Numerical Solutions & Analysis for PDEs&ODEs

Zihang Zhang (ZJU), Lizhang Chen(BJTU)

Faculty Advisor: Dr. Zhilin Li (NCSU)

Abstract

❑ **Infinite domain problems** can be applied broadly and effectively in our daily life such as temperature distribution in physical phenomena and option pricing in financial mathematics.

❑ **Sturm–Liouville theory** is the theory of real second-order linear ordinary differential equations of a certain form. Sturm–Liouville equations have been widely researched and its numerical solutions can have good properties.

❑ **Partial Differential Equations (PDEs)** can be applied broadly and effectively in our daily life such as temperature distribution in physical phenomena and option pricing in financial mathematics.

❑ **The heat equation is a parabolic PDE** that describes the distribution of heat (or variation in temperature) in a given region over time.

Objectives & Applications

❑ Validate some finite difference methods to solve one dimensional Stum-Liouville problems and use them to solve one dimensiional finite domain problems.

❑ Transform a classical infinite domain ordinary differencial equations into a Sturm-Liouville problem, generate a method for numerical solutions, analyze and minimize the error.

❑ Develop and validate some finite difference methods to solve parabolic PDEs such as the heat equation numerically. And we build a model to solve the special clothing design for high temperature work.

Models

❑ **Infinite domain problem in 1D**

$$u''(x) = f(x), a < x < +\infty$$

❑ **Boundary conditons**

$$u(a) = ua, \lim_{x\to\infty} u(x) = u\inf$$

❑ **Sturm-Liouville problem**

$$\frac{d}{dx}\left(p(x)\frac{du(x)}{dx}\right) + q(x)u(x) = f(x)$$

❑ **Boundary conditons**

$$u(a) = \alpha, u(b) = \beta$$

❑ **Heat Equation in 1D**

$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial t^2}, a < x < b, 0 < t$$

❑ **Boundary conditions**

$$u(a,t) = g_1(t), u(b,t) = g_2(t)$$

❑ **Initial Condition:**

$$u(x,0) = u_0(x)$$

Methods

❑ **Finite difference method**

$$\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2} = f(x_i)$$

,where $u_{i+1} = u_i + h$

Applying Taylor expansion, we can see the pattern is second order convergent.

❑ **Splitting method**

Split the interval $[a, +\infty)$ into two intervals:

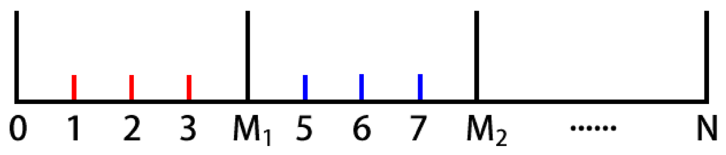
$[a, b]$ and $[b, +\infty)$, and in the second interval,

Let $t = \frac{1}{x}$, then the original equation $u''(x) = f(x), a < x < +\infty$

transforms to $$\frac{d^2u(t)}{dt^2} + \frac{2}{t}\frac{du(t)}{dt} = \frac{f(\frac{1}{t})}{t^4}$$

,which is a transformation of a Sturm-Liouville equation having boundary $u(0) = \lim_{x\to\infty} u(x) = u\inf$

❑ **Finite-Difference Approximations to the Heat Conduction Equation**



$$A_1 u_{0,j+1} + B_1 u_{1,j+1} + C_1 u_{2,j+1} = v'_{1,j}$$
$$\cdots = \cdots$$
$$A_{M_1-1} u_{M_1-2,j+1} + B_{M_1-1} u_{M_1-1,j+1} + C_{M_1-1} u_{M_1,j+1} = v_{M_1-1,j}$$
$$A_{M_1} u_{M_1-1,j+1} + B_{M_1} u_{M_1,j+1} + C_{M_1} u_{M_1+1,j+1} = v_{M_1,j}$$
$$A_{M_1+1} u_{M_1,j+1} + B_{M_1+1} u_{M_1+1,j+1} + C_{M_1+1} u_{M_1+2,j+1} = v_{M_1+1,j}$$
$$\cdots = \cdots$$
$$A_{M_2} u_{M_2-1,j+1} + B_{M_2} u_{M_2,j+1} + C_{M_2} u_{M_2+1,j+1} = v_{M_2,j}$$
$$\cdots = \cdots$$
$$A_N u_{N-1,j+1} + B_N u_{N,j+1} + C_N u_{N+1,j+1} = v'_{N,j}$$

$$\begin{bmatrix} B_1 & C_1 & & & & & & & \\ A_2 & B_2 & C_2 & & & & & & \\ & A_3 & B_3 & C_3 & & & & & \\ & & A_{M_1} & B_{M_1} & C_{M_1} & & & & \\ & & & \ddots & \ddots & \ddots & & & \\ & & & & A_{M_2} & B_{M_2} & C_{M_2} & & \\ & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & A_N & B_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_{M_1} \\ \vdots \\ u_{M_2} \\ \vdots \\ u_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_M \\ \vdots \\ v_{M_2} \\ \vdots \\ v_N \end{bmatrix}$$

Results

❑ **Infinite domain problem**

Splitting method

Solve the equation for

$$u(x) = 1 + e^{-x}$$

Boundaries :

$$u(1) = 1 + 1/e, \lim_{x\to\infty} u = 1$$

Choose b=10

Switch n1,n2

Plot infinite norm of error Against the maximum of h.

The plot proves the error converges by second order.

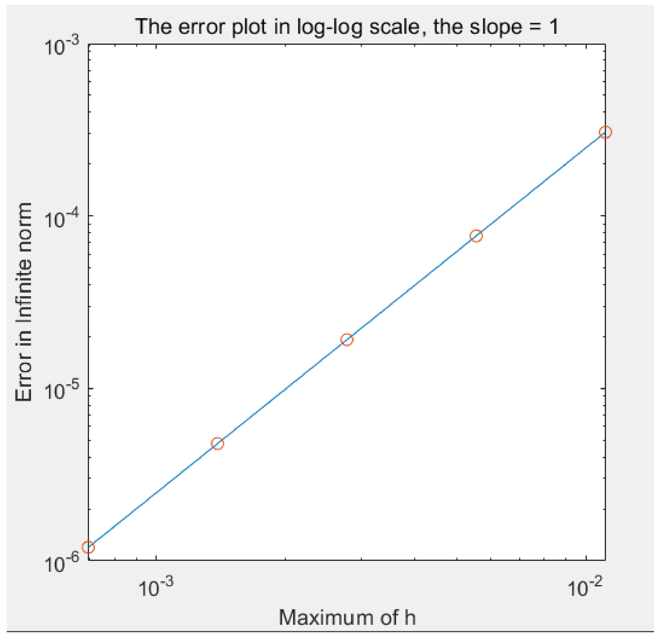


Figure 1. Error plot in log-log scale

Results

❑ **Error analysis**

Three parts of different truncation error

Interval $[a,b]$: $e_1 = \frac{1}{12}u''(x)h_1^2$

Interval $[b,+\infty)$: $e_2 = \frac{1}{12}u''''(t)h_2^2 + \frac{1}{3t}u'''(t)h_2^2$

Point b : $e3 = -\frac{1}{6}\alpha_1 u'''(b)h_1^3 + \frac{1}{6}\alpha_3 u'''(b)\left(\frac{b^2h_2}{1-bh_2}\right)^2$

Compare all the three errors, we prove in theory the error converges by second order. Switch b to see when the error is the smallest. Fix n1=100, n2=50, a=1, Switch b between 0 and 100, We see at point b=4.49 the error reaches its bottom.

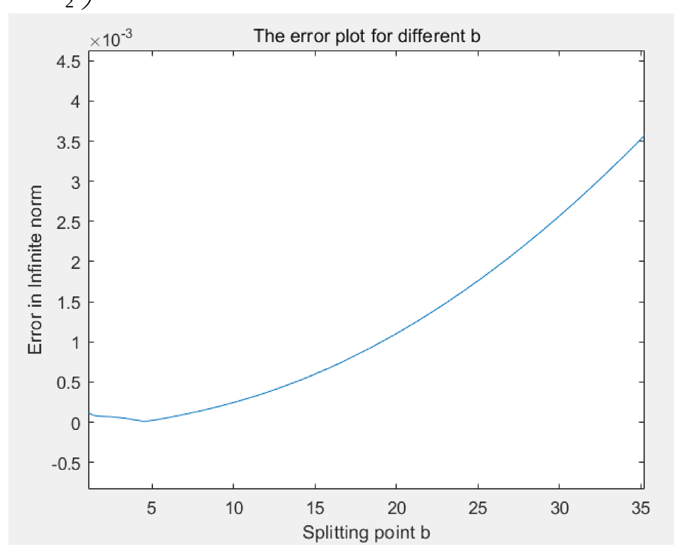


Figure 2. Error plot for different b

❑ **The numerical solution of heat equation**

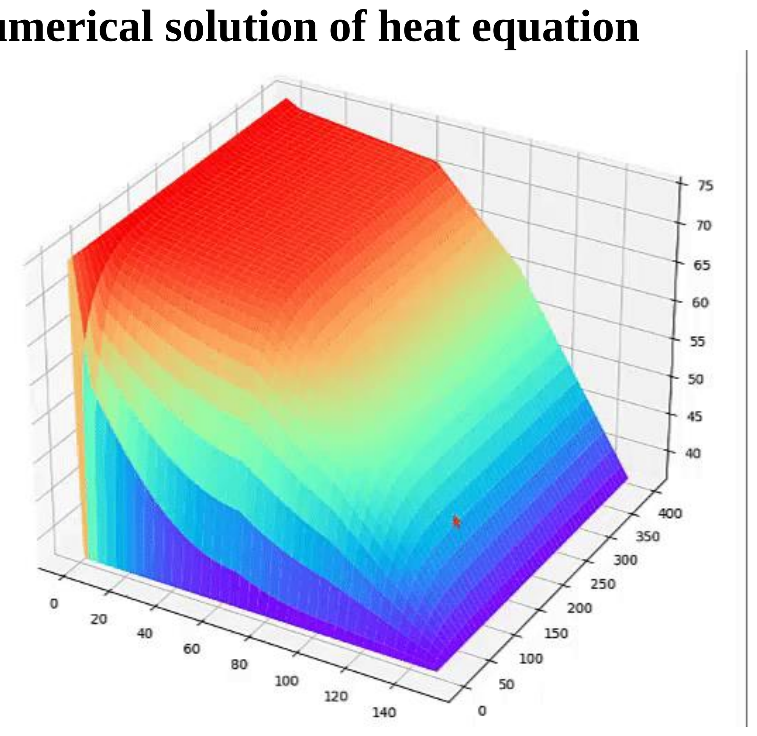


Figure 3. Temperature changes with time goes

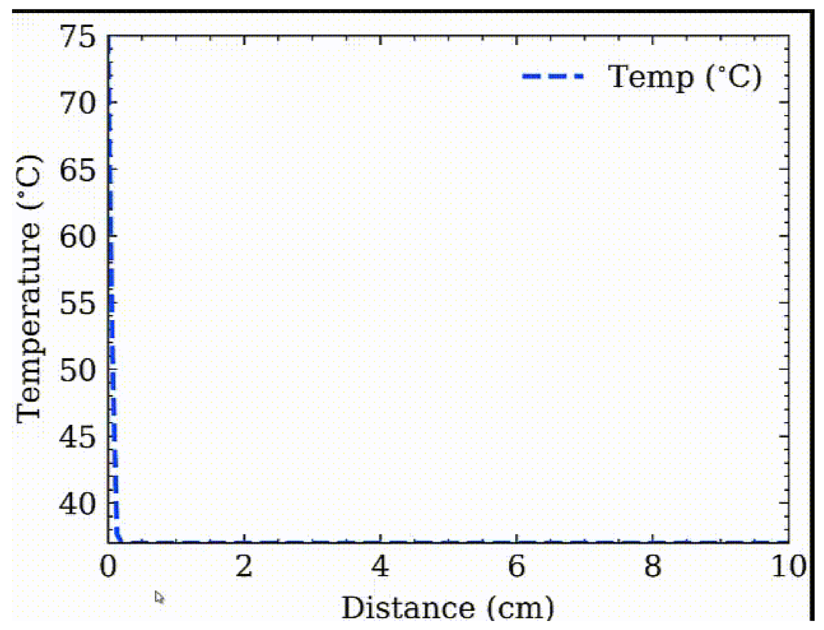


Figure 4. Temperature distribution with respect to time and space

Future Work

❑ Carry out furthur and deeper error analysis and work out a general code to generate the numerical solution for the infinite domain problem which minimize the overall error.

❑ Extend the splitting method further into infinite domain partial differential equations and utilize polar coordinates to generate a solution.

❑ Combine our mathematical learning with financial problems like put option with infinite domain.

Contact

[Name] Zihang Zhang
[Organization] School of Mathematics, Zhejiang University
[Address] 2 Huaifang North Road, Fengtai District, Beijing
[Email] zzhleon@163.com
[Phone] 086-17787248696

[Name] Lizhang Chen
[Organization] School of Science, Beijing Jiaotong University
[Address] No.3 Shangyuancun, Haidian District,Beijing 100044 P. R. China
[Email] lz.chen0807@gmail.com
[Phone] 086-18310588966

References

Li, Z., Qiao, Z., & Tang, T. (2018). *Numerical solution of differential equations: Introduction to finite difference and finite element methods.* Cambridge, UK: Cambridge University Press.