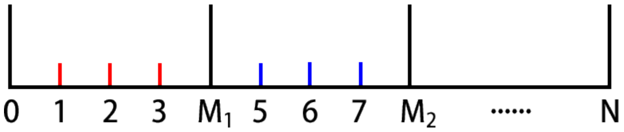
Finite-Difference Approximations to the Heat Conduction Equation



$$u_t = a^2 u_{xx} \tag{1}$$

$$u_{xx} = \frac{u(x-h) - 2u(x) + u(x+h)}{h^2} \equiv h^{-2}\left(u_{n-1} - 2u_n + u_{n+1}\right) \tag{2}$$

$$\frac{u(t+\Delta t) - u(t)}{\Delta t} = \frac{a^2}{2}\left[u_{xx}(t+\Delta t) + u_{xx}(t)\right] \tag{3}$$

$$\frac{u_{j+1} - u_j}{\Delta t} = \frac{a^2}{2}\left(u_{xx,j+1} + u_{xx,j}\right) \tag{4}$$

$$u_{j+1} - \frac{a^2 \Delta t}{2} u_{xx,j+1} = u_j + \frac{a^2 \Delta t}{2} u_{xx,j} \tag{5}$$

$$u_{n,j+1} - \frac{a^2 \Delta t}{2h^2}\left[u_{n-1,j+1} - 2u_{n,j+1} + u_{n+1,j+1}\right] = u_n(t) + \frac{a^2 \Delta t}{2h^2}\left[u_{n-1,j} - 2u_{n,j} + u_{n+1,j}\right] \tag{6}$$

$$A_n u_{n-1,j+1} + B_n u_{n,j+1} + C_n u_{n+1,j+1} = D_n u_{n-1,j} + E_n u_{n,j} + F_n u_{n+1,j} \tag{7}$$

where

$$\begin{aligned} A_n &= C_n \equiv -\frac{a^2 \Delta t}{2h^2} \\ B_n &\equiv 1 + \frac{a^2 \Delta t}{h^2} \\ D_n &= F_n \equiv \frac{a^2 \Delta t}{2h^2} \\ E_n &\equiv 1 - \frac{a^2 \Delta t}{h^2} \end{aligned} \tag{8}$$

Define

$$D_n u_{n-1,j} + E_n u_{n,j} + F_n u_{n+1,j} \equiv v_{n,j} \tag{9}$$

Then Eq. (7) becomes

$$A_n u_{n-1,j+1} + B_n u_{n,j+1} + C_n u_{n+1,j+1} = v_{n,j} \tag{10}$$

Consider the boundary condition

$$\frac{\kappa_1}{h}\left(u_{M-1,j+1} - u_{M,j+1}\right) - \frac{\kappa_2}{h}\left(u_{M,j+1} - u_{M+1,j+1}\right) = 0 \tag{11}$$

$$\frac{\kappa_1}{h}u_{M-1,j+1}-\frac{\left(\kappa_1+\kappa_2\right)}{h}u_{M,j+1}+\frac{\kappa_2}{h}u_{M+1,j+1}=0 \tag{12}$$

Define

$$\begin{aligned}
A_M &\equiv \frac{\kappa_1}{h} \\
B_M &\equiv -\frac{\left(\kappa_1+\kappa_2\right)}{h} \\
C_M &\equiv \frac{\kappa_2}{h} \\
v_{M,j} &\equiv 0
\end{aligned} \tag{13}$$

Then eq. (12) becomes

$$A_M u_{M-1,j+1}+B_M u_{M,j+1}+C_M u_{M+1,j+1}=v_{M,j} \tag{14}$$

For the whole equation set,

$$\begin{aligned}
A_1 u_{0,j+1}+B_1 u_{1,j+1}+C_1 u_{2,j+1} &= v'_{1,j} \\
\cdots &= \cdots \\
A_{M_1-1}u_{M_1-2,j+1}+B_{M_1-1}u_{M_1-1,j+1}+C_{M_1-1}u_{M_1,j+1} &= v_{M_1-1,j} \\
A_{M_1}u_{M_1-1,j+1}+B_{M_1}u_{M_1,j+1}+C_{M_1}u_{M_1+1,j+1} &= v_{M_1,j} \\
A_{M_1+1}u_{M_1,j+1}+B_{M_1+1}u_{M_1+1,j+1}+C_{M_1+1}u_{M_1+2,j+1} &= v_{M_1+1,j} \\
\cdots &= \cdots \\
A_{M_2}u_{M_2-1,j+1}+B_{M_2}u_{M_2,j+1}+C_{M_2}u_{M_2+1,j+1} &= v_{M_2,j} \\
\cdots &= \cdots \\
A_N u_{N-1,j+1}+B_N u_{N,j+1}+C_N u_{N+1,j+1} &= v'_{N,j}
\end{aligned} \tag{15}$$

Define

$$\begin{aligned}
v_{1,j} &\equiv v'_{1,j}-A_1 u_{0,j+1} \\
v_{N,j} &\equiv v'_{N,j}-C_N u_{N+1,j+1}
\end{aligned} \tag{16}$$

Then eq. (15) can be rewritten as a matrix equation

$$\begin{bmatrix}
B_1 & C_1 & & & & & & & \\
A_2 & B_2 & C_2 & & & & & & \\
 & A_3 & B_3 & C_3 & & & & & \\
 & & A_{M_1} & B_{M_1} & C_{M_1} & & & & \\
 & & & \ddots & \ddots & \ddots & & & \\
 & & & & A_{M_2} & B_{M_2} & C_{M_2} & & \\
 & & & & & \ddots & \ddots & \ddots & \\
 & & & & & & A_N & B_N &
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_{M_1} \\ \vdots \\ u_{M_2} \\ \vdots \\ u_N \end{bmatrix}
=
\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_M \\ \vdots \\ v_{M_2} \\ \vdots \\ v_N \end{bmatrix} \tag{17}$$

The steps to solve the tridiagonal matrix

$$\begin{bmatrix} B_1 & C_1 & & & \\ A_2 & B_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & A_N & B_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix} \tag{18}$$

Step 1:

$$\begin{bmatrix} B_1 \frac{A_2}{B_1} & C_1 \frac{A_2}{B_1} & & & \\ A_2 & B_2 & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & A_N & B_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} \frac{A_2}{B_1} v_1 \\ v_2 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix} \tag{19}$$

Step 2:

$$\begin{bmatrix} B_1 \frac{A_2}{B_1} & C_1 \frac{A_2}{B_1} & & & \\ 0 & B_2 - C_1 \frac{A_2}{B_1} & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & A_N & B_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} \frac{A_2}{B_1} v_1 \\ v_2 - \frac{A_2}{B_1} v_1 \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix} \tag{20}$$

Step 3:

$$\begin{bmatrix} B_1 & C_1 & & & \\ 0 & B_2' & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & A_{N-1} & B_{N-1} & C_{N-1} \\ & & & A_N & B_N \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2' \\ \vdots \\ v_{N-1} \\ v_N \end{bmatrix} \tag{21}$$

Step 4: Repeat step 1 to step 3 to the *n*th row

$$\begin{bmatrix} B_1 & C_1 & & & \\ 0 & B_2' & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & B_{N-1}' & C_{N-1} \\ & & & 0 & B_N' \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2' \\ \vdots \\ v_{N-1}' \\ v_N' \end{bmatrix} \tag{22}$$

Step 5:

$$\begin{bmatrix} B_1 & C_1 & & & \\ 0 & B_2' & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & B_{N-1}' & C_{N-1} \\ & & & 0 & C_{N-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2' \\ \vdots \\ v_{N-1}' \\ \dfrac{C_{N-1}}{B_N'} v_N' \end{bmatrix} \tag{23}$$

Step 6:

$$\begin{bmatrix} B_1 & C_1 & & & \\ 0 & B_2' & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \\ & & & 0 & C_{N-1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2' \\ \vdots \\ \dfrac{1}{B_{N-1}'}\left(v_{N-1}' - \dfrac{C_{N-1}}{B_N'} v_N' \right) \\ \dfrac{C_{N-1}}{B_N'} v_N' \end{bmatrix} \tag{24}$$

Step 7:

$$\begin{bmatrix} B_1 & C_1 & & & \\ 0 & B_2' & C_2 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \\ & & & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2' \\ \vdots \\ v_{N-1}'' \\ v_{N-1}'' \end{bmatrix} \tag{25}$$

Step 8: Repeat step 5 to step 8 to the first row

$$\begin{bmatrix} 1 & 0 & & & \\ 0 & 1 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 1 & 0 \\ & & & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix} = \begin{bmatrix} v_1'' \\ v_2'' \\ \vdots \\ v_{N-1}'' \\ v_{N-1}'' \end{bmatrix} \tag{26}$$

Then we get the solution of equation set

$$u_n = v_n'', \qquad n = 1, 2, \cdots, N \tag{27}$$