}^n = \{(x_1, \cdots, x_n) : x_i \in \mathbb{R}\}$

$\mathrm{SL(n,R)} = \{\ \mathrm{Q} \in GL(n,\mathbb{R}) : det(Q) = 1\}\ PSL(n,\mathbb{R})$

Special Orthogonal Group SO(3)

Definition: The special orthogonal group SO(3) of unital rotations is defined as

$$SO(3) = \{r \in GL(3, \mathbb{R}) : r^\top r = 1, \det r = 1\}$$

We know that a harmonic function f is one that satisfies Laplace's equation:

$$\sum_{i=0}^{n} \frac{\partial^2 f}{\partial x_i^2} = 0$$

With a little algebraic manipulation, for n=3 we can rewrite Laplace's equation in terms of spherical coordinates as:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 f}{\partial\varphi^2} = 0$$

Spherical Harmonics

Definition: A function f defined on $SO(3)$, which satisfies Laplace's equation is called a **spherical harmonic**.
Definition: A function, $f : \mathbb{R}^n \to \mathbb{R}$ or $\mathbb{C}$ is called homogeneous of degree k iff $f(tx) = t^k f(x)\ \forall x \in \mathbb{R}^n, t > 0$.
Definition: Let $\mathcal{P}_k(n+1)$ denote the space of homogeneous polynomials of degree k in $n+1$ variables. Then, we can define

$$\mathcal{H}_k(n+1) = \{f \in \mathcal{P}_k(n+1) : \Delta f = 0\}$$

Which then leads us to define $\mathcal{H}_k(S^n)$, the space of spherical harmonics, which is the restriction of $\mathcal{H}_k(n+1)$ onto $SO(n)$.

Link Between Spherical Harmonics And Representations Of SO(3)

As it turns out, the spaces $\mathcal{H}_k(S^n)$ defined previously end up generating all possible irreducible representations of $SO(3)$. We will build up to this using the following theorems and lemmas.

Proposition: Let G be a compact group. If V is a submodule of the G-module U, then there is a submodule W such that $U = V \oplus W$. Each G-module is a direct sum of irreducible submodules.
Theorem:(Shur's lemma) Let G be any group and let V and W be irreducible G-modules. Then

1. Every morphism $V \to W$ is either trivial or an isomorphism
2. Every morphism f : $V \to V$ is of the form f(v) = λ v for $\lambda \in \mathbb{C}$
3. Dim(Hom(V,W)) = 1 if $V \cong W$ else Dim(Hom(V,W)) = 0

Link Between Spherical Harmonics And Representations Of SO(3)

Definition: The **character** χ_V of a representation V is the function :

$$\chi_V : G \to \mathbb{C}, \quad g \mapsto Tr(l_g)$$

Where $l_g : V \to V$ is the linear map $v \mapsto gv$. If V is irreducible then χ_V is an irreducible character.

Theorem: A representation is determined up to isomorphism by its character.

Proof: Let $V = \bigoplus_j n_j V(j)$ be a decomposition of V into irreducible representations. Then, $\chi_v = \sum_j n_j \chi_{v(j)}$ where $n_j =< \chi_v, \chi_{v(j)} >$.

Note that we are using the inner product $< \chi_v, \chi_w >= \int_g \chi_v(g) \bar{\chi}_w(g) dg$ which has orthogonal properties.

$SU(2)/\mathbb{Z}_2 \cong SO(3)$

We define the special unitary group of dimention 2 as

$$SU(2) = \left\{ \begin{pmatrix} \alpha & -\overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix} : \quad \alpha, \beta \in \mathbb{C}, |\alpha|^2 + |\beta|^2 = 1 \right\}$$

We know that the algebra over SU(2) is generated by

$$E_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad E_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

Thus, consider the map $\Phi : SU(2) \to SO(3)$ such that a matrix $M \mapsto \Phi_M(N) := MNM^{-1}$ for all $N \in \mathfrak{so}(2)$. We take without proof that this is a surjective homomorphism, and its kernel turns out to be the matrices such that $AE_iA^{-1} = E_i$ for i=1,2,3. This yields that $\ker(\Phi) = \{\pm I\} \cong \mathbb{Z}_2$. It follows that $SU(2)/\mathbb{Z}_2 \cong SO(3)$.

Irreducible Representations of SU(2) and SO(3)

There is a link between irreducible representations of the two groups.
Definition: The representations of $SU(n), U(n)$ and $GL(n, \mathbb{C})$ on $\mathbb{C}$ in which elements of the groups simply act by matrix multiplication are called the **standard** representations.
Letting V_1 be the standard representation for SU(2), we use the fact that every other irreducible representation of SU(2), V_n, will have the nth symmetric powers of V_1, as their representation space.

Letting V_n , be the space of homogeneous polynomials of degree n in two variables z_1 and z_2. The dimension of V_n is n + 1. Viewing polynomials as functions on $\mathbb{C}^2$, we obtain a left action of $GL(2, \mathbb{C})$ and hence SU(2) on the polynomials by letting $(gP)z = P(zg)$ where

Irreducible Representations of SU(2) and SO(3)

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad P \in \mathbb{C}[z_1, z_2] \quad z = (z_1, z_2)$$

And so zg is computed to be $(az_1 + cz_2, bz_1 + dz_2)$. Each g acts as a homogeneous linear transformation $\implies$ the subspaces $V_n \subset \mathbb{C}[z_1, z_2]$ are SU(2) invariant. We pick the basis $\{f_k = z_1^k z_2^{n-k} \quad 0 \leqslant k \leqslant n\}$ in order to show that the V_n are irreducible representations.

Proposition: The V_n are irreducible

Proof: It suffices to show that each SU(2)-equivariant endomorphism A of V_n is a multiple of the identity. So let A be equivariant and, for $a \in U(1)$, set

$$g = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in SU(2)$$

Irreducible Representations of SU(2) and SO(3)

Then $g_a f_k = a^{2k-n} f_k$ and $g_a A f_k = A g_a f_k = A a^{2k-n} fk = a^{2k-n} A f_k$. Now choose a such that all the powers a^{2k-n}, $0 \leqslant k \leqslant n$ are distinct. It is not difficult to verify that, with a chosen this way, the a^{2k-n} eigenspace of g_n in V_n is generated by f_k. Thus $Af_k = c_k f_k$ for some $c_k \in \mathbb{C}$. We now consider the real rotations

$$r_t = \begin{pmatrix} cos(t) & -sin(t) \\ sin(t) & cos(t) \end{pmatrix} \in SU(2) \quad t \in \mathbb{R}$$

And then we get that

$$\begin{aligned} A r_t f_k &= A(z_1 cos(t) + z_2 sin(t))^n \\ &= \sum_k \binom{n}{k} cos^k(t) sin^{n-k}(t) A f_k \\ &= \sum_k \binom{n}{k} cos^k(t) sin^{n-k}(t) c_k f_k \end{aligned}$$

Irreducible Representations of SU(2) and SO(3)

But also we have that

$$r_t A f_k = \sum_k \binom{n}{k} cos^k(t) sin^{n-k}(t) c_n f_k$$

This implies that the coefficients $c_n = c_k \implies A = c_n * I$. $\square$

Now, if we let $e(t) = \begin{pmatrix} e^{it} & 0 \\ 0 & e^{-it} \end{pmatrix}$, we know that every element in SU(2) is conjugate to a diagonal matrix $\implies$ they're conjugate to e(t) for some t. Further, $e(t)$ and $e(s)$ are conjugate iff $s \equiv \pm t (mod 2\pi)$. Thus for some class function $q : SU(2) \to \mathbb{C}$, q_e where $t \mapsto q \circ e(t)$ is an even 2π periodic function. The space of continuous class functions may then be identified with the space of 2π periodic functions from $\mathbb{R}$ to $\mathbb{C}$. The character χ_n of V_n at $e(t)$ has value

$$\sum_{k=0}^{n} e^{i(n-2k)t}$$

Irreducible Representations of SU(2) and SO(3)

When $t = k * \pi$ $k \in \mathbb{Z}$, the sum simply evaluates to $\frac{sin(n+1)}{sin(t)}$. We denote this function by $\kappa_n(t)$ and by a theorem called the addition theorem we get the identity

$$\kappa_n(t) = cos(nt) + \kappa_{n-1}cos(t)$$

This means that the κ_i generate the same vector space as $\{cos(kt) : k \in \mathbb{N}\}$. This is well known from Fourier Analysis to be uniformly dense in the space of 2π periodic functions $\implies$ the characters χ_n are uniformly dense in the space of class functions of SU(2).
Prop: Every irreducible unitary representation of SU(2) is isomorphic to some V_n.
Proof: Suppose the irreducible representation W with character χ is different from all the V_n. By orthogonality of characters $< \chi, \chi_n >= 0$ and $< \chi, \chi >= 1$. Contradiction because the χ_n generate a dense subspace.

Irreducible Representations of SU(2) and SO(3)

Now, we make use of the existence of the morphism Φ mentioned earlier from SU(2) to SO(3) whose kernel is $\{\pm I\}$. If W is an irreducible representation of SO(3), then the corresponding representation $\Phi^* W$ of SU(2) is irreducible and -I acts as the identity. Conversely, if -I acts as the identity on the SU(2) representation V, then we obtain an associated representation of SO(3). Therefore, there is a correspondence between the representations of SO(3) with those V_n of SU(2) for which -I acts as the identity.
We know that -I acts as multiplication by -1^n on V_n, thus the V_{2n} yield the irreducible representations W_n, of SO(3). Note that W_n has dimension $2n + 1$. We will show that the W_n, can be realized as suitable SO(3)-invariant spaces of polynomials, i.e. the spherical harmonics.

$$\Phi(e(t)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(2t) & -sin(2t) \\ 0 & sin(2t) & cos(2t) \end{pmatrix} = R(2t)$$

Thus the character χ_{W_n} at $R(t) \equiv \chi_{2n}$ at $e(\frac{t}{2})$ which is $\sum_0^{2n} e^{i(n-k)t}$. We now restrict our attention back to $\mathcal{P}_l(3)$ the complex vector space of homogeneous polynomials in three variables of degree l, viewed as functions on $\mathbb{R}^3$ and $\mathcal{H}_l(3)$, the harmonics functions.

Lemma: $\text{Dim}(\mathcal{P}_l(3)) = \frac{(l+1)(l+2)}{2}$ and $\text{Dim}(\mathcal{H}_l(3)) = 2l+1$.

Proof: We pick the basis $\{x_1^p x_2^q x_3^r \;:\; p+q+r = l\}$ of $\mathcal{P}_l(3)$. By a combinatorial argument, there are k+1 ways of choosing non negative triple (p, q, r) such that their sum is l.

$$\implies Dim(\mathcal{P}_l(3)) = \sum_{k=0}^{l} (k+1) = \frac{(l+1)(l+2)}{2}$$

Frame Title

Then, any polynomial $p \in \mathcal{P}_l(3)$ can be written as

$$p(x_1, x_2, x_3) = \sum_{k=0}^{l} \frac{x_1^k}{k!} p_k(x_2, x_3)$$

where p_k is homogeneous of degree $l - k$. Thus,

$$\Delta p = \sum_{k=0}^{l-2} \frac{x_1^k}{k!} p_{k+2} + \sum_{k=0}^{l} \frac{x_1^k}{k!} \left(\frac{\partial^2 p_k}{\partial x_2^2} + \frac{\partial^2 p_k}{\partial x_3^2} \right)$$

And so $\Delta p = 0$ iff

$$p_{k+2} = -\left(\frac{\partial^2 p_k}{\partial x_2^2} + \frac{\partial^2 p_k}{\partial x_3^2} \right) \quad 0 \leqslant k \leqslant l - 2$$

Thus elements of $\mathcal{H}_l(3)$ are uniquely determined by p_0 and p_1. This implies:

$$Dim(\mathcal{H}_l(3)) = Dim(\mathcal{P}_l(2)) + Dim(\mathcal{P}_{l-1}(2)) = (l + 1) + l = 2l + 1$$

Frame Title

Lemma: The action of the laplace operator on the space of C^∞ functions from $\mathbb{R}^3 \to \mathbb{C}$ commutes with the action of SO(3).
Corollary: $\mathcal{H}_l(3)$ is an SO(3) invariant subspace of $\mathcal{P}_l(3)$.
Now, since $\text{Dim}(\mathcal{H}_l(3))) = 2l + 1 = \text{Dim}(W_l)$. It is natural to ask whether or not is is also an irreducible representation.
Proposition: The space $\mathcal{H}_l(3)$ of harmonic polynomials of degree l is an irreducible SO(3) module.
Proof: We in fact show that $\mathcal{H}_l(3) \cong W_l$. Suppose that we have a decomposition of $\mathcal{H}_l(3)$ into irreducible modules as follows:

$$\mathcal{H}_l(3) \cong \oplus_v W_{n_v}$$

Then, we show that $\exists n_v \geqslant l$.

Let $T \subset SO(3)$ be the subset of real rotation matrices defined as $R(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & cos(2t) & -sin(2t) \\ 0 & sin(2t) & cos(2t) \end{pmatrix}$. We have already computed the character value of W_n, at R(t), and thus we know that the character value of $\mathcal{H}_l(3)$ at R(t) is a linear combination of e^{itk} for $|k| \leqslant \max\{n_v\}$ So, we need to find a T invariant subspace of $\mathcal{H}_l(3)$ on which R(t) acts via multiplication by $e^{\pm itk}$. Simply consider $f_l(x_1, x_2, x_3) = (x_2 + ix_3)^l \in \mathcal{H}_l(3)$. We get that

$$\begin{aligned} R(t)f_l(x_1, x_2, x_3) &= (x_2 cos(t) + x_3 sin(t) + i(x_2(-sin(t)) + x_3 cos(t))^l \\ &= e^{-ilt} f_l(x_1, x_2, x_3) \end{aligned}$$

And we are done.
Now, all that is left to remark is that every polynomial $f \in \mathcal{H}_l(3)$ is uniquely determined by its restriction on the Riemann sphere S^2, and so, $\mathcal{H}_l(S^2)$, the spherical harmonics, determine every representation of SO(3).